APPLICATION OF GROUP ANALYSIS IN GAS KINETICS

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1 Introduction

A full description of gas evolution as a system of particles supposes a study of individual molecule trajectories. But a very small volume contains a large number of molecules. For example, $V = 1cm^3$ of air contains $N = 2.687 \times 10^{19}$ molecules under normal conditions (Avogadro hypothesis). If we were able to solve equations for each molecule we could answer any question about the behavior of gas. However this is impossible, because the number N is very big. In continuum mechanics we adopt a macroscopic viewpoint: we ignore all the fine details of the molecular or atomic structure and, for the purpose of study, we replace the microscopic medium with a hypothetical continuum in which the basic values are replaced by average values. The kinetic theory approach can be regarded as an intermediate link between the microscopic gas model (set of individual particles) and the macroscopic (continuum models): all average values are described with the help of a theoretical–probability approach. The distinguishing line between the macroscopic model and microscopic one is based on the Knudsen number:

(a) continuum ($Kn < 0.1$);
(b) rarefied gas ($0.1 < Kn < 5$);
(c) free molecular flow ($Kn > 5$).

The typical evolution equation in kinetic theory is a nonlinear Boltzmann equation. This equation, published more than 125 years ago, describes the evolution of distribution function of molecules which interact through binary elastic collisions.

According to kinetic theory the state of the gas at time $t \geq 0$ is characterized by one particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ of its molecules on space coordinates $\mathbf{x} \in R^3$ and velocity $\mathbf{v} \in R^3$. The space–temporal evolution of the distribution function is described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} = I(f, f), \qquad (1)$$

with nonlinear collision integral

$$I(f, f) = \int d\mathbf{w}\ d\mathbf{n}\ B(u, \frac{\mathbf{un}}{u})[f(\mathbf{v}^*)f(\mathbf{w}^*) - f(\mathbf{v})f(\mathbf{w})],$$

where $\mathbf{w} \in R^3$, $d\mathbf{w}$ is an element of volume R^3, $\mathbf{n}$ is a unit vector, $d\mathbf{n} = sin(\theta)d\theta d\varepsilon$ is an element of unit sphere in R^3 ($0 \leq \theta \leq \pi, 0 \leq \varepsilon \leq 2\pi$), $\mathbf{u} = \mathbf{v} - \mathbf{w}$ is a relative velocity of particles in collision, $u = |\mathbf{u}|$, $B = u\sigma(u, \frac{\mathbf{un}}{u})$, and $\sigma(u, \frac{\mathbf{un}}{u})$ is a cross section. The initial velocities $\mathbf{v}, \mathbf{w}$ and the final velocities of two molecules in collision are connected by the usual dynamic relationships:

$$\mathbf{v}^* = \frac{1}{2}(\mathbf{v} + \mathbf{w} + u\mathbf{n}),\ \mathbf{w}^* = \frac{1}{2}(\mathbf{v} + \mathbf{w} - u\mathbf{n}).$$

The main obstacle to the application of Lie's infinitesimal technique to the Boltzmann equation is nonlocality and the large number of independent variables. For the (pseudo) Maxwell's

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molecules ($\sigma(u,\theta) = g_\gamma(\theta)$) the Boltzmann equation can be simplified by the Fourier transformation [1] $\varphi = \int d\mathbf{v} e^{i\mathbf{wv}} f(\mathbf{v})$ and has following representation

$$\frac{\partial \varphi}{\partial t} + i\frac{\partial^2 \varphi}{\partial \mathbf{x} \partial \mathbf{w}} = \int d\mathbf{n}\, g_0(\mathbf{n})[\varphi(\mathbf{w}')\varphi(\mathbf{v}') - \varphi(\mathbf{0})\varphi(\mathbf{w})], \tag{2}$$

where $\mathbf{w}' = (\mathbf{w} + |\mathbf{w}|\mathbf{n})/2,\ \mathbf{v}' = (\mathbf{w} - |\mathbf{w}|\mathbf{n})/2$.

Although the Boltzmann equation has been studied for a long time, only one exact solution in closed form with a nonzero collision integral has been obtained to date, the so–called BKW–solution (constructed independently by Bobylev [1, 2] and Krook and Wu [3, 4]). The discovery of the BKW–solution sparked renewed interest in the construction of exact solutions to the Boltzmann equation and its models (a review of respective results can be found in [5, 6, 7]).

In mathematical physics there are some methods of constructing exact solutions. One of them is based on symmetries of a given equation: group analysis [8, 9]. The goal of this paper is to illustrate an application of group analysis in gas kinetics.

2 Approaches of application group analysis to integro–differential equations

Group analysis was developed especially for differential equations. The application of it to integro–differential equations presents some difficulties. The main one arises from the integral (nonlocal) terms present in these equations. There are several ways by which one can overcome these difficulties, from among which we point out the following[2]:

(1) by finding a representation of an admissible group or a solution (on the basis of priory assumptions);

(2) by studying a system of moments – a method of moments;

(3) by transforming the original integro-differential equation into a differential equation;

(4) by constructing determining equations and finding their solutions – a direct method.

The first approach supposes an a priori choice of a form of symmetries or a solution on the basis of some assumptions about representation. This approach is the simplest and the most efficient. A well known BKW–solution of the Boltzmann equation was found in this way. For the Boltzmann equation this approach was applied in [10, 11]. The approach used in [12, 13, 14, 1, 15] can be related to the first approach above. The main problem in this approach is to discover a representation of the admissible group (or solution).

In the second approach (a method of moments) a transition to an infinite system of differential equations (system of moment equations) is made. After that for each differential equation of the system, containing a finite number of terms, classical group analysis (for differential equations) is applied. Then the process of limitation is carried out. The first application of this approach for finding an admissible Lie group was done in [16] and then it was used for one of model of the Boltzmann equation in [17, 18]. There are some problems in the application of this approach. One of them is that for some equations the construction of the moment system is impossible.

In the third approach, as in the previous one, a transformation of initial integro–differential equations to differential is made. After that a classical algorithm of group analysis is applied

[2]Some discussion of applications of group analysis to integro–differential equations can be found in [7].

to the differential equations. The results of works [4, 19, 20] are obtained by this approach. In this way the same question as in the previous approaches, about the completeness of the Lie group, can not be solved.

It is worth noting that no heuristic expedient permits a complete answer to the question: what is the widest Lie group of transformation, admissible by a given nonlocal equation. In the first three approaches listed above one needs to prove that the constructed admissible Lie group is a full group, because, in general case, either loss of some invariant properties or obtaining new ones is possible[3]. For the completeness of description group properties of equations with nonlocal operators it is necessary to use successive approach of group analysis: constructing of the determining equations and finding their solutions. Such an approach for studying group properties of integro–differential equations was proposed in [10, 21] and later was applied in [22, 23, 24, 25]. For other models this approach was used in [26, 27, 28]. Another advantage of this approach is the possibility of applying Lie–Bäclund transformations, conditional symmetries and other methods to integro–differential equations (there are some trivial examples of such applications). The main difficulty of this approach is in finding the general solution of determining equations (in differential equations a splitting process leads to an overdetermined system of equations that helps to solve determining equations).

Here we illustrate a successive application of the direct method.

3 The direct method

Let us consider an abstract equation with nonlocal operators, in particular, integral:

$$\Phi(x,u) = 0. \tag{3}$$

Here u is a vector of dependent variables, x is a vector of independent variables. We consider one–parametric Lie group $G^1(X)$ of transformations

$$x' = f^x(x,u;a), \quad u' = f^u(x,u;a) \tag{4}$$

with generator

$$X = \zeta^u(x,u)\partial_u + \xi^x(x,u)\partial_x.$$

Definition. A one–parametric Lie group G^1 of transformations (4) is a symmetry group admissible by the equation (3) if G^1 converts every solution $u(x)$ of (3) into a solution $u_a(x)$ of the same equation.

The transformed solution is

$$u_a(x') = f^u(x,u(x);a)$$

with substituted $x = \psi^x(x';a)$ which is found from the relation $x' = f^x(x,u(x);a)$.

The equation, determining an admissible Lie group (4) is

$$\left(\frac{\partial}{\partial a}\Phi(x,u_a(x))\right)_{|a=0} = 0. \tag{5}$$

[3]For example, the one–parametric semigroup of integral transformations [1] is transferred by the Fourier transformation to the group of point transformations.

The practical construction of determining equations for integro-differential equations is performed by using the canonical Lie–Bäcklund's representation of the generator X:

$$\tilde{X} = \tilde{\zeta}^{\alpha}\partial_{u^{\alpha}}, \quad \tilde{\zeta}^{\alpha} = \zeta^{u^{\alpha}}(x,u) - \xi^{x_{\beta}}(x,u)u^{\alpha}_{x_{\beta}}$$

and acting on the equation (3), where the derivatives $\partial_{u^{\alpha}}$ should be understood in terms of Frechet derivatives. Here we should remember that the determining equations (5) have to be satisfied for arbitrary solution of the original equations (3).

The obtaining of the determining equations for integro–differential equations like for differential equations should not present problems. The main difficulty is in the finding of the general solutions of the determining equations. Here the important circumstance is the knowledge of properties of solutions of the original equations.

The main advantage of the direct method is in ability to answer on the question about completeness of an admissible group.

Remark. For a system of differential equations (without nonlocal terms) the determining equations (5) coincide with the determining equations constructed by usual way (after some trivial simplifications[4]).

4 Example of application of the direct method

Let us consider an application of the direct method to the Fourier–image of the spatially homogeneous and isotropic Boltzmann equation

$$\Phi \equiv \frac{\partial\varphi(x,t)}{\partial t} + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\ ds = 0. \tag{6}$$

The group generator is written in the form

$$X = \xi(x,t,\varphi)\partial_x + \eta(x,t,\varphi)\partial_t + \zeta(x,t,\varphi)\partial_{\varphi}.$$

In this example, the determining equation (5) has the following form:

$$\begin{gathered} D_t\psi(x,t) + \psi(0,t)\varphi(x,t) + \psi(x,t)\varphi(0,t) - \\ -2\int_0^1 \varphi(x(1-s)s,t)\psi(xs,t)\ ds = 0, \end{gathered} \tag{7}$$

where $\varphi(x,t)$ is an arbitrary solution of (6), D_t is the total differentiation with respect to t and the function $\psi(x,t)$ is determined by an infinitesimal generator X in accordance with the transition to the canonical Lie–Bäcklund operator:

$$\psi(x,t) = \zeta(x,t,\varphi(x,t)) - \xi(x,t,\varphi(x,t))\varphi_x(x,t) - \eta(x,t,\varphi(x,t))\varphi_t(x,t).$$

The approach for constructing the general solution of determining equation (7) is the following one. We restrict the determining equation to the subset of solutions of equation (6) determined by the initial conditions

$$\varphi(x,t_0) = bx^n \tag{8}$$

[4]These simplifications are connected with a consideration of the infinitesimal generator X as Lie–Bäcklund canonical generator

at the given (arbitrary) moment $t = t_0$[5]. Here n is a positive integer. More precisely, we let $t = t_0$ in the determining equation (7), and substitute the function $\phi(x,t)$ and its derivatives obtained from equations (6) and (8). We consider the resulting equation at an arbitrary initial time t_0 which is denoted again by t. Accordingly, equation (7) is written in terms of the following functions:

$$\hat{\xi}(x,t) = \xi(bx^n, x, t), \quad \hat{\eta}(x,t) = \eta(bx^n, x, t),$$
$$\hat{\zeta}(x,t) = \zeta(bx^n, x, t), \quad \hat{\zeta}_\varphi(x,t) = \zeta_\varphi(bx^n, x, t),$$
$$\hat{\zeta}_t(x,t) = \zeta_t(bx^n, x, t), \quad \hat{\zeta}_x(x,t) = \zeta_x(bx^n, x, t), \ \ldots$$

Then we solve equation (7) by letting $n = 0, 1, 2, \ldots$ in the initial condition (8), and simultaneously varying the parameter b.

We proceed now to the calculations. The coefficients of infinitesimal generator X are assumed to be locally analytic functions. Hence they can be represented by the Taylor series with respect to φ:

$$\xi(x,t,\varphi) = \sum_{l\geq 0} q_l(x,t)\varphi^l, \quad \eta(x,t,\varphi) = \sum_{l\geq 0} r_l(x,t)\varphi^l,$$
$$\zeta(x,t,\varphi) = \sum_{l\geq 0} p_l(x,t)\varphi^l,$$

Let $n = 0$ in initial data (8). Then the determining equation (7) has the form

$$\hat{\zeta}(x,t) + b(\hat{\zeta}(O,t) + \hat{\zeta}(x,t)) - 2b\int_0^1 \hat{\zeta}(xs,t)\ ds.$$

It follows:

$$\frac{\partial p_0}{\partial t} = 0, \quad \frac{\partial p_{l+1}}{\partial t}(x,t) + p_l(x,t) + p_l(0,t) - 2\int_0^1 p_l(xs,t)\ ds = 0 \tag{9}$$

with $l = 0, 1, \ldots$.

Let $n \geq 1$ in equation (8). Then

$$\varphi_t = P_n b^2 x^{2n}, \quad \varphi_x = nbx^{n-1}, \quad \varphi_{tt} = Q_n b^3 x^{3n}, \quad \varphi_{tx} = 2nP_n b^2 x^{2n-1},$$

and the determining equation (7) yields:

$$\begin{aligned} \hat{\zeta}_t + b[-nx^{n-1}\hat{\xi}_t + x^n\hat{\zeta}(0,t) - 2x^n\int_0^1 (1-s)^n\hat{\zeta}(xs,t)\ ds\ ] + \\ +b^2[-P_n x^{2n}\hat{\eta}_t + P_n x^{2n}\hat{\zeta}_\varphi - 2nP_n x^{2n-1}\hat{\xi} - \delta_{n1}\hat{\xi}(0,t) + \\ 2nx^{2n-1}\int_0^1 (1-s)^n s^{n-1}\hat{\xi}(xs,t)\ ds\ ] + \\ +b^3[-nP_n x^{2n-1}\hat{\xi}_\varphi - Q_n x^{3n}\hat{\eta} + 2P_n x^{3n}\int_0^1 (1-s)^n s^{2n}\hat{\eta}(xs,t)\ ds\ ] - \\ -b^4[P_n^2 x^{4n}\hat{\eta}_\varphi\ ] = 0 \end{aligned} \tag{10}$$

where

$$P_n = \frac{(n!)^2}{(2n+1)!}, \quad Q_n = 2P_n\frac{(2n)!n!}{(3n+1)!},$$

[5]Solvability of the Cauchy problem (6)–(8) is proved (see, for example, [6]).

Now we treat b as an arbitrary parameter. Accordingly, we split equation (10) into a series of equations by equating to zero the coefficients of $b^k,\ k=0,1,\dots$ in the left-hand side of equation (10). The resulting system of equations, together with equations (9), solve the determining equation (7).

It follows from the expansions of functions $\xi(x,t,\varphi),\quad \eta(x,t,\varphi),\quad \zeta(x,t,\varphi)$ that for $k=0$ the corresponding coefficient in the left–hand side of equation (10) vanishes in virtue of the first equation (9).

For $k=1$, equation (10) yields:

$$x[-p_0(x,t)+2\int_0^1(1-(1-s)^n)p_0(xs,t)\ ds]-n\frac{\partial q_0(x,t)}{\partial t}=0.$$

Since n is arbitrary, it follows:

$$p_0(x,t)=0,\quad \frac{\partial q_0(x,t)}{\partial t}=0.$$

Hence $z\hat{eta}(0,t)=0$.

Similarly, one obtains for $k=2$:

$$x[-p_1(x,t)-p_1(0,t)+2\int_0^1(1-(1-s)^n s^n)p_1(xs,t)\ ds-P_n\frac{\partial r_0(x,t)}{\partial t}+$$

$$+P_np_1(x,t)]-n\frac{\partial q_1(x,t)}{\partial t}-2nP_nq_0(x,t)+2n\int_0^1(1-s)^n s^{n-1})q_0(xs,t)\ ds=0.$$

Whence

$$p_1(x,t)=c_0+c_1x,\quad \frac{\partial q_1(x,t)}{\partial t}=0,\quad q_0(x,t)=c_x,\quad \frac{\partial r_0(x,t)}{\partial t}=-c_0$$

where c_0,c_1,c_2 are arbitrary constants.

For $k=3$, one has

$$x^{n+1}[-p_2(x,t)-p_2(0,t)+2\int_0^1(1-(1-s)^n s^{2n})p_2(xs,t)\ ds-$$

$$-P_n\frac{\partial r_1(x,t)}{\partial t}+2P_np_2(x,t)+2P_n\int_0^1(1-s)^n s^{2n})r_0(xs,t)\ ds-Q_nr_0(x,t)\ ]+$$

$$+x^n[-n\frac{\partial q_2(x,t)}{\partial t}-2nP_nq_1(x,t)+2n\int_0^1(1-s)^n s^{2n-1})q_1(xs,t)\ ds-$$

$$-nP_nq_1(x,t)=0.$$

Whence

$$q_1(x,t)=0,\quad \frac{\partial q_2(x,t)}{\partial t}=0,\quad p_2(x,t)=0,\quad \frac{\partial r_1(x,t)}{\partial t}=0,\quad r_0(x,t)=-c_0t+c_3.$$

where c_3 is an arbitrary constant.

For $k=4+l,\ l=0,1,\dots$, equation (10) yields

$$x^{n+1}[\frac{\partial p_{\alpha+4}(x,t)}{\partial t}-2\int_0^1(1-s)^n s^{(3\alpha)n}p_{3+\alpha}(xs,t)\ ds+$$

$$(3+\alpha)P_n p_{3+\alpha}(x,t) - P_n\frac{\partial r_{2+\alpha}(x,t)}{\partial t} - (\alpha+1)P_n^2 r_{\alpha+1}(x,t)+$$
$$+2P_n\int_0^1 (1-s)^n s^{(3+\alpha)n})r_{\alpha+1}(xs,t)\ ds - Q_n r_{\alpha+1}(x,t)\ ]+$$
$$x^n[-n\frac{\partial q_{\alpha+3}(x,t)}{\partial t} - 2nP_n q_{\alpha+2}(x,t) + 2n\int_0^1 (1-s)^n s^{(\alpha+3)n-1} q_{\alpha+2}(xs,t)\ ds -$$
$$n(\alpha+2)P_n q_{\alpha+2}(x,t) = 0.$$

Whence

$$p_{l+3}(x,t)=0,\quad q_{l+2}(x,t)=0,\quad r_{l+1}(x,t)=0,\quad (l=0,1,\ldots).$$

It follows from the above equations that

$$\xi = c_3 x,\quad \eta = c_1 - c_4 t,\quad \zeta = (c_2 x + c_4)\varphi$$

with arbitrary constants c_1, c_2, c_3, c_4.

Thus, equation (6) admits the four–dimensional Lie algebra L^4 spanned by the generators

$$X_1 = \partial_t,\ X_2 = x\varphi\partial_\varphi,\ X_3 = x\partial_x,\ X_4 = \varphi\partial_\varphi - t\partial_t.$$

An optimal system of one–dimensional subalgebras of L^4 is

$$X_1,\ X_4 + cX_3,\ X_2 - X_1,\ X_4 \pm X_2,\ X_1 + X_3, \tag{11}$$

where c is arbitrary constant. Invariant solutions, corresponding to this system of subalgebras have following form:

$$X_1:\quad \varphi = g(y),\ y = x;$$
$$X_4 + cX_3:\quad \varphi = g(y),\quad y = xt^c;$$
$$X_2 - X_1:\quad \varphi = e^{-xt}g(y),\quad y = x;$$
$$X_1 + X_3:\quad \varphi = g(y),\quad y = xe^{-t};$$
$$X_4 \pm X_2:\quad \varphi = t^{-(1\pm x)}g(y),\quad y = x.$$

Different invariant solutions are obtained by substitution of these expressions into equation (6). After that original equation is reduced to the equation with one independent variable. For example, so called class of the BKW–solutions is obtained as invariant solution with respect to subalgebra $X_1 + X_3$[6]. Factor system in this case is

$$-y\frac{dg(y)}{dy} + g(y)g(0) = \int_0^1\ g(ys)g(y(1-s))\ ds \tag{12}$$

and BKW–solution is $g = 6e^y(1-y)$.

The same approach can be applied to the Fourier–image of the Boltzmann equations, describing homogeneous relaxation of N–component gas mixture with the Maxwell's molecules interaction and the Smolukhovskiy kinetic equation of coagulation. Here we gives these results.

[6]In physical literature a representation of BKW–solution, which can be obtained as invariant solution with respect to similar subalgebra $X_2 - X_3 + c^{-1}X_1$ is usually used.

An admissible full group of the Fourier–image of the Boltzmann equations

$$\frac{\partial \varphi_i(x_i,t)}{\partial t} = \sum_{j=1}^{N} \nu_{ij} \int_0^1 ds[\varphi_i(x_i(1-\varepsilon_{ij}s),t)\varphi_j(x_i\varepsilon_{ij}s),t)-$$

$$\varphi_i(x_i,t)\varphi_j(0,t)], (i=1,2).$$

corresponds to Lie algebra with generators [25]

$$X_1 = \partial_t,\ X_2 = \sum_\alpha x_\alpha \varphi_\alpha \partial_{\varphi_\alpha},\ X_3 = \sum_\alpha x_\alpha \partial_{x_\alpha},\ X_4 = \sum_\alpha \varphi_\alpha \partial_{\varphi_\alpha} - t\partial_t.$$

Here $\varepsilon_{ij}, \nu_{ij}$ are constants.

For some kernels of coagulation, the equation (6) is structurally close to the Fourier–image of the Smolukhovskiy's kinetic equation of coagulation

$$\frac{\partial \varphi_i(x,\tau)}{\partial \tau} = x^\gamma \int_0^1 s^{k_1}(1-s)^{k_2}\varphi(x(1-s),\tau)\varphi(xs),\tau)\ ds,$$

which is encountered in the kinetic theory of disperse systems, such as atmospheric aerosols, colloid solutions, suspensions.

An admissible group of the last equation consists of transformations, corresponding to generators [23]

$$X_1 = \partial_\tau,\ X_2 = x\varphi\partial\varphi,\ X_3 = x\partial_x - \gamma\varphi\partial_\varphi,\ X_4 = \varphi\partial_\varphi - \tau\partial_\tau.$$

5 Kinetic Vlasov equation in a problem of high–frequency one–dimensional vibrations of collisionless plasma

The problem mentioned in the heading of this section is described by the following system:

$$f_t + vf_x - E_v = 0,\ E_v = 0,\ E_t = \int vf\ dv,\ E_x = 1 - \int f\ dv, \tag{13}$$

where $f = f(t,x,v), E = E(t,x,v)$ and integration is carried out over the one–dimensional velocity space R^1.

In [16] for (13) the admissible Lie group G^5 was constructed. Its infinitesimal generators are

$$\begin{aligned} X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 &= x\partial_x + v\partial_v + E\partial_E - f\partial_f, \\ X_4 &= \cos t\, \partial_x - \sin t\, \partial_v + \cos t\, \partial_E, \\ X_5 &= \sin t\, \partial_x + \cos t\, \partial_v + \sin t\, \partial_E, \end{aligned} \tag{14}$$

This group was obtained on the basis of the infinite system of moment equations derived from (13). Using the direct method for (13), it can be shown that the group G^5 is the full Lie group of transformations admissible by the system (13).

Let $u = (f,E,x,t,v)$ and let the one-parametric Lie group of transformations admissible by (13) has an infinitesimal generator

$$X = \zeta_1\partial_E + \zeta_1\partial_f + \xi\partial_x + \eta\partial_t + z\partial_t,$$

The acting of the generator X on the former two equations of the system (13) is defined in the usual form (as for differential equations)

$$[D_t\tilde{\zeta}_2 + vD_x\tilde{\zeta}_2 - ED_v\tilde{\zeta}_2 - f_v\tilde{\zeta}_1]_{|(S)} = 0,\ [D_v\tilde{\zeta}_1]_{|(S)} = 0, \tag{15}$$

where D_t, D_x, D_v denote the total derivatives with respect to variables t, x, v; the subscript (13) means that the corresponding relation is considered in all the solutions of the system (13),

$$\tilde{\zeta}_1 = \zeta_1 - \xi E_x - \eta E_t - zE_v,\ \tilde{\zeta}_2 = \zeta_2 - \xi f_x - \eta f_t - zf_v.$$

The result of the acting of X in the latter two equations of (13) is derived in accordance with the scheme given in this section as follows:

$$[D_t\tilde{\zeta}_1 - \int v\tilde{\zeta}_2\, dv]_{|(S)} = 0,\ [D_x\tilde{\zeta}_1 - \int \tilde{\zeta}_2\, dv]_{|(S)} = 0. \tag{16}$$

Since the initial functions are chosen arbitrarily, equations (15) are split with respect to derivatives E_t, E_x, f_v, f_x. The solution of the split equations has the form

$$\begin{gathered}\zeta_1 = E(c_1 - \eta'(t)) - \varphi''(t) - x\eta''(t),\ \zeta_2 = \psi(f), \\ z = x\eta'(t) + \varphi'(t) + c_1 v,\ \xi = x(c_1 + \eta'(t)) + \varphi(t),\end{gathered} \tag{17}$$

where $\eta(t), \varphi(t), \psi(f)$ are arbitrary functions, c_i are arbitrary constants.

Integration of equations (16), taking into account the solution (17), gives the following relations:

$$\begin{gathered}-E_x(c_1 + 2\eta') + c_1 + \int \psi(f)\, dv = 0, \\ -E_t(c_1 + 2\eta') - 3E\eta'' - \varphi''' - x\eta'' - \varphi' - \int v\psi(f)\, dv = 0.\end{gathered} \tag{18}$$

If $f = 0$ is put in these relations, then, in order that the integral $\int \psi(0)\, dv$ be bounded, it is necessary that $\psi(0) = 0$ and, therefore, also

$$\eta' = 0,\ \varphi''' + \varphi' = 0,$$

that is

$$\eta = c_2,\ \varphi = c_3 + c_4 \sin t + c_5 \cos t. \tag{19}$$

By substituting the relations (19) into equations (18), we obtain the following

$$\int (c_1 f + \psi(f))\, dv = 0,\ \int v(c_1 f + \psi(f))\, dv = 0.$$

Simple reasoning shows that the necessary condition of correctness of the latter relations for the arbitrary initial function $f(t, x, v) = f_0(v)$ is

$$\psi(f) = -c_1 f.$$

Comparison shows that the five–dimensional Lie group of transformations obtained here by means of the direct method coincides with the group G^5 (14) and, therefore, the latter is the full Lie group admissible by (13).

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