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# Accurate evaluation of pionium wave functions

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A suitable numerical approach based on Sturmian functions is employed to solve the pionium problem for both local and nonlocal potentials. The approach accounts for both the short-ranged strong interaction and the long-ranged Coulomb force and provides accurately the wave function and binding energy of pionium. It is found that the ground-state pionium wave function in realistic pion-pion strong interactions might be considerably different from the hydrogen-like one at a small distance.

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Pionium is a hadronic atom of $\pi^+$ and $\pi^-$ mixed with a small part of the $\pi^0\pi^0$ component. It is bound mainly by the Coulomb force, but affected by strong interaction at a small distance. Pionium decays predominantly into $\pi^0\pi^0$ via a strong interaction, which probes the low-energy interactions of the pions. It has been believed that pionium might be employed to test more accurately the predictions of chiral perturbation theory, which is an effective theory of QCD at law energies. The investigation of pionium has recently become of particular interest due to the pionium experiment DIRAC that aims to measure the pionium lifetime with 10% accuracy [1,2].

In order to be able to extract precise pion-pion scattering length data, the relation between lifetime and scattering lengths has to be known reliably and with high accuracy. The nonrelativistic formula of the pionium lifetime in the lowest order of electromagnetic interactions was first evaluated by Deser *et al.* [3] and later reanalyzed by others [4]. It reads as

$$\Gamma_0 = \frac{2}{9}\frac{64\pi p}{M^3}|\psi(0)|^2|a_0 - a_2|^2, \tag{1}$$

where $M$ is the mass of the $\pi\pi$ system, $p$ is the center-of-mass momentum of the $\pi^0$ in the pionium system, $\psi(0)$ is the pionium S-wave function at the origin ($r=0$), and $a_0$ and $a_2$ are the S-wave $\pi\pi$ scattering lengths for isospin $I=0$ and 2, respectively. In the approximation of the pionium wave function $\psi$ in Eq. (1) to the hydrogen-like wave function, one derives the chiral perturbation result at leading order [4],

$$\Gamma = \frac{2}{9}\alpha^3 p|a_0 - a_2|^2, \tag{2}$$

where $\alpha$ is the fine structure constant.

An evaluation of the relativistic, strong interaction, and higher-order electromagnetic corrections to the nonrelativistic formula in Eq. (1) has recently been done in the frameworks of quantum field theory and chiral perturbation theory. These works have led to similar estimates, of the order of 6%, of these corrections [5–8]. The strong interaction of the pion-pion system with respect to the bound state wave equation has been treated perturbatively though the methods employed in these works are quite different.

However, it is arguable to treat the pion-pion strong interaction as a small perturbation as well as to approximate in the lowest order the pionium wave function to the hydrogen-like one at a small distance. In the work we tackle this issue by evaluating accurately the pionium wave functions in realistic pion-pion strong interactions.

A correct treatment of pionium must include the coupling of the $\pi^+\pi^-$ and $\pi^0\pi^0$ configurations. The dynamical equations of the ($\pi^+\pi^-$, $\pi^0\pi^0$) system take the form

$$(E - H^0_{\pi^+\pi^-})\psi_{\pi^+\pi^-} = (V_c + V_{11})\psi_{\pi^+\pi^-} + V_{12}\psi_{\pi^0\pi^0}, \tag{3}$$

$$(E - H^0_{\pi^0\pi^0})\psi_{\pi^0\pi^0} = V_{22}\psi_{\pi^0\pi^0} + V_{21}\psi_{\pi^+\pi^-}, \tag{4}$$

where $E$ is the energy of the ($\pi^+\pi^-$, $\pi^0\pi^0$) system, $H^0_{\pi^+\pi^-}$ and $H^0_{\pi^0\pi^0}$ are, respectively, the free energies of the $\pi^\pm$ and $\pi^0$, $V_C$ is the Coulomb interaction between $\pi^+$ and $\pi^-$, and $V_{ij}$ are the strong interactions of the system. The strong interactions take, for example, for pionium in S waves the form in the isospin basis,

$$V = \begin{pmatrix} \frac{2}{3}V^0 + \frac{1}{3}V^2 & \frac{\sqrt{2}}{3}(V^2 - V^0) \\ \frac{\sqrt{2}}{3}(V^2 - V^0) & \frac{1}{3}V^0 + \frac{2}{3}V^2 \end{pmatrix}, \tag{5}$$

where $V^0$ and $V^2$ are, respectively, the isospin 0 and 2 strong interactions of the $\pi\pi$ system. In principle, one could solve Eq. (5) through expanding the pionium wave functions $\psi_{\pi^+\pi^-}$ and $\psi_{\pi^0\pi^0}$ in any complete set of orthonormal functions. The complete set of harmonic oscillator wave functions is widely applied to bound state problems since they have analytical froms both in coordinate and momentum spaces. Bound state problems with only the strong interaction or only the Coulomb force can be well solved in the regime of harmonic oscillator wave functions, by choosing the oscillator length being of order 1 fm or 100 fm, respectively. However, the harmonic oscillator wave function approach fails to describe hadronic atoms that are dominated by the long-ranged Coulomb force and influenced by the short-ranged strong interaction. The reason is that two very different oscillator lengths are involved to account for the short-ranged strong interaction and the long-ranged Coulomb force.

 

The pionium problem is more difficult than the more popular protonium problem in term of evaluating the wave functions since the Bohr radius of pionium is much larger than the one of protonium. The protonium has been successfully investigated [9] in a numerical approach based on Sturmian functions [10]. The numerical method is much more powerful, accurate and much easier to use than all other methods applied to this problem in history. It can be applied to solve the $\bar{N}N$ bound state problem for local and nonlocal potentials, accounting for both the strong short-range nuclear interaction and the long-range Coulomb force and provides directly the wave function and binding energy of protonium and $\bar{N}N$ deep bound states [9].

For future work the so-called inverse iteration method [11] might be employed to study the pionium problem. The method has been proven economic and powerful in studying other eigenvalue problems.

Because almost all bound-state hydrogentic wave functions are close to zero energy, the innermost zero of the functions are insensitive to the principal quantum number. This accounts for the fact that the bound hydrogen functions do not form a complete set; the continuum is needed to analyze the region between the origin and the limiting first zero. Unlike hydrogen functions, the first node of the Sturmian functions continues to move closer to the origin with increasing the principal number $n$. This is the key point why a short-ranged nuclear force can easily be taken into account for hadronic atoms by using complete sets of the Sturmian functions.

In this work we tackle the pionium problem in the Sturmian function method in two model interactions.

Model A: The interactions of the $\pi\pi$ system are derived in the meson-exchange model [12]. The low and medium energy data of the pion-pion and pion-Kaon scattering processes are well reproduced in the model. The interaction is in momentum space, energy dependent, and nonlocal.

Model B: The potentials $V^I$ of the $\pi\pi$ system are the ones employed for calculating the electromagnetic corrections in low-energy pion-nucleon scattering [13,14] and for studying the influence of the hadronic interaction on pionium wave functions [15]. The potentials are independent of both the energy $E$ of the system and pion masses, and reproduce very well the phase shifts given by two-loop chiral perturbation theory [16].

To guarantee the accuracy of numerical method employed here, we have followed in our numerical calculations the procedures as follows.

(1) Employ a hadronic potential to the pionium problem. The interaction should be strong enough to provide at least one deep bound state for the $\pi\pi$ system. Solve the pionium problem in any numerical method (for example expanding the pionium wave function in the complete basis of the harmonic oscillator functions or Sturmian functions) to obtain the binding energy and wave function of the deep bound states of the $\pi\pi$ system.

(2) Solve the pure Coulomb pionium problem in the complete basis of Sturmian functions, with the length parameter $b$ as large as possible, to numerically reproduce the analytical hydrogen-like wave functions in high accuracy.

(3) Solve the pionium problem in (1) in the complete basis of Sturmian functions employed in (2). The binding energy and wave function in (1) must be accurately reproduced in the present basis, or one has to employ a larger basis with larger $b$ in (2).

A basis worked out by following the steps mentioned above enables one to accurately evaluate the binding energy and wave function of pionium. In the work of [9], the Sturmian function method has been compared with the traditionally used method, namely the Numerov approach [17], applied to the $\bar{N}N$ atomic problem in the Kohno-Weise potential [18]. The binding energies, widths, and wave functions at short distance presented in Ref. [17] for the states $^1S_0$, $^3P_0$, $^3S_1$ and $^3SD_1$ are well reproduced in the Sturmian function approach.

For a comparison we have resolved the simplified pionium problem in Ref. [19] where the $\pi^0\pi^0$ component is ignored and the strong interaction is simply a Yakawa form. It is quite an easy job to solve such a simplified pionium problem in the Sturmian function method. The relative correction to the binding energy of the pionium ground state due to strong interaction is derived in our work as

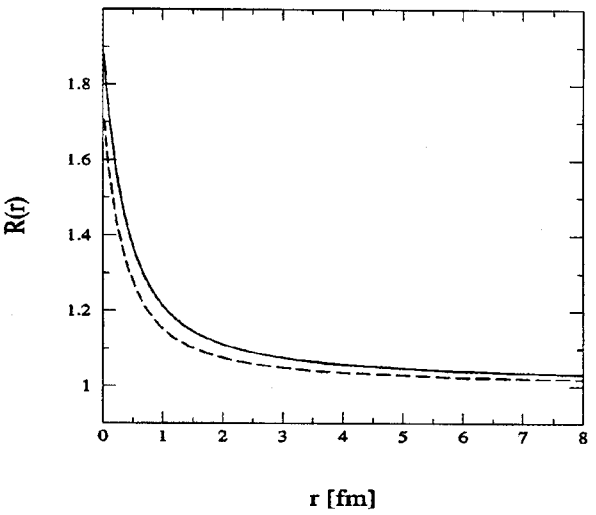


FIG. 1. Ratios of the 1 $s$ pionium state wave to the 1 $s$ hydrogen-like wave function in our calculation (solid line) and the work [19] (long-dashed line).

TABLE I. Energy shifts of the 1 $s$ pionium state, compared to the pure Coulomb interaction level. A negative $\Delta E$ means the energy level is pulled down by the strong interaction.

| | Model A | Model B (Case I) | Model B (Case II) | Model B (Case III) |
|---|---|---|---|---|
| $\Delta E$ (eV) | −0.1455 | −3.8872 | −3.0718 | −3.2484 |

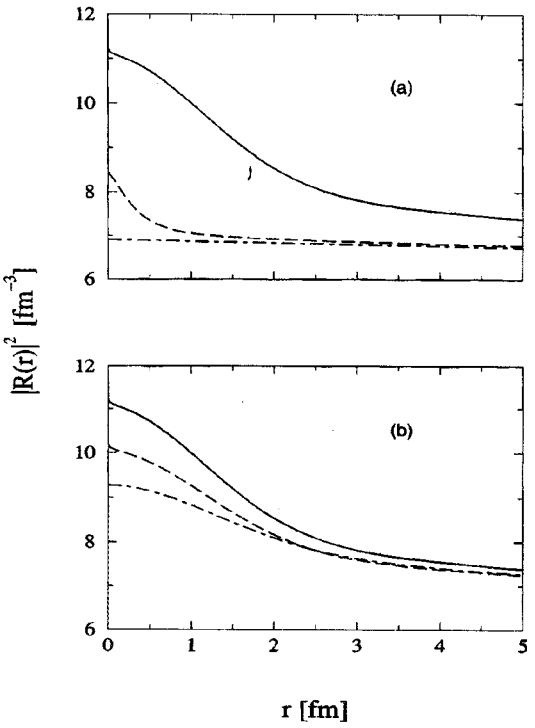


FIG. 2. Squared 1 $s$ radial wave functions for $\pi^+\pi^-$ component of the pionium, long-dashed line in (a) for Model A, solid lines in both (a) and (b) for the first case of Model B (relativistic, with $\pi^0\pi^0$ configuration), long-dashed line in (b) for the second case of Model B (relativistic, without $\pi^0\pi^0$ configuration), and the dashed-dotted line in (b) for the third case of Model B (nonrelativistic, with a $\pi^0\pi^0$ configuration). The pure Coulomb interaction wave function is plotted as the dashed-dotted line in (a). All the wave functions have been multiplied by a factor $10^8$.

$$\eta = \frac{E - E_0}{E_0} = 0.0021543718, \tag{6}$$

where $E$ and $E_0$ are, respectively, the ground state binding energies of the pionium and the $\pi\pi$ system with only the Coulomb interaction. The pionium 1 $s$ energy level is slightly shifted lower compared to the pure Coulomb interaction. The approach employed here is so powerful that the binding energy of the pionium ground state can be evaluated to an accuracy of better than $10^{-8}$. The relative correction derived in the work [19] is $\eta = 0.0020256$ [20]. We would say that the method employed in the work [19] needs to be improved since an accuracy of $10^{-4}$ guaranteed in [19] is not good enough for the pionium problem. But, as realized by the authors [19], the calculation of the pionium wave functions is not a simple problem. Shown in Fig. 1 are the ratios $\psi_{\pi^+\pi^-}(r)/\psi_c(r)$, derived, respectively, in our calculation and the work of [19], of the 1 $s$ pionium state wave function to the 1 $s$ hydrogen-like wave function. The pionium ground state wave function in our work is a little bit bigger than that in the work [19] at short distances, which is consistent with that our binding energy of the pionium ground state is larger.

Listed in Table I are the energy shifts of the pionium ground state, due to the strong interactions in both Model A and Model B, form the pure Coulomb-interaction $\pi\pi$ system. The accuracy of the numerical evaluation here is better than $10^{-8}$. The energy level is pulled lower, hence the strong interactions are attractive for the $\pi\pi$ system in both Model A and Model B. However, the hadronic interactions are not strong enough to form a deep bound state. This may mislead one to think that the wave function of the $\pi\pi$ system without a deep bound state can be well approximated with the hydrogen-like wave function.

To show the relativistic effect on the system and the importance of the coupling of the $\pi^+\pi^-$ and $\pi^0\pi^0$ configurations, we have considered three cases in Model B, that is, treating the system relativistically (the relativistic Schrödinger equation is solved), with and without including the coupling of the $\pi^+\pi^-$ and $\pi^0\pi^0$ configurations; and treating the system nonrelativistically (the normal Schrödinger equation is solved), with including the coupling of the $\pi^+\pi^-$ and $\pi^0\pi^0$ configurations. It is found in Table I that both the relativistic effect and the coupling of the $\pi^+\pi^-$ and $\pi^0\pi^0$ configurations are not negligible. This is also shown by the wave functions below. For Model A we consider only the relativistic case with the $\pi^0\pi^0$ coupling included since the effective potential derived in the meson-exchange theory is relativistic.

Shown in Fig. 2 are the 1 $s$ radial wave functions for the $\pi^+\pi^-$ component of pionium in both Model A and Model B. For a comparison the 1 $s$ hydrogen-like wave function is also plotted in Fig. 2(a). In Fig. 2(b) are the wave functions derived in the three cases of Model B. It is clear that the ground state pionium wave functions are considerably different from the hydrogen-like one at small distances in any model employed here. One may derive from Fig. 2(a) the ratio of the squared pionium wave function to the squared hydrogen-like wave function at the origin ($r=0$),

$$\frac{|\psi_{\pi^+\pi^-}(0)|^2}{|\psi_c(0)|^2} = \begin{cases} 1.22, & \text{Model A;} \\ 1.62, & \text{Model B.} \end{cases} \tag{7}$$

The interactions in Model A and Model B are realistic, the low-energy data of the pion-pion scattering processes are well reproduced in these models. Therefore, the results here are around the truth.

We conclude that it might not be reasonable to treat the pion-pion strong interaction as a small perturbation as well as to approximate in the lowest order the pionium wave function to the hydrogen-like one at a small distance.

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