ว่าด้วยระบบผู้ล่า-ผู้ถูกล่าบางระบบและตัวแบบโรคระบาด ที่มีโครงสร้างเป็นระยะ และผลของการดล

นายอ้ายเหลิน โจว

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์
มหาวิทยาลัยเทคโนโลยีสุรนารี
ปีการศึกษา 2559

ON SOME PREDATOR-PREY SYSTEMS AND EPIDEMIC MODELS WITH STAGE STRUCTURE AND IMPULSIVE EFFECTS

Airen Zhou

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

Suranaree University of Technology

Academic Year 2016

ON SOME PREDATOR-PREY SYSTEMS
AND EPIDEMIC MODELS WITH STAGE
STRUCTURE AND IMPULSIVE EFFECTS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

Thesis Examining Committee

(Asst. Prof. Dr. Eckart Schulz)
Chairperson

(Prof. Dr. Pairote Sattayatham)
Member (Thesis Advisor)

(Prof. Dr. Jianjun Jiao)
Member

(Asst. Prof. Dr. Arjuna Chaiyasena)
Member

(Asst. Prof. Dr. Benjawan Rodjanadid)
Member

(Prof. Dr. Sukit Limpijumnong)
Vice Rector for Academic Affairs
and Innovation

(Prof. Dr. Santi Maensiri)
Dean of Institute of Science

อ้ายเหลิน โจว : ว่าด้วยระบบผู้ล่า-ผู้ถูกล่าบางระบบและตัวแบบโรคระบาดที่มีโครงสร้างเป็นระยะและผลของการดล (ON SOME PREDATOR-PREY SYSTEMS AND EPIDEMIC MODELS WITH STAGE STRUCTURE AND IMPULSIVE EFFECTS) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร.ไพโรจน์ สัตยธรรม, 103 หน้า.

ระบบผู้ล่า-ผู้ถูกล่าเป็นตัวแบบที่สำคัญมากในเรื่องพลวัตของประชากรซึ่งมีพลวัตมากมาย และในเร็ว ๆ นี้ ตัวแบบนี้ก็ได้รับความสนใจอย่างกว้างขวางในวิชานิเวศวิทยา ในส่วนแรกของวิทยานิพนธ์นี้จะมีการพิจารณาตัวแบบผู้ล่า-ผู้ถูกล่าพร้อมกับการแพร่ของการดลและการปลดปล่อยประชากรผู้ล่า ตัวแบบที่กำลังพิจารณานี้จะเป็นตัวแบบผู้ล่า-ผู้ถูกล่าสำหรับสองบริเวณ ซึ่งเชื่อมโยงด้วยการแพร่ของประชากรผู้ล่า มันได้แสดงถึงพัฒนาการของประชากร ข้อสรุปในส่วนนี้ก็คือ การแพร่ของการดลและการปลดปล่อยผู้ล่าจะให้กลยุทธ์พื้นฐานที่น่าเชื่อถือสำหรับการจัดการสัตว์ที่รบกวน

ตัวแบบโรคระบาด SIR เป็นตัวแบบเป็นที่นิยมตัวแบบหนึ่งในวิชาโรคระบาด การให้วัคซีนเป็นช่วง ๆ เป็นกลยุทธที่สำคัญในการกำจัดโรคระบาด ได้มีการนำเสนอตัวแบบโรคระบาดพร้อมด้วยโครงสร้างเป็นระยะและการให้วัคซีนเป็นช่วง ๆ ไว้ในส่วนที่สองของวิทยานิพนธ์ฉบับนี้ ผลลัพธ์ที่ได้จะให้กลยุทธ์ที่น่าเชื่อถือในการป้องกันการเกิดโรคระบาด

สาขาวิชาคณิตศาสตร์
ปีการศึกษา 2559

ลายมือชื่อนักศึกษา Airen Zhou
ลายมือชื่ออาจารย์ที่ปรึกษา
ลายมือชื่ออาจารย์ที่ปรึกษาร่วม Jianjun Jiao

AIREN ZHOU : ON SOME PREDATOR-PREY SYSTEMS AND EPIDEMIC MODELS WITH STAGE STRUCTURE AND IMPULSIVE EFFECTS. THESIS ADVISOR : PROF. PAIROTE SATTAYATHAM, Ph.D. 103 PP.

PREDATOR-PREY MODEL/ SIR EPIDEMIC MODEL/ STAGE STRUCTURE/ IMPULSIVE DIFFUSION/ EXTINCTION/ PERMANENCE

The predator-prey system is a very important model in population dynamics, which exhibits abundant dynamics. It has received extensive attention in ecology in recent years. A predator-prey model with impulsive diffusion and release on a predator population is considered in the first part of this dissertation. This predator-prey model applies to two regions, which are connected by diffusion of the predator population. It portrays the evolvement of population. It is concluded that the impulsive diffusion and releasing predator provide a reliable tactic basis for pest management.

The SIR epidemic model is one of the most popular epidemic models in epidemiology. Pulse vaccination is an important strategy to eradicate an infectious disease. An SIR epidemic model with stage structure and pulse vaccination is also presented in the second part of this dissertation. The results obtained provide a reliable tactic basis for preventing the disease from spreading.

School of Mathematics

Academic Year 2016

Student's Signature Airen Zhou

Advisor's Signature P Sattayatham

Co-Advisor's Signature Jianjun Jiao

ACKNOWLEDGEMENTS

I am profoundly grateful to my thesis advisor and co-advisor, Prof. Dr. Pairote Sattayatham and Prof. Dr. Jianjun Jiao. Without their long-term guidance, encouragement, support and patient help, this thesis could not have been carried out.

I am deeply indebted to all the teachers who taught or helped me during my studies at Suranaree University of Technology. They are Asst. Prof. Dr. Eckart Schulz, Prof. Dr. Sergey V. Meleshko, Assoc. Prof. Dr. Prapasri Asawakun, Asst. Prof. Dr. Arjuna Chaiyasena, and Asst. Prof. Dr. Benjawan Rodjanadid.

Many thanks go to all the lecturers in the School of Mathematics for all the lectures I attended while studying.

I have also benefited from my seniors and juniors at Suranaree University of Technology and my colleagues at Guizhou University of Finance and Economics, to whom I would like to express my sincere thanks at this opportunity.

Thanks are due to all my friends who always have given their friendly help.

Last but not least, I would like to cordially thank my parents, my wife, and my daughter for their understanding, endurance and moral support during all these years.

Airen Zhou

CONTENTS

CONTENTS (Continued)

LIST OF FIGURES

LIST OF FIGURES (Continued)

Figure **Page**

CHAPTER I

INTRODUCTION

In this chapter, we discuss some models of population dynamics. We give an introduction to the predator-prey model and the epidemic model, where we follow the standard works (Gopalsamy, 1992; Hassell, 1978; Driver, 1977; Hethcote, 1989). The final goal of this chapter will be the derivation of the necessity for our studies.

1.1 Remarks on Models of Single Species Dynamics

Mathematical descriptions of ecological systems may be made for two quite different purposes, one practical and the other theoretical (Maynard Smith, 1974).

We begin with some remarks on models of single species dynamics (Gopalsamy, 1992). Most of the differential equation models of population dynamics have been derived starting from the following simple format

$$\frac{dN(t)}{dt} = \left\{ \begin{array}{c} \text{an individual's contribution} \\ \text{to population change in unit time} \end{array} \right\} N(t) \tag{1.1}$$

where $N(t)$ denotes the density of a population (or biomass) of a single species at time t. Subsequently, one makes an assumption regarding the factor inside the braces in (1.1). In particular, if one assumes that an individual's contribution to the change in population in unit time is denoted by a function, say $f(N)$, defined suitably for all $t > 0, N \geq 0$, then one obtains from (1.1) the so called Kolmogorov formulation in the form

$$\frac{dN(t)}{dt} = f\left(t, N(t)\right) N(t). \tag{1.2}$$

Various choices of f together with some ecologically plausible assumptions such as the temporal constancy of the environment and density dependent effects in f lead to several well known ordinary differential equations of population dynamics. For instance, if $f(t, N) \equiv r$ (a positive constant) one obtains the Malthusian formulation

$$\frac{dN(t)}{dt} = rN(t) \tag{1.3}$$

and if one assumes $f(t, N) \equiv r - (r/K)N$ for some positive constants r and K, one gets the familiar logistic equation

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right). \tag{1.4}$$

The above logistic equation implies a monotonic approach as $t \to \infty$ of the population density to the steady state $N(t) \equiv K$.

1.2 Review on Predator-Prey Models

The predator-prey system is an important population model and has been been studied by many authors (Hassell, 1978; Jiao, Cai and Chen, 2011; Wang and Chen, 1997; Berezovskaya, Song and Castillo-Chavez, 2010; Georgescu and Hsieh, 2007). For instance, three types of predator-prey interactions have been observed by Hassell (1978): (1) those where the prey becomes extinct (e.g. Gause, 1934; Luchinbill, 1973); (2) those where both populations oscillate out of phase with each other (e.g. Huffaker, 1958; Huffaker, Shea and Herman, 1963); and (3) those where both populations persist, but fluctuate less regularly (e.g. Utida, 1957; Burnett, 1977).

There is a well-known model (Driver, 1977), which was invented and studied by Lotka (1925) and Volterra (1926). Let $x(t)$ be the population at time t of some animal species called prey and let $y(t)$ be the population of a predator species which lives on these prey. We assume that $x(t)$ would increase at a rate proportional to $x(t)$ if the prey were left alone, i.e., we would have $x'(t) = a_1 x(t)$, where $a_1 > 0$. However the predators are hungry, and the rate at which each of them eats prey is limited only by his ability to find prey. Thus we shall assume that the activities of the predators reduce the growth rate of $x(t)$ by an amount proportional to the product $x(t)y(t)$, i.e.,

$$x'(t) = a_1 x(t) - b_1 x(t)y(t),$$

where b_1 is another positive constant.

Now let us also assume that the predators are completely dependent on the prey as their food supply. If there were no prey, we assume $y'(t) = -a_2 y(t)$, where $a_2 > 0$, i.e., the predator species would die out exponentially. However, given food the predators breed at a rate proportional to their number and to the amount of food available to them. Thus we consider the pair of equations

$$\begin{cases} x'(t) = a_1 x(t) - b_1 x(t)y(t) \\ y'(t) = -a_2 y(t) + b_2 x(t)y(t), \end{cases} \tag{1.5}$$

where a_1, a_2, b_1 and b_2 are positive constants. This well-known model is the most primitive predator-prey population model.

From the book of Cushing (1987), we know that in order to gain a better understanding of the dynamics of biological populations, theoretical biologists and applied mathematicians have modified classical models and modeling methodologies in many ways. All mathematical models make simplifying assumptions, of course, and there is a relentless trade-off between biological accuracy and mathematical tractability. One way to view many of the simplifying assumptions made in population models is with regard to various uniformities and homogeneities that are either explicitly or implicitly postulated. For example, two common simplifications concern homogeneities in space and time. There exists now a rather large and growing body of literature on predator-prey models for biological populations (Berezovskaya, Song and Castillo-Chavez, 2010; Chen, Huang, Ruan and Wang, 2013; Jiao, Pang, Chen and Luo, 2008; Jiao and Chen, 2012).

1.3 Review on Epidemic Model

Mass immunization is often used as a tool to control the transmission of epidemics. Pulse vaccination is an important and effective strategy for the elimination of infectious diseases, and has been considered in much literature (Agur, Cojocaru, Mazor, Anderson and Danon, 1993; Lu, Chi and Chen, 2002; Meng and Chen, 2008b; Shulgin, Stone and Agur, 1998; Stone, Shulgin and Agur, 2000; Gao, Teng and Xie, 2009; Gakkhar and Negi, 2008).

In the classical epidemiological model, the population is usually divided into disjoint three classes, which we can refer to the paper of Hethcote (1989). The susceptible class consists of those individuals who can incur the disease but are not yet infective. The infective class consists of those who are transmitting the disease to others. The removed class consists of those who are removed from the susceptible-infective interaction by recovery with immunity, isolation, or death. The fractions of the total population in these classes are denoted by $S(t)$, $I(t)$ and $R(t)$, respectively.

Hethcote (1989) investigated an SIR model of an endemic disease with vital dynamics as follows:

$$
\begin{cases}
(NS(t))' = -\lambda SNI + \mu N - \mu NS, \\
(NI(t))' = \lambda SNI - \gamma NI - \mu NI, \\
(NR(t))' = \gamma NI - \mu NR, \\
NS(0) = NS_0 > 0, NI(0) = NI_0 \geq 0, \\
NR(0) = NR_0 \geq 0, \\
NS(t) + NI(t) + NR(t) = N,
\end{cases}
\tag{1.6}
$$

where the contact rate λ, the removal rate constant γ and the death rate constant μ are positive constants.

If each equation in (1.6) is divided by N, then the IVP in terms of $S(t)$ and $I(t)$ is

$$
\begin{cases}
S'(t) = -\lambda SI + \mu - \mu S, \\
I'(t) = \lambda SI - \gamma I - \mu I, \\
S(0) = S_0 > 0, I(0) = I_0 \geq 0.
\end{cases}
\tag{1.7}
$$

For more details of SIS and SIR models, we refer to Hethcote (1989), Gao et al., (2006).

Epidemiological models are now widely used as more epidemiologists realize the role that modeling can play in basic understanding and policy development (e.g. Gao et al., 2009; Gao, Zhang and He, 2013; Xu, 2013, 2014).

1.4 Stage-Structured Population Model and Impulsive Effects

In the real world, there are many species whose individual members have a life history that takes them through two stages, immature and mature. That is, stage structure was introduced in biological models, and stage-structured models have attracted much attention in recent decades. A systematic consideration for stage-structured models can be found in the literature (Nisbet et al., 1986; Nisbet and Gurney, 1984; Nisbet et al., 1989). In 1990, Aiello and Freedman (Aiello and Freedman, 1990) considered a single species model with stage structure and discrete delay, showed that there exists a globally asymptotically stable equilibrium for this model. An excellent survey on the dynamics of stage-structured population models was made in (Liu and Chen, 2002). Predator-prey models with stage structure for the predator have also received considerable attention in recent years. For example, we can refer to (Georgescu and Hsieh, 2007; Xiao and Chen, 2004; Liu and Beretta, 2006; Aiello et al., 1992; Xu, 2011; Wang and Chen, 1997; Wang et al., 2001).

It is known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, i.e., differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems, which have been introduced into population dynamics lately (Lakshmikantham, Bainov and Simeonov, 1989; Liu and Chen, 2003). Impulsive equations have been studied in many investigations (Jiao et al., 2008b; Jiao and Chen, 2008).

Thus, there is a need to investigate stage-structured population models with impulse.

CHAPTER II

PRELIMINARIES

In this chapter, we start discussing the concept of persistence and extinction, an important concept for biological systems. Then, we review the mathematical theory and techniques that are important in the study of nonlinear difference equations and systems, where we follow the work (Allen, 2007; Edelstein-Keshet, 1988; Cantrell and Cosner, 1996; Cao and Gard, 1998). The main objective of this chapter is to pave the way for our study.

2.1 Concept of Persistence and Extinction

Basically, persistence of a system means no state of the system approaches zero, that is, there can be no extinction of any of the populations that make up the biological system (Allen, 2007).

Definition 2.1. Given a system of differential equations, $\dfrac{dX}{dt} = F(X,t)$, $X(0) = X_0$, where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$, the system is said to be persistent if for any positive initial conditions $X_0 > 0$, the solution $X(t)$ satisfies

$$\liminf_{t\to\infty} x_i(t) > 0$$

for $i = 1, 2, \cdots, n$.

There are other definitions of persistence that either weaken or strengthen the previous definition. For example, the system is said to be weakly persistent if

$$\limsup_{t\to\infty} x_i(t) > 0$$

for $i = 1, 2, \cdots, n$; uniformly persistent if there exists $\delta > 0$ such that

$$\liminf_{t\to\infty} x_i(t) > \delta$$

for $i = 1, 2, \cdots, n$; permanent if there exists a time $T > 0$ and a compact set K in the interior of the positive cone, $R_i^n = \{(x_1, x_2, \cdots, x_n) \in R^n | x_i > 0, i = 1, 2, \cdots, n\}$ such that $X(t) \in K$ for $t > T$.

Weak persistence and persistence are generally not very good indications of population survival because solutions may be initial condition dependent. For example, in the case of persistence, there could be a set of initial conditions $\{X_0^k\}_{k=1}^{\infty}$ such that the corresponding solution $X^k(t) = (x_i^k(t))$ satisfies

$$\epsilon_k > \liminf_{t\to\infty} x_i^k(t) > 0,$$

where $\epsilon_k \to 0$ as $k \to \infty$ for some i. Even uniform persistence and permanence may not be very good measures of survival since solutions may approach very close to the extinction boundaries if δ is small or the compact set K is close to the extinction boundaries. Another more reasonable type of persistence criterion is referred to as practical persistence. Practical persistence requires that the bounds on the solutions be specified a priori. Given $L_i > 0$ and $M_i > 0$, solution $x_i(t)$ exhibit practical persistence if $0 < L_i < \liminf_{t\to\infty} x_i(t) \leq \limsup_{t\to\infty} x_i(t) \leq M_i, i = 1, 2, \cdots, n$ (Cantrell and Cosner, 1996; Cao and Gard, 1998).

In general, practical persistence implies permanence. Persistence implies weak persistence. If solutions are uniformly bounded, $\limsup_{t\to\infty} x_i(t) < M, i = 1, \cdots, n$, then uniform persistence and permanence are equivalent. If a system has a globally stable equilibrium in R_+^n, then it is permanent. The converse of this statement is not true. If a system is permanent, it may not have a globally stable equilibrium. Further discussion and examples of systems that are permanent or persistent may be found in Hofbauer and Sigmund (1988, 1998) or Freedman and

Moson (1990).

2.2 Discrete Dynamical System

Some mathematical problems of interest in nonlinear difference equations include identification of equilibrium and periodic solutions and analyses of the stability of these types of solutions. Equilibrium solutions are biologically interesting because they represent "resting states" or "stationary states" of the system. The zero solution is often an equilibrium solution. If the zero solution is stable, then the system may approach zero. However, if a positive solution is an equilibrium solution and it is stable, then for initial values close to this equilibrium, solutions approach it. In population dynamics, the zero equilibrium represents population extinction and a positive equilibrium represents survival of the population. The zero equilibrium is often not a desired state, unless, for example, the state represents the proportion of the population that is infected or a population of pests.

It is important to distinguish between local and global stability. Local stability of an equilibrium implies that solutions approach the equilibrium only if they are initially close to it. For example, if the initial population size is very small and the zero equilibrium is stable, then extinction of the population may occur. However, if the initial population size is large, then local stability of the zero equilibrium tells nothing about population extinction. Global stability of an equilibrium is much stronger. Global stability implies that regardless of the initial population size, solutions approach the equilibrium. We state conditions for local stability and global stability of an equilibrium in the case of a scalar difference equation, where only one state is modeled such as population size. In addition, we state conditions for local stability of an equilibrium when several states are modeled by first-order difference equations or when one state is modeled by a second-order or higher-order difference equation. These latter conditions are known as the Jury conditions (Allen, 2007).

First, we concentrate on first-order equations and systems.

2.2.1 Basic Definitions and Notation

Definition 2.2. For the first-order difference equation,

$$x_{t+1} = f(x_t), \tag{2.1}$$

an equilibrium solution or steady-state solution is a constant solution $\bar{x}$ to the difference equation, that is, a solution $\bar{x}$ satisfying

$$\bar{x} = f(\bar{x}). \tag{2.2}$$

For the first-order system, $X_{t+1} = F(X_t)$, an equilibrium solution or a steady-state solution is a constant solution $\bar{X}$ satisfying

$$\bar{X} = f(\bar{X}). \tag{2.3}$$

solution $\bar{x}$ satisfying (2.2) or $\bar{X}$ satisfying (2.3) are also called fixed points of the function f or F, respectively.

The term "equilibrium solution" or "steady-state solution" is often shortened to "equilibrium" or "steady-state." For the two-dimensional, first-order system,

$$x_{t+1} = f(x_t, y_t),$$
$$y_{t+1} = g(x_t, y_t),$$

an equilibrium solution is a solution $(\bar{x}, \bar{y})$ such that $\bar{x} = f(\bar{x}, \bar{y})$ and $\bar{y} = g(\bar{x}, \bar{y})$. An equilibrium solution for a higher-order difference equation $f(x_{t+k}, \cdots, x_{t+1}, x_t) = 0$ is a solution $\bar{x}$ satisfying $f(\bar{x}, \cdots, \bar{x}, \bar{x}) = 0$.

For convenience, we introduce an alternate notation for the solution at time t, x_t in (2.1). The solution can be expressed in terms of the initial value x_0. Denote $f(f(x_0)) = f^2(x_0)$. In general,

$$x_t = f(f(\cdots f(x_0)\cdots)) = f^t(x_0),$$

where the superscript t represents the number of time steps or iterations beginning from the initial value x_0.

Solutions to the difference equation (2.1) may exhibit periodic behavior.

Definition 2.3. A periodic solution of period $m > 1$ of the difference equation (2.1) is a real-valued solution $\bar{x}_k$ satisfying

$$f^m(\bar{x}_k) = \bar{x}_k \quad \text{and} \quad f^i(\bar{x}_k) \neq \bar{x}_k \quad \text{for} \quad i = 1, 2, \ldots, m-1.$$

An m-cycle is a set of points $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_m\}$, where for each $k = 1, \cdots, m$, $\bar{x}_k$ is a periodic solution of period m. The set $\{\bar{x}_1, f(\bar{x}_1), \cdots, f^{m-1}(\bar{x}_1)\}$ is called the periodic orbit of $\bar{x}_1$. A periodic solution of period m of the first order system $X_{t+1} = F(X_t)$ is a real-valued vector $\bar{X}_k$ satisfying

$$F^m(\bar{X}_k) = \bar{X}_k \quad \text{and} \quad F^i(\bar{X}_k) \neq \bar{X}_k \quad \text{for} \quad i = 1, 2, \ldots, m-1.$$

An m-cycle is a set of vectors $\{\bar{X}_1, F(\bar{X}_1), \cdots, F^{m-1}(\bar{X}_1)\}$ is called the periodic orbit of $\bar{X}_1$.

If $\bar{x}_k, k = 1, \cdots, m-1$ is a periodic solution, then each $\bar{x}_k$ is a fixed point of the functions f^m, f^{2m}, f^{3m}, and so on (or $\bar{X}_k$ is a fixed point of the functions F^m, F^{2m}, F^{3m}, and so on). In addition, Definition 2.3, implies that a solution of period m is the smallest value such that $f^m(\bar{x}_k) = \bar{x}_k$ or $F^m(\bar{X}_k) = \bar{X}_k$.

Next, we define the local stability of an equilibrium. An equilibrium is called locally asymptotically stable if for any small perturbation away from the equilibrium, the solution returns to the equilibrium value. In mathematical terminology,

Definition 2.4. An equilibrium solution $\bar{x}$ of (2.1) is locally stable if, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - \bar{x}| < \delta$, then

$$|x_t - \bar{x}| = |f^t(x_0) - \bar{x}| < \epsilon \quad \text{for every} \quad t \geq 0.$$

If $\bar{x}$ is not stable it is said to be unstable. The equilibrium solution $\bar{x}$ is locally attracting if there exists $\gamma > 0$ such that for all $|x_0 - \bar{x}| < \gamma$.

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} f^t(x_0) = \bar{x}.$$

The equilibrium solution $\bar{x}$ is locally asymptotically stable if it is locally stable and locally attracting.

The convergence behavior for a first-order difference equation of the form (2.1) that is locally asymptotically stable may take one of two forms, either convergent oscillations or convergent exponential solutions. If the solution values tend to amplify themselves and do not converge to the equilibrium no matter how small the value of ϵ, then the equilibrium is unstable. Such instability may appear as divergent oscillations or divergent exponential solutions. The case where solutions do not converge toward or diverge away from the equilibrium is sometimes referred to as neutral stability (stable, but not asymptotically stable).

In the next sections, we give criteria for determining local stability for first-order equations and systems.

2.2.2 Local Stability in First-Order Equations

In a study of local stability, first equilibrium solutions are identified, then linearization techniques are applied to determine the behavior of solutions near the equilibrium. If the equilibrium is stable for any set of initial conditions, then this type of stability is referred to as global stability. Some techniques for determining global stability of first-order difference equations are studied in the next section. In the particular case of linear difference equations or linear first-order systems, it will be seen that local and global stability are equivalent (Allen, 2007).

Suppose that the difference equation (2.1) has an equilibrium at $\bar{x}$. The equilibrium is translated to the origin by defining a new variable,

$$u_t = x_t - \bar{x}.$$

Then u_{t+1} satisfies

$$u_{t+1} = x_{t+1} - \bar{x} = f(x_t) - \bar{x} = f(u_t + \bar{x}) - f(\bar{x}) = g(u_t), \tag{2.4}$$

where $g(u) = f(u + \bar{x}) - f(\bar{x})$. The equilibrium $\bar{x}$ in the original system has been translated to zero in the new system. Note that zero is a fixed point of g iff $\bar{x}$ is a fixed point of f. In addition, zero is a locally stable (unstable or locally asymptotically stable) fixed point of g iff $\bar{x}$ is a locally stable (unstable or locally asymptotically stable) fixed point of f.

To find conditions for local asymptotic stability of $\bar{x}$, we assume f has a continuous second-order derivative in some interval I containing $\bar{x}$. Then Taylor's Theorem with remainder can be applied,

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{f''(\xi)}{2!}(x - \bar{x})^2$$

for some $\xi \in I$. For $(x - \bar{x}_t)$ sufficiently small, the following linear approximation is valid: $f(x_t) - \bar{x} \approx f'(\bar{x})(x_t - \bar{x})$ or $u_{t+1} \approx f'(\bar{x})u_t$. We refer to this latter approximation as the linear approximation to the difference equation (2.1) at the equilibrium $\bar{x}$:

$$u_{t+1} = f'(\bar{x})u_t. \tag{2.5}$$

If x_0 is sufficiently close to $\bar{x}$ then the dynamics of u_t are determined by the linearization (2.5). The value of $f'(\bar{x})$ determines whether $\bar{x}$ is locally asymptotically stable or unstable. If $|f'(\bar{x})| > 1$, then u_t will not approach 0 (and x_t will not approach $\bar{x}$), and if $|f'(\bar{x})| < 1$, then u_t approaches 0 (and x_t approaches $\bar{x}$). There is exponential convergence if $0 < f'(x) < 1$ and oscillatory convergence if $-1 < f'(x) < 0$. We have the following theorem.

Theorem 2.1. *Assume f' is continuous on an open interval I containing $\bar{x}$ and $\bar{x}$ is a fixed point of f. Then $\bar{x}$ is a locally asymptotically stable equilibrium of $x_{t+1} = f(x_t)$ if*

$$|f'(\bar{x})| < 1$$

and unstable if

$$|f'(\bar{x})| > 1.$$

A rigorous proof of Theorem 2.1 is based on the Mean Value Theorem and only requires that f' be continuous (Allen, 2007).

The criterion for stability in Theorem 2.1 can be applied to periodic solutions. In the case of a periodic solution of period m, the function $f^m(x)$ is used instead of $f(x)$.

Theorem 2.2. *Suppose f' is continuous on an open interval I and the m-cycle,*

$$\{\bar{x}_1, f(\bar{x}_1), \cdots, f^{m-1}(\bar{x}_1)\}$$

of the difference equation (2.1) is contained in I. Then the m-cycle is locally asymptotically stable if

$$\left| \frac{d[f^m(\bar{x}_k)]}{dx} \right| < 1 \tag{2.6}$$

for some k and unstable if

$$\left|\frac{d[f^m(\bar{x}_k)]}{dx}\right| > 1 \tag{2.7}$$

for some k.

2.2.3 Global Stability in First-Order Equations

Global stability of an equilibrium removes the restrictions on the initial conditions. In global asymptotic stability, solutions approach the equilibrium for all initial conditions. However, because of our applications to biological systems, we consider only positive initial conditions. In addition, we distinguish between global attractivity and global asymptotic stability.

Definition 2.5. Suppose $\bar{x}$ is an equilibrium of the difference equation

$$x_{t+1} = f(x_t), \tag{2.8}$$

where $f : [0, a) \to [0, a)$, $0 < a \leq \infty$. Then $\bar{x}$ is said to be globally attractive if for all initial conditions $x_0 \in (0, a)$, $\lim_{t\to\infty} x_t = \bar{x}$. The equilibrium $\bar{x}$ is said to be globally asymptotically stable if $\bar{x}$ is globally attractive and if $\bar{x}$ is locally stable.

Globally attractive equilibria are locally attractive, and therefore globally asymptotically stable equilibria are locally asymptotically stable. Sedaghat (1997) proved that if the map f is continuous, then a globally attracting equilibrium must be locally asymptotically stable. Thus, for a continuous map f, global attractivity is equivalent to global asymptotic stability (Allen, 2007).

In the global definitions given in Definition 2.5, it is assumed that solutions are nonnegative; the initial conditions and f are restricted to the interval $[0, a)$. For biological models, this is a reasonable assumption. It is often the case in biological models that zero is an equilibrium, $f(0) = 0$. If there is an additional positive equilibrium. $f(\bar{x}) = \bar{x}$, a question of interest is whether the zero or positive equilibrium is globally asymptotically stable. An analogous definition for global asymptotic stability of an equilibrium $\bar{X}$ for first-order systems, $X_{t+1} = F(X_t)$, can be stated.

First, we make some assumptions about the function f.

(i) f is a continuous function on $[0, a), 0 < a \leq \infty$.

(ii) $f : [0, a) \to [0, a), 0 < a \leq \infty$.

Because of assumption (i), continuity of f, global asymptotic stability and global attractivity are equivalent.

The following result shows global asymptotic stability of the origin; solutions approach zero (extinction).

Theorem 2.3. *If the function f of (2.8) satisfies (i), (ii), and $0 < f(x) < x$ for all $x \in (0, a)$, then the origin is globally asymptotically stable.*

2.2.4 Stability in First-Order Systems

We provide a simple criterion for verifying local asymptotic stability of an equilibrium solution to a first-order system of difference equations. We derive criteria for the equilibrium $(\bar{x}, \bar{y})$ of the following two-dimensional first-order system to be locally asymptotically stable.

$$\begin{aligned} x_{t+1} &= f(x_t, y_t), \\ y_{t+1} &= g(x_t, y_t). \end{aligned} \tag{2.9}$$

Before we state the stability result for system (2.9), we linearize the system about the equilibrium.

We assume that f and g have continuous second-order partial derivatives in an open set containing the equilibrium $(\bar{x}, \bar{y})$. Then applying a Taylor series expansion about the equilibrium for the function $f(x,y)$ yields

$$\begin{aligned} f(x,y) &= f(\bar{x},\bar{y}) + \frac{\partial f(\bar{x},\bar{y})}{\partial x}(x-\bar{x}) + \frac{\partial f(\bar{x},\bar{y})}{\partial y}(y-\bar{y}) \\ &\quad + \frac{\partial^2 f(\bar{x},\bar{y})}{\partial x^2}\frac{(x-\bar{x})^2}{2!} + \frac{\partial^2 f(\bar{x},\bar{y})}{\partial y^2}\frac{(y-\bar{y})^2}{2!} + \cdots, \end{aligned}$$

where the notation

$$\frac{\partial f(\bar{x},\bar{y})}{\partial x} = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(\bar{x},\bar{y}))}$$

has been used.

Denote $u = x - \bar{x}$ and $v = y - \bar{y}$. Then we have

$$\begin{aligned} f(x,y) &\approx f(\bar{x},\bar{y}) + \frac{\partial f(\bar{x},\bar{y})}{\partial x}(x-\bar{x}) + \frac{\partial f(\bar{x},\bar{y})}{\partial y}(y-\bar{y}) \\ &= \bar{x} + \frac{\partial f(\bar{x},\bar{y})}{\partial x}u + \frac{\partial f(\bar{x},\bar{y})}{\partial y}v. \end{aligned}$$

A similar approximation for g yields

$$g(x,y) \approx \bar{y} + \frac{\partial g(\bar{x},\bar{y})}{\partial x}u + \frac{\partial g(\bar{x},\bar{y})}{\partial y}v.$$

The linearization of the system (2.9) about the equilibrium $(\bar{x},\bar{y})$, where $u_t = x_t - \bar{x}$ and $v_t = y_t - \bar{y}$, is given by

$$X_{t+1} = JX_t,$$

where $X_t = (u_t, v_t)^T$ and J is the Jacobian matrix of $(f,g)^T$ evaluated at the equilibrium $(\bar{x},\bar{y})$,

$$J = \begin{pmatrix} \dfrac{\partial f(\bar{x},\bar{y})}{\partial x} & \dfrac{\partial f(\bar{x},\bar{y})}{\partial y} \\ \dfrac{\partial g(\bar{x},\bar{y})}{\partial x} & \dfrac{\partial g(\bar{x},\bar{y})}{\partial y} \end{pmatrix}.$$

Let the Jacobian matrix J be denoted by (a_{ij}), where the elements (a_{ij}) represent the partial derivatives of f or g evaluated at the equilibrium point. Recall that the eigenvalues of the Jacobian matrix $J = (a_{ij})$ are found by solving $\det(J - \lambda I) = 0$ or

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0.$$

in a simplified form, the characteristic equation is

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0 \quad \text{or} \quad \lambda^2 - \mathrm{Tr}(J)\lambda + \det(J) = 0.$$

There is an easy check for local stability in the two-dimensional case. Local stability depends on the values of the trace and the determinant of the Jacobian matrix. The proof follows Edelstein-Keshet (1988).

Theorem 2.4. *Assume the functions $f(x, y)$ and $g(x, y)$ have continuous first-order partial derivatives in x and y on some open set in R^2 that contains the point $(\bar{x}, \bar{y})$. Then the equilibrium point $(\bar{x}, \bar{y})$ of the nonlinear system*

$$x_{t+1} = f(x_t, y_t), \quad y_{t+1} = g(x_t, y_t),$$

is locally asymptotically stable if the eigenvalues of the Jacobian matrix J evaluated at the equilibrium satisfy $|\lambda_j| < 1$ iff

$$|\,Tr(J)| < 1 + det(J) < 2. \tag{2.10}$$

The equilibrium is unstable if some $|\lambda_j| > 1$, that is, if any one of three inequalities is satisfied,

$$Tr(J) > 1 + det(J). \quad Tr(J) < -1 - det(J), \quad or \quad det(J) > 1. \tag{2.11}$$

CHAPTER III

A PREDATOR-PREY MODEL WITH IMPULSIVE DIFFUSION AND RELEASE OF PREDATOR POPULATION

In this chapter, we investigate a predator-prey model with impulsive diffusion and release of the predator population. This is a predator-prey model for two regions, which are connected by diffusion of the predator population. It portrays the evolvement of population. We expect to obtain some dynamical properties of the investigated system. We also expect that the impulsive diffusion and predator release will provide a tactic reliable basis for pest management.

3.1 Introduction

The warfare between human and pests has sustained for thousands of years. In the past few decades, man has adopted a variety advanced and modern weapons, for instance chemical pesticides, biological pesticides, remote sensing and measure, computers, atomic energy, etc. Some brilliant achievements have been obtained. However, the warfare will never be over. Although a great number of pesticides can be used to control pests, insect pests impairing crops are increasingly resistant to pesticides. With pesticides employed, the surviving pests breed a large number of pests with resistance to pesticides. So the pesticide is rendered ineffective to some degree, so that insect pests will continue to thrive. In addition, the chemical pesticides kill not only pests but also their natural enemies. Therefore, insect pests are rampant again, and the effect of chemical control has been undermined. Furthermore, the practice proves that long-term adopting chemical control may lead to disastrous results, for example, environmental contamination and toxicosis of the man and animals and so on.

The use of a natural enemy to suppress pests is one of the most important approaches in pest control. Biological control (Caltagirone and Doutt, 1989; DeBach, 1964; DeBach and Rosen, 1991; Barclay, 1982; Murray, 1989; Freedman, 1976; Grasman et al., 2001; Liu and Chen, 2003) is one of the reduction in pest populations through the actions of other living organisms, often called natural enemies or beneficial species. It is the purposeful introduction and establishment of one or more natural enemies from region of origin of an exotic pest, specifically for the purpose of suppressing the abundance of the pest in a new target region to a level at which it no longer causes economic damage. Jiao et al. (2007) analyzed the dynamics of a stage-structured Holling mass defence predator-prey model with impulsive perturbations on predators

$$\begin{cases} \left.\begin{aligned} x_1'(t) &= rx_2(t) - re^{-w\tau_1}x_2(t-\tau_1) - wx_1(t), \\ x_2'(t) &= re^{-w\tau_1}x_2(t-\tau_1) - \frac{\beta x_2(t)}{1+ax_2+bx_2^2}x_3(t) - d_3x_2(t) - d_4x_2^2(t), \\ x_3'(t) &= \frac{k\beta x_2(t)}{1+ax_2+bx_2^2}x_3(t) - dx_3(t), \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} \Delta x_1(t) &= 0, \\ \Delta x_2(t) &= 0, \\ \Delta x_3(t) &= \mu, \end{aligned}\right\} t = n\tau, n = 1, 2, \ldots, \\ (\varphi_1(\zeta), \varphi_2(\zeta), \varphi_3(\zeta)) \in C_+ = C([-\tau_1, 0], R_+^3), \varphi_i(0) > 0, i = 1, 2, 3, \end{cases} \tag{3.1}$$

where $x_1(t)$ and $x_2(t)$ represent the immature and mature pest densities, respectively, and $x_3(t)$ denotes the density of the nature enemy. The biological meanings of the parameters can be seen in (Jiao et al., 2007).

The dispersal is a ubiquitous phenomenon in the natural world. It is important for us to understand the ecological and evolutionary dynamics of populations mirrored by the large number of mathematical models devoted to it in the scientific literature (Levin, 1994; Allen, 1983; Song and Chen, 2002; Cui and Chen, 2001). In recent years, the analysis of these models has focused on the coexistence of populations and local (or global) stability of equilibria (Beretta and Takeuchi, 1998; Beretta and Takeuchi, 1987; Freedman and Takeuchi, 1989a; Freedman, 1987; Freedman et al., 1986; Freedman and Takeuchi, 1989b; Hui and Chen, 2005). Spatial factors play a fundamental role in the persistence and stability of the population, although complete results have not yet been obtained even in the simplest one-species case. Whereas the population dynamics with the effects of spatial heterogeneity is modeled by a diffusion process, most previous papers focused on the population dynamical system modeled by ordinary differential equations. But in practice, it is often the case that diffusion occurs at a regular pulse. For example, when winter comes, birds migrate between patches in search for a better environment, whereas they do not diffuse in other seasons, and the excursion of foliage seeds occurs at fixed period of time every year. Thus, impulsive diffusion provides a more natural description. Lately theories of impulsive differential equations (Bainov and Simeonov, 1993) have been introduced into population dynamics. Jiao et al. (2010) propose to investigate the dynamical behaviors of a stage-structured predator-prey model with prey impulsively diffusing between two patches

$$
\begin{cases}
\left.\begin{aligned}
\frac{dx_1(t)}{dt} &= x_1(t)(a_1 - b_1 x_1(t)),\\
\frac{dx_2(t)}{dt} &= x_2(t)(a_2 - b_2 x_2(t)) - \alpha x_2(t) y_2(t),\\
\frac{dy_1(t)}{dt} &= k\alpha x_2(t) y_2(t) - k\alpha e^{-w\tau_1} x_2(t-\tau_1) y_2(t-\tau_1) - w y_1(t),\\
\frac{dy_2(t)}{dt} &= k\alpha e^{-w\tau_1} x_2(t-\tau_1) y_2(t-\tau_1) - d y_2(t),
\end{aligned}\right\} t \neq n\tau,\\
\left.\begin{aligned}
\Delta x_1(t) &= d_1(x_2(t) - x_1(t)),\\
\Delta x_2(t) &= d_2(x_1(t) - x_2(t)),\\
\Delta y_1(t) &= 0,\\
\Delta y_2(t) &= 0,
\end{aligned}\right\} t = n\tau, n = 1, 2, \ldots,
\end{cases}
\tag{3.2}
$$

where we suppose that the system is composed of two patches connected by diffusion and occupied by a single species; $x_i\,(i = 1, 2)$ is the density of species in the ith patch, and $y_1(t)$ and $y_2(t)$ represent the densities of the immature individual predator and mature individual predator at time t in the second patch. The biological meanings of parameters can be seen in (Jiao et al., 2010).

The organization of this chapter is as follows. In the next section, we introduce the model and background concepts. In Section 3, some important lemmas are presented. We give the globally asymptotically stable conditions of the prey-extinction boundary periodic solution of system (3.3) and the permanent condition of system (3.3) in Section 4. Simulation analysis and brief discussion are given in the last section to conclude this chapter.

3.2 The Model

In this chapter, we establish a predator-prey model with periodic impulsive diffusion and periodic release of predator population:

$$
\left\{
\begin{array}{l}
\left.
\begin{array}{l}
\dfrac{dx_1(t)}{dt} = x_1(t)(a_1 - b_1 x_1(t)) - \dfrac{\beta_1 x_1(t) y_1(t)}{\sigma_1 + x_1(t)}, \\
\dfrac{dy_1(t)}{dt} = \dfrac{k_1 \beta_1 x_1(t) y_1(t)}{\sigma_1 + x_1(t)} - d_1 y_1(t), \\
\dfrac{dx_2(t)}{dt} = x_2(t)(a_2 - b_2 x_2(t)) - \dfrac{\beta_2 x_2(t) y_2(t)}{\sigma_2 + x_2(t)}, \\
\dfrac{dy_2(t)}{dt} = \dfrac{k_2 \beta_2 x_2(t) y_2(t)}{\sigma_2 + x_2(t)} - d_2 y_2(t),
\end{array}
\right\} t \neq (n+l)\tau, t \neq (n+1)\tau, \\
\left.
\begin{array}{l}
\Delta x_1(t) = 0, \\
\Delta y_1(t) = D(y_2(t) - y_1(t)), \\
\Delta x_2(t) = 0, \\
\Delta y_2(t) = D(y_1(t) - y_2(t)),
\end{array}
\right\} t = (n+l)\tau, n \in Z_+, \\
\left.
\begin{array}{l}
\Delta x_1(t) = 0, \\
\Delta y_1(t) = \mu_1, \\
\Delta x_2(t) = 0, \\
\Delta y_2(t) = \mu_2,
\end{array}
\right\} t = (n+1)\tau, n \in Z_+,
\end{array}
\right.
\tag{3.3}
$$

where we suppose that the system is composed of two patches connected by diffusion. These two patches are separated by rivers or highways or railways. The predator population can traverse the rivers or highways or railways, whereas the prey population cannot. In this system, $x_i(t)$ and $y_i(t)$ represent the sizes of prey and predator populations in patch $i\,(i = 1, 2)$ at time t, $a_i > 0$ represents the intrinsic growth rate of the prey population in patch $i\,(i = 1, 2)$, and $b_i > 0$ represents the coefficient of the intraspecific competition of the prey population in patch $i\,(i=1,2)$. The predator consumes the prey according to Holling type-II functional response

$$\frac{\beta_i x_i(t)}{\sigma_i + x_i(t)}(i=1,2)$$

with the half-saturation constant σ_i in patch $i\,(i=1,2)$ at time t. $k_i\,(i=1,2)$ is the rate of conversion of nutrients into the reproduction of the predator in patch $i\,(i=1,2)$, $d_i\,(i=1,2)$ represents the death in patch $i\,(i=1,2)$. The pulse diffusion occurs every τ period (τ is a positive constant), the system evolves from its initial state without being further affected by diffusion until the next pulse appears; $\Delta y_i((n+l)\tau) = y_i((n+l)\tau^+) - y_i((n+l)\tau)$, where $y_i((n+l)\tau^+)$ represents the density of population in the ith patch immediately after the nth diffusion pulse at time $t=(n+l)\tau$, whereas $y_i((n+l)\tau)$ represents the density of population in the ith patch before the nth diffusion pulse at time $t=(n+l)\tau, 0<l<1, n\in Z_+$, $0<D<1$ represents the diffusive rate between the patches, $\Delta y_i((n+1)\tau) = y_i((n+1)\tau^+) - y_i((n+1)\tau)$, and $\mu_i\,(i=1,2)$ represents the releasing amount of predator population at $t=(n+1)\tau, n\in Z_+$ in patch $i\,(i=1,2)$.

3.3 The Lemmas

The solution of (3.3), denoted by $X(t) = (x_1(t), y_1(t), x_2(t), y_2(t))^T$, is a piecewise continuous function $X{:}R_+ \to R_+^4$, $X(t)$ is continuous on $(n\tau, (n+l)\tau]$ and $((n+l)\tau, (n+1)\tau]$, $n\in Z_+$, and $X(n\tau^+) = \lim_{t\to n\tau^+} X(t)$, $X((n+l)\tau^+) = \lim_{t\to (n+l)\tau^+} X(t)$ exist. Obviously, the global existence and uniqueness of solutions of (3.3) is guaranteed by the smoothness properties of f, the mapping defined by the right-hand side of system (3.3) (Bainov and Simeonov, 1993).

Let $V: R_+ \times R_+^4 \to R_+$. Then V is said to belong to class V_0 if

(i) V is continuous in $(n\tau, (n+l)\tau] \times R_+^4$ and $((n+l)\tau, (n+1)\tau] \times R_+^4$, for all $z\in R_+^4, n\in Z_+$, and $V(n\tau^+, z) = \lim_{(t,y)\to(n\tau^+,z)} V(t,y)$ and $V((n+l)\tau^+, z)$

$= \lim_{(t,y)\to((n+l)\tau^+,y)} V(t,y)$ exist.

(ii) V is locally Lipschitzian in z.

Definition 3.1. If $V \in V_0$, then, for $(t,z) \in (n\tau,(n+l)\tau] \times R_+^4$ and $((n+l)\tau,(n+1)\tau] \times R_+^4$, the upper right derivative of $V(t,z)$ with respect to the impulsive differential system (3.3) is defined as

$$D^+V(t,z) = \limsup_{h\to 0} \frac{1}{h}[V(t+h, z+hf(t,z)) - V(t,z)].$$

Since $\frac{dx_i(t)}{dt} = 0$ when $x_i(t) = 0$, $\frac{dy_i(t)}{dt} = 0$ when $y_i(t) = 0$, and $\Delta y_i(t) = \mu_i > 0$ when $t = (n+1)\tau$, we easily obtain the following lemma.

Lemma 3.1. *Suppose that $X(t)$ is a solution of (3.3) with $X(0^+) \geq 0$. Then $X(t) \geq 0$ for $t \geq 0$, and further $X(t) > 0$ $(t \geq 0)$ for $X(0^+) > 0$.*

Lemma 3.2. *(Lakshmikantham et al., 1989) Let the function $m \in PC'[R_+, R]$ satisfy the inequalities*

$$\begin{cases} m'(t) \leq p(t)m(t) + q(t), t \geq t_0, t \neq t_k, k = 1,2,\ldots, \\ m(t_k^+) \leq d_k m(t_k) + b_k, t = t_k, \end{cases} \tag{3.4}$$

where $p, q \in C[R_+, R]$ and $d_k \geq 0$ and b_k are constants. Then

$$\begin{aligned} m(t) \leq\ & m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^t p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^t p(s)ds\right)\right) b_k \\ & + \int_{t_0}^t \prod_{s<t_k<t} d_k \exp\left(\int_s^t p(\sigma)d\sigma\right) q(s)ds, \quad t \geq t_0. \end{aligned}$$

Now, we show that all solutions of (3.3) are uniformly ultimately bounded.

Lemma 3.3. *There exists a constant $M > 0$ such that $x_i(t) \leq M$, $y_i(t) \leq M$ $(i = 1,2)$ for each solution $(x_1(t), y_1(t), x_2(t), y_2(t))$ of (3.3) with all t large enough.*

Proof. Define

$$V(t) = k_1x_1(t) + y_1(t) + k_2x_2(t) + y_2(t),$$

and $\lambda = \min_{i=1,2}\{d_i\}$. When $t \neq n\tau, t \neq (n+l)\tau$, we have

$$\begin{aligned}
& D^+V(t) + \lambda V(t) \\
= \;& k_1x_1(t)[(a_1+\lambda) - b_1x_1(t)] - (d_1-\lambda)y_1(t) \\
& + k_2x_2(t)[(a_2+\lambda) - b_2x_2(t)] - (d_2-\lambda)y_2(t) \\
\leq \;& k_1x_1(t)[(a_1+\lambda) - b_1x_1(t)] + k_2x_2(t)[(a_2+\lambda) - b_2x_2(t)] \\
= \;& -k_1b_1\Big(x_1(t) - \frac{a_1+\lambda}{2b_1}\Big)^2 + \frac{k_1(a_1+\lambda)^2}{4b_1} \\
& -k_2b_2\Big(x_2(t) - \frac{a_2+\lambda}{2b_2}\Big)^2 + \frac{k_2(a_2+\lambda)^2}{4b_2} \\
\leq \;& \frac{k_1(a_1+\lambda)^2}{4b_1} + \frac{k_2(a_2+\lambda)^2}{4b_2} \triangleq \zeta.
\end{aligned}$$

When $t = n\tau$, we have

$$\begin{aligned}
V(n\tau^+) &= k_1x_1(n\tau^+) + y_1(n\tau^+) + k_2x_2(n\tau^+) + y_2(n\tau^+) \\
&= k_1x_1(n\tau) + y_1(n\tau) + \mu_1 + k_2x_2(n\tau) + y_2(n\tau) + \mu_2 \\
&= k_1x_1(n\tau) + y_1(n\tau) + k_2x_2(n\tau) + y_2(n\tau) + \mu_1 + \mu_2 \\
&= V(n\tau) + (\mu_1 + \mu_2).
\end{aligned}$$

When $t = (n+l)\tau$, we have

$$\begin{aligned}
& V((n+l)\tau^+) \\
= \;& k_1x_1((n+l)\tau^+) + y_1((n+l)\tau^+) + k_2x_2((n+l)\tau^+) + y_2((n+l)\tau^+) \\
= \;& k_1x_1((n+l)\tau) + (1-D)y_1((n+l)\tau) + Dy_2((n+l)\tau) + k_2x_2((n+l)\tau) \\
& + Dy_1((n+l)\tau) + (1-D)y_2((n+l)\tau) \\
= \;& k_1x_1((n+l)\tau) + y_1((n+l)\tau) + k_2x_2((n+l)\tau) + y_2((n+l)\tau) \\
= \;& V((n+l)\tau).
\end{aligned}$$

By Lemma 3.2, for $t \in (n\tau, (n+1)\tau]$, we have

$$\begin{aligned} V(t) &\leq V(0^+)e^{-\lambda t} + \frac{\zeta}{\lambda}(1-e^{-\lambda t}) + (\mu_1+\mu_2)\frac{e^{-\lambda(t-\tau)}}{1-e^{\lambda\tau}} + (\mu_1+\mu_2)\frac{e^{\lambda\tau}}{e^{\lambda\tau}-1} \\ &\to \frac{\zeta}{\lambda} + (\mu_1+\mu_2)\frac{e^{\lambda\tau}}{e^{\lambda\tau}-1} \text{ as } t\to\infty. \end{aligned}$$

So $V(t)$ is uniformly ultimately bounded. Hence, by the definition of $V(t)$ we have that there exists a constant $M>0$ such that $x_i(t) \leq M, y_i(t) \leq M\,(i=1,2)$ for t large enough. The proof is complete. □

If $x_i(t) = 0\,(i=1,2)$, then we have the subsystem of (3.3)

$$\begin{cases} \left.\begin{aligned} \frac{dy_1(t)}{dt} &= -d_1y_1(t), \\ \frac{dy_2(t)}{dt} &= -d_2y_2(t), \end{aligned}\right\} t \neq (n+l)\tau, t\neq (n+1)\tau, \\ \left.\begin{aligned} \Delta y_1(t) &= D(y_2(t)-y_1(t)), \\ \Delta y_2(t) &= D(y_1(t)-y_2(t)), \end{aligned}\right\} t = (n+l)\tau, \\ \left.\begin{aligned} \Delta y_1(t) &= -\mu_1, \\ \Delta y_2(t) &= -\mu_2, \end{aligned}\right\} t = (n+1)\tau, n=1,2,\dots. \end{cases} \tag{3.5}$$

We obtain the analytic solution of (3.5) between pulses as follows:

$$\begin{cases} y_1(t) = \begin{cases} y_1(n\tau^+)e^{-d_1(t-n\tau)}, t\in (n\tau,(n+l)\tau], \\ y_1((n+l)\tau^+)e^{-d_1(t-(n+l)\tau)}, t\in ((n+l)\tau,(n+1)\tau], \end{cases} \\ y_2(t) = \begin{cases} y_2(n\tau^+)e^{-d_2(t-n\tau)}, t\in (n\tau,(n+l)\tau], \\ y_2((n+l)\tau^+)e^{-d_2(t-(n+l)\tau)}, t\in ((n+l)\tau,(n+1)\tau]. \end{cases} \end{cases} \tag{3.6}$$

Considering the third and fourth equations of (3.5), we have

$$\begin{cases} y_1((n+l)\tau^+) = (1-D)e^{-d_1l\tau}y_1(n\tau^+) + De^{-d_2l\tau}y_2(n\tau^+), \\ y_2((n+l)\tau^+) = De^{-d_1l\tau}y_1(n\tau^+) + (1-D)e^{-d_2l\tau}y_2(n\tau^+). \end{cases} \tag{3.7}$$

In fact, from $\Delta y_1(t) = D(y_2(t) - y_1(t)), t = (n+l)\tau$, we have

$$y_1((n+l)\tau^+) - y_1((n+l)\tau) = D[y_2((n+l)\tau) - y_1((n+l)\tau)]$$

i.e.,

$$y_1((n+l)\tau^+) = (1-D)y_1((n+l)\tau) + Dy_2((n+l)\tau).$$

From (3.6), we have

$$y_1((n+l)\tau) = y_1(n\tau^+)e^{-d_1 l\tau}, \quad y_2((n+l)\tau) = y_2(n\tau^+)e^{-d_2 l\tau},$$

Hence,

$$y_1((n+l)\tau^+) = (1-D)e^{-d_1 l\tau}y_1(n\tau^+) + De^{-d_2 l\tau}y_2(n\tau^+).$$

From $\Delta y_2(t) = D(y_1(t) - y_2(t)), t = (n+l)\tau$, we have

$$y_2((n+l)\tau^+) - y_2((n+l)\tau) = D[y_1((n+l)\tau) - y_2((n+l)\tau)],$$

i.e.,

$$y_2((n+l)\tau^+) = Dy_1((n+l)\tau) + (1-D)y_2((n+l)\tau).$$

Together with

$$y_1((n+l)\tau) = y_1(n\tau^+)e^{-d_1 l\tau}, \quad y_2((n+l)\tau) = y_2(n\tau^+)e^{-d_2 l\tau},$$

we have

$$y_2((n+l)\tau^+) = De^{-d_1 l\tau}y_1(n\tau^+) + (1-D)e^{-d_2 l\tau}y_2(n\tau^+).$$

Considering the fifth and sixth equations of (3.5), we also have

$$\begin{cases} y_1((n+1)\tau^+) = y_1((n+l)\tau^+)e^{-d_1(1-l)\tau} + \mu_1, \\ y_2((n+1)\tau^+) = y_2((n+l)\tau^+)e^{-d_2(1-l)\tau} + \mu_2. \end{cases} \tag{3.8}$$

In fact, from $\Delta y_1(t) = \mu_1, t = (n+1)\tau$, we have

$$y_1((n+1)\tau^+) = y_1((n+1)\tau) + \mu_1.$$

From (3.6), $y_1((n+1)\tau) = y_1((n+l)\tau^+)e^{-d_1(1-l)\tau}$, then

$$y_1((n+1)\tau^+) = y_1((n+l)\tau^+)e^{-d_1(1-l)\tau} + \mu_1.$$

Similarly, from $\Delta y_2(t) = \mu_2, t = (n+1)\tau$, we have

$$y_2((n+1)\tau^+) = y_2((n+1)\tau) + \mu_2.$$

From (3.6), $y_2((n+1)\tau) = y_2((n+l)\tau^+)e^{-d_2(1-l)\tau}$. Thus,

$$y_2((n+1)\tau^+) = y_2((n+l)\tau^+)e^{-d_2(1-l)\tau} + \mu_2.$$

Substituting (3.7) into (3.8), we have the stroboscopic map of (3.5)

$$\begin{cases} y_1((n+1)\tau^+) = (1-D)e^{-d_1\tau}y_1(n\tau^+) + De^{-[d_1(1-l)+d_2 l]\tau}y_2(n\tau^+) + \mu_1, \\ \\ y_2((n+1)\tau^+) = De^{-[d_1 l+d_2(1-l)]\tau}y_1(n\tau^+) + (1-D)e^{-d_2\tau}y_2(n\tau^+) + \mu_2. \end{cases} \tag{3.9}$$

In fact,

$$\begin{aligned} y_1((n+1)\tau^+) &= y_1((n+l)\tau^+)e^{-d_1(1-l)\tau} + \mu_1 \\ &= [(1-D)e^{-d_1 l\tau}y_1(n\tau^+) + De^{-d_2 l\tau}y_2(n\tau^+)]e^{-d_1(1-l)\tau} + \mu_1 \\ &= (1-D)e^{-d_1\tau}y_1(n\tau^+) + De^{-[d_1(1-l)\tau+d_2 l]\tau}y_2(n\tau^+) + \mu_1, \end{aligned}$$

and

$$\begin{aligned} y_2((n+1)\tau^+) &= y_2((n+l)\tau^+)e^{-d_2(1-l)\tau} + \mu_2 \\ &= [De^{-d_1 l\tau}y_1(n\tau^+) + (1-D)e^{-d_2 l\tau}y_2(n\tau^+)]e^{-d_2(1-l)\tau} + \mu_2 \\ &= De^{-[d_1 l+d_2(1-l)]\tau}y_1(n\tau^+) + (1-D)e^{-d_2\tau}y_2(n\tau^+) + \mu_2. \end{aligned}$$

System (3.9) has one fixed point

$$\begin{cases} y_1^* = \dfrac{\mu_2 B_1 + \mu_1(1-B_2)}{(1-A_1)(1-B_2) - A_2 B_1} > 0, \\ y_2^* = \dfrac{\mu_1 A_2 + \mu_2(1-A_1)}{(1-A_1)(1-B_2) - A_2 B_1} > 0, \end{cases} \tag{3.10}$$

where

$$A_1 = (1-D)e^{-d_1\tau} < 1,$$

$$B_1 = De^{-[d_1(1-l)+d_2 l]\tau} < 1,$$

$$A_2 = De^{-[d_1 l+d_2(1-l)]\tau} < 1,$$

$$B_2 = (1-D)e^{-d_2\tau} < 1.$$

Lemma 3.4. *The fixed point (y_1^*, y_2^*) of (3.9) is globally asymptotically stable.*

Proof. For convenience, we denote $(y_1^n, y_2^n) = (y_1(n\tau^+), y_2(n\tau^+))$. The linear form of (3.9) can be written as

$$\begin{pmatrix} y_1^{n+1} \\ \\ y_2^{n+1} \end{pmatrix} = M \begin{pmatrix} y_1^n \\ \\ y_2^n \end{pmatrix}. \tag{3.11}$$

Obviously, the near dynamics of (y_1^*, y_2^*) is determined by linear system (3.11). The stability of (y_1^*, y_2^*) is determined by the eigenvalue of M less than 1. If M satisfies the Jury criterion (Jury, 1974), then we know that the eigenvalue of M is less than 1,

$$1 - \mathrm{tr}M + \det M > 0. \tag{3.12}$$

We easily see that (y_1^*, y_2^*) is a unique fixed point of (3.9) and

$$M = \begin{pmatrix} A_1 & B_1 \\ \\ A_2 & B_2 \end{pmatrix}. \tag{3.13}$$

Since

$$
\begin{aligned}
& 1-\operatorname{tr} M+\det M \\
= \ & 1-(A_1+B_2)+(A_1B_2-A_2B_1) \\
= \ & (1-A_1)(1-B_2)-A_2B_1 \\
= \ & [1-(1-D)e^{-d_1\tau}]\times[1-(1-D)e^{-d_2\tau}]-De^{-[d_1l+d_2(1-l)]\tau}\cdot De^{-[d_1(1-l)+d_2l]\tau} \\
= \ & [1-(1-D)e^{-d_1\tau}]\times[1-(1-D)e^{-d_2\tau}]-D^2e^{-(d_1+d_2)\tau} \\
= \ & (1-e^{-d_1\tau})\times(1-e^{-d_2\tau})+De^{-d_2\tau}(1-e^{-d_1\tau})+De^{-d_1\tau}(1-e^{-d_2\tau}) \\
> \ & 0,
\end{aligned}
$$

by the Jury criterion, (y_1^*, y_2^*) is locally stable, and then, it is globally asymptotically stable. This completes the proof. □

Lemma 3.5. *The periodic solution $(\widetilde{y_1(t)}, \widetilde{y_2(t)})$ of system (3.5) is globally asymptotically stable, where*

$$
\begin{cases}
\widetilde{y_1(t)} = \begin{cases} y_1^* e^{-d_1(t-n\tau)}, t\in(n\tau,(n+l)\tau], \\ y_1^{**}e^{-d_1(t-(n+l)\tau)}, t\in((n+l)\tau,(n+1)\tau], \end{cases} \\
\widetilde{y_2(t)} = \begin{cases} y_2^* e^{-d_2(t-n\tau)}, t\in(n\tau,(n+l)\tau], \\ y_2^{**}e^{-c_2(t-(n+l)\tau)}, t\in((n+l)\tau,(n+1)\tau], \end{cases}
\end{cases}
\tag{3.14}
$$

and where y_1^* and y_2^* are determined as in (3.10), and y_1^{**} and y_2^{**} are defined as

$$
\begin{cases}
y_1^{**}=(1-D)e^{-d_1l\tau}y_1^*+De^{-d_2l\tau}y_2^*, \\
y_2^{**}=De^{-d_1l\tau}y_1^*+(1-D)e^{-d_2l\tau}y_2^*.
\end{cases}
\tag{3.15}
$$

3.4 The Dynamics

Theorem 3.6. *If*

$$D < \frac{1}{2} \tag{3.16}$$

and

$$\max_{i=1,2}\left\{a_i\tau - \frac{\beta_i[y_i^*(1-e^{-d_il\tau}) + y_i^{**}(1-e^{-d_i(1-l)\tau})]}{\sigma_i d_i}\right\} < 0 \quad (i = 1, 2), \tag{3.17}$$

then the prey-extinction boundary periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ *of (3.3) is globally asymptotically stable, where* y_i^* $(i = 1, 2)$ *and* y_i^{**} $(i = 1, 2)$ *are defined by (3.10) and (3.15).*

Proof. First, we prove the local stability of the prey-extinction boundary periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ of (3.3). Defining $x_1(t) = x_1(t), y_{11}(t) = y_1(t) - \widetilde{y_1(t)}, x_2(t) = x_2(t), y_{12}(t) = y_2(t) - \widetilde{y_2(t)}$, we have the following linear system which is similar to (3.3) and which has one periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$:

$$\begin{pmatrix} \frac{dx_1(t)}{dt} \\ \frac{dy_{11}(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dy_{12}(t)}{dt} \end{pmatrix} = \begin{pmatrix} a_1 - \frac{\beta_1\widetilde{y_1(t)}}{\sigma_1} & 0 & 0 & 0 \\ \frac{k_1\beta_1\widetilde{y_1(t)}}{\sigma_1} & -d_1 & 0 & 0 \\ 0 & 0 & a_2 - \frac{\beta_2\widetilde{y_2(t)}}{\sigma_2} & 0 \\ 0 & 0 & \frac{k_2\beta_2\widetilde{y_2(t)}}{\sigma_2} & -d_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ y_{11}(t) \\ x_2(t) \\ y_{12}(t) \end{pmatrix}.$$

It is easy to obtain the fundamental matrix

$$\Phi(t) = \begin{pmatrix} \exp\left[\int_0^t (a_1 - \frac{\beta_1\widetilde{y_1(s)}}{\sigma_1})ds\right] & 0 & 0 & 0 \\ \exp\left[\int_0^t \frac{k_1\beta_1\widetilde{y_1(s)}}{\sigma_1}ds\right] & \exp(-d_1t) & 0 & 0 \\ 0 & 0 & \exp\left[\int_0^t (a_2 - \frac{\beta_2\widetilde{y_2(s)}}{\sigma_2})ds\right] & 0 \\ 0 & 0 & \exp\left[\int_0^t \frac{k_2\beta_2\widetilde{y_2(s)}}{\sigma_2}ds\right] & \exp(-d_2t) \end{pmatrix}.$$

The linearization of the fifth, sixth, seventh, and eighth equations of (3.3) is

$$\begin{pmatrix} x_1((n+l)\tau^+) \\ y_{11}((n+l)\tau^+) \\ x_2((n+l)\tau^+) \\ y_{12}((n+l)\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-D & 0 & D \\ 0 & 0 & 1 & 0 \\ 0 & D & 0 & 1-D \end{pmatrix} \begin{pmatrix} x_1((n+l)\tau) \\ y_{11}((n+l)\tau) \\ x_2((n+l)\tau) \\ y_{12}((n+l)\tau) \end{pmatrix}.$$

The linearization of the ninth, tenth, eleventh, and twelfth equations of (3.3) is

$$\begin{pmatrix} x_1((n+1)\tau^+) \\ y_{11}((n+1)\tau^+) \\ x_2((n+1)\tau^+) \\ y_{12}((n+1)\tau^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1((n+1)\tau) \\ y_{11}((n+1)\tau) \\ x_2((n+1)\tau) \\ y_{12}((n+1)\tau) \end{pmatrix}.$$

The stability of the periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ is determined by the eigenvalues of

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-D & 0 & D \\ D & 0 & 1 & 0 \\ 0 & D & 0 & 1-D \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Phi(\tau),$$

which are

$$\begin{aligned}
\lambda_1 &= \exp\left[\int_0^\tau \left(a_1 - \frac{\beta_1 \widetilde{y_1(s)}}{\sigma_1}\right) ds\right], \\
\lambda_2 &= \left|\frac{(1-D)(K_1+K_3) + \sqrt{(1-D)^2(K_1+K_3)^2 - 4(1-2D)K_1K_3}}{2}\right| \\
&\leq \left|\frac{(1-D)(K_1+K_3) + \sqrt{(1+D)^2(K_1+K_3)^2}}{2}\right| \\
&\leq \left|\frac{(K_1+K_3)}{2}\right| < 1, \\
\lambda_3 &= \exp\left[\int_0^\tau \left(a_2 - \frac{\beta_2 \widetilde{y_2(s)}}{\sigma_2}\right) ds\right],
\end{aligned}$$

and

$$\begin{aligned}
\lambda_4 &= \left|\frac{(1-D)(K_1+K_3) - \sqrt{(1-D)^2(K_1+K_3)^2 - 4(1-2D)K_1K_3}}{2}\right| \\
&\leq \left|\frac{(1-D)(K_1+K_3) - \sqrt{(1-D)^2(K_1-K_3)^2}}{2}\right| \\
&= \left|\frac{(1-D)(K_1+K_3) - (1-D)|K_1-K_3|}{2}\right| \\
&\leq (1-D)\max\{K_1, K_3\} < 1,
\end{aligned}$$

where $K_1 = e^{-d_1\tau} < 1$, $K_3 = e^{-d_2\tau} < 1$, and condition (3.16) holds. According to conditions (3.16), (3.17), and the Floquet theory (Bainov and Simeonov, 1993), if

$$\exp\left[\int_0^\tau \left(a_i - \frac{\beta_i \widetilde{y_i(s)}}{\sigma_i}\right) ds\right] < 1 \quad (i = 1, 2),$$

then

$$\lambda_1 < 1$$

and

$$\lambda_3 < 1,$$

and thus the prey-extinction boundary periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ of (3.3) is locally stable.

In fact, we can compute

$$\exp\left[\int_0^\tau \left(a_i - \frac{\beta_i \widetilde{y_i(s)}}{\sigma_i}\right) ds\right] < 1 \quad (i = 1, 2).$$

To see this, consider

$$\widetilde{y_1(t)} = \begin{cases} y_1^* e^{-d_1(t-n\tau)}, t \in (n\tau, (n+l)\tau], \\ \\ y_1^{**} e^{-d_1(t-(n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau]. \end{cases}$$

For $t \in (0, l\tau] \cup (l\tau, \tau]$, taking $n = 0$, we have

$$\widetilde{y_1(t)} = \begin{cases} y_1^* e^{-d_1 t}, t \in (0, l\tau], \\ \\ y_1^{**} e^{-d_1(t-l\tau)}, t \in (l\tau, \tau]. \end{cases}$$

We need to show that

$$\begin{aligned} & \exp\left[\int_0^\tau (a_1 - \frac{\beta_1 \widetilde{y_1(s)}}{\sigma_1}) ds\right] \\ = \; & \exp[a_1\tau - \frac{\beta_1}{\sigma_1}\int_0^\tau \widetilde{y_1(s)} ds] < 1. \end{aligned}$$

For this, it suffices to show that

$$a_1\tau - \frac{\beta_1}{\sigma_1}\int_0^\tau \widetilde{y_1(s)} ds < 0.$$

Since

$$\int_0^\tau \widetilde{y_1(s)} ds = \int_0^{l\tau} \widetilde{y_1(s)} ds + \int_{l\tau}^\tau \widetilde{y_1(s)} ds,$$

and

$$\begin{aligned} \int_0^{l\tau} \widetilde{y_1(s)} ds &= \int_0^{l\tau} y_1^* e^{-d_1 s} ds = -\frac{y_1^*}{d_1} e^{-d_1 s}\Big|_{s=0}^{s=l\tau} \\ &= -\frac{y_1^*}{d_1}(e^{-d_1 l\tau} - 1) \\ &= \frac{y_1^*}{d_1}(1 - e^{-d_1 l\tau}), \end{aligned}$$

and

$$\begin{aligned} \int_{l\tau}^\tau \widetilde{y_1(s)} ds &= \int_{l\tau}^\tau y_1^{**} e^{-d_1(s-l\tau)} ds = -\frac{y_1^{**}}{d_1} e^{-d_1(s-l\tau)}\Big|_{s=l\tau}^{s=\tau} \\ &= -\frac{y_1^{**}}{d_1}[e^{-d_1(1-l)\tau} - 1] \\ &= \frac{y_1^{**}}{d_1}(1 - e^{-d_1(1-l)\tau}), \end{aligned}$$

then

$$\begin{aligned}\int_0^\tau \widetilde{y_1(s)}ds &= \int_0^{l\tau} \widetilde{y_1(s)}ds + \int_{l\tau}^\tau \widetilde{y_1(s)}ds \\ &= \frac{y_1^*}{d_1}(1-e^{-d_1 l\tau}) + \frac{y_1^{**}}{d_1}(1-e^{-d_1(1-l)\tau}) \\ &= \frac{y_1^*(1-e^{-d_1 l\tau}) + y_1^{**}(1-e^{-d_1(1-l)\tau})}{d_1}.\end{aligned}$$

By

$$a_1\tau - \frac{\beta_1}{\sigma_1}\int_0^\tau \widetilde{y_1(s)}ds < 0,$$

we have

$$a_1\tau - \frac{\beta_1[y_1^*(1-e^{-d_1 l\tau}) + y_2^{**}(1-e^{-d_1(1-l)\tau})]}{\sigma_1 d_1} < 0.$$

Similarly,

$$\widetilde{y_2(t)} = \begin{cases} y_2^* e^{-d_2(t-n\tau)}, t \in (n\tau, (n+l)\tau], \\ \\ y_2^{**} e^{-d_2(t-(n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau]. \end{cases}$$

For $t \in (0, l\tau] \cup (l\tau, \tau]$, taking $n = 0$, we have

$$\widetilde{y_2(t)} = \begin{cases} y_2^* e^{-d_2 t}, t \in (0, l\tau], \\ \\ y_2^{**} e^{-d_2(t-l\tau)}, t \in (l\tau, \tau]. \end{cases}$$

We need to show that

$$\begin{aligned} & \exp\left[\int_0^\tau (a_2 - \frac{\beta_2 \widetilde{y_2(s)}}{\sigma_2})ds\right] \\ = \ & \exp[a_2\tau - \frac{\beta_2}{\sigma_2}\int_0^\tau \widetilde{y_2(s)}ds] < 1, \end{aligned}$$

or equivalently,

$$a_2\tau - \frac{\beta_2}{\sigma_2}\int_0^\tau \widetilde{y_2(s)}ds < 0.$$

Since

$$\int_0^\tau \widetilde{y_2(s)}ds = \int_0^{l\tau} \widetilde{y_2(s)}ds + \int_{l\tau}^\tau \widetilde{y_2(s)}ds,$$

and

$$\begin{aligned}
\int_0^{l\tau} \widetilde{y_2(s)}ds &= \int_0^{l\tau} y_2^* e^{-d_2 s} ds = -\frac{y_2^*}{d_2} e^{-d_2 s}\Big|_{s=0}^{s=l\tau} \\
&= -\frac{y_2^*}{d_2}[e^{-d_2 l\tau} - 1] \\
&= \frac{y_2^*}{d_2}(1 - e^{-d_2 l\tau}),
\end{aligned}$$

and

$$\begin{aligned}
\int_{l\tau}^{\tau} \widetilde{y_2(s)}ds &= \int_{l\tau}^{\tau} y_2^{**} e^{-d_2(s-l\tau)} ds = -\frac{y_2^{**}}{d_2} e^{-d_2(s-l\tau)}\Big|_{s=l\tau}^{s=\tau} \\
&= -\frac{y_2^{**}}{d_2}[e^{-d_2(1-l)\tau} - 1] \\
&= \frac{y_2^{**}}{d_2}(1 - e^{-d_2(1-l)\tau}),
\end{aligned}$$

then

$$\begin{aligned}
\int_0^{\tau} \widetilde{y_2(s)}ds &= \int_0^{l\tau} \widetilde{y_2(s)}ds + \int_{l\tau}^{\tau} \widetilde{y_2(s)}ds \\
&= \frac{y_2^*}{d_2}(1 - e^{-d_2 l\tau}) + \frac{y_2^{**}}{d_2}(1 - e^{-d_2(1-l)\tau}) \\
&= \frac{y_2^*(1 - e^{-d_2 l\tau}) + y_2^{**}(1 - e^{-d_2(1-l)\tau})}{d_2}.
\end{aligned}$$

By

$$a_2\tau - \frac{\beta_2}{\sigma_2}\int_0^{\tau} \widetilde{y_2(s)}ds < 0,$$

we have

$$a_2\tau - \frac{\beta_2[y_2^*(1 - e^{-d_2 l\tau}) + y_2^{**}(1 - e^{-d_2(1-l)\tau})]}{\sigma_2 d_2} < 0.$$

Thus,

$$\max_{i=1,2}\left\{a_i\tau - \frac{\beta_i[y_i^*(1 - e^{-d_i l\tau}) + y_i^{**}(1 - e^{-d_i(1-l)\tau})]}{\sigma_i d_i}\right\} < 0 \quad (i = 1, 2).$$

In the following, we will prove the global attraction. By condition (3.17) we can choose $\varepsilon > 0$ such that

$$\rho_i = \exp\left[\int_0^{\tau}\left(a_i - \frac{\beta_i(\widetilde{y_i(s)} - \varepsilon)}{\sigma_i}\right)ds\right] < 1 \quad (i = 1, 2).$$

From the second and fourth equations of (3.3) we notice that $\frac{dy_i(t)}{dt} \geq -d_i y_i(t)\,(i = 1, 2)$. Then, we consider the following impulsive comparative differential equation:

$$\begin{cases} \left.\begin{aligned} \frac{dy_{21}(t)}{dt} &= -d_1 y_{21}(t), \\ \frac{dy_{22}(t)}{dt} &= -d_2 y_{22}(t), \end{aligned}\right\} t \neq (n+l)\tau, t \neq (n+1)\tau, \\ \left.\begin{aligned} \Delta y_{21}(t) &= D(y_{22}(t) - y_{21}(t)), \\ \Delta y_{22}(t) &= D(y_{21}(t) - y_{22}(t)), \end{aligned}\right\} t = (n+l)\tau, \\ \left.\begin{aligned} \Delta y_{21}(t) &= \mu_1, \\ \Delta y_{22}(t) &= \mu_2, \end{aligned}\right\} t = (n+1)\tau. \end{cases} \tag{3.18}$$

From Lemma 3.5 and the comparison theorem of impulsive equations [see Theorem 3.1.1 in (Lakshmikantham et al., 1989)] we have $y_1(t) \geq y_{21}(t)$, $y_2(t) \geq y_{22}(t)$, and $y_{21}(t) \to \widetilde{y_1(t)}$, $y_{22}(t) \to \widetilde{y_2(t)}$ as $t \to \infty$. Then

$$\begin{cases} y_1(t) \geq y_{21}(t) \geq \widetilde{y_1(t)} - \varepsilon, \\ y_2(t) \geq y_{22}(t) \geq \widetilde{y_2(t)} - \varepsilon, \end{cases} \tag{3.19}$$

for t large enough. For convenience, we may assume that (3.19) holds for all $t \geq 0$. From (3.3) and (3.19) we get

$$\frac{dx_i(t)}{dt} \leq \left[a_i - \frac{\beta_i(\widetilde{y_i(t)} - \varepsilon)}{\sigma_i}\right] x_i(t) \quad (i = 1, 2). \tag{3.20}$$

So $x_i((n+1)\tau) \leq x_i(n\tau^+) \exp[\int_{n\tau}^{(n+1)\tau} (a_i - \frac{\beta_i(\widetilde{y_i(s)} - \varepsilon)}{\sigma_i}) ds]\,(i = 1, 2)$. Hence $x_i(n\tau) \leq x_i(0^+)\rho_i^n\,(i = 1, 2)$ and $x_i(n\tau) \to 0\,(i = 1, 2)$ as $n \to \infty$; therefore $x_i(t) \to 0\,(i = 1, 2)$ as $t \to \infty$.

Next, we will prove that $y_i(t) \to \widetilde{y_i(t)}\,(i = 1, 2)$ as $t \to \infty$. For $\varepsilon_1 > 0$, there must exist a $t_0 > 0$ such that $0 < x_i(t) < \varepsilon_1\,(i = 1, 2)$ for all $t \geq t_0$. Without loss of generality, we may assume that $0 < x_i(t) < \varepsilon_1$ for all $t \geq 0$. For system (3.3), we have

$$-d_i y_i(t) \leq \frac{dy_i(t)}{dt} \leq -\left(d_i - \frac{k_i\beta_i\varepsilon_1}{\sigma_i+\varepsilon_1}\right) y_i(t) \quad (i=1,2), \tag{3.21}$$

and then we have $y_{21}(t) \leq y_1(t) \leq y_{31}(t)$, $y_{22}(t) \leq y_2(t) \leq y_{32}(t)$, and $y_{21}(t) \to \widetilde{y_1(t)}, y_{22}(t) \to \widetilde{y_2(t)}$, $y_{31}(t) \to \widetilde{y_{31}(t)}, y_{32}(t) \to \widetilde{y_{32}(t)}$ as $t \to \infty$, where $(y_{21}(t), y_{22}(t))$ and $(y_{31}(t), y_{32}(t))$ are the solutions of (3.18) and

$$\begin{cases} \left.\begin{aligned} \frac{dy_{31}(t)}{dt} &= -(d_1 - \frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})y_{31}(t), \\ \frac{dy_{32}(t)}{dt} &= -(d_2 - \frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})y_{32}(t), \end{aligned}\right\} t \neq (n+l)\tau, t \neq (n+1)\tau, \\ \left.\begin{aligned} \Delta y_{31}(t) &= D(y_{32}(t) - y_{31}(t)), \\ \Delta y_{32}(t) &= D(y_{31}(t) - y_{32}(t)), \end{aligned}\right\} t = (n+l)\tau, \\ \left.\begin{aligned} \Delta y_{31}(t) &= \mu_1, \\ \Delta y_{32}(t) &= \mu_2, \end{aligned}\right\} t = (n+1)\tau, \end{cases} \tag{3.22}$$

respectively.

$$\begin{cases} \widetilde{y_{31}(t)} = \begin{cases} y_{31}^* e^{-(d_1 - \frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})(t-n\tau)}, t \in (n\tau, (n+l)\tau], \\ y_{31}^{**} e^{-(d_1 - \frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})(t-(n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau], \end{cases} \\ \widetilde{y_{32}(t)} = \begin{cases} y_{32}^* e^{-(d_2 - \frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})(t-n\tau)}, t \in (n\tau, (n+l)\tau], \\ y_{32}^{**} e^{-(d_2 - \frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})(t-(n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau], \end{cases} \end{cases} \tag{3.23}$$

where y_{31}^* and y_{32}^* are determined as

$$\begin{cases} y_{31}^* = \dfrac{\mu_2 B_{31} + \mu_1(1-B_{32})}{(1-A_{31})(1-B_{32}) - A_{32}B_{31}} > 0, \\ y_{32}^* = \dfrac{\mu_1 A_{32} + \mu_2(1-A_{31})}{(1-A_{31})(1-B_{32}) - A_{32}B_{31}} > 0, \end{cases} \tag{3.24}$$

and y_{31}^{**} and y_{32}^{**} are defined as

$$\begin{cases} y_{31}^{**} = (1-D)e^{-(d_1 - \frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})l\tau} y_{31}^* + De^{-(d_2 - \frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})l\tau} y_{32}^*, \\ y_{32}^{**} = De^{-(d_1 - \frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})l\tau} y_{31}^* + (1-D)e^{-(d_2 - \frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})l\tau} y_{32}^*, \end{cases} \tag{3.25}$$

where

$$\begin{aligned}
A_{31} &= (1-D)e^{-(d_1-\frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})\tau} < 1,\\
B_{31} &= De^{-[(d_1-\frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})(1-l)+(d_2-\frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})l]\tau} < 1,\\
A_{32} &= De^{-[(d_1-\frac{k_1\beta_1\varepsilon_1}{\sigma_1+\varepsilon_1})l+(d_2-\frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})(1-l)]\tau} < 1,\\
B_{32} &= (1-D)e^{-(d_2-\frac{k_2\beta_2\varepsilon_1}{\sigma_2+\varepsilon_1})\tau} < 1.
\end{aligned}$$

For any $\varepsilon_2 > 0$, there exists $t_1, t > t_1$, such that

$$\widetilde{y_{21}(t)} - \varepsilon_2 < y_1(t) < \widetilde{y_{31}(t)} + \varepsilon_2$$

and

$$\widetilde{y_{22}(t)} - \varepsilon_2 < y_2(t) < \widetilde{y_{32}(t)} + \varepsilon_2.$$

Letting $\varepsilon_1 \to 0$, we have

$$\widetilde{y_1(t)} - \varepsilon_2 < y_1(t) < \widetilde{y_1(t)} + \varepsilon_2$$

and

$$\widetilde{y_2(t)} - \varepsilon_2 < y_2(t) < \widetilde{y_2(t)} + \varepsilon_2$$

for t large enough, which implies $y_1(t) \to \widetilde{y_1(t)}$ and $y_2(t) \to \widetilde{y_2(t)}$ as $t \to \infty$. This completes the proof. □

The next work is to investigate the permanence of system (3.3).

Definition 3.2. System (3.3) is said to be permanent if there are constants m, $M > 0$ (independent of the initial value) and a finite time T_0 such that for all solutions $(x_1(t), y_1(t), x_2(t), y_2(t))$ with any initial values $x_1(0^+) > 0$, $y_1(0^+) > 0$, $x_2(0^+) > 0$, $y_2(0^+) > 0$, we have $m \le x_1(t) \le M$, $m \le y_1(t) \le M$, $m \le x_2(t) \le M$, $m \le y_2(t) \le M$ for all $t \ge T_0$. Here T_0 may depend on the initial values $(x_1(0^+), y_1(0^+), x_2(0^+), y_2(0^+))$.

Theorem 3.7. *If*

$$\min_{i=1,2}\left\{a_i\tau-\frac{\beta_i[y_i^*(1-e^{-d_il\tau})+y_i^{**}(1-e^{-d_i(1-l)\tau})]}{\sigma_id_i}\right\}>0 \quad (i=1,2), \tag{3.26}$$

then system (3.3) is permanent, where $y_i^*\,(i=1,2)$ and $y_i^{**}\,(i=1,2)$ are defined by (3.10) and (3.15), respectively.

Proof. Suppose $(x_1(t),y_1(t),x_2(t),y_2(t))$ is a solution of (3.3) with $x_1(0)>0,y_1(0)>0,x_2(0)>0,y_2(0)>0$. By Lemma 3.3 there exists a constant $M>0$ such that $x_1(t)\le M,y_1(t)\le M,x_2(t)\le M,y_2(t)\le M$ for t large enough. From (3.3) and Theorem 3.6 we have $y_i(t)>\widetilde{y_i(t)}-\varepsilon_2>y_i^*e^{-d_il\tau}+y_i^{**}e^{-d_i(1-l)\tau}\triangleq m_i\,(i=1,2)$ for ε_2 small enough. So we only need to find $m_3>0$ and ε_3 such that $x_i(t)>m_3$ for t large enough. Otherwise, we can select $m_4>0$ small enough satisfying $m_4<\frac{\sigma_id_i}{k_i\beta_i-d_i}\,(d_i<k_i\beta_i)$ and prove $x_i(t)<m_4$ cannot hold for $t\ge 0$. Suppose the contrary. By condition (3.26), choosing ε_3 small enough, we can obtain

$$\delta_i=a_i\tau-\frac{\beta_i[y_{4i}^*(1-e^{-(d_i-\frac{k_i\beta_im_4}{\sigma_i+m_4})l\tau})+y_{4i}^{**}(1-e^{-(d_i-\frac{k_i\beta_im_4}{\sigma_i+m_4})(1-l)\tau})]}{\sigma_i(d_i-\frac{k_i\beta_im_4}{\sigma_i+m_4})}-\frac{\beta_i\varepsilon_3}{\sigma_i}\tau>0$$

with $y_{4i}^*\,(i=1,2)$ and $y_{4i}^{**}\,(i=1,2)$ defined as in (3.30) and (3.31)below. Then,

$$\begin{cases}\left.\begin{aligned}\frac{dy_1(t)}{dt}&<-(d_1-\frac{k_1\beta_1m_4}{\sigma_1+m_4})y_1(t),\\ \frac{dy_2(t)}{dt}&<-(d_2-\frac{k_2\beta_2m_4}{\sigma_2+m_4})y_2(t),\end{aligned}\right\} t\ne(n+l)\tau, t\ne(n+1)\tau,\\ \left.\begin{aligned}\Delta y_1(t)&=D(y_2(t)-y_1(t)),\\ \Delta y_2(t)&=D(y_1(t)-y_2(t)),\end{aligned}\right\} t=(n+l)\tau,\\ \left.\begin{aligned}\Delta y_1(t)&=\mu_1,\\ \Delta y_2(t)&=\mu_2,\end{aligned}\right\} t=(n+1)\tau.\end{cases} \tag{3.27}$$

By Lemma 3.5 we have $y_1(t)\le y_{41}(t)$, $y_2(t)\le y_{42}(t)$ and $y_{41}(t)\to$

$\overline{y_{41}(t)}, y_{42}(t) \to \overline{y_{42}(t)}, t \to \infty$, where $(y_{41}(t), y_{42}(t))$ is the solution of

$$\begin{cases} \left.\begin{aligned} \frac{dy_{41}(t)}{dt} &= -(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})y_{41}(t), \\ \frac{dy_{42}(t)}{dt} &= -(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})y_{42}(t), \end{aligned}\right\} t \neq (n+l)\tau, t \neq (n+1)\tau, \\ \left.\begin{aligned} \Delta y_{41}(t) &= D(y_{42}(t) - y_{41}(t)), \\ \Delta y_{42}(t) &= D(y_{41}(t) - y_{42}(t)), \end{aligned}\right\} t = (n+l)\tau, \\ \left.\begin{aligned} \Delta y_{41}(t) &= \mu_1, \\ \Delta y_{42}(t) &= \mu_2, \end{aligned}\right\} t = (n+1)\tau, \end{cases} \tag{3.28}$$

with

$$\begin{cases} \overline{y_{41}(t)} = \begin{cases} y_{41}^* e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(t - n\tau)}, t \in (n\tau, (n+l)\tau], \\ y_{41}^{**} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(t - (n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau], \end{cases} \\ \overline{y_{42}(t)} = \begin{cases} y_{42}^* e^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})(t - n\tau)}, t \in (n\tau, (n+l)\tau], \\ y_{42}^{**} e^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})(t - (n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau], \end{cases} \end{cases} \tag{3.29}$$

where y_{41}^* and y_{42}^* are determined as

$$\begin{cases} y_{41}^* = \dfrac{\mu_2 B_{41} + \mu_1(1 - B_{42})}{(1 - A_{41})(1 - B_{42}) - A_{42}B_{41}} > 0, \\ y_{42}^* = \dfrac{\mu_1 A_{42} + \mu_2(1 - A_{41})}{(1 - A_{41})(1 - B_{42}) - A_{42}B_{41}} > 0, \end{cases} \tag{3.30}$$

and y_{41}^{**}, y_{42}^{**} are defined as

$$\begin{cases} y_{41}^{**} = (1 - D)e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})l\tau} y_{41}^* + De^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})l\tau} y_{42}^*, \\ y_{42}^{**} = De^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})l\tau} y_{41}^* + (1 - D)e^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})l\tau} y_{42}^*, \end{cases} \tag{3.31}$$

where

$$\begin{aligned}
A_{41} &= (1-D)e^{-(d_1-\frac{k_1\beta_1 m_4}{\sigma_1+m_4})\tau} < 1,\\
B_{41} &= De^{-[(d_1-\frac{k_1\beta_1 m_4}{\sigma_1+m_4})(1-l)+(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})l]\tau} < 1,\\
A_{42} &= De^{-[(d_1-\frac{k_1\beta_1 m_4}{\sigma_1+m_4})l+(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})(1-l)]\tau} < 1,\\
B_{42} &= (1-D)e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})\tau} < 1.
\end{aligned}$$

Therefore, there exist $T_1 > 0$ and $\varepsilon_3 > 0$ such that

$$y_1(t) \le y_{41}(t) \le \overline{y_{41}(t)} + \varepsilon_3$$

and

$$y_2(t) \le y_{42}(t) \le \overline{y_{42}(t)} + \varepsilon_3.$$

Then,

$$\frac{dx_i(t)}{dt} \ge \left[a_i - \frac{\beta_i(\overline{y_{4i}(t)} + \varepsilon_3)}{\sigma_i}\right] x_i(t) \quad (i = 1, 2), \tag{3.32}$$

for $t \ge T_1$. Let $N_1 \in N$ and $N_1\tau > T_1$. Integrating (3.32) on $(n\tau, (n+1)\tau), n \ge N_1$, we have

$$\begin{aligned}
x_i((n+1)\tau) &\ge x_i(n\tau^+)\exp\left(\int_{n\tau}^{(n+1)\tau}\left[a_i - \frac{\beta_i(\overline{y_{4i}(t)}+\varepsilon_3)}{\sigma_i}\right]dt\right)\\
&= x_i(n\tau)e^{\delta_i} \quad (i = 1, 2).
\end{aligned}$$

Then, $x_i((N_1+k)\tau) \ge x_i(N_1\tau^+)e^{k\delta_i} \to \infty$ as $k \to \infty$, which is a contradiction to the boundedness of $x_i(t)\,(i = 1, 2)$. Hence, there exists $t_1 > 0$ such that $x_i(t) \ge m_3\,(i = 1, 2)$. This completes the proof. □

Now, we give more details about the computation of the δ_i. We need to $\delta_i > 0$, where

$$\begin{aligned}
\delta_i &= \int_0^\tau [a_i - \frac{\beta_i(\overline{y_{4i}(t)} + \varepsilon_3)}{\sigma_i}]dt\\
&= \int_0^\tau [a_i - \frac{\beta_i\overline{y_{4i}(t)}}{\sigma_i} - \frac{\beta_i}{\sigma_i}\varepsilon_3]dt\\
&= a_i\tau - \frac{\beta_i}{\sigma_i}\int_0^\tau \overline{y_{4i}(t)}dt - \frac{\beta_i\varepsilon_3}{\sigma_i}\tau.
\end{aligned}$$

First, we compute $\int_0^{\tau} \overline{y_{41}(t)}dt$.

$$\begin{cases} \overline{y_{41}(t)} = \begin{cases} y_{41}^{*} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(t-n\tau)}, t \in (n\tau, (n+l)\tau], \\ y_{41}^{**} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(t-(n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau], \end{cases} \\ \overline{y_{42}(t)} = \begin{cases} y_{42}^{*} e^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})(t-n\tau)}, t \in (n\tau, (n+l)\tau], \\ y_{42}^{**} e^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})(t-(n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau]. \end{cases} \end{cases}$$

For $t \in (0, \tau] = (0, l\tau] \cup (l\tau, \tau]$, taking $n = 0$, we have,

$$\begin{cases} \overline{y_{41}(t)} = \begin{cases} y_{41}^{*} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})t}, t \in (0, l\tau], \\ y_{41}^{**} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(t-l\tau)}, t \in (l\tau, \tau], \end{cases} \\ \overline{y_{42}(t)} = \begin{cases} y_{42}^{*} e^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})t}, t \in (0, l\tau], \\ y_{42}^{**} e^{-(d_2 - \frac{k_2\beta_2 m_4}{\sigma_2 + m_4})(t-l\tau)}, t \in (l\tau, \tau]. \end{cases} \end{cases}$$

Since $\int_0^{\tau} \overline{y_{41}(t)}dt = \int_0^{l\tau} \overline{y_{41}(t)}dt + \int_{l\tau}^{\tau} \overline{y_{41}(t)}dt$, and

$$\begin{aligned} \int_0^{l\tau} \overline{y_{41}(t)}dt &= \int_0^{l\tau} y_{41}^{*} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})t} dt \\ &= -\frac{y_{41}^{*}}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4}} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})t} \Big|_{t=0}^{t=l\tau} \\ &= -\frac{y_{41}^{*}}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4}} [e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})l\tau} - 1] \\ &= \frac{y_{41}^{*}}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4}} [1 - e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})l\tau}], \end{aligned}$$

and

$$\begin{aligned} \int_{l\tau}^{\tau} \overline{y_{41}(t)}dt &= \int_{l\tau}^{\tau} y_{41}^{**} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(t-l\tau)} dt \\ &= -\frac{y_{41}^{**}}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4}} e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(t-l\tau)} \Big|_{t=l\tau}^{t=\tau} \\ &= -\frac{y_{41}^{**}}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4}} [e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(1-l)\tau} - 1] \\ &= \frac{y_{41}^{**}}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4}} [1 - e^{-(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1 + m_4})(1-l)\tau}], \end{aligned}$$

then,

$$
\begin{aligned}
\delta_1 &= a_1\tau - \frac{\beta_1}{\sigma_1}\int_0^{\tau}\overline{y_{41}(t)}dt - \frac{\beta_1\varepsilon_3}{\sigma_1}\tau \\
&= a_1\tau - \frac{\beta_1}{\sigma_1}\Big[\int_0^{l\tau}\overline{y_{41}(t)}dt + \int_{l\tau}^{\tau}\overline{y_{41}(t)}dt\Big] - \frac{\beta_1\varepsilon_3}{\sigma_1}\tau \\
&= a_1\tau - \frac{\beta_1}{\sigma_1}\Bigg[\frac{y_{41}^*}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1+m_4}}\Big(1 - e^{-(d_1-\frac{k_1\beta_1 m_4}{\sigma_1+m_4})l\tau}\Big) \\
&\quad + \frac{y_{41}^{**}}{d_1 - \frac{k_1\beta_1 m_4}{\sigma_1+m_4}}\Big(1 - e^{-(d_1-\frac{k_1\beta_1 m_4}{\sigma_1+m_4})(1-l)\tau}\Big)\Bigg] \\
&\quad - \frac{\beta_1\varepsilon_3}{\sigma_1}\tau \\
&= a_1\tau \\
&\quad - \frac{\beta_1[y_{41}^*(1 - e^{-(d_1-\frac{k_1\beta_1 m_4}{\sigma_1+m_4})l\tau}) + y_{41}^{**}(1 - e^{-(d_1-\frac{k_1\beta_1 m_4}{\sigma_1+m_4})(1-l)\tau})]}{\sigma_1(d_1 - \frac{k_1\beta_1 m_4}{\sigma_1+m_4})} \\
&\quad - \frac{\beta_1\varepsilon_3}{\sigma_1}\tau \\
&> 0.
\end{aligned}
$$

In a similar way, we compute $\int_0^{\tau}\overline{y_{42}(t)}dt$.

Since $\int_0^{\tau}\overline{y_{42}(t)}dt = \int_0^{l\tau}\overline{y_{42}(t)}dt + \int_{l\tau}^{\tau}\overline{y_{42}(t)}dt$, and

$$
\begin{aligned}
\int_0^{l\tau}\overline{y_{42}(t)}dt &= \int_0^{l\tau} y_{42}^* e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})t}dt \\
&= -\frac{y_{42}^*}{d_2 - \frac{k_2\beta_2 m_4}{\sigma_2+m_4}} e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})t}\Big|_{t=0}^{t=l\tau} \\
&= -\frac{y_{42}^*}{d_2 - \frac{k_2\beta_2 m_4}{\sigma_2+m_4}}[e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})l\tau} - 1] \\
&= \frac{y_{42}^*}{d_2 - \frac{k_2\beta_2 m_4}{\sigma_2+m_4}}[1 - e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})l\tau}],
\end{aligned}
$$

and

$$\begin{aligned}
\int_{l\tau}^{\tau} \overline{y_{42}(t)}dt &= \int_{l\tau}^{\tau} y_{42}^{**} e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})(t-l\tau)}dt \\
&= -\frac{y_{42}^{**}}{d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4}} e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})(t-l\tau)}\Big|_{t=l\tau}^{t=\tau} \\
&= -\frac{y_{42}^{**}}{d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4}}[e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})(1-l)\tau}-1] \\
&= \frac{y_{42}^{**}}{d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4}}[1-e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})(1-l)\tau}],
\end{aligned}$$

then,

$$\begin{aligned}
\delta_2 &= a_2\tau - \frac{\beta_2}{\sigma_2}\int_0^{\tau}\overline{y_{42}(t)}dt - \frac{\beta_2\varepsilon_3}{\sigma_2}\tau \\
&= a_2\tau - \frac{\beta_2}{\sigma_2}\Big[\int_0^{l\tau}\overline{y_{42}(t)}dt + \int_{l\tau}^{\tau}\overline{y_{42}(t)}dt\Big] - \frac{\beta_2\varepsilon_3}{\sigma_2}\tau \\
&= a_2\tau - \frac{\beta_2}{\sigma_2}\Bigg[\frac{y_{42}^{*}}{d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4}}\Big(1-e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})l\tau}\Big) \\
&\quad +\frac{y_{42}^{**}}{d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4}}\Big(1-e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})(1-l)\tau}\Big)\Bigg] \\
&\quad -\frac{\beta_2\varepsilon_3}{\sigma_2}\tau \\
&= a_2\tau \\
&\quad -\frac{\beta_2[y_{42}^{*}(1-e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})l\tau}) + y_{42}^{**}(1-e^{-(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})(1-l)\tau})]}{\sigma_2(d_2-\frac{k_2\beta_2 m_4}{\sigma_2+m_4})} \\
&\quad -\frac{\beta_2\varepsilon_3}{\sigma_2}\tau \\
&> 0.
\end{aligned}$$

Therefore,

$$\delta_i = a_i\tau - \frac{\beta_i[y_{4i}^{*}(1-e^{-(d_i-\frac{k_i\beta_i m_4}{\sigma_i+m_4})l\tau}) + y_{4i}^{**}(1-e^{-(d_i-\frac{k_i\beta_i m_4}{\sigma_i+m_4})(1-l)\tau})]}{\sigma_i(d_i-\frac{k_i\beta_i m_4}{\sigma_i+m_4})} - \frac{\beta_i\varepsilon_3}{\sigma_i}\tau > 0,$$

and thus

$$\min_{i=1,2}\{a_i\tau - \frac{\beta_i[y_i^*(1-e^{-d_il\tau})+y_i^{**}(1-e^{-d_i(1-l)\tau})]}{\sigma_i d_i}\} > 0\,(i=1,2).$$

3.5 Simulation Analysis and Discussion

In this chapter, we have established a predator-prey model with impulsive diffusion and release of predator population. We have proved that all solutions of the investigated system are uniformly ultimately bounded. By Theorem 3.6, if $D < \frac{1}{2}$ and

$$\max_{i=1,2}\left\{a_i\tau - \frac{\beta_i[y_i^*(1-e^{-d_il\tau})+y_i^{**}(1-e^{-d_i(1-l)\tau})]}{\sigma_i d_i}\right\} < 0 \quad (i=1,2),$$

then the prey-extinction boundary periodic solution $(0, \widetilde{y_1(t)}, 0, \widetilde{y_2(t)})$ of system (3.3) is globally asymptotically stable. By Theorem 3.7, if

$$\min_{i=1,2}\left\{a_i\tau - \frac{\beta_i[y_i^*(1-e^{-d_il\tau})+y_i^{**}(1-e^{-d_i(1-l)\tau})]}{\sigma_i d_i}\right\} > 0 \quad (i=1,2),$$

then system (3.3) is permanent.

3.5.1 The Dynamical Behaviors Influenced by Parameter D

Let $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.5, \mu_2 = 0.3, d_1 = 0.3, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.1$. Then conditions (3.16) and (3.17) are obviously satisfied, and thus the prey-extinction periodic solution of system (3.3) is globally asymptotically stable (see Figure 3.1).

Also assume that $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.5, \mu_2 =$

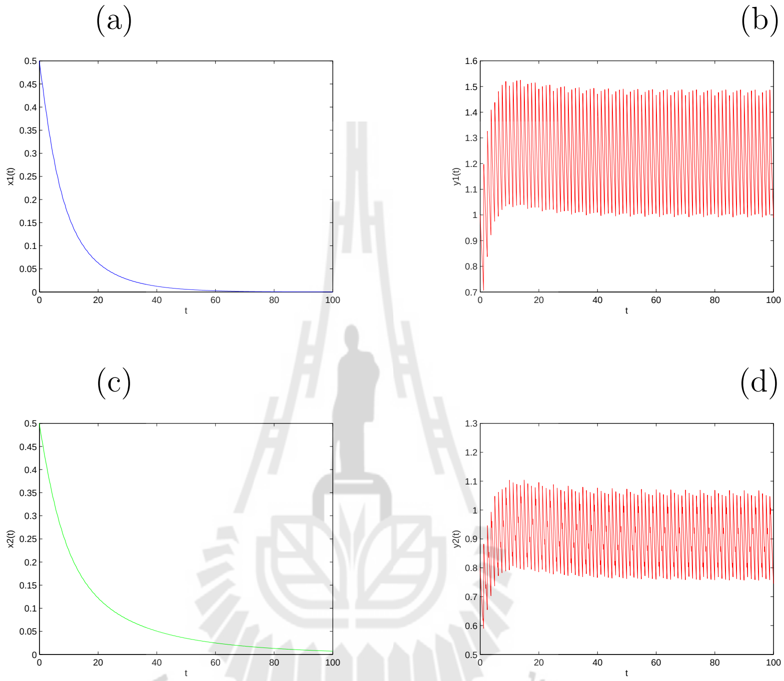

Figure 3.1 Globally asymptotically stable prey-extinction periodic solution of system (3.3) with $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.5, \mu_2 = 0.3, d_1 = 0.3, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.1$. (a) Time-series of $x_1(t)$; (b) Time-series of $y_1(t)$; (c) Time-series of $x_2(t)$; (d) Time-series of $y_2(t)$.

$0.3, d_1 = 0.3, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.95$. Then condition (3.26) is obviously satisfied, and system (3.3) is permanent (see Figure 3.2).

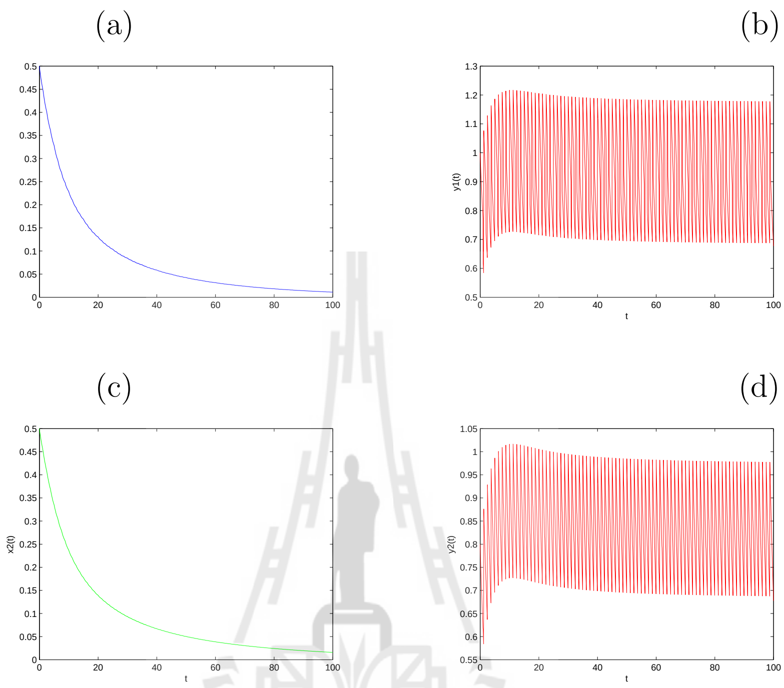

Figure 3.2 The permanence for system (3.3) with $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.5, \mu_2 = 0.3, d_1 = 0.3, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.95$. (a) Time-series of $x_1(t)$; (b) Time-series of $y_1(t)$; (c) Time-series of $x_2(t)$; (d) Time-series of $y_2(t)$.

From (3.17) and (3.26) we can calculate that there exists one threshold D^*, which satisfies

$$\max_{i=1,2}\left\{a_i\tau - \frac{\beta_i[y_i^*(1-e^{-d_il\tau}) + y_i^{**}(1-e^{-d_i(1-l)\tau})]}{\sigma_i d_i}\right\} < 0 \quad (i = 1, 2)$$

or

$$\min_{i=1,2}\left\{a_i\tau - \frac{\beta_i[y_i^*(1-e^{-d_il\tau}) + y_i^{**}(1-e^{-d_i(1-l)\tau})]}{\sigma_i d_i}\right\} > 0 \quad (i = 1, 2).$$

If $D < D^*$, the prey population will go to extinction; if $D > D^*$, the population will be permanent.

3.5.2 The Dynamical Behaviors Influenced by Parameters μ_1 and μ_2

In this subsection, we always assume that $\mu = \mu_1 = \mu_2$. Assume that $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.4, \mu_2 = 0.4, d_1 = 0.4, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.2$. Then conditions (3.16) and (3.17) are obviously satisfied, and the prey-extinction periodic solution of system (3.3) is globally asymptotically stable (see Figure 3.3).

Also assume that $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.3, \mu_2 = 0.3, d_1 = 0.3, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.2$. Then condition (3.26) is obviously satisfied, and system (3.3) is permanent (see Figure 3.4).

We can calculate that there exists at least one threshold μ^* such that if $\mu > \mu^*$, then the prey population will go to extinction, and if $\mu < \mu^*$, then the population will be permanent.

From the simulations we discover that an increasing diffusive rate of predator population will counteract the pest management. We conclude that the impulsive diffusion and releasing predator model provides a reliable tactic basis for pest management.

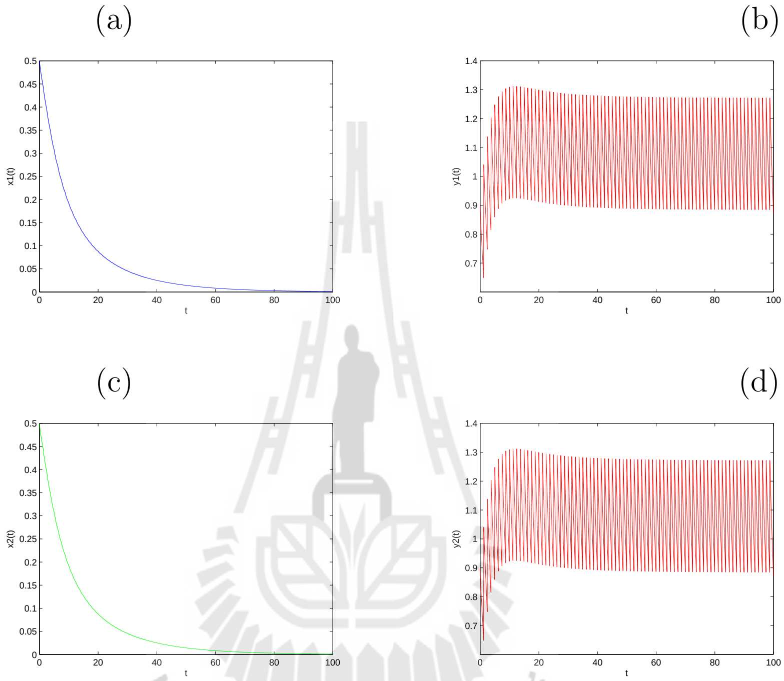

Figure 3.3 Globally asymptotically stable prey-extinction periodic solution of system (3.3) with $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.4, \mu_2 = 0.4, d_1 = 0.4, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.2$, (a) Time-series of $x_1(t)$; (b) Time-series of $y_1(t)$; (c) Time-series of $x_2(t)$; (d) Time-series of $y_2(t)$.

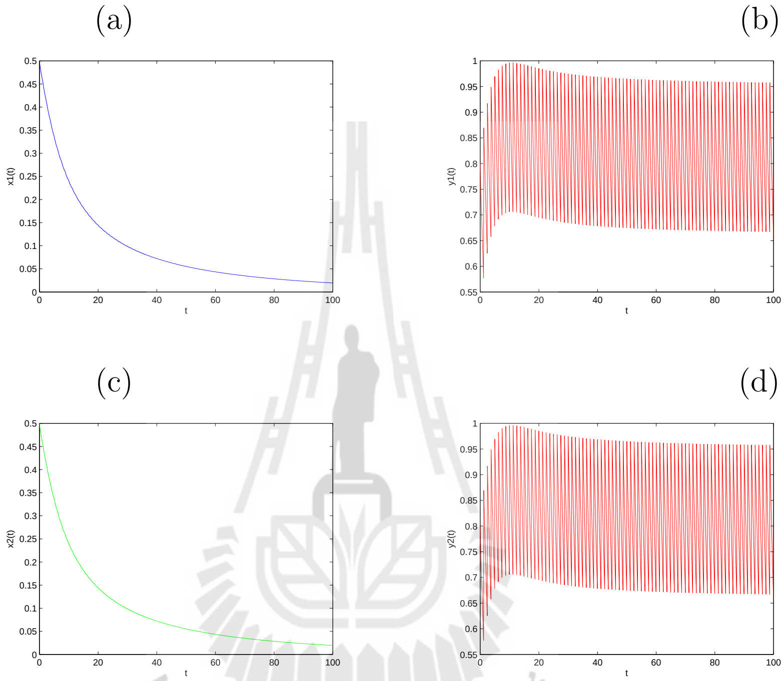

Figure 3.4 The permanence for system (3.3) with $x_1(0) = 0.5, y_1(0) = 0.5, x_2(0) = 0.5, y_2(0) = 0.5, a_1 = 0.1, b_1 = 0.2, a_2 = 0.1, b_2 = 0.2, \beta_1 = 0.5, \beta_2 = 5, k_1 = 0.5, k_2 = 5, \mu_1 = 0.3, \mu_2 = 0.3, d_1 = 0.3, d_2 = 0.3, \sigma_1 = 3.5, \sigma_2 = 3.5, \tau = 1, l = 0.25, D = 0.2$, (a) Time-series of $x_1(t)$; (b) Time-series of $y_1(t)$; (c) Time-series of $x_2(t)$; (d) Time-series of $y_2(t)$.

CHAPTER IV

AN SIR EPIDEMIC MODEL WITH STAGE STRUCTURE AND PULSE VACCINATION

The present chapter is to introduce birth pulse of the population, state structure and pulse vaccination into the SIR epidemic model and obtain some important qualitative properties for the investigated system. As a matter of fact, pulse birth is used in an epidemic model. To the best of our knowledge, no such research has been conducted before.

4.1 Introduction

The SIR (susceptible, infectious, recovered) epidemic model is one of the most popular epidemic models in epidemiology; it was initially proposed by Kermack and Mckendrick (Kermack and Mckendrick, 1927; Kermack and Mckendrick, 1932; Kermack and Mckendrick, 1933; Kermack and Mckendrick, 1937). Since then, SIR models have been considered by many researchers (Shulgin, Stone and Agur, 1998; d'Onofrio, 2005; Gao, Teng and Xie, 2009; Lu, Chi and Chen, 2002; Meng and Chen, 2008b; Buonomo et al., 2008; Xu and Ma, 2009; Zhang and Suo, 2010; Liu et al., 2012; Yuan et al., 2014; Liu, 2015; Jiao et al, 2015). For details of a simple SIR model, we can refer to the books of Hethcote (1989) and Anderson and May (1992).

In addition, Gao et al. (2009) have investigated a delayed SIR epidemic model with pulse vaccination. They conclude that the infection-free periodic solution is globally attractive and the system is permanent. Meng and Chen (2008b) studied the SIR epidemic model with both vertical and horizontal transmission, analyzed some dynamical behaviors, such as the infection-free equilibrium, the positive equilibrium, the permanence, global asymptotic behavior and so on, and obtained some important qualitative properties.

Recently, pulse vaccination strategy, a new vaccination strategy against measles, has been proposed. Its theoretical study was started by Agur et al. in (Agur, Cojocaru, Mazor, Anderson and Danon, 1993). Furthermore, a lot of original work has been done in (Shulgin, Stone and Agur, 1998; Stone, Shulgin and Agur, 2000; Gao et al., 2006; Agur, Cojocaru, Mazor, Anderson and Danon, 1993; d'Onofrio, 2005; Gao, Teng and Xie, 2009; Lu, Chi and Chen, 2002; Meng and Chen, 2008b).

In the real world, individual members of many species experience two stages of life, immature and mature ones. Stage-structured population models have received great attention, and many stage-structured models have been studied in recent years (Li and Wang, 2005; Xiao and Chen, 2001; Wang and Chen, 1997; Song and Chen, 2001; Xiao and Chen, 2003; Aiello and Freedman, 1990; Aiello et al., 1992).

Theories of impulsive differential equations have been introduced into population dynamics lately (Liu, 1995; Lakshmikantham et al., 1989; Bainov and Simeonov, 1993; Liu and Chen, 2003). Impulsive equations are found in almost every domain of applied science and have been studied in many investigations (Jiao and Chen, 2008; Lakshmikantham et al., 1989; Bainov and Simeonov, 1993; Jiao et al., 2008a; Jiao et al., 2008b). They generally describe phenomena which are subject to steep or instantaneous changes.

Motivated by the above studies, our study is to investigate transmission dynamics of an SIR epidemic model with stage structure and pulse vaccination. We assume that the matured population approaches a steady state, if there is no disease infection and all matured individuals are susceptible. We assume full immunity of recovered individuals; that is to say, those individuals are no longer susceptible after they have recovered.

4.2 The Model

In this chapter, we consider an SIR epidemic model with stage structure and pulse vaccination:

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(t)}{dt} &= -(c+d_1)S_1(t), \\ \frac{dS_2(t)}{dt} &= cS_1(t) - d_2S_2(t) - \beta S_2(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S_2(t)I(t) - (r+d_3)I(t), \\ \frac{dR(t)}{dt} &= rI(t) - d_4R(t), \end{aligned}\right\} t \neq n\tau, \quad t \neq (n+l)\tau, \\ \left.\begin{aligned} \Delta S_1(t) &= S_2(t)(a - bS_2(t)), \\ \Delta S_2(t) &= 0, \\ \Delta I(t) &= 0, \\ \Delta R(t) &= 0, \end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\ \left.\begin{aligned} \Delta S_1(t) &= 0, \\ \Delta S_2(t) &= -\mu S_2(t), \\ \Delta I(t) &= 0, \\ \Delta R(t) &= \mu S_2(t), \end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2, \ldots, \end{cases} \tag{4.1}$$

where $S_1(t), S_2(t)$ represent the numbers of the immature and the mature of the susceptibles, and $I(t), R(t)$ represent the numbers of the infectious, and the recovered, respectively. c is called the rate of the immature susceptible turning into the mature susceptible. d_1, d_2, d_3, d_4, respectively denote the natural death rate of the immature susceptible, the mature susceptible, the infectious and the recovered. β is the average number of adequate contacts of a mature infectious individual per unit time. r stands for the recovery rate of the mature infectious individual. The mature susceptible is birth pulse with intrinsic rate of natural increase and density dependence rate of the mature susceptible denoted by a, b, respectively. The pulse birth and pulse vaccination occurs every τ period (τ is a positive constant). $\Delta S_2(t) = S_2(t^+) - S_2(t)$. $\mu(0 < \mu < 1)$ is the proportion of the successful vaccinations which is called pulse vaccination rate, at $t = (n + l)\tau$, $0 < l < 1$, $n \in Z_+$. $\Delta S_1(t) = S_1(t^+) - S_1(t)$, and $S_2(t)(a - bS_2(t))$ represents the birth effort of the mature susceptible at $t = n\tau, n \in Z_+$.

In this chapter, we assume:

(i) The susceptible is infertile after being infected; that is to say, the infectious and the recovered have no ability to reproduce.

(ii) The immature susceptible is immune to the disease for taking from their parent population; that is to say, the immature susceptible achieves temporary immunity.

As the first, second, and third equations do not comprise $R(t)$, we can simplify system (4.1) as follows:

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(t)}{dt} &= -(c+d_1)S_1(t), \\ \frac{dS_2(t)}{dt} &= cS_1(t) - d_2S_2(t) - \beta S_2(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S_2(t)I(t) - (r+d_3)I(t), \end{aligned}\right\} t \neq n\tau, \quad t \neq (n+l)\tau, \\ \left.\begin{aligned} &\Delta S_1(t) = S_2(t)(a - bS_2(t)), \\ &\Delta S_2(t) = 0, \\ &\Delta I(t) = 0, \end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\ \left.\begin{aligned} &\Delta S_1(t) = 0, \\ &\Delta S_2(t) = -\mu S_2(t), \\ &\Delta I(t) = 0, \end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2, \ldots. \end{cases} \tag{4.2}$$

This is equivalent to system (4.1).

4.3 Some Lemmas

Before discussing the main results, we will introduce some definitions, notations and lemmas. Denote by $f = (f_1, f_2, f_3, f_4)$ the map defined by the right-hand side of system (4.1), the solution of (4.1), denoted by $z(t) = (S_1(t), S_2(t), I(t), R(t))^T$, is a piecewise continuous function $z : R_+ \to R_+^4$, where $R_+ = [0, \infty)$, $R_+^4 = \{z \in R^4 : z > 0\}$, $z(t)$ is continuous on $(n\tau, (n+l)\tau] \times R_+^4$ and $((n+l)\tau, (n+1)\tau] \times R_+^4$ $(n \in Z_+, 0 < l < 1)$. According to Lakshmikantham et al. (1989) and Bainov and Simeonov (1993), the global existence and uniqueness of solutions of system (4.1) is guaranteed by the smoothness properties of f, the mapping defined by the right-hand side of system (4.1).

Let $V : R_+ \times R_+^4 \to R_+$. Then V is said to belong to class V_0 if:

(i) V is continuous in $(n\tau, (n+l)\tau] \times R_+^4$ and $((n+l)\tau, (n+1)\tau] \times R_+^4$, for all $z \in R_+^4$, $n \in Z_+$, and $\lim_{(t,y)\to((n+l)\tau^+,z)} V(t,y) = V((n+l)\tau^+, z)$ and $\lim_{(t,y)\to((n+1)\tau^+,z)} V(t,y) = V((n+1)\tau^+, z)$ exist.

(ii) V is locally Lipschitzian in z.

Definition 4.1. If $V \in V_0$, then, for $(t,z) \in (n\tau, (n+l)\tau] \times R_+^4$ and $((n+l)\tau, (n+1)\tau) \times R_+^4$, the upper right derivative of $V(t,z)$ with respect to the impulsive differential system (4.1) is defined as

$$D^+V(t,z) = \lim_{h\to 0} \sup \frac{1}{h}[V(t+h, z+hf(t,z)) - V(t,z)].$$

Lemma 4.1. *(see (Lakshmikantham et al., 1989), Theorem 1.4.1) Let the function $m \in PC'[R_+, R]$ satisfy the inequalities*

$$\begin{cases} m'(t) \leq p(t)m(t) + q(t),\ t \neq t_k,\ k = 1, 2, \ldots, \\ m(t_k^+) \leq d_k m(t_k) + b_k,\ t = t_k,\ t \geq t_0, \end{cases} \tag{4.3}$$

where $p, q \in C[R_+, R]$ and $d_k \geq 0$ and b_k are constants. Then

$$\begin{aligned} m(t) \ \leq\ & m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) b_k \\ & + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(\sigma)d\sigma\right) q(s)ds, \quad t \geq t_0. \end{aligned}$$

Lemma 4.2. *There exists a constant $M > 0$ such that $S_1(t) \leq M$, $S_2(t) \leq M$, $I(t) \leq M$, $R(t) \leq M$ for each solution $(S_1(t), S_2(t), I(t), R(t))$ of system (4.1) with t large enough.*

Proof. Define $V(t) = S_1(t) + S_2(t) + I(t) + R(t)$, $d = \min\{d_1, d_2, d_3, d_4\}$. When

$t \neq (n+l)\tau, t \neq (n+1)\tau$, we have

$$\begin{aligned}
D^+V(t) + dV(t) &= -cS_1(t) - d_1S_1(t) + cS_1(t) - d_2S_2(t) - \beta S_2(t)I(t) \\
&\quad +\beta S_2(t)I(t) - (r+d_3)I(t) + rI(t) - d_4R(t) \\
&\quad +dS_1(t) + dS_2(t) + dI(t) + dR(t) \\
&= -(d_1 - d)S_1(t) - (d_2 - d)S_2(t) \\
&\quad -(d_3 - d)I(t) - (d_4 - d)R(t) \leq \delta \leq 0.
\end{aligned}$$

When $t = (n+l)\tau$, we have

$$\begin{aligned}
V((n+l)\tau^+) &= S_1((n+l)\tau^+) + S_2((n+l)\tau^+) + I((n+l)\tau^+) + R((n+l)\tau^+) \\
&= S_1((n+l)\tau) + (1-\mu)S_2((n+l)\tau) + I((n+l)\tau) \\
&\quad +R((n+l)\tau) + \mu S_2((n+l)\tau) \\
&= S_1((n+l)\tau) + S_2((n+l)\tau) + I((n+l)\tau) + R((n+l)\tau) \\
&= V((n+l)\tau).
\end{aligned}$$

When $t = (n+1)\tau$, we have

$$\begin{aligned}
V((n+1)\tau^+) &= S_1((n+1)\tau^+) + S_2((n+1)\tau^+) + I((n+1)\tau^+) + R((n+1)\tau^+) \\
&= [S_1((n+1)\tau) + S_2((n+1)\tau)(a - bS_2((n+1)\tau))] + S_2((n+1)\tau) \\
&\quad +I((n+1)\tau) + R((n+1)\tau) \\
&= S_1((n+1)\tau) + S_2((n+1)\tau)(a - bS_2((n+1)\tau)) + S_2((n+1)\tau) \\
&\quad +I((n+1)\tau) + R((n+1)\tau) \\
&= V((n+1)\tau) + S_2((n+1)\tau)(a - bS_2((n+1)\tau)) \\
&\leq V((n+1)\tau) + \frac{a^2}{4b}.
\end{aligned}$$

We introduce the notation $\xi = \frac{a^2}{4b} > 0$. Then by Lemma 4.1., for $t \in (n\tau, (n+1)\tau]$, we have

$$\begin{aligned} V(t) &\leq V(0)\exp(-dt) + \int_0^t \delta\exp(-d(t-s))ds + \sum_{0<n\tau<t} \xi\exp(-d(t-n\tau)) \\ &< V(0)\exp(-dt) + \frac{\delta}{d}(1-\exp(-dt)) + \frac{\xi\exp(-d(t-\tau))}{1-\exp(d\tau)} + \frac{\xi\exp(d\tau)}{\exp(d\tau)-1} \\ &\to \frac{\delta}{d} + \frac{\xi\exp(d\tau)}{\exp(d\tau)-1} \quad \text{as} \quad t\to\infty. \end{aligned}$$

So $V(t)$ is uniformly ultimately bounded. Hence, by the definition of $V(t)$ we see that there exists a constant $M > 0$, such that $S_1(t) \leq M$, $S_2(t) \leq M$, $I(t) \leq M$, $R(t) \leq M$ for t large enough. □

We choose the following notation:

$$\Omega^* = \frac{(c+d_1-d_2)[1+e^{-(c+d_1-d_2)\tau} - e^{-(c+d_1)\tau} - e^{d_2\tau}] + ac[1-e^{-(c+d_1-d_2)\tau}]}{(c+d_1-d_2)[1-e^{-(c+d_1)\tau}] + ac[1-e^{-(c+d_1-d_2)l\tau}]}.$$

If $I(t) = 0$, then we have the following subsystem of (4.2):

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(t)}{dt} &= -(c+d_1)S_1(t), \\ \frac{dS_2(t)}{dt} &= cS_1(t) - d_2S_2(t), \end{aligned}\right\} t\neq n\tau, \quad t\neq(n+l)\tau, \\ \left.\begin{aligned} &\Delta S_1(t) = S_2(t)(a-bS_2(t)), \\ &\Delta S_2(t) = 0, \end{aligned}\right\} t = n\tau, \quad n=1,2,\ldots, \\ \left.\begin{aligned} &\Delta S_1(t) = 0, \\ &\Delta S_2(t) = -\mu S_2(t), \end{aligned}\right\} t=(n+l)\tau, \quad n=1,2,\ldots. \end{cases} \tag{4.4}$$

We easily obtain the analytic solution of system (4.4) between pulses as follows:

$$\begin{cases} S_1(t) = S_1(n\tau^+)e^{-(c+d_1)(t-n\tau)}, \quad t\in(n\tau,(n+1)\tau], \\ S_2(t) = \begin{cases} e^{-d_2(t-n\tau)}\left[S_2(n\tau^+) + \frac{cS_1(n\tau^+)(1-e^{-(c+d_1-d_2)(t-n\tau)})}{c+d_1-d_2}\right], \quad t\in(n\tau,(n+l)\tau], \\ e^{-d_2(t-(n+l)\tau)}\left[S_2((n+l)\tau^+) + \dfrac{cS_1((n+l)\tau^+)(1-e^{-(c+d_1-d_2)(t-(n+l)\tau)})}{c+d_1-d_2}\right], \\ \qquad t\in((n+l)\tau,(n+1)\tau]. \end{cases} \end{cases} \tag{4.5}$$

In effect, since $\dfrac{dS_1(t)}{dt} = -cS_1(t) - d_1S_1(t) = -(c+d_1)S_1(t)$, then $\dfrac{dS_1(t)}{S_1(t)} = -(c+d_1)dt, t \in (n\tau, (n+1)\tau]$, i.e., $\dfrac{dS_1(u)}{S_1(u)} = -(c+d_1)du, u \in (n\tau, (n+1)\tau]$, integrating with respect to u from $n\tau$ to t on both sides, we have

$$\begin{aligned}
\int_{n\tau}^{t} \frac{dS_1(u)}{S_1(u)} &= -\int_{n\tau}^{t} (c+d_1)du \\
&\Rightarrow \ln S_1(u)\Big|_{u=n\tau}^{u=t} = -(c+d_1)(t-n\tau) \\
&\Rightarrow \ln S_1(t) - \ln S_1(n\tau^+) = -(c+d_1)(t-n\tau) \\
&\Rightarrow \ln \frac{S_1(t)}{S_1(n\tau^+)} = -(c+d_1)(t-n\tau) \\
&\Rightarrow \frac{S_1(t)}{S_1(n\tau^+)} = e^{-(c+d_1)(t-n\tau)} \\
&\Rightarrow S_1(t) = S_1(n\tau^+)e^{-(c+d_1)(t-n\tau)}, \quad t \in (n\tau, (n+1)\tau].
\end{aligned}$$

Since $\dfrac{dS_2(t)}{dt} = cS_1(t) - d_2S_2(t), \quad t \in (n\tau, (n+l)\tau]$, then

$$\begin{aligned}
\frac{dS_2(t)}{dt} &= cS_1(n\tau^+)e^{-(c+d_1)(t-n\tau)} - d_2S_2(t) \\
&\Rightarrow \frac{dS_2(t)}{dt} + d_2S_2(t) = cS_1(n\tau^+)e^{-(c+d_1)(t-n\tau)} \\
&\Rightarrow e^{d_2t}S_2'(t) + d_2e^{d_2t}S_2(t) = cS_1(n\tau^+)e^{-(c+d_1)(t-n\tau)}e^{d_2t} \\
&\Rightarrow \frac{d}{dt}[e^{d_2t}S_2(t)] = cS_1(n\tau^+)e^{-(c+d_1)t+(c+d_1)n\tau}e^{d_2t} \\
&\Rightarrow d[e^{d_2t}S_2(t)] = cS_1(n\tau^+)e^{(c+d_1)n\tau}e^{-(c+d_1-d_2)t}dt \\
&\Rightarrow d[e^{d_2u}S_2(u)] = cS_1(n\tau^+)e^{(c+d_1)n\tau}e^{-(c+d_1-d_2)u}du, \quad u \in (n\tau, (n+l)\tau].
\end{aligned}$$

Integrating with respect to u from $n\tau$ to t on both sides,

$$\begin{aligned}
&\int_{n\tau}^{t} d(e^{d_2 u} S_2(u)) \\
= \quad & cS_1(n\tau^+)e^{(c+d_1)n\tau}\int_{n\tau}^{t} e^{-(c+d_1-d_2)u}du \\
\Rightarrow \quad & e^{d_2 t}S_2(t) - e^{d_2 n\tau}S_2(n\tau^+) = -\frac{cS_1(n\tau^+)e^{(c+d_1)n\tau}}{c+d_1-d_2}[e^{-(c+d_1-d_2)t} - e^{-(c+d_1-d_2)n\tau}] \\
\Rightarrow \quad & e^{d_2 t}S_2(t) = e^{d_2 n\tau}S_2(n\tau^+) + \frac{cS_1(n\tau^+)e^{(c+d_1)n\tau}}{c+d_1-d_2}(e^{-(c+d_1-d_2)n\tau} - e^{-(c+d_1-d_2)t}) \\
\Rightarrow \quad & e^{d_2 t}S_2(t) = e^{d_2 n\tau}\Bigg[S_2(n\tau^+) \\
& \qquad\qquad + \frac{cS_1(n\tau^+)e^{(c+d_1-d_2)n\tau}}{c+d_1-d_2}(e^{-(c+d_1-d_2)n\tau} - e^{-(c+d_1-d_2)t})\Bigg] \\
\Rightarrow \quad & e^{d_2 t}S_2(t) = e^{d_2 n\tau}\left[S_2(n\tau^+) + \frac{cS_1(n\tau^+)}{c+d_1-d_2}(1 - e^{-(c+d_1-d_2)(t-n\tau)})\right] \\
\Rightarrow \quad & S_2(t) = e^{-d_2(t-n\tau)}\left[S_2(n\tau^+) + \frac{cS_1(n\tau^+)(1 - e^{-(c+d_1-d_2)(t-n\tau)})}{c+d_1-d_2}\right], \\
& \qquad\qquad\qquad\qquad\qquad\qquad t \in (n\tau, (n+l)\tau].
\end{aligned}$$

Similarly, since $\frac{dS_2(t)}{dt} = cS_1(t) - d_2S_2(t), \quad t \in ((n+l)\tau, (n+1)\tau])$, and $S_1(t) = S_1((n+l)\tau^+)e^{-(c+d_1)(t-(n+l)\tau)}, t \in ((n+l)\tau, (n+1)\tau]$, then

$$\frac{dS_2(t)}{dt} = cS_1((n+l)\tau^+)e^{-(c+d_1)(t-(n+l)\tau)} - d_2S_2(t),$$

and we have,

$$
\begin{aligned}
& \frac{dS_2(t)}{dt} + d_2S_2(t) \\
& = cS_1((n+l)\tau^+)e^{-(c+d_1)(t-(n+l)\tau)} \\
\Rightarrow \quad & e^{d_2t}S_2'(t) + d_2e^{d_2t}S_2(t) \\
& = cS_1((n+l)\tau^+)e^{-(c+d_1)(t-(n+l)\tau)}e^{d_2t} \\
\Rightarrow \quad & e^{d_2t}S_2'(t) + d_2e^{d_2t}S_2(t) = cS_1((n+l)\tau^+)e^{-(c+d_1)(t-(n+l)\tau)}e^{d_2t} \\
\Rightarrow \quad & \frac{d}{dt}[e^{d_2t}S_2(t)] = cS_1((n+l)\tau^+)e^{-(c+d_1)(t-(n+l)\tau)}e^{d_2t} \\
\Rightarrow \quad & d[e^{d_2t}S_2(t)] = cS_1((n+l)\tau^+)e^{(c+d_1)(n+l)\tau}e^{-(c+d_1)t}e^{d_2t}dt \\
\Rightarrow \quad & d[e^{d_2t}S_2(t)] = cS_1((n+l)\tau^+)e^{(c+d_1)(n+l)\tau}e^{-(c+d_1-d_2)t}dt \\
\Rightarrow \quad & d[e^{d_2u}S_2(u)] = cS_1((n+l)\tau^+)e^{(c+d_1)(n+l)\tau}e^{-(c+d_1-d_2)u}du, \\
\Rightarrow \quad & \int_{(n+l)\tau}^{t} d[e^{d_2u}S_2(u)] = cS_1((n+l)\tau^+)e^{(c+d_1)(n+l)\tau}\int_{(n+l)\tau}^{t} e^{-(c+d_1-d_2)u}du \\
\Rightarrow \quad & e^{d_2t}S_2(t) - e^{d_2(n+l)\tau}S_2((n+l)\tau^+) \\
& = -\frac{cS_1((n+l)\tau^+)e^{(c+d_1)(n+l)\tau}}{c+d_1-d_2}\left[e^{(c+d_1-d_2)t} - e^{(c+d_1-d_2)(n+l)\tau}\right] \\
\Rightarrow \quad & e^{d_2t}S_2(t) = e^{d_2(n+l)\tau}S_2((n+l)\tau^+) \\
& \qquad + \frac{cS_1((n+l)\tau^+)e^{(c+d_1)(n+l)\tau}}{c+d_1-d_2}\left[e^{(c+d_1-d_2)(n+l)\tau} - e^{(c+d_1-d_2)t}\right] \\
\Rightarrow \quad & e^{d_2t}S_2(t) = e^{d_2(n+l)\tau}\Bigg[S_2((n+l)\tau^+) \\
& \qquad + \frac{cS_1((n+l)\tau^+)e^{(c+d_1-d_2)(n+l)\tau}}{c+d_1-d_2}(e^{-(c+d_1-d_2)(n+l)\tau} - e^{-(c+d_1-d_2)t})\Bigg] \\
\Rightarrow \quad & e^{d_2t}S_2(t) = e^{d_2(n+l)\tau}\left[S_2((n+l)\tau^+) + \frac{cS_1((n+l)\tau^+)}{c+d_1-d_2}(1 - e^{-(c+d_1-d_2)(t-(n+l)\tau)})\right] \\
\Rightarrow \quad & S_2(t) = e^{-d_2(t-(n+l)\tau)}\Bigg[S_2((n+l)\tau^+) \\
& \qquad + \frac{cS_1((n+l)\tau^+)(1 - e^{-(c+d_1-d_2)(t-(n+l)\tau)})}{c+d_1-d_2}\Bigg], t \in ((n+l)\tau, (n+1)\tau].
\end{aligned}
$$

Considering the fourth, fifth, seventh, and eighth equations of system (4.2), we have the stroboscopic map of (4.2)

$$\begin{cases} S_1((n+1)\tau^+) = \left[e^{-(c+d_1)\tau} + \dfrac{ac\zeta}{c+d_1-d_2}\right] S_1(n\tau^+) + a(1-\mu)e^{-d_2\tau}S_2(n\tau^+) \\ \qquad\qquad -b\left[\dfrac{c\zeta}{c+d_1-d_2}S_1(n\tau^+) + (1-\mu)e^{-d_2\tau}S_2(n\tau^+)\right]^2, \\ S_2((n+1)\tau^+) = \dfrac{c\zeta}{c+d_1-d_2}S_1(n\tau^+) + (1-\mu)e^{-d_2\tau}S_2(n\tau^+), \end{cases} \tag{4.6}$$

where $\zeta = e^{-d_2\tau}[(1-\mu)(1-e^{-(c+d_1-d_2)l\tau}) + e^{-(c+d_1-d_2)l\tau} - e^{-(c+d_1-d_2)\tau}] > 0$.

In fact, since $\Delta S_2(t) = -\mu S_2(t), t = (n+l)\tau$, then $\Delta S_2((n+l)\tau) = S_2((n+l)\tau^+) - S_2((n+l)\tau) = -\mu S_2((n+l)\tau)$, i.e., $S_2((n+l)\tau^+) = (1-\mu)S_2((n+l)\tau)$. By

$$S_2(t) = e^{-d_2(t-n\tau)}\left[S_2(n\tau^+) + \frac{cS_1(n\tau^+)(1-e^{-(c+d_1-d_2)(t-n\tau)})}{c+d_1-d_2}\right],\ t\in(n\tau,(n+l)\tau],$$

we have

$$S_2((n+l)\tau) = e^{-d_2l\tau}\left[S_2(n\tau^+) + \frac{cS_1(n\tau^+)(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}\right].$$

Thus,

$$\begin{aligned} S_2((n+l)\tau^+) &= (1-\mu)S_2((n+l)\tau) \\ &= (1-\mu)e^{-d_2l\tau}\left[S_2(n\tau^+) + \frac{cS_1(n\tau^+)(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}\right]. \end{aligned}$$

Since $\Delta S_1(t) = 0, t = (n+l)\tau$, then $\Delta S_1((n+l)\tau) = S_1((n+l)\tau^+) - S_1((n+l)\tau) = 0$, i.e., $S_1((n+l)\tau^+) = S_1((n+l)\tau)$, also,

$$S_1(t) = S_1(n\tau^+)e^{-(c+d_1)(t-n\tau)},\ t\in(n\tau,(n+1)\tau],$$

then

$$S_1((n+l)\tau) = S_1(n\tau^+)e^{-(c+d_1)l\tau},$$

$$S_1((n+1)\tau) = S_1(n\tau^+)e^{-(c+d_1)\tau}.$$

Therefore,

$$S_1((n+l)\tau^+) = S_1((n+l)\tau) = S_1(n\tau^+)e^{-(c+d_1)l\tau}.$$

Since $\Delta S_1(t) = S_2(t)(a - bS_2(t)), t = n\tau$, then

$$\begin{aligned}\Delta S_1((n+1)\tau) &= S_1((n+1)\tau^+) - S_1((n+1)\tau) \\ &= S_2((n+1)\tau)(a - bS_2((n+1)\tau)).\end{aligned}$$

Thus,

$$\begin{aligned}S_1((n+1)\tau^+) &= S_1((n+1)\tau) + S_2((n+1)\tau)(a - bS_2((n+1)\tau)) \\ &= S_1((n+1)\tau) + aS_2((n+1)\tau) - bS_2^2((n+1)\tau).\end{aligned}$$

By

$$S_2(t) = e^{-d_2(t-(n+l)\tau)}\left[S_2(n+l)\tau^+) + \frac{cS_1((n+l)\tau^+)(1 - e^{-(c+d_1-d_2)(t-(n+l)\tau)})}{c+d_1-d_2}\right],$$
$$t \in ((n+l)\tau, (n+1)\tau].$$

we have

$$S_2((n+1)\tau) = e^{-d_2(1-l)\tau}\left[S_2(n+l)\tau^+) + \frac{cS_1((n+l)\tau^+)(1 - e^{-(c+d_1-d_2)(1-l)\tau})}{c+d_1-d_2}\right].$$

Therefore,

$$
\begin{aligned}
& S_2((n+1)\tau) \\
= \; & e^{-d_2(1-l)\tau}\Bigg[(1-\mu)e^{-d_2 l\tau}\Big(S_2(n\tau^+) + \frac{cS_1(n\tau^+)(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}\Big) \\
& + \frac{cS_1(n\tau^+)e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2)(1-l)\tau})}{c+d_1-d_2}\Bigg] \\
= \; & e^{-d_2(1-l)\tau}\Bigg[(1-\mu)e^{-d_2 l\tau}S_2(n\tau^+) + \frac{c(1-\mu)S_1(n\tau^+)e^{-d_2 l\tau}(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2} \\
& + \frac{cS_1(n\tau^+)e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2)(1-l)\tau})}{c+d_1-d_2}\Bigg] \\
= \; & (1-\mu)e^{-d_2\tau}S_2(n\tau^+) + e^{-d_2\tau}\Bigg[\frac{c(1-\mu)S_1(n\tau^+)(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2} \\
& + \frac{cS_1(n\tau^+)e^{-(c+d_1-d_2)l\tau}(1-e^{-(c+d_1-d_2)(1-l)\tau})}{c+d_1-d_2}\Bigg] \\
= \; & (1-\mu)e^{-d_2\tau}S_2(n\tau^+) + e^{-d_2\tau}\Bigg[\frac{c(1-\mu)S_1(n\tau^+)(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2} \\
& + \frac{cS_1(n\tau^+)e^{-(c+d_1-d_2)l\tau} - cS_1(n\tau^+)e^{-(c+d_1-d_2)\tau}}{c+d_1-d_2}\Bigg] \\
= \; & (1-\mu)e^{-d_2\tau}S_2(n\tau^+) + \frac{cS_1(n\tau^+)e^{-d_2\tau}}{c+d_1-d_2}\Bigg[(1-\mu)(1-e^{-(c+d_1-d_2)l\tau}) \\
& + e^{-(c+d_1-d_2)l\tau} - e^{-(c+d_1-d_2)\tau}\Bigg].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& S_2((n+1)\tau) \\
= \; & \frac{cS_1(n\tau^+)e^{-d_2\tau}}{c+d_1-d_2}\Bigg[(1-\mu)(1-e^{-(c+d_1-d_2)l\tau}) + e^{-(c+d_1-d_2)l\tau} - e^{-(c+d_1-d_2)\tau}\Bigg] \\
& + (1-\mu)e^{-d_2\tau}S_2(n\tau^+),
\end{aligned}
$$

i.e.,

$$
S_2(n+1)\tau) = \frac{c\zeta}{c+d_1-d_2}S_1(n\tau^+) + (1-\mu)e^{-d_2\tau}S_2(n\tau^+),
$$

where $\zeta = e^{-d_2\tau}[(1-\mu)(1-e^{-(c+d_1-d_2)l\tau}) + e^{-(c+d_1-d_2)l\tau} - e^{-(c+d_1-d_2)\tau}] > 0$.

Since $\Delta S_2(t) = 0, t = n\tau, n \in Z_+$, then

$$\begin{aligned} S_2(n+1)\tau^+) &= S_2((n+1)\tau) \\ &= \frac{c\zeta}{c+d_1-d_2} S_1(n\tau^+) + (1-\mu)e^{-d_2\tau} S_2(n\tau^+), \end{aligned}$$

where $\zeta = e^{-d_2\tau}[(1-\mu)(1-e^{-(c+d_1-d_2)l\tau}) + e^{-(c+d_1-d_2)l\tau} - e^{-(c+d_1-d_2)\tau}] > 0$.

Since $S_1((n+1)\tau^+) = S_1((n+1)\tau) + aS_2((n+1)\tau) - bS_2^2((n+1)\tau)$, then

$$\begin{aligned} S_1((n+1)\tau^+) &= S_1(n\tau^+)e^{-(c+d_1)\tau} \\ &\quad + \frac{ac\zeta}{c+d_1-d_2} S_1(n\tau^+) + a(1-\mu)e^{-d_2\tau} S_2(n\tau^+) \\ &\quad - b\left[\frac{c\zeta}{c+d_1-d_2} S_1(n\tau^+) + (1-\mu)e^{-d_2\tau} S_2(n\tau^+)\right]^2 \\ &= \left[e^{-(c+d_1)\tau} + \frac{ac\zeta}{c+d_1-d_2}\right] S_1(n\tau^+) + a(1-\mu)e^{-d_2\tau} S_2(n\tau^+) \\ &\quad - b\left[\frac{c\zeta}{c+d_1-d_2} S_1(n\tau^+) + (1-\mu)e^{-d_2\tau} S_2(n\tau^+)\right]^2. \end{aligned}$$

If we choose $A = e^{-(c+d_1)\tau} + \dfrac{ac\zeta}{c+d_1-d_2} > 0$, $B = a(1-\mu)e^{-d_2\tau} > 0$, $C = \dfrac{c\zeta}{c+d_1-d_2}$, $D = (1-\mu)e^{-d_2\tau}$, $A < 1$, and $0 < D < 1$, the following two equivalence relations are found by calculation

$$\begin{aligned} \mu < \Omega^* &\Leftrightarrow 1 - A - D + AD - BC < 0, \\ \mu > \Omega^* &\Leftrightarrow 1 - A - D + AD - BC > 0. \end{aligned}$$

From $1 - A - D + AD - BC < 0$, we have

$$1 - e^{-(c+d_1)\tau} - \tfrac{ac\zeta}{c+d_1-d_2} - (1-\mu)e^{-d_2\tau} + (e^{-(c+d_1)\tau} + \tfrac{ac\zeta}{c+d_1-d_2})(1-\mu)e^{-d_2\tau}$$

$$-a(1-\mu)e^{-d_2\tau}\tfrac{c\zeta}{c+d_1-d_2} < 0$$

$$\Longleftrightarrow 1 - e^{-(c+d_1)\tau} - \tfrac{ac\zeta}{c+d_1-d_2} - (1-\mu)e^{-d_2\tau} + e^{-(c+d_1)\tau}e^{-d_2\tau}(1-\mu) < 0$$

$$\Longleftrightarrow (c+d_1-d_2) - (c+d_1-d_2)e^{-(c+d_1)\tau} - ac\zeta - (c+d_1-d_2)e^{-d_2\tau}(1-\mu)$$

$$+(c+d_1-d_2)e^{-(c+d_1+d_2)\tau}(1-\mu) < 0$$

$$\Longleftrightarrow [ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}](1-\mu)$$

$$> (c+d_1-d_2) - (c+d_1-d_2)e^{-(c+d_1)\tau} - ace^{-d_2\tau}e^{-(c+d_1-d_2)l\tau} + ace^{-d_2\tau}e^{-(c+d_1-d_2)\tau}$$

$$\Longleftrightarrow ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}$$

$$-[ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}]\mu$$

$$> (c+d_1-d_2) - (c+d_1-d_2)e^{-(c+d_1)\tau} - ace^{-d_2\tau}e^{-(c+d_1-d_2)l\tau} + ace^{-d_2\tau}e^{-(c+d_1-d_2)\tau}$$

$$\Longleftrightarrow [ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}]\mu$$

$$< ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}$$

$$-(c+d_1-d_2) + (c+d_1-d_2)e^{-(c+d_1)\tau} + ace^{-d_2\tau}e^{-(c+d_1-d_2)l\tau} - ace^{-d_2\tau}e^{-(c+d_1-d_2)\tau}$$

$$\Longleftrightarrow [ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}]\mu$$

$$< ace^{-d_2\tau} - ace^{-d_2\tau}e^{-(c+d_1-d_2)l\tau} + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}$$

$$-(c+d_1-d_2) + (c+d_1-d_2)e^{-(c+d_1)\tau} + ace^{-d_2\tau}e^{-(c+d_1-d_2)l\tau} - ace^{-d_2\tau}e^{-(c+d_1-d_2)\tau}$$

$$\Longleftrightarrow [ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}]\mu$$

$$< ace^{-d_2\tau} + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}$$

$$-(c+d_1-d_2) + (c+d_1-d_2)e^{-(c+d_1)\tau} - ace^{-d_2\tau}e^{-(c+d_1-d_2)\tau}$$

$$\Longleftrightarrow [ace^{-d_2\tau}(1-e^{-(c+d_1-d_2)l\tau}) + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau}]\mu$$

$$< ace^{-d_2\tau} + (c+d_1-d_2)e^{-d_2\tau} - (c+d_1-d_2)e^{-(c+d_1+d_2)\tau} - (c+d_1-d_2)$$

$$+(c+d_1-d_2)e^{-(c+d_1)\tau} - ace^{-(c+d_1)\tau}.$$

Hence,

$$\begin{aligned}
\mu &< \frac{(c+d_1-d_2)[1+e^{-(c+d_1+d_2)\tau}-e^{-(c+d_1)\tau}-e^{-d_2\tau}]-ace^{-d_2\tau}[1-e^{-(c+d_1-d_2)\tau}]}{(c+d_1-d_2)[e^{-(c+d_1+d_2)\tau}-e^{-d_2\tau}]-ace^{-d_2\tau}[1-e^{-(c+d_1-d_2)l\tau}]}\\
&= \frac{(c+d_1-d_2)[1-e^{-(c+d_1)\tau}-e^{d_2\tau}+e^{-(c+d_1-d_2)\tau}]+ac[1-e^{-(c+d_1-d_2)\tau}]}{(c+d_1-d_2)[1-e^{-(c+d_1)\tau}]+ac[1-e^{-(c+d_1-d_2)l\tau}]}\\
&= \frac{(c+d_1-d_2)[1+e^{-(c+d_1-d_2)\tau}-e^{-(c+d_1)\tau}-e^{d_2\tau}]+ac[1-e^{-(c+d_1-d_2)\tau}]}{(c+d_1-d_2)[1-e^{-(c+d_1)\tau}]+ac[1-e^{-(c+d_1-d_2)l\tau}]}.
\end{aligned}$$

Set

$$\Omega^* = \frac{(c+d_1-d_2)[1+e^{-(c+d_1-d_2)\tau}-e^{-(c+d_1)\tau}-e^{d_2\tau}]+ac[1-e^{-(c+d_1-d_2)\tau}]}{(c+d_1-d_2)[1-e^{-(c+d_1)\tau}]+ac[1-e^{-(c+d_1-d_2)l\tau}]},$$

then

$$\mu < \Omega^* \quad \Leftrightarrow \quad 1-A-D+AD-BC<0$$

and

$$\mu > \Omega^* \quad \Leftrightarrow \quad 1-A-D+AD-BC>0.$$

The two fixed points of (4.6) are obtained as $G_1(0,0)$ and $G_2(S_1^*,S_2^*)$, where

$$\begin{cases} S_1^* = \dfrac{(1-D-A+AD-BC)(-1+D)}{bC^2}, & \mu<\Omega^*,\\ S_2^* = \dfrac{-(1-D-A+AD-BC)}{bC}, & \mu<\Omega^*. \end{cases} \tag{4.7}$$

In fact, from (4.6), we have

$$\begin{cases} S_1((n+1)\tau^+) = AS_1(n\tau^+)+BS_2(n\tau^+)-b[CS_1(n\tau^+)+DS_2(n\tau^+)]^2,\\ S_2((n+1)\tau^+) = CS_1(n\tau^+)+DS_2(n\tau^+). \end{cases}$$

Let $f(S_1,S_2) = S_1, g(S_1,S_2) = S_2$, we have $S_1(n+1)\tau^+) = f(S_1(n\tau^+),S_2(n\tau^+)) = S_1(n\tau^+) = S_1,\ S_2(n+1)\tau^+) = g(S_1(n\tau^+),S_2(n\tau^+)) = S_2(n\tau^+) = S_2$. Thus,

$$\begin{cases} S_1 = AS_1+BS_2-b[CS_1+DS_2]^2,\\ S_2 = CS_1+DS_2, \end{cases}$$

So that

$$\begin{cases} S_1 = AS_1 + BS_2 - bS_2^2, \\ CS_1 = (1-D)S_2, \end{cases}$$

and

$$\begin{cases} CS_1 = ACS_1 + BCS_2 - bCS_2^2, \\ CS_1 = (1-D)S_2, \end{cases}$$

and hence

$$(1-D)S_2 = A(1-D)S_2 + BCS_2 - bCS_2^2,$$

one can get $S_2 = 0$, or $(1-D) = A(1-D) + BC - bCS_2$.

From $(1-D) = A(1-D) + BC - bCS_2$, we can get

$$S_2 = \frac{-(1-D-A+AD-BC)}{bC}.$$

Since, $S_1 = \frac{1-D}{C}S_2$, then as $S_2 = 0$, we have $S_1 = 0$; as $S_2 = \frac{-(1-D-A+AD-BC)}{bC}$, we have

$$S_1 = \frac{(1-D-A+AD-BC)(-1+D)}{bC^2}.$$

Therefore, the two fixed points of (4.6) are obtained as $G_1(0,0)$ and $G_2(S_1^*, S_2^*)$, where

$$\begin{cases} S_1^* = \dfrac{(1-D-A+AD-BC)(-1+D)}{bC^2}, \quad \mu < \mu^*, \\ S_2^* = \dfrac{-(1-D-A+AD-BC)}{bC}, \quad \mu < \mu^*. \end{cases}$$

Lemma 4.3. *(i) If $\mu > \Omega^*$, then the fixed point $G_1(0,0)$ is globally asymptotically stable. (ii) If $\mu < \Omega^*$, then the fixed point $G_2(S_1^*, S_2^*)$ is globally asymptotically stable.*

Proof. This proof is similar to Lemma 3.3 of (Jiao, Cai and Chen, 2011). For convenience, denote $(S_1^n, S_2^n) = (S_1(n\tau^+), S_2(n\tau^+))$. The linear form of (4.6) can be written as

$$\begin{pmatrix} S_1^{n+1} \\ S_2^{n+1} \end{pmatrix} = M \begin{pmatrix} S_1^n \\ S_2^n \end{pmatrix}. \tag{4.8}$$

Obviously, the near dynamics of $G_1(0,0)$ and $G_2(S_1^*, S_2^*)$ are determined by linear system (4.8). The stabilities of $G_1(0,0)$ and $G_2(S_1^*, S_2^*)$ are determined by the eigenvalue of M less than 1. If M satisfies the Jury criterion (Jury, 1974), we know that the eigenvalue of M is less than 1,

$$1 - \mathrm{tr}M + \det M > 0. \tag{4.9}$$

(i) If $\mu > \Omega^*$, namely $1 - D - A + AD - BC > 0$, $G_1(0,0)$ is the unique fixed point of system of (4.6), we have

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4.10}$$

Calculating $1 - \mathrm{tr}M + \det M = 1 - (A + D) + (AD - BC) > 0$, and from the Jury criterion, $G_1(0,0)$ is locally stable, and then it is globally asymptotically stable.

(ii) If $\mu < \Omega^*$, say $1 - A - D + AD - BC < 0$, $G_1(0,0)$ is unstable. For $1 - A - D + AD - BC < 0$, $G_2(S_1^*, S_2^*)$ exists, and

$$M = \begin{pmatrix} A - 2b(CS_1^* + DS_2^*)C & B - 2b(CS_1^* + DS_2^*)D \\ C & D \end{pmatrix}. \tag{4.11}$$

Also

$$
\begin{aligned}
& 1-\operatorname{tr} M+\det M \\
= \; & 1-\{[A-2b(CS_1^*+DS_2^*)C]+D\} \\
& +\{[A-2b(CS_1^*+DS_2^*)C]\times D-[B-2b(CS_1^*+DS_2^*)D]\times C\} \\
= \; & 1-A+2b(CS_1^*+DS_2^*)C-D \\
& +[AD-2b(CS_1^*+DS_2^*)CD-BC+2b(CS_1^*+DS_2^*)DC] \\
= \; & 1-A-D+2b(CS_1^*+DS_2^*)C+AD-BC \\
= \; & (1-A-D+AD-BC)+2b\times \\
& \left(C\frac{(1-D-A+AD-BC)(-1+D)}{bC^2}+D\frac{-(1-D-A+AD-BC)}{bC}\right)C \\
= \; & (1-A-D+AD-BC) \\
& +2\left((1-D-A+AD-BC)(-1+D)-D(1-D-A+AD-BC)\right) \\
= \; & (1-A-D+AD-BC)-2(1-A-D+AD-BC) \\
= \; & -(1-A-D+AD-BC)>0.
\end{aligned}
$$

From the Jury criterion, $G_2(S_1^*, S_2^*)$ is locally stable, and then it is globally asymptotically stable. This completes the proof. $\square$

Lemma 4.4. *(i) If* $\mu > \Omega^*$*, then the trivial periodic solution* $(0,0)$ *of system (4.4) is globally asymptotically stable.*

(ii) If $\mu < \Omega^*$*, then the periodic solution* $(\widetilde{S_1(t)}, \widetilde{S_2(t)})$ *of system (4.4) is globally asymptotically stable, where*

$$
\begin{cases}
\widetilde{S_1(t)} = S_1^* e^{-(c+d_1)(t-n\tau)}, \; t\in(n\tau,(n+1)\tau], \\
\widetilde{S_2(t)} = \begin{cases}
e^{-d_2(t-n\tau)}\left[S_2^*+\dfrac{cS_1^*(1-e^{-(c+d_1-d_2)(t-n\tau)})}{c+d_1-d_2}\right], \quad t\in(n\tau,(n+l)\tau], \\
e^{-d_2(t-(n+l)\tau)}\left[(1-\mu)e^{-d_2 l\tau}(S_2^*+\dfrac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2})\right. \\
\left. +\dfrac{cS_1^* e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2)(t-(n+l)\tau)})}{c+d_1-d_2}\right], \; t\in((n+l)\tau,(n+1)\tau],
\end{cases}
\end{cases}
\tag{4.12}
$$

in which S_1^*, S_2^* are determined as in (4.7).

4.4 The Dynamics

In this section, for system (4.2) there obviously exists an infection-free periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$. First, we prove that the infection-free periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$ of system (4.2) is globally asymptotically stable. After that, we prove that system (4.2) is permanent.

Theorem 4.5. *If*

$$\mu < \Omega^*,$$

$$\tau > \frac{1}{c+d_1}\ln(1+a),$$

and

$$\begin{aligned}\mu \quad > \quad & \left[\frac{S_2^*(1-e^{-d_2\tau})}{d_2} + \frac{cS_1^*(1-e^{-d_2\tau})}{d_2(c+d_1-d_2)} - \frac{cS_1^*(1-e^{-(c+d_1)\tau})}{(c+d_1)(c+d_1-d_2)} - \frac{(r+d_3)\tau}{\beta}\right] \\ & \times \left[(e^{-d_2 l\tau} - e^{-d_2\tau})\left(\frac{S_2^*}{d_2} + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau)})}{d_2(c+d_1-d_2)}\right)\right]^{-1},\end{aligned}$$

then the infection-free periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$ of system (4.2) is globally asymptotically stable, where S_1^*, S_2^* are defined by (4.7).

Proof. First of all, we prove the local stability. Defining $Z_1(t) = S_1(t) - \widetilde{S_1(t)}$, $Z_2(t) = S_2(t) - \widetilde{S_2(t)}$, $I(t) = I(t)$, we have the following linearly similar system for (4.2):

$$\begin{pmatrix} \frac{dZ_1(t)}{dt} \\ \frac{dZ_2(t)}{dt} \\ \frac{dI(t)}{dt} \end{pmatrix} = \begin{pmatrix} -(c+d_1) & 0 & 0 \\ c & -d_2 & -\beta\widetilde{S_2(t)} \\ 0 & 0 & \beta\widetilde{S_2(t)} - (r+d_3) \end{pmatrix} \begin{pmatrix} Z_1(t) \\ Z_2(t) \\ I(t) \end{pmatrix}.$$

It is easy to obtain the fundamental matrix

$$\Phi(t)=\begin{pmatrix} \exp[-(c+d_1)t] & 0 & 0 \\ * & \exp(-d_2 t) & \dagger \\ 0 & 0 & \exp[\int_0^t(\beta\widetilde{S_2(s)}-(r+d_3))ds] \end{pmatrix}.$$

There is no need to calculate the exact forms of $*$, $\dagger$, as they are not required in the analysis that follows. The linearization of the fourth, fifth, and sixth equations of system (4.2) is

$$\begin{pmatrix} Z_1((n+1)\tau^+) \\ Z_2((n+1)\tau^+) \\ I((n+1)\tau^+) \end{pmatrix}=\begin{pmatrix} 1+a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Z_1((n+1)\tau) \\ Z_2((n+1)\tau) \\ I((n+1)\tau) \end{pmatrix}.$$

The linearization of the seventh, eighth, and ninth equations of system (4.2) is

$$\begin{pmatrix} Z_1((n+l)\tau^+) \\ Z_2((n+l)\tau^+) \\ I((n+l)\tau^+) \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\mu & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} Z_1((n+l)\tau) \\ Z_2((n+l)\tau) \\ I((n+l)\tau) \end{pmatrix}.$$

The stability of the infection-free periodic solution $(\widetilde{S_1(t)},\widetilde{S_2(t)},0)$ is determined by the eigenvalues of

$$M=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\mu & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1+a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Phi(\tau),$$

which are

$$\lambda_1=(1+a)\exp[-(c+d_1)\tau],$$

$$\lambda_2=(1-\mu)e^{-d_2\tau}<1,$$

and

$$\lambda_3 = \exp\left[\int_0^\tau (\beta\widetilde{S_2(s)} - (r+d_3))ds\right].$$

According to the conditions of this theorem, we easily know that $(1+a)\exp[-(c+d_1)\tau] < 1$, and $\exp\left[\int_0^\tau(\beta\widetilde{S_2(s)} - (r+d_3))ds\right] < 1$, then $\lambda_1 < 1$, and $\lambda_3 < 1$. From the Floquet theory (Klausmeier, 2008), the infection-free solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$ of system (4.2) is locally stable.

In fact, $|\lambda_1| < 1$, i.e., $\lambda_1 < 1$, since

$$\begin{aligned}
& (1+a)\exp[-(c+d_1)\tau] < 1,\\
\Rightarrow\ & e^{-(c+d_1)\tau} < \frac{1}{1+a},\\
\Rightarrow\ & -(c+d_1)\tau < \ln\frac{1}{1+a},\\
\Rightarrow\ & -(c+d_1)\tau < -\ln(1+a),\\
\Rightarrow\ & (c+d_1)\tau > \ln(1+a),\\
\Rightarrow\ & \tau > \frac{1}{c+d_1}\ln(1+a).
\end{aligned}$$

In fact, $\lambda_3 < 1$. Since

$\exp[\int_0^\tau (\beta\widetilde{S_2(s)} - (r+d_3))ds] < 1$, i.e., $\exp[-(r+d_3)\tau + \beta\int_0^\tau \widetilde{S_2(s)}ds] < 1$, need to $-(r+d_3)\tau + \beta\int_0^\tau \widetilde{S_2(s)}ds < 0$, need to $\beta\int_0^\tau \widetilde{S_2(s)}ds < (r+d_3)\tau$, i.e., $\int_0^\tau \widetilde{S_2(s)}ds < \frac{(r+d_3)\tau}{\beta}$.

From

$$\widetilde{S_2(t)} = \begin{cases} e^{-d_2(t-n\tau)}\left[S_2^* + \dfrac{cS_1^*(1-e^{-(c+d_1-d_2)(t-n\tau)})}{c+d_1-d_2}\right], & t\in(n\tau,(n+l)\tau],\\ e^{-d_2(t-(n+l)\tau)}\left[(1-\mu)e^{-d_2 l\tau}(S_2^* + \dfrac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2})\right. & \\ \quad\left.+\dfrac{cS_1^* e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2)(t-(n+l)\tau)})}{c+d_1-d_2}\right], & t\in((n+l)\tau,(n+1)\tau]. \end{cases}$$

For $t \in (0, \tau]$, $n = 0$, i.e., $(0, \tau] = (0, l\tau] \cup (l\tau, \tau]$, we have

$$\widetilde{S_2(t)} = \begin{cases} e^{-d_2 t}\left[S_2^* + \dfrac{cS_1^*(1 - e^{-(c+d_1-d_2)t})}{c + d_1 - d_2}\right], & t \in (0, l\tau], \\ e^{-d_2(t-l\tau)}\Bigg[(1-\mu)e^{-d_2 l\tau}(S_2^* + \dfrac{cS_1^*(1 - e^{-(c+d_1-d_2)l\tau})}{c + d_1 - d_2}) & \\ \qquad + \dfrac{cS_1^* e^{-(c+d_1)l\tau}(1 - e^{-(c+d_1-d_2)(t-l\tau)})}{c + d_1 - d_2}\Bigg], & t \in (l\tau, \tau]. \end{cases}$$

Since $\int_0^\tau \widetilde{S_2(s)}ds = \int_0^{l\tau} \widetilde{S_2(s)}ds + \int_{l\tau}^\tau \widetilde{S_2(s)}ds$, then we have

$$\begin{aligned}
& \int_0^{l\tau} \widetilde{S_2(s)}ds \\
= \;& \int_0^{l\tau} e^{-d_2 s}\left[S_2^* + \frac{cS_1^*(1 - e^{-(c+d_1-d_2)s})}{c + d_1 - d_2}\right] ds \\
= \;& \int_0^{l\tau} e^{-d_2 s} S_2^* ds + \int_0^{l\tau} \frac{cS_1^* e^{-d_2 s}(1 - e^{-(c+d_1-d_2)s})}{c + d_1 - d_2} ds \\
= \;& \int_0^{l\tau} e^{-d_2 s} S_2^* ds + \int_0^{l\tau} \frac{cS_1^* e^{-d_2 s} - cS_1^* e^{-(c+d_1)s}}{c + d_1 - d_2} ds \\
= \;& -\frac{S_2^*}{d_2} e^{-d_2 s}\Big|_{s=0}^{s=l\tau} + \int_0^{l\tau} \frac{cS_1^*}{c + d_1 - d_2} e^{-d_2 s} ds - \frac{cS_1^*}{(c + d_1 - d_2)} \int_0^{l\tau} e^{-(c+d_1)s} ds \\
= \;& -\frac{S_2^*}{d_2}(e^{-d_2 l\tau} - 1) - \frac{cS_1^*}{d_2(c + d_1 - d_2)} e^{-d_2 s}\Big|_{s=0}^{s=l\tau} \\
& + \frac{cS_1^*}{(c + d_1)(c + d_1 - d_2)} e^{-(c+d_1)s}\Big|_{s=0}^{s=l\tau} \\
= \;& \frac{S_2^*}{d_2}(1 - e^{-d_2 l\tau}) - \frac{cS_1^*}{d_2(c + d_1 - d_2)}(e^{-d_2 l\tau} - 1) \\
& + \frac{cS_1^*}{(c + d_1)(c + d_1 - d_2)}(e^{-(c+d_1)l\tau} - 1) \\
= \;& \frac{S_2^*}{d_2}(1 - e^{-d_2 l\tau}) + \frac{cS_1^*}{d_2(c + d_1 - d_2)}(1 - e^{-d_2 l\tau}) \\
& + \frac{cS_1^*}{(c + d_1)(c + d_1 - d_2)}(e^{-(c+d_1)l\tau} - 1) \\
= \;& \frac{S_2^*}{d_2}(1 - e^{-d_2 l\tau}) + \frac{cS_1^*}{d_2(c + d_1 - d_2)}(1 - e^{-d_2 l\tau}) \\
& - \frac{cS_1^*}{(c + d_1)(c + d_1 - d_2)}(1 - e^{-(c+d_1)l\tau}).
\end{aligned}$$

And

$$
\begin{aligned}
&\int_{l\tau}^{\tau} \widetilde{S_2(s)} ds = \\
&\int_{l\tau}^{\tau} e^{-d_2(s-l\tau)} \Big[(1-\mu) e^{-d_2 l\tau} (S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}) \\
&\qquad + \frac{cS_1^* e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2)(s-l\tau)})}{c+d_1-d_2} \Big] ds \\
= \;& \int_{l\tau}^{\tau} (1-\mu) e^{-d_2 s} (S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}) ds + \int_{l\tau}^{\tau} \frac{cS_1^* e^{-(c+d_1)l\tau} e^{-d_2(s-l\tau)}}{c+d_1-d_2} ds \\
& - \int_{l\tau}^{\tau} \frac{e^{-d_2(s-l\tau)} e^{-(c+d_1-d_2)(s-l\tau)} cS_1^* e^{-(c+d_1)l\tau}}{c+d_1-d_2} ds \\
= \;& (S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2})(1-\mu)(-\frac{1}{d_2}) e^{-d_2 s} \Big|_{s=l\tau}^{s=\tau} \\
& + \frac{cS_1^* e^{-(c+d_1)l\tau}}{c+d_1-d_2} (-\frac{1}{d_2}) e^{-d_2(s-l\tau)} \Big|_{s=l\tau}^{s=\tau} - \int_{l\tau}^{\tau} \frac{cS_1^* e^{-(c+d_1)s}}{c+d_1-d_2} ds \\
= \;& \frac{(1-\mu)}{d_2} (S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}) (e^{-d_2 l\tau} - e^{-d_2\tau}) \\
& + \frac{cS_1^* e^{-(c+d_1)l\tau}}{c+d_1-d_2} \frac{1}{d_2} (1 - e^{-d_2(1-l)\tau}) - \frac{cS_1^*}{c+d_1-d_2} (-\frac{1}{c+d_1}) e^{-(c+d_1)s} \Big|_{s=l\tau}^{s=\tau} \\
= \;& \frac{(1-\mu)}{d_2} (S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}) (e^{-d_2 l\tau} - e^{-d_2\tau}) \\
& + \frac{cS_1^* e^{-(c+d_1)l\tau}}{d_2(c+d_1-d_2)} (1 - e^{-d_2(1-l)\tau}) + \frac{cS_1^*(e^{-(c+d_1)\tau} - e^{-(c+d_1)l\tau})}{(c+d_1)(c+d_1-d_2)} \\
= \;& (1-\mu) \left(\frac{S_2^*}{d_2} + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{d_2(c+d_1-d_2)} \right) \left(e^{-d_2 l\tau} - e^{-d_2\tau} \right) \\
& + \frac{cS_1^* e^{-(c+d_1)l\tau}}{d_2(c+d_1-d_2)} \left(1 - e^{-d_2(1-l)\tau} \right) \\
& + \frac{cS_1^*}{(c+d_1)(c+d_1-d_2)} \left(e^{-(c+d_1)\tau} - e^{-(c+d_1)l\tau} \right).
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\tau} \widetilde{S_2(s)}ds &= \int_0^{l\tau} \widetilde{S_2(s)}ds + \int_{l\tau}^{\tau} \widetilde{S_2(s)}ds \\
&= \frac{S_2^*}{d_2}(1-e^{-d_2 l\tau}) + \frac{cS_1^*}{d_2(c+d_1-d_2)}(1-e^{-d_2 l\tau}) \\
&\quad -\frac{cS_1^*}{(c+d_1)(c+d_1-d_2)}(1-e^{-(c+d_1)l\tau}) \\
&\quad +(1-\mu)\left(\frac{S_2^*}{d_2}+\frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{d_2(c+d_1-d_2)}\right)\left(e^{-d_2 l\tau}-e^{-d_2\tau}\right) \\
&\quad +\frac{cS_1^* e^{-(c+d_1)l\tau}}{d_2(c+d_1-d_2)}\left(1-e^{-d_2(1-l)\tau}\right) \\
&\quad +\frac{cS_1^*}{(c+d_1)(c+d_1-d_2)}\left(e^{-(c+d_1)\tau}-e^{-(c+d_1)l\tau}\right). \\
&= \frac{S_2^*}{d_2}(1-e^{-d_2 l\tau}) + \frac{cS_1^*}{d_2(c+d_1-d_2)}(1-e^{-d_2 l\tau}) \\
&\quad -\frac{cS_1^*}{(c+d_1)(c+d_1-d_2)} + \frac{cS_1^* e^{-(c+d_1)l\tau}}{d_2(c+d_1-d_2)} \\
&\quad -\frac{cS_1^* e^{-(c+d_1-d_2)l\tau}e^{-d_2\tau}}{d_2(c+d_1-d_2)} + \frac{cS_1^* e^{-(c+d_1)\tau}}{(c+d_1)(c+d_1-d_2)} \\
&\quad +\left(e^{-d_2 l\tau}-e^{-d_2\tau}\right)\left(\frac{S_2^*}{d_2}+\frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{d_2(c+d_1-d_2)}\right) \\
&\quad -\mu\left(e^{-d_2 l\tau}-e^{-d_2\tau}\right)\left(\frac{S_2^*}{d_2}+\frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{d_2(c+d_1-d_2)}\right) \\
&= \frac{S_2^*}{d_2}-\frac{S_2^*}{d_2}e^{-d_2 l\tau}+\frac{cS_1^*}{d_2(c+d_1-d_2)}-\frac{cS_1^* e^{-d_2 l\tau}}{d_2(c+d_1-d_2)} \\
&\quad -\frac{cS_1^*}{(c+d_1)(c+d_1-d_2)}+\frac{cS_1^* e^{-(c+d_1)l\tau}}{d_2(c+d_1-d_2)} \\
&\quad -\frac{cS_1^* e^{-(c+d_1-d_2)l\tau}e^{-d_2\tau}}{d_2(c+d_1-d_2)}+\frac{cS_1^* e^{-(c+d_1)\tau}}{(c+d_1)(c+d_1-d_2)} \\
&\quad +\frac{S_2^*}{d_2}e^{-d_2 l\tau}-\frac{S_2^*}{d_2}e^{-d_2\tau}+\frac{cS_1^* e^{-d_2 l\tau}}{d_2(c+d_1-d_2)}-\frac{cS_1^* e^{-(c+d_1)l\tau}}{d_2(c+d_1-d_2)} \\
&\quad -\frac{cS_1^* e^{-d_2\tau}}{d_2(c+d_1-d_2)}+\frac{cS_1^* e^{-(c+d_1-d_2)l\tau}e^{-d_2\tau}}{d_2(c+d_1-d_2)} \\
&\quad -\mu\left(e^{-d_2 l\tau}-e^{-d_2\tau}\right)\left(\frac{S_2^*}{d_2}+\frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{d_2(c+d_1-d_2)}\right) \\
&= \frac{S_2^*}{d_2}(1-e^{-d_2\tau})+\frac{cS_1^*(1-e^{-d_2\tau})}{d_2(c+d_1-d_2)}+\frac{cS_1^*(e^{-(c+d_1)\tau}-1)}{(c+d_1)(c+d_1-d_2)} \\
&\quad -\mu\left(e^{-d_2 l\tau}-e^{-d_2\tau}\right)\left(\frac{S_2^*}{d_2}+\frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{d_2(c+d_1-d_2)}\right) \\
&< \frac{(r+d_3)\tau}{\beta}.
\end{aligned}
$$

That is,

$$\frac{S_2^*(1-e^{-d_2\tau})}{d_2}+\frac{cS_1^*(1-e^{-d_2\tau})}{d_2(c+d_1-d_2)}-\frac{cS_1^*(1-e^{-(c+d_1)\tau})}{(c+d_1)(c+d_1-d_2)}-\frac{(r+d_3)\tau}{\beta}$$
$$< \mu\left(e^{-d_2 l\tau}-e^{-d_2\tau}\right)\left(\frac{S_2^*}{d_2}+\frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{d_2(c+d_1-d_2)}\right).$$

This implies that

$$\mu > \left[\frac{S_2^*(1-e^{-d_2\tau})}{d_2}+\frac{cS_1^*(1-e^{-d_2\tau})}{d_2(c+d_1-d_2)}-\frac{cS_1^*(1-e^{-(c+d_1)\tau})}{(c+d_1)(c+d_1-d_2)}-\frac{(r+d_3)\tau}{\beta}\right]$$
$$\times\left[(e^{-d_2 l\tau}-e^{-d_2\tau})\left(\frac{S_2^*}{d_2}+\frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau)})}{d_2(c+d_1-d_2)}\right)\right]^{-1}.$$

Therefore, according to the conditions of this theorem, we easily know that $\exp[\int_0^\tau(\beta\widetilde{S_2(s)}-(r+d_3))ds]<1$, i.e., $\lambda_3<1$.

The following task is to prove the global attractivity; choose $\varepsilon>0$ such that

$$\rho=\exp\left[\int_0^\tau\left(\beta(\widetilde{S_2(s)}+\varepsilon)-(r+d_3)\right)ds\right]<1.$$

From the second equation of system (4.2), we notice that $\dfrac{dS_2(t)}{dt}\le cS_1(t)-d_2S_2(t)$, so we consider the following impulsive differential equation

$$\begin{cases}\left.\begin{aligned}\frac{dS_{11}(t)}{dt}&=-cS_{11}(t)-d_1S_{11}(t),\\ \frac{dS_{12}(t)}{dt}&=cS_{11}(t)-d_2S_{12}(t),\end{aligned}\right\} t\neq n\tau,\quad t\neq(n+l)\tau,\\ \left.\begin{aligned}\Delta S_{11}(t)&=S_{12}(t)(a-bS_{12}(t)),\\ \Delta S_{12}(t)&=0,\end{aligned}\right\} t=n\tau,\quad n=1,2,\ldots,\\ \left.\begin{aligned}\Delta S_{11}(t)&=0,\\ \Delta S_{12}(t)&=-\mu S_{12}(t),\end{aligned}\right\} t=(n+l)\tau,\quad n=1,2,\ldots.\end{cases} \tag{4.13}$$

From Lemma 4.4 and the comparison theorem of impulsive equations [see (Lakshmikantham et al., 1989), Theorem 3.1.1], we have $S_1(t)\le S_{11}(t), S_2(t)\le S_{12}(t)$, and $S_{11}(t) \to \widetilde{S_1(t)}$, $S_{12}(t) \to \widetilde{S_2(t)}$ as $t \to \infty$; that is,

$$\begin{cases} S_1(t) \le S_{11}(t) \le \widetilde{S_1(t)} + \varepsilon, \\ \\ S_2(t) \le S_{12}(t) \le \widetilde{S_2(t)} + \varepsilon, \end{cases} \tag{4.14}$$

for t large enough. For convenience, we may assume that (4.14) holds for all $t \ge 0$. From (4.2) and (4.14), we get

$$\begin{cases} \dfrac{dI(t)}{dt} \le [\beta(\widetilde{S_2(t)} + \varepsilon) - (r + d_3)]I(t), \quad t \ne n\tau, \quad t \ne (n+l)\tau, \\ \\ \Delta I(t) = 0, \quad t = n\tau, \quad t = (n+l)\tau. \end{cases} \tag{4.15}$$

So $I(t) \le I(0^+)\exp[\int_0^t (\beta(\widetilde{S_2(t)} + \varepsilon) - (r + d_3))ds]$, thus $I((n+1)\tau) \le I(n\tau^+) \times \exp[\int_{n\tau}^{(n+1)\tau} (\beta(\widetilde{S_2(t)} + \varepsilon) - (r + d_3))ds]$, hence $I(n\tau) \le I(0^+)\rho^n$ and $I(n\tau) \to 0$ as $n \to \infty$. Therefore, $I(t) \to 0$ as $t \to \infty$.

Next we prove that $S_1(t) \to \widetilde{S_1(t)}$, $S_2(t) \to \widetilde{S_2(t)}$ as $t \to \infty$. Since $\forall \varepsilon > 0$, we have $0 < I(t) < \varepsilon$ for all $t \ge 0$, then, for system (4.2), we have

$$cS_1(t) - (d_2 + \beta\varepsilon)S_2(t) \le \frac{dS_2(t)}{dt} \le cS_1(t) - d_2S_2(t), \tag{4.16}$$

then we have $S_{21}(t) \le S_1(t) \le S_{31}(t)$, $S_{22}(t) \le S_2(t) \le S_{32}(t)$, and $S_{21}(t) \to \widetilde{S_{21}}(t)$, $S_{22}(t) \to \widetilde{S_{22}}(t)$, $S_{31}(t) \to \widetilde{S_1}(t)$, $S_{32}(t) \to \widetilde{S_2}(t)$, as $t \to \infty$. Meanwhile $(S_{21}(t), S_{22}(t))$ and $(S_{31}(t), S_{32}(t))$ are the solutions to

$$\begin{cases} \left.\begin{aligned} \frac{dS_{21}(t)}{dt} &= -cS_{21}(t) - d_1S_{21}(t), \\ \frac{dS_{22}(t)}{dt} &= cS_{21}(t) - (d_2 + \beta\varepsilon)S_{22}(t), \end{aligned}\right\} t \ne n\tau, \quad t \ne (n+l)\tau, \\ \left.\begin{aligned} \Delta S_{21}(t) &= S_{22}(t)(a - bS_{22}(t)), \\ \\ \Delta S_{22}(t) &= 0, \end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\ \left.\begin{aligned} \Delta S_{21}(t) &= 0, \\ \\ \Delta S_{22}(t) &= -\mu S_{22}(t), \end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2 \ldots, \end{cases} \tag{4.17}$$

and

$$\begin{cases} \left.\begin{aligned} \frac{dS_{31}(t)}{dt} &= -cS_{31}(t) - d_1 S_{31}(t), \\ \frac{dS_{32}(t)}{dt} &= cS_{31}(t) - d_2 S_{32}(t), \end{aligned}\right\} t \neq n\tau, \quad t \neq (n+l)\tau, \\ \left.\begin{aligned} \Delta S_{31}(t) &= S_{32}(t)(a - bS_{32}(t)), \\ \Delta S_{32}(t) &= 0, \end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\ \left.\begin{aligned} \Delta S_{31}(t) &= 0, \\ \Delta S_{32}(t) &= -\mu S_{32}(t), \end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2, \ldots, \end{cases} \tag{4.18}$$

respectively. Here $(\widetilde{S_{21}(t)}, \widetilde{S_{22}(t)})$ can be expressed as

$$\begin{cases} \widetilde{S_{21}(t)} = S_{21}^* e^{-(c+d_1)(t-n\tau)},\ t \in (n\tau, (n+1)\tau], \\ \widetilde{S_{22}(t)} = \begin{cases} e^{-(d_2+\beta\varepsilon)(t-n\tau)} \left[S_{22}^* + \frac{cS_{21}^*(1-e^{-(c+d_1-d_2-\beta\varepsilon)(t-n\tau)})}{c+d_1-d_2-\beta\varepsilon}\right],\ t \in (n\tau, (n+l)\tau], \\ e^{-(d_2+\beta\varepsilon)(t-(n+l)\tau)} \Bigg[(1-\mu) e^{-(d_2+\beta\varepsilon)l\tau} (S_{22}^* + \frac{cS_{21}^*(1-e^{-(c+d_1-d_2-\beta\varepsilon)l\tau})}{c+d_1-d_2-\beta\varepsilon}) \\ \quad + \frac{cS_{21}^* e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2-\beta\varepsilon)(t-(n+l)\tau)})}{c+d_1-d_2-\beta\varepsilon}\Bigg],\ t \in ((n+l)\tau, (n+1)\tau], \end{cases} \end{cases} \tag{4.19}$$

where

$$\begin{cases} S_{21}^* = \dfrac{(1 - D_1 - A_1 + A_1 D_1 - B_1 C_1)(-1 + D_1)}{bC_1^2}, \quad \mu < \widetilde{\Omega^*}, \\ S_{22}^* = \dfrac{-(1 - D_1 - A_1 + A_1 D_1 - B_1 C_1)}{bC_1}, \quad \mu < \widetilde{\Omega^*}, \end{cases} \tag{4.20}$$

and $\zeta_1 = e^{-(d_2+\beta\varepsilon)\tau}[(1-\mu)(1-e^{-(c+d_1-d_2-\beta\varepsilon)l\tau}) + e^{-(c+d_1-d_2-\beta\varepsilon)l\tau} - e^{-(c+d_1-d_2-\beta\varepsilon)\tau}] > 0$. $A_1 = e^{-(c+d_1)\tau} + \dfrac{ac\zeta_1}{c+d_1-d_2-\beta\varepsilon} > 0$, $B_1 = a(1-\mu)e^{-(d_2+\beta\varepsilon)\tau} > 0$, $C_1 = \dfrac{c\zeta_1}{c+d_1-d_2-\beta\varepsilon}$, $D_1 = (1-\mu)e^{-(d_2+\beta\varepsilon)\tau}$, $A_1 < 1$, $0 < D_1 < 1$, and

$$\widetilde{\Omega^*} = \frac{(c+d_1-d_2-\beta\varepsilon)[1+e^{-(c+d_1-d_2-\beta\varepsilon)\tau} - e^{-(c+d_1)\tau} - e^{(d_2+\beta\varepsilon)\tau}] + ac[1-e^{-(c+d_1-d_2-\beta\varepsilon)\tau}]}{(c+d_1-d_2-\beta\varepsilon)[1-e^{-(c+d_1)\tau}] + ac[1-e^{-(c+d_1-d_2-\beta\varepsilon)l\tau}]}.$$

Therefore, for any $\varepsilon_1 > 0$, there exists $t_1, t > t_1$, such that

$$\widetilde{S_{21}(t)} - \varepsilon_1 < S_1(t) < \widetilde{S_1(t)} + \varepsilon_1$$

and

$$\widetilde{S_{22}(t)} - \varepsilon_1 < S_2(t) < \widetilde{S_2(t)} + \varepsilon_1.$$

Letting $\varepsilon \to 0$, we have

$$\widetilde{S_1(t)} - \varepsilon_1 < S_1(t) < \widetilde{S_1(t)} + \varepsilon_1$$

and

$$\widetilde{S_2(t)} - \varepsilon_1 < S_2(t) < \widetilde{S_2(t)} + \varepsilon_1$$

for t large enough, which implies that $S_1(t) \to \widetilde{S_1(t)}$, $S_2(t) \to \widetilde{S_2(t)}$ as $t \to \infty$. This completes the proof. □

The next work is to investigate the permanence of system (4.2). Before starting this work, we should give the following definition.

Definition 4.2. System (4.2) is said to be permanent if there are constants $m, M > 0$ (independent of the initial value) and a finite time T_0, such that for all solutions $(S_1(t), S_2(t), I(t))$ with any initial values $S_1(0^+) > 0, S_2(0^+) > 0, I(0^+) > 0$, we have $m \leq S_1(t) \leq M, m \leq S_2(t) \leq M, m \leq I(t) \leq M$ for all $t \geq T_0$. Here T_0 may depend on the initial values $(S_1(0^+), S_2(0^+), I(0^+))$.

Theorem 4.6. *If*

$$\mu < \Omega^*,$$

$$\tau < \frac{1}{c+d_1}\ln(1+a),$$

and

$$\begin{aligned} \mu \;<\; & \left[\frac{S_2^*(1-e^{-d_2\tau})}{d_2} + \frac{cS_1^*(1-e^{-d_2\tau})}{d_2(c+d_1-d_2)} - \frac{cS_1^*(1-e^{-(c+d_1)\tau})}{(c+d_1)(c+d_1-d_2)} - \frac{(r+d_3)\tau}{\beta}\right] \\ & \times \left[(e^{-d_2 l\tau} - e^{-d_2\tau})\left(\frac{S_2^*}{d_2} + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau)})}{d_2(c+d_1-d_2)}\right)\right]^{-1}, \end{aligned} \tag{4.21}$$

then system (4.2) is permanent, where S_1^*, S_2^* are defined by (4.7).

Proof. Let $(S_1(t), S_2(t), I(t))$ be a solution of (4.2) with $S_1(0) > 0, S_2(0) > 0, I(0) > 0$. By Lemma 4.2, we have proved there exists a constant $M > 0$, such that $S_1(t) \le M, S_2(t) \le M, I(t) \le M$ for t large enough.

From the proof of Theorem 4.5, we know that $S_1(t) > \widetilde{S_1(t)} - \varepsilon_1, S_2(t) > \widetilde{S_2(t)} - \varepsilon_1$ for t large enough, and $\varepsilon_1 > 0$. So, $S_1(t) \ge S_1^* e^{-(c+d_1)\tau} - \varepsilon_1 = m_2$, and

$$
\begin{aligned}
& S_2(t) \\
\ge\ & e^{-d_2 l\tau}\left[S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)\tau})}{c+d_1-d_2}\right] \\
& + e^{-d_2(1-l)\tau}\left[(1-\mu)e^{-d_2\tau}\left(S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)l\tau})}{c+d_1-d_2}\right)\right. \\
& \left. + \frac{cS_1^* e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2)(1-l)\tau})}{c+d_1-d_2}\right] \\
\ge\ & e^{-d_2 l\tau}\left[S_2^* + \frac{cS_1^*(1-e^{-(c+d_1-d_2)\tau})}{c+d_1-d_2}\right] + e^{-d_2(1-l)\tau} \\
& \times\left[(1-\mu)e^{-d_2\tau}S_2^* + \frac{cS_1^*[(1-\mu)e^{-d_2\tau} + e^{-(c+d_1)\tau}](1-e^{-(c+d_1-d_2)(1-l)\tau})}{c+d_1-d_2}\right] - \varepsilon_1 \\
=\ & m_2',
\end{aligned}
$$

for t large enough, where S_1^* and S_2^* are defined by (4.7). Thus, we only need to find $m_1 > 0$ such that $I(t) \ge m_1$ for t large enough. We will do it in the following two steps.

1° Prove that $I(t) \ge m_1$, for t large enough. Otherwise, we can select $m_3 > 0$ small enough, and prove $I(t) < m_3$ cannot hold for $t \ge 0$. By condition

(4.21), we can obtain

$$\begin{aligned}
\sigma &= \frac{S_{42}^*}{d_2+\beta m_3}(1-e^{-(d_2+\beta m_3)l\tau}) \\
&\quad + \frac{cS_{41}^*}{(d_2+\beta m_3)(c+d_1-d_2-\beta m_3)}(1-e^{-(d_2+\beta m_3)l\tau}) \\
&\quad - \frac{cS_{41}^*}{(c+d_1)(c+d_1-d_2-\beta m_3)}(1-e^{-(c+d_1)l\tau}) \\
&\quad + (1-\mu)(e^{-(d_2+\beta m_3)l\tau}-e^{-(d_2+\beta m_3)\tau})\Big(\frac{S_{42}^*}{d_2+\beta m_3}+\frac{cS_{41}^*(1-e^{-(c+d_1-d_2-\beta m_3)l\tau})}{(d_2+\beta m_3)(c+d_1-d_2-\beta m_3)}\Big) \\
&\quad + \frac{cS_{41}^* e^{-(c+d_1)l\tau}}{(d_2+\beta m_2)(c+d_1-d_2-\beta m_3)}(1-e^{-(d_2+\beta m_3)(1-l)\tau}) \\
&\quad + \frac{cS_{41}^*}{(c+d_1)(c+d_1-d_2-\beta m_3)}(e^{-(c+d_1)\tau}-e^{-(c+d_1)l\tau}) - \frac{(r+d_3)\tau}{\beta} \\
&> 0.
\end{aligned}$$

By Lemma 4.4, we have $S_1(t) \geq S_{41}(t), S_2(t) \geq S_{42}(t)$, and $S_{41}(t) \to \widetilde{S_{41}(t)}$, $S_{42}(t) \to \widetilde{S_{42}(t)}$, $t \to \infty$, where $(S_{41}(t), S_{42}(t))$ is the solution to

$$\begin{cases}
\left.\begin{aligned}
\frac{dS_{41}(t)}{dt} &= -cS_{41}(t) - d_1 S_{41}(t), \\
\frac{dS_{42}(t)}{dt} &= cS_{41}(t) - (d_2+\beta m_3)S_{42}(t),
\end{aligned}\right\} t \neq n\tau, \quad t \neq (n+l)\tau, \\
\left.\begin{aligned}
\Delta S_{41}(t) &= S_{42}(t)(a - bS_{42}(t)), \\
\Delta S_{42}(t) &= 0,
\end{aligned}\right\} t = n\tau, \quad n = 1, 2, \ldots, \\
\left.\begin{aligned}
\Delta S_{41}(t) &= 0, \\
\Delta S_{42}(t) &= -\mu S_{42}(t),
\end{aligned}\right\} t = (n+l)\tau, \quad n = 1, 2, \ldots,
\end{cases} \tag{4.22}$$

with

$$\begin{cases}
\widetilde{S_{41}(t)} = S_{41}^* e^{-(c+d_1)(t-n\tau)}, t \in (n\tau, (n+1)\tau], \\
\widetilde{S_{42}(t)} = \begin{cases}
e^{-(d_2+\beta m_3)(t-n\tau)}\left[S_{42}^* + \frac{cS_{41}^*(1-e^{-(c+d_1-d_2-\beta m_3)(t-n\tau)})}{c+d_1-d_2-\beta m_3}\right], t \in (n\tau, (n+l)\tau], \\
e^{-(d_2+\beta m_3)(t-(n+l)\tau)}\Big[(1-\mu)e^{-(d_2+\beta m_3)l\tau}(S_{42}^* + \frac{cS_{41}^*(1-e^{-(c+d_1-d_2-\beta m_3)l\tau})}{c+d_1-d_2-\beta m_3}) \\
+\frac{cS_{41}^* e^{-(c+d_1)l\tau}(1-e^{-(c+d_1-d_2-\beta m_3)(t-(n+l)\tau)})}{c+d_1-d_2-\beta m_3}\Big], t \in ((n+l)\tau, (n+1)\tau].
\end{cases}
\end{cases} \tag{4.23}$$

Here S^*_{41} and S^*_{42} are determined as

$$\begin{cases} S^*_{41} = \dfrac{(1-D_2-A_2+A_2D_2-B_2C_2)(-1+D_2)}{bC_2^2}, & \mu<\Omega^{**} \\ S^*_{42} = \dfrac{-(1-D_2-A_2+A_2D_2-B_2C_2)}{bC_2}, & \mu<\Omega^{**}, \end{cases} \tag{4.24}$$

and $\zeta_2 = e^{-(d_2+\beta m_3)\tau}[(1-\mu)(1-e^{-(c+d_1-d_2-\beta m_3)l\tau}) + e^{-(c+d_1-d_2-\beta m_3)l\tau} - e^{-(c+d_1-d_2-\beta m_3)\tau}] > 0$. $A_2 = e^{-(c+d_1)\tau} + \dfrac{ac\zeta_2}{c+d_1-d_2-\beta m_3} > 0$, $B_2 = a(1-\mu)e^{-(d_2+\beta m_3)\tau} > 0$, $C_2 = \dfrac{c\zeta_2}{c+d_1-d_2-\beta m_3}$, $D_2 = (1-\mu)e^{-(d_2+\beta m_3)\tau}$, $A_2<1$, $0<D_2<1$, and

$$\Omega^{**} = \frac{(c+d_1-d_2-\beta m_3)[1+e^{-(c+d_1-d_2-\beta m_3)\tau}-e^{-(c+d_1)\tau}-e^{(d_2+\beta m_3)\tau}]+ac[1-e^{-(c+d_1-d_2-\beta m_3)\tau}]}{(c+d_1-d_2-\beta m_3)[1-e^{-(c+d_1)\tau}]+ac[1-e^{-(c+d_1-d_2-\beta m_3)l\tau}]}.$$

Therefore, there exist $T_1>0$ and $\varepsilon_3>0$, such that

$$S_1(t) \geq S_{41}(t) \geq \widetilde{S_{41}(t)} - \varepsilon_3$$

and

$$S_2(t) \geq S_{42}(t) \geq \widetilde{S_{42}(t)} - \varepsilon_3.$$

Then

$$\frac{dI(t)}{dt} \geq [\beta(\widetilde{S_{42}(t)} - \varepsilon_3) - (r+d_3)]I(t), \tag{4.25}$$

for $t \geq T_1$. Let $N_1 \in N$ and $N_1\tau > T_1$. Integrating (4.25) on $(n\tau,(n+1)\tau]$, $n \geq N_1$, we have

$$I((n+1)\tau) \geq I(n\tau^+)\exp\Big(\int_{n\tau}^{(n+1)\tau}[\beta(\widetilde{S_{42}(t)}-\varepsilon_3)-(r+d_3)]dt\Big) = I(n\tau)e^{\sigma},$$

then $I((N_1+k)\tau) \geq I(N_1\tau^+)e^{k\sigma} \to \infty$, as $k\to\infty$, which is a contradiction to the boundedness of $I(t)$. Hence, there exists a $t_1>0$, such that $I(t_1)\geq m_3$.

2° If $I(t)\geq m_3$ for all $t\geq t_1$, and let $m_1=m_3$ then our aim is obtained. Otherwise, let $t^* = \inf_{t\geq t_1}\{I(t)<m_3\}$, there are two possible cases for t^*. In the following, we will apply the ideas of Meng and Chen (2008b) to complete the remaining proof.

Case (I): $t^* = n_1\tau$, $n_1 \in N$. Then $I(t) \geq m_3$ for $t \in [t_1, t^*)$ and $x(t^*) = m_3$. Select $n_2, n_3 \in N$, such that

$$n_2\tau > T_1, \;\; e^{(n_2+1)\sigma_1\tau}e^{n_3\sigma} > 1,$$

where $\sigma_1 = \beta m_2^{'} - (r + d_3) < 0$. Let $T = n_2\tau + n_3\tau$, we claim that there must be a $t_2 \in (t^*, t^* + T]$, such that $I(t_2) > m_3$. Otherwise, (i.e., $\forall t \in (t^*, t^* + T]$, $I(t) \leq m_3$) consider (4.22) with $S_{41}(n_1\tau^+) = S_1(n_1\tau^+), S_{42}(n_1\tau^+) = S_2(n_1\tau^+)$, for $t \in (n\tau, (n+1)\tau)$ and $n_1 \leq n \leq n_1 + n_2 + n_3$, we have

$$S_1(t) \geq S_{41}(t) \geq \widetilde{S_{41}(t)} - \varepsilon_3$$

and

$$S_2(t) \geq S_{42}(t) \geq \widetilde{S_{42}(t)} - \varepsilon_3,$$

for $t^* + n_2\tau \leq t \leq t^* + T$. This implies (4.25) holds for $t^* + n_2\tau \leq t \leq t^* + T$. As in step 1, we have

$$I(t^* + T) \geq I(t^* + n_2\tau)e^{n_3\sigma}.$$

The third equation of (4.2) gives

$$\frac{dI(t)}{dt} \geq I(t)[\beta m_2^{'} - (r + d_3)] = \sigma_1 I(t), \tag{4.26}$$

for $t \in [t^*, t^* + n_2\tau]$.

Integrating on $[t^*, t^* + n_2\tau]$, we have

$$I(t^* + n_2\tau) \geq m_3 e^{\sigma_1 n_2\tau}.$$

Then

$$\begin{aligned} I(t^* + T) &\geq I(t^* + n_2\tau)e^{n_3\sigma} \geq m_3 e^{\sigma_1 n_2\tau}e^{n_3\sigma} \\ &\geq m_3 e^{\sigma_1(n_2+1)\tau}e^{n_3\sigma} > m_3, \end{aligned}$$

which is a contradiction.

Let $\bar{t} = \inf_{t \geq t^*}\{I(t) > m_3\}$, thus $I(t) \leq m_3$ for $t \in [t^*, \bar{t}]$, $I(\bar{t}) = m_3$, since $I(t)$ is continuous and $I(t)$ is not affected by the impulsive effect. For $t \in (t^*, \bar{t}]$, suppose $t \in (t^* + (p-1)\tau, t^* + p\tau], p \in N$, and $p \leq n_2 + n_3$, by (4.26) we have

$$\begin{aligned}
I(t) &\geq I(t^* + (p-1)\tau)e^{\sigma_1(t-(t^*+(p-1)\tau))} \\
&\geq I(t^*)e^{(p-1)\sigma_1\tau}e^{\sigma_1\tau} = I(t^*)e^{p\sigma_1\tau} \\
&\geq m_3 e^{p\sigma_1\tau} \geq m_3 e^{(n_2+n_3)\sigma_1\tau} = m_1' \quad (\text{clearly}, m_3 > m_1'),
\end{aligned}$$

hence, we have $I(t) \geq m_1'$ for $t \in (t^*, \bar{t})$. For $t > \bar{t}$, the same arguments can be presented, since $I(\bar{t}) \geq m_3$.

Case (II): $t^* \neq n\tau, n \in N$. Then $I(t) \geq m_3$ for $t \in [t_1, t^*]$ and $I(t^*) = m_3$, suppose $t^* \in (n_4\tau, (n_4+1)\tau)$, $n_4 \in N$. There are two possible cases for $t \in (t^*, (n_4+1)\tau)$.

Case (IIa): $I(t) \leq m_3$ for all $t \in (t^*, (n_4+1)\tau)$. We claim that there must be a $t_2' \in [(n_4+1)\tau, (n_4+1)\tau + T]$, such that $I(t_2') > m_3$. Otherwise, i.e., $\forall t \in [(n_4+1)\tau, (n_4+1)\tau + T]$, we have $I(t) \leq m_3$. Consider (4.22) with $S_{41}((n_4+1)\tau^+) = S_1((n_4+1)\tau^+)$, $S_{42}((n_4+1)\tau^+) = S_2((n_4+1)\tau^+)$, one can get

$$S_1(t) \geq S_{41}(t) \geq \widetilde{S_{41}}(t) - \varepsilon_3, \quad S_2(t) \geq S_{42}(t) \geq \widetilde{S_{42}}(t) - \varepsilon_3,$$

for $t \in (n\tau, (n+1)\tau]$ and $n_4 + 1 \leq n \leq n_4 + 1 + n_2 + n_3$. Similarly, we have

$$I((n_4 + 1 + n_2 + n_3)\tau) \geq I((n_4 + 1 + n_2)\tau)e^{n_3\sigma}.$$

Since $I(t) \leq m_3$ for $t \in (t^*, (n_4+1)\tau)$, (4.26) holds on $[t^*, (n_4+1+n_2)\tau]$, so we have

$$I((n_4 + 1 + n_2)\tau) \geq m_3 e^{(n_2+1)\sigma_1\tau}.$$

In fact, since $t \leq (n_4+1+n_2)\tau, n_4\tau \leq t^* \leq (n_4+1)\tau$, $\sigma_1 < 0$, then $n_2\tau \leq t - t^* \leq (n_2+1)\tau$, $e^{(t-t^*)\sigma_1\tau} \geq e^{(n_2+1)\sigma_1\tau}$. Thus,

$$I((n_4 + 1 + n_2)\tau) \geq I(t^*)e^{(t-t^*)\sigma_1\tau} \geq m_3 e^{(n_2+1)\sigma_1\tau}.$$

Therefore,

$$\begin{aligned} I((n_4+1+n_2+n_3)\tau) &\geq I((n_4+1+n_2)\tau)e^{n_3\sigma} \\ &\geq m_3 e^{(n_2+1)\sigma_1\tau} e^{n_3\sigma} > m_3, \end{aligned}$$

which is a contradiction. Let $\bar{t} = \inf_{t>t^*}\{I(t) > m_3\}$, then $I(t) \leq m_3$ for $t \in (t^*, \bar{t})$ and $I(\bar{t}) = m_3$. For $t \in (t^*, \bar{t})$, suppose $t \in (n_4\tau + (p'-1)\tau, n_4\tau + p'\tau]$, $p' \in N, p' \leq 1 + n_2 + n_3$, we have

$$\begin{aligned} I(t) &\geq I(t^*)e^{(t-t^*)\sigma_1} \geq I(t^*)e^{\sigma_1\tau} \geq I(t^*)e^{p'\sigma_1\tau} \\ &\geq I(t^*)e^{(1+n_2+n_3)\sigma_1\tau} \geq m_3 e^{(1+n_2+n_3)\sigma_1\tau}. \end{aligned}$$

Let $m_1 = m_3 e^{(1+n_2+n_3)\sigma_1\tau} < m_3 e^{(n_2+n_3)\sigma_1\tau} = m_1'(\text{clearly}, m_3 > m_1)$, hence, $I(t) \geq m_1$ for $t \in (t^*, \bar{t})$. For $t > \bar{t}$, the same arguments can be presented, since $I(\bar{t}) \geq m_1$.

Case (IIb): Suppose that there exists a $t \in (t^*, (n_4+1)\tau)$, such that $I(t) > m_3$. Let $t^{**} = \inf_{t>t^*}\{I(t) > m_3\}$, then $I(t) \leq m_3$ for $t \in (t^*, t^{**})$ and $x(t^{**}) = m_3$. For $t \in (t^*, t^{**})$, (4.26) holds true, integrating (4.26) over (t^*, t^{**}), we have

$$I(t) \geq I(t^*)e^{\sigma_1(t-t^*)} \geq m_3 e^{\sigma_1\tau} > m_3 e^{(1+n_2+n_3)\sigma_1\tau} = m_1.$$

Since $I(t^{**}) \geq m_3$, for $t > t^{**}$, the same arguments can be presented. Hence $I(t) \geq m_1$ for all $t \geq t_1$. This completes the proof. □

4.5 Discussion

In this chapter, we have considered an SIR epidemic model with state structure and pulse vaccination at different fixed moments. We have proved that all solutions of system (4.2) are uniformly ultimately bounded. The conditions for the global asymptotic stability of the infection-free periodic solution of system (4.2) are given, and the permanence of system (4.2) is also obtained.

From the conditions of Theorems 4.5 and 4.6, we know that there exists a threshold τ_0. If $\tau > \tau_0$, the infection-free periodic solution $(\widetilde{S_1(t)}, \widetilde{S_2(t)}, 0)$ of system (4.2) is globally asymptotically stable. If $\tau < \tau_0$, system (4.2) is permanent. That is, improving the proportion of vaccinations and enlarging the period of birth pulse, the disease will die out. If the period of pulse vaccination is suitable, system (4.2) will be permanent. This means after some period of time the disease will come to be endemic. The results obtained provide a reliable tactic basis for preventing the disease from spreading.

CHAPTER V

CONCLUSIONS

This dissertation is devoted to the investigation of population dynamics, which includes two models.

In the fist part, we establish a predator-prey model with periodic impulsive diffusion and periodic release of predator population. The model comprises two regions, which are connected by diffusion of predator population, and portrays the evolvement of population. We prove that all solutions of the investigated system are uniformly ultimately bounded. We also prove that there exists globally asymptotically stable prey-extinction boundary periodic solution. The condition for permanence is obtained. Simulations are also employed to verify our results. It is discovered that increasing the diffusive rate of the predator population will counteract the pest management. We conclude that the impulsive diffusion and releasing predator provide a reliable tactic basis for pest management.

In the second part, we investigate an SIR epidemic model with stage structure and pulse vaccination. By using the discrete dynamical system determined by stroboscopic map, we obtain the conditions for the global asymptotical stability of the infection-free periodic solution of the studied system. Permanence conditions of the investigated system are also given. The results indicate that pulse vaccination rate plays an important role in eradicating the disease. It provides a reliable tactic basis for preventing the disease from spreading.

REFERENCES

REFERENCES

Allen, L. J. S. (1983). Persistence and extinction in single-species reaction-diffusion models. **Bull. Math. Biol.** 45: 209–227.

Allen, L. J. S. (2007). **An Introduction to Mathematical Biology**. New Jersey: Prentice Hall.

Agur, Z., Cojocaru, L., Mazor, G., Anderson, R., and Danon, Y. (1993). Pulse mass measles vaccination across age cohorts. **Proc. Natl. Acad. Sci. USA.** 90: 11698–11702.

Aiello, W. G., and Freedman, H. I. (1990). A time-delay model of single-species growth with stage structure. **Math. Biosci.** 101: 139–153.

Aiello, W. G., Freedman, H. I., and Wu, J. (1992). Analysis of a model representing stage structured population growth with state-dependent time delay. **SIAM J. Appl. Math.** 52: 855–869.

Anderson, R. M., and May, R. M. (1992). **Infectious Diseases of Humans: Dynamics and Control**. Oxford: Oxford University Press.

Bainov, D., and Simeonov, P. (1993). **Impulsive Differential Equations: Periodic Solution and Applications**. New York: Longman Scientific and Technical.

Barclay, H. J. (1982). Models for pest control using predator release, habitat management and pesticide release in combination. **J. Appl. Ecol.** 19: 337–348.

Beretta, E., and Takeuchi, Y. (1987). Global stability of single species diffusion Volterra models with continuous time delays. **Bull. Math. Biol.** 49: 431–448.

Beretta, E., and Takeuchi, Y. (1998). Global asymptotic stability of Lotka-Volterra diffusion models with continuous time delays. **SIAM J. Appl. Math.** 48: 627–651.

Berezovskaya, F. S., Song, B., and Castillo-Chavez, C. (2010). Role of prey dispersal and refuges on predator-prey dynamics. **SIAM. J. Appl. Math.** 70(6): 1821–1839.

Buonomo, B., d'Onofrio, A., and Lacitignola, D. (2008). Global stability of an SIR epidemic model with information dependent vaccination. **Math. Biosci.** 216: 9–16.

Burnett, T. (1977). Biological models of two acarine predators of the grain mite, Acarus siro L. **Can. J. Zool**. 55(8): 1312–1323.

Caltagirone, L., and Doutt, R. (1989). The history of the vedalia beetle importation to California and its impact on the development of biological control. **Ann. Rev. Entomol.** 34: 1–16.

Cantrell, R. S., and Cosner, C. (1996). Practical persistence in ecological models via comparison methods. **Proc. Roy. Soc. Edinburgh**. 126(02): 247–272.

Cao, Y., and Gard, T. C. (1998). Practical persistence for differential delay models of population interactions. **Elec. J. Diff. Equs.** Conference 01: 41–53.

Chen, J., Huang, J., Ruan, S., and Wang, J. (2013). Bifurcations of invariant tori in predator-prey models with seasonal prey harvesting. **SIAM J. Appl. Math.** 73(5): 1876–1905.

Cui, J., and Chen, L. (2001). Permanence and extinction in logistic and Lotka-Volterra system with diffusion. **J. Math. Anal. Appl.** 258: 512–535.

DeBach, P. (1964). **Biological Control of Insect Pests and Weeds**. New York: Rheinhold.

DeBach, P., and Rosen, D. (1991). **Biological Control by Natural Enemies** (2nd ed). Cambridge: Cambridge University Press.

d'Onofrio, A. (2005) On pulse vaccination strategy in the SIR epidemic model with vertical transmission. **Appl. Math. Lett.** 18: 729–732.

Driver, R. D. (1977). **Ordinary and Delay Differential Equations**. New York: Springer-Verlag.

Edelstein-Keshet, L. (1988). **Mathematical Models in Biology.** New York: The Random House/Birkhauser Mathematics Series.

Freedman, H. J. (1976). Graphical stability, enrichment, and pest control by a natural enemy. **Math. Biosci.** 31: 207–225.

Freedman, H. I. (1987). Single species migration in two habitats: persistence and extinction. **Math. Model.** 8: 778–780.

Freedman, H. I., and Moson, P. (1990). Persistence definitions and their connections. **Proc. Amer. Math. Soc.** 109(4): 1025–1033.

Freedman, H. I., Rai, B., and Waltman, P. (1986). Mathematical models of population interactions with dispersal II: differential survival in a change of habitat. **J. Math. Anal. Appl.** 115: 140–154.

Freedman, H. I., and Takeuchi, Y. (1989a). Global stability and predator dynamics in a model of prey dispersal in a patchy environment. **Nonlinear Anal.** 13: 993–1002.

Freedman, H. I., and Takeuchi, Y. (1989b). Predator survival versus extinction as a function of dispersal in a predator-prey model with patchy environment. **Appl. Anal.** 31: 247–1266.

Gakkhar, S., and Negi, K. (2008). Pulse vaccination in SIRS epidemic model with non-monotonic incidence rate. **Chaos Solitons Fractals.** 35: 626–638.

Gao, S., Chen, L., Nieto, J. J., and Torres, A. (2006). Analysis of a delayed epidemic model with pulse vaccination and saturation incidence. **Vaccine** 24: 6037–6045.

Gao, S, Teng, Z., and Xie, D. (2009). Analysis of a delayed SIR epidemic model with pulse vaccination. **Chaos Solitons Fractals.** 40: 1004–1011.

Gao, S., Zhang, F., and He, Y. (2013). The effects of migratory bird population in a nonautonomous ecoepidemiological model. **Appl. Math. Mod.** 37(6): 3903–3916.

Gause, G. F. (1934). **The Struggle for Existence**. Baltimore: Williams and Wilkins (Reprinted, 1964, by Hafner, New York).

Georgescu, P., and Hsieh, Y. H. (2007). Global dynamics of a predator-prey model with stage structure for the predator. **SIAM J. Appl. Math.** 67: 1379–1395.

Gopalsamy, K. (1992). **Stability and Oscillations in Delay Differential Equations of Population Dynamics.** Dordrecht/Norwell: Kluwer Academic.

Grasman, J., Van Herwaarden OA, Hemerik, L., and Van Lenteren, J. C. (2001). A two-component model of host-parasitoid interactions: determination of the size of inundative releases of parasitoids in biological pest control. **Math. Biosci.** 169: 207–216.

Gurney, S. C., Nisbet, R. M., and Blythe, S. P. (1984). The systematic formulation of models of stage-structured populations. In **The Dynamics of Physiologically Structured Populations** (Lecture Notes in Biomath. 68, pp. 474–494). Berlin: Springer.

Hassell, M. P. (1978). **The Dynamics of Arthropod Predator-prey Systems**. Princeton: Princeton University Press.

Hethcote, H. W. (1989). Three basic epidemiological models. In **Applied Mathematical Ecology** (pp. 119–144). Berlin: Springer.

Hofbauer, J., and Sigmund, K. (1988). **The Theory of Evolution and Dynamical Systems**. Cambridge: Cambridge University Press.

Hofbauer, J., and Sigmund, K. (1998). **Evolutionary Games and Population Dynamics**. Cambridge: Cambridge University Press.

Huffaker, C. B. (1958). Experimental studies on predation: dispersion factors and predator-prey oscillations. **Hilgardia.** 27: 343–383.

Huffaker, C. B., Shea, K. P., and Herman, S. G. (1963). Experimental studies on redation. Complex dispersion and levels of food in an acarine predator-prey interaction. **Hilgardia**. 34: 305–329.

Hui, J., and Chen, L. (2005). A single species model with impulsive diffusion. **Acta. Math. Appl. Sin-E.** 21: 43–48.

Jiao, J., Cai, S., and Chen, L. (2011). Analysis of a stage-structured predator-prey system with birth pulse and impulsive harvesting at different moments. **Nonlinear Anal. RWA.** 12: 2232–2244.

Jiao, J., Cai, S., and Li, L. (2015). Dynamics of an SIR model with vertical transmission and impulsive dispersal. **J. Appl. Math. Comput.** doi:10.1007/s12190-015-0934-2.

Jiao, J., and Chen, L. (2008). Global attractivity of a stage-structure variable coefficients predator-prey system with time delay and impulsive perturbations on predators. **Int. J. Biomath.** 1: 197–208.

Jiao, J., and Chen, L. (2012). The genic mutation on dynamics of a predator-prey system with impulsive effect. **Nonl. Dyn.** 70(1): 141–153.

Jiao, J., Chen, L., Cai, S., and Wang, L. (2010). Dynamics of a stage-structured predator-prey model with prey impulsively diffusing between two patches. **Nonlinear Anal. RWA.** 11: 2748–2756.

Jiao, J., Chen, L., and Luo, G. (2008a). An appropriate pest management SI model with biological and chemical control concern. **Appl. Math. Comput.** 196: 285–293.

Jiao, J., Meng, X., and Chen, L. (2007). A stage-structured Holling mass defence predator-prey model with impulsive perturbations on predators. **Appl. Math. Comput.** 189: 1448–1458.

Jiao, J., Pang, G., Chen, L., and Luo, G. (2008b). A delayed stage-structured predator-prey model with impulsive stocking on prey and continuous harvesting on predator. **Appl. Math. Comput.** 195: 316–325.

Jury, E. L. (1974). **Inners and Stability of Dynamics System**. New York: Wiley.

Kermack, W. O., and Mckendrick, A. G. (1927). A contribution to the mathematical theory of epidemics. **Proc. R. Soc. A** 115(772): 700–721.

Kermack, W. O., and Mckendrick, A. G. (1932). Contributions to the mathematical theory of epidemics. II. The problem of endemicity. **Proc. R. Soc. A** 138(834): 55–83.

Kermack, W. O., and Mckendrick, A. G. (1933). Contributions to the mathematical theory of epidemics. III. Further studies of the problem of endemicity. **Proc. R. Soc. A** 141(843): 94–122.

Kermack, W. O., and Mckendrick, A. G. (1937). Contributions to the mathematical theory of epidemics IV. Analysis of experimental epidemics of the virus disease mouse ectromelia. **J. Hyg.** 37: 172–187.

Klausmeier, C. A. (2008). Floquet theory: a useful tool for understanding nonequilibrium dynamics. **Theor. Ecol.** 1: 153–161.

Lakshmikantham, V., Bainov, D., and Simeonov, P. (1989). **Theory of Impulsive Differential Equations**. Singapore: World Scientific.

Levin, S. A. (1994). Dispersion and population interaction. **Am. Nat.** 108: 207–228.

Li, X., and Wang, W. (2005). A discrete epidemic model with stage structure. **Chaos Solitons Fractals.** 26: 947–958.

Liu, X. (1995). Impulsive stabilization and applications to population growth models. **J. Math.** 25: 381–395.

Liu, L. (2015). A delayed SIR model with general nonlinear incidence rate. **Adv. Differ. Equ.** 2015, 329.

Liu, S., and Beretta, E. (2006). A stage-structured predator-prey model of Beddington-DeAngelis type. **SIAM J. Appl. Math.** 66: 1101–1129.

Liu, L., Cai, W., and Wu, Y. (2012). Global dynamics for an SIR patchy model with susceptibles dispersal. **Adv. Differ. Equ.** 2012, 131.

Liu, X., and Chen, L. (2003). Complex dynamics of Holling type II Lotka-Volterra predator-prey system with impulsive perturbations on the predator. **Chaos Solitons Fractals.** 16: 311–320.

Liu, S., Chen, L., and Agarwal, R. (2002). Recent progress on stage-structured population dynamics. **Math. Comput. Model.** 36: 1319–1360.

Lotka, A. J. (1925). **Elements of Physical Biology**. Baltimore: Williams and Wilkins, Baltimore (Reissued as Elements of Mathematical Biology by Dover, 1956).

Luchinbill, L. S. (1973). Coexistence in laboratory populations of Paramecium aurelia and its predator Didinium nasutum. **Ecology.** 54: 1320–1327.

Lu, Z., Chi, X., and Chen, L. (2002). The effect of constant and pulse vaccination on SIR epidemic model with horizontal and vertical transmission. **Math. Comput. Model.** 36: 1039–1057.

Magal, P., and Ruan, S., eds. (2008). **Structured Population Models in Biology and Epidemiology.** Lecture Notes in Math, 1936. Berlin Heidelberg: Springer-Verlag.

Maynard Smith, J. (1974). **Models in Ecology**. Cambridge, England: Cambridge University Press.

Meng, X., and Chen, L. (2008a). Permanence and global stability in an impulsive Lotka-Volterra N-species competitive system with both discrete delays and continuous delays. **Int. J. Biomath.** 1: 179–196.

Meng, X., and Chen, L. (2008b). The dynamics of a new SIR epidemic model concerning pulse vaccination strategy. **Appl. Math. Comput.** 197: 582–597.

Murray, J. D. (1989). **Mathematical Biology**. New York: Springer-Verlag.

Nisbet, R. M., and Gurney, W. S. C. (1984). "Stage-structure" models of uniform larval competition. in **Mathematical Ecology** (Lecture Notes in Biomath. 54, pp. 97–113). Berlin: Springer.

Nisbet, R. M., Gurney, W. S. C., and Metz, J. A. J. (1989) Stage structure models applied in evolutionary ecology. In **Applied Mathematical Ecology** (pp.428–449) Berlin: Springer.

Sedaghat, H. (1997). The impossibility of unstable, globally attracting fixed points for continuous mappings of the line. **Amer. Math. Monthly.** 104: 356–358.

Shulgin, B., Stone, L., and Agur, Z. (1998). Pulse vaccination strategy in the SIR epidemic model. **Bull. Math. Biol.** 60: 1123–1148.

Song, X., and Chen, L. (2001). Optimal harvesting and stability for a two species competitive system with stage-structure. **Math. Biosci.** 170: 173–186.

Song, X., and Chen, L. (2002). Uniform persistence and global attractivity for nonautonomous competitive systems with dispersion. **J. Syst. Sci. Complex.** 15: 307–314.

Stone, L., Shulgin, B., and Agur, Z. (2000). Theoretical examination of the pulse vaccination policy in the SIR epidemic model. **Math. Comput. Model.** 31: 207–215.

Utida, S. (1957). Cyclic fluctuations of population density intrinsic to the host parasite system. **Ecology**. 38: 442–449.

Volterra, V. (1926). Variazioni e fluttuazioni del numero d'individui in specie animali conviventi. **Mem. Acd. Lincei.** 2: 31–113 (Translation in: Chapman, R. N. 1931. Animal Ecology, pp. 409–448. McGraw-Hill, New York).

Wang, W., and Chen, L. (1997). A predator prey system with stage-structure for predator. **Comput. Math. Appl.** 33: 83–91.

Wang, W., Mulone, G., Salemi, F., and Salone, V. (2001). Permanence and stability of a stage-structured predator-prey model. **J. Math. Anal. Appl.** 262: 499–528.

Xiao, Y., and Chen, L. (2001). Analysis of a SIS epidemic model with stage structure and a delay. **Commun. Nonlinear Sci. Numer. Simul.** 6: 35–39.

Xiao, Y., and Chen, L. (2003). On an SIS epidemic model with stage-structure. **J. Syst. Sci. Complex.** 16: 275–288.

Xiao, Y., and Chen, L. (2004). Global stability of a predator-prey system with stage structure for the predator. **Acta Math. Sin.** (Engl. Ser.) 20: 63–70.

Xu, R. (2011). Global dynamics of a predator-prey model with time delay and stage structure for the prey. **Nonlinear Anal. RWA.** 12: 2151–2162.

Xu, R. (2013). Mathematical analysis of the global dynamics of an eco-epidemiological model with time delay. **J. Fran. Inst.** 350(10): 3342–3364.

Xu, R. (2014). Global dynamics of an SEIRI epidemiological model with time delay. **Appl. Math. Comp.** 232: 436–444.

Xu, R., and Ma, Z. (2009). Global stability of a SIR epidemic model with nonlinear incidence rate and time delay. **Nonlinear Anal., Real World Appl.** 10: 3175–3189.

Yuan, X., Xue, Y., and Liu, M. (2014). Global stability of an SIR model with two susceptible groups on complex networks. **Chaos Solitons Fractals.** 59: 42–50.

Zhang, Z., and Suo, Y. (2010). Qualitative analysis of a SIR epidemic model with saturated treatment rate. **J. Appl. Math. Comput.** 34: 177–194.

CURRICULUM VITAE

NAME: Airen Zhou **GENDER**: Male

NATIONALITY: Chinese **DATE OF BIRTH**: Nov. 15, 1978

PLACE OF BIRTH: Jiangxi, China **E-MAIL**: zhouairen@163.com

AFFILIATION AND ADDRESS: School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

EDUCATIONS:

- College Graduate Certificate in Mathematics Education, Shangrao Normal University, Jiangxi, China, 2000.
- Master of Science in Applied Mathematics, Guizhou University, Guizhou, China, 2008.

CONFERENCE:

Dynamical analysis of a population model with stage structure and impulsive effects. **The 41^{st} Congress on Science and Technology of Thailand (STT41)**. November 6-8, 2015, Suranaree University of Technology, Nakhon Ratchasima, Thailand.

PUBLICATIONS:

[1] Zhou, A., Sattayatham, P., Jiao, J. (2016). Analysis of a predator-prey model with impulsive diffusion and releasing on predator population. **Advances in Difference Equations**. DOI: 10.1186/s13662-016-0834-2.

[2] Zhou, A., Sattayatham, P., Jiao, J. (2016). Dynamics of an SIR epidemic model with stage structure and pulse vaccination. **Advances in Difference Equations**. DOI: 10.1186/s13662-016-0853-z.