SYMMETRY AND MULTIPLE WAVES IN GAS DYNAMICS

S.V.Meleshko[1]

1 Introduction

Modern technology requires a deeper knowledge of the behavior of real physical phenomena. Today the main way of studying physical processes and obtaining new knowledge is by mathematical modelling: using efficient mathematical modelling allows us to reduce the time used in investigation and obtaining of new results.

Mathematical models of real world phenomenon are formulated as algebraic, partial differential or integral equations (or a combination of them). These equations are constructed on the basis of our knowledge of physical phenomena. After the construction of equations the study of their properties is necessary. At this stage symmetry analysis plays a significant role. In parallel with group analysis a method known as a method of degenerate hodograph was developed. Firstly degenerate hodograph solutions were applied to problems of gas dynamics. Examples of this include as the problem of gas motions behind two semiinfinite pistons which move at an angle to each other in a plane; flows in two–sided angles; gas effusion into a vacuum on a tilt wall; and the cumulation of energy under nonshock compression of gas.

Here we discuss multiple waves and their connections with symmetry properties. Applications are considered in terms of gas dynamics equations [2].

2 Gas dynamics equations

In a three–dimensional isentropic case the gas dynamics equations are

$$\frac{du_i}{dt} + \frac{\partial\theta}{\partial x_i} = 0, \quad (i = 1, 2, 3), \tag{1}$$
$$\frac{d\theta}{dt} + \kappa\theta \mathrm{div}\ u = 0.$$

Here (u_1, u_2, u_3) is a velocity vector, $\theta = c^2/\kappa, \kappa = \gamma - 1$, $c = A(S_0)\rho^{\frac{\gamma-1}{2}}$ is a sound speed, γ is a polytropic gas exponent and $d/dt \equiv \partial/\partial t + u_\alpha \partial/\partial x_\alpha$ (summation with respect to repeat Greece index is assumed).

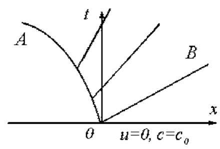

Fig.1.

[1]Suranaree University of Technology, Thailand

[2]Review of works in this area done before 1983 can be found in [1]. Here only references having direct relation to the discussing subject are given.

Let us consider an infinitely long tube extending along the x–axis. We assume that gas fills the half tube $x > 0$ with a piston situated at $x = 0$, and that the gas is in motionless state with a constant density ρ_0. At $t = 0$, we start pulling the piston to the left so that the piston follows a path $x = x(t)$ for $t > 0$ in the x, t plane (fig. 1). As a result the gas is set into motion.

The solution of this problem can be considered as the continuous one–dimensional motion of gas. In this case gas dynamics equations can be reduced to the equations written in the Riemann invariants

$$\frac{\partial r}{\partial t} + (u + c)\frac{\partial r}{\partial x} = 0, \quad \frac{\partial l}{\partial t} + (u - c)\frac{\partial l}{\partial x} = 0.$$

where $r = u - \frac{2}{\gamma-1}c$, $l = u + \frac{2}{\gamma-1}c$.

Since the gas is originally motionless with uniform density ρ_0, there is a constant state I in the x, t plane where the $\frac{dx}{dt} = u \pm c$ characteristics have a constant slope $\pm c_0 = c(\rho_0)$.

The Riemann invariants in adjoint to axis x domain are $r = -\frac{2}{\gamma-1}c_0$ and $l = +\frac{2}{\gamma-1}c_0$. Because the l–invariant is constant along characteristics $\frac{dx}{dt} = u - c$, then it is constant into the adjoining region AOB. It allows us to construct the solution of the piston problem. Application of the Riemann waves can be found in modern supersonic wind tunnels.

Definition. A solution with a constant Riemann invariant is called a simple (Riemann) wave.

Another representation for the simple wave is $l = l(r)$. A generalization of this case for three–dimensional gas dynamics equations is

$$u_i = u_i(\theta), \ (i = 1, 2, 3).$$

So, a simple wave is a solution in which all dependent functions are functions only one unknown function. A generalization from the simple wave is a multiple wave. A multiple wave is such solution in which finite relationships between dependent variables are assumed.

3 Multiple waves (definitions and basic facts)

A multiple wave of rank r in the field G of the space $x_1, x_2, \ldots, x_n$ for a homogeneous system of quasilinear differential equations

$$\sum_{\alpha=1}^{n} A_\alpha(u)\frac{\partial u}{\partial x_\alpha} = 0 \tag{2}$$

is called a solution $u^i = u^i(x_1, x_2, \ldots, x_n)$, $(i = 1, 2, \ldots, m)$ such that a rank of the Jacobi matrix on it

$$\frac{\partial(u_1, u_2, \ldots, u_m)}{\partial(x_1, x_2, \ldots, x_n)}$$

is equal to r [1]. Here A_α are $N \times m$ matrixes of elements $a^\alpha_{ij}(u)$, $(i = 1, 2, \ldots, N; j = 1, 2, \ldots, m; \alpha = 1, 2, \ldots, n)$.

Depending on the value of r a multiple wave is named simple ($r = 1$), double ($r = 2$) or triple ($r = 3$) wave. The value $r = 0$ corresponds to uniform flow with constant u_i, $(i = 1, 2, \ldots, m)$, and $r = n$ corresponds to a general case of nondegenerate solutions. Multiple waves of all ranks compose a class of degenerate hodograph solutions.

The singularity of the Jacobi matrix means that the functions $u_i(x)$ $(i = 1, 2, \dots, m)$ are functionally dependent (hodograph is degenerated), with $m-r$ number of functional constraints

$$u_i = \Phi_i(\lambda^1, \lambda^2, \dots, \lambda^r), (i = 1, 2, \dots, m). \tag{3}$$

The variables $\lambda^1(u), \lambda^2(u), \dots, \lambda^r(u)$ are called parameters of the wave. The solutions with degenerate hodograph are a generalization of the travelling waves: the wave parameters of the travelling waves are linear forms of independent variables. To find the r–multiple wave it is necessary to substitute the representation (3) into the system (2). We get an overdetermined system of differential equations for the wave parameters $\lambda^i(x)$, $(i = 1, 2, \dots, r)$ which should be studied for compatibility.

The main problem of the theory of solutions with degenerate hodograph is getting a closed system of equations in the space of dependent variables (hodograph), establishing the arbitrariness of the general solution and determining flow in the physical space.

Equations of the system (2) are not changed under transformations[3].

$$x_i' = ax_i + b_i, \quad (i = 1, 2, \dots, n), \tag{4}$$

which compose a group of transformations G^{n+1}.

From a group analysis point of view a r–multiple wave is a partially invariant solution [2] with respect to G^{n+1} . Among partially invariant solutions, those irreducible to invariant take a special place. They are connected with the problem of constructing invariant multiple waves which is easier than the problem for constructing partially invariant solutions. That is, for an invariant r–multiple wave the wave parameters can be chosen from two types up to equivalence transformations: the first kind of waves are waves with parameters

$$\lambda^1 = x_{n-r+1}, \ \lambda^2 = x_{n-r+2}, \ \dots, \ \lambda^r = x_n,$$

the second kind has wave parameters

$$\lambda^1 = x_{n-r}/x_n, \ \lambda^2 = x_{n-r+1}/x_n, \ \dots, \ \lambda^r = x_{n-1}/x_n.$$

The equivalence transformations consist of linear replacement of the independent variables $x' = Vx$ with nondegenerate $n \times n$ square matrix V. Moreover, more difficult analysis of compatibility of obtained overdetermined systems is required for partially invariant solutions than for invariant solutions. Therefore it is worthwhile to ascertain a form of an irreducible wave a priori. In common cases this problem is difficult and unsolved. Only some sufficient conditions of reducibility are known [2, 3]. The practical significance of these conditions is based on the fact that during the process of formation of compatibility conditions for the multiple wave parameters it is necessary to veto reduction.

In the studying of solutions with degenerate hodograph, the question of compatibility of overdetermined system is very important. Since a general analysis of compatibility of occurred systems is difficult, it was made under additional assumptions about multiple waves. Originally there were geometric and kinematic conditions: either potential flows [4] or rectilinearity of level lines [5]. Other confinements were constructed on the basis of the algebraic structure of the

[3]The widest admissible Lie group of the system (2) can be wider than the group G^{n+1}.

system (2), connected with so–called simple integral elements of the system (2) (see, for example, [6]).

Because in any case the analysis of compatibility of overdetermined system have to be done, then a classification of solutions according to availability of function arbitrariness in the Cauchy problem is more natural from a compatibility theory point of view. A study of solutions, having functional arbitrariness, is based on the property of compatible systems of equations that after a finite number of extensions they comes to involution. If a system of differential equations is in involution, then a functional arbitrariness of solution is determined by the Cartan characters, which are connected by definite way with higher parametric derivatives. For an existence of solutions, having functional arbitrariness, it is required that the rank of the matrix composed from coefficients of high derivatives is not equal to the number of all high derivatives (in any extension). This approach has been developed in [7, 8, 9].

4 Simple waves

According to the definition of the simple wave such solution has a representation

$$u^i = u^i(\lambda), \ (i = 1, 2, \dots, m), \tag{5}$$

where $\lambda = \lambda(x_1, x_2, \dots, x_n)$ is a wave parameter.

Substituting (5) into the original system (2), we have overdetermined homogeneous system of quasilinear differential equations for the function λ:

$$c_{i\alpha}(\lambda)\frac{\partial \lambda}{\partial x_\alpha} = 0, \ (i = 1, 2, \dots, m). \tag{6}$$

Here $c_{ik} = a^k_{i\beta} u'_\beta$ and prime means derivative with respect to the wave parameter λ.

The structure of the solution of system (6) depends on the matrix C, which is composed from the coefficients $c_{ik}(\lambda)$. Equations on function $u(\lambda)$ when there exists a nontrivial simple wave are called by equations of simple waves. The system (6) admits nontrivial solution if only if

$$r \equiv rang\ C < min(n, m).$$

In this case without a loss of generality the system (6) can be written in a form

$$\frac{\partial \lambda}{\partial x_i} = \sum_{\alpha=r+1}^{n} b_{i\alpha}(\lambda)\frac{\partial \lambda}{\partial x_\alpha}, \ (i = 1, 2, \dots, r). \tag{7}$$

The description of all solutions of (7) is in the following theorem.

Theorem. A solution of system (7) is implicitly given by formula

$$\lambda = f(x_{r+1} + \sum_{\alpha=1}^{r} x_\alpha b_{\alpha r+1}, \dots, x_n + \sum_{\alpha=1}^{r} x_\alpha b_{\alpha n}), \tag{8}$$

where f is arbitrary mapping from R^{n-r} in R.

Proof of this theorem consists of in consecutive finding of the general solution of system (7). The surfaces in R^n on which $\lambda = const$, called level surfaces, compose r–dimensional planes.

The most often encountered case in applications is $r = n - 1$. In this case the representation of the solution of the system (6) can be reduced to

$$\sum_{\alpha=1}^{n} x_\alpha \Delta_\alpha(\lambda) = F(\lambda), \tag{9}$$

where $F(\lambda)$ is an arbitrary function, $\Delta_i, (i = 1, 2, \ldots, n)$ are known $(n-1)$ – order minors of the matrix C, which is a function of λ.

Let us consider the simple wave of equations described an isentropic movement of polytropic gas (1) [10]. We can choose θ as a wave parameter. Then the equations (1) accept a form

$$u_i' \frac{d\theta}{dt} + \frac{\partial \theta}{\partial x_i} = 0, \ (i = 1, 2, 3), \tag{10}$$
$$\frac{d\theta}{dt} + \kappa\theta u_\alpha' \frac{\partial \theta}{\partial x_\alpha} = 0.$$

and the matrix C is $(x_4 \equiv t)$:

$$C = \begin{pmatrix} u_1 u_1' + 1 & u_2 u_1' & u_3 u_1' & u_1' \\ u_1 u_2' & u_2 u_2' + 1 & u_3 u_2' & u_2' \\ u_1 u_3' & u_2 u_3' & u_3 u_3' + 1 & u_3' \\ u_1 + \kappa\theta u_1' & u_2 + \kappa\theta u_2' & u_3 + \kappa\theta u_3' & 1 \end{pmatrix}.$$

For an existence of a nontrivial simple wave it is necessary to require that det $(C) = 0$, that is

$$(u_1')^2 + (u_2')^2 + (u_3')^2 = 1/(\kappa\theta).$$

Because rang $C = 3$, then the system (10) becomes

$$\frac{\partial\theta/\partial x_1}{u_1'} = \frac{\partial\theta/\partial x_2}{u_2'} = \frac{\partial\theta/\partial x_3}{u_3'} = -\frac{\partial\theta/\partial t}{(1 + u_\alpha u_\alpha')}$$

with the general integral

$$x_\alpha u_\alpha' - t(1 + u_\alpha u_\alpha') = F(\theta).$$

Changing the wave parameter θ on $\tau = 2\sqrt{\theta/\kappa}$, we get

$$u_1(\tau) = \int \sin\psi(\tau) \cos\phi(\tau)\ d\tau, \quad u_2(\tau) = \int \sin\psi(\tau) \sin\phi(\tau)\ d\tau,$$

$$u_3(\tau) = \int \cos\psi(\tau)\ d\tau,$$

where $\psi(\tau)$, $\phi(\tau)$ are arbitrary functions of τ.

Special cases of simple waves are obtained under additional assumptions. In particular, for the stationary simple wave $(\partial\theta/\partial t = 0)$ we have

$$1 + u_\alpha u_\alpha' = 0.$$

It corresponds to the Bernoulli integral

$$u_\alpha u_\alpha + 2\theta = M^2 = const.$$

And level surfaces of functions u_i and θ are planes in the space of the independent variable x_1, x_2, x_3:

$$x_\alpha u'_\alpha = F(\theta).$$

Applications of simple waves are well known in the classical gas dynamics. They are: unsteady motion of a plane piston in gas (Riemann wave) and steady flow past a smooth profile (Prandtl–Mayer wave). More general problems about flow past some surfaces by simple waves are valid. For example, steady flow past unwrapped surfaces can be described by a simple wave. Parametrically these surfaces Γ in the space $R^3(x)$ are given by

$$x_i = q_i(s) + \nu p_i(s), \ (i = 1, 2, 3),$$

where $q_i(s), p_i(s)$ are functions satisfying to equations

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ p'_1 & p'_2 & p'_3 \\ q'_1 & q'_2 & q'_3 \end{vmatrix} = 0, \ (p_1, p_2, p_3) \times (q'_1, q'_2, q'_3) \neq 0.$$

Theorem [11]. An arbitrary, sufficiently smooth expanding surface Γ, which is not a plane, can be passed over by a simple wave.

5 Double waves

For the double wave the parametric representation of solution has a form

$$u_i = u_i(\lambda, \mu), \ (i = 1, 2, \dots, m) \tag{11}$$

with wave parameters $\lambda = \lambda(x), \ \mu = \mu(x)$. In the result of substitution the representation (11) into the system (1) we get an overdetermined system

$$A_\alpha(u_\lambda \lambda_\alpha + u_\mu \mu_\alpha) = 0. \tag{12}$$

where $(\lambda_i \equiv \partial\lambda/\partial x_i, \ \mu_i \equiv \partial\mu/\partial x_i, \ i = 1, 2, \dots, n)$.

The system (12) must be studied for compatibility. In the general case, it is difficult to analyze its compatibility. As already mentioned above, because it is easier to solve the problem of compatibility for invariant double waves, therefore it is helpful to ascertain a form of double waves, which are not reducible to invariant double waves. Only sufficient conditions providing the reduction of a double wave to an invariant solution are known. These conditions can be formulated as following.

Theorem [2]. If in the homogeneous system of quasilinear equations (1), which is a result of formation of compatibility conditions, the number of the independent equations $N = 2n - 1$, then a double wave is the invariant double wave. The wave parameters of the double wave can be chosen one of two types (up to the equivalence transformations): either $\lambda = x_1, \quad \mu = x_2$; or $\lambda = x_1/x_3, \ \mu = x_2/x_3$.

We demonstrate a practical application of the theorem on the plane irrotational isentropic gas flow[4]:

$$\frac{du_i}{dt}+\frac{\partial\theta}{\partial x_i}=0,\quad (i=1,2), \tag{13}$$
$$\frac{d\theta}{dt}+\kappa\theta\mathrm{div}\ u=0,\quad \frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}=0.$$

Here we assume that wave parameters are u_1 and u_2.

We are looking for irreducible double wave. The theorem prohibits $N=5$ independent homogeneous quasilinear first order equations.

In the result of substituting $\theta(u_1,u_2)$ into (13) a system, consisting of from four homogeneous quasilinear differential equations is obtained ($\theta_i\equiv\partial\theta/\partial u_i,\ \psi_i=\theta_i^2-\kappa\theta\ \ (i=1,2)$):

$$S_i\equiv\frac{du_i}{dt}+\theta_\alpha\frac{\partial u_\alpha}{\partial x_i}=0,\quad (i=1,2), \tag{14}$$
$$S_3\equiv\psi_\alpha\frac{\partial u_\alpha}{\partial x_\alpha}+2\theta_1\theta_2\frac{\partial u_2}{\partial x_1}=0,\quad S_4\equiv\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}=0.$$

Substituting derivatives $\partial u_i/\partial x_j,\ \ (i=1,2,3;j=0,1,2)$, found from the system (14) through parametric derivatives $\partial u_i/\partial x_1,\ \ (i=1,2)$, into expression

$$D_0S_3+(u_\alpha+\theta_\alpha)D_\alpha S_3-\psi_\alpha D_\alpha S_\alpha-2\theta_1\theta_2D_1S_2=0,$$

we get a homogeneous square–law form with respect to parametric derivatives $\partial u_i/\partial x_1,\ (i=1,2)$:

$$Mb_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_1}\frac{\partial u_\beta}{\partial x_1}=0, \tag{15}$$

where $D_i,\ (i=0,1,2)$ is the total differentiation with respect to the independent variables x_i $(i=0,1,2,3)$, $x_0\equiv t$ and the factors of the form are

$$b_{ii}=-\psi_i^2, b_{21}=b_{12}=(\psi_1\psi_2-2\theta_1^2\theta_2^2),$$

$$M=\psi_2(1+\theta_{11})-2\theta_1\theta_2\theta_{12}+\psi_1(1+\theta_{22}).$$

If at least one of the factors b_{ij} $(i,j=1,2)$ is not equal to zero, then an equation (15) is the fifth quasilinear homogeneous equation, that is prohibited by the theorem. Therefore for irreducible double wave it is necessary to consider $b_{ij}=0$ $(i,j=1,2)$, that gives[5]

$$M=0.$$

The condition $M=0$ provides a compatibility (moreover, even involution) of system (13) with two arbitrary functions of one argument.

[4]Full classification of irreducible plane double wave with functional arbitrereiness was done in [7].
[5]This double wave were obtained in [4].

6 Double waves of systems with $2n-2$ quasilinear equations

We consider double waves with $n=3$ independent variables when the wave parameters satisfy to four homogeneous quasilinear equations of the first order

$$\sum_{\alpha=1}^{3}(\lambda_\alpha p_j^\alpha(\lambda,\mu)+\mu_\alpha q_j^\alpha(\lambda,\mu))=0,(j=1,2,3,4). \tag{16}$$

Such kind of systems often arise in the process of the classification of double waves. We will classify irreducible double waves of (16) having functional arbitrariness [12].

1. **Transformation of equivalence.** Let $u=(\lambda,\mu)$ be the parameters of a double wave and let $\lambda_i=\partial\lambda/\partial x_i$, $\mu_i=\partial\mu/\partial x_i$, and $u_i=(\lambda_i,\mu_i)$ $(i=1,2,3)$. For system (16) the property of being homogeneous and autonomous is invariant under the following equivalence transformations:

(a) the choice of wave parameters $\lambda'=L(\lambda,\mu)$, $\mu'=M(\lambda,\mu)$;

(b) a non-singular linear transformation of the independent variables.

By virtue of the double wave condition $rank\ \partial(\lambda,\mu)/\partial(x_1,x_2,x_3)=2$ it can be shown by means of equivalence transformation that any system (16) of four independent equations can be reduced to one of the following two types: either

$$\begin{gathered}\lambda_1=0,\quad \lambda_2=0,\quad \mu_3=0\\ \lambda_3+a(\lambda,\mu)\mu_1+b(\lambda,\mu)\mu_2=0\quad (a^2+b^2\neq 0)\end{gathered} \tag{17}$$

or

$$u_3=Au_1,\ u_2=Bu_1. \tag{18}$$

Here $A=(a_{ij}(\lambda,\mu)),B=(b_{ij}(\lambda,\mu))$ are 2×2 square matrices.

Theorem. All systems (17) having solutions with functional arbitrariness, are equivalent to the system

$$\lambda=x_3,\quad \mu_1+g(\mu)\mu_2=-1.$$

For the system (18) firstly we note that

$$D_2(u_3-Au_1)-D_3(u_2-Bu_1)\equiv Gu_{11}-C<u_1,u_1>=0.$$

Here $G=AB-BA$ with elements

$$g_{11}=-g_{22}=a_{12}b_{21}-a_{21}b_{12}$$

$$g_{12}=a_{12}(b_{22}-b_{11})-b_{12}(a_{22}-a_{11}),\quad g_{21}=-a_{21}(b_{22}-b_{11})+b_{21}(a_{22}-a_{11})$$

C is a bilinear mapping, whose coordinates are determined by A and B and their derivatives with respect to λ and μ.

If $det\ G\neq 0$, then the solution of system (18) can have at most a constant arbitrariness. So, by virtue of our assumptions $det\ G=0$, i.e.

$$\begin{aligned}a_{12}a_{21}(b_{22}-b_{11})^2-(a_{12}b_{21}+a_{21}b_{12})(b_{22}-b_{11})(a_{22}-a_{11})+\\ b_{12}b_{21}(a_{22}-a_{11})^2-\Delta^2=0\end{aligned}$$

$$(\Delta = a_{12}a_{21} - b_{12}b_{21})$$

Theorem. Apart from equivalence transformations, system (18) has solutions with an arbitrary function that cannot be reduced to invariant ones only if matrix A has real eigenvalues or conditions

$$b_{21} = \frac{a_{21}b_{12}}{a_{12}}, \quad b_{22} = \xi + b_{11} \quad (\xi \equiv \frac{(a_{22} - a_{11})}{a_{12}})$$
$$\frac{\partial b_{12}}{\partial \lambda} - \frac{\partial b_{11}}{\partial \mu} + \frac{b_{12}}{a_{12}}(\frac{\partial a_{11}}{\partial \mu} - \frac{\partial a_{12}}{\partial \lambda}) = 0$$
$$\frac{\partial b_{11}}{\partial \lambda} - \frac{a_{21}}{a_{12}}\frac{\partial b_{12}}{\partial \mu} + \xi\frac{\partial b_{11}}{\partial \mu} + \frac{b_{12}}{a_{12}}(\frac{a_{21}}{a_{12}}\frac{\partial a_{12}}{\partial \mu} - \frac{\partial a_{11}}{\partial \lambda} - \xi\frac{\partial a_{11}}{\partial \mu}) = 0$$

are satisfied.

Remark. The classification of double waves considered in all papers known for us can be reduced to analyzing the solutions of system (18) when a matrix A has real eigenvalues. In many of these publications this property is not pointed out explicitly. It is a consequence of the following. The classification of double waves involves transferring to the hodograph space $x_1 = P(\lambda, \mu, x_3), \quad x_2 = Q(\lambda, \mu, x_3)$, followed by obtaining a second–order degenerate algebraic equation in $\partial P/\partial x_3$ and $\partial Q/\partial x_3$, which splits into the product of two linear forms. It can be shown that this is only possible if matrix A has real eigenvalues.

Theorem. Let a matrix A in (18) has real eigenvalues. Then systems of the form (18), having solutions with an arbitrary function that are irreducible to invariant ones are equivalent to one of the following systems

a) with coefficients $(b_{21}(\partial b_{21}/\partial \lambda) \neq 0)$

$$a_{11} = -\lambda, \; b_{11} = 0, \; a_{22} = -\lambda - b_{21}(\frac{\partial b_{21}}{\partial \lambda})^{-1}, b_{22} = b_{21}\frac{\partial b_{21}}{\partial \mu}(\frac{\partial b_{21}}{\partial \lambda})^{-1},$$

and the general solution

$$\lambda = x_1/x_3, \quad \Phi(\mu - b_{21}x_2/x_3, b_{21}/x_3) = 0$$

$$a_{11} = 0, \; b_{11} = -\lambda, \; a_{22} = 1/\phi_\lambda, \; b_{21} = 1, \; b_{22} = -\lambda + (\phi_\mu + \psi' e^{-\mu})/\phi_\lambda$$

and the general solution

$$\lambda = x_1/x_2, \; \Phi((x_3/x_2 + \phi)e^\mu + \psi, \; x_2 e^{-\mu}) = 0$$

c) with coefficients

$$a_{11} = 0, \; b_{11} = 0, \quad a_{22} = 1/\phi_\lambda, \quad b_{21} = 1, \quad b_{22} = \phi_\mu/\phi_\lambda$$

and the general solution

$$\lambda = X_1, \quad \Phi(\mu - x_2, \; x_3 + \phi) = 0$$

d) with coefficients, satisfying to conditions

$$(a_{22} - a_{11})\frac{\partial b_{11}}{\partial \mu} - (b_{22} - b_{11})\frac{\partial a_{11}}{\partial \mu} = 0$$
$$(a_{22} - a_{11})\frac{\partial b_{22}}{\partial \lambda} - (b_{22} - b_{11})\frac{\partial a_{22}}{\partial \lambda} = 0,$$

and the general solution having two arbitrary functions of one argument.

Here $\Phi = \Phi(\xi_1, \xi_2)$, $\phi = \phi(\lambda, \mu)$, $\psi = \psi(\mu)$ are arbitrary functions and $\Phi_{\xi_2} \neq 0$. In case (d) system (18) is said to be written in terms of Riemann invariants.

Remark. The wide application of multiple waves in multi–dimensional gas dynamics allows us to suppose a successful application of the degenerate hodograph method to obtaining of the exact solutions in the theory of plasticity, where from the solutions with degenerate hodograph only simple waves for hyperbolic systems with two independent variables were used. If number of independent variables more than two, it is known only separate examples of simple [13] and double waves [14] plasticity theory. In [15, 16] double waves with functional arbitrariness for Prandtl–Reis equations of rigid–plastic body

$$\frac{\partial v_\alpha}{\partial x_a} = 0, \quad \frac{\partial v_i}{\partial t} = \frac{\partial \sigma}{\partial x_i} + \frac{\partial S_{i\alpha}}{\partial x_\alpha}, (i = 1, 2, 3),$$
$$\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = 2\Psi S_{ij} \ (i = 1, 2, 3),$$
$$S_{\alpha\beta} S_{\alpha\beta} = 2k^2,$$

were constructed. Here (S_{ij}) is a deviator of strength tensor, $v = (v_1, v_2, v_3)$ the vector of displacement speed.

7 Triple wave of isentropic potential gas flows

Let us consider isentropic potential gas flows

$$\frac{d\mathbf{u}}{dt} + \nabla\theta = 0, \quad \frac{d\theta}{dt} - \kappa\theta div\ \mathbf{u} = 0, \quad rot\ \mathbf{u} = 0. \qquad (19)$$

For the triple waves in which $\theta = \theta(u_1, u_2, u_3)$ it is necessary and sufficient that the condition

$$-\psi_1\theta_{23}^2 - \psi_2\theta_{13}^2 - \psi_3\theta_{12}^2 + 2\theta_1\theta_2\theta_{13}\theta_{23} + 2\theta_1\theta_3\theta_{12}\theta_{23} + 2\theta_2\theta_3\theta_{12}\theta_{13} -$$
$$-2\theta_1\theta_2\theta_{12}(1 + \theta_{33}) - 2\theta_1\theta_3\theta_{13}(1 + \theta_{22}) - 2\theta_2\theta_3\theta_{23}(1 + \theta_{11}) +$$
$$+\psi_1(1 + \theta_{22})(1 + \theta_{33}) + \psi_2(1 + \theta_{11})(1 + \theta_{33}) + \psi_3(1 + \theta_{11})(1 + \theta_{22}) = 0,$$

are fulfilled [17]. Here $\theta_i = \partial\theta/\partial u_i$, $\theta_{ij} = \partial^2\theta/\partial u_i \partial u_j$, $\psi_i = \theta_i^2 - \kappa\theta$. If $u_3 = \Phi(u_1, u_2)$, then necessary and sufficient conditions of existence of the triple wave are

$$\Phi_{12}^2 - \Phi_{11}\Phi_{22} = 0,$$

where $\Phi_{ij} = \partial^2\Phi/\partial u_i \partial u_j$. These triple waves are solutions with one function of three arguments arbitrariness. The necessity of these conditions has been revealed in [18]. For the first case for $\kappa(\kappa - 1) < 0$ he has also found particular solution [19]: $\theta = (a_0 + a_\alpha u_\alpha)^2$. Here a_i, $(i = 0, 1, 2, 3)$ are constants and $a_\alpha a_\alpha = -3\kappa/(4(\kappa - 1))$.

Acknowledgment

The work was prepared for *THE JOINT ISAMM/FRD INTER-DISCIPLINARY WORKSHOP ON SYMMETRY ANALYSIS AND MATHEMATICAL MODELLING*, December 1998, University of North-West, Mmabatho. It is pleasure to thank Professor Nail H. Ibragimov for the invitation me on this Workshop. I also acknowledge the financial support from the Foundation for Research Development of South Africa for arranging my visit of this Workshop and "Institute for Symmetry Analysis and Mathematical Modelling" in University of the North-West, South Africa.

References

[1] Sidorov A.F., Shapeev V.P. and Yanenko N.N. The method of differential constraints and its applications in gas dynamics. Nauka, Novosibirsk. 1984.

[2] Ovsiannikov L.V. Invariant–group solutions of hydrodynamic equations. Trudy II Vsesouzn. s'ezda po teoret. i pricl. mechan. vyp. 2, Nauka, Moscow. 1965, P.302–305.

[3] Ovsiannikov L.V. Group analysis of differential equations. Nauka, Moscow. 1978. (English translation, Ames, W.F., Ed., published by Academic Press, New York, 1982).

[4] Pogodin Yu.Ya., Suchkov V.A. and Yanenko N.N. On travelling waves of gas dynamics equations. Dokl. Acad. Nauk S.S.S.R., 1958. 119, 3. P.443–445.

[5] Sidorov A.F. and Yanenko N.N. To the question about unsteady plane flows of polytropic gas with straight characteristics. Dokl. Acad. Nauk S.S.S.R., 1958. 123, 5. P.832–834.

[6] Kucharczyk P., Peradzynski Z., Zavistowska E. Unsteady multidimensional isentropic flows described by linear Riemann invariants. Arch. Of Mech. 1973. 25, 2. P.319 - 350.

[7] Meleshko S.V. On classification of plane isoentropic double wave type gas flow. Prikl. matem. and mech., 1985. 49, 3. P.406–410. (English transl. in J.Appl.Maths Mechs, 1989, 49, 3.)

[8] Meleshko S.V. On nonisoentropic steady space and plane nonsteady double waves. Prikl. matem. and mech., 1989, 53, 2. pp.255-260. (English transl. in J.Appl.Maths Mechs, 1989, 53, 2.)

[9] Meleshko S.V. Classification of the solutions with degenerate hodograph of the gas dynamics and plasticity equations. Doctoral thesis, 1991. Sverdlovsk.

[10] Yanenko N.N. Multiple waves of quasilinear systems of equations. Dokl. Acad. Nauk S.S.S.R., 1956. 109, 1. P.44–47.

[11] Burnat M. The Cauchy problem for compressible flows of simple wave type. Arch. mech. stos., 1962. 14, 3/4. P. 313–341.

[12] Meleshko S.V. Homogeneous Autonomous Systems with Three Independent Variables. Prikl. matem. and mech., 1994, 58, 5. pp.97-102. (English transl. in J.Appl.Maths Mechs, 1994, v.58, No.5, pp.857-863.).

[13] Annin B.D., Bytev O., Senashov S.I. Group properties of the elasticity and plasticity equations: Nauka, 1985.– 142.

[14] Peradzynski Z. Hyperbolic flows in ideal plasticity. Arch. Mech. 1975. 27, 1. P.141 –156.

[15] Meleshko S.V. Double waves in ideal rigid-plastic body with plane deformation. Journ.prikl.mech.i techn.phiz, 1990, 2. pp.131-136.

[16] Meleshko S.V. On solutions with degenerate hodograph of the quasisteady equations of the theory of plasticity with Mises flow condition. Journ.prikl.mech.i techn.phiz, 1991, 1. pp.82–88.

[17] Meleshko S.V. On potential polytropic gas flows of the triple waves. Prikl. matem. and mech., 1990, 54, 5. pp.775-779. (English transl. in J.Appl.Maths Mechs, 1990, v.54, No.5.)

[18] Sidorov A.F. On unsteady potential motions of politropic gas with degenerate hodograph. Prikl. matem. and mech., 1959. 23, 5. P.940–943.

[19] Sidorov A.F. On exact solutions of triple wave type gas dynamics equations. Dokl. Acad. Nauk S.S.S.R., 1970. 194, 4. P.782–785.