GROUP ANALYSIS OF INTEGRO-DIFFERENTIAL EQUATIONS FOR ONE-DIMENSIONAL VISCOELASTIC MATERIALS WITH MEMORY

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สำหรับวัสดุยืดหยุ่นหนืดหนึ่งมิติที่มีความจำ

นายหลงเฉา โจว

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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หลงเฉา โจว : การวิเคราะห์กลุ่มของสมการอินทิโกรดิฟเฟอเรนเชียลสำหรับวัสดุยืดหยุ่นหนืดหนึ่งมิติที่มีความจำ (GROUP ANALYSIS OF INTEGRO-DIFFERENTIAL EQUATIONS FOR ONE-DIMENSIONAL VISCOELASTIC MATERIALS WITH MEMORY) อาจารย์ที่ปรึกษา: ศาสตราจารย์ ดร.เซอร์เก เมเลชโก, 129 หน้า.

วิทยานิพนธ์นี้มีจุดสำคัญ คือ การประยุกต์ใช้วิธีการวิเคราะห์กลุ่มหาผลเฉลยของสมการอินทิโกรดิฟเฟอเรนเชียล 2 ระบบสมการ ที่ใช้อธิบายพฤติกรรมของวัสดุยืดหยุ่นหนืดหนึ่งมิติที่มีความจำ ทั้งนี้ได้ใช้แนวคิดที่พึ่งถูกพัฒนาล่าสุดสำหรับการคำนวณหากลุ่มลีที่ถูกแอดมิทโดยสมการอินทิโกรดิฟเฟอเรนเชียล

ระบบสมการแรกที่ศึกษา คือ ระบบสมการอินทิโกรดิฟเฟอเรนเชียลซึ่งเกี่ยวเนื่องกับการผ่อนปรนความเค้นแบบไม่เชิงเส้น สมการกำหนดของระบบสมการดังกล่าวได้ถูกสร้างขึ้นและหาผลเฉลยเพื่อให้เกิดการจำแนกกลุ่มอย่างบริบูรณ์ แต่ละกลุ่มที่ถูกแอดมิทโดยระบบสมการดังกล่าวจะถูกนำไปใช้หาผลเฉลยยืนยงต่าง ๆ ของระบบสมการ

ระบบสมการที่สองที่ได้ศึกษาในวิทยานิพนธ์นี้ คือ ระบบสมการที่สมนัยกับตัวแบบของการยืดหยุ่นหนืดตามอุณหภูมิแบบเชิงเส้น จากการศึกษาทำให้ได้สมการกำหนด และในการหาผลเฉลยของสมการกำหนดนี้ ได้แบ่งการพิจารณาเป็น 4 กรณี ซึ่งนำไปสู่การได้กลุ่มสมมาตรของแต่ละกรณีข้างต้น

สาขาวิชาคณิตศาสตร์	ลายมือชื่อนักศึกษา Longqiao Zhou
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LONGQIAO ZHOU : GROUP ANALYSIS OF INTEGRO-DIFFERENTIAL EQUATIONS FOR ONE-DIMENSIONAL VISCOELASTIC MATERIALS WITH MEMORY. THESIS ADVISOR : PROF. SERGEY V. MELESHKO, Ph.D. 129 PP.

VISCOELASTIC MATERIALS/THERMOVISCOELASTICITY/ADMITTED LIE GROUPS/GROUP CLASSIFICATION/INVARIANT SOLUTIONS

We focus on the application of the group analysis method to two systems of integro-differential equations describing behavior of one-dimensional viscoelastic materials with memory. Recently developed approaches for calculating an admitted Lie group of integro-differential equations were used in this thesis.

The first system considered is a system of integro-differential equations related with nonlinear stress relaxation. The determining equations of these equations are constructed and the complete group classification is derived by solving the determining equations. Using the admitted groups, invariant solutions of the system are also presented.

The second system studied in the thesis is a system of equations corresponding to a linear thermoviscoelastics model. The determining equations for this system are also obtained. To solve these equations, four different cases of the system are studied. Finally, the symmetry groups of each case are obtained.

School of Mathematics	Student's Signature Longqiao Zhou
Academic Year 2015	Advisor's Signature

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CHAPTER I

INTRODUCTION

Many important physical processes in nature are governed by differential equations. Nonlinearity and the presence of a large number of variables in the initial equations are sources of significant mathematical difficulties in the analysis of the solutions of these equations. Frequently, it is virtually impossible to give explicit solutions, and while a multitude of numerical methods has been developed to obtain approximate solutions, there remains intense interest in finding exact solutions. Each solution has value, first, as the exact description of the real process in the framework of a given model; secondly, as a model to compare various numerical methods; thirdly, as a basis to improve the models used.

1.1 The method used in this thesis

One of the methods for obtaining exact solutions of differential equations is the Lie group analysis method. The general theory of applications of Lie groups to differential equations is discussed in numerous books and papers, in particular in (Ovsiannikov, 1982; Bluman and Kumei, 1989; Olver, 1993; Ibragimov, 1999). In the literature, this approach has been applied to many mathematical physics problems, for example, (Ibragimov, 1996) and references therein.

The classical Lie group theory provides a universal tool for calculating the admitted Lie group for a system of differential equations. However, one encounters several difficulties when trying to apply this method to integro-differential equations. The main difficulty comes from their nonlocal terms (integral terms).

Since the definition of an admitted Lie group given for partial differential equations cannot be applied to integro-differential equations, this concept requires further investigation. A regular method for calculating an admitted Lie group of integro-differential equations was introduced in (Grigoriev and Meleshko, 1986; Grigoriev and Meleshko, 1987; Grigoriev et al., 2010), where a Lie group admitted by integro-differential equations was defined as a Lie group satisfying determining equations.

The way of obtaining determining equations for integro-differential equations is obtained in a similar (and not more difficult) way as used for differential equations. The main difficulty in obtaining an admitted Lie group of integro-differential equations consists in solving the determining equations, because they are also of integro-differential type. Usually, a solution of the determining equations of a system of partial differential equations is obtained by its simplification, where the main simplification is derived by splitting the determining equations. It should be noted that contrary to differential equations, the way of splitting determining equations of integro-differential equations depends on the equations studied. There are several methods to split determining equations of integro-differential equations (Grigoriev and Meleshko, 1986; Grigoriev et al., 2010; Meleshko, 1988; Kovalev et al., 1992). One of these methods was first used in (Meleshko, 1988) for finding the admitted Lie group of the system of equations that describes one-dimensional motion of a viscoelastic medium. Other applications of the method can be found in (Kovalev et al., 1992; Grigoriev and Meleshko, 1990; Ozer, 2003a; Ozer, 2003b).

1.2 The background of the studied models

Behavior of many materials under applied load can be approximated by specifying a relation between the applied load or stress σ and the resultant deformation or strain e. This relation is described by constitutive equations. For example, in the case of elastic materials the constitutive equation, identified as Hooke's law, states that stress is proportional to strain, i.e., $\sigma = Ee$ (E is called Young's modulus). In the case of viscous materials, the relation states that the stress is proportional to strain rate, i.e., $\sigma = \eta\dot{e}$ (η is called the viscosity). Here $\dot{e}$ is the derivative of e with respect to t.

The class of materials that exhibit the characteristics of elastic as well as viscous materials are known as viscoelastic materials (Rabotnov, 1977; Lokshin and Suvorova, 1982; Lakes, 1998; Reddy, 2007; Borcherdt, 2009; Shames and Cozzarelli, 1997; Renardy et al., 1987). The description of the behavior of viscoelastic materials is more complicated than that of either elastic materials or viscous fluids. In an elastic materials, the stress depends on the present value of the strain, while the stress in a viscous fluid is determined by the velocity gradient. All materials exhibit some viscoelastic response, and in some applications, even a small viscoelastic response can be significant. In general there are two alternative forms used to represent the stress-strain-time relations of viscoelastic materials.

First, one considers a differential form of the constitutive equations, where the stress is a function of the present values of both the strain and the velocity gradient. Usual one-dimensional models such as Kelvin–Voigt, Maxwell and their generalizations are widely used in the theory of linear viscoelastic materials.

Generalizations of the Kelvin–Voigt and Maxwell models to the nonlinear and three-dimensional case can be found in (Destrade and Saccomandi, 2005; Destrade and Saccomandi, 2009; Filograna et al., 2009; Beatty and Zhou, 1994; Pucci and Saccomandi, 2010; Pucci and Saccomandi, 2011)*. Here we also mention the approaches of (Godunov, 1978; Godunov and Romenskii, 1998) and (Sokolovskii, 1948; Malvern, 1951). In (Godunov, 1978; Godunov and Romenskii, 1998), the formulation of a nonlinear Maxwell model using equations for the metric tensor of effective elastic deformation in Eulerian coordinates is developed. The Sokolovski-Malvern model (Sokolovskii, 1948; Malvern, 1951) is applied for describing visco-plastic longitudinal waves in bars (Cristescu, 2007).

The model considered in the thesis belongs to the second alternative form where the constitutive equations include integrals. Experiments show that there are some viscoelastic materials (e.g. polymers, suspensions and emulsions) which cannot be described using a differential form of the constitutive equations. Such materials have memory: the material response is not only determined by the current state, but is also determined by all past states. The models including past history have an integral representation. The advantage of using an integral representation over a differential form lies in the flexibility of representing the actually measured viscoelastic material properties. The integral representation can also be extended readily to describe the behavior of aging materials.

Knowledge of the viscoelastic response of a material is based on measurements. Experiments for the measurements can roughly be separated into two types:

(a) experiments where the strain holds a constant value (stress relaxation test);

(b) experiments where a constant stress is applied to a specimen (creep test).

Creep is a slow, continuous deformation of a material under constant stress. Stress

*See also literature therein.

relaxation is the gradual decrease of stress when the material is held at constant strain. Creep and stress relaxation phenomena are common to many viscoelastic materials. According to this classification, models applied for describing the behavior of viscoelastic materials also fall into two types: stress relaxation and creep models.

The constitutive equation relating stress to strain describing stress relaxation behavior of one-dimensional linearly viscoelastic materials is given by

$$\sigma(t) = Y(t)e(0) + \int_0^t Y(t-\tau)\dot{e}(\tau)\, d\tau, \tag{1.1}$$

where the variable t is the present time, τ is the past time, $\sigma(t)$ denotes stress as a function of time, $e(t)$ denotes strain as a function of time, $Y(t)$ is called the relaxation modulus (or relaxation function), which does not depend on the spatial coordinate. The stress $\sigma(t)$ depends on the earlier history of the strain $e(t)$ via the strain rate $\dot{e}(\tau)$. Here the spatial coordinate x is omitted.

Some alternative forms of (1.1) are

$$\sigma(t) = Y(0)e(t) + \int_0^t \dot{Y}(t-\tau)e(\tau)\, d\tau,$$

and

$$\sigma(t) = Y(0)e(t) + \int_0^t \dot{Y}(\tau)e(t-\tau)\, d\tau. \tag{1.2}$$

Equation (1.2) represents a viscoelastic material with the memory property: the variable σ depends on not only the current value of the variable e, but also the past history of e.

It is important to note that the use of $Y(t-\tau)$ in equation (1.1) implies that the shape of the relaxation function is invariant with respect to a shift in the time origin. Thus, the stress response in material subjected to some applied strain at $t=\tau$ is identical to the response that would have occurred if the same strain had been applied at $t = 0$. A material satisfying this condition is said to be nonaging. For aging materials $Y(t-\tau)$ in equation (1.1) is replaced by a more general material function $Y(t,\tau)$ which is a function of two variables. Discussions of aging materials can be found in (Hodge, 1995; Struik, 1978).

Linear constitutive equations are suitable for describing the behavior of real materials for small deformations only. However, in most viscoelastic materials, the actual behavior exhibits nonlinearity. Thus it is necessary to develop constitutive equations for nonlinear viscoelastic materials.

The constitutive equations derived by Volterra (Volterra, 1930) and Green-Rivlin (Green and Rivlin, 1957) belong to the first type (creep models). In the one-dimensional case and for unaging materials, they can be written in the following form

$$e(t) = \int_{-\infty}^{t} J_1(t-\tau_1)\, d\sigma(\tau_1) + \int_{-\infty}^{t}\int_{-\infty}^{t} J_2(t-\tau_1, t-\tau_2)\, d\sigma(\tau_1) d\sigma(\tau_2) + \dots \quad (1.3)$$

where $e(t)$ is strain, $\sigma(t)$ is stress and $J_i(t-\tau_i)$ is called a creep function.

Since the form of equation (1.3) is very general, for practical purposes the following constitutive equations were used:

(a) Rabotnov equation (Rabotnov, 1977)

$$\varphi(e(t)) = \sigma(t) + \int_{-\infty}^{t} K(t-\tau)\, \sigma(\tau)\, d\tau$$

(b) Leaderman-Rozovski equation (Leaderman, 1943; Rozovskii, 1955)

$$e(t) = \psi(\sigma(t)) + \int_{-\infty}^{t} K(t-\tau)\, G(\sigma(\tau))\, d\tau$$

It was shown that these two models give similar approximation of experimental results (Lokshin and Suvorova, 1982).

The first studied model in the thesis is a stress relaxation model

$$\varphi(\sigma(t)) = e(t) + \int_0^t H(t,\tau)e(\tau)\, d\tau,\ \varphi'(\sigma) \neq 0, \tag{1.4}$$

which is an alternative to Rabotnov's model allowing for aging.

Further extension of linear viscoelasticity is inclusion of thermodynamics in a model. The general framework of the theory of thermodynamics of simple materials with fading memory was proposed by (Coleman, 1964a; Coleman, 1964b; Day, 1972; Fabrizio and Morro, 1992; Amendola et al., 2012).

For viscoealstic materials with fading memory the constitutive assumption states that the free anergy ψ, the stress $\mathbf{T}$, specific entropy η and the heat flux q at time t depend on the history of the deformation gradient $\mathbf{F}$, the history of the temperature θ, and the present value of the temperature gradient $\nabla\theta$. The thermodynamics process obeys the laws of balances of momentum and energy; the first and second laws of thermodynamics. Roughly speaking, the second law of thermodynamics is viewed as the basic tool for an a priori characterization of the description of the material behavior. Much research on thermodynamics (in particular, thermoviscoelasticity) has been undertaken in (Gurtion and Herrera, 1965; Gurtion and Hrusa, 1988; Brandon and Hrusa, 1988; Fabrizio and Morro, 1992; Lazzari and Vuk, 1992; Hajar and Blanc, 1998), where attempts to produce constitutive equations, to find the thermodynamic restrictions on the constitutive equations and to establish some theorems for the equations were presented.

A thermodynamical model for a body composed of an inhomogeneous isotropic linear thermoviscoelastic material was established by (Navarro, 1978), whose constitutive equations are an extension of linear viscoelasticity to the non-isothermal situation. He studied the existence, uniqueness, and asymptotic behavior of solutions and obtained very interesting theorems. Other research on the existence, uniqueness and asymptotic stability of solutions of thermodynamics models can be found in (Foutsitzi et al., 1997; Liu and Zheng, 1996; Giorgi and Naso, 2000; Foutsitzi et al., 2003).

1.3 Objectives and organization of this thesis

The main objective of the thesis is to apply the Lie group analysis method to two systems of integro-differential equations describing behavior of one-dimensional viscoelastic materials with memory.

The thesis is divided into six chapters, its content can be summarized as follows. Because the main tool used in the thesis is the Lie group analysis method, Chapter II is devoted to an outline of some concepts of group analysis.

Chapter III presents the two models of one-dimensional viscoelastic materials with memory which are extensions of linear viscoelasticity. The first considered model is a nonlinear stress relaxation model. The second model is a linear thermoviscoelastics model. Some studies and restrictions of these two models are also discussed in this chapter.

The results of group analysis of integro-differential equations related with nonlinear stress relaxation are described in Chapter IV. Determining equations are constructed by applying the definition of admitted Lie group of integro-differential equations. In the first step, a classifying equation is obtained. Studying the classifying equation, the complete group classification is derived. Using the admitted Lie groups, invariant solutions are found.

Chapter V deals with symmetry groups of equations for the linear thermoviscoelastics model. For solving determining equations, four different cases are studied. Finally, the symmetry groups of each case are obtained.

Lastly, the conclusion is presented in Chapter VI.

CHAPTER II

GROUP ANALYSIS

In this chapter we review the basic concepts from Lie group analysis: continuous transformation groups and their generators, Lie algebra of generators, invariant solutions, admitted Lie groups of differential equations and admitted Lie groups of intgro-differential equations.

2.1 One-parameter transformation groups

Many elements of group analysis are based on the consideration of one-parameter transformation groups, to which this section is devoted. Continuous groups G are determined by a first-order linear differential operator (called infinitesimal generator). The important theoretical fact is that there is a one-to-one correspondence between a group G and its generator. This section also contains an introduction to the theory of prolonged generators and Lie-Bäcklund operators.

2.1.1 Definition of a transformation group

Let V be an open set in $Z = R^N(z)$, and $\triangle$ be a symmetric interval in R^1. Assume that the point transformations

$$\bar{z}_i = g^i(z; a), \qquad (i = 1, 2, \cdots, N) \tag{2.1}$$

are invertible. Here $z \in V \subset Z$ and $a \in \triangle$ is a parameter. It is also convenient to use the notion $g_a(z) = g(z; a)$.

Definition 1 *A set of transformations (2.1) is called a local one-parameter Lie group of transformations G^1 if it has the following properties:*

1. *$g_0(z) = z$ for any $z \in V$;*

2. *$g_a(g_b(z)) = g_{a+b}(z)$ for all $a, b, a+b \in \triangle$ and $z \in V$;*

3. *if $a \in \triangle$ and $g_a(z) = z$ for all $z \in V$, then $a = 0$.*

A Lie group G^1 is called a continuous group of the class C^k if the function $g(z, a)$ belongs to the class $C^k(V)$. In applied group analysis all functions are considered to be sufficiently many times continuously differentiable.

2.1.2 Generator of a one-parameter group

Consider a local one-parameter Lie group G^1 of transformations (2.1), expand the functions $g^i(z, a)$ into the Taylor series in the parameter a in a neighborhood of $a = 0$. Taking into account the initial condition which is property 1 of definition 1, we arrive at what is called the infinitesimal transformation of the group G^1:

$$\bar{z}^i \approx z^i + a\zeta^i(z), \quad i = 1, \ldots, N, \tag{2.2}$$

where

$$\zeta^i(z) = \frac{\partial g^i(z, a)}{\partial a} \mid_{a=0} . \tag{2.3}$$

The infinitesimal transformation (2.2)-(2.3) defines the tangent vector

$$\zeta(z) = (\zeta^1(z), \zeta^2(z), \ldots, \zeta^N(z))$$

at a point z to the curve described by the transformed points $\bar{z}$, and is therefore called the tangent vector field of the group G^1. The tangent vector field is often written as a first-order linear differential operator

$$X = \zeta^i(z)\partial_{z_i}, \tag{2.4}$$

Notice that the summation convention is adopted in which there is summation over repeated upper and lower indices.

2.1.3 The Lie equations

Given an infinitesimal transformation (2.2), or generator (2.4), the transformations (2.1) of the corresponding one-parameter group G^1 are completely defined by solving the following system of first-order ordinary differential equations:

$$\frac{d\bar{z}}{da} = \zeta(\bar{z}), \qquad \bar{z}|_{a=0} = z. \tag{2.5}$$

Equations (2.5) are called Lie equations.

Theorem 1 (Lie) *Let $\zeta \in C^k(V)$ with $\zeta(z_0) \neq 0$ for some $z_0 \in V$. The solution of the Cauchy problem (2.5) generates a local Lie group of transformations with the infinitesimal generator $X = \zeta^i(z)\partial_{z_i}$.*

The Lie theorem establishes a one-to-one correspondence between Lie group of transformations (2.1) and the infinitesimal generator $X = \zeta^i(z)\partial_{z_i}$.

2.1.4 Prolongation of Lie group transformations

Application of Lie group analysis for differential equations requires introduction of functions which depend not only on the independent variables and the dependent variables, but also on derivatives of finite orders. Hence, the question arises: how do the point transformations act on derivatives? The transformations of derivatives under the action of the point transformations, regarded as a change of variables, is well-known from Calculus.

The space $Z = R^n(x) \times R^m(u)$ is prolonged by introducing the variables $p = (p^j_\alpha)$. The space J^l of the variables

$$x = (x_1, x_2, \ldots, x_n), \quad u = (u^1, u^2, \ldots, u^m), \quad p = (p^j_\alpha)\ (j = 1, 2, \ldots, m;\ |\alpha| \leq l).$$

is called an $l - th$ prolongation of the space Z. This space can be provided with a manifold structure. Here and below the following notions are used:

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$
$$\alpha, i = (\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \cdots, \alpha_n),$$

The variable p^j_α plays the role of a derivative:

$$p^j_\alpha = \frac{\partial^{|\alpha|} u^j}{\partial x_\alpha} = \frac{\partial^{|\alpha|} u^j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

Let G be a one-parameter group of transformations of independent variables $x = (x_1, x_2, \ldots, x_n)$ and dependent variables $u = (u^1, u^2, \ldots, u^m)$:

$$\bar{x}_i = f^i(x, u; a), \quad f^i\ |_{a=0} = x_i, \quad i = 1, 2, \ldots, n, \tag{2.6}$$

$$\bar{u}^j = \varphi^j(x, u; a), \quad \varphi^j\ |_{a=0} = u^j, \quad j = 1, 2, \ldots, m. \tag{2.7}$$

The generator of the group G is written in the form

$$X = \xi^i(x, u)\partial_{x_i} + \eta^j(x, u)\partial_{u^j}, \tag{2.8}$$

$$\xi^i(x, u) = \frac{\partial f^i}{\partial a}(x, u; 0), \quad \eta^j(x, u) = \frac{\partial \varphi^j}{\partial a}(x, u; 0). \tag{2.9}$$

Equations (2.6), (2.7) yield the rule for the change of derivatives:

$$\bar{p}^j_k D_i(f^k) = D_i(\varphi^j), \tag{2.10}$$

where D_i is the operator of the total differentiation with respect to x_i is defined by

$$D_i = \partial_{x_i} + u_i^k \partial_{u^k} + u_{ij}^k \partial_{u_j^k} + \cdots , \quad i = 1, 2, \ldots , n. \tag{2.11}$$

Upon solving equation (2.10) with respect to $\bar{p}_i^j$, we obtain the transformation of the first derivatives,

$$\bar{p}_i^j = P_i^j(x, u, p_{(1)}, a).$$

Extensions to second and higher order derivatives are obtained by further differentiating equation (2.10). An extension of a group of transformations (2.6), (2.7) to derivatives of any order is again a one-parameter group and is called an extended point transformation group.

Set

$$\bar{p}_i^j = p_i^j + a\eta_i^j.$$

By differentiating equation (2.10) with respect to the parameter a and using property 1 of a Lie group, we obtain the first prolongation formula:

$$\eta_i^j = D_i\eta^j - p_k^j D_i \xi^k.$$

Hence, the extended generator is

$$X_{(1)} = \xi^i \partial_{x_i} + \eta^j \partial_{u^j} + \eta_i^j \partial_{p_i^j}.$$

Similarly, we have the following definition

Definition 2 *The generator*

$$X_{(l)} = X + \sum_{j,\alpha} \eta_\alpha^j \partial_{p_\alpha^j} \tag{2.12}$$

with the coefficients

$$\eta_{\alpha,k}^j = D_k \eta_\alpha^j - \sum_i p_{\alpha,i}^j D_k \xi^i$$

is called the $l-th$ prolongation of the generator X.

2.2 Admitted Lie group of a system of differential equations

2.2.1 Invariant of Lie group

In applications, the study of most equations is closely connected with a regularly assigned manifold. The concept of invariant manifolds, relative to transformations, is studied in this subsection.

Consider a system of s equations in R^N:

$$F_k(z) = 0, \qquad k = 1, 2, \ldots, s, \tag{2.13}$$

where $z = (z_1, z_2, \ldots, z_N) \in R^N$ and $s < N$. If the Jacobian matrix $\|\frac{\partial F}{\partial z}\|$ is of rank s:

$$rank \left\| \frac{\partial F_k(z)}{\partial z_i} \right\| = s \tag{2.14}$$

at all points z satisfying equations (2.13), then the locus of solutions z of system of equations (2.13) is an $(N-s)$-dimensional manifold $M \subset R^N$.

Definition 3 *The system (2.13) is said to be invariant with respect to a Lie group G^1 of transformations*

$$\bar{z_i} = g^i(z; a), \qquad (i = 1, 2, \cdots, N)$$

if each solution $z = (z_1, z_2, \ldots, z_N)$ of system (2.13) is mapped to a solution $\bar{z} = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N)$ of the same system, i.e.

$$F_k(\bar{z}) = 0, \qquad k = 1, 2, \ldots, s.$$

Geometrically, the invariance of (2.13) means that each point z on the surface M (the manifold defined by (2.13)) is moved by G^1 along the surface M, i.e., $z \in M$ implies that $\bar{z} \in M$. The manifold M is also called an invariant manifold with respect to G^1.

In the following theorem the necessary and sufficient condition that some system is an invariant of the group is presented.

Theorem 2 *The system of equations (2.13) is invariant with respect to a Lie group G^1 with an infinitesimal generator X if and only if*

$$XF_k(z) \mid_M = 0, \quad k = 1, 2, \dots, s, \tag{2.15}$$

where the symbol $|_M$ means that the equations $XF_k(z)$ are considered on the manifold M.

2.2.2 Admitted Lie group

For the purpose of group analysis it is convenient to view the differential equation S as a manifold. According to Lie's suggestion, one says that S admits a group (or generator) if S is contained in an invariant manifold of a Lie group. In the present subsection is devoted to finding and studying the set of all groups G^1, which are admitted by the differential equation S.

A system of $l-th$ order differential equations is considered as a manifold in J^l. Let this manifold be assigned by the equations

$$S^k(x, u, p) = 0, \quad (k = 1, 2, \dots, s).$$

The system and the manifold

$$(S) = \{(x, u, p) \in J^l | S^k(x, u, p) = 0, \quad (k = 1, 2, \dots, s)\}, \tag{2.16}$$

are denoted by (S). It is assumed that (S) is a regularly assigned manifold.

Definition 4 *A Lie group of transformations G^1 (2.6), (2.7) with the generator X (2.8) is admitted by the system (S) if the manifold (S) is an invariant manifold with respect to the prolonged group $G_{(l)}$ of the Lie group G^1. The generator X of the admitted Lie group G^1 is also called admitted by the system (S).*

This definition and Theorem of the previous subsection provide the following infinitesimal criterion for admitted Lie groups of differential equations.

Theorem 3 *The system of differential equations (S) is invariant under the group with an infinitesimal generator X if and only if*

$$X_{(l)}S(x,u,p)\mid_{(S)}=0, \tag{2.17}$$

where $X_{(l)}$ is the prolonged infinitesimal generator of X, and the symbol $|_{(S)}$ means that the equations $X_{(l)}S(x,u,p)$ are considered on the manifold (S).

Equations (2.17) are called the determining equations and will be denoted by the symbol DE. They are considered as equations with respect to the unknown coefficients $\xi^i(x,u)$, $\eta^j(x,u)$.

The first important property of the system DE is that all equations are linear and homogeneous relative to the coefficients $\xi^i(x,u)$, $\eta^j(x,u)$. Since the coefficients of the generator X do not depend on the derivatives p^k_α, the determining equations DE can be split with respect to the parametric derivatives.

The second important property of the determining equations is that the vector space of its solutions is closed under the commutator. Hence, all admitted generators of given differential equations form a Lie algebra. The algebraic structure of the admitted Lie group is used to obtain invariant solutions which is described latter. The main feature of invariant solutions is that they reduce the number of the independent variables.

2.3 Symmetry groups of integro-differential equations

As we mentioned in the Chapter I, the main difficulties when applying group analysis to integro-differential equations arise from the nonlocal terms; a direct transference of the known scheme of the group analysis method to integro-differential equations is impossible. It is thus necessary to develop a method for constructing determining equations defining a Lie group admitted by the studied integro-differential equations.

A newly developed definition of admitted Lie group for equations with nonlocal operators was introduced by (Grigoriev and Meleshko, 1987) (also see (Grigoriev et al., 2010; Meleshko, 2005)), where a Lie group admitted by integro-differential equations was defined as a Lie group satisfying determining equations. The determining equations are obtained by the action of a canonical Lie-Bäcklund operator.

2.3.1 Lie-Bäcklund operators

Definition 5 *A Lie-Bäcklund operator is defined by the formal sum*

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^k \frac{\partial}{\partial u^k} + \zeta^k_i \frac{\partial}{\partial u^k_i} + \zeta^k_{i_1 i_2} \frac{\partial}{\partial u^k_{i_1 i_2}} + \dots, \tag{2.18}$$

where ξ^i and η^k depend on the independent and dependent variables and finite number of derivatives. Other coefficients are determined by the prolongation for-

mulae:

$$\zeta_i^k = D_i\Big(\eta^k - \xi^j u_j^k\Big) + \xi^j u_{ij}^k,$$
$$\zeta_{i_1 i_2}^k = D_{i_1} D_{i_2}\Big(\eta^k - \xi^j u_j^k\Big) + \xi^j u_{j i_1 i_2}^k, \dots \tag{2.19}$$

The Lie-Bäcklund operator (2.18) is often written in the abbreviated form

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^k \frac{\partial}{\partial u^k} + \cdots . \tag{2.20}$$

The set of all Lie-Bäcklund operators is an infinite dimensional Lie algebra and it is denoted by $L_{\mathcal{B}}$, This Lie algebra is endowed with the following properties:

(1) $D_i \in L_{\mathcal{B}}$. In other words, the total differentiation is a Lie-Bäcklund operators. Furthermore,

$$X_* = \xi_*^i D_i \in L_{\mathcal{B}} \tag{2.21}$$

for any $\xi_*^i \in \mathcal{A}$.

(2) Let L_* be the set of all Lie-Bäcklund operators of the form (2.21), then L_* is an ideal of $L_{\mathcal{B}}$.

(3) Two operations X_1, $X_2 \in L_{\mathcal{B}}$ are said to be equivalent (i.e., $X_1 \sim X_2$) if $X_1 - X_2 \in L_*$.

Definition 6 *Lie-Bäcklund operator with* $\xi^i = 0 \quad (i = 1, 2, \dots, n)$ *is called a canonical Lie-Bäcklund operator.*

Remark. Any Lie-Bäcklund operator X is equivalent to a canonical Lie-Bäcklund operator. Namely, $X \sim \bar{X}$, where

$$\bar{X} = X - \xi^i D_i = \bar{\eta}^k \frac{\partial}{\partial u^k} + ..., \qquad \bar{\eta}^k = \eta^k - \xi^j u_j^k.$$

2.3.2 Definition of a Lie group admitted by integro-differential equations

Let us consider an abstract system of integro-differential equations:

$$\Phi(x, u) = 0. \tag{2.22}$$

Here as above, u is the vector of the dependent variables and x is the vector of the independent variables. Let $G^1(X)$ be a one-parameter Lie group of transformations

$$\bar{x} = f^x(x, u; a), \qquad \bar{u} = f^u(x, u; a) \tag{2.23}$$

with the generator

$$X = \xi^i(x, u)\partial_{x_i} + \eta^j(x, u)\partial_{u^j},$$

transforms a solution $u_0(x)$ of equations (2.22) into the solution $u_a(x)$ of the same equations. The transformed function $u_a(x)$ is

$$u_a(\bar{x}) = f^u(x, u(x); a),$$

where $x = \psi^x(\bar{x}; a)$ is substituted into this expression. The function $\psi^x(\bar{x}; a)$ is found from the relation $\bar{x} = f^x(x, u(x); a)$. Differentiating the equations $\Phi(x, u_a(x))$ with respect to the group parameter a and evaluating the result for the value $a = 0$, we obtain the equations

$$\left(\frac{\partial}{\partial a}\Phi(x, u_a(x))\right)_{|a=0} = 0.$$

These equations coincide with the equations

$$(\bar{X}\Phi)(x, u_0(x)) = 0 \tag{2.24}$$

obtained by the action of the canonical Lie-Bäcklund operator $\bar{X}$, which is equivalent to the generator X:

$$\bar{X} = \bar{\eta}^j \partial_{u^j} + \cdots ,$$

with $\bar{\eta}^j = \eta^j(x,u) - \xi^i(x,u)u_i^j$. Equation (2.24) can be constructed without requiring the property that the Lie group should transform a solution into a solution. This allows the following definition of an admitted Lie group.

Definition 7 *A one-parameter Lie group $G^1(X)$ of transformations (2.23) is a Lie group admitted by (2.22), or a symmetry group of (2.22), if $G^1(X)$ satisfies the equations (2.24) for any solution $u_0(x)$ of (2.22). Equations (2.24) are called the determining equations.*

Remark. As a rule, equations or systems considered along nonlocal operators also include operators or equations with partial derivatives. Hence, the definition of an admitted Lie group for equations with nonlocal terms has to be consistent with the definition of an admitted Lie group of partial differential equations. It is worth to note that for a system of differential equations (without integral terms) the determining equations (2.24) coincide with the determining equations (2.17).

The way of obtaining determining equations of integro-differential equations is similar (and not more difficult) to the way used for differential equations. Notice that the determining equations of integro-differential equations are integro-differential equations.

The main difficulty in finding an admitted Lie group of integro-differential equations consists of solving the determining equations. For solving the determining equations we need to study the problem of existence of a solution of a Cauchy problem. Although in applications of the group analysis method to partial differential equations there is no special attention to the problem of existence of solutions of a Cauchy problem, this problem plays one of the key roles in solving determining equations: it allows splitting them. In partial differential equations there is no special study of the Cauchy problem because either a studied system of equations is of Cauchy-Kovalevskaya type, or a geometrical approach is used for finding an admitted Lie group.

2.3.3 Group classification

Most systems of equations include arbitrary elements: constants and functions of the independent and dependent variables. These elements specify a process. The group classification problem consists of finding all principal Lie groups admitted by a system of equations. Part of these groups is admitted for all arbitrary elements. This part is called a kernel of admitted Lie groups. Another part depends on the specification of arbitrary elements. This part contains extensions of the kernel.

2.4 Lie algebra of generators

Consider two generators

$$X_1 = \xi_1^i(z)\partial_{z_i}, \quad X_2 = \xi_2^i(z)\partial_{z_i}. \tag{2.25}$$

Definition 8 *The commutator* $[X_1, X_2]$ *of generators (2.25) is a generator defined by the formula*

$$[X_1, X_2] = X_1X_2 - X_2X_1,$$

or equivalently

$$[X_1, X_2] = [X_1(\xi_2^i(z)) - X_2(\xi_1^i(z))]\partial_{z_i}. \tag{2.26}$$

It follows from (2.26) that the commutator satisfies the following algebraic properties:

(1) The commutator is bilinear. This means that for any generators X_1, X_2, X_3 and any numbers a, b, the following identities hold:

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3],$$

$$[X_1, aX_2 + bX_3] = a[X_1, X_2] + b[X_1, X_3],$$

(2) The commutator is skew-symmetric. This means that for any generators X_1, X_2, the identity

$$[X_1, X_2] = -[X_2, X_1],$$

is satisfied.

(3) The commutator satisfies the Jacobi identity. that is to say that for any generators X_1, X_2, X_3 the relation

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$$

is satisfied.

Definition 9 *A vector space L of generators is called a Lie algebra if it is closed under the commutator, i.e., $[X_1, X_2] \in L$ for any $X_1, X_2 \in L$.*

The dimension of the Lie algebra L is the dimension of the vector space L. We use the symbol L_r to denote an $r-$dimensional Lie algebra.

Theorem 4 *If a system S admits generators X_1 and X_2, then the system S admits their commutator $[X_1, X_2]$.*

This theorem means that the vector space of all admitted generators is a Lie algebra (admitted by the system S). This algebra is called a principal algebra. To construct exact solutions we uses subalgebras of the admitted algebra.

Definition 10 *A vector subspace N of a Lie algebra L is called a subalgebra of L if*

$$[X, Y] \in N, \quad for\ any\ X, Y \in N.$$

Definition 11 *A vector subspace I of a Lie algebra L is called a ideal of L if*

$$[X, Y] \in I, \quad for\ any\ X \in L,\ Y \in I.$$

Definition 12 *Two Lie algebra of generators L and L' are similar if there exists a change of variable that transforms one into the other.*

Hence, if Lie algebras L and L' are similar, then the generators $X = \zeta^\beta(z)\partial_{z_\beta} \in L$ and $\hat{X} = \hat{\zeta}^\beta(\hat{z})\partial_{\hat{z}_\beta} \in L'$ of these algebras are related by the formula

$$\hat{\zeta}^\beta(\hat{z}) = X(q^\beta(z))\Big|_{z=q^{-1}(\hat{z})}.$$

A linear one-to-one mapping F of a Lie algebra L onto a Lie algebra L' is called an isomorphism (algebra L and L' are said to be isomorphic) if for any $X, Y \in L$, the equality

$$F([X, Y]) = [F(X), F(Y)]'$$

holds, where the symbol $[\ ,\ ]$ and $[\ ,\ ]'$ are used to denote the commutator in L and L', respectively. An isomorphism of Lie algebra L onto itself is called an automorphism of L.

The set of all subalgebras can be classified with respect to automorphisms. These constructions are studied in the next section.

2.5 Classification of subalgebras

Consider an $r-$dimensional Lie algebra L_r. Let X_1, X_2,...,X_r be a basis of the vector space L_r. Any $X \in L$ is their linear combination. In particular,

$[X_i, X_j] \in L$, hence

$$[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma, \quad \alpha, \beta = 1, 2, \ldots, r. \tag{2.27}$$

The constant coefficients $c^\gamma_{\alpha\beta}$ are called the structure constants of the Lie algebra L_r for the basis X_α.

In applications, it is convenient to use relations (2.27) written in the form of a table of commutators whose entry at the intersection of the X_α row with the X_β column is $[X_\alpha, X_\beta]$.

If $X = x^\alpha X_\alpha$ and $Y = y^\beta X_\beta$ belong to L_r, then

$$[X, Y] = x^\alpha y^\beta [X_\alpha, X_\beta] = x^\alpha y^\beta c^\gamma_{\alpha\beta} X_\gamma.$$

Therefore, for the coordinates $x = (x^1, x^2, \ldots, x^r)$ and $y = (y^1, y^2, \ldots, y^r)$ of the generators X and Y, we can define an operation of commutation $[x, y]$:

$$[x, y]^\gamma = x^\alpha y^\beta c^\gamma_{\alpha\beta}, \quad (\gamma = 1, 2, \ldots, r).$$

With this operation the vector space $R^r(x)$ becomes a Lie algebra. This Lie algebra is also denoted by L_r.

An automorphism $A_y(t)$ of R^r is defined by

$$\hat{x} = A_y(t)x,$$

where $\hat{x} = (\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^r)$ is the solution of the equations

$$\frac{d}{dt}\hat{x}^\gamma = \hat{x}^\alpha y^\beta c^\gamma_{\alpha\beta}, \quad \hat{x}^\gamma_{|t=0} = x^\gamma.$$

The set of all automorphisms $A_y(t)$ is called a set of inner automorphisms of the Lie algebra L_r. This set is denoted by $\mathrm{Int}(L_r)$. Any subalgebra $L_s \subset L_r$ is transformed by $A_y(t)$ into a similar subalgebra. Similar subalgebras of the same dimension composes a class and select a representative of each class.

Definition 13 *A set of the representatives of all these classes is called an optimal system of subalgebras.*

Thus, an optimal system of subalgebras of a Lie algebra L with inner automorphisms $A = \mathrm{Int}(L)$ is a set of subalgebras $\Theta_A(L)$ such that

(1) No two elements of this set can be transformed into each other by inner automorphisms of the Lie algebra L;

(2) Any subalgebra of the Lie algebra L can be transformed into one of the subalgebras of the set $\Theta_A(L)$.

In group analysis the problem of finding all inner automorphisms is reduced to the problem for finding inner automorphisms A_k for the canonical basis vectors $y = e^k$, $(k = 1, 2, \ldots, r)$ in R^r:

$$\frac{d}{dt}\hat{x}^\gamma = \hat{x}^\alpha c^\gamma_{\alpha k}, \quad \hat{x}^\gamma_{|t=0} = x^\gamma, \quad (\gamma = 1, 2, \ldots, r).$$

The inner automorphisms A_k corresponds to the Lie group of transformations with the generator

$$x^\alpha c^\gamma_{\alpha k}\partial_{x^\gamma}.$$

2.6 Invariant solutions

Definition 14 *Let a system (S) of differential equations admits a group G, and let H be a subgroup of G. A solution $u = U(x)$ of system (S) is called an $H-$invariant solution if the manifold $u = U(x)$ is an invariant manifold with respect to any transformation of the group H.*

Let a system (S) of differential equations admit a group G, and let H_r be its $r-$parameter subgroup generated by

$$X_k = \xi_k^i(x,u)\partial_{x_i} + \eta_k^j(x,u)\partial_{u^j}, \qquad k = 1, 2, \ldots, r. \tag{2.28}$$

Let $r_* = rank\|\xi_k^i, \eta_k^j\|$. Hence, H_r has $m+n-r_*$ functionally independent invariants

$$J_1(x,u), \ldots, J_{m+n-r_*}(x,u).$$

Suppose that the Jacobian of J with respect to u is of rank m, without loss of generality we can choose the first m invariants such that

$$rank\left\|\frac{\partial J_\alpha(x,u)}{\partial u^j}\right\| = m, \tag{2.29}$$

where $\alpha, j = 1, \ldots, m$. Hence, setting

$$\mu^\alpha = J_\beta(x,u), \quad \lambda^\beta = J_{m+\nu}(x,u), \qquad \alpha = 1, \ldots, m; \quad \nu = 1, \ldots, n-r_*, \tag{2.30}$$

we can write the invariant solution of system (S) in the form

$$\mu^\alpha = \Phi^\alpha(\lambda^1, \ldots, \lambda^{n-r_*}), \qquad \alpha = 1, \ldots, m. \tag{2.31}$$

The functions Φ^α are determined by a system of differential equations, denoted by S/H_r. The system S/H_r is called a reduced system, the number $n-r_*$ is the number of independent variables in factor system S/H_r and it is called a rank of the invariant solution.

Remark. In most applications, we can choose invariants (2.30) such that the invariants λ^β $(\nu = 1, \ldots, n-r_*)$ don't depend on the dependent variables u^j. This facilitates the calculation of invariant solutions. The condition (2.29) guarantees that equations (2.31) can be solved for u^j. Substituting them into original system (S), we can obtain reduced system S/H_r.

The algorithm of seeking invariant solutions of differential equations consists of the following four steps.

1st step. Find the admitted Lie group of the differential equations by solving the determining equation. All admitted Lie groups form a Lie algebra.

2nd step. Choose a subalgebra and find invariants of the subalgebra.

3rd step. Construct a representation of an invariant.

4th step. Substituting the representation into the original system of differential equations, we obtain equations with reduced number of independent variables.

CHAPTER III

MATHEMATICAL MODELS

In this chapter, we study one-dimensional viscoelastic materials with memory, considering two different models: a nonlinear stress relaxation model and a linear thermoviscoelastic model.

3.1 Nonlinear stress relaxation model

3.1.1 Fundamental equations

Consider longitudinal motion of a homogeneous one-dimensional body, e.g. a bar of uniform cross-section. If the body is incompressible, then we can assume that $\rho = 1$, where ρ is the mass density. We denote by $u(t, x)$ the displacement at time t of the particle with reference position x (x is Lagrangian coordinate, and $x + u(t, x)$ is the position at time t of this particle). The strain $e(t, x)$ and velocity $v(t, x)$ given by

$$e = \frac{\partial u}{\partial x}, \quad v = \frac{\partial u}{\partial t},$$

satisfy the kinematic compatibility relation

$$e_t = v_x. \tag{3.1}$$

The equation of balance of linear momentum, in the absence of body force, has the form

$$\sigma_x = v_t, \tag{3.2}$$

where $\sigma(t, x)$ is the stress.

The first model considered in this chapter which describes the stress relaxation behavior of one-dimensional viscoelastic materials (allowing aging) is given by the system consisting of two partial differential equations and an integral equation:

$$v_t = \sigma_x, \quad e_t = v_x, \quad \varphi(\sigma) = e + \int_0^t H(t,\tau)e(\tau)\, d\tau,\ \varphi'(\sigma) \neq 0. \qquad (3.3)$$

Here time t and reference position x are independent variables, while the stress σ, the velocity v, and the strain e are dependent variables, $H(t,\tau)$ is the kernel of relaxation and does not depend on x, $\varphi(\sigma)$ is a sufficiently smooth function of the stress. If $\varphi(\sigma)$ is a linear function, then system (3.3) describes the linear behavior of a viscoelastic material.

3.1.2 Discussions on the integral kernel

The relaxation modulus is one of the main characteristics of a material: each material has its own value of relaxation modulus, so choosing different kernels $H(t,\tau)$, system (3.3) describes the behavior of different materials. For example, if the kernel is identically zero, i.e., $H(t,\tau) = 0$, then system (3.3) becomes the nonlinear wave equation

$$u_{tt} = \phi'(u_x)u_{xx},$$

which is called a nonlinear viscoelastic one-dimensional Kelvin model (Chirkunov, 1984). Here the function ϕ is the inverse function of φ.

Reduction of system (3.3) to a system of differential equations is also possible for some more complicated kernels*. For example, if the kernel has the form $H(t,\tau) = k(t)p(\tau)$, then the integral equation of system (3.3) becomes the differ-

*Examples of integro-differential equations modeling materials with memory which can be reduced to systems of partial differential equations and their exact solutions can be found in (Pucci and Saccomandi, 2015).

ential equation

$$e_t(t,x) = \varphi'(\sigma)\sigma_t(t,x) + [h(t) - k(t)p(t)]e(t,x) - h(t)\varphi(\sigma),$$

where $h(t) = k'(t)/k(t)$. In particular, if the kernel is an exponential function, $H(t,\tau) = Ke^{-R(t-\tau)/\eta}$, where K, R and η are constant with $K = -R^2/\eta$, then system (3.3) is reduced to the following system of partial differential equations

$$v_t = \sigma_x,\ e_t = v_x,\ e_t = \varphi'(\sigma)\sigma_t - \frac{R}{\eta}(1-R)e - \frac{R}{\eta}\varphi(\sigma). \tag{3.4}$$

System (3.4) describes stress relaxation response of the Maxwell model. More details about the Maxwell model and some other models can be found in (Nowick and Berry, 1972; James and Kasif, 1989).

If the kernel has the following form

$$H(t,\tau) = \mathcal{H}(t-\tau) - \mathcal{H}(t-\tau-1), \quad (t > 1),$$

where $\mathcal{H}$ is the unit Heaviside step function, then system (3.3) becomes a system of delay partial differential equations

$$v_t = \sigma_x, \quad e_t = v_x, \quad \varphi'(\sigma)\sigma_t = e_t + e(t) - e(t-1).$$

Research of delay partial differential equation can be found in (Ibragimov et al., 2011).

If system (3.3) of integro-differential equations can be reduced to a system of differential equations, then the kernel $H(t,\tau)$ is called a degenerated kernel. In the thesis, we apply group analysis to the integro-differential equations (3.3), excluding degenerated kernels from the study. In particular, it is assumed that $H \neq 0$ and $H(t,\tau) \neq k(t)p(\tau)$.

3.2 Linear thermoviscoelastic model

In this section we present the constitutive equations defining linear thermoviscoelastic material, as well as two special models studied in the thesis.

3.2.1 Three-dimensional case

Consider a body occupying a fixed bounded domain Ω in a suitable reference configuration, and assume that the reference configuration is a natural state, namely a state with zero stress and constant absolute base temperature Θ_0. Let $\mathbf{X}$ be the position vector of a material point, $\mathbf{u}(\mathbf{X}, t)$ be the displacement field and $\Theta(\mathbf{X}, t) - \Theta_0 = \theta(\mathbf{X}, t)$ be the temperature difference from Θ_0. If the material is homogeneous, then under hypotheses of small deformations and small variations of temperature with respect to the given natural state, we assume that $\rho = 1$, where ρ is the mass density.

We consider a specific Helmoltz free energy functional depending upon both displacement and temperature difference histories in quadratic manner ((Navarro, 1978))

$$\begin{aligned}
\psi(t) \quad = \quad & \frac{1}{2}\nabla\mathbf{u}(t)\cdot\mathbf{G}_\infty\nabla\mathbf{u}(t) - \theta(t)\mathbf{L}_\infty\cdot\nabla\mathbf{u}(t) - \frac{1}{2}\frac{c_\infty}{\Theta_0}\theta^2(t) \\
& -\frac{1}{2}\int_0^\infty[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(t-s)]\cdot\dot{\mathbf{G}}(s)[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(t-s)]ds \\
& +\int_0^\infty[\theta(t) - \theta(t-s)]\dot{\mathbf{L}}(s)\cdot[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(t-s)]ds \\
& +\frac{1}{2\Theta_0}\int_0^\infty\dot{c}(s)[\theta(t) - \theta(t-s)]^2ds,
\end{aligned} \tag{3.5}$$

where the dependence on $\mathbf{X}$ is omitted for convenience. The material properties $\mathbf{G}(s)$, $\mathbf{L}(s)$, and $c(s)$, $s \geq 0$, are the relaxation tensor fields of fourth, second, and zero order, respectively. In addition, we assume that they do not depend on the position vector $\mathbf{X}$. The limits

$$\mathbf{G}_\infty = \lim_{s\to\infty} \mathbf{G}(s), \qquad \mathbf{L}_\infty = \lim_{s\to\infty} \mathbf{L}(s), \qquad c_\infty = \lim_{s\to\infty} c(s),$$

are called the equilibrium elasticity modulus, the equilibrium stress-temperature tensor, and the equilibrium specific heat. The tensors $\mathbf{G}$ and $\mathbf{L}$ are assumed to be symmetric

$$\mathbf{G}(s) = \mathbf{G}^T(s), \qquad \mathbf{L}(s) = \mathbf{L}^T(s), \ \ for\ all\ s \geq 0,$$

where the symmetry of the tensor $\mathbf{G}$ means that

$$G_{ijkl}(s) = G_{klij}(s), \quad for\ any\ i, j, k, l.$$

The Clausius-Duhem inequality (second law of thermodynamics) assures that the stress tensor $\mathbf{T}$ and the entropy η are determined by the free energy (more details can be found in (Coleman, 1964b; Gurtin, 1990)). In assumptions of small deformations these relations are

$$\mathbf{T} = \frac{\partial\psi}{\partial\mathbf{E}}, \quad \eta = -\frac{\partial\psi}{\partial\theta},$$

where $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is Green's strain tensor. Hence, the constitutive equations for the stress tensor $\mathbf{T}$ and the specific entropy η are

$$\begin{aligned} \mathbf{T}(t) &= \mathbf{G}_\infty\nabla\mathbf{u}(t) - \int_0^\infty \dot{\mathbf{G}}(s)[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(t-s)]ds \\ &\quad -\mathbf{L}_\infty\theta(t) + \int_0^\infty \dot{\mathbf{L}}(s)[\theta(t) - \theta(t-s)]ds, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \Theta_0\eta(t) &= \Theta_0\left[\mathbf{L}_\infty\cdot\nabla\mathbf{u}(t) - \int_0^\infty \dot{\mathbf{L}}(s)\cdot[\nabla\mathbf{u}(t) - \nabla\mathbf{u}(t-s)]ds\right] \\ &\quad +c_\infty\theta(t) - \int_0^\infty \dot{c}(s)[\theta(t) - \theta(t-s)]ds. \end{aligned} \tag{3.7}$$

There are the following alternative forms for the stress tensor $\mathbf{T}$ and the specific entropy η

$$\mathbf{T}(t) = \mathbf{G}_0 \nabla \mathbf{u}(t) + \int_0^t \dot{\mathbf{G}}(t-s) \nabla \mathbf{u}(s) ds - \mathbf{L}_0 \theta(t) - \int_0^t \dot{\mathbf{L}}(t-s) \theta(s) ds, \qquad (3.8)$$

$$\Theta_0 \eta(t) = \Theta_0 \left[\mathbf{L}_0 \cdot \nabla \mathbf{u}(t) + \int_0^t \dot{\mathbf{L}}(t-s) \cdot \nabla \mathbf{u}(s) ds \right] + c_0 \theta(t) + \int_0^t \dot{c}(t-s) \theta(s) ds, \qquad (3.9)$$

where $\mathbf{G}_0$, $\mathbf{L}_0$, c_0 are the values of the functions $\mathbf{G}(s)$, $\mathbf{L}(s)$, $c(s)$ at the point $s = 0$, respectively.

We complete the set of constitutive equations with the Fourier heat conduction law for the heat flux vector:

$$\mathbf{q}(t) = -\mathbf{K} \nabla \theta(t), \qquad (3.10)$$

where $\mathbf{K}$ is a constant tensor called a thermal conductivity tensor. The Clausius-Duhem inequality implies that the thermal conductivity tensor $\mathbf{K}$ is positive semi-definite.

Using assumptions of small deformations and small variations of the temperature with respect to the natural state, we only retain the first-order in $\nabla \mathbf{u}$ and θ. Hence we assume that $h = \Theta_0 \dot{\eta}$, where h is the rate at which heat is absorbed per unit of volume, and it is defined by

$$\rho h = \rho \dot{e} - \mathbf{T} : \mathbf{L},$$

where e is the internal energy pre unit mass, $\mathbf{L}$ is the velocity gradient. Therefore, the equations of balances of linear momentum and energy, in the absence of body forces and radiant heating, become

$$\ddot{\mathbf{u}} = div_x \mathbf{T}, \qquad (3.11)$$

$$\Theta_0 \dot{\eta}(t) + div_x \mathbf{q} = 0. \qquad (3.12)$$

3.2.2 One-dimensional case

Now we restrict our attention to a linear thermoviscoelastic bar with memory, occupying the reference configuration $\Omega = [0, \tau]$. In the one-dimensional case equations (3.8)-(3.10) are reduced to the following constitutive relations

$$\begin{cases} \sigma(t) = G_0 e(t) + \displaystyle\int_0^t \dot{G}(t-s)e(s)ds - L_0\theta(t) - \int_0^t \dot{L}(t-s)\theta(s)ds, \\ \Theta_0\eta(t) = \Theta_0\left[L_0 e(t) + \displaystyle\int_0^t \dot{L}(t-s)e(s)ds\right] + c_0\theta(t) + \int_0^t \dot{c}(t-s)\theta(s)ds, \\ q(t) = -\kappa\theta_x(t), \end{cases} \tag{3.13}$$

where the material coordinate x is omitted, σ is the axial stress, $e = u_x$ the axial deformation, the constitutive functions $G(s)$, $L(s)$, $c(s)$ do not depend on x, the constants G_0, L_0, c_0 are values of the functions $G(s)$, $L(s)$, $c(s)$ at the point $s = 0$, which are assumed to be positive, the constant κ is also assumed to be positive.

Now we match (3.13) with motion, energy balance equations involving zero sources (i.e. $b = r = 0$) and the kinematic compatibility relation, namely

$$\sigma_x = v_t, \qquad \Theta_0\dot{\eta} + q_x = 0, \qquad e_t = v_x,$$

where v is the velocity defined by $v = u_t$. As a consequence, we obtain the following system of integro-differential equations

$$\begin{gathered} \sigma_x = v_t, \qquad e_t = v_x, \qquad w_t = \kappa\theta_{xx}, \\ \sigma = G_0 e + \int_0^t \dot{G}(t-s)e(s)ds - L_0\theta - \int_0^t \dot{L}(t-s)\theta(s)ds, \\ w = \Theta_0\left[L_0 e + \int_0^t \dot{L}(t-s)e(s)ds\right] + c_0\theta + \int_0^t \dot{c}(t-s)\theta(s)ds, \end{gathered} \tag{3.14}$$

where the new variable w is defined by $w = \Theta_0\eta$.

3.2.3 Dimensional analysis

Equations (3.14) contain a large number of variables and parameters, thus it is difficult to study equations (3.14) directly. Dimensional analysis is a method for reducing the number and complexity of variables and parameters. This method is widely used in engineering and mechanics in the literature, for example (Bridgman, 1922; Sedov, 1993). In this subsection, model (3.14) is simplified by applying dimensional analysis to it. For this end, it is rewritten as

$$\begin{aligned}
&\bar{\sigma}_{\bar{x}} = \bar{v}_{\bar{t}}, \quad \bar{e}_{\bar{t}} = \bar{v}_{\bar{x}}, \quad \bar{w}_{\bar{t}} = \kappa\bar{\theta}_{\bar{x}\bar{x}}, \\
&\bar{\sigma} = G_0\bar{e} + \int_0^{\bar{t}} \dot{G}(\bar{t}-\bar{s})\bar{e}(\bar{s})d\bar{s} - L_0\bar{\theta} - \int_0^{\bar{t}} \dot{L}(\bar{t}-\bar{s})\bar{\theta}(\bar{s})d\bar{s}, \\
&\bar{w} = \Theta_0\left[L_0\bar{e} + \int_0^{\bar{t}} \dot{L}(\bar{t}-\bar{s})\bar{e}(\bar{s})d\bar{s}\right] + c_0\bar{\theta} + \int_0^{\bar{t}} \dot{c}(\bar{t}-\bar{s})\bar{\theta}(\bar{s})d\bar{s}.
\end{aligned} \tag{3.15}$$

Let us consider the following transformation

$$\bar{t} = t_0 t, \quad \bar{x} = x_0 x, \quad \bar{v} = v_0 v, \quad \bar{\sigma} = \sigma_0\sigma, \quad \bar{e} = e_0 e, \quad \bar{\theta} = \theta_0\theta, \quad \bar{w} = w_0 w, \tag{3.16}$$

where t_0, x_0, v_0, σ_0, e_0, θ_0 and w_0 are positive dimensional numbers. Applying transformation (3.16) to equations (3.15), they are transformed to

$$\begin{aligned}
&\frac{\sigma_0}{x_0}\sigma_x = \frac{v_0}{t_0}v_t, \quad \frac{e_0}{t_0}e_t = \frac{v_0}{x_0}v_x, \quad \frac{w_0}{t_0}w_t = \frac{\kappa\theta_0}{x_0^2}\theta_{xx}, \\
&\sigma = \frac{G_0e_0}{\sigma_0}e + \frac{t_0e_0}{\sigma_0}\int_0^t \dot{G}(t_0(t-s))e(s)ds - \frac{L_0\theta_0}{\sigma_0}\theta - \frac{t_0\theta_0}{\sigma_0}\int_0^t \dot{L}(t_0(t-s))\theta(s)ds, \\
&w = \frac{\Theta_0L_0e_0}{w_0}e + \frac{\Theta_0t_0e_0}{w_0}\int_0^t \dot{L}(t_0(t-s))e(s)ds + \frac{c_0\theta_0}{w_0}\theta + \frac{t_0\theta_0}{w_0}\int_0^t \dot{c}(t_0(t-s))\theta(s)ds.
\end{aligned} \tag{3.17}$$

Let

$$t_0 = e_0 = \frac{\kappa}{\Theta_0L_0^2}, \quad v_0^2 = x_0^2 = \frac{\kappa^2}{\Theta_0c_0L_0^2}, \quad \sigma_0 = \frac{\kappa}{c_0}, \quad \theta_0 = \frac{\kappa}{c_0L_0}, \quad w_0 = \frac{\kappa}{L_0},$$

we obtain the equations

$$\sigma_x = v_t, \qquad e_t = v_x, \qquad w_t = \theta_{xx},$$

$$\sigma = Ee + \int_0^t \tilde{G}(t-s)e(s)ds - \theta - \int_0^t \tilde{L}(t-s)\theta(s)ds, \tag{3.18}$$

$$w = e + \int_0^t \tilde{L}(t-s)e(s)ds + \theta + \int_0^t \tilde{c}(t-s)\theta(s)ds,$$

where the number E is defined by

$$E = \frac{G_0 c_0}{\Theta_0 L_0^2},$$

and the functions $\tilde{G}(t-s)$, $\tilde{L}(t-s)$ and $\tilde{c}(t-s)$ are

$$\tilde{G}(t-s) = \frac{\kappa c_0}{\Theta_0^2 L_0^4}\dot{G}(t_0(t-s)),$$

$$\tilde{L}(t-s) = \frac{\kappa}{\Theta_0 L_0^3}\dot{L}(t_0(t-s)), \qquad \tilde{c}(t-s) = \frac{\kappa}{\Theta_0 c_0 L_0^2}\dot{c}(t_0(t-s)).$$

3.2.4 Basic equations and two special cases

The second model which describes thermoviscoelastic bar with memory (allowing for aging) is established by the following system

$$\sigma_x = v_t, \qquad e_t = v_x, \qquad \theta_{xx} = w_t,$$

$$\sigma = Ee + \int_0^t G(t,s)e(s)ds - \theta - \int_0^t L(t,s)\theta(s)ds, \tag{3.19}$$

$$w = e + \int_0^t L(t,s)e(s)ds + \theta + \int_0^t c(t,s)\theta(s)ds.$$

Here time t and distance x are the independent variables, the stress σ, the velocity v, the strain e, the temperature θ, and w are the dependent variables, $G(t,s)$, $L(t,s)$ and $c(t,s)$ are the relaxation functions.

Group analysis of system (3.19) of linear thermoviscoelasticity will be discussed in Chapter V. A way of splitting determining equations of integro-differential equations depends on the equations studied. For some special relations among the material functions $G(t,s)$, $L(t,s)$, $c(t,s)$, we need to develop different ways for splitting the determining equations. Two special cases of system (3.19) are of particular interest. The first case defines behavior of linear thermoviscoelastic material described by equations (3.19) without kernel $L(t,s)$. These equations have the form

$$\begin{gathered}\sigma_x = v_t, \qquad e_t = v_x, \qquad \theta_{xx} = w_t, \\ \sigma = Ee - \theta + \int_0^t G(t,s)e(s)ds, \qquad w = e + \theta + \int_0^t c(t,s)\theta(s)ds.\end{gathered} \tag{3.20}$$

To avoid that these equations are reduced to partial differential equations, we assume that $G(t,s) \neq 0$ and $c(t,s) \neq 0$.

The second case is obtained by letting $G(t,s) = h_1(t)L(t,s)$ and $c(t,s) = q_1(t)L(t,s)$ in system (3.19) for some functions $h_1(t)$, $q_1(t)$. In this case, equations (3.19) become

$$\begin{gathered}\sigma_x = v_t, \qquad e_t = v_x, \qquad \theta_{xx} = w_t, \\ \sigma = Ee + h_1 \int_0^t L(t,s)e(s)ds - \theta - \int_0^t L(t,s)\theta(s)ds, \\ w = e + \int_0^t L(t,s)e(s)ds + \theta + q_1 \int_0^t L(t,s)\theta(s)ds.\end{gathered} \tag{3.21}$$

For this model, we will exclude the kernel $L(t,s) = 0$ from the study, because this kernel is a degenerate one. There exist more complicated kernels $L(t,s)$ such that equations (3.21) are degenerated. For example, if $1 + h_1 q_1 \neq 0$ and the kernel $L(t,s)$ is a function of separable variables, i.e.,

$$L(t,s) = k(t)p(s),$$

with some function $k(t)$, $p(s)$, we can show that equations (3.21) can be reduced to partial differential equations.

CHAPTER IV

GROUP CLASSIFICATION OF NONLINEAR STRESS RELAXATION EQUATIONS

In this chapter we focus on the application of the group analysis method to the equations describing the stress relaxation behavior of one-dimensional viscoelastic materials

$$v_t = \sigma_x, \quad e_t = v_x, \quad \varphi(\sigma) = e + \int_0^t H(t,\tau)e(\tau)\, d\tau, \quad \varphi'(\sigma) \neq 0, \tag{4.1}$$

which are presented as equations (3.3) in Chapter III. Here time t and distance x are independent variables, the stress σ, the velocity v and the strain e are dependent variables, the relaxation function $H(t,\tau)$ does not depend on the variable x, $\varphi(\sigma)$ is a sufficiently smooth function of the stress. If $\varphi(\sigma)$ is a linear function, then system (4.1) describes linear behavior of a viscoelastic material.

As for differential equations, an admitted Lie group of integro-differential equations (4.1) is defined by determining equations. The general theory of constructing determining equations for integro-differential equations can be found in Chapter II.

In the first section of this chapter we obtain a classifying equation by solving the determining equations. In the section following it, the complete group classification of equations (4.1) with respect to the arbitrary function $\varphi(\sigma)$ and kernel $H(t,\tau)$ is studied. Invariant solutions of the system of equations (4.1) are discussed in the last section.

4.1 Admitted Lie groups

The determining equations of system (4.1) are integro-differential equations for the coordinates of the infinitesimal generator

$$X = \xi\partial_x + \eta\partial_t + \zeta^v\partial_v + \zeta^\sigma\partial_\sigma + \zeta^e\partial_e,$$

where the coefficients $\xi, \eta, \zeta^v, \zeta^\sigma$ and ζ^e are functions depending on the variables t, x, v, σ and e. In order to construct the determining equations, one has to use the canonical Lie-Bäcklund operator (Ibragimov, 1999) which is equivalent to the generator X:

$$\overline{X} = (\zeta^v - \xi v_x - \eta v_t)\partial_v + (\zeta^\sigma - \xi\sigma_x - \eta\sigma_t)\partial_\sigma + (\zeta^e - \xi e_x - \eta e_t)\partial_e + \cdots$$

Applying the prolonged canonical Lie-Bäcklund operator to system (4.1), the determining equations are

$$\left(\frac{\partial}{\partial t}\widehat{\zeta^v}(t,x) - \frac{\partial}{\partial x}\widehat{\zeta^\sigma}(t,x)\right)_{|(S)} = 0, \quad \left(\frac{\partial}{\partial t}\widehat{\zeta^e}(t,x) - \frac{\partial}{\partial x}\widehat{\zeta^v}(t,x)\right)_{|(S)} = 0, \tag{4.2}$$

$$\left(\varphi'\widehat{\zeta^\sigma}(t,x) - \widehat{\zeta^e}(t,x) - \int_0^t H(t,\tau)\widehat{\zeta^e}(\tau,x)\,d\tau\right)_{|(S)} = 0. \tag{4.3}$$

Here

$$\widehat{\zeta^v}(h_1) = \zeta^v(h_2) - \xi(h_2)v_x(h_1) - \eta(h_2)v_t(h_1),$$

$$\widehat{\zeta^\sigma}(h_1) = \zeta^\sigma(h_2) - \xi(h_2)\sigma_x(h_1) - \eta(h_2)\sigma_t(h_1),$$

$$\widehat{\zeta^e}(h_1) = \zeta^e(h_2) - \xi(h_2)e_x(h_1) - \eta(h_2)e_t(h_1),$$

where for the sake of simplicity of the presentation we have denoted

$$h_1 = (t,x), \qquad h_2 = (t,x,v(t,x),\sigma(t,x),e(t,x));$$

and the subscript $|(S)$ means that the expression is satisfied for any solution of system (4.1).

We assume that there exists a solution to the Cauchy problem consisting of equations (4.1) with the initial data*

$$e(t,x_0)=e_0(t),\quad v(t,x_0)=v_0(t),\quad \sigma(t,x_0)=\sigma_0(t),$$

where $e_0(t)$, $v_0(t)$, are arbitrary functions, and initial function $\sigma_0(t)$ satisfies the equation

$$\varphi(\sigma_0(t))=e_0(t)+\int_0^t H(t,\tau)e_0(\tau)\,d\tau. \tag{4.4}$$

The main derivatives of the functions $e(t,x)$, $v(t,x)$, $\sigma(t,x)$ at the point $x=x_0$ are found from equations (4.1)

$$\begin{gathered}\sigma_x(t,x_0)=v_t(t,x_0)=v_0'(t),\quad v_x(t,x_0)=e_t(t,x_0)=e_0'(t),\\ \sigma_t(t,x_0)=\frac{g(t)}{\varphi'(\sigma_0(t))},\quad e_x(t,x_0)=f(t),\end{gathered} \tag{4.5}$$

where

$$g(t)=e_0'(t)+H(t,t)e_0(t)+\int_0^t H_t(t,\tau)e_0(\tau)\,d\tau; \tag{4.6}$$

and the function $f(t)$ satisfies the second kind Volterra integral equation

$$\varphi'(\sigma_0(t))v_0'(t)=f(t)+\int_0^t H(t,\tau)f(\tau)\,d\tau. \tag{4.7}$$

Remark. Using a fixed-point theorem, we can prove that equation (4.7) has a unique solution.

*Mathematical studies of initial and boundary value problems modeling materials with memory can be found in (Renardy et al., 1987).

4.1.1 Determining equations (4.2)

Now we begin the study of the determining equations from equations (4.2) which are related with partial differential equations of system (4.1).

Substituting the main derivatives v_t, σ_t, e_t, v_x, σ_x, e_x (4.5) into the determining equations (4.2), considered at the point $x = x_0$ defined in (4.5), we obtain

$$\begin{aligned} & v_0'(\zeta_v^v - \eta_t - \zeta_\sigma^\sigma + \xi_x - \eta_e e_0') + (v_0')^2(-\eta_v + \xi_\sigma) + f(-\zeta_e^\sigma + \eta_e \frac{g}{\varphi'}) + v_0' f \xi_e \\ & + \zeta_\sigma^v \frac{g}{\varphi'} + \zeta_t^v - \zeta_x^\sigma + \eta_x \frac{g}{\varphi'} + e_0' \left(\zeta_e^v - \zeta_v^\sigma + \eta_v \frac{g}{\varphi'} - \xi_\sigma \frac{g}{\varphi'} - \xi_t - \xi_e e_0' \right) = 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} & v_0'(\zeta_v^e - \zeta_\sigma^v + \eta_x + \xi_\sigma e_0') + (v_0')^2 \eta_\sigma + f(-\xi_\sigma \frac{g}{\varphi'} - \zeta_e^v - \xi_t) + v_0' f(-\xi_v + \eta_e) \\ & + \zeta_\sigma^e \frac{g}{\varphi'} + \zeta_t^e - \zeta_x^v + e_0'(\zeta_e^e - \eta_\sigma \frac{g}{\varphi'} - \eta_t - \zeta_v^v + \xi_x + e_0'(-\eta_e + \xi_v)) = 0. \end{aligned} \tag{4.9}$$

For splitting the determining equations let us show that the variables v_0, v_0' and f are functionally independent. In fact, fix the function $e_0(t)$ (but arbitrary) and consider the following initial data

$$v_0(t) = a_1 + \int_{t_0}^t u_1(s)\, ds,$$

where

$$u_1(s) = \frac{1}{\varphi'(\sigma_0(s))}[u_2(s) + \int_0^s H(s,\tau) u_2(\tau)\, d\tau], \quad u_2(s) = a_2 + a_3(t_0 - s)^n.$$

Here t_0 is a fixed (but arbitrary) time, $n \geq 1$ is an integer, and a_1, a_2 and a_3 are arbitrary constants. We have that

$$v_0'(t) = u_1(t) = \frac{1}{\varphi'(\sigma_0(t))}[u_2(t) + \int_0^t H(t,\tau) u_2(\tau)\, d\tau]$$

or

$$\varphi'(\sigma_0(t)) v_0'(t) = u_2(t) + \int_0^t H(t,\tau) u_2(\tau)\, d\tau.$$

Because of the uniqueness of solution of equation (4.7), we find that

$$f(t) = a_2 + a_3(t_0 - t)^n.$$

Notice that at $t = t_0$, we have

$$\begin{aligned} v_0(t_0) &= a_1, \quad f(t_0) = a_2, \\ v_0'(t_0) &= \frac{1}{\varphi'(\sigma_0(t_0))}[a_2 + a_2\int_0^{t_0} H(t_0,\tau)\, d\tau + a_3\int_0^{t_0} H(t_0,\tau)(t_0-\tau)^n\, d\tau]. \end{aligned} \tag{4.10}$$

Since the set of functions $(t_0 - \tau)^n$ is total in the space $L_2[0, t_0]$, and t_0 is such that $H(t_0, \tau) \neq 0$, there exists n for which $\int_0^{t_0} H(t_0,\tau)(t_0-\tau)^n d\tau \neq 0$. Hence, for given values $v_0(t_0)$, $v_0'(t_0)$, $f(t_0)$ we can solve equations (4.10) with respect to the coefficients a_1, a_2, a_3. This means that the variables v_0, v_0' and f are functionally independent.

By virtue of the functional independence of the variables v_0, v_0' and f, we can split determining equations (4.8) and (4.9) with respect to them. The splitting gives

$$\begin{gathered} \xi_v = \xi_e = \xi_\sigma = 0, \quad \eta_v = \eta_e = \eta_\sigma = 0, \quad \zeta^v_e = -\xi_t, \\ \xi_x - \eta_t = \zeta^\sigma_\sigma - \zeta^v_v, \quad \zeta^\sigma_e = 0, \quad \zeta^e_v - \zeta^v_\sigma = -\eta_x, \end{gathered} \tag{4.11}$$

$$\begin{gathered} (\zeta^v_\sigma + \eta_x)g + (\zeta^v_t - \zeta^\sigma_x)\varphi' = (2\xi_t + \zeta^\sigma_v)\varphi' e_0', \\ \zeta^e_\sigma g + (\zeta^e_t - \zeta^v_x)\varphi' = (\eta_t + \zeta^v_v - \xi_x - \zeta^e_e)\varphi' e_0'. \end{gathered} \tag{4.12}$$

Further splitting can be performed with respect to the variables e_0, e_0', g and $\varphi'(\sigma_0)$. Let us show that they are also functionally independent. Consider the initial data for fixed time t_0

$$e_0(t) = a_1 + a_2(t - t_0) + (t - t_0)^2[a_3(t - t_0)^{n_1} + a_4(t - t_0)^{n_2}], \tag{4.13}$$

where $n_i \geq 1$ $(i = 1, 2)$ are integer numbers. Hence, the variables $e_0(t_0)$, $e_0'(t_0)$,

$g(t_0)$, $\varphi(\sigma_0(t_0))$ have the following expression

$$
\begin{aligned}
e_0(t_0) &= a_1, \quad e_0'(t_0) = a_2, \\
g(t_0) &= a_2 + a_1 H(t_0,t_0) + a_1 \int_0^{t_0} H_t(t_0,\tau)\,d\tau + a_2 \int_0^{t_0} H_t(t_0,\tau)(\tau - t_0)\,d\tau \\
&\quad + a_3 \int_0^{t_0} H_t(t_0,\tau)(\tau - t_0)^{n_1+2}\,d\tau + a_4 \int_0^{t_0} H_t(t_0,\tau)(\tau - t_0)^{n_2+2}\,d\tau, \\
\varphi(\sigma_0(t_0)) &= a_1 + a_1 \int_0^{t_0} H(t_0,\tau)\,d\tau + a_2 \int_0^{t_0} H(t_0,\tau)(\tau - t_0)\,d\tau \\
&\quad + a_3 \int_0^{t_0} H(t_0,\tau)(\tau - t_0)^{n_1+2}\,d\tau + a_4 \int_0^{t_0} H(t_0,\tau)(\tau - t_0)^{n_2+2}\,d\tau.
\end{aligned}
\tag{4.14}
$$

The Jacobian of $e_0(t_0)$, $e_0'(t_0)$, $g(t_0)$, $\varphi(\sigma_0(t_0))$ with respect to a_1, a_2, a_3 and a_4 is

$$
\begin{aligned}
\Delta &= \left(\int_0^{t_0} H(t_0,\tau)(\tau - t_0)^{n_2+2}\,d\tau\right)\left(\int_0^{t_0} H_t(t_0,\tau)(\tau - t_0)^{n_1+2}\,d\tau\right) \\
&\quad - \left(\int_0^{t_0} H(t_0,\tau)(\tau - t_0)^{n_1+2}\,d\tau\right)\left(\int_0^{t_0} H_t(t_0,\tau)(\tau - t_0)^{n_2+2}\,d\tau\right).
\end{aligned}
$$

Because of

$$
\frac{\partial}{\partial a_i}\varphi'(\sigma_0(t_0)) = \frac{\partial}{\partial a_i}\varphi'(\varphi^{-1}(\varphi(\sigma_0(t_0)))) = \frac{\varphi''(\sigma_0(t_0))}{\varphi'(\sigma_0(t_0))}\frac{\partial \varphi(\sigma_0(t_0))}{\partial a_i}, \quad i = 1,2,3,4,
$$

then the Jacobian of $e_0(t_0)$, $e_0'(t_0)$, $g(t_0)$, $\varphi'(\sigma_0(t_0))$ with respect to a_1, a_2, a_3 and a_4 is

$$
\Delta_1 = \frac{\varphi''(\sigma_0(t_0))}{\varphi'(\sigma_0(t_0))}\Delta.
$$

If $\Delta = 0$ for any integer numbers n_i $(i = 1,2)$, then by virtue of $H(t_0,\tau) \neq 0$ we obtain that there exists a function $h(t_0)$ such that

$$
H_t(t_0,\tau) = h(t_0)H(t_0,\tau). \tag{4.15}
$$

Solving equation (4.15) and since t_0 is arbitrary, the kernel has the form

$$
H(t,\tau) = k(t)p(\tau), \tag{4.16}
$$

with some functions $k(t)$ and $p(\tau)$. Since this is a degenerate kernel, there exist two integer numbers n_1, n_2 such that $\Delta \neq 0$. This means that the variables e_0, e_0', g and $\varphi(\sigma_0)$ are functionally independent.

If $\varphi'' \neq 0$, then $\Delta_1 \neq 0$. Equations (4.12) can be split with respect to e_0, e_0', g and $\varphi'(\sigma_0)$. Splitting, the overdetermined system of equations

$$\begin{gathered}\zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \quad 2\xi_t + \zeta^\sigma_v = 0, \quad \zeta^e_\sigma = 0, \\ \zeta^e_t - \zeta^v_x = 0, \quad \eta_t + \zeta^v_v - \xi_x - \zeta^e_e = 0.\end{gathered} \tag{4.17}$$

is obtained.

If $\varphi'' = 0$, then equations (4.12) can also be split. In this case we use arbitrariness of e_0, e_0' and g. Splitting also gives system (4.17).

Integrating the overdetermined system of equations (4.11), (4.17), we find

$$\begin{gathered}\xi = t(c_1x + c_2) + c_3x^2 + c_5x + c_6, \quad \eta = x(c_3t + c_4) + c_1t^2 + c_7t + c_8, \\ \zeta^v = -e(c_1x + c_2) - \sigma(c_3t + c_4) - v(2c_1t + 2c_3x + c_5 - c_9) + \lambda_{xt}, \\ \zeta^\sigma = -\sigma(3c_1t + c_3x + c_7 - c_9) - 2v(c_1x + c_2) + \lambda_{tt}, \\ \zeta^e = -e(c_1t + 3c_3x + 2c_5 - c_7 - c_9) - 2v(c_3t + c_4) + \lambda_{xx}.\end{gathered} \tag{4.18}$$

Here c_i, $(i = 1, 2, \ldots, 9)$ are arbitrary constants, and $\lambda(t, x)$ is an arbitrary function of two arguments.

4.1.2 Determining equation (4.3)

For the study of the determining equation (4.3), which is related with the integral equation, it is convenient to introduce

$$
\begin{gathered}
z_0 = \zeta^\sigma + 2v\xi_t = -\sigma(3c_1 t + c_3 x + c_7 - c_9) + \lambda_{tt}, \\
z_1 = \zeta^e + 2v\eta_x = -e(c_1 t + 3c_3 x + 2c_5 - c_7 - c_9) + \lambda_{xx}.
\end{gathered}
\tag{4.19}
$$

By virtue of (4.5) and (4.19) we have at point $x = x_0$ that

$$
\widehat{\zeta^\sigma} = z_0 - 2v_0\xi_t - \xi v_0' - \eta\frac{g}{\varphi'}, \qquad \widehat{\zeta^e} = z_1 - 2v_0\eta_x - \xi f - \eta e_0'.
$$

Determining equation (4.3) becomes

$$
\begin{gathered}
\varphi' z_0 - z_1 - \int_0^t H(t,\tau) z_1(\tau)\, d\tau \\
-2v_0\xi_t\varphi' - \xi\varphi' v_0' + 2v_0\eta_x + \xi f + \int_0^t H(t,\tau)\xi(\tau) f(\tau)\, d\tau \\
-\eta g + \eta e_0' + 2\int_0^t H(t,\tau) v_0(\tau)\eta_x(\tau)\, d\tau + \int_0^t H(t,\tau)\eta(\tau) e_0'(\tau)\, d\tau = 0.
\end{gathered}
\tag{4.20}
$$

Integrating by parts

$$
\begin{aligned}
\int_0^t H(t,\tau)\eta(\tau) e_0'(\tau)\, d\tau &= H(t,t)\eta(t)e_0(t) - H(t,0)\eta(0)e_0(0) \\
&\quad - \int_0^t H_\tau(t,\tau)\eta(\tau)e_0(\tau)\, d\tau - \int_0^t H(t,\tau)\eta_t(\tau)e_0(\tau)\, d\tau.
\end{aligned}
$$

and using (4.6) and (4.7), determining equation (4.20) reduces to the equation

$$
\begin{gathered}
\underline{\varphi' z_0 - z_1 - \int_0^t H(t,\tau) z_1(\tau)\, d\tau} + 2v_0(\eta_x - \varphi'\xi_t) + \int_0^t H(t,\tau) f(\tau)[\xi(\tau) - \xi(t)]\, d\tau \\
+2\int_0^t H(t,\tau) v_0(\tau)\eta_x(\tau)\, d\tau \underline{- H(t,0)\eta(0)e_0(0) - \int_0^t H(t,\tau) e_0(\tau)\eta_t(\tau)\, d\tau} \\
\underline{- \int_0^t e_0(\tau)[H_t(t,\tau)\eta(t) + H_\tau(t,\tau)\eta(\tau)]\, d\tau} = 0.
\end{gathered}
\tag{4.21}
$$

We can show that the underlined part of equation (4.21) vanishes. In fact, choosing the initial data $v_0(t) = 0$, we obtain from equation (4.21) that

$$\begin{aligned} &\varphi' z_0 - z_1 - \int_0^t H(t,\tau) z_1(\tau)\, d\tau \\ &-H(t,0)\eta(0)e_0(0) - \int_0^t H(t,\tau) e_0(\tau)\eta_t(\tau)\, d\tau \\ &- \int_0^t e_0(\tau)[H_t(t,\tau)\eta(t) + H_\tau(t,\tau)\eta(\tau)]\, d\tau = 0. \end{aligned} \tag{4.22}$$

Hence, equation (4.21) becomes

$$2v_0(\eta_x - \varphi'\xi_t) + \int_0^t H(t,\tau) f(\tau)[\xi(\tau) - \xi(t)]\, d\tau + 2\int_0^t H(t,\tau) v_0(\tau)\eta_x(\tau)\, d\tau = 0. \tag{4.23}$$

If one chooses the initial function $v_0(t) = \frac{1}{2}$, then $f(t) = 0$, and equation (4.23) gives

$$\eta_x - \varphi'\xi_t + \int_0^t H(t,\tau)\eta_x(\tau)\, d\tau = 0. \tag{4.24}$$

Substituting the coefficients (4.18) into (4.24), and splitting it with respect to x, we obtain

$$c_1 = 0, \tag{4.25}$$

$$-c_2\varphi' + (c_3 t + c_4) + \int_0^t H(t,\tau)(c_3\tau + c_4)\, d\tau = 0. \tag{4.26}$$

Using (4.4), equation (4.22) reduces to the equation

$$\begin{aligned} &\varphi'[\lambda_{tt} - (c_3 x + c_7 - c_9)\sigma_0] + (2c_3 x + 2c_5 - 2c_7 - c_9)\varphi \\ &- \int_0^t H(t,\tau)\lambda_{xx}(\tau)\, d\tau - \lambda_{xx} - (c_4 x + c_8)H(t,0)e_0(0) \\ &+(c_3 x + c_7)e_0 - \int_0^t e_0(\tau) z_2(t,\tau,x)\, d\tau = 0, \end{aligned} \tag{4.27}$$

where

$$z_2(t,\tau,x) = [(c_3 t + c_4)x + c_7 t + c_8]H_t(t,\tau) + [(c_3\tau + c_4)x + c_7\tau + c_8]H_\tau(t,\tau).$$

We can derive that there exists a function $f_1(t,x)$ such that

$$z_2(t,\tau,x) = f_1(t,x)H(t,\tau). \tag{4.28}$$

In fact, let us consider the function[†]

$$e_0(\tau) = a_1 + a_2\tau + \tau(t-\tau)[a_3\psi_1(\tau) + a_4\psi_2(\tau)].$$

Then we have

$$\begin{aligned}
e_0(0) &= a_1, \quad e_0(t) = a_1 + a_2 t,\\
\varphi(\sigma_0) &= a_1\left(1 + \int_0^t H(t,\tau)\,d\tau\right) + a_2\left(t + \int_0^t \tau H(t,\tau)\,d\tau\right)\\
&\quad + a_3\int_0^t \tau(t-\tau)\psi_1(\tau)H(t,\tau)\,d\tau\\
&\quad + a_4\int_0^t \tau(t-\tau)\psi_2(\tau)H(t,\tau)\,d\tau,\\
\int_0^t e_0(\tau)z_2(t,\tau,x)\,d\tau &= a_1\int_0^t z_2(t,\tau,x)\,d\tau + a_2\int_0^t \tau z_2(t,\tau,x)\,d\tau\\
&\quad + a_3\int_0^t \tau(t-\tau)\psi_1(\tau)z_2(t,\tau,x)\,d\tau\\
&\quad + a_4\int_0^t \tau(t-\tau)\psi_2(\tau)z_2(t,\tau,x)\,d\tau.
\end{aligned} \tag{4.29}$$

If there exist functions $\psi_i(\tau), (i=1,2)$ such that the determinant

$$\begin{aligned}
\Delta_2 &= \left(\int_0^t \tau(t-\tau)\psi_1(\tau)H(t,\tau)\,d\tau\right)\left(\int_0^t \tau(t-\tau)\psi_2(\tau)z_2(t,\tau,x)\,d\tau\right)\\
&\quad - \left(\int_0^t \tau(t-\tau)\psi_2(\tau)H(t,\tau)\,d\tau\right)\left(\int_0^t \tau(t-\tau)\psi_1(\tau)z_2(t,\tau,x)\,d\tau\right)
\end{aligned}$$

is not equal to zero, then for the given values $e_0(0)$, $e_0(t)$, $\varphi(\sigma_0)$, $\int_0^t e_0(\tau)z_2(t,\tau,x)\,d\tau$ equations (4.29) are solved, with respect to the coefficients

[†]For the sake of simplicity we use here the general form of the functions $\psi_i(\tau)$, $(i=1,2)$ compare with (4.13).

a_1, a_2, a_3, a_4. Thus, the variables $e_0(0)$, $e_0(t)$, $\varphi(\sigma_0)$, $\int_0^t e_0(\tau)z_2(t,\tau,x)\,d\tau$ are functionally independent. Then we can split equation (4.27) with respect to those variables. Splitting equation (4.27), we obtain contradictory relations. Hence, $\Delta_2 = 0$ for all functions $\psi_1(\tau), \psi_2(\tau) \in L_2[0,t]$. By virtue of $H(t,\tau) \neq 0$ we obtain that there exists a function $f_1(t,x)$ such that $z_2(t,\tau,x) = f_1(t,x)H(t,\tau)$.

Substituting (4.28) into (4.27), the following equation

$$\varphi'[\lambda_{tt} - (c_3x + c_7 - c_9)\sigma_0] + (2c_3x + 2c_5 - 2c_7 - c_9 - f_1)\varphi$$

$$- \int_0^t H(t,\tau)\lambda_{xx}(\tau)\,d\tau - \lambda_{xx} - (c_4x + c_8)H(t,0)e_0(0) + (c_3x + c_7 + f_1)e_0 = 0. \tag{4.30}$$

is obtained. Splitting it with respect to $e_0(0)$, $e_0(t)$, $\sigma_0(t)$, we obtain

$$c_3x + c_7 + f_1 = 0, \tag{4.31}$$

$$(c_4x + c_8)H(t,0) = 0, \tag{4.32}$$

$$\varphi'[\lambda_{tt} - (c_3x + c_7 - c_9)\sigma_0] + (3c_3x + 2c_5 - c_7 - c_9)\varphi = \lambda_{xx} + \int_0^t H(t,\tau)\lambda_{xx}(\tau)\,d\tau. \tag{4.33}$$

Furthermore, splitting equation (4.32) and (4.28) with respect to x, we have

$$c_4H(t,0) = 0, \quad c_8H(t,0) = 0, \tag{4.34}$$

$$(c_3t + c_4)H_t(t,\tau) + (c_3\tau + c_4)H_\tau(t,\tau) = -c_3H(t,\tau), \tag{4.35}$$

and

$$(c_7t + c_8)H_t(t,\tau) + (c_7\tau + c_8)H_\tau(t,\tau) = -c_7H(t,\tau). \tag{4.36}$$

From equations (4.34)-(4.36) we derive that $c_4 = 0$ and $c_8 = 0$. Indeed, if $c_4^2 + c_8^2 \neq 0$, then uniqueness of the Cauchy problem of equation (4.35) or (4.36) with the initial data $H(t,0) = 0$ implies that $H(t,\tau) = 0$, which contradicts assumptions about the function $H(t,\tau)$.

Equations (4.35) and (4.36) reduce to the equations

$$c_i[tH_t(t,\tau)+\tau H_\tau(t,\tau)+H(t,\tau)]=0, \quad (i=3,7).$$

The general solution of the equation

$$tH_t(t,\tau)+\tau H_\tau(t,\tau)+H(t,\tau)=0$$

has the form $H(t,\tau)=\frac{1}{t}R(\frac{\tau}{t})$. The kernels of this type are excluded from the study, because they have a singularity at the time $t=0$. Hence

$$c_3=0, \quad c_7=0.$$

Equation (4.26) gives that

$$c_2=0.$$

Therefore,

$$c_i=0, \quad (i=1,2,3,4,7,8),$$

and the classifying equation is

$$\varphi'(\lambda_{tt}+c_9\sigma)+(2c_5-c_9)\varphi=\lambda_{xx}+\int_0^t H(t,\tau)\lambda_{xx}(\tau)\,d\tau. \qquad (4.37)$$

4.2 Group classification of equations (4.1)

In this section, the group classification of equations (4.1) with respect to the function $\varphi(\sigma)$ and kernel $H(t,\tau)$ is obtained. For classification we consider the classifying equation (4.37).

4.2.1 Arbitrary function $\varphi(\sigma)$

Splitting equation (4.37) with respect to φ, we obtain

$$\lambda_{tt} = 0, \quad c_5 = 0, \quad c_9 = 0,$$

$$\lambda_{xx} + \int_0^t H(t,\tau)\lambda_{xx}(\tau)\, d\tau = 0. \tag{4.38}$$

Since $\lambda_{tt} = 0$ then

$$\lambda(t,x) = a(x)t + b(x)$$

with some functions $a(x)$ and $b(x)$. Substituting this $\lambda(t,x)$ into equation (4.38), equation (4.38) becomes

$$a''(x)[t + \int_0^t \tau H(t,\tau)\, d\tau] + b''(x)[1 + \int_0^t H(t,\tau)\, d\tau] = 0. \tag{4.39}$$

Equation (4.39) implies that

$$b''(x) = 0, \quad a''(x)[t + \int_0^t \tau H(t,\tau)\, d\tau] = 0. \tag{4.40}$$

Because the Volterra integral equation of second kind,

$$f(t) + \int_0^t H(t,\tau) f(\tau)\, d\tau = 0$$

has unique solution $f(t) = 0$, then $t + \int_0^t \tau H(t,\tau)\, d\tau \neq 0$. The last equation of (4.40) gives that $a''(x) = 0$. Hence, $a(x)$, $b(x)$ and $\lambda(t,x)$ have the following form

$$a(x) = a_0 x + a_1, \quad b(x) = b_0 x + b_1, \quad \lambda(t,x) = (a_0 x + a_1)t + b_0 x + b_1,$$

where a_0, a_1, b_0, b_1 are constant. Thus, the components of the infinitesimal generator are as follows,

$$\xi = c_6, \quad \eta = 0, \quad \zeta^v = a_0, \quad \zeta^\sigma = 0, \quad \zeta^e = 0,$$

with arbitrary constants c_6 and a_0. Therefore, the kernel of the admitted Lie groups of equations (4.1) is defined by the generators

$$X_1 = \partial_x, \quad X_2 = \partial_v. \tag{4.41}$$

4.2.2 Linear function $\varphi(\sigma)$

Let $\varphi(\sigma) = E\sigma + E_1$, where $E \neq 0$. Without loss of generality we can assume that $E_1 = 0$. In fact, if $E_1 \neq 0$, then we consider $\bar{\sigma} = \sigma + \dfrac{E_1}{E}$ for which system (4.1) becomes

$$v_t = \bar{\sigma}_x, \quad e_t = v_x, \quad E\bar{\sigma} = e + \int_0^t H(t,\tau)e(\tau,x)\, d\tau.$$

Substituting $\varphi(\sigma) = E\sigma$ into the classifying equation (4.37), we have

$$2Ec_5\sigma + E\lambda_{tt} - \lambda_{xx} - \int_0^t H(t,\tau)\lambda_{xx}(\tau)\, d\tau = 0.$$

Splitting it with respect to σ, we obtain

$$c_5 = 0, \quad E\lambda_{tt} = \lambda_{xx} + \int_0^t H(t,\tau)\lambda_{xx}(\tau)\, d\tau.$$

Therefore, system (4.1) along the generators X_1, X_2 also admits the generators

$$X_3 = v\partial_v + \sigma\partial_\sigma + e\partial_e, \quad X_4 = \lambda_{tx}\partial_v + \lambda_{tt}\partial_\sigma + \lambda_{xx}\partial_e, \tag{4.42}$$

where $\lambda(t,x)$ is a solution of the equation

$$E\lambda_{tt} = \lambda_{xx} + \int_0^t H(t,\tau)\lambda_{xx}(\tau)\, d\tau.$$

4.2.3 Nonlinear function $\varphi(\sigma)$, ($\varphi''(\sigma) \neq 0$)

Differentiating equation (4.37) with respect to σ, equation

$$\varphi''(\lambda_{tt} + c_9\sigma) + 2c_5\varphi' = 0$$

is obtained. Differentiating this equation with respect to t and x, we have

$$\varphi''\lambda_{ttt} = 0, \qquad \varphi''\lambda_{ttx} = 0.$$

The condition $\varphi'' \neq 0$ implies that

$$\lambda_{tt} = c_{10}, \tag{4.43}$$

where c_{10} is constant.

Substituting (4.43) into (4.37), it becomes

$$\varphi'(c_{10} + c_9\sigma) + (2c_5 - c_9)\varphi = \lambda_{xx} + \int_0^t H(t,\tau)\lambda_{xx}(\tau)\, d\tau. \tag{4.44}$$

Since the left hand side term of equation (4.44) does not depend on t and x, and the right hand side term of equation (4.44) does not depend on σ, then there exists a constant c_{11} such that

$$\varphi'(c_{10} + c_9\sigma) + (2c_5 - c_9)\varphi = c_{11}, \tag{4.45}$$

and

$$\lambda_{xx} + \int_0^t H(t,\tau)\lambda_{xx}(\tau)\, d\tau = c_{11}. \tag{4.46}$$

Substituting the general solution of equation (4.43)

$$\lambda(t,x) = \frac{c_{10}}{2}t^2 + a(x)t + b(x)$$

into (4.46), we get that

$$a''(x)[t + \int_0^t \tau H(t,\tau)\ d\tau] + b''(x)[1 + \int_0^t H(t,\tau)\ d\tau] = c_{11}, \tag{4.47}$$

where $a(x)$ and $b(x)$ are some functions.

For $t = 0$, equation (4.47) reduces to the equation

$$b''(x) = c_{11}.$$

Since $t + \int_0^t \tau H(t,\tau)\ d\tau \neq 0$, we obtain that

$$a''(x) = -c_{11} r(t), \tag{4.48}$$

where

$$r(t) = \frac{\int_0^t H(t,\tau)\ d\tau}{t + \int_0^t H(t,\tau)\tau\ d\tau}.$$

This implies that

$$a''(x) = c_{12}, \quad -c_{11} r(t) = c_{12}, \tag{4.49}$$

where c_{12} is constant.

The further study depends on equation (4.45). All functions $\varphi(\sigma)$ satisfying equation (4.45) have the following two forms $\varphi(\sigma) = \alpha \exp(\gamma\sigma) + \beta$, $(\alpha\gamma \neq 0)$ and $\varphi(\sigma) = \alpha\sigma^\beta + \gamma$, $(\alpha\beta(\beta - 1) \neq 0)$. Here, α, β and γ are constant. In addition, we need to proceed with the analysis of the kernel $H(t,\tau)$.

Case of function $\varphi(\sigma) = \alpha \exp(\gamma\sigma) + \beta$**,** $(\alpha\gamma \neq 0)$

From equation (4.45) we get that

$$c_5 = \gamma k, \quad c_{10} = -2k, \quad c_9 = 0, \quad c_{11} = 2\gamma\beta k,$$

where k is constant.

For arbitrary kernel $H(t,\tau)$, equations (4.49) implies that $c_{11} = 0$ and $c_{12} = 0$. Extensions of the kernel of admitted Lie algebras only occur for $\beta = 0$. The extension of the kernel of admitted Lie algebras consists of the generator

$$X_5 = \gamma(x\partial_x - v\partial_v - 2e\partial_e) - 2\partial_\sigma.$$

The kernel $H(t,\tau)$ satisfies the equation

$$\int_0^t H(t,\tau)\, d\tau = 0. \tag{4.50}$$

Then we have

$$c_{12} = 0, \quad \lambda_{tx} = c_{13}, \quad \lambda_{xx} = c_{11}.$$

Therefore, the extension of the kernel of admitted Lie algebras is defined by the generator

$$X_6 = \gamma[x\partial_x - v\partial_v - 2(e-\beta)\partial_e] - 2\partial_\sigma.$$

If the kernel $H(t,\tau)$ is such that $r(t) = 1/K$ or

$$t + \int_0^t (\tau - K)H(t,\tau)\, d\tau = 0, \tag{4.51}$$

where K is constant. Then

$$Kc_{12} = -c_{11}, \quad \lambda_{tx} = c_{12}x + c_{13}, \quad \lambda_{xx} = c_{12}t + c_{11},$$

and the extension of the kernel of admitted Lie algebras is defined by the generator

$$X_7 = \gamma[Kx\partial_x - (Kv + 2\beta x)\partial_v - 2(Ke + \beta t - \beta K)\partial_e] - 2K\partial_\sigma.$$

Case of function $\varphi(\sigma) = \alpha\sigma^{\beta} + \gamma$**,** $(\alpha\beta(\beta - 1) \neq 0)$

Substituting the representation of the function $\varphi(\sigma)$ into equation (4.45), one has

$$c_5 = \frac{1-\beta}{2}c_9, \quad c_{10} = 0, \quad c_{11} = -\beta\gamma c_9.$$

For arbitrary kernel $H(t,\tau)$ there is an additional generator

$$X_8 = (1-\beta)x\partial_x + (1+\beta)v\partial_v + 2\sigma\partial_\sigma + 2\beta e\partial_e$$

for $\gamma = 0$.

If the kernel $H(t,\tau)$ satisfies equation (4.50), then there is an additional generator

$$X_9 = (1-\beta)x\partial_x + (1+\beta)v\partial_v + 2\sigma\partial_\sigma + 2\beta(e-\gamma)\partial_e.$$

If the kernel $H(t,\tau)$ satisfies equation (4.51), then the additional generator is

$$X_{10} = K(1-\beta)x\partial_x + [K(1+\beta)v + 2\beta\gamma x]\partial_v + 2K\sigma\partial_\sigma + 2\beta(Ke + \gamma t - K\gamma)\partial_e.$$

4.3 Invariant solutions for arbitrary kernel $H(t,\tau)$

The study presented in this section is aimed at constructing invariant solutions of equation (4.1). For each obtained function $\varphi(\sigma)$, we study the admitted Lie algebra. Choosing a subalgebra from an optimal system of subalgebras, finding invariants of the subalgebra, and assuming dependence between these invariants, we obtain the representation of invariant solutions. Substituting the representation into equations (4.1) we construct equations with reduced number of independent variable.

4.3.1 Invariant solutions with $\varphi(\sigma) = E\sigma,\ (E \neq 0)$

Consider the admitted Lie algebra L_3 consisting of the generators X_1, X_2, X_3. The commutator table of the Lie algebra L_3 is

	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	X_2
X_3	0	$-X_2$	0

Using this table of commutators, the inner automorphisms are obtained

$$
\begin{aligned}
A_2: &\quad \hat{x}_2 = x_2 - a_2 x_3, \\
A_3: &\quad \hat{x}_2 = x_2 e^{a_3},
\end{aligned}
$$

where only changeable coordinates of the inner automorphisms are presented, a_2, a_3 are parameters of automorphisms A_2, A_3, respectively. An optimal system of one-dimensional subalgebras is

$$\{X_1\}, \quad \{X_1 \pm X_2\}, \quad \{X_3 + \mu X_1\}, \quad \{X_2\},$$

where μ is a constant.

Invariants of the subalgebra $\{X_1\}$ are t, v, σ and e. Hence, an invariant solution has the representation

$$v = V(t), \quad \sigma = U(t), \quad e = W(t).$$

Substitution of this representation of a solution into equations (4.1) with $\varphi = E\sigma$ gives

$$V_t(t) = 0, \quad W_t(t) = 0, \quad EU(t) = W(t) + \int_0^t H(t,\tau)W(\tau)\, d\tau.$$

Therefore, this invariant solution is

$$v = a, \quad e = b, \quad \sigma = \frac{b}{E}[1 + \int_0^t H(t,\tau)\ d\tau],$$

where a, b are arbitrary constants.

A representation of an invariant solution corresponding to the subalgebra $\{X_1 \pm X_2\}$ is

$$v = V(t) \pm x, \quad \sigma = U(t), \quad e = W(t).$$

Substituting this representation of a solution into equations (4.1), the reduced equations are obtained

$$V_t(t) = 0, \quad W_t(t) = \pm 1, \quad EU(t) = W(t) + \int_0^t H(t,\tau)W(\tau)\ d\tau.$$

Therefore, this invariant solution is

$$v = a \pm x, \quad e = b \pm t, \quad \sigma = \frac{1}{E}[b \pm t + \int_0^t H(t,\tau)(b \pm \tau)\ d\tau],$$

where a, b are arbitrary constants.

Considering invariants corresponding to the subalgebra $\{X_3 + \mu X_1\}$, one can note that there are no invariant solutions for $\mu = 0$. For $\mu \neq 0$ the invariants are

$$t, \quad v\exp(-y), \quad \sigma\exp(-y), \quad e\exp(-y) \qquad (y = \frac{x}{\mu}).$$

Invariant solutions have the representation

$$v = V(t)\exp(y), \quad \sigma = U(t)\exp(y), \quad e = W(t)\exp(y).$$

Substituting this representation into equations (4.1), the reduced equations are

$$V_t(t) = \frac{1}{\mu}U(t), \quad W_t(t) = \frac{1}{\mu}V(t), \quad EU(t) = W(t) + \int_0^t H(t,\tau)W(\tau)\ d\tau. \quad (4.52)$$

The latter system (4.52) is a system with the single independent variable t.

4.3.2 Invariant Solutions with $\varphi(\sigma) = \alpha\exp(\gamma\sigma),\ \gamma \neq 0$

For the function $\varphi = \alpha\exp(\gamma\sigma),\ \gamma \neq 0$, the commutator table of the Lie algebra $\{X_1,\ X_2,\ X_5\}$ is

	X_1	X_2	X_5
X_1	0	0	γX_1
X_2	0	0	$-\gamma X_2$
X_5	$-\gamma X_1$	γX_2	0

The set of automorphisms is defined by the commutator table:

$$\begin{aligned} &A_1: \quad \hat{x}_1 = x_1 - \gamma a_1 x_3, \\ &A_2: \quad \hat{x}_2 = x_2 + \gamma a_2 x_3, \\ &A_3: \quad \hat{x}_1 = x_1 e^{\gamma a_3}, \quad \hat{x}_2 = x_2 e^{-\gamma a_3}, \end{aligned}$$

where only changeable coordinates of the inner automorphisms are presented, a_1, a_2, a_3 are parameters of automorphisms A_1, A_2, A_3, respectively.

The optimal system of one-dimensional subalgebras of this Lie algebra consists of the subalgebras

$$\{X_5\}, \quad \{X_2 + \mu X_1\}, \quad \{X_1\},$$

where μ is constant.

Invariants of the subalgebra $\{X_5\}$ are $t,\ xv,\ \sigma + \dfrac{2}{\gamma}\ln x,\ x^2 e$. Invariant solution has the representation

$$v = x^{-1}V(t), \quad \sigma = U(t) - \frac{2}{\gamma}\ln x, \quad e = x^{-2}W(t).$$

Substituting this representation into equations (4.1) with $\varphi = \alpha \exp(\gamma\sigma),\ \gamma \neq 0$, they become

$$V_t(t) = -\frac{2}{\gamma}, \quad W_t(t) = -V(t), \quad \alpha \exp(\gamma U(t)) = W(t) + \int_0^t H(t,\tau)W(\tau)\ d\tau.$$

Hence,

$$V(t) = -\frac{2t}{\gamma} + a, \quad W(t) = \frac{t^2}{\gamma} - at + b,$$

and $U(t)$ satisfies the equation

$$\alpha \exp(\gamma U(t)) = \frac{t^2}{\gamma} - at + b + \int_0^t H(t,\tau)(\frac{\tau^2}{\gamma} - a\tau + b)\ d\tau,$$

where a, b are arbitrary constants.

In the case of the subalgebra $\{X_2 + \mu X_1\}$ $(\mu \neq 0)$ the representation of an invariant solution is

$$v = \frac{x}{\mu} + V(t), \quad \sigma = U(t), \quad e = W(t).$$

Substituting this representation into equations (4.1), one has

$$V_t(t) = 0, \quad W_t(t) = \frac{1}{\mu}, \quad \alpha \exp(\gamma U(t)) = W(t) + \int_0^t H(t,\tau)W(\tau)\ d\tau.$$

Hence,

$$v = \frac{x}{\mu} + V(t) = \frac{x}{\mu} + a, \quad e = W(t) = \frac{t}{\mu} + b,$$

and $U(t)$ satisfies the equation

$$\alpha \exp(\gamma U(t)) = \frac{t}{\mu} + b + \int_0^t H(t,\tau)(\frac{\tau}{\mu} + b)\ d\tau,$$

where a, b are arbitrary constants. An invariant solution of the subalgebra $\{X_1\}$ has the form

$$v = V(t), \quad \sigma = U(t), \quad e = W(t).$$

Equations (4.1) become

$$v = V(t) = a, \quad e = W(t) = b, \quad \alpha \exp(\gamma U(t)) = b[1 + \int_0^t H(t,\tau)\ d\tau],$$

where a, b are arbitrary constants.

4.3.3 Invariant Solutions with $\varphi(\sigma) = \alpha\sigma^{\beta}$, $(\alpha\beta(\beta-1) \neq 0)$

For the function $\varphi(\sigma) = \alpha\sigma^{\beta}$, $(\alpha\beta(\beta-1) \neq 0)$, the admitted Lie algebra is $\{X_1,\ X_2,\ X_8\}$ and the commutator table of this Lie algebra is

	X_1	X_2	X_8
X_1	0	0	k_1X_1
X_2	0	0	k_2X_2
X_8	$-k_1X_1$	$-k_2X_2$	0

where $k_1 = 1-\beta$ and $k_2 = 1+\beta$. The set of automorphisms is defined by the commutator table:

$$A_1: \quad \hat{x}_1 = x_1 - k_1a_1x_3,$$
$$A_2: \quad \hat{x}_2 = x_2 - k_2a_2x_3,$$
$$A_3: \quad \hat{x}_1 = x_1e^{k_1a_3}, \quad \hat{x}_2 = x_2e^{k_2a_3},$$

where only changeable coordinates of the inner automorphisms are presented, a_1, a_2, a_3 are parameters of automorphisms A_1, A_2, A_3, respectively.

Automorphisms of this Lie algebra depend on β. The optimal system of one-dimensional subalgebras consists of the subalgebras:

$$\beta = -1: \ \{X_8 + pX_2\}, \quad \{X_1 + pX_2\}, \quad \{X_2\};$$

$$\beta \neq -1: \ \{X_8\}, \quad \{X_1 + pX_2\}, \quad \{X_2\},$$

where p is constant.

Case $\beta = -1$

For the subalgebra $\{X_8 + pX_2\}$ the invariants are

$$t, \quad v - \frac{p}{2}\ln x, \quad x^{-1}\sigma, \quad xe.$$

A representation of invariant solutions has the form

$$v = \frac{p}{2}\ln x + V(t), \quad \sigma = xU(t), \quad e = x^{-1}W(t),$$

where functions U, V, W have to satisfy the equations

$$V_t(t) = U(t), \quad W_t(t) = \frac{p}{2}, \quad \alpha(U(t))^{-1} = W(t) + \int_0^t H(t,\tau)W(\tau)\,d\tau.$$

For the subalgebra $\{X_1 + pX_2\}$, a representation of invariant solutions is

$$v = px + V(t), \quad \sigma = U(t), \quad e = W(t).$$

Substituting this representation into equations (4.1), one obtains

$$v = px + a, \quad e = pt + b,$$

and $U(t)$ satisfies the equation

$$\alpha(U(t))^{-1} = pt + b + \int_0^t H(t,\tau)(p\tau + b)\,d\tau,$$

where a, b are arbitrary constants.

Case $\beta \neq -1$

For the subalgebra $\{X_8\}$, the invariants are

$$t, \quad x^{-\gamma_1}v, \quad x^{-\gamma_2}\sigma, \quad x^{-\gamma_3}e.$$

where $\gamma_1 = \dfrac{1+\beta}{1-\beta}$, $\gamma_2 = \dfrac{2}{1-\beta}$ and $\gamma_3 = \dfrac{2\beta}{1-\beta}$. Invariant solutions have the form

$$v = x^{\gamma_1}V(t), \quad \sigma = x^{\gamma_2}U(t), \quad e = x^{\gamma_3}W(t),$$

where functions U, V, W have to satisfy the equations

$$V_t(t) = \gamma_2 U(t), \quad W_t(t) = \gamma_1 V(t), \quad \alpha(U(t))^\beta = W(t) + \int_0^t H(t,\tau)W(\tau)\ d\tau.$$

For the subalgebra $\{X_1 + pX_2\}$, a representation of invariant solutions is

$$v = px + V(t), \quad \sigma = U(t), \quad e = W(t).$$

Substituting this representations into equations (4.1), one obtains

$$V(t) = a, \quad W(t) = pt + b,$$

and $U(t)$ satisfies

$$\alpha(U(t))^\beta = pt + b + \int_0^t H(t,\tau)(p\tau + b)\ d\tau,$$

where a, b are arbitrary constants.

CHAPTER V

SYMMETRY GROUPS OF EQUATIONS FOR ONE-DIMENSIONAL LINEAR THERMOVISCOELASTICITY

In this chapter, we deal with applications of the group analysis method to the model describing a thermoviscoelastic bar with memory (allowing for aging). The considered system is (see (3.19))

$$\begin{cases} \sigma_x = v_t, \quad e_t = v_x, \quad \theta_{xx} = w_t, \\ \sigma = Ee + \int_0^t G(t,s)e(s)ds - \theta - \int_0^t L(t,s)\theta(s)ds, \\ w = e + \int_0^t L(t,s)e(s)ds + \theta + \int_0^t c(t,s)\theta(s)ds. \end{cases} \tag{5.1}$$

Here time t and distance x are the independent variables, the stress σ, the velocity v, the strain e, the temperature θ, and w are the dependent variables, $G(t,s)$, $L(t,s)$ and $c(t,s)$ are relaxation functions.

5.1 Initial functions and their derivatives

The Cauchy problem for equations (5.1) is stated with the following initial data:

$$v(t,x_0) = v_0(t), \quad e(t,x_0) = e_0(t), \quad \theta(t,x_0) = \theta_0(t), \quad \theta_x(t,x_0) = \theta_1(t). \tag{5.2}$$

Here $v_0(t)$, $e_0(t)$, $\theta_0(t)$, $\theta_1(t)$ are arbitrary functions and the initial data $\sigma(t,x_0)$ and $w(t,x_0)$ are

$$\sigma(t,x_0) = f_0(t) - g_0(t), \quad w(t,x_0) = f_1(t) + g_1(t),$$

where

$$f_0(t) = Ee_0(t) + \int_0^t G(t,s)e_0(s)ds, \quad f_1(t) = e_0(t) + \int_0^t L(t,s)e_0(s)ds,$$

$$g_0(t) = \theta_0(t) + \int_0^t L(t,s)\theta_0(s)ds, \quad g_1(t) = \theta_0(t) + \int_0^t c(t,s)\theta_0(s)ds.$$

By virtue of the initial conditions (5.2) and equations (5.1) we can find the main derivatives of the functions $v(t,x)$, $\sigma(t,x)$, $e(t,x)$ $\theta(t,x)$ and $w(t,x)$ at $x = x_0$:

$$\begin{gathered} \sigma_x(t,x_0) = v_t(t,x_0) = v_0'(t), \quad v_x(t,x_0) = e_t(t,x_0) = e_0'(t), \\ \sigma_t(t,x_0) = f_0'(t) - g_0'(t), \quad e_x(t,x_0) = e_1(t), \quad \theta_t(t,x_0) = \theta_0'(t), \\ w_t(t,x_0) = f_1'(t) + g_1'(t), \quad w_x(t,x_0) = k_1(t) + p_1(t). \end{gathered} \tag{5.3}$$

Here the functions $k_1(t)$, $p_1(t)$ have the expressions

$$k_1(t) = e_1(t) + \int_0^t L(t,s)e_1(s)ds, \quad p_1(t) = \theta_1(t) + \int_0^t c(t,s)\theta_1(s)ds,$$

and the function $e_1(t)$ is the solution of the Volterra integral equation of second kind

$$Ee_1(t) + \int_0^t G(t,s)e_1(s)ds = v_0'(t) + p_0(t), \tag{5.4}$$

where

$$p_0(t) = \theta_1(t) + \int_0^t L(t,s)\theta_1(s)ds.$$

Setting

$$k_0(t) = Ee_1(t) + \int_0^t G(t,s)e_1(s)ds,$$

we have the relation

$$v_0'(t) = k_0(t) - p_0(t). \tag{5.5}$$

For studying the determining equation of the third equation in system (5.1) we need to consider the main second order derivatives of the functions $v(t,x)$, $\sigma(t,x)$, $e(t,x)$ $\theta(t,x)$ and $w(t,x)$ at $x = x_0$:

$$\begin{gathered}\sigma_{xx}(t,x_0) = e_0''(t), \quad v_{xx}(t,x_0) = e_1'(t), \quad e_{xx} = e_2(t), \quad \theta_{tx}(t,x_0) = \theta_1'(t), \\ \theta_{xx}(t,x_0) = f_1'(t) + g_1'(t), \quad w_{xx}(t,x_0) = k_3(t) + f_3(t) + g_3(t).\end{gathered} \tag{5.6}$$

Here the functions $k_3(t)$, $f_3(t)$, $g_3(t)$ are

$$k_3(t) = e_2(t) + \int_0^t L(t,s)e_2(s)ds,$$

$$f_3(t) = f_1'(t) + \textstyle\int_0^t c(t,s)f_1'(s)ds, \quad g_3(t) = g_1'(t) + \int_0^t c(t,s)g_1'(s)ds,$$

and the function $e_2(t)$ satisfies the Volterra integral equation of second kind

$$Ee_2(t) + \int_0^t G(t,s)e_2(s)ds = e_0''(t) + f_2(t) + g_2(t), \tag{5.7}$$

where

$$f_2(t) = f_1'(t) + \int_0^t L(t,s)f_1'(s)ds, \quad g_2(t) = g_1'(t) + \int_0^t L(t,s)g_1'(s)ds.$$

Denoting

$$k_2(t) = Ee_2(t) + \int_0^t G(t,s)e_2(s)ds,$$

we have

$$e_0''(t) = k_2(t) - f_2(t) - g_2(t). \tag{5.8}$$

The Cauchy problem written above contains a large of number of functions $\theta_i(t)$, $p_i(t)$, $e_j(t)$, $k_r(t)$, $f_r(t)$, $g_r(t)$, $(i = 0,1;\ j = 0,1,2;\ r = 0,1,2,3)$. In order to solve the determining equations of system (5.1) we need to understand relations between them. Only four of the functions can be arbitrarily chosen, they are $e_1(t)$, $\theta_1(t)$, $e_0(t)$ and $\theta_0(t)$. The remaining functions are determined by these four functions. For example, the functions $k_0(t)$ and $k_1(t)$ only depend on $e_1(t)$, and the functions $p_0(t)$, $p_1(t)$ only depend on $\theta_1(t)$. As integral equation (5.4) has a unique solution, the arbitrariness of $e_1(t)$ is determined by the arbitrariness of $v_0'(t)$ when $p_0(t)$ is fixed. To split the determining equations of system (5.1), the following properties are used.

Proposition 1 *Let $y(t)$ be an arbitrary function. If $L(t,s) \neq 0$, then the variables y, y', y'' and $\int_0^t L(t,s)y(s)ds$ are functionally independent.*

Proof Let the initial function $y(s)$ have the form

$$y(s) = a_1 + a_2(s-t_0) + \frac{a_3}{2}(s-t_0)^2 + a_4(s-t_0)^{n+3}, \quad n \geq 1.$$

Here t_0 is a fixed (but arbitrary) time, and a_i, $(i = 1,2,3,4)$ are arbitrary constants. Thus, at $t = t_0$, we have

$$\begin{aligned} y(t_0) &= a_1, \quad y'(t_0) = a_2, \quad y''(t_0) = a_3, \\ \int_0^{t_0} L(t_0,s)y(s)ds &= a_1\int_0^{t_0} L(t_0,s)ds + a_2\int_0^{t_0} L(t_0,s)(s-t_0)ds \\ &\quad + \frac{a_3}{2}\int_0^{t_0} L(t_0,s)(s-t_0)^2 ds + a_4\int_0^{t_0} L(t_0,s)(s-t_0)^{n+3}ds. \end{aligned} \tag{5.9}$$

Since the set of functions $(s-t_0)^n$ is complete in the space $L_2[0,t_0]$, and t_0 is such that $L(t_0,s) \neq 0$, there exists n for which $\int_0^{t_0} L(t_0,s)(t_0-s)^n ds \neq 0$. Hence, for given values $y(t_0)$, $y'(t_0)$, $y''(t_0)$ and $\int_0^{t_0} L(t_0,s)y(s)ds$ we can solve equations (5.9) with respect to the coefficients a_1, a_2, a_3, a_4. This means that the variables y, y', y'' and $\int_0^{t_0} L(t_0,s)y(s)ds$ are functionally independent.

Proposition 2 *Let $y(t)$ be an arbitrary function. If $L(t,s) \neq 0$ and $G(t,s) \neq$*

$h(t)L(t,s)$ for any function $h(t)$, then the variables y, y', y'', $\int_0^t G(t,s)y(s)ds$ and $\int_0^t L(t,s)y(s)ds$ are functionally independent.

Proof Choose the following initial data

$$y(s) = a_1 + a_2(s-t_0) + \frac{a_3}{2}(s-t_0)^2 + (s-t_0)^3[a_4(s-t_0)^{n_1} + a_5(s-t_0)^{n_2}].$$

Here t_0 is a fixed (but arbitrary) time, $n_i \geq 1$ $(i = 1,2)$ are integer numbers, and a_j, $(j = 1,2,3,4,5)$ are arbitrary constants. Hence, we find at $t = t_0$

$$\begin{aligned}
y(t_0) &= a_1, \quad y'(t_0) = a_2, \quad y''(t_0) = a_3, \\
\int_0^{t_0} G(t_0,s)y(s)ds &= a_1\int_0^{t_0} G(t_0,s)ds \\
&\quad +a_2\int_0^{t_0} G(t_0,s)(s-t_0)ds \\
&\quad +\frac{a_3}{2}\int_0^{t_0} G(t_0,s)(s-t_0)^2ds \\
&\quad +a_4\int_0^{t_0} G(t_0,s)(s-t_0)^{n_1+3}ds \\
&\quad +a_5\int_0^{t_0} G(t_0,s)(s-t_0)^{n_2+3}ds, \qquad (5.10) \\
\int_0^{t_0} L(t_0,s)y(s)ds &= a_1\int_0^{t_0} L(t_0,s)ds \\
&\quad +a_2\int_0^{t_0} L(t_0,s)(s-t_0)ds \\
&\quad +\frac{a_3}{2}\int_0^{t_0} L(t_0,s)(s-t_0)^2ds \\
&\quad +a_4\int_0^{t_0} L(t_0,s)(s-t_0)^{n_1+3}ds \\
&\quad +a_5\int_0^{t_0} L(t_0,s)(s-t_0)^{n_2+3}ds.
\end{aligned}$$

The Jacobian of $y(t_0)$, $y'(t_0)$, $y''(t_0)$, $\int_0^{t_0} G(t_0,s)y(s)ds$, $\int_0^{t_0} L(t_0,s)y(s)ds$ with respect to a_1, a_2, a_3, a_4 and a_5 is

$$\begin{aligned}\Delta &= \left(\int_0^{t_0} G(t_0,s)(s-t_0)^{n_1+2}ds\right)\left(\int_0^{t_0} L(t_0,s)(s-t_0)^{n_2+2}ds\right)\\ &\quad -\left(\int_0^{t_0} G(t_0,s)(s-t_0)^{n_2+2}ds\right)\left(\int_0^{t_0} L(t_0,s)(s-t_0)^{n_1+2}ds\right).\end{aligned}$$

If $\Delta = 0$ for all integer numbers n_i $(i = 1, 2)$, then, by virtue of $L(t_0, s) \neq 0$, we get that there exists a function $h(t_0)$ such that

$$G(t_0, s) = h(t_0)L(t_0, s),$$

which contradicts the assumptions. Hence, there exist two integer numbers n_1, n_2 such that $\Delta \neq 0$, and for the given values $y(t_0)$, $y'(t_0)$, $y''(t_0)$, $\int_0^{t_0} G(t_0,s)y(s)ds$, $\int_0^{t_0} L(t_0,s)y(s)ds$ we can solve equations (5.10) with respect to the coefficients a_1, a_2, a_3, a_4, a_5. This means that the variables y, y', y'', $\int_0^{t} G(t,s)y(s)ds$ and $\int_0^{t} L(t,s)y(s)ds$ are functionally independent.

5.2 Determining equations

A symmetry group of equations (5.1) is assumed in the form

$$X = \xi\partial_x + \eta\partial_t + \zeta^v\partial_v + \zeta^\sigma\partial_\sigma + \zeta^e\partial_e + \zeta^\theta\partial_\theta + \zeta^w\partial_w,$$

where the coordinates $\xi, \eta, \zeta^v, \zeta^\sigma$, ζ^e, ζ^θ and ζ^w are functions of the seven variables t, x, v, σ, e, θ, w.

The canonical Lie-Bäcklund operator which is equivalent to the generator X is

$$\begin{aligned}\overline{X} &= (\zeta^v - \xi v_x - \eta v_t)\partial_v + (\zeta^\sigma - \xi\sigma_x - \eta\sigma_t)\partial_\sigma\\ &\quad +(\zeta^e - \xi e_x - \eta e_t)\partial_e + (\zeta^\theta - \xi\theta_x - \eta\theta_t)\partial_\theta + (\zeta^w - \xi w_x - \eta w_t)\partial_w + \cdots\end{aligned}$$

According to the definition of admitted Lie group for integro-differential equations, the determining equations for equations (5.1) are

$$\left(\frac{\partial}{\partial t}W^v(t,x) - \frac{\partial}{\partial x}W^\sigma(t,x)\right)_{|(S)} = 0, \tag{5.11}$$

$$\left(\frac{\partial}{\partial t}W^e(t,x) - \frac{\partial}{\partial x}W^v(t,x)\right)_{|(S)} = 0, \tag{5.12}$$

$$\left(\frac{\partial^2}{\partial x^2}W^\theta(t,x) - \frac{\partial}{\partial t}W^w(t,x)\right)_{|(S)} = 0, \tag{5.13}$$

$$\begin{aligned}&\Big(W^\sigma(t,x) - EW^e(t,x) + W^\theta(t,x)\\ &\quad - \int_0^t G(t,s)W^e(s,x)\,ds + \int_0^t L(t,s)W^\theta(s,x)\,ds\Big)_{|(S)} = 0,\end{aligned} \tag{5.14}$$

$$\begin{aligned}&\Big(W^w(t,x) - W^e(t,x) - W^\theta(t,x)\\ &\quad - \int_0^t L(t,s)W^e(s,x)\,ds - \int_0^t c(t,s)W^\theta(s,x)\,ds\Big)_{|(S)} = 0,\end{aligned} \tag{5.15}$$

where

$$W^v(h_1) = \zeta^v(h_2) - \xi(h_2)v_x(h_1) - \eta(h_2)v_t(h_1),$$

$$W^\sigma(h_1) = \zeta^\sigma(h_2) - \xi(h_2)\sigma_x(h_1) - \eta(h_2)\sigma_t(h_1),$$

$$W^e(h_1) = \zeta^e(h_2) - \xi(h_2)e_x(h_1) - \eta(h_2)e_t(h_1),$$

$$W^\theta(h_1) = \zeta^\theta(h_2) - \xi(h_2)\theta_x(h_1) - \eta(h_2)\theta_t(h_1),$$

$$W^w(h_1) = \zeta^w(h_2) - \xi(h_2)w_x(h_1) - \eta(h_2)w_t(h_1),$$

for the sake of simplicity of the presentation we have denoted

$$h_1 = (t,x), \qquad h_2 = (t,x,v(t,x),\sigma(t,x),e(t,x),\theta(t,x),w(t,x));$$

and the subscript $|(S)$ means that the expression is satisfied for any solution of system (5.1).

According to Propositions 1 and 2, the study of determining equations (5.11)-(5.15) is separated into the following four different cases:

(a) $L(t,s) \neq 0$ and $c(t,s) \neq q_1(t)L(t,s)$ for any function $q_1(t)$;

(b) $L(t,s) \neq 0$, $c(t,s) = q_1(t)L(t,s)$ for some function $q_1(t)$, but there is no a function $h_1(t)$ such that $G(t,s) = h_1(t)L(t,s)$;

(c) $L(t,s) = 0$;

(d) $c(t,s) = q_1(t)L(t,s)$ and $G(t,s) = h_1(t)L(t,s)$ for some functions $q_1(t)$, $h_1(t)$.

For each case, the algorithm of seeking solutions of the determining equations of system (5.1) consists of two steps. The first step is to solve determining equations (5.11)-(5.13) which are related with partial differential equations (PDE) of system (5.1). The second step is to simplify determining equations (5.14) and (5.15) which are related with integral equations (IDE) of system (5.1) by substituting the results obtained in the first step, and to solve the simplified equations.

Let us study determining equations (5.11) and (5.12). Substituting the derivatives (5.3) and relations (5.5) into the determining equations (5.11) and (5.12), considered at point x_0, these equations are

$$\begin{gathered}
(\xi_\sigma - \eta_v)(p_0)^2 - \xi_\theta p_0\theta_1 + \xi_w p_0 p_1 + (\xi_\sigma - \eta_v)(k_0)^2 + \xi_e e_1 k_0 + \xi_w k_0 k_1 + \xi_\theta k_0\theta_1 \\
+2(\eta_v - \xi_\sigma)k_0 p_0 - \xi_e e_1 p_0 - \xi_w k_1 p_0 - \xi_w k_0 p_1 + (\eta_\theta f_0' - \eta_\theta g_0' - \zeta_\theta^\sigma)\theta_1 + [\eta_e e_0' \\
+\eta_w f_1' + \eta_\theta\theta_0' + \eta_w g_1' - (\zeta_v^v - \zeta_\sigma^\sigma + \xi_x - \eta_t)]p_0 - (\eta_w f_0' - \eta_w g_0' - \zeta_w^\sigma)p_1 \\
-[\eta_e e_0' + \eta_w f_1' + \eta_\theta\theta_0' + \eta_w g_1' - (\zeta_v^v - \zeta_\sigma^\sigma + \xi_x - \eta_t)]k_0 \\
+(\eta_e f_0' - \eta_e g_0' - \zeta_e^\sigma)e_1 + (\eta_w f_0' - \eta_w g_0' - \zeta_w^\sigma)k_1 - \xi_e(e_0')^2 + (\eta_v - \xi_\sigma)e_0' f_0' \\
-\xi_w e_0' f_1' - [\xi_\theta\theta_0' + (\eta_v - \xi_\sigma)g_0' + \xi_w g_1' - (\zeta_e^v - \zeta_v^\sigma - \xi_t)]e_0' \\
+(\zeta_\sigma^v + \eta_x)f_0' + \zeta_w^v f_1' + \zeta_\theta^v\theta_0' - (\zeta_\sigma^v + \eta_x)g_0' + \zeta_w^v g_1' + (\zeta_t^v - \zeta_x^\sigma) = 0,
\end{gathered} \tag{5.16}$$

and

$$\eta_\sigma p_0^2 - \eta_\theta p_0\theta_1 + \eta_w p_0 p_1 + \eta_\sigma k_0^2 + (\eta_e - \xi_v)e_1 k_0 + \eta_w k_0 k_1 + \eta_\theta k_0\theta_1 - \eta_w k_0 p_1$$
$$-2\eta_\sigma k_0 p_0 - (\eta_e - \xi_v)e_1 p_0 - \eta_w k_1 p_0 + (\xi_\theta e_0' - \zeta_\theta^v)\theta_1 - [\xi_\sigma e_0' - (\zeta_\sigma^v - \zeta_v^e - \eta_x)]p_0$$
$$-(\xi_w e_0' - \zeta_w^v)p_1 + [\xi_\sigma e_0' - (\zeta_\sigma^v - \zeta_v^e - \eta_x)]k_0 + (\xi_w e_0' - \zeta_w^v)k_1$$
$$-[\xi_\sigma f_0' + \xi_w f_1' + \xi_\theta \theta_0' - \xi_\sigma g_0' + \xi_w g_1' + (\zeta_e^v + \xi_t)]e_1 + (\xi_v - \eta_e)(e_0')^2$$
$$-\eta_\sigma e_0' f_0' - \eta_w e_0' f_1' + [\eta_\sigma g_0' - \eta_\theta \theta_0' - \eta_w g_1' + (\zeta_e^e - \zeta_v^v - \eta_t + \xi_x)]\, e_0'$$
$$+\zeta_\sigma^e f_0' + \zeta_w^e f_1' + \zeta_\theta^e \theta_0' - \zeta_\sigma^e g_0' + \zeta_w^e g_1' + (\zeta_t^e - \zeta_x^v) = 0. \tag{5.17}$$

Determining equation (5.13) is cumbersome. However, further analysis of equations (5.16) and (5.17) leads us to the property that the coefficients ξ and η do not depend on the variables v, σ, e, θ, w in each of the case (a)-(d). For the sake of completeness we present here equation (5.13) where this property is applied:

$$-2\eta_x\theta_1' + \zeta_{\theta\theta}^\theta \theta_1^2 + \zeta_{ww}^\theta p_1^2 + \zeta_{\sigma\sigma}^\theta p_0^2 - 2\zeta_{\sigma\theta}^\theta \theta_1 p_0 + 2\zeta_{\sigma w}^\theta p_0 p_1 - 2\zeta_{\theta w}^\theta \theta_1 p_1 + \zeta_v^\theta e_1' + \zeta_{ee}^\theta e_1^2$$
$$+\zeta_{\sigma\sigma}^\theta k_0^2 + \zeta_{ww}^\theta k_1^2 + 2\zeta_{\sigma e}^\theta e_1 k_0 + 2\zeta_{ew}^\theta e_1 k_1 + 2\zeta_{\sigma w}^\theta k_0 k_1 - 2\zeta_{\sigma e}^\theta p_0 e_1 + 2\zeta_{e\theta}^\theta \theta_1 e_1$$
$$-2\zeta_{ew}^\theta p_1 e_1 - 2\zeta_{\sigma\sigma}^\theta p_0 k_0 + 2\zeta_{\sigma\theta}^\theta \theta_1 k_0 - 2\zeta_{\sigma w}^\theta p_1 k_0 - 2\zeta_{\sigma w}^\theta p_0 k_1 + 2\zeta_{\theta w}^\theta \theta_1 k_1 - 2\zeta_{ww}^\theta p_1 k_1$$
$$+[2\zeta_{v\theta}^\theta e_0' + (2\zeta_{x\theta}^\theta - \xi_{xx})]\theta_1 - [2\zeta_{v\sigma}^\theta e_0' + (2\zeta_{x\sigma}^\theta - \zeta_v^w)]p_0 - [2\zeta_{vw}^\theta e_0' + (2\zeta_{xw}^\theta + \xi_t)]p_1$$
$$+(2\zeta_{ve}^\theta e_0' + 2\zeta_{xe}^\theta)e_1 + [2\zeta_{v\sigma}^\theta e_0' + (2\zeta_{x\sigma}^\theta - \zeta_v^w)]k_0 + [2\zeta_{vw}^\theta e_0' + (2\zeta_{xw}^\theta + \xi_t)]k_1$$
$$-(\zeta_\theta^w + \eta_{xx})\theta_0' + \zeta_\sigma^w g_0' + (\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t)g_1' + \zeta_w^\theta g_3 + \zeta_{vv}^\theta (e_0')^2 + \zeta_w^\theta k_3 + \zeta_e^\theta e_2$$
$$+\zeta_\sigma^\theta e_0'' + (2\zeta_{xv}^\theta - \zeta_e^w)e_0' - \zeta_\sigma^w f_0' + (\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t)f_1' + \zeta_w^\theta f_3 + (\zeta_{xx}^\theta - \zeta_t^w) = 0. \tag{5.18}$$

5.3 Case (a)

In this case, it is assumed that $L(t,s) \neq 0$ and $c(t,s) \neq q_1(t)L(t,s)$ for any function $q_1(t)$. According to Proposition 2 and definitions of the variables p_0, p_1, the variables θ_1, θ_1', p_0 and p_1 can be taken arbitrary.

5.3.1 Solving determining equations for PDE

The arbitrariness of θ_1, θ_1', p_0 and p_1 allows us to split the determining equations (5.16) and (5.17) with respect to the variables θ_1, p_0 and p_1. The following overdetermined system of equations is obtained

$$\xi_v = \xi_\sigma = \xi_e = \xi_\theta = \xi_w = 0, \quad \eta_v = \eta_\sigma = \eta_e = \eta_\theta = \eta_w = 0,$$

$$\zeta^\sigma_\theta = \zeta^\sigma_w = 0, \quad \zeta^v_\theta = \zeta^v_w = 0, \tag{5.19}$$

$$\zeta^e_v - \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_v - \zeta^\sigma_\sigma - \eta_t + \xi_x = 0,$$

$$-\zeta^\sigma_e e_1 + (\zeta^v_e - \zeta^\sigma_v - \xi_t)e_0' + (\zeta^v_\sigma + \eta_x)f_0' - (\zeta^v_\sigma + \eta_x)g_0' + (\zeta^v_t - \zeta^\sigma_x) = 0, \tag{5.20}$$

$$\begin{gathered} -(\zeta^v_e + \xi_t)e_1 + (\zeta^e_e - \zeta^v_v - \eta_t + \xi_x)e_0' + \zeta^e_\sigma f_0' + \zeta^e_w f_1' \\ +\zeta^e_\theta \theta_0' - \zeta^e_\sigma g_0' + \zeta^e_w g_1' + (\zeta^e_t - \zeta^v_x) = 0. \end{gathered} \tag{5.21}$$

By virtue of arbitrariness of e_0, θ_0 and e_1, equations (5.20)-(5.21) are simplified

$$\begin{gathered} \zeta^\sigma_e = 0, \quad \zeta^v_e + \xi_t = 0, \quad \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \\ \zeta^e_t - \zeta^v_x = 0, \quad \zeta^v_e - \zeta^\sigma_v - \xi_t = 0, \end{gathered} \tag{5.22}$$

$$(\zeta^e_e - \zeta^v_v - \eta_t + \xi_x)e_0' + \zeta^e_\sigma f_0' + \zeta^e_w f_1' = 0, \tag{5.23}$$

$$\zeta^e_\theta \theta_0' - \zeta^e_\sigma g_0' + \zeta^e_w g_1' = 0. \tag{5.24}$$

As noted, the property that ξ, η do not depend on the variables v, σ, e, θ, w, reduces determining equation (5.13) to equation (5.18). Splitting equation

(5.18) with respect to θ_1, θ_1', p_0 and p_1, we derive the equations

$$\eta_x = 0, \quad \zeta^\theta_{\sigma\sigma} = \zeta^\theta_{\sigma\theta} = \zeta^\theta_{\sigma w} = 0, \quad \zeta^\theta_{\theta\theta} = \zeta^\theta_{\theta w} = 0, \quad \zeta^\theta_{ww} = 0, \tag{5.25}$$

$$\begin{gathered} 2\zeta^\theta_{e\theta}e_1 + 2\zeta^\theta_{v\theta}e_0' + (2\zeta^\theta_{x\theta} - \xi_{xx}) = 0, \\ 2\zeta^\theta_{\sigma e}e_1 + 2\zeta^\theta_{v\sigma}e_0' + (2\zeta^\theta_{x\sigma} - \zeta^w_v) = 0, \\ 2\zeta^\theta_{ew}e_1 + 2\zeta^\theta_{vw}e_0' + (2\zeta^\theta_{xw} + \xi_t) = 0, \end{gathered} \tag{5.26}$$

$$\begin{gathered} \zeta^\theta_v e_1' + \zeta^\theta_{ee}e_1^2 + 2\zeta^\theta_{\sigma e}e_1k_0 + 2\zeta^\theta_{ew}e_1k_1 + (2\zeta^\theta_{ve}e_0' + 2\zeta^\theta_{xe})e_1 \\ +[2\zeta^\theta_{v\sigma}e_0' + (2\zeta^\theta_{x\sigma} - \zeta^w_v)]k_0 + [2\zeta^\theta_{vw}e_0' + (2\zeta^\theta_{xw} + \xi_t)]k_1 - \zeta^w_\theta\theta_0' \\ +\zeta^w_\sigma g_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)g_1' + \zeta^\theta_w g_3 + \zeta^\theta_\sigma e_0'' + (2\zeta^\theta_{xv} - \zeta^w_e)e_0' - \zeta^w_\sigma f_0' \\ +(\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)f_1' + \zeta^\theta_w f_3 + \zeta^\theta_w k_3 + \zeta^\theta_e e_2 + (\zeta^\theta_{xx} - \zeta^w_t) = 0. \end{gathered} \tag{5.27}$$

According to the arbitrariness of e_1 and e_0', equations (5.26) imply that

$$\begin{gathered} \zeta^\theta_{e\theta} = \zeta^\theta_{\sigma e} = \zeta^\theta_{ew} = 0, \quad \zeta^\theta_{v\theta} = \zeta^\theta_{v\sigma} = \zeta^\theta_{vw} = 0, \\ 2\zeta^\theta_{x\theta} - \xi_{xx} = 0, \quad 2\zeta^\theta_{x\sigma} - \zeta^w_v = 0, \quad 2\zeta^\theta_{xw} + \xi_t = 0. \end{gathered} \tag{5.28}$$

Then equation (5.27) can be split:

$$\zeta^\theta_v = 0, \quad \zeta^\theta_{ee} = 0, \quad \zeta^\theta_{xe} = 0, \quad \zeta^\theta_{xx} - \zeta^w_t = 0, \tag{5.29}$$

$$\begin{gathered} -\zeta^w_\theta\theta_0' + \zeta^w_\sigma g_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)g_1' + \zeta^\theta_w g_3 + \zeta^\theta_w k_3 + \zeta^\theta_e e_2 \\ +\zeta^\theta_\sigma e_0'' - \zeta^w_e e_0' - \zeta^w_\sigma f_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)f_1' + \zeta^\theta_w f_3 = 0. \end{gathered} \tag{5.30}$$

In order to study equation (5.30), consider the initial data $e_0(t) = 0$ for which the variables e_0'', e_0', f_0', f_1', f_2, f_3 vanish. Hence, equation (5.30) is reduced to the equation

$$-\zeta^w_\theta\theta_0' + \zeta^w_\sigma g_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)g_1' + \zeta^\theta_w g_3 + \zeta^\theta_w \bar{k}_3 + \zeta^\theta_e \bar{e}_2 = 0, \tag{5.31}$$

where the function $\bar{e}_2(t)$ satisfies the equation

$$E\bar{e}_2(t) + \int_0^t G(t,s)\bar{e}_2(s)ds = g_2(t), \tag{5.32}$$

and the function $\bar{k}_3(t)$ has the form

$$\bar{k}_3(t) = \bar{e}_2(t) + \int_0^t L(t,s)\bar{e}_2(s)ds. \tag{5.33}$$

Further splitting determining equations (5.23), (5.24) and (5.30) can be performed with respect to the variables which are associated with e_0 and θ_0. To this end, we consider two cases: $L_t(t,s) \neq 0$ and $L_t(t,s) = 0$.

Case: $L_t(t,s) \neq 0$

We divide this case into the following three subcases.

1^0. Case: $c_t(t,s) \neq q_2(t)L_t(t,s)$ for any function $q_2(t)$.

We can show that the variables θ_0', g_0' and g_1' can be arbitrarily chosen. In fact, using proposition 2, we have that the variables θ_0', θ_0, $\int_0^t L_t(t,s)\theta_0(s)ds$, $\int_0^t c_t(t,s)\theta_0(s)ds$ are functionally independent. Notice that

$$g_0'(t) = \theta_0'(t) + L(t,t)\theta_0(t) + \int_0^t L_t(t,s)\theta_0(s)ds,$$

and

$$g_1'(t) = \theta_0'(t) + c(t,t)\theta_0(t) + \int_0^t c_t(t,s)\theta_0(s)ds,$$

it is possible to show that the variables θ_0', g_0' and g_1' are also functionally independent.

Therefore, we can split equation (5.24) with respect to θ_0', g_0', g_1':

$$\zeta_\sigma^e = \zeta_\theta^e = \zeta_w^e = 0, \tag{5.34}$$

Then, equation (5.23) becomes

$$\zeta^e_e - \zeta^v_v - \eta_t + \xi_x = 0. \tag{5.35}$$

Because of the property that the derivatives θ_0', g_0', g_1' can be arbitrarily chosen, and the variables g_3, $\bar{e}_2$, $\bar{k}_3$ only depend on g_1', after splitting equation (5.31) with respect to θ_0', g_0' and g_1', we get

$$\zeta^w_\sigma = \zeta^w_\theta = 0, \tag{5.36}$$

$$(\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)g_1' + \zeta^\theta_w g_3 + \zeta^\theta_w \bar{k}_3 + \zeta^\theta_e \bar{e}_2 = 0. \tag{5.37}$$

By virtue of the arbitrariness of g_1', the variables g_1', g_2 and g_3 are also functionally independent in case (a). Because $\bar{k}_3$ and $\bar{e}_2$ are determined by g_2, equation (5.37) yields the equations

$$\zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = 0. \tag{5.38}$$

Substituting (5.36) and (5.38) into (5.30), we obtain

$$\zeta^\theta_\sigma = \zeta^w_e = 0. \tag{5.39}$$

Consequently, splitting determining equations (5.16) and (5.17), we obtain the following overdetermined system of equations

$$\begin{gathered}
\xi_v = \xi_\sigma = \xi_e = \xi_\theta = \xi_w = 0, \quad \eta_v = \eta_\sigma = \eta_e = \eta_\theta = \eta_w = 0, \\
\zeta^v_\theta = \zeta^v_w = 0, \quad \zeta^\sigma_e = \zeta^\sigma_\theta = \zeta^\sigma_w = 0, \quad \zeta^e_\sigma = \zeta^e_\theta = \zeta^e_w = 0, \quad \zeta^v_e + \xi_t = 0, \\
\zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \quad \zeta^e_t - \zeta^v_x = 0, \quad \zeta^e_v - \zeta^v_\sigma + \eta_x = 0, \\
\zeta^v_e - \zeta^\sigma_v - \xi_t = 0, \quad \zeta^v_v - \zeta^\sigma_\sigma - \eta_t + \xi_x = 0, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = 0.
\end{gathered} \tag{5.40}$$

Splitting determining equations (5.18), we have

$$\xi_t = 0, \quad \eta_x = 0, \quad \zeta^\theta_v = \zeta^\theta_\sigma = \zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^\theta_{\theta\theta} = 0,$$

$$\zeta^w_v = \zeta^w_\sigma = \zeta^w_e = \zeta^w_\theta = 0, \quad 2\zeta^\theta_{x\theta} - \xi_{xx} = 0, \tag{5.41}$$

$$\zeta^\theta_{xx} - \zeta^w_t = 0, \quad \zeta^\theta_\theta - \zeta^w_w + 2\xi_x + \eta_t = 0,$$

Integrating the overdetermined system of equations (5.40) and (5.41), we obtain

$$\xi = c_1 x + c_2, \quad \eta = c_3 t + c_4, \quad \zeta^v = -v(c_1 - c_5) + \lambda_{xt},$$

$$\zeta^\sigma = -\sigma(c_3 - c_5) + \lambda_{tt}, \quad \zeta^e = -e(2c_1 - c_3 - c_5) + \lambda_{xx}. \tag{5.42}$$

$$\zeta^\theta = c_6\theta + \mu_t, \quad \zeta^w = (c_6 - 2c_1 + c_3)w + \mu_{xx}.$$

Here c_i, $(i = 1, 2, \ldots, 6)$ are arbitrary constants and $\lambda = \lambda(t, x)$, $\mu = \mu(t, x)$ are arbitrary functions of two arguments.

2^0. Case: $c_t(t,s) = q_2(t)L_t(t,s)$ for some function $q_2(t)$ but there is no function $h_2(t)$ such that $G_t(t,s) = h_2(t)L_t(t,s)$.

For this subcase, using Proposition 2, we can verify that the variables e_0', f_0' and f_1' are functionally independent. Splitting equation (5.23) with respect to e_0', f_0', f_1' and simplifying equation (5.24), we derive that the solution of determining equations (5.16) and (5.17) is the same as in the previous case (5.40).

Using Proposition 1 for solving equation (5.30), the condition $L_t(t,s) \neq 0$ and the arbitrariness of $\theta_0(t)$ imply that the variables θ_0', θ_0, $\int_0^t L_t(t,s)\theta_0(s)ds$ are functionally independent. Notice that

$$g_0'(t) = \theta_0'(t) + L(t,t)\theta_0(t) + \int_0^t L_t(t,s)\theta_0(s)ds,$$

and

$$g_1'(t) = \theta_0'(t) + c(t,t)\theta_0(t) + q_2(t)\int_0^t L_t(t,s)\theta_0(s)ds.$$

The dependence of the variables θ_0', g_0', g_1' is determined by the coefficients $q_2(t)$ and $c(t,t)$. In fact, we have the following relations.

If $q_2(t) = 0$ and $c(t,t) \neq 0$, then the variables θ_0', g_0' and g_1' are still functionally independent. This means that equations (5.41) are still valid.

If $q_2(t) = 0$ and $c(t,t) = 0$, then $\theta_0' = g_1'$, and substituting this relation into equation (5.31) and splitting it with respect to g_0', g_1', system (5.41) is again derived.

Now assume that $q_2(t) \neq 0$. It is convenient to introduce

$$l(t) = \frac{1}{q_2(t)}.$$

The relation between g_0' and g_1' is

$$g_0'(t) = (1 - l(t))\theta_0'(t) + [L(t,t) - l(t)c(t,t)]\theta_0(t) + l(t)g_1'(t).$$

Substituting this relation into equation (5.31), we get that

$$\begin{gathered}[\zeta_\sigma^w - \zeta_\theta^w - l(t)\zeta_\sigma^w]\theta_0' + \zeta_\sigma^w[L(t,t) - l(t)c(t,t)]\theta_0 + \\ [(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) + l(t)\zeta_\sigma^w]g_1' + \zeta_w^\theta g_3 + \zeta_w^\theta \bar{k}_3 + \zeta_e^\theta = 0,\end{gathered} \tag{5.43}$$

Splitting equation (5.43) with respect to θ_0', θ_0 and g_1', we have

$$\zeta_\sigma^w[L(t,t) - l(t)c(t,t)] = 0, \tag{5.44}$$

$$\zeta_\sigma^w - \zeta_\theta^w - l(t)\zeta_\sigma^w = 0, \tag{5.45}$$

$$[(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) + l(t)\zeta_\sigma^w]g_1' + \zeta_w^\theta g_3 + \zeta_w^\theta \bar{k}_3 + \zeta_e^\theta \bar{e}_2 = 0. \tag{5.46}$$

Furthermore, splitting equation (5.46) with respect to g_1', g_2 and g_3, we find

$$\zeta_e^\theta = \zeta_w^\theta = 0, \quad (\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) + l(t)\zeta_\sigma^w = 0, \tag{5.47}$$

Equations (5.45) and (5.47) lead equation (5.30) to the equation

$$\zeta^{\theta}_{\sigma}e''_0 - \zeta^{w}_{e}e'_0 - \zeta^{w}_{\sigma}f'_0 - l(t)\zeta^{w}_{\sigma}f'_1 = 0. \tag{5.48}$$

Notice that

$$f'_0(t) = Ee'_0(t) + G(t,t)e_0(t) + \int_0^t G_t(t,s)e_0(s)ds,$$

$$f'_1(t) = e'_0(t) + L(t,t)e_0(t) + \int_0^t L_t(t,s)e_0(s)ds.$$

Because of $G_t(t,s) \neq h_2(t)L_t(t,s)$ for any function $h_2(t)$, the variables e''_0, e'_0, f'_0 and f'_1 are functionally independent. This means that equation (5.48) can be split:

$$\zeta^{\theta}_{\sigma} = \zeta^{w}_{e} = \zeta^{w}_{\sigma} = 0. \tag{5.49}$$

From equations (5.45) and (5.47), it follows that

$$\zeta^{w}_{\theta} = 0, \qquad \zeta^{\theta}_{\theta} - \zeta^{w}_{w} - 2\xi_x + \eta_t = 0.$$

Therefore, we also obtain system (5.41).

Consequently, because systems (5.40) and (5.41) are still valid, the coefficients $\xi, \eta, \zeta^v, \zeta^\sigma$, ζ^e, ζ^θ and ζ^w of the infinitesimal generator have form (5.42).

3^0. Case: $c_t(t,s) = q_2(t)L_t(t,s)$ and $G_t(t,s) = h_2(t)L_t(t,s)$ for some functions $q_2(t)$, $h_2(t)$.

For convenience, we introduce the functions

$$\bar{f}(t) = \int_0^t L_t(t,s)e_0(s)ds, \qquad \bar{g}(t) = \int_0^t L_t(t,s)\theta_0(s)ds. \tag{5.50}$$

Then

$$\begin{gathered} f'_0 = Ee'_0 + G(t,t)e_0 + h_2\bar{f}, \qquad f'_1 = e'_0 + L(t,t)e_0 + \bar{f}, \\ g'_0 = \theta'_0 + L(t,t)\theta_0 + \bar{g}, \qquad g'_1 = \theta'_0 + c(t,t)\theta_0 + q_2\bar{g}. \end{gathered} \tag{5.51}$$

Substituting these expressions into (5.23) and (5.24), they reduce to the equations

$$
\begin{gathered}
[(\zeta^e_e - \zeta^v_v - \eta_t + \xi_x) + E\zeta^e_\sigma + \zeta^e_w]e_0' \\
+[G(t,t)\zeta^e_\sigma + L(t,t)\zeta^e_w]e_0 + (h_2\zeta^e_\sigma + \zeta^e_w)\bar{f} = 0,
\end{gathered} \tag{5.52}
$$

$$
(\zeta^e_\theta - \zeta^e_\sigma + \zeta^e_w)\theta_0' - [L(t,t)\zeta^e_\sigma - c(t,t)\zeta^e_w]\theta_0 - (\zeta^e_\sigma - q_2\zeta^e_w)\bar{g} = 0. \tag{5.53}
$$

By virtue of $L_t(t,s) \neq 0$, the variables e_0', e_0, $\bar{f}$ are functionally independent, and the variables θ_0', θ_0, $\bar{g}$ are also functionally independent. Splitting equations (5.52) and (5.53) with respect to e_0', e_0, $\bar{f}$ and θ_0', θ_0, $\bar{g}$, respectively, we obtain that

$$
(\zeta^e_e - \zeta^v_v - \eta_t + \xi_x) + E\zeta^e_\sigma + \zeta^e_w = 0, \quad \zeta^e_\theta - \zeta^e_\sigma + \zeta^e_w = 0, \tag{5.54}
$$

$$
\begin{gathered}
\zeta^e_\sigma - q_2\zeta^e_w = 0, \quad h_2\zeta^e_\sigma + \zeta^e_w = 0, \\
G(t,t)\zeta^e_\sigma + L(t,t)\zeta^e_w = 0, \quad L(t,t)\zeta^e_\sigma - c(t,t)\zeta^e_w = 0.
\end{gathered} \tag{5.55}
$$

Notice that system (5.55) consists of a homogeneous linear equations for ζ^e_σ and ζ^e_w. it is natural to consider equations (5.55) in matrix form

$$
A\begin{pmatrix} \zeta^e_\sigma \\ \zeta^e_w \end{pmatrix} = 0,
$$

where

$$
A = \begin{pmatrix} 1 & -q_2(t) \\ h_2(t) & 1 \\ G(t,t) & L(t,t) \\ L(t,t) & -c(t,t) \end{pmatrix}.
$$

If $rank(A) = 2$, then system (5.55) has only the trivial solution, that is

$$
\zeta^e_\sigma = \zeta^e_w = 0.
$$

It follows from equations (5.54) that

$$\zeta^e_\theta = 0, \qquad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = 0.$$

This means that system (5.40) is also obtained.

If $rank(A) = 1$, then equations (5.54) and (5.55) yield:

$$\zeta^e_\sigma = q_2\zeta^e_w, \quad \zeta^e_\theta = (q_2 - 1)\zeta^e_w, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = -(Eq_2 + 1)\zeta^e_w. \tag{5.56}$$

Therefore, the determining equations (5.16) and (5.17) can be split to the following overdetermined system of equations

$$\begin{gathered}
\xi_v = \xi_\sigma = \xi_e = \xi_\theta = \xi_w = 0, \quad \eta_v = \eta_\sigma = \eta_e = \eta_\theta = \eta_w = 0, \\
\zeta^v_\theta = \zeta^v_w = 0, \quad \zeta^\sigma_e = \zeta^\sigma_\theta = \zeta^\sigma_w = 0, \quad \zeta^e_\sigma = q_2\zeta^e_w, \quad \zeta^e_\theta = (q_2 - 1)\zeta^e_w, \\
\zeta^v_e + \xi_t = 0, \quad \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \quad \zeta^e_t - \zeta^v_x = 0, \\
\zeta^e_v - \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_e - \zeta^\sigma_v - \xi_t = 0, \\
\zeta^v_v - \zeta^\sigma_\sigma - \eta_t + \xi_x = 0, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = -(Eq_2 + 1)\zeta^e_w.
\end{gathered} \tag{5.57}$$

For studying equation (5.30), notice that equations (5.44), (5.45), (5.47) and (5.48) are still valid in this case. By substitution of expressions (5.51) into equation (5.48) we obtain that

$$\begin{gathered}
\zeta^\theta_\sigma e_0'' - [\zeta^w_e + (E + l(t))\zeta^w_\sigma]e_0' \\
-\zeta^w_\sigma[G(t, t + l(t)L(t, t))]e_0 - \zeta^w_\sigma(h_2(t) + l(t))\bar{f} = 0.
\end{gathered} \tag{5.58}$$

Splitting this equation with respect to e_0'', e_0', e_0 and $\bar{f}$, we have the equations

$$\zeta^\theta_\sigma = 0,$$

$$\zeta^w_e + (E + l(t))\zeta^w_\sigma = 0, \tag{5.59}$$

$$\zeta^w_\sigma[G(t, t + l(t)L(t, t))] = 0, \quad \zeta^w_\sigma(h_2(t) + l(t)) = 0. \tag{5.60}$$

Equations (5.44) and (5.60) imply that if $rank(A) = 2$, then $\zeta^w_\sigma = 0$. From equations (5.45) (5.47) and (5.59), it follows that

$$\zeta^w_e = \zeta^w_\theta = 0, \qquad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = 0.$$

That is, system (5.41) is still valid for the present case.

If $rank(A) = 1$, then $h_2(t) = -l(t)$. This leads to the equations

$$\zeta^w_e = (h_2 - E)\zeta^w_\sigma, \quad \zeta^w_\theta = (1 + h_2)\zeta^w_\sigma, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = h_2\zeta^w_\sigma.$$

Therefore, we obtain

$$\begin{gathered}
\xi_t = 0, \quad \eta_x = 0, \quad \zeta^\theta_v = \zeta^\theta_\sigma = \zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^\theta_{\theta\theta} = 0, \\
\zeta^w_v = 0, \quad \zeta^w_e = (h_2 - E)\zeta^w_\sigma, \quad 2\zeta^\theta_{x\theta} - \xi_{xx} = 0, \quad \zeta^\theta_{xx} - \zeta^w_t = 0, \\
\zeta^w_\theta = (1 + h_2)\zeta^w_\sigma, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = h_2\zeta^w_\sigma.
\end{gathered} \tag{5.61}$$

Summarizing, we have the following results:

(a) If $rank(A) = 2$, then the determining equations (5.16)-(5.18) can be split to the system of equations (5.40) and (5.41). Moreover, the general solution of the system of equations (5.40) and (5.41) is the system of equations (5.42).

(b) If $rank(A) = 1$, then the overdetermined system of equations (5.57) and (5.61) are obtained. We can show that in this case $q_2' \neq 0$. Indeed, if $q_2' = 0$, that is, $q_2(t)$ is constant, say α, then $h_2(t)$ is constant and $c_t(t, s) = \alpha L_t(t, s)$. Hence, the function $c(t, s)$ has the form

$$c(t, s) = \alpha L(t, s) + m(s),$$

for some function $m(s)$. Because of $rank(A) = 1$, we have the relation $c(t,t) = \alpha L(t,t)$, which implies that $m(s) = 0$. Thus

$$c(t,s) = \alpha L(t,s),$$

which contradicts the assumptions. As $q_2' \neq 0$, the general solution of the system of equations (5.57) and (5.61) is the system of equations (5.42).

In other words, the system of equations (5.42) constitutes a solution of the determining equations (5.16)-(5.18) for the present case.

Case: $L_t(t,s) = 0$

Since $L_t(t,s) = 0$ (i.e. $L(t,s) = L(s)$), the variables f_0', f_1', g_0' and g_1' can be written in the forms

$$f_0'(t) = Ee_0'(t) + G(t,t)e_0(t) + \tilde{f}(t), \quad f_1'(t) = e_0'(t) + L(t)e_0(t),$$

$$g_0'(t) = \theta_0'(t) + L(t)\theta_0(t), \quad g_1'(t) = \theta_0'(t) + c(t,t)\theta_0(t) + \tilde{g}(t),$$

where

$$\tilde{f}(t) = \int_0^t G_t(t,s)e_0(s)ds, \quad \tilde{g}(t) = \int_0^t c_t(t,s)\theta_0(s)ds.$$

Substituting these expressions into equations (5.23) and (5.24), they are rewritten in the forms

$$[(\zeta_e^e - \zeta_v^v - \eta_t + \xi_x) + E\zeta_\sigma^e + \zeta_w^e]e_0' + [G(t,t)\zeta_\sigma^e + L(t)\zeta_w^e]e_0 + \zeta_\sigma^e\tilde{f} = 0, \tag{5.62}$$

$$(\zeta_\theta^e - \zeta_\sigma^e + \zeta_w^e)\theta_0' - [L(t)\zeta_\sigma^e - c(t,t)\zeta_w^e]\theta_0 + \zeta_w^e\tilde{g} = 0, \tag{5.63}$$

and equation (5.31) becomes

$$\begin{gathered}(-\zeta_\theta^w + \zeta_\sigma^w)\theta_0' + \zeta_\sigma^w L(t)\theta_0 \\ +(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t)g_1' + \zeta_w^\theta g_3 + \zeta_w^\theta \bar{k}_3 + \zeta_e^\theta \bar{e}_2 = 0.\end{gathered} \tag{5.64}$$

For studying equations (5.62)-(5.64), we also divide this case into three subcases as follows.

1^0. Case: $c_t(t,s) \neq 0$.

Notice that the variables θ_0', θ_0, $\tilde{g}$ are functionally independent, equation (5.63) can be split with respect to θ_0', θ_0, $\tilde{g}$ to the three equations

$$\zeta^e_w = 0, \quad L(t)\zeta^e_\sigma = 0, \quad \zeta^e_\theta - \zeta^e_\sigma = 0,$$

Since $L(t) \neq 0$, then $\zeta^e_\sigma = 0$. So we find

$$\zeta^e_\sigma = \zeta^e_\theta = \zeta^e_w = 0,$$

which lead equation (5.62) to the equation

$$\zeta^e_e - \zeta^v_v - \eta_t + \xi_x = 0.$$

Finally, we also derive that system (5.40).

Since the variables g_3, $\bar{k}_3$, $\bar{e}_2$ only depend on g_1', splitting equation (5.64) with respect to θ_0', θ_0, g_1', we get

$$\zeta^w_\sigma - \zeta^w_\theta = 0, \quad L(t)\zeta^w_\sigma = 0, \tag{5.65}$$

$$(\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)g_1' + \zeta^\theta_w g_3 + \zeta^\theta_w \bar{k}_3 + \zeta^\theta_e \bar{e}_2 = 0, \tag{5.66}$$

Equations (5.65) imply that

$$\zeta^w_\sigma = \zeta^w_\theta = 0. \tag{5.67}$$

For splitting equation (5.66), notice that the variables g_1', g_2, g_3 can be arbitrarily chosen and the variables $\bar{k}_3$, $\bar{e}_2$ are only determined by g_2. Splitting (5.66) with respect to g_1', g_2, g_3, we derive

$$\zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^\theta_\theta - \zeta^w_w + 2\xi_x - \eta_t = 0, \tag{5.68}$$

Equations (5.67) and (5.68) lead (5.30) to the equations

$$\zeta^{\theta}_{\sigma} = \zeta^{w}_{e} = 0.$$

That means that the system of equations (5.41) is also established.

As a result of the system of equations (5.40) and (5.41), equations (5.42) also constitute a solution of the determining equations (5.16)-(5.18) in the present case.

2^0. Case: $c_t(t,s) = 0$, but $G_t(t,s) \neq 0$.

Similar to the way of solving equations (5.62) and (5.63) for the previous case, splitting equation (5.62) with respect to e_0', e_0, $\tilde{f}$, and reducing equation (5.63), we also get that system (5.40).

Since the condition $c_t(t,s) = 0$, (i.e. $c(t,s) = c(s)$) implies that $\tilde{g}$ vanishes, then $\theta_0' = g_1' - c(t)\theta_0$. Because of this relation and the arbitrariness of θ_0, g_1', equation (5.64) can be split to the equations

$$L(t)\zeta^{w}_{\sigma} + c(t)(\zeta^{w}_{\theta} - \zeta^{w}_{\sigma}) = 0, \tag{5.69}$$

$$\zeta^{\theta}_{e} = \zeta^{\theta}_{w} = 0, \quad \zeta^{w}_{\sigma} - \zeta^{w}_{\theta} + (\zeta^{\theta}_{\theta} - \zeta^{w}_{w} - 2\xi_x + \eta_t) = 0. \tag{5.70}$$

Hence, equation (5.30) becomes

$$\zeta^{\theta}_{\sigma} e_0'' - \zeta^{w}_{e} e_0' - \zeta^{w}_{\sigma} f_0' + (\zeta^{\theta}_{\theta} - \zeta^{w}_{w} - 2\xi_x + \eta_t) f_1' = 0. \tag{5.71}$$

By virtue of the relations $G_t(t,s) \neq 0$, $L(t) \neq 0$ and Proposition 1, it is possible to show that the variables e_0'', e_0', f_0', f_1' can be arbitrarily chosen. This means that we can split equation (5.71) with respect to e_0'', e_0', f_0', f_1'. Splitting it, we obtain

$$\zeta^{\theta}_{\sigma} = \zeta^{w}_{e} = \zeta^{w}_{\sigma} = 0, \quad \zeta^{\theta}_{\theta} - \zeta^{w}_{w} - 2\xi_x + \eta_t = 0.$$

From the latter equations and equation (5.69), it follows that $\zeta^{w}_{\theta} = 0$. That is, system (5.41) is also established.

In other words, for this case, equations (5.42) form a solution of the determining equations (5.16)-(5.18).

3^0.Case: $c_t(t,s) = 0$ and $G_t(t,s) = 0$.

For this case, we have that $G(t,s) = G(s)$ and $\tilde{f}$, $\tilde{g}$ vanish. Splitting equation (5.62) and (5.63) with respect to e_0', e_0, and θ_0', θ_0, respectively, we find that the relations

$$G(t)\zeta_\sigma^e + L(t)\zeta_w^e = 0, \quad L(t)\zeta_\sigma^e - c(t)\zeta_w^e = 0, \tag{5.72}$$

$$(\zeta_e^e - \zeta_v^v - \eta_t + \xi_x) + E\zeta_\sigma^e + \zeta_w^e = 0, \quad \zeta_\theta^e - \zeta_\sigma^e + \zeta_w^e = 0, \tag{5.73}$$

Equations (5.72) consists of a homogeneous linear equations for ζ_σ^e and ζ_w^e, consider its matrix form

$$B\begin{pmatrix} \zeta_\sigma^e \\ \zeta_w^e \end{pmatrix} = 0,$$

where

$$B = \begin{pmatrix} G(t) & L(t) \\ L(t) & -c(t) \end{pmatrix}.$$

If $rank(B) = 2$, (i.e. $L^2(t) + c(t)G(t) \neq 0$), then the solution of equations (5.72) is

$$\zeta_\sigma^e = \zeta_w^e = 0.$$

From this solution and equations (5.73), it follows that equations (5.40) are still valid.

If $rank(B) = 1$, then $L^2(t) + c(t)G(t) = 0$, equations (5.72) and (5.73) imply that the relation

$$\zeta_\sigma^e = r\zeta_w^e, \quad \zeta_\theta^e = (r-1)\zeta_w^e, \quad \zeta_e^e - \zeta_v^v - \eta_t + \xi_x = -(Er+1)\zeta_w^e,$$

where

$$r(t) = \frac{c(t)}{L(t)}.$$

Therefore, we have the following overdetermined system of equations

$$\xi_v = \xi_\sigma = \xi_e = \xi_\theta = \xi_w = 0, \quad \eta_v = \eta_\sigma = \eta_e = \eta_\theta = \eta_w = 0,$$

$$\zeta^v_\theta = \zeta^v_w = 0, \quad \zeta^\sigma_e = \zeta^\sigma_\theta = \zeta^\sigma_w = 0, \quad \zeta^e_\sigma = r\zeta^e_w, \quad \zeta^e_\theta = (r-1)\zeta^e_w,$$

$$\zeta^v_e + \xi_t = 0, \quad \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \quad \zeta^e_t - \zeta^v_x = 0, \tag{5.74}$$

$$\zeta^e_v - \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_e - \zeta^\sigma_v - \xi_t = 0,$$

$$\zeta^v_v - \zeta^\sigma_\sigma - \eta_t + \xi_x = 0, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = -(Er+1)\zeta^e_w.$$

For studying equation (5.64), notice that equations (5.69)-(5.71) are still valid in this case, and the variables f_0', f_1' have the forms

$$f_0' = Ee_0' + G(t)e_0, \quad f_1' = e_0' + L(t)e_0.$$

Substituting these expressions into equation (5.71), and splitting it with respect to e_0'', e_0', e_0, we get that

$$\zeta^\theta_\sigma = 0,$$

$$G(t)\zeta^w_\sigma - L(t)(\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t) = 0, \tag{5.75}$$

$$(\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t) - \zeta^w_e - E\zeta^w_\sigma = 0. \tag{5.76}$$

From equations (5.69) and (5.70), it follows that

$$\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = \zeta^w_\theta - \zeta^w_\sigma, \tag{5.77}$$

$$L(t)\zeta^w_\sigma + c(t)(\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t) = 0. \tag{5.78}$$

If $rank(B) = 2$, then the homogeneous linear equations (5.75) and (5.78) for ζ^w_σ and $\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t$ only have the trivial solution, that is

$$\zeta^w_\sigma = 0, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = 0.$$

Moreover, equations (5.76) and (5.77) become

$$\zeta^w_e = \zeta^w_\theta = 0.$$

This means that system (5.41) is also obtained.

If $rank(B) = 1$, then $L^2(t) + c(t)G(t) = 0$, and equations (5.75), (5.76), (5.77) imply the equations

$$\zeta^\theta_\theta - \zeta^w_w + 2\xi_x - \eta_t = k\zeta^w_\sigma, \quad \zeta^w_e = (k - E)\zeta^w_\sigma, \quad \zeta^w_\theta = (k+1)\zeta^w_\sigma,$$

where

$$k(t) = \frac{G(t)}{L(t)} = -\frac{1}{r(t)}.$$

Therefore, we obtain the following overdetermined system of equations for solving determining equation (5.18):

$$\begin{gathered}
\xi_t = 0, \quad \eta_x = 0, \quad \zeta^\theta_v = \zeta^\theta_\sigma = \zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^\theta_{\theta\theta} = 0, \\
\zeta^w_v = 0, \quad \zeta^w_e = (k - E)\zeta^w_\sigma, \quad \zeta^w_\theta = (1 + k)\zeta^w_\sigma, \qquad (5.79) \\
2\zeta^\theta_{x\theta} - \xi_{xx} = 0, \quad \zeta^\theta_{xx} - \zeta^w_t = 0, \quad \zeta^\theta_\theta - \zeta^w_w + 2\xi_x - \eta_t = k\zeta^w_\sigma.
\end{gathered}$$

If $r' = 0$, then $r(t)$ is constant (say γ), we have $c(s) = \gamma L(s)$ which contradicts the assumptions. Hence, we can assume that $r' \neq 0$. The general solution of the overdetermined system of equations (5.74) and (5.79) is the system of equations (5.42).

5.3.2 Solving determining equations for IDE

We new study the remaining determining equations (5.14) and (5.15) which are related with the integral parts of system (5.1). From equations (5.3), it follows that at point $x = x_0$

$$W^\sigma = \zeta^\sigma - \xi v_0' - \eta(f_0' - g_0'), \qquad W^e = \zeta^e - \xi e_1 - \eta e_0',$$

$$W^\theta = \zeta^\theta - \xi\theta_1 - \eta\theta_0', \qquad W^w = \zeta^w - \xi(k_1 + p_1) - \eta(f_1' + g_1').$$

Notice that ξ does not depend on t and the relation (5.5), thus the determining equation (5.14) reduce to the equation

$$\begin{gathered}\zeta^\sigma - E\zeta^e + \zeta^\theta - \int_0^t G(t,s)\zeta^e(s)ds + \int_0^t L(t,s)\zeta^\theta(s)ds \\ -\eta(f_0' - g_0') + E\eta e_0' + \int_0^t G(t,s)\eta(s)e_0'(s)ds \\ -\eta\theta_0' - \int_0^t L(t,s)\eta(s)\theta_0'(s)ds = 0.\end{gathered} \tag{5.80}$$

Using the expression

$$\begin{aligned} f_0' - g_0' &= Ee_0' - \theta_0' + G(t,t)e_0 \\ &\quad -L(t,t)\theta_0 + \int_0^t G_t(t,s)e_0(s)ds - \int_0^t L_t(t,s)\theta_0(s)ds, \end{aligned}$$

and integrating by parts for $\int_0^t G(t,s)\eta(s)e_0'(s)ds$, $\int_0^t L(t,s)\eta(s)\theta_0'(s)ds$, determining equation (5.80) can be rewritten in the form

$$\begin{gathered}\zeta^\sigma - E\zeta^e + \zeta^\theta - \int_0^t G(t,s)\zeta^e(s)ds + \int_0^t L(t,s)\zeta^\theta(s)ds \\ -G(t,0)\eta(0)e_0(0) - \int_0^t G(t,s)\eta_t(s)e_0(s)ds - \int_0^t Z_1(t,s)e_0(s)ds \\ +L(t,0)\eta(0)\theta_0(0) + \int_0^t L(t,s)\eta_t(s)\theta_0(s)ds + \int_0^t Z_2(t,s)\theta_0(s)ds = 0.\end{gathered} \tag{5.81}$$

where

$$Z_1(t,s) = G_t(t,s)\eta(t) + G_s(t,s)\eta(s),$$
$$Z_2(t,s) = L_t(t,s)\eta(t) + L_s(t,s)\eta(s). \tag{5.82}$$

Similarly, determining equation (5.15) can be reduced to the equation

$$\begin{aligned}
&\zeta^w - \zeta^e - \zeta^\theta - \int_0^t L(t,s)\zeta^e(s)ds - \int_0^t c(t,s)\zeta^\theta(s)ds \\
&-L(t,0)\eta(0)e_0(0) - \int_0^t L(t,s)\eta_t(s)e_0(s)ds - \int_0^t Z_2(t,s)e_0(s)ds \\
&-c(t,0)\eta(0)\theta_0(0) - \int_0^t c(t,s)\eta_t(s)\theta_0(s)ds - \int_0^t Z_3(t,s)\theta_0(s)ds = 0,
\end{aligned} \tag{5.83}$$

where

$$Z_3(t,s) = c_t(t,s)\eta(t) + c_s(t,s)\eta(s). \tag{5.84}$$

By substitution of equations (5.42) into equation (5.81), we obtain that

$$\begin{aligned}
&\lambda_{tt} - E\lambda_{xx} + \mu_t - \int_0^t G(t,s)\lambda_{xx}(s)ds + \int_0^t L(t,s)\mu_t(s)ds \\
&+2E(c_1 - c_3)e_0 - c_4G(t,0)e_0(0) + (c_3 - c_5 + c_6)\theta_0 + c_4L(t,0)\theta_0(0) \\
&+(2c_1 - 3c_3)\int_0^t G(t,s)e_0(s)ds - \int_0^t Z_1(t,s)e_0(s)ds \\
&+(2c_3 - c_5 + c_6)\int_0^t L(t,s)\theta_0(s)ds + \int_0^t Z_2(t,s)\theta_0(s)ds = 0.
\end{aligned} \tag{5.85}$$

Here the relation

$$\sigma_0 = Ee_0 + \int_0^t G(t,s)e_0(s)ds - \theta_0 - \int_0^t L(t,s)\theta_0(s)ds,$$

is used.

Since equation (5.85) has to be satisfied for any initial functions $e_0(t)$ and $\theta_0(t)$, choosing initial functions $e_0(t) = 0$ and $\theta_0(t) = 0$, equation (5.85) becomes

$$\lambda_{tt} - E\lambda_{xx} + \mu_t - \int_0^t G(t,s)\lambda_{xx}(s)ds + \int_0^t L(t,s)\mu_t(s)ds = 0. \tag{5.86}$$

If we choose the initial function $\theta_0(t) = 0$, then equation (5.85) yields:

$$\begin{aligned} &2E(c_1 - c_3)e_0 - c_4 G(t,0)e_0(0) \\ &+(2c_1 - 3c_3)\int_0^t G(t,s)e_0(s)ds - \int_0^t Z_1(t,s)e_0(s)ds = 0. \end{aligned} \tag{5.87}$$

Finally, equation (5.85) reduces to the equation

$$\begin{aligned} &(c_3 - c_5 + c_6)\theta_0 + c_4 L(t,0)\theta_0(0) \\ &+(2c_3 - c_5 + c_6)\int_0^t L(t,s)\theta_0(s)ds + \int_0^t Z_2(t,s)\theta_0(s)ds = 0. \end{aligned} \tag{5.88}$$

For further study, it is necessary to consider two cases: $G(t,s) \neq 0$ and $G(t,s) = 0$.

Case: $G(t,s) \neq 0$

If $G(t,s) \neq 0$, then we can derive that there exists a function $K_1(t)$ such that

$$Z_1(t,s) = K_1(t)G(t,s). \tag{5.89}$$

In fact, consider the initial function

$$e_0(s) = a_1 + a_2 s + s(t-s)[a_3(t-s)^{n_1} + a_4(t-s)^{n_2}].$$

Then we have

$$\begin{aligned} e_0(0) &= a_1, \quad e_0(t) = a_1 + a_2 t, \\ \int_0^t G(t,s)e_0(s)ds &= a_1\int_0^t G(t,s)ds + a_2\int_0^t sG(t,s)ds \\ &\quad + a_3\int_0^t s(t-s)^{n_1+1}G(t,s)ds + a_4\int_0^t s(t-s)^{n_2+1}G(t,s)ds, \\ \int_0^t Z_1(t,s)e_0(s)ds &= a_1\int_0^t Z_1(t,s)ds + a_2\int_0^t sZ_1(t,s)ds \\ &\quad + a_3\int_0^t s(t-s)^{n_1+1}Z_1(t,s)ds + a_4\int_0^t s(t-s)^{n_2+1}Z_1(t,s)ds. \end{aligned}$$

If there exist numbers $n_i, (i = 1, 2)$ such that the determinant

$$\begin{aligned}\Delta_2 &= \left(\int_0^t s(t-s)^{n_1+1}G(t,s)ds\right)\left(\int_0^t s(t-s)^{n_2+1}Z_1(t,s)ds\right)\\ &\quad -\left(\int_0^t s(t-s)^{n_2+1}G(t,s)ds\right)\left(\int_0^t s(t-s)^{n_1+1}Z_1(t,s)ds\right)\end{aligned}$$

is not equal to zero, then for the given values $\int_0^t Z_1(t,s)e_0(s)ds$, $\int_0^t G(t,s)e_0(s)ds$, $e_0(t)$, $e_0(0)$, we can solve the latter four equations with respect to the coefficients a_1, a_2, a_3, a_4. Thus, the variables $e_0(0)$, $e_0(t)$, $\int_0^t G(t,s)e_0(s)ds$, $\int_0^t Z_1(t,s)e_0(s)ds$ are functionally independent. This means that we can split equation (5.87) with respect to them. Splitting equation (5.87), we obtain contradictory relations. Hence, $\Delta_2 = 0$ for all numbers n_1 and n_2. By virtue of $G(t,s) \neq 0$ we obtain that there exists a function $K_1(t)$ such that $Z_1(t,s) = K_1(t)G(t,s)$.

From this relation, equation (5.87) is rewritten in the form

$$2E(c_1 - c_3)e_0 - c_4G(t,0)e_0(0) + (2c_1 - 3c_3 - K_1(t))\int_0^t G(t,s)e_0(s)ds = 0. \tag{5.90}$$

Splitting (5.90) with respect to $e_0(0)$, $e_0(t)$, $\int_0^t G(t,s)e_0(s)ds$, we obtain

$$c_1 = c_3, \qquad K_1 = -c_3, \tag{5.91}$$

$$c_4G(t,0) = 0. \tag{5.92}$$

From equations (5.89) and (5.92) we derive that $c_4 = 0$. Indeed, if $c_4 \neq 0$, then by virtue of uniqueness of the solution of Cauchy problem of equation (5.89) with the initial data $G(t,0) = 0$, we get that $G(t,s) = 0$, which contradicts the assumption.

Thus, equation (5.89) becomes

$$c_3[tG_t(t,s) + sG_s(t,s) + G(t,s)] = 0.$$

The general solution of the equation

$$tG_t(t,s) + sG_s(t,s) + G(t,s) = 0$$

has the form $G(t,s) = \frac{1}{t}R(\frac{s}{t})$. The kernels of this type are excluded from the study, because they have a singularity at the time $t = 0$. Hence

$$c_3 = 0.$$

From the equalities $c_3 = c_4 = 0$, equation (5.88) reduces to the equation

$$(c_6 - c_5)[\theta_0 + \int_0^t L(t,s)\theta_0(s)ds] = 0.$$

Notice that this equation has to be satisfied for any initial function $\theta_0(t)$. This is only possible if

$$c_5 = c_6.$$

Case: $G(t,s) = 0$

If $G(t,s) = 0$, then $L(t,s) \neq 0$. Similar to the previous case ($G(t,s) \neq 0$) we have that there exists a function $K_2(t)$ such that

$$Z_2(t,s) = K_2(t)L(t,s). \tag{5.93}$$

Hence, equation (5.88) is rewritten in the form

$$\begin{gathered}(c_3 - c_5 + c_6)\theta_0 + c_4L(t,0)\theta_0(0) \\ +(2c_3 - c_5 + c_6 + K_2(t))\int_0^t L(t,s)\theta_0(s)ds = 0.\end{gathered} \tag{5.94}$$

Splitting equation (5.94) with respect to θ_0, $\theta_0(0)$ and $\int_0^t L(t,s)\theta_0(s)ds$, we obtain

$$c_3 = c_5 - c_6, \qquad K_2 = -c_3, \qquad c_4L(t,0) = 0. \tag{5.95}$$

Similarly to the previous case, we can get that $c_3 = c_4 = 0$, and equation (5.87) becomes

$$c_1[Ee_0 + \int_0^t G(t,s)e_0(s)ds] = 0,$$

which implies that $c_1 = 0$.

In other words, in both cases we derive that

$$c_1 = c_3 = c_4 = 0, \qquad c_5 = c_6.$$

Substituting these equalities and system (5.42) into determining equation (5.83), it can be reduced to the equation

$$\mu_{xx} - \lambda_{xx} - \mu_t - \int_0^t L(t,s)\lambda_{xx}(s)ds - \int_0^t c(t,s)\mu_t(s)ds = 0. \tag{5.96}$$

Thus, the components of the infinitesimal generator are as follows

$$\begin{gathered} \xi = c_2, \quad \eta = 0, \quad \zeta^v = c_6 v + \lambda_{xt}, \quad \zeta^\sigma = c_6\sigma + \lambda_{tt}, \\ \zeta^e = c_6 e + \lambda_{xx}, \quad \zeta^\theta = c_6\theta + \mu_t, \quad \zeta^w = c_6 w + \mu_{xx}, \end{gathered} \tag{5.97}$$

Therefore, we obtain that the symmetry groups of equations (5.1) correspond to the Lie algebra L_3 with generators

$$\begin{gathered} X_1 = \partial_x, \\ X_2 = v\partial_v + \sigma\partial_\sigma + e\partial_e + \theta\partial_\theta + w\partial_w, \\ X_3 = \lambda_{tx}\partial_v + \lambda_{tt}\partial_\sigma + \lambda_{xx}\partial_e + \mu_t\partial_\theta + \mu_{xx}\partial_w, \end{gathered} \tag{5.98}$$

where $\lambda(t,x)$, $\mu(t,x)$ is a solution of system (5.86) with (5.96).

5.4 Case (b)

For case (b): $L(t,s) \neq 0$, $c(t,s) = q_1(t)L(t,s)$ for some function $q_1(t)$, but there is no a function $h_1(t)$ such that $G(t,s) = h_1(t)L(t,s)$, using Proposition 1, we can prove that the variables e_1, e_1', k_0 and k_1 are functionally independent. Splitting determining equations (5.16) and (5.17) with respect to e_1, k_0 and k_1, equations (5.19), (5.22)-(5.24) are still valid. By splitting equation (5.18) with respect to e_1, e_1', k_0 and k_1, we also derive equations (5.25), (5.28)-(5.30). Next, the process of solving determining equations is similar to the process presented in the case (a), we only need a small amount of modifications to obtain the same results which is system (5.42).

Because the result obtained in this case is consistent with the result found in the case (a) for solving determining equations (5.11), (5.13), when substituting the result into determining equations (5.14) and (5.15), it does not cause change. Therefore, for this case, the admitted Lie group generators are still X_1, X_2, X_3.

5.5 Case (c)

The linear thermoviscoelastic equations (5.1) without the kernel $L(t,s)$ ($L(t,s) = 0$) has the form (also see (3.20))

$$\begin{gathered} \sigma_x = v_t, \qquad e_t = v_x, \qquad \theta_{xx} = w_t, \\ \sigma = Ee - \theta + \int_0^t G(t,s)e(s)ds, \qquad w = e + \theta + \int_0^t c(t,s)\theta(s)ds. \end{gathered} \tag{5.99}$$

In this case, we have that $k_1 = e_1$, $p_0 = \theta_1$. Because $G(t,s) \neq 0$ and $c(t,s) \neq 0$, it is possible to verify that the variables e_1', e_1, k_0 are functionally independent, and the variables θ_1', θ_1, p_1 are also functionally independent. After simplifying determining equations (5.16) and (5.17) by substituting into the relations $k_1 = e_1$,

$p_0 = \theta_1$, and splitting them with respect to e_1, k_0, θ_1, p_1, the overdetermined system of equations (5.19), (5.22)-(5.24) is also obtained. The property that ξ, η do not depend on the variables v, σ, e, θ and w guarantees equation the determining equation (5.18) is still valid in this case. Splitting equation (5.18) with respect to e_1', e_1, k_0, θ_1', θ_1, p_1, we derive equations (5.25), (5.28)-(5.30).

Notice that the following relations hold in this case:

$$f_2 = f_1' = e_0', \quad g_0' = \theta_0', \quad g_2 = g_1', \quad k_3 = e_2.$$

Equations (5.23), (5.24) and (5.30) reduce to the equations

$$[(\zeta_e^e - \zeta_v^v - \eta_t + \xi_x) + \zeta_w^e]e_0' + \zeta_\sigma^e f_0' = 0, \tag{5.100}$$

$$(\zeta_\theta^e - \zeta_\sigma^e)\theta_0' + \zeta_w^e g_1' = 0, \tag{5.101}$$

$$\begin{aligned}&(\zeta_\sigma^w - \zeta_\theta^w)\theta_0' + (\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t)g_1' + \zeta_w^\theta g_3 + (\zeta_w^\theta + \zeta_e^\theta)e_2 \\ &+\zeta_\sigma^\theta e_0'' + [(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) - \zeta_e^w]e_0' - \zeta_\sigma^w f_0' + \zeta_w^\theta f_3 = 0.\end{aligned} \tag{5.102}$$

Since $c_t(t,s) = 0$ and $c(t,t) = 0$ imply that $c(t,s) = 0$, then $c_t(t,s) \neq 0$ or $c(t,t) \neq 0$. In both cases the variables θ_0' and g_1' can take arbitrary values by choosing the proper initial function $\theta_0(t)$. Similarly, the variables e_0' and f_0' can also take arbitrary values by choosing the proper initial function $e_0(t)$. Splitting equations (5.100) and (5.101) with respect to e_0', f_0' and θ_0', g_1', respectively, we also obtain (5.34), (5.35). This means that equations (5.40) are also established.

In order to solve equation (5.102), consider initial data $e_0(t) = 0$ for which the variables e_0'', e_0', f_0' and f_3 vanish. Then equation (5.102) becomes

$$(\zeta_\sigma^w - \zeta_\theta^w)\theta_0' + (\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t)g_1' + \zeta_w^\theta g_3 + (\zeta_w^\theta + \zeta_e^\theta)\tilde{e}_2 = 0, \tag{5.103}$$

where function $\tilde{e}_2$ satisfies the equation

$$E\tilde{e}_2(t) + \int_0^t G(t,s)\tilde{e}_2(s)ds = g_1'(t). \tag{5.104}$$

Splitting equation (5.103) with respect to θ_0' and g_1', we get that

$$\zeta_\sigma^w - \zeta_\theta^w = 0, \tag{5.105}$$

and

$$(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t)g_1' + \zeta_w^\theta g_3 + (\zeta_w^\theta + \zeta_e^\theta)\tilde{e}_2 = 0. \tag{5.106}$$

The latter equation implies that

$$\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t = 0, \quad \zeta_w^\theta = \zeta_e^\theta = 0. \tag{5.107}$$

Equation (5.102), when substituted into (5.105) and (5.107), reduce to the equation

$$\zeta_\sigma^\theta e_0'' - \zeta_e^w e_0' - \zeta_\sigma^w f_0' = 0. \tag{5.108}$$

Splitting equation (5.108) with respect to e_0'', e_0' and f_0', we have

$$\zeta_\sigma^\theta = \zeta_e^w = \zeta_\sigma^w = 0. \tag{5.109}$$

This means that system (5.41) is also derived.

Consequently, the solution of determining equations (5.16)-(5.18) is still system (5.42) in the present case. Therefore, by studying determining equations (5.14) and (5.15), the admitted generators X_1, X_2, X_3 are obtained.

5.6 Case (d)

A special system of system (5.1) is established by letting $G(t,s) = h_1(t)L(t,s)$ and $c(t,s) = q_1(t)L(t,s)$ for some functions $h_1(t)$, $q_1(t)$. It can be written as follows: (also see (3.21))

$$\begin{gathered}
\sigma_x = v_t, \qquad e_t = v_x, \qquad \theta_{xx} = w_t, \\
\sigma = Ee + h_1 \int_0^t L(t,s)e(s)ds - \theta - \int_0^t L(t,s)\theta(s)ds, \\
w = e + \int_0^t L(t,s)e(s)ds + \theta + q_1 \int_0^t L(t,s)\theta(s)ds.
\end{gathered} \tag{5.110}$$

5.6.1 Solving determining equations for PDE

Since $G(t,s) = h_1(t)L(t,s)$ and $c(t,s) = q_1(t)L(t,s)$, then the variables k_0, e_1, k_1, θ_1, p_0, p_1 have the relations

$$k_0 = (E - h_1)e_1 + h_1 k_1, \quad p_1 = (1 - q_1)\theta_1 + q_1 p_0. \tag{5.111}$$

Substituting these relations into equations (5.16) and (5.17), and splitting them with respect to e_1, k_1, θ_1, p_0, we obtain the following equations

$$\begin{gathered}
\eta_v - \xi_\sigma = q_1 \xi_w, \quad \xi_\theta = (1 - q_1)\xi_w, \quad \xi_e = q_1(E - h_1)\xi_w, \\
\eta_\sigma = -q_1 \eta_w, \quad \eta_\theta = (1 - q_1)\eta_w, \quad \eta_e - \xi_v = q_1(E - h_1)\eta_w,
\end{gathered} \tag{5.112}$$

$$(1 - h_1 q_1)\xi_w = 0, \qquad (1 - h_1 q_1)\eta_w = 0, \tag{5.113}$$

$$\begin{gathered}
z_1 - (1 - q_1)z_3 = 0, \qquad z_2 - q_1 z_3 = 0, \\
(E - h_1)z_2 - z_4 = 0, \qquad -h_1 z_2 + z_3 = 0,
\end{gathered} \tag{5.114}$$

$$\begin{gathered}
r_1 - (1 - q_1)r_3 = 0, \qquad r_2 + q_1 r_3 = 0, \\
(E - h_1)r_2 - r_4 = 0, \qquad h_1 r_2 + r_3 = 0,
\end{gathered} \tag{5.115}$$

$$\begin{gathered}
-\xi_e (e_0')^2 + (\eta_v - \xi_\sigma)e_0' f_0' - \xi_w e_0' f_1' - [\xi_\theta \theta_0' + (\eta_v - \xi_\sigma)g_0' + \xi_w g_1' - (\zeta_e^v - \zeta_v^\sigma \\
-\xi_t)]e_0' + (\zeta_\sigma^v + \eta_x)f_0' + \zeta_w^v f_1' + \zeta_\theta^v \theta_0' - (\zeta_\sigma^v + \eta_x)g_0' + \zeta_w^v g_1' + (\zeta_t^v - \zeta_x^\sigma) = 0,
\end{gathered} \tag{5.116}$$

$$(\xi_v - \eta_e)(e_0')^2 - \eta_\sigma e_0' f_0' - \eta_w e_0' f_1' + [\eta_\sigma g_0' - \eta_\theta \theta_0' - \eta_w g_1'$$
$$+(\zeta_e^e - \zeta_v^v - \eta_t + \xi_x)]e_0' + \zeta_\sigma^e f_0' + \zeta_w^e f_1' + \zeta_\theta^e \theta_0' \tag{5.117}$$
$$-\zeta_\sigma^e g_0' + \zeta_w^e g_1' + (\zeta_t^e - \zeta_x^v) = 0,$$

where

$$z_1 = \eta_\theta f_0' - \eta_\theta g_0' - \zeta_\theta^\sigma,$$
$$z_2 = \eta_e e_0' + \eta_w f_1' + \eta_\theta \theta_0' + \eta_w g_1' - (\zeta_v^v - \zeta_\sigma^\sigma + \xi_x - \eta_t),$$
$$z_3 = \eta_w f_0' - \eta_w g_0' - \zeta_w^\sigma, \qquad z_4 = \eta_e f_0' - \eta_e g_0' - \zeta_e^\sigma, \tag{5.118}$$
$$r_1 = \xi_\theta e_0' - \zeta_\theta^v, \qquad r_2 = \xi_\sigma e_0' - (\zeta_\sigma^v - \zeta_v^e - \eta_x),$$
$$r_3 = \xi_w e_0' - \zeta_w^v, \qquad r_4 = \xi_\sigma f_0' + \xi_w f_1' + \xi_\theta \theta_0' - \xi_\sigma g_0' + \xi_w g_1' + (\zeta_e^v + \xi_t).$$

For further studying these equations, we need to consider the following two cases: $1 - h_1 q_1 \neq 0$ and $1 - h_1 q_1 = 0$.

Case: $1 - h_1 q_1 \neq 0$

For this case, equations (5.113) imply that

$$\xi_w = 0, \qquad \eta_w = 0.$$

Then equations (5.112) become

$$\xi_\theta = \xi_e = 0, \quad \eta_\sigma = \eta_\theta = 0, \quad \eta_v - \xi_\sigma = 0, \quad \eta_e - \xi_v = 0. \tag{5.119}$$

Notice that equations (5.114) consist of homogeneous linear equations for z_i ($i = 1, 2, 3, 4$), and the system of equations (5.115) is also composed of homogeneous linear equations for r_i ($i = 1, 2, 3, 4$). Because the determinants of the coefficient matrices are both $1-h_1q_1$ which is not equal to zero, we have

$$z_1 = z_2 = z_3 = z_4 = 0, \qquad r_1 = r_2 = r_3 = r_4 = 0. \tag{5.120}$$

By virtue of arbitrariness of e_0' and $f_0' - g_0'$, from equations (5.118)-(5.120), it follows that

$$\begin{gathered}\xi_\sigma = 0, \quad \eta_e = 0, \quad \zeta^v_\theta = \zeta^v_w = 0, \quad \zeta^\sigma_e = \zeta^\sigma_\theta = \zeta^\sigma_w = 0, \\ \zeta^v_e + \xi_t = 0, \quad \zeta^e_v - \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_v - \zeta^\sigma_\sigma - \eta_t + \xi_x = 0.\end{gathered} \tag{5.121}$$

Moreover, equations (5.116) and (5.117) reduce to the equations

$$\zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \quad \zeta^e_t - \zeta^v_x = 0, \quad \zeta^v_e - \zeta^\sigma_v - \xi_t = 0, \tag{5.122}$$

$$(\zeta^e_e - \zeta^v_v - \eta_t + \xi_x)e_0' + \zeta^e_\sigma f_0' + \zeta^e_w f_1' = 0, \tag{5.123}$$

$$\zeta^e_\theta \theta_0' - \zeta^e_\sigma g_0' + \zeta^e_w g_1' = 0. \tag{5.124}$$

Notice that the relations $f_0 = (E-h_1)e_0 + h_1 f_1$ and $g_1 = (1-q_1)\theta_0 + q_1 g_0$ hold in this case. The variables f_0', g_1' have the forms

$$\begin{gathered}f_0' = (E-h_1)e_0' + h_1 f_1' - h_1' e_0 + h_1' f_1, \\ g_1' = (1-q_1)\theta_0' + q_1 g_0' - q_1'\theta_0 + q_1' g_0.\end{gathered} \tag{5.125}$$

Substituting them into equations (5.123) and (5.124), they become

$$\begin{gathered}[(\zeta^e_e - \zeta^v_v - \eta_t + \xi_x) + (E-h_1)\zeta^e_\sigma]e_0' \\ +(\zeta^e_w + h_1\zeta^e_\sigma)f_1' - h_1'\zeta^e_\sigma e_0 + h_1'\zeta^e_\sigma f_1 = 0,\end{gathered} \tag{5.126}$$

$$[\zeta^e_\theta + (1-q_1)\zeta^e_w]\theta_0' - (\zeta^e_\sigma - q_1\zeta^e_w)g_0' - q_1'\zeta^e_w\theta_0 + q_1'\zeta^e_w g_0 = 0. \tag{5.127}$$

If there is no function $k(t)$ such that $L_t(t,s) = k(t)L(t,s)$, then the variables e_0', e_0, f_1', f_1 are functionally independent and the variables θ_0', θ_0, g_0', g_0 are also functionally independent. Splitting equations (5.126) and (5.127) with respect to e_0', e_0, f_1', f_1 and θ_0', θ_0, g_0', g_0, respectively, we get that

$$(\zeta^e_e - \zeta^v_v - \eta_t + \xi_x) + (E - h_1)\zeta^e_\sigma = 0, \quad \zeta^e_\theta + (1 - q_1)\zeta^e_w = 0, \tag{5.128}$$

$$h_1\zeta^e_\sigma + \zeta^e_w = 0, \quad \zeta^e_\sigma - q_1\zeta^e_w = 0, \quad h_1'\zeta^e_\sigma = 0, \quad q_1'\zeta^e_w = 0. \tag{5.129}$$

If $L_t(t,s) = k(t)L(t,s)$ for some function $k(t)$, then f_1' has the expression

$$f_1' = e_0' + e_0 - k(t)e_0 + k(t)f_1,$$

Substituting this expression into equations (5.126) and (5.127), and splitting them with respect to e_0', e_0, f_1, we also derive equations (5.128) and (5.129).

Set

$$C = \begin{pmatrix} h_1 & 1 \\ 1 & -q_1 \\ h_1' & 0 \\ 0 & q_1' \end{pmatrix}.$$

If $rank(C) = 2$, then the homogeneous linear system of equations (5.129) only has trivial solution, that is

$$\zeta^e_\sigma = \zeta^e_w = 0.$$

From equations (5.128), it follows that

$$\zeta^e_\theta = 0, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = 0.$$

This means that equations (5.40) have also been obtained.

If $rank(C) = 1$, then equations (5.128) and (5.129) imply that the following relations hold:

$$\zeta^e_\sigma = q_1\zeta^e_w, \quad \zeta^e_\theta = (q_1 - 1)\zeta^e_w,$$

$$\zeta^e_e - \zeta^v_v - \eta_t + \xi_x = -(Eq_1 + 1)\zeta^e_w.$$

Therefore, we obtain

$$\xi_v = \xi_\sigma = \xi_e = \xi_\theta = \xi_w = 0, \quad \eta_v = \eta_\sigma = \eta_e = \eta_\theta = \eta_w = 0,$$

$$\zeta^v_\theta = \zeta^v_w = 0, \quad \zeta^\sigma_e = \zeta^\sigma_\theta = \zeta^\sigma_w = 0, \quad \zeta^e_\sigma = q_1\zeta^e_w, \quad \zeta^e_\theta = (q_1-1)\zeta^e_w,$$

$$\zeta^v_e + \xi_t = 0, \quad \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \quad \zeta^e_t - \zeta^v_x = 0, \tag{5.130}$$

$$\zeta^e_v - \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_e - \zeta^\sigma_v - \xi_t = 0,$$

$$\zeta^v_v - \zeta^\sigma_\sigma - \eta_t + \xi_x = 0, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = -(Eq_1+1)\zeta^e_w.$$

Because ξ, η do not depend on v, σ, e, θ, w, the determining equation (5.18) is still valid. To solve determining equation (5.18) in this case, substituting relations (5.111) into (5.18), and then, splitting equation (5.18) with respect to e_1', e_1, k_1, θ_1', θ_1, p_0, we get that

$$\zeta^\theta_v = 0, \quad \eta_x = 0, \quad \zeta^\theta_{xx} - \zeta^w_t = 0, \tag{5.131}$$

$$\begin{cases} h_1^2\zeta^\theta_{\sigma\sigma} + 2h_1\zeta^\theta_{\sigma w} + \zeta^\theta_{ww} = 0, \\ \zeta^\theta_{\sigma\sigma} + 2q_1\zeta^\theta_{\sigma w} + q_1^2\zeta^\theta_{ww} = 0, \\ h_1\zeta^\theta_{\sigma\sigma} + (1+h_1q_1)\zeta^\theta_{\sigma w} + q_1\zeta^\theta_{ww} = 0, \end{cases} \tag{5.132}$$

$$\begin{cases} h_1(E-h_1)\zeta^\theta_{\sigma\sigma} + h_1\zeta^\theta_{\sigma e} + (E-h_1)\zeta^\theta_{\sigma w} + \zeta^\theta_{ew} = 0, \\ (E-h_1)\zeta^\theta_{\sigma\sigma} + \zeta^\theta_{\sigma e} + q_1(E-h_1)\zeta^\theta_{\sigma w} + q_1\zeta^\theta_{ew} = 0, \end{cases} \tag{5.133}$$

$$\begin{cases} \zeta^\theta_{\sigma\theta} - (1-q_1)\zeta^\theta_{\sigma w} + q_1\zeta^\theta_{\theta w} - q_1(1-q_1)\zeta^\theta_{ww} = 0, \\ h_1\zeta^\theta_{\sigma\theta} - h_1(1-q_1)\zeta^\theta_{\sigma w} + \zeta^\theta_{\theta w} - (1-q_1)\zeta^\theta_{ww} = 0, \end{cases} \tag{5.134}$$

$$\begin{cases} (E-h_1)^2\zeta^\theta_{\sigma\sigma} + 2(E-h_1)\zeta^\theta_{\sigma e} + \zeta^\theta_{ee} = 0, \\ \zeta^\theta_{\theta\theta} - 2(1-q_1)\zeta^\theta_{\theta w} + (1-q_1)^2\zeta^\theta_{ww} = 0, \\ (E-h_1)\zeta^\theta_{\sigma\theta} + \zeta^\theta_{e\theta} - (E-h_1)(1-q_1)\zeta^\theta_{\sigma w} - (1-q_1)\zeta^\theta_{ew} = 0, \end{cases} \tag{5.135}$$

$$\begin{cases} h_1(2\zeta^\theta_{x\sigma} - \zeta^w_v) + (2\zeta^\theta_{xw} + \xi_t) = 0, \\ (2\zeta^\theta_{x\sigma} - \zeta^w_v) + q_1(2\zeta^\theta_{xw} + \xi_t) = 0, \end{cases} \tag{5.136}$$

$$\begin{cases} 2\zeta^\theta_{xe} + (E-h_1)(2\zeta^\theta_{x\sigma} - \zeta^w_v) = 0, \\ (2\zeta^\theta_{x\theta} - \xi_{xx}) - (1-q_1)(2\zeta^\theta_{xw} + \xi_t) = 0, \end{cases} \tag{5.137}$$

and

$$\begin{aligned} &-\zeta^w_\theta\theta_0' + \zeta^w_\sigma g_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)g_1' + \zeta^\theta_w g_3 + \zeta^\theta_w k_3 + \zeta^\theta_e e_2 \\ &+\zeta^\theta_\sigma e_0'' - \zeta^w_e e_0' - \zeta^w_\sigma f_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)f_1' + \zeta^\theta_w f_3 = 0. \end{aligned} \tag{5.138}$$

Since $1 - h_1q_1 \neq 0$, equations (5.132) and (5.136) imply that

$$\zeta^\theta_{\sigma\sigma} = \zeta^\theta_{\sigma w} = \zeta^\theta_{ww} = 0, \quad 2\zeta^\theta_{x\sigma} - \zeta^w_v = 0, \quad 2\zeta^\theta_{xw} + \xi_t = 0. \tag{5.139}$$

From equations (5.133)-(5.135) and (5.137), we can derive that

$$\begin{aligned} \zeta^\theta_{\sigma e} = \zeta^\theta_{\sigma\theta} = 0, \quad \zeta^\theta_{ee} = \zeta^\theta_{e\theta} = \zeta^\theta_{ew} = 0, \\ \zeta^\theta_{\theta\theta} = \zeta^\theta_{\theta w} = 0, \quad \zeta^\theta_{xe} = 0, \quad 2\zeta^\theta_{x\theta} - \xi_{xx} = 0. \end{aligned} \tag{5.140}$$

For studying equation (5.138), consider initial data $e_0(t) = 0$, for which e_0'', e_0', f_0', f_1', f_2 and f_3 vanish. Then equation (5.138) becomes

$$-\zeta^w_\theta\theta_0' + \zeta^w_\sigma g_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t)g_1' + \zeta^\theta_w g_3 + \zeta^\theta_w \bar{k}_3 + \zeta^\theta_e \bar{e}_2 = 0, \tag{5.141}$$

where the function $\bar{e}_2$ is a solution of the equation

$$E\bar{e}_2(t) + h_1(t)\int_0^t L(t,s)\bar{e}_2(s)ds = g_2(t), \tag{5.142}$$

and the function $\bar{k}_3(t)$ has the form

$$\bar{k}_3(t) = \bar{e}_2(t) + \int_0^t L(t,s)\bar{e}_2(s)ds. \tag{5.143}$$

Because the variables g_3, $\bar{k}_3$ and $\bar{e}_2$ are determined by g_1', we hope to retain the variable g_1', it is necessary to consider two cases: $q_1 = 0$ and $q_1 \neq 0$.

If $q_1 = 0$, then $g_3 = g_1' = \theta_0'$, and

$$g_2 = \theta_0' + \int_0^t L(t,s)\theta_0'(s)ds.$$

Equation (5.141) can be rewritten as the form

$$\zeta_\sigma^w g_0' + [(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) - \zeta_\theta^w + \zeta_w^\theta]\theta_0' + \zeta_w^\theta \bar{k}_3 + \zeta_e^\theta \bar{e}_2 = 0, \tag{5.144}$$

Notice that

$$g_0'(t) = \theta_0'(t) + L(t,t)\theta_0(t) + \int_0^t L_t(t,s)\theta_0(s)ds.$$

If $L_t(t,s) \neq 0$, then the variables θ_0' and g_0' can take arbitrary values. If $L_t(t,s) = 0$, then $L(t,t) \neq 0$, because of $L(t,s) \neq 0$. The variables θ_0' and g_0' are still arbitrary. So after splitting equation (5.144) with respect to θ_0' and g_0', we have

$$\zeta_\sigma^w = 0, \tag{5.145}$$

$$[(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) - \zeta_\theta^w + \zeta_w^\theta]\theta_0' + \zeta_w^\theta \bar{k}_3 + \zeta_e^\theta \bar{e}_2 = 0. \tag{5.146}$$

Since $L(t,s) \neq 0$, the variables θ_0' and g_2 are arbitrary. By virtue of $\bar{k}_3$, $\bar{e}_2$ are determined by g_2, and $\bar{k}_3$, $\bar{e}_2$ are functionally independent, from equation (5.146), we find

$$\zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = \zeta^w_\theta. \tag{5.147}$$

Substituting equations (5.145) and (5.147) into equation (5.138), equation (5.138) becomes

$$\zeta^\theta_\sigma e_0'' - \zeta^w_e e_0' + (\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t) f_1' = 0. \tag{5.148}$$

Splitting this equation with respect to e_0'', e_0' and f_1', the following equalities are derived:

$$\zeta^\theta_\sigma = \zeta^w_e = 0, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = \zeta^w_\theta = 0. \tag{5.149}$$

Therefore, equations (5.131), (5.139), (5.140), (5.145), (5.147) and (5.149) imply the overdetermined system of equations (5.41).

Now assume that $q_1 \neq 0$. It is convenient to introduce

$$l_1(t) = \frac{1}{q_1(t)}.$$

The relation among θ_0, g_0 and g_1 is

$$g_0 = (1 - l_1)\theta_0 + l_1 g_1,$$

Then the variable g_0' has the form

$$g_0' = (1 - l_1)\theta_0' + l_1 g_1' - l_1'\theta_0 + l_1' g_1.$$

Substituting this relation into equation (5.141), it becomes

$$\begin{gathered} [-\zeta^w_\theta + (1 - l_1)\zeta^w_\sigma]\theta_0' - l_1'\zeta^w_\sigma \theta_0 + l_1'\zeta^w_\sigma g_1 \\ +[(\zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t) + l_1\zeta^w_\sigma] g_1' + \zeta^\theta_w g_3 + \zeta^\theta_w \bar{k}_3 + \zeta^\theta_e \bar{e}_2 = 0. \end{gathered} \tag{5.150}$$

If $c_t(t, s) \neq R(t)c(t, s)$ for any function $R(t)$. then the variables θ_0', θ_0, g_1' and g_1 are functionally independent. So we can split equation (5.150) with respect to θ_0', θ_0, g_1' and g_1. After splitting we obtain

$$l_1'\zeta_\sigma^w = 0, \tag{5.151}$$

$$\zeta_\theta^w = (1 - l_1)\zeta_\sigma^w, \tag{5.152}$$

$$[(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) + l_1\zeta_\sigma^w]g_1' + \zeta_w^\theta g_3 + \zeta_w^\theta \bar{k}_3 + \zeta_e^\theta \bar{e}_2 = 0. \tag{5.153}$$

If $c_t(t,s) = R(t)c(t,s)$ for some non-zero function $R(t)$, then we have $c(t,t) \neq 0$ and the relation

$$g_1 = \theta_0 + \frac{1}{R(t)}[g_1' - \theta_0' - c(t,t)\theta_0].$$

Splitting equations (5.150) with respect to θ_0', θ_0, g_1', when substituted into last relation, the equations (5.151)-(5.153) are also found.

If $c_t(t,s) = 0$, then $c(t,t) \neq 0$ and the relation $\theta_0' = g_1' - c(t,t)\theta_0$ holds. Substituting this relation into (5.150), and splitting it with respect to θ_0, g_1, g_1', also give the equations (5.151)-(5.153).

Because $c(t,s) = q_1(t)L(t,s)$, then $g_3 = (1 - q_1)g_1' + q_1 g_2$, and equation (5.153) can be rewritten to the equation

$$[(\zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t) + l_1\zeta_\sigma^w + (1 - q_1)\zeta_w^\theta]g_1' + q_1\zeta_w^\theta g_2 + \zeta_w^\theta \bar{k}_3 + \zeta_e^\theta \bar{e}_2 = 0, \tag{5.154}$$

which implies that

$$\zeta_e^\theta = \zeta_w^\theta = 0, \quad \zeta_\theta^\theta - \zeta_w^w - 2\xi_x + \eta_t = -l_1\zeta_\sigma^w. \tag{5.155}$$

Substituting equations (5.125), (5.152) and (5.155) into equation (5.138), it becomes

$$\zeta_\sigma^\theta e_0'' - [\zeta_e^w + (E - h_1)\zeta_\sigma^w]e_0' - (h_1 + l_1)\zeta_\sigma^w f_1' + h_1'\zeta_\sigma^w e_0 - h_1'\zeta_\sigma^w f_1 = 0. \tag{5.156}$$

From the latter equation, it follows that

$$\zeta^{\theta}_{\sigma} = 0, \qquad \zeta^{w}_{e} + (E - h_1)\zeta^{w}_{\sigma} = 0, \tag{5.157}$$

$$(h_1 + l_1)\zeta^{w}_{\sigma} = h_1'\zeta^{w}_{\sigma} = 0. \tag{5.158}$$

Because the condition $rank(C) = 2$ implies that at least one in h_1', l_1', $h_1 + l_1$ is not equal to zero, from equations (5.151) and (5.158), we can deduce that $\zeta^{w}_{\sigma} = 0$. Furthermore,

$$\zeta^{w}_{e} = \zeta^{w}_{\theta} = 0, \qquad \zeta^{\theta}_{\theta} - \zeta^{w}_{w} - 2\xi_x + \eta_t = 0. \tag{5.159}$$

Therefore, the overdetermined system of equations (5.41) is also derived from equations (5.131), (5.139), (5.140), (5.152), (5.155), (5.157) and (5.159).

If $rank(C) = 1$, then $h_1' = l_1' = h_1 + l_1 = 0$, and we obtain the following equations

$$\begin{gathered}
\xi_t = 0, \quad \eta_x = 0, \quad \zeta^{\theta}_{v} = \zeta^{\theta}_{\sigma} = \zeta^{\theta}_{e} = \zeta^{\theta}_{w} = 0, \quad \zeta^{\theta}_{\theta\theta} = 0, \\
\zeta^{w}_{v} = 0, \quad \zeta^{w}_{e} = (h_1 - E)\zeta^{w}_{\sigma}, \quad 2\zeta^{\theta}_{x\theta} - \xi_{xx} = 0, \quad \zeta^{\theta}_{xx} - \zeta^{w}_{t} = 0, \\
\zeta^{w}_{\theta} = (1 + h_1)\zeta^{w}_{\sigma}, \quad \zeta^{\theta}_{\theta} - \zeta^{w}_{w} + 2\xi_x - \eta_t = h_1\zeta^{w}_{\sigma}.
\end{gathered} \tag{5.160}$$

Consequently, for solving determining equations (5.11)-(5.13) in case: $1 - h_1q_1 \neq 0$, we obtain the following results:

(1) If $rank(C) = 2$, then we have equations (5.40) and (5.41). The system of equations (5.42) is the solution of equations (5.40) and (5.41).

(2) If $rank(C) = 1$, then we derive the overdetermined system of equations

(5.130) and (5.160); The general solution of this system is

$$\xi = c_1x + c_2, \quad \eta = c_3t + c_4, \quad \zeta^v = -v(c_1 - c_5) + \lambda_{xt},$$

$$\zeta^\sigma = -\sigma(c_3 - c_5) + \lambda_{tt}, \quad \zeta^e = -e(2c_1 - c_3 - c_5) + H(x,\tau) + \lambda_{xx}. \tag{5.161}$$

$$\zeta^\theta = c_6\theta + \mu_t, \quad \zeta^w = (c_6 - 2c_1 + c_3)w + R(x,\nu) + \mu_{xx}.$$

Here c_i, $(i = 1, 2, \ldots, 6)$ are arbitrary constants, $H(x,\tau)$, $R(x,\nu)$ and $\lambda = \lambda(t,x)$, $\mu = \mu(t,x)$ are arbitrary functions of two arguments, where

$$\tau = q_1\sigma - (Eq_1 + 1)e + (q_1 - 1)\theta + w,$$

$$\nu = \sigma + (h_1 - E)e + (1 + h_1)\theta - h_1w.$$

Because $rank(C) = 1$ implies the relation $1 + h_1q_1 = 0$, using this relation and the last two equations of original system (5.110), we have $\tau = \nu = 0$. This means that the functions $H(x,\tau)$ and $R(x,\nu)$ only depend on x, denote $H(x,\tau) = H(x)$ and $R(x,\nu) = R(x)$. We can assume that $H(x) = R(x) = 0$. Indeed, If $H(x) \neq 0$ and $R(x) \neq 0$, then we consider any functions $\bar{\lambda}(t,x)$ and $\bar{\mu}(t,x)$ having the form

$$\bar{\lambda}(t,x) = \lambda(t,x) + \bar{H}(x), \qquad \bar{\mu}(t,x) = \mu(t,x) + \bar{R}(x),$$

where the functions $\bar{H}(x)$ and $\bar{R}(x)$ satisfy $\bar{H}_{xx}(x) = H(x)$, $\bar{R}_{xx}(x) = R(x)$. Therefore, system (5.161) becomes system (5.42).

Case: $1 - h_1q_1 = 0$

In this case, by arbitrariness of e_0 and θ_0, equations (5.114) imply that

$$\eta_e = (Eq_1 - 1)\eta_w,$$

$$\zeta^\sigma_\theta = (1 - q_1)\zeta^\sigma_w, \quad \zeta^v_v - \zeta^\sigma_\sigma + \xi_x - \eta_t = q_1\zeta^\sigma_w, \quad \zeta^\sigma_e = (Eq_1 - 1)\zeta^\sigma_w, \tag{5.162}$$

and

$$\eta_e e_0' + \eta_w f_1' - q_1 \eta_w f_0' = 0, \quad \eta_\theta \theta_0' + \eta_w g_1' + q_1 \eta_w g_0' = 0. \tag{5.163}$$

From equations (5.163), we can deduce that $\eta_w = 0$. In fact, since $\eta_e = (Eq_1 - 1)\eta_w$ and $\eta_\theta = (1 - q_1)\eta_w$, then equations (5.163) become

$$\eta_w[(Eq_1 - 1)e_0' + f_1' - q_1 f_0'] = 0, \quad \eta_w[(1 - q_1)\theta_0' + g_1' + q_1 g_0'] = 0, \tag{5.164}$$

If $\eta_w \neq 0$, then

$$(Eq_1 - 1)e_0' + f_1' - q_1 f_0' = 0, \tag{5.165}$$

$$(1 - q_1)\theta_0' + g_1' + q_1 g_0' = 0. \tag{5.166}$$

Notice that the relations (5.125), equations (5.165) and (5.166) can be reduced to

$$q_1 h_1'(e_0 - f_1) = 0, \tag{5.167}$$

$$2(1 - q_1)\theta_0' + 2q_1 g_0' - q_1'\theta_0 + q_1' g_0 = 0. \tag{5.168}$$

From equation (5.167), by arbitrariness of $e_0 - f_1$, it follows that $q_1 = 0$ or $h_1' = 0$. If $q_1 = 0$, then equation (5.168) implies that $\theta_0' = 0$ which is impossible. If $h_1' = 0$, then $q_1' = 0$. Hence, equation (5.168) becomes $(1 - q_1)\theta_0' + q_1 g_0' = 0$. Since θ_0' and g_0' are arbitrary, so we get that $1 - q_1 = 0$ and $q_1 = 0$, it is also impossible. Therefore, $\eta_w = 0$. Furthermore, $\eta_\sigma = \eta_e = \eta_\theta = 0$ and $\xi_v = 0$.

Similarly, equations (5.115) yield:

$$\eta_v = 0, \quad \xi_\sigma = \xi_e = \xi_\theta = 0,$$

$$\zeta_\theta^v = (1 - q_1)\zeta_w^v, \quad \zeta_\sigma^v - \zeta_v^e - \eta_x = -q_1 \zeta_w^v, \quad \zeta_\sigma^v + \xi_t = -(Eq_1 - 1)\zeta_w^v. \tag{5.169}$$

Thus, using arbitrariness of e_0 and θ_0, equations (5.116) and (5.117) are split as follows:

$$\zeta_t^v - \zeta_x^\sigma = 0, \quad \zeta_t^e - \zeta_x^v = 0, \tag{5.170}$$

$$(\zeta_e^v - \zeta_v^\sigma - \xi_t)e_0' + (\zeta_\sigma^v + \eta_x)f_0' + \zeta_w^v f_1' = 0,$$
$$\zeta_\theta^v \theta_0' - (\zeta_\sigma^v + \eta_x)g_0' + \zeta_w^v g_1' = 0, \tag{5.171}$$

$$(\zeta_e^e - \zeta_v^v - \eta_t + \xi_x)e_0' + \zeta_\sigma^e f_0' + \zeta_w^e f_1' = 0,$$
$$\zeta_\theta^e \theta_0' - \zeta_\sigma^e g_0' + \zeta_w^e g_1' = 0. \tag{5.172}$$

Substituting the relations (5.125) into equations (5.171), we derive

$$[(\zeta_e^v - \zeta_v^\sigma - \xi_t) + (E - h_1)(\zeta_\sigma^v + \eta_x)]e_0' + [h_1(\zeta_\sigma^v + \eta_x) + \zeta_w^v]f_1'$$
$$-h_1'(\zeta_\sigma^v + \eta_x)e_0 + h_1'(\zeta_\sigma^v + \eta_x)f_1 = 0, \tag{5.173}$$

$$[\zeta_\theta^v + (1 - q_1)\zeta_w^v]\theta_0' - [(\zeta_\sigma^v + \eta_x) - q_1\zeta_w^v]g_0' - q_1'\zeta_w^v\theta_0 + q_1'\zeta_w^v g_0 = 0. \tag{5.174}$$

Because $1 + h_1q_1 \neq 0$ and the general solution of equation $L_t(t,s) = k(t)L(t,s)$ has the form $L(t,s) = m(t)r(s)$ with some functions $m(t)$, $r(s)$, which is a degenerate kernel, we can assume that $L_t(t,s) \neq k(t)L(t,s)$ for any function $k(t)$. Hence, the variables e_0', e_0, f_1', f_1 are functionally independent, and θ_0', θ_0, g_0', g_0 are also functionally independent. Splitting equations (5.173) and (5.174) with respect to e_0', e_0, f_1', f_1 and θ_0', θ_0, g_0', g_0, respectively, we obtain

$$(\zeta_e^v - \zeta_v^\sigma - \xi_t) + (E - h_1)(\zeta_\sigma^v + \eta_x) = 0, \quad \zeta_\theta^v + (1 - q_1)\zeta_w^v = 0, \tag{5.175}$$

$$h_1(\zeta_\sigma^v + \eta_x) + \zeta_w^v = 0, \quad (\zeta_\sigma^v + \eta_x) - q_1\zeta_w^v = 0, \quad h_1'(\zeta_\sigma^v + \eta_x) = 0, \quad q_1'\zeta_w^v = 0. \tag{5.176}$$

Since $1 - h_1q_1 = 0$, then $rank(C) = 2$. From equations (5.176), we get

$$\zeta_\sigma^v + \eta_x = \zeta_w^v = 0,$$

which leads equations (5.175) to

$$\zeta_\theta^v = 0, \qquad \zeta_e^v - \zeta_v^\sigma - \xi_t = 0.$$

Similarly, for studying equations (5.172), we can obtain that

$$\zeta^e_\sigma = \zeta^e_\theta = \zeta^e_w = 0, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = 0.$$

Therefore, for studying determining equations (5.16) and (5.17), we obtain the following equations

$$\begin{gathered}
\xi_v = \xi_\sigma = \xi_e = \xi_\theta = \xi_w = 0, \quad \eta_v = \eta_\sigma = \eta_e = \eta_\theta = \eta_w = 0, \\
\zeta^v_\theta = \zeta^v_w = 0, \quad \zeta^e_\sigma = \zeta^e_\theta = \zeta^e_w = 0, \quad \zeta^\sigma_\theta = (1 - q_1)\zeta^\sigma_w, \\
\zeta^\sigma_e = (Eq_1 - 1)\zeta^\sigma_w, \quad \zeta^v_e + \xi_t = 0, \quad \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_t - \zeta^\sigma_x = 0, \\
\zeta^e_t - \zeta^v_x = 0, \quad \zeta^e_v - \zeta^v_\sigma + \eta_x = 0, \quad \zeta^v_e - \zeta^\sigma_v - \xi_t = 0, \\
\zeta^v_v - \zeta^\sigma_\sigma + \xi_x - \eta_t = q_1\zeta^\sigma_w, \quad \zeta^e_e - \zeta^v_v - \eta_t + \xi_x = 0.
\end{gathered} \tag{5.177}$$

Since ξ, η do not depend on v, σ, e, θ, w, the determining equation (5.18) is still valid in the present case. Thus, equations (5.131)-(5.138) still hold. Because the process of solving equation (5.138) does not depend on the condition $1-h_1q_1 \neq 0$, the solution of equation (5.138) is the same with the previous case. Notice that the condition $1 - h_1q_1 = 0$ implies that $q_1 \neq 0$ and $rank(C) = 2$, equation (5.138) can be reduced to the equations

$$\zeta^\theta_\sigma = \zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^w_\sigma = \zeta^w_e = \zeta^w_\theta = 0, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = 0. \tag{5.178}$$

As a result of these equations, equations (5.132)-(5.134) are trivial, equations (5.135)-(5.137) imply that

$$(Eq_1 - 1)\xi_t = 0, \tag{5.179}$$

$$\zeta^\theta_{\theta\theta} = 0, \qquad \zeta^w_v = q_1\xi_t, \qquad 2\zeta^\theta_{x\theta} - \xi_{xx} = (1 - q_1)\xi_t. \tag{5.180}$$

Therefore, if $Eq_1 - 1 \neq 0$, then we obtain system (5.41). If $Eq_1 - 1 = 0$, then we derive

$$\zeta^\theta_v = \zeta^\theta_\sigma = \zeta^\theta_e = \zeta^\theta_w = 0, \quad \zeta^\theta_{\theta\theta} = 0, \quad \zeta^w_\theta = \zeta^w_\sigma = \zeta^w_e = 0,$$

$$\eta_x = 0, \quad 2\zeta^\theta_{x\theta} - \xi_{xx} = (1 - q_1)\xi_t, \quad \zeta^w_v = q_1\xi_t, \tag{5.181}$$

$$\zeta^\theta_{xx} - \zeta^w_t = 0, \quad \zeta^\theta_\theta - \zeta^w_w - 2\xi_x + \eta_t = 0.$$

Consequently, for solving determining equations (5.11)-(5.13), we obtain the following results:

(1) If $Eq_1 - 1 \neq 0$, then we have the overdetermined system of equations (5.177) and (5.41). Integrating this system, we obtain

$$\xi = c_1 x + c_2, \quad \eta = c_3 t + c_4, \quad \zeta^v = -v(c_1 - c_5) + \lambda_{xt},$$

$$\zeta^\sigma = -\sigma(c_3 - c_5) + K(t, \tau_1) + \lambda_{tt}, \quad \zeta^e = -e(2c_1 - c_3 - c_5) + \lambda_{xx}, \tag{5.182}$$

$$\zeta^\theta = c_6\theta + \mu_t, \quad \zeta^w = (c_6 - 2c_1 + c_3)w + \mu_{xx}.$$

Here c_i, $(i = 1, 2, \ldots, 6)$ are arbitrary constants, $K(t, \tau_1)$ and $\lambda = \lambda(t, x)$, $\mu = \mu(t, x)$ are arbitrary functions of two arguments, where

$$\tau_1 = -q_1\sigma + (1 - q_1)\theta + (Eq_1 - 1)e + w.$$

Remark: For this result, we can state two assumptions: 1^0 q_1 is constant; 2^0 $K(t, \tau_1)$ satisfies $K(t, 0) = 0$. In fact, if $q_1' \neq 0$, then $K(t, \tau_1) = 0$. That is equations (5.182) becomes the equations (5.42). For assumption 2^0, if $K(t, 0) \neq 0$, we consider the following functions:

$$\bar{K}(t, \tau_1) = K(t, \tau_1) - K(t, 0), \qquad \bar{\lambda}(t, x) = \lambda(t, x) + k(t),$$

where the function $k(t)$ satisfies equation $k_{tt}(t) = K(t, 0)$.

(2) If $Eq_1 - 1 = 0$, we derive the overdetermined system of equations (5.177) and (5.181). Integrating them, we obtain

$$\xi = c_7 t + c_1 x + c_2, \quad \eta = c_3 t + c_4,$$

$$\zeta^v = -c_7 e - (c_1 - c_5)v + \lambda_{xt}, \quad \zeta^e = -(2c_1 - c_3 - c_5)e + \lambda_{xx},$$

$$\zeta^\sigma = -(c_3 - c_5)\sigma - 2c_7 v + K_1(t, \nu_1) + \lambda_{tt}, \tag{5.183}$$

$$\zeta^\theta = (\frac{1 - q_1}{2} c_7 x + c_6)\theta + \mu_t,$$

$$\zeta^w = c_7 q_1 v + (\frac{1 - q_1}{2} c_7 x + c_6 - 2c_1 + c_3)w + \mu_{xx}.$$

Here c_i, $(i = 1, 2, \ldots, 7)$ are arbitrary constants, $K_1(t, \nu_1)$ and $\lambda = \lambda(t, x)$, $\mu = \mu(t, x)$ are arbitrary functions of two arguments, where

$$\nu_1 = -q_1\sigma + (1 - q_1)\theta + w.$$

Similar to the previous remark, we can assume that $K_1(t, \nu_1)$ satisfies condition $K_1(t, 0) = 0$.

In other words, the following three different results are obtained by solving determining equations for PDE:

Result 1: system (5.42) for the general case;

Result 2: system (5.182) for case: $1 - h_1 q_1 = 0$, q_1 is constant, but $Eq_1 - 1 \neq 0$;

Result 3: system (5.183) for case: $1 - h_1 q_1 = 0$, $Eq_1 - 1 = 0$.

5.6.2 Solving determining equations for IDE

Because there are three different cases in solving determining equations (5.11) and (5.13) which are related with the partial differential equations of system (5.110), we consider each of the three cases for studying determining equations

(5.14) and (5.15) which are related with integral equations of system (5.110).

For result 1: system (5.42)

For the general case, we have the system (5.42). Omitting the calculations akin to those done in the Sect. 5.3.2, we present the final expression for the group generator:

$$X_1 = \partial_x, \qquad X_2 = v\partial_v + \sigma\partial_\sigma + e\partial_e + \theta\partial_\theta + w\partial_w,$$
$$X_3 = \lambda_{tx}\partial_v + \lambda_{tt}\partial_\sigma + \lambda_{xx}\partial_e + \mu_t\partial_\theta + \mu_{xx}\partial_w, \tag{5.184}$$

where $\lambda(t,x)$, $\mu(t,x)$ is a solution of the equations

$$\lambda_{tt} - E\lambda_{xx} + \mu_t - h_1\int_0^t L(t,s)\lambda_{xx}(s)ds + \int_0^t L(t,s)\mu_t(s)ds = 0, \tag{5.185}$$

$$\mu_{xx} - \lambda_{xx} - \mu_t - \int_0^t L(t,s)\lambda_{xx}(s)ds - q_1\int_0^t L(t,s)\mu_t(s)ds = 0, \tag{5.186}$$

which are the reduced equations of system (5.86) with (5.96) when substituting into the relations $G(t,s) = h_1(t)L(t,s)$, $c(t,s) = q_1(t)L(t,s)$.

For result 2: system (5.182)

For this result, we have $1 - h_1q_1 = 0$, q_1 is constant, but $Eq_1 - 1 \neq 0$. Since ξ also does not depend on t in this solution, then equations (5.81)-(5.84) are still suitable for the present case. The substitution of equations (5.182) into equation (5.81) gives

$$\begin{gathered} K(t,\tau_1) + \lambda_{tt} - E\lambda_{xx} + \mu_t - h_1\int_0^t L(t,s)\lambda_{xx}(s)ds + \int_0^t L(t,s)\mu_t(s)ds \\ +2E(c_1-c_3)e_0 - c_4h_1L(t,0)e_0(0) + (c_3-c_5+c_6)\theta_0 + c_4L(t,0)\theta_0(0) \\ +(2c_1-3c_3)h_1\int_0^t L(t,s)e_0(s)ds - h_1\int_0^t Z_2(t,s)e_0(s)ds \\ +(2c_3-c_5+c_6)\int_0^t L(t,s)\theta_0(s)ds + \int_0^t Z_2(t,s)\theta_0(s)ds = 0, \end{gathered} \tag{5.187}$$

where

$$Z_2(t,s) = L_t(t,s)(c_3t + c_4) + L_s(t,s)(c_3s + c_4),$$

$$\tau_1 = -q_1\sigma_0 + (1-q_1)\theta_0 + (Eq_1 - 1)e_0 + w_0 = 2\left(\theta_0 + q_1\int_0^t L(t,s)\theta_0(s)ds\right). \tag{5.188}$$

Here, the relation $1-h_1q_1 = 0$ and the last two equations of original system (5.110) are used.

By virtue of the arbitrariness of e_0, θ_0, and $K(t,0) = 0$, equation (5.187) is reduced to the three equations:

$$\lambda_{tt} - E\lambda_{xx} + \mu_t - h_1\int_0^t L(t,s)\lambda_{xx}(s)ds + \int_0^t L(t,s)\mu_t(s)ds = 0, \tag{5.189}$$

$$\begin{aligned} &2E(c_1 - c_3)e_0 - c_4h_1L(t,0)e_0(0) \\ &+(2c_1 - 3c_3)h_1\int_0^t L(t,s)e_0(s)ds - h_1\int_0^t Z_2(t,s)e_0(s)ds = 0, \end{aligned} \tag{5.190}$$

$$\begin{aligned} &K(t,\tau_1) + (c_3 - c_5 + c_6)\theta_0 + c_4L(t,0)\theta_0(0) \\ &+(2c_3 - c_5 + c_6)\int_0^t L(t,s)\theta_0(s)ds + \int_0^t Z_2(t,s)\theta_0(s)ds = 0. \end{aligned} \tag{5.191}$$

Because of $L(t,s) \neq 0$ and $h_1 \neq 0$, then equation (5.190) yields:

$$c_1 = c_3 = c_4 = 0,$$

which leads equation (5.191) to

$$K(t,\tau_1) + (c_6 - c_5)[\theta_0 + \int_0^t L(t,s)\theta_0(s)ds] = 0. \tag{5.192}$$

According to the expression (5.188), equation (5.192) can be rewritten in the form

$$K(t,\tau_1) = \frac{(c_5 - c_6)}{2q_1}\tau_1 + (c_5 - c_6)\left(1 - \frac{1}{q_1}\right)\theta_0. \tag{5.193}$$

Since the variables θ_0 and τ_1 are functionally independent, then this equation implies that

$$(c_5 - c_6)\left(1 - \frac{1}{q_1}\right) = 0. \tag{5.194}$$

If $q_1 \neq 1$, then $c_5 = c_6$. Therefore, we obtain the admitted Lie groups corresponding to the generators X_1, X_2, X_3.

If $q_1 = 1$, then

$$K(t, \tau_1) = \frac{(c_5 - c_6)}{2}\tau_1. \tag{5.195}$$

Since $c_1 = c_3 = c_4 = 0$, the components of the infinitesimal generator for η, ζ^e, ζ^θ, ζ^ω have the forms

$$\eta = 0, \quad \zeta^e = c_5 e + \lambda_{xx}, \quad \zeta^\theta = c_6\theta + \mu_t, \quad \zeta^w = c_6 w + \mu_{xx}. \tag{5.196}$$

Substituting these expressions into equation (5.83), equation (5.83) and last equation of the original system (5.110) yield equation (5.186) and

$$c_5 = c_6.$$

Hence, $K(t, \tau_1) = 0$. Therefore, the admitted Lie groups corresponding to the generators X_1, X_2, X_3 are also obtained for the present case.

For result 3: system (5.183)

This result is presented in the case: $1 - h_1 q_1 = 0$, $Eq_1 - 1 = 0$. For this model, we have the expressions

$$G(t, s) = EL(t, s), \qquad c(t, s) = \frac{1}{E}L(t, s),$$

and the system (5.183) is satisfied. It is convenient to write

$$z_0 = \zeta^\sigma + 2v\xi_t = -(c_4 - c_6)\sigma + K_1(t, \nu_1) + \lambda_{tt}. \tag{5.197}$$

By evaluating some integrals by parts, the expanded form of determining equation (5.14) is

$$\begin{aligned}
&z_0 - E\zeta^e + \zeta^\theta - E\int_0^t L(t,s)\zeta^e(s)ds + \int_0^t L(t,s)\zeta^\theta(s)ds \\
&-2v_0\xi_t + E\int_0^t L(t,s)[\xi(s)-\xi(t)]e_1(s)ds - \int_0^t L(t,s)[\xi(s)-\xi(t)]\theta_1(s)ds \\
&-EL(t,0)\eta(0)e_0(0) - E\int_0^t L(t,s)\eta_t(s)e_0(s)ds - E\int_0^t Z_2(t,s)e_0(s)ds \\
&+L(t,0)\eta(0)\theta_0(0) + \int_0^t L(t,s)\eta_t(s)\theta_0(s)ds + \int_0^t Z_2(t,s)\theta_0(s)ds = 0,
\end{aligned} \tag{5.198}$$

Because of the arbitrariness of the functions $v_0(t)$ and $\theta_1(t)$, the last equation can be simplified to two equations (let $v_0(t) = \theta_1(t) = 0$)

$$\begin{aligned}
&z_0 - E\zeta^e + \zeta^\theta - E\int_0^t L(t,s)\zeta^e(s)ds + \int_0^t L(t,s)\zeta^\theta(s)ds \\
&-EL(t,0)\eta(0)e_0(0) - E\int_0^t L(t,s)\eta_t(s)e_0(s)ds - E\int_0^t Z_2(t,s)e_0(s)ds \\
&+L(t,0)\eta(0)\theta_0(0) + \int_0^t L(t,s)\eta_t(s)\theta_0(s)ds + \int_0^t Z_2(t,s)\theta_0(s)ds = 0,
\end{aligned} \tag{5.199}$$

$$\begin{aligned}
&-2v_0\xi_t + E\int_0^t L(t,s)[\xi(s)-\xi(t)]e_1(s)ds \\
&-\int_0^t L(t,s)[\xi(s)-\xi(t)]\theta_1(s)ds = 0,
\end{aligned} \tag{5.200}$$

where

$$Z_2(t,s) = L_t(t,s)\eta(t) + L_s(t,s)\eta(s).$$

For simplifying equation (5.200), we choose initial functions $v_0(t) = \dfrac{1}{2}$ and $\theta_1(t) = 0$ (then $e_1(t)$ vanishes). Equation (5.200) becomes $\xi_t = 0$, which implies that

$$c_7 = 0.$$

Substituting system (5.183) (when $c_7 = 0$) into equation (5.199), it is simplified to the form

$$K_1(t,\nu_1)+\lambda_{tt}-E\lambda_{xx}+\mu_t-E\int_0^t L(t,s)\lambda_{xx}(s)ds+\int_0^t L(t,s)\mu_t(s)ds$$
$$+2E(c_1-c_3)e_0-c_4EL(t,0)e_0(0)$$
$$+(2c_1-3c_3)E\int_0^t L(t,s)e_0(s)ds-E\int_0^t Z_2(t,s)e_0(s)ds \tag{5.201}$$
$$+(c_3-c_5+c_6)\theta_0+c_4L(t,0)\theta_0(0)$$
$$+(2c_3-c_5+c_6)\int_0^t L(t,s)\theta_0(s)ds+\int_0^t Z_2(t,s)\theta_0(s)ds=0,$$

where

$$\nu_1=-\frac{1}{E}\sigma_0+\left(1-\frac{1}{E}\right)\theta_0+w_0=2\left(\theta_0+\frac{1}{E}\int_0^t L(t,s)\theta_0(s)ds\right). \tag{5.202}$$

Using the arbitrariness of e_0 and θ_0, from equation (5.201), it follows that

$$\lambda_{tt}-E\lambda_{xx}+\mu_t-E\int_0^t L(t,s)\lambda_{xx}(s)ds+\int_0^t L(t,s)\mu_t(s)ds=0, \tag{5.203}$$

$$\begin{gathered}2(c_1-c_3)e_0-c_4L(t,0)e_0(0)\\+(2c_1-3c_3)\int_0^t L(t,s)e_0(s)ds-\int_0^t Z_2(t,s)e_0(s)ds=0,\end{gathered} \tag{5.204}$$

$$\begin{gathered}K_1(t,\nu_1)+(c_3-c_5+c_6)\theta_0+c_4L(t,0)\theta_0(0)\\+(2c_3-c_5+c_6)\int_0^t L(t,s)\theta_0(s)ds+\int_0^t Z_2(t,s)\theta_0(s)ds=0.\end{gathered} \tag{5.205}$$

Because of $L(t,s)\neq 0$, then equation (5.204) gives:

$$c_1=c_3=c_4=0.$$

Hence, equation (5.205) reduces to

$$K_1(t,\nu_1)+(c_6-c_5)[\theta_0+\int_0^t L(t,s)\theta_0(s)ds]=0. \tag{5.206}$$

According to the expression (5.202), equation (5.206) can be rewritten in the form

$$K_1(t, \nu_1) = \frac{E(c_5 - c_6)}{2}\nu_1 + (c_5 - c_6)\,(1 - E)\,\theta_0. \tag{5.207}$$

Since the variables θ_0 and ν_1 are functionally independent, then this equation implies that

$$(c_5 - c_6)(1 - E) = 0. \tag{5.208}$$

If $E \neq 1$, then $c_5 = c_6$. Therefore equations (5.110) admit Lie groups corresponding to the generators X_1, X_2, X_3. If $E = 1$, then $K_1(t, \nu_1) = \dfrac{(c_5 - c_6)}{2}\nu_1$.

Since $c_1 = c_3 = c_4 = c_7 = 0$, equation (5.83) is still valid. This equation and last equation of the original system (5.110) yield equation (5.186) and

$$c_5 = c_6.$$

Hence, $K_1(t, \nu_1) = 0$. Therefore, the admitted Lie groups corresponding to the generators X_1, X_2, X_3 are also obtained for the present case.

CHAPTER VI

CONCLUSION

In this thesis, we have studied two models which describe the behavior of one-dimensional viscoelastic materials with memory by using the Lie group analysis method. These models are presented in integro-differential forms, that is, the constitutive equations include nonlocal terms (integral terms). The classical Lie group analysis method could not be used for these equations. A recently developed approach for calculating an admitted Lie group of integro-differential equations was applied in the thesis.

The first model considered in the thesis is a nonlinear stress relaxation model, which is described by a system of integro-differential equations (4.1). Determining equations of system (4.1) were constructed by applying the definition of admitted Lie group for integro-differential equations. The general solution of the determining equations gave us a complete group classification of equations (4.1) with respect to the function $\varphi(\sigma)$ and the kernel $H(t,\tau)$. The group classification separates all models into three classes: (a) the linear function $\varphi(\sigma) = E\sigma$; (b) the function $\varphi(\sigma) = \alpha\exp(\gamma\sigma) + \beta$, $(\alpha\gamma \neq 0)$; (c) the function $\varphi(\sigma) = \alpha\sigma^{\beta} + \gamma$, $(\alpha\beta(\beta-1) \neq 0)$. Along with the group classification, representations of all invariant solutions and reduced equations are constructed.

The second model studied in the thesis is the model extending linear viscoelastics model to non-isotermal situation (5.1). The determining equations for system (5.1) are still integro-differential. To solve these equations two propositions are proven. According to these propositions, the study is separated into equations four different cases: (a) $L(t,s) \neq 0$ and $c(t,s) \neq q_1(t)L(t,s)$ for any function $q_1(t)$; (b) $L(t,s) \neq 0$, $c(t,s) = q_1(t)L(t,s)$ for some function $q_1(t)$, but there is no a function $h_1(t)$ such that $G(t,s) = h_1(t)L(t,s)$; (c) $L(t,s) = 0$; (d) $c(t,s) = q_1(t)L(t,s)$ and $G(t,s) = h_1(t)L(t,s)$ for some functions $q_1(t)$, $h_1(t)$. In each case, the symmetry groups of equations (5.1) correspond to the Lie algebra with generators X_1, X_2, X_3.

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