OPTION PRICING MODEL FOR JUMP-DIFFUSION WITH STOCHASTIC VOLATILITY

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แบบจำลองการประเมินราคาออปชันสำหรับการแพร่อย่างกระโดด ที่มีความผันผวนแบบสโทแคสติก

นางสาวนธิยา มากะเต

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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OPTION PRICING MODEL FOR JUMP-DIFFUSION WITH STOCHASTIC VOLATILITY

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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นนธิยา มากะเต : แบบจำลองการประเมินราคาออปชันสำหรับการแพร่อย่างกระโดดที่มีความผันผวนแบบสโทแคสติก (OPTION PRICING MODEL FOR JUMP-DIFFUSION WITH STOCHASTIC VOLATILITY) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร.ไพโรจน์ สัตยธรรม, 86 หน้า.

วิทยานิพนธ์ฉบับนี้เสนอการพิจารณาตัวแบบสำหรับการแพร่อย่างกระโดดที่มีความผันผวนแบบสโทแคสติกโดยรวมการกระโดดในความผันผวนแบบสโทแคสติก ในลำดับแรกได้กล่าวถึงพลศาสตร์ของราคาสินทรัพย์โดยที่สินทรัพย์มีการเคลื่อนที่แบบบราวเนียนเรขาคณิตพร้อมด้วยกระบวนการปัวซงเชิงประกอบที่มีความผันผวนแบบสโทแคสติกตามตัวแบบเฮสตันโดยมีการรวมการกระโดดในความผันผวนแบบสโทแคสติก ต่อมาศึกษาราคาสินทรัพย์ที่เคลื่อนที่การแพร่แบบกระโดดที่มีการผันกลับค่าเฉลี่ยและมีความผันผวนแบบสโทแคสติกซึ่งรวมการกระโดดในความผันผวนแบบสโทแคสติก ต่อจากนั้นได้มีการนำเสนอวิธีการสร้างสูตรสำหรับสิทธิเลือกที่จะซื้อแบบยุโรปด้วยวิธีหาฟังก์ชันลักษณะเฉพาะของสินทรัพย์อ้างอิง ซึ่งสามารถแสดงได้สูตรที่ชัดเจน ในท้ายที่สุดได้แสดงวิถีของตัวอย่างของแบบจำลองเปรียบเทียบกับข้อมูลที่แท้จริง

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STOCHASTIC VOLATILITY / JUMP-DIFFUSION MODEL/ OPTION PRICING/ CHARACTERISTIC FUNCTION/ MEAN REVERTING PROCESS

In this thesis, a jump-diffusion model with stochastic volatility and additional jumps in volatility is considered. Firstly, we present the dynamics of the asset price, in which the asset price follows a geometric Brownian motion combined with a compound Poisson process and the stochastic volatility follows the Heston model with jumps incorporated into the stochastic volatility. Secondly, the asset price follows mean reverting jump-diffusion with jumps in stochastic volatility. The formula of the European option is calculated by using the technique based on the characteristic function of an underlying asset which can be expressed in an explicit formula. Finally, a simulation example shows a sample path of the model as compared to the actual data.

School of Mathematics	Student's Signature________________
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CHAPTER I

INTRODUCTION

Financial derivatives are a kind of risk management instrument. A derivative's value depends on the price changes in some of the underlying assets. Many forms of financial derivatives instruments exist in the financial markets. Among them, the three most fundamental financial derivatives instruments are: forward contracts, futures, and options. If the underlying assets are stocks, bonds, foreign exchange rates and commodities etc., then the corresponding risk management instruments are: stock futures (options), bond futures (options), currency futures (options) and commodity futures (options) etc. In risk management of the underlying assets using financial derivatives, the basic strategy is hedging, i.e., the trader holds two positions of equal amounts but opposite directions, one in the underlying markets, and the other in the derivatives markets, simultaneously. This risk management strategy is based on the following reasoning: it is believed that under normal circumstances, prices of underlying assets and their derivatives change roughly in the same direction with basically the same magnitude; hence losses in the underlying assets (derivatives) markets can be offset by gains in the derivatives (underlying assets) markets; therefore losses can be prevented or reduced by combining the risks due to the price changes. The subject of this thesis is the pricing of financial derivatives and risk management by hedging.

1.1 Introduction to the Option Pricing Problem

An option is an agreement that the holder can buy from, or sell to, the seller of the option at a specified future time a certain amount of an underlying asset at a specified price. But the holder is under no obligation to exercise the contract. The holder of an option has the right, but not the obligation, to carry out the agreement according to the terms specified in the agreement. In an option contract, the specified price is called *the exercise price* or *strike price*, the specified date is called *the expiration date* or *maturity date* and the action to perform the buying or selling of the asset according to the option contract is called *exercise*. According to buying or selling an asset, options have the following types:

- *call option* is a contract to buy at a specified future time a certain amount of an underlying asset at a specified price.
- *put option* is a contract to sell at a specified future time a certain amount of an underlying asset at a specified price.

According to terms on exercise in the contract, options have the following types:

- *European options* can be exercised only on the expiration date.
- *American options* can be exercised on or prior to the expiration date.

Denote by K and T the strike price and expiration date respectively; then an option's payoff (value) $C(T, S)$ at expiration date is:

$$(S_T - K)^+ = \max(S_T - K, 0) \qquad \text{(call option)}$$

$$(K - S_T)^+ = \max(K - S_T, 0) \qquad \text{(put option)}$$

where S_T denotes the price of the underlying asset at the expiration date $t = T$. An option is a contingent claim. Take a call option as example. If S_T, the underlying asset's price at expiration date, is higher than the strike price K, then the holder of the option can exercise the rights to buy the asset at the strike price K (to gain profits). Otherwise, the option is a worthless. That is

$$C(T, S_T) = \begin{cases} S_T - K & \text{if} \quad S_T > K \\ 0 & \text{otherwise.} \end{cases}$$

In the case of $S_T > K$, the option is called ”in the money”. It is said to be ”out of the money” if $S_T < K$. If $S_T = K$, it is ”at the money”. Similarly, the payoff function is $(K - S_T)^+$ for a European put option.

The price paid for a contingent claim is called *the premium.* When the option is traded on an organized market, the premium is quoted by the market. Otherwise, the problem is to price the option. Also, even if the option is traded on an organized market, it can be interesting to detect some possible abnormalities in the market.

Taking into account the premium , the total gain of the option holder at its expiration date is

[Total gain] = [Gain of the option at expiration] - [Premium]

i.e.,

$$\text{Total gain} = (S_T - K)^+ - \text{premium} \qquad \text{(call option)}$$

$$\text{Total gain} = (K - S_T)^+ - \text{premium} \qquad \text{(put option)}$$

As a derived security, the price of an option varies with the price of its underlying asset. Since the underlying asset is a risky asset, its price is a random variable.

Therefore the price of any option derived from it is also random. However, once the price of the underlying asset is set, the price of its derived security (option) is also determined, i.e., if the price of an underlying asset at time t is S_t, the price of the option is C_t, then there exists a function $C(t, S)$ such that

$$C_t = C(t, S_t)$$

where $C(t, S)$ is a deterministic function of two variables. Our task is to determine this function by establishing a model of partial differential equations.

C_T, an option's value at expiration date, is already set, which is the option's payoff:

$$C_T = \begin{cases} (S_T - K)^+ & \text{(call option)} \\ (K - S_T)^+ & \text{(put option)} \end{cases}$$

The problem of option pricing :

1. To find $C = C(t, S)$, $(0 < S < \infty, 0 < t < T)$, such that

$$C(T, S) = \begin{cases} (S_T - K)^+ & \text{(call option)} \\ (K - S_T)^+ & \text{(put option)} \end{cases}$$

   In particular, if a stock's price at the option's initial date $t = 0$ is S_0, we want to know how much to pay for the premium.

2. How do we model the underlying asset on a stock price?

1.2 Random Walk of Asset Prices

In the research on option pricing, the dynamics of the asset price is usually represented by its relative change, $\frac{dS_t}{S_t}$, called return. The most common model, geometric Brownian motion model (GBM), says that the return of the asset price is made up of two parts as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1.1}$$

where μ, known as the drift, marks the average rate of growth, and σ is called volatility that keeps the information of the standard deviation of the return. The first part μdt reflects a predictable, deterministic and anticipated return which is similar to the return of the investment in banks. The second part σdW_t simulates the random change in the asset price in response to external effects, such as uncertain events. The quantity dW_t contains the information of the randomness of the asset price and is known as Wiener process or Brownian motion. It is a random variable which follows a normal distribution, with mean zero and variance dt. This means that dW_t can be written as $dW_t = \phi\sqrt{dt}$. Here ϕ is a random variable with a standardized normal distribution. Its probability density function is given by

$$f(\phi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2 d\phi}$$

for $-\infty < \phi < \infty$.

Equation (1.1) is known as *Black-Scholes model* or *diffusion model.*

By using Itô's formula (for details see Section 2.6), equation (1.1) implies that

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right). \tag{1.2}$$

1.3 Outline of the Thesis

To attain the major objective, we give a brief outline of how we intend to proceed and what each chapter contains. The thesis is organized as follows.

In Chapter II, we introduce some notation, terminology and some mathematical tools which are used in the main theorems.

In Chapter III, we consider the problem of finding a closed-form formula for a European call option where the underlying asset follows jump-diffusion and the stochastic volatility follows mean reverting process with jump. This formula will be useful for option pricing rather than an estimation of it as appeared in Eraker's work (2003).

In Chapter IV, we consider the problem of finding a closed-form formula for a European call option where the asset price follows mean reverting jump-diffusion and the stochastic volatility has jumps. We briefly discuss model descriptions for option pricing. Deriving a formula for a characteristic function is presented. A closed-form formula for a European call option in terms of characteristic functions is presented.

The conclusion of the thesis is presented in the last chapter.

CHAPTER II

PRELIMINARIES

2.1 The Black-Scholes Model

In 1973, Black and Scholes tackled the problem of pricing and hedging a European option on a non-dividend paying stock. We briefly explain the main result. Firstly, we make the following assumptions.

- There is no arbitrage opportunity.
- The risk free interest rate is deterministic and equal to $r > 0$.
- The transactions do not incur any fees or costs (i.e. frictionless market).
- The underlying security does not pay a dividend.
- Under the real world measure the stock price process follows geometric Brownian motion (1.1).

Suppose that the above assumptions hold. Standard derivative pricing theory offers two ways for computing the fair value $C(t, S_t)$ of a European call option at time $t \leq T$. Under the partial differential equation (PDE) approach the function $C(t, S)$ is computed by solving the PDE,

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} + rs\frac{\partial C}{\partial s} - rC = 0, \quad \text{for } t \in [0, T]. \tag{2.1}$$

This is the famous Black-Scholes PDE of European call option.

In order to obtain a unique solution for the Black-Scholes PDE we must consider final and boundary conditions. We will restrict our attention to a European call option, $C(t,s)$. At maturity, $t=T$, a call option is worth:

$$C(T,s)=\max(S_T-K,0)$$

where K is the exercise price. So this will serve as the final condition.

The asset price boundary conditions are applied at $s=0$ and also as $s\to\infty$.

If $s=0$ then ds is also zero and therefore s can never change. This implies at $s=0$ we have:

$$C(t,0)=0.$$

Obviously, if the asset price increase without bound as $s\to\infty$, the value of the option becomes that of the asset:

$$C(t,s)\approx s,\quad s\to\infty.$$

The European call option $C(t,S_t)$ is computed by solving the final boundary value problem:

$$\begin{cases} \frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 C}{\partial s^2}+rs\frac{\partial C}{\partial s}-rC=0, \quad \text{for } t\in[0,T] \\ C(t,0)=0 \\ C(t,s)\approx s, s\to\infty \\ C(T,s)=\max(S_T-K,0). \end{cases} \tag{2.2}$$

Alternatively, the value $C(t,S_t)$ can be computed as the expectation of the discounted payoff under the risk-neutral measure Q, the so-called *risk-neutral pricing* approach. Under Q, the process S_t satisfies the stochastic differential equation (SDE)

$$\frac{dS_t}{S_t}=rdt+\sigma d\tilde{W}_t \tag{2.3}$$

for a standard Brownian motion $\tilde{W}_t$. In particular, the drift μ in equation (1.1) has been replaced by risk-free interest rate r. The risk-neutral pricing rule now states that

$$C(t,S_t) = E_Q\left[e^{-r(T-t)}\max\left(S_T-K,0\right)|\mathcal{F}_t\right] \tag{2.4}$$

where E_Q denotes expectation with respect to Q.

To obtain the analytical formula for the option price, we compute this expectation which is in fact the computation of an integral.

$$\begin{aligned} C(t,S_t) &= E_Q\left[e^{-r(T-t)}\max\left(S_T-K,0\right)|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\int_K^\infty (S_T-K)f_{S_T}(s)ds, \end{aligned} \tag{2.5}$$

where $f_{S_T}(\cdot)$ is the probability density function of S_T under the risk-neutral probability.

The solution of PDE (2.2), or the risk-neutral value of stock price obtained from equation (2.5), is given by

$$C(t,S_t;r,\sigma,T,K) = S_t\Phi(d_1)-Ke^{-r(T-t)}\Phi(d_2), \tag{2.6}$$

where

$$\begin{aligned} d_1 &= \frac{\ln S_t-\ln K+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1-\sigma\sqrt{T-t} \end{aligned}$$

and Φ is the cumulative distribution function for the standard normal distribution. The equation (2.6) is known as Black-Scholes formula for a European call option. Similarly, the price for a European put option is:

$$P(t,S_t;r,\sigma,T,K) = Ke^{-r(T-t)}\Phi(-d_2)-S_t\Phi(-d_1).$$

2.2 Characteristic Functions of a Random Variable

The characteristic function of a random variable is the Fourier transform of its distribution. Many probabilistic properties of random variables correspond to analytical properties of their characteristic functions, making this concept very useful for studying random variables.

Definition 2.1. *(A Characteristic function)*
The characteristic function of the $\Re^d$-valued random variable X is the function $\psi_X : \Re^d \to \Re$ defined by

$$\psi_X(t) = E[e^{itX}] = E\left[cos(tX)\right] + iE\left[sin(tX)\right]. \tag{2.7}$$

Let F_X be the distribution function of X. Then

$$\psi_X(t) = E(e^{itX}) = \int_{-\infty}^{+\infty} e^{itx} dF(x)$$

so that ψ is the Fourier transform of F.

The characteristic function of a random variable determines the probability distribution: two variables with the same characteristic function are identically distributed. A characteristic function is always continuous and verifies

$$\psi_X(0) = 1$$

$$|\psi_X(t)| \leq 1$$

$$\psi_{aX+b}(t) = e^{itb}\psi_X(at).$$

Normal Distribution

The normal distribution, $N(\mu, \sigma^2)$ is (one of) the most important distributions. As seen before, its characteristic function is given by:

$$\psi(z; \mu, \sigma^2) = e^{i\mu z - \frac{1}{2}\sigma^2 z^2}$$

and the density function is:

$$f(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}.$$

The normal, by definition, is symmetric around its mean, has a skewness equal to 0 and a kurtosis equal to 3.

Example 2.1. *(Poisson Characteristic Function)*

For a Poisson distribution $P(\lambda)$, we can define the probability mass and characteristic function as:

$$f(k) := P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\psi_X(z) = e^{-\lambda(1-e^{iz})}.$$

2.3 Arbitrage and Martingales

In the theory of option pricing, one fundamental and essential concept is arbitrage. Formally speaking, it states that there is never any opportunity to make an instantaneous risk-free profit. More correctly, such opportunities cannot exist for a significant length of time before prices move to eliminate them. Almost all financial theories assume the existence of risk-free investments that give a guaranteed return with no chance of default, e.g. a government bond or a deposit in a bank. The greatest risk-free return that one can make on a portfolio of assets is the same as the return if the equivalent amount of cash were placed in a bank. In the definition of arbitrage, the key words are instantaneous and risk-free. This means, by investing in equities, one can probably beat the bank, but this cannot be certain, if one wants a greater return, one must accept a greater risk. In the binomial model, if r is the spot rate and the stock price process can be represented as

$$S_0 = s$$
$$S_1 = sZ$$

where Z is a stochastic variable defined as

$$\begin{cases} Z = u & \text{with probability } p_u \\ Z = d & \text{with probability } p_d \end{cases}$$

where $u > d$ and $p_u + p_d = 1$, then free of arbitrage results in

$$d \leq (1 + r) \leq u.$$

The arbitrage theory leads to the definition of the risk-neutral measure, or martingale measure: a probability measure Q is called *martingale* if the following condition holds

$$S_0 = \frac{1}{1+r} E_Q[S_1].$$

The risk-neutral measure is the basis of the valuation in this thesis.

The principle of risk-neutral valuation just replaces the drift of the historical (real) asset process with the risk-free interest rate. For a better comparison, we recall both processes.

Historical asset process:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Risk-neutral asset process:

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t$$

An implication of the risk-neutral process is that the discounted asset price is a martingale. We will briefly show there exists an equivalent martingale measure between the historical asset process and its risk-neutral counterpart. In other words, the risk-neutral valuation implies an equivalent martingale measure. We first define a martingale and an equivalent martingale measure in more details.

Definition 2.2. *A stochastic process X_t is a martingale based on a filtration $F = (\mathcal{F}_t)_{t\geq 0}$ if it satisfies the following three conditions*

1. X_t *is* $\mathcal{F}_t-$ *measurable.*

2. $E[X_t|\mathcal{F}_t] < \infty$.

3. $E[X_s|\mathcal{F}_t] = X_t, \quad s \geq t$.

A stochastic process is always associated with a measure that characterizes the distribution law of increments. We denote P and Q as the measures for the historical and the risk-neutral processes respectively. In fact, the measure Q of a risk-neutral process with respect to the measure P for any event is always continuous, this relation establishes an equivalence between two measures. Generally, given a measurable space (Ω, Σ), two measures P and Q are equivalent, if

$$P(A) > 0 \Rightarrow Q(A) > 0, \quad \forall A \in \Sigma$$

and

$$P(A) = 0 \Rightarrow Q(A) = 0.$$

Using two equivalent measures, we could define a Radon-Nikodym derivative,

$$M = \frac{dQ}{dP},$$

which enables us to change a measure to another. It follows immediately

$$\begin{aligned} E_P[XM] &= \int_\Omega XMdP \\ &= \int_\Omega XdQ \\ &= E_Q[X]. \end{aligned}$$

The Girsanov theorem gives us some concrete instructions to change the measures for an Itô process.

Theorem 2.1. *(The Girsanov theorem)*
Given a measurable space (Ω, F, P), and an Itô process X_t,

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t.$$

Denote M_t as a (an exponential) martingale under the measure P,

$$M_t = \exp\left(-\frac{1}{2}\int_0^t \gamma^2(u)du + \int_0^t \gamma(u)dW_M(u)\right)$$

with

$$E_P[M_t] = 1.$$

Additionally, W and W_M are correlated with $dWdW_M = \rho dt$. Then we have the following results:

1. *M_t defines a Radon-Nikodym derivative*

$$M_t = \frac{dP^*}{dP}(t)$$

2. *If we define*

$$dW_t^* = dW_t - \gamma_t(dW_t dW_{M_t}) = dW_t - \rho\gamma_t dt,$$

   it is then a new Brownian motion in (Ω, F, P^*).

3. The Itô process X_t may take a new form under P^*,

$$\begin{aligned} X_t &= a(X_t,t)dt + b(X_t,t)dW_t^* \\ &= a(X_t,t)dt + b(X_t,t)[dW_t - \rho\gamma_t dt] \\ &= [a(X_t,t) - \rho\gamma_t b(X_t,t)]dt + b(X_t,t)dW_t. \end{aligned}$$

We apply the Girsanov theorem to verify the equivalent measures between the historical stock process and the risk-neutral stock process. To this end, we construct a Radon-Nikodym derivative M_t as follows

$$\begin{aligned} M_t &= \frac{dQ}{dP}(t) \\ &= \exp\left(-\frac{1}{2}\int_0^t \gamma^2 du + \int_0^t \gamma dW_u\right) \end{aligned}$$

with

$$\gamma = \frac{\mu - r}{\sigma}.$$

Thus we have $W_t = W_M(t)$. The term γ may be interpreted as an excess return measured in volatility. Therefore under the measure Q the Brownian motion W_t^* is equal to

$$\begin{aligned} dW_t^* &= dW_t - \gamma dt \\ &= dW_t - \frac{\mu - r}{\sigma}dt, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma[dW_t - \frac{\mu - r}{\sigma}dt] \\ &= rdt + \sigma dW_t \end{aligned}$$

that is identical to the risk-neutral process. In this sense, the measure Q is called the *risk-neutral measure*, and is equivalent to the historical statistical measure P.

2.4 The Shortcomings of the Black-Scholes Model

Since the Black-Scholes model uses the geometric Brownian motion, there are shortcomings of this model, such as

(1) the asymmetric leptokurtic features, that is, the return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution

(2) the volatility smile, that is, the implied volatility is not a constant as assumed in Black-Scholes model

(3) the large random fluctuations such as crashes and rallies.

Therefore, many financial engineering studies have been undertaken to modify and improve the Black-Scholes formula to explain some or all of the above three empirical phenomena. The supporting details will be discussed later in the thesis.

We note that the tail of the distribution is where the extreme values occur. Empirical distributions for stock prices and returns have found that extreme values are more likely than would be predicted by the normal distribution. This means that, between periods where the market exhibits relatively modest changes in prices and returns, there will be periods where there are changes that are much higher (crashes and booms) than predicted by the normal distribution. This is not only of concern to financial theorists, but also to practitioners. However, heavy or fat tails can help explain larger price fluctuations for stocks over short time periods than can be explained by changes in fundamental economic variable.

2.4.1 Implied Volatility

It is possible to deduce the *implied volatility* of call (or put) options by solving the reverse Black-Scholes equation, that is, find the volatility that would equal the Black-Scholes price to the market price of the option. This is a good way to see how derivatives markets perceive the underlying volatility.

More precisely, using Black-Scholes option pricing, call options C are a function of $C(t, S_t)$ where t is the time at which C is being priced, T is the expiration date, r is the risk free rate of return, and K is the strike price. Note that all the independent variables are observable except σ. Since the quoted option price C^{obs} is observable, using the Black-Scholes formula we can therefore calculate or imply the volatility that is consistent with the quoted options prices and observed variables. We can therefore define implied volatility I by:

$$C_{BS}(t, S; r, I, T, K) = C^{obs},$$

where C_{BS} is the option price calculated by the Black-Scholes equation (2.6).

Implied volatility surfaces are graphs plotting I for each call option strike K and expiration T. Theoretically options whose underlying asset is governed by geometric Brownian motion should have a flat implied volatility surface, since volatility is a constant; however in practice the implied volatility surface is not flat and I varies with K and T.

Implied volatility plotted against strike prices from empirical data tends to vary in a "u-shaped" relationship, known as the *volatility smile*, with the lowest value normally at $S = K$ (called "at the money" options). The opposite graph shape to a volatility smile is known as a volatility frown due to its shape. The smile curve has become a prominent feature since the 1987 October crash (see for instance Bates (2000)).

2.4.2 Jump-Diffusion

Some authors try to explain the volatility smile and the leptokurticity by changing the underlying stock distribution from a diffusion process to a jump-diffusion process. For jump-diffusion models, the normal evolution of price is given by a diffusion process, punctuated by jumps at random intervals. Here the jumps represent rare events crashes and large drawdowns. Such an evolution can be represented by modeling the price as a Lévy process with a nonzero Gaussian component and a jump part, which is a compound Poisson process with finitely many jumps in every time interval. Merton (1976) was first to actually introduce jumps in the stock distribution. The Merton jump-diffusion model with Gaussian jumps (known as an exponential Lévy model introduced by Merton (1976)) is given by

$$S_t = S_0 \exp\left(\mu t + \sigma W_t + \sum_{n=1}^{N_t} Y_n\right)$$

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, and independent jumps $Y_n \sim N(m, \delta^2)$. The Poisson process and the jumps are assumed to be independent of the Brownian motion. The use of the Poisson process is economically motivated by two assumptions: the number of crashes in non overlapping time interval should be independent and the chance of occurrence of one crash should be roughly proportional to the length of the time interval.

In analogy to the Black-Scholes model, the parameter μ stands in the Merton model for the expected stock return and σ is the volatility of regular shocks to the shock return. The jumps component can be interpreted as a model for crashes. The parameter λ is the expected number of crashes per year, m and δ^2 determine the distribution of a single jump. Kou (2002), suggests that the distribution of jump size is an asymmetric exponential with a density of the form f

$$f(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y\geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y<0\}}, \quad \eta_1 > 1, \eta_2 > 0$$

where $p, q \geq 0, p + q = 1$, represent the probabilities of upward and downward jumps. The requirement $\eta_1 > 1$ is needed to ensure that $E[Y] < \infty$ and $E[S_t] < \infty$; it essentially means that the average upward jump cannot exceed 100%, which is quite reasonable. For notational simplicity and in order to get analytical solutions for various option pricing problems, the drift μ and the volatility σ are assumed to be constants, and the Brownian motion and jumps are assumed to be one-dimensional. Ramezani and Zeng (2002) independently propose the double exponential jump-diffusion model from an econometric viewpoint as a way of improving the empirical fit of Mertons normal jump-diffusion model to stock price data.

2.5 Stochastic Volatility

In this section we present the literature review of stochastic volatility and extensions of the stochastic volatility model.

2.5.1 The Volatility Problem

Although the Black-Scholes formula is successful in explaining stock option prices (Black and Scholes (1973)) and Merton (1976), it does have known biases (Rubinstein (1895)). Its performance does not work on foreign currency options (Melino (1990) and Turnbull (1991)). The Black-Scholes model assumes that stock return is normally distributed and the volatility of the return is constant. Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) were the first to model volatility as a deterministic function of time and stock price. The deterministic volatility can be fitted to observed option prices to obtain an implied price process for the underlying asset. Dumas et al. (1998) study S&P 500 options, they conclude that deterministic volatility is unreliable and not useful for valuation and risk management.

2.5.2 Historical Volatility

Let $S_1, S_2, ..., S_N$ be a sequence of empirical asset price data and R_i be the log return observed on a given time. We have

$$R_i = \ln\left(\frac{S_{i+1}}{S_i}\right).$$

The mean return $\overline{R}$ is calculated by

$$\overline{R} = \frac{1}{N}\sum_{i=1}^{N} R_i.$$

To estimate the historical volatility $\hat{\sigma}$ we calculate the annualized standard deviation of the log returns;

$$\hat{\sigma} = \sqrt{\frac{252}{N-1}\sum_{i=1}^{N}(R_i - \overline{R})^2}$$

where N is the number of observations. The factor 252 is determined by the approximately 252 business days in a year. We can consider more or less than one year for a historical volatility. Parkinson instead of daily closing prices, considered the high and the low prices of the underlying asset an that day and used

$$R_i = \ln\left(\frac{S_{i+1}^{high}}{S_i^{low}}\right).$$

The volatility would then be

$$\sigma_{parkinson} = \sqrt{\frac{1}{4\ln 2}\frac{252}{N-1}\sum_{i=1}^{N}(R_i - \overline{R})^2}.$$

2.5.3 Stochastic Volatility Models

The Black-Scholes model assumes that the volatility is constant over a given time interval and unaffected by the changes in the stock price.

Several different stochastic processes have been suggested for the volatility, i.e.,

- *Ornstein* − *Uhlenbeck*(OU) process:

$$d\sigma_t = -\alpha\sigma_t dt + \beta dW_t^v.$$

The OU process has a closed - form solution

$$\sigma_t = \sigma_0 e^{-\alpha t} + \beta\int_0^t e^{-\alpha(t-s)}dW_s^v.$$

- Cox Ingersoll Ross (CIR) process: The volatility of the underlying asset return $v_t = \sigma_t^2$ satisfies the following:

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}W_t^v$$

with $\omega = \beta^2, \theta = 2\alpha$, and $\xi = 2\beta$.

- The GARCH process:

$$dv_t = (\omega - \theta v_t)dt + \xi v_t dW_t^v.$$

- The 3/2 process:

$$dv_t = (\omega v_t^2 - \theta v_t)dt + \xi v_t^{3/2}dW_t^v.$$

The Heston Model

Heston (1993) proposed a stochastic volatility model that allowed volatility to vary in time. Assume that the asset price follows the diffusion

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t}dW_t^S$$

where μ is the rate of return of the asset, v_t is the volatility of asset returns, W_t^S is Brownian motion corresponding to the asset price S_t. The volatility process follows mean reverting process:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v$$

where κ is rate of reversion, θ is long run mean, σ is volatility of volatility and W_t^v is Brownian motion corresponding to the volatility v_t. W_t^v has correlation ρ with W_t^S that is

$$dW_t^S dW_t^v = \rho dt.$$

If $2\kappa\theta > 0$, the process is larger than zero (see Cox, Ingersoll and Ross (1985) or Feller (1951)). If $\rho > 0$, the volatility will increase as the asset return increases. This make the right tail and squeezes the left tail so that the distribution has a fat right tail. There is evidence that the correlation between asset return and implied volatility is negative, which is known as the leverage effect. That is, ρ affects the skewness of the distribution. A phenomenon known as volatility clustering means that large price variations are more likely to be followed by large price variations and vice versa. The mean reversion parameter κ can be represented by the degree of volatility clustering. The advantage of the Heston model is the closed form solution for European call options.

2.5.4 A Stochastic Volatility Model with Jump: the Bates Model

Bates (1996) extended the Heston stochastic volatility model by adding proportional log normal jump to the underlying asset model. Bates model has the following form:

$$dS_t = S_t\left(\mu dt + \sqrt{v_t}dW_t^S\right) + S_{t-}dZ_t, \tag{2.8}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v, \tag{2.9}$$

where S_t, $v_t, \kappa, \theta, \sigma, W_t^S, W_t^v$ are defined in Heston model. S_{t-} means the value of the process before the jump is used on the left-hand side of the formula. Z_t is a compound Poisson process with intensity λ and log normal distribution of jump size such that if k is its jump size then

$$\ln(1+k) \sim N(\ln(1+\overline{k}) - \frac{1}{2}\delta^2, \delta^2), \tag{2.10}$$

where $\overline{k}$ is the mean of the jump size. The parameter $\mu = r - \lambda\overline{k}$ under the risk-neutral probability. Assume the log asset price $L_t = \ln S_t$. Applying Itô's lemma, we obtain

$$dL_t = (r - \lambda\overline{k} - \frac{1}{2}v_t)dt + \sqrt{v}_t dW_t^S + d\widetilde{Z}_t, \tag{2.11}$$

where $\widetilde{Z}_t$ is a compound Poisson process with intensity λ and normal distribution of jump sizes. This model can also be viewed as a generalization of the Merton jump-diffusion model allowing for stochastic volatility and generalization of Heston model allowing for jump in underlying asset. Jumps in the log-price have to be normal in this model. One can replace the normal distribution by any other distribution for the jump size, provided that the characteristic function is computable.

2.6 Itô Stochastic Calculus

This section provides a brief exposition of all definitions and tools used for readers familiar with stochastic calculus.

A *stochastic process* is a sequence of random variables $X = (X_t)_{t\geq 0}$ on some probability space (Ω, F, P). Note that, by abuse of the standard notation, whenever we write $t \geq 0$ that means $t \in [0, T]$. A stochastic process X induces a *probability transition function* of the form

$$P[X_{t+1} = s_{t+1} | X_t = s_t, ..., X_0 = s_0].$$

That is the probability that the state at future time $t+1$ is s_{t+1}, given that the states at past times $t, ..., 0$ were $s_t, ..., s_0$, respectively.

A *Markov process* is a stochastic process such that for all t, for all $s_0, ..., s_t, s_{t+1}$,

$$P[X_{t+1} = s_{t+1} | X_t = s_t, ..., X_0 = s_0] = P[X_{t+1} = s_{t+1} | X_t = s_t].$$

This equation is the *Markov property*, sometimes called the *memoryless property*; it implies that probability transition to future state, such as s_{t+1} depends only on the present state s_t, but are independent of the remote past, $s_{t-1}, ..., s_0$.

A stochastic process is called a *Gaussian process*, if $X_t \sim N(\mu_t, \sigma_t^2)$ for all t. A Gaussian process is fully characterized by its mean and covariance function.

2.6.1 Standard Brownian Motion

A *standard Brownian motion* process or a *Wiener process* $(W_t)_{t \geq 0}$ is a stochastic process on $[0, \infty)$ defined on a probability space (Ω, F, P) such that:

1. It starts at zero, i.e., $W_0 = 0$.
2. It has stationary, independent increments, i.e.,

   $$W_{t+u} - W_t, \quad \forall u > 0$$

   are stationary and independent.
3. For every $t > 0, W_t$ has a normal $N(0, t)$ distribution.
4. It has continuous sample paths: no jumps.

Stationary increments of the condition mean that the distributions of increments $W_{t+u} - W_t$ do not depend on the time t, but they depend on the time-distance u of two observations. For example, if one models a log stock price $\log S_t$ as a Brownian motion(with drift) process

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right)$$

the distribution of increment in 2009 for the next one year $\log(S_{2009+1})-\log(S_{2009})$ is the same as in 2020, $\log(S_{2020+1}) - \log(S_{2020})$:

$$\log(S_{2009+1}) - \log(S_{2009}) \overset{d}{=} \log(S_{2020+1}) - \log(S_{2020}).$$

Recall that the conditional probability of the event A given B is assuming $P(B) > 0$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If A and B are independent events:

$$P(A|B) = P(A).$$

The property of independent increments means that when modeling a log stock price $\log S_t$ as a Brownian motion (with drift) process, the probability distribution of a log stock price in year 2010 is not affected by whatever happens in year 2009 in the stock price (such as stock price crash):

$$P\left(\log(S_{2010+1}) - \log(S_{2010})|\log(S_{2009+1}) - \log(S_{2009})\right)$$

$$= P\left(\log(S_{2010+1}) - \log(S_{2010})\right).$$

The Brownian motion (Wiener process) has three properties which make it of fundamental importance to the theory of stochastic process: it is Gaussian, a Markov process, and a martingale. Let $W = (W_t(\omega))_{t\geq 0}$ denote a Brownian motion, in which t is the time and each ω is a particle; then $W_t(\omega)$ represents the position of that particle at time t. One can show that except on a set of probability zero, every sample path ($W_t(\omega)$ as a function of t for fixed ω) is continuous but of unbounded variation on every compact time set.

Proposition 2.2. *(Property of Brownian motion)*

1. *Martingale property: Brownian motion is one of the simplest example of a martingale. We have, for all* $0 \leq s \leq t$*,*

$$E[W_t|\mathcal{F}_s] = E[W_t|W_s] = W_s.$$

2. *Path property: Brownian motion has continuous paths, i.e.,* $W = (W_t)_{t\geq 0}$ *is a continuous function of* t*. However, the paths of Brownian motion are very erratic. Moreover, the paths of Brownian motion are of infinite variation, i.e., their variation is infinite on every interval. Another property , we have that*

$$P\left(\sup_{t\geq 0} W_t = +\infty \text{ and } \inf_{t\geq 0} W_t = -\infty\right) = 1.$$

This means that the Brownian path will keep oscillating between positive and negative values.

3. *Scaling Property: for every* $c \neq 0$, $\tilde{W} = \left(\tilde{W}_t = cW_{t/c^2}\right)_{t\geq 0}$ *is also Brownian motion.*

2.6.2 Itô Integral

Since the sample paths are of unbounded variation on every compact set, they cannot be differentiable in the Stieltjes integral sense. Although Stieltjes integration with respect to the paths of the Brownian motion is not possible, the differential dW does have an intuitive interpretation. Engineers think of dW as white noise, and using generalized functions, one can define the quantity dW rigorously. Wiener gave meaning to dW in his definition of what is called the *Wiener integral*, but in such integrals the integrands are functions of time only (certain functions). It was Itô (1944) who first defined an integral for random integrands with respect to the Brownian motion. Itô used his integral to represent a large class of diffusions as solutions of stochastic differential equations (SDE). In 1953, Doob extended Itô's work on integration by using martingales instead of Brownian motion. The integral was so constructed that integration with respect to a martingale yields a martingale.

The best known extension of the Itô integral is the *semimartingale integral*. If all the paths of an adapted process are right continuous and of finite variation on compact time sets, we call the process a *VF process* . If V is a VF process and H is a bounded predictable process (H is $\mathcal{F}_t - measurable$) then, for each fixed ω, we denote by $\int_0^t H_s(\omega)dV_s(\omega)$ the Lebesgue-Stieltjes integral.

A stochastic process is a local martingale if certain integrability conditions in the definition of a martingale are relaxed. A stochastic process X is a *semimartingale* if X can be written in the form

$$X = L + V$$

where L is a local martingale and V is a VF process. If H is a bounded, predictable process, one can the define $\int_0^t H_s dX_s$ by

$$\int_0^t H_s dX_s = \int_0^t H_s dL_s + \int_0^t H_s dV_s$$

We will refer to this stochastic integral as the *semimartingale integral*. In fact the semimartingale form the largest class of processes for which the Itô integral formula can be defined.

2.6.3 Poisson and Compound Poisson Process

Definition 2.3. *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=0}^{n} \tau_i$. The process $(N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n} \tag{2.12}$$

is called a Poisson process with intensity λ.

The Poisson process is therefore defined as a counting process: it counts the number of random times (T_n) which occur between 0 and t, where $(T_n - T_{n-1})_{n\geq 1}$ is an independent and identically distributed(i.i.d.) sequence of exponential variables.

Proposition 2.3. *Let* $(N_t)_{t\geq 0}$ *be a Poisson process.*

1. *For any* $t > 0$, N_t *is almost surely finite.*

2. *For any* ω, *the sample path* $t \to N_t(\omega)$ *is piecewise constant and increase by jumps of size 1.*

3. *The sample path* $t > 0$, N_t *are right continuous with left limits (cadlag).*

4. *For any* $t > 0$, $N_{t-} = N_t$ *with probability 1.*

5. N_t *is continuous in probability:*

$$\forall t > 0,\ N_s \underset{s\to t}{\overset{P}{\to}} N_t. \tag{2.13}$$

6. *For any* $t > 0$, N_t *follows a Poisson distribution with parameter* λt*:*

$$\forall n \in \mathbb{N},\ P(N_t = n) = e^{\lambda t}\frac{(\lambda t)^n}{n!}. \tag{2.14}$$

7. *The characteristic function of* N_t *is given by*

$$E[e^{iuN_t}] = \exp \lambda t(e^{iu} - 1),\ \forall u \in \Re. \tag{2.15}$$

8. N_t *has independent increments: for any* $t_1 < \cdots < t_n$,

$$N_{t_n} - N_{t_{n-1}}, ..., N_{t_2} - N_{t_1}, N_{t_1}$$

are independent random variables.

9. The increments of N are homogeneous: for any $t > s$,

$$N_t - N_s$$

has the same distribution as N_{t-s}.

10. N_t has the Markov property: for any $t > s$,

$$E[f(N_t)|N_u, u \leq s] = E[f(N_t)|N_s]. \tag{2.16}$$

Proof. See Cont (2004). □

Definition 2.4. *(Compound Poisson process)*
A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process X_t defined as

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where jumps sizes Y_i are i.i.d. with distribution f and N_t is a Poisson process with intensity λ, independent from $(Y_i)_{i>1}$.

The following properties of a compound Poisson process are deduced from the definition:

1. The sample paths of X are cadlag piecewise constant functions.

2. The jump times $(T_i)_{i>1}$ have the same law as the jump times of the Poisson process N_t: they can be expressed as partial sums of independent exponential random variables with parameter λ.

3. The jump sizes $(Y_i)_{i>1}$ are independent and identically distributed with law f.

The Poisson process itself can be seen as a compound Poisson process on $\Re$ such that $Y_i = 1$. This explains the origin of term "compound Poisson" in the definition.

Definition 2.5. *(Lévy Process)*
A cadlag stochastic process $(X_t)_{t>0}$ on (Ω, F, P) with values in $\Re^d$ such that $X_0 = 0$ is called a Lévy process if it possesses the following properties:

1. *Independent increments: for every increasing sequence of times $t_0, ..., t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent.*

2. *Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.*

3. *Stochastic continuity: $\forall \varepsilon > 0,\ \lim_{h \to 0} P\left(|X_{t+h} - X_t| \geq \varepsilon\right) = 0$.*

Proposition 2.4. *$(X_t)_{t\geq 0}$ is compound Poisson process if and only if it is a Lévy process and its sample paths are piecewise constant functions.*

Proof. see Cont (2004). □

Definition 2.6. *(Poisson random measure)*
Let (Ω, F, P) be a probability space, $E \subset \Re^d$ and μ a given (positive) Radon measure μ on a measurable space $(E, \mathcal{E})$. A Poisson random measure on E with intensity measure μ is an integer valued random measure:

$$M : \Omega \times \mathcal{E} \to \mathbb{N}$$

$$(\omega, A) \mapsto M(\omega, A),$$

such that

1. *For (almost all) $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on E: for any bounded measurable $A \subset E$, $M(A) < \infty$ is an integer valued random variable.*

2. For each measurable set $A \subset E$, $M(\cdot, A) = M(A)$ is a Poisson random variable with parameter $\mu(A)$:

$$\forall k \in \mathbb{N}, \quad P\left(M(A) = k\right) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}.$$

3. For disjoint measurable sets $A_1, ..., A_n \in \mathcal{E}$, the variables $M(A_1), ..., M(A_n)$ are independent.

Cont (2004) proved that for any Radon measure μ on $E \subset \Re^d$, there exists a Poisson random measure M on E with intensity μ. Consequently, any Poisson random measure on E can be represented as a counting measure associated with a random sequence of points in E, i.e., there exists $(T_n(\omega))_{n\geq 1}$, such that

$$\forall A \in \mathcal{E}, \;\; M(\omega, A) = \sum_{n\geq 1} 1_A(T_n(\omega)) = \#\left\{n \geq 1, \; T_n(\omega) \in A\right\}. \qquad (2.17)$$

Define a random variable $T_n = \sum\limits_{i=1}^{n} \tau_i$ where $(\tau_i)_{i\geq 1}$ is a sequence of independent exponential random variables with parameter λ. Moreover, by equation (2.17), the Poisson process may be expressed in terms of the Poisson random measure M in the following:

$$N_t(\omega) = M(\omega, [0, t]) = \int_{[0,t]} M(\omega, ds)$$

where ds is the Lebesgue area element on $[0, t]$.

For every compound Poisson process $(X_t)_{t\geq 0}$ on $\Re^d$ with intensity λ and jump size distribution f, its jump measure

$$J_X(B) = \#\{(t, X_t - X_{t-}) \in B\}$$

is a Poisson random measure on $\Re^d \times [0, \infty)$ with intensity measure

$$\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt$$

where B is a measurable subset of $\Re^d \times [0,\infty]$ and ν is Lévy measure of the compound Poisson process. This implies that every compound Poisson process can be represented in the following form:

$$X_t = \sum_{s\in[0,t]} \Delta X_s = \int_{\Re^d\times[0,t]} x J_X(dx \times ds)$$

where J_X is a Poisson random measure with intensity measure $\nu(dx)dt$. Let E be a measurable subset on $\Re$. For a measurable function $f : [0,t] \times E \to \Re^d$, one can construct an integral with respect to the Poisson random measure M, given by the random variable

$$\int_{E\times[0,T]} f(y,t)M(\cdot, dydt) = \sum_{n\geq 1} f(Y_n(\cdot), T_n(\cdot)).$$

2.6.4 The Itô Formula and its Extensions

We now review Itô's formula and its extensions.

Lemma 2.5. *(Itô's formula)*
Assume that the process $X = (X_t)_{t\geq 0}$ has stochastic differential equation given by

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t$$

where $\mu(t,X_t)$ and $\sigma(t,X_t)$ are adapted processes, and let f be a $C^{1,2}$- function. Defined the process $Y_t = f(t,X_t)$. Then Y has a stochastic differential given by

$$df(t,X_t) = \left\{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right\} dt + \sigma\frac{\partial f}{\partial x}dW_t. \tag{2.18}$$

Note that the term $\mu\frac{\partial f}{\partial x}$, for example, is shorthand notation for

$$\mu\left(t,X_t\right)\frac{\partial f}{\partial x}\left(t,X_t\right),$$

and correspondingly for the other terms.

In fact Itô's formula provides a derivative chain rule for stochastic functions, clarifying the relationship between a stochastic process and a function of that stochastic process. Itô's formula has many extensions. The following Itô' formulas are the key step in establishing the main theorem of our thesis (for the proof see Cont (2004)).

Lemma 2.6. *(Itô's formula for jump-diffusion processes)*
Let $X = (X_t)_{t\geq 0}$ be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process given by

$$X_t = X_0 + \int_0^t \mu\,(s, X_s)ds + \int_0^t \sigma\,(s, X_s)dW_s + \sum_{i=1}^{N_t} \Delta X_i,$$

where $\mu(s, X_s)$ and $\sigma(s, X_s)$ are continuous non anticipating processes with

$$E\left[\int_0^T \sigma^2\,(t, X_t)\,dt\right] < \infty.$$

Then, for any $C^{1,2}$- function, $f : [0,T] \times \Re \to \Re$, the process $Y_t = f(t, X_t)$ can be represented as:

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \left[\frac{\partial f}{\partial s}\,(s, X_s) + \mu(s, X_s)\frac{\partial f}{\partial x}\,(s, X_s)\right] ds \\ &+ \frac{1}{2}\int_0^t \sigma^2(s, X_s)\frac{\partial^2 f}{\partial x^2}(t, X_t)ds \\ &+ \int_0^t \sigma\,(s, X_s)\frac{\partial f}{\partial x}(s, X_s)dW_s \\ &+ \sum_{\{n\geq 1, T_n \leq t\}} [f(X_{T_n-} + \Delta X_n) - f(X_{T_n-})] \end{aligned}$$

In differential notation

$$df(t, X_t) = \left\{\frac{\partial f}{\partial t}(t, X_t) + \mu(t, X_t)\frac{\partial f}{\partial x}(t, X_t) + \frac{\sigma^2(t, X_t)}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\right\} dt$$
$$+ \sigma(t, X_t)\frac{\partial f}{\partial x}(t, X_t)dW_t + f(t, X_{t-} + \Delta X_t) - f(t, X_{t-}).$$

Note that a non anticipating process is also called an *adapted process*: $(X_t)_{t\in[0,T]}$ is said to be $\mathcal{F}_t$-adapted, if the random variable X_t is $\mathcal{F}_t$-measurable.

Theorem 2.7. *(Feynman-Kac Theorem)*

Let a, b and g be smooth, bounded functions. Let X solve the stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

and let

$$u(x,t) = E[g(X_T)|X_t = x].$$

Then u is a solution of

$$u_t + au_x + \frac{1}{2}b^2 u_{xx} = 0$$

$$u(x,T) = g(x)$$

for $t < T$.

CHAPTER III

OPTION PRICING UNDER STOCHASTIC VOLATILITY WITH JUMP

In this Chapter, we would like to consider the problem of finding a closed-form formula for a European call option where the underlying asset and volatility follow the Bates model by adding jumps in volatility. This formula will be useful for option pricing rather than an estimation of it as appeared in Eraker's work (2003).

3.1 Introduction

Eraker, Johannes and Polson (2003) extend Bates' work by incorporating jumps in volatility, and their model is given by

$$\begin{aligned} dS_t &= S_t \left(\mu dt + \sqrt{v_t} dW_t^S \right) + S_{t-} Y_t dN_t^S & (3.1) \\ dv_t &= \kappa(\theta - v_t)dt + \sigma \sqrt{v_t} dW_t^v + Z_t dN_t^v. & (3.2) \end{aligned}$$

Eraker et al. (2003) develop a likelihood-based estimation strategy and provide estimates of parameters, spot volatility, jump times, and jump sizes using S&P 500 and Nasdaq 100 index returns. Moreover, they examine the volatility structure of the S&P and Nasdaq indices and indicate that models with jumps in volatility are preferred over those without jumps in volatility. But they did not provide a closed-form formula for the price of a European call option.

3.2 Description of the Model

It is assumed that a risk-neutral probability measure exists. The asset price S_t under this measure follows a jump-diffusion process, and the volatility follows a pure mean reverting and square root diffusion process with jump, i.e., our models are governed by the following dynamics:

$$dS_t = S_t\left((r-\lambda^S m)dt + \sqrt{v_t}dW_t^S\right) + S_{t-}Y_t dN_t^S \quad (3.3)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v + Z_t dN_t^v \quad (3.4)$$

where $S_t, v_t, \kappa, \theta, \sigma, W_t^S$ and W_t^v are defined as in the Bates model, r is the risk-free interest rate, N_t^S and N_t^v are independent Poisson processes with constant intensities λ^S and λ^v respectively. Y_t is the proportional jump size of the asset price with density $\phi_Y(y)$ and $E[Y_t] := m < \infty$ and Z_t is the jump size of the volatility with density $\phi_Z(z)$. Moreover, we assume that the jump processes N_t^S and N_t^v are independent of standard Brownian motions W_t^S and W_t^v .

3.3 Partial Integro-Differential Equations

Consider the process $\overrightarrow{X_t} = \left(X_t^{(1)}, X_t^{(2)}\right)$ where $X_t^{(1)}$ and $X_t^{(2)}$ are processes in $\Re$ and satisfy the following equations:

$$dX_t^{(1)} = f_1(X_t^{(1)}, X_t^{(2)}, t)dt + g_1(X_t^{(1)}, X_t^{(2)}, t)dW_t^{(1)} + X_{t-}^{(1)}Y_t dN_t^{(1)} \quad (3.5)$$

$$dX_t^{(2)} = f_2(X_t^{(1)}, X_t^{(2)}, t)dt + g_2(X_t^{(1)}, X_t^{(2)}, t)dW_t^{(2)} + Z_t dN_t^{(2)} \quad (3.6)$$

where $f_1,\ g_1,\ f_2$ and g_2 are all continuously differentiable, W_t^S and W_t^v are standard Brownian motions with $Corr[dW_t^{(1)}, dW_t^{(2)}] = \rho$, N_t^S and N_t^v are independent Poisson processes with constant intensities $\lambda^{(1)}$ and $\lambda^{(2)}$ respectively.

Since every compound Poisson process can be represented as an integral form of a Poisson random measure (Cont (2004)) then the last term on the right hand side of equation (3.5) and equation (3.6) can be written as follows:

$$\int_0^t X_{s-}^{(1)} Y_s dN_s^{(1)} = \sum_{n=1}^{N_t^{(1)}} X_{n-}^{(1)} Y_n = \int_0^t \int_{\Re} X_{s-}^{(1)} q J_{\mathrm{Q}}(ds\, dq)$$

$$\int_0^t Z_s dN_s^{(2)} = \sum_{n=1}^{N_t^{(2)}} Z_n = \int_0^t \int_{\Re} r J_R(ds\, dr)$$

where Y_n are i.i.d. random variables with density $\phi_Y(y)$ and J_{Q} is a Poisson random measure of the process $\mathrm{Q}_t = \sum_{n=1}^{N_t^{(1)}} Y_n$ with intensity measure $\lambda^{(1)}\phi_Y(dq)dt$, Z_n are i.i.d. random variables with density $\phi_Z(z)$ and J_R is a Poisson random measure of the process $R_t = \sum_{n=1}^{N_t^{(2)}} Z_n$ with intensity measure $\lambda^{(2)}\phi_Z(dr)dt$.

Let $U(x_1, x_2)$ be a bounded real-valued function and twice continuously differentiable with respect to x_1 and x_2 and

$$u(x_1, x_2, t) = E\left[U(X_T^{(1)}, X_T^{(2)}) | X_t^{(1)} = x_1, X_t^{(2)} = x_2\right]. \tag{3.7}$$

By the two dimensional Dynkin's formula (Hanson (2007)), u is a solution of the partial integro-differential equation (PIDE)

$$\begin{aligned} 0 \;=\; & \frac{\partial u(x_1, x_2, t)}{\partial t} + \bar{\mathcal{A}} u(x_1, x_2, t) \\ & + \lambda^{(1)} \int_{\Re} [u(x_1 + y, x_2, t) - u(x_1, x_2, t)] \phi_Y(y) dy \\ & + \lambda^{(2)} \int_{\Re} [u(x_1, x_2 + z, t) - u(x_1, x_2, t)] \phi_Z(z) dz \end{aligned}$$

subject to the final condition $u(x_1, x_2, T) = U(x_1, x_2)$. The notation $\bar{\mathcal{A}}$ is defined by

$$\begin{aligned} \bar{\mathcal{A}} u(x_1, x_2, t) \;=\; & f_1 \frac{\partial u(x_1, x_2, t)}{\partial x_1} + f_2 \frac{\partial u(x_1, x_2, t)}{\partial x_2} + \frac{1}{2} g_1^2 \frac{\partial^2 u(x_1, x_2, t)}{\partial x_1^2} \\ & + \rho g_1 g_2 \frac{\partial^2 u(x_1, x_2, t)}{\partial x_1 \partial x_2} + \frac{1}{2} g_2^2 \frac{\partial^2 u(x_1, x_2, t)}{\partial x_2^2}. \end{aligned} \tag{3.8}$$

3.4 A Closed-Form Formula for European Call Options

Let C denote the price at time t of a European style call option on the current price of the underlying asset S_t with strike price K and expiration time T.

The terminal payoff of a European call option on the underlying stock S_t with strike price K is

$$\max(S_T - K, 0).$$

This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \leq K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate r is constant over the lifetime of the option, the price of the European call at time t is equal to the discounted conditional expected payoff

$$\begin{aligned}
C(S_t, v_t, t; K, T) &= e^{-r(T-t)} E_{\mathcal{M}}[\max(S_T - K, 0)|S_t, v_t] \\
&= e^{-r(T-t)} \left(\int_K^\infty S_T P_{\mathcal{M}}(S_T|S_t, v_t) dS_T - K \int_K^\infty P_{\mathcal{M}}(S_T|S_t, v_t) dS_T \right) \\
&= S_t \left(\frac{1}{e^{r(T-t)} S_t} \int_K^\infty S_T P_{\mathcal{M}}(S_T|S_t, v_t) dS_T \right) - K e^{-r(T-t)} \int_K^\infty P_{\mathcal{M}}(S_T|S_t, v_t) dS_T \\
&= S_t \left(\frac{1}{E_{\mathcal{M}}[S_T|S_t, v_t]} \int_K^\infty S_T P_{\mathcal{M}}(S_T|S_t, v_t) dS_T \right) - K e^{-r(T-t)} \int_K^\infty P_{\mathcal{M}}(S_T|S_t, v_t) dS_T \\
&= S_t P_1(S_t, v_t, t; K, T) - K e^{-r(T-t)} P_2(S_t, v_t, t; K, T) \qquad (3.9)
\end{aligned}$$

where $E_{\mathcal{M}}$ is the expectation with respect to the risk-neutral probability measure $\mathcal{M}$, $P_{\mathcal{M}}(S_T|S_t, v_t)$ is the corresponding conditional density given (S_t, v_t) and

$$P_1(S_t, v_t, t; K, T) = \left(\int_K^\infty S_T P_{\mathcal{M}}(S_T|S_t, v_t) dS_T \right) / E_{\mathcal{M}}[S_T|S_t, v_t].$$

Note that P_1 is the risk-neutral probability that $S_T > K$ (since the integrand is nonnegative and the integral over $(0,\infty)$ is one), and finally that

$$P_2(S_t, v_t, t; K, T) = \left(\int_K^\infty P_{\mathcal{M}}(S_T|S_t, v_t) dS_T \right) = \Pr \text{ob}(S_T > K|S_t, v_t)$$

is the risk-neutral in-the-money probability. Moreover,

$$E_{\mathcal{M}}[S_T|S_t, v_t] = e^{r(T-t)} S_t$$

for $t \geq 0$.

Assume that the asset price S_t and the volatility v_t satisfy equations (3.3) and (3.4) respectively. We would like to compute the price of a European call option with strike price K and maturity T. To do this, we make a change of variable from S_t to $L_t = \ln S_t$, i.e., where S_t satisfies equation (3.3) and its inverse $S_t = e^{L_t}$. Denote $k = \ln K$ the logarithm of the strike price. By the jump-diffusion chain rule, $\ln S_t$ satisfies the SDE

$$d \ln S_t = (r - \lambda^S m - \frac{v_t}{2})dt + \sqrt{v_t} dW_t^S + \ln(1 + Y_t) dN_t^S. \qquad (3.10)$$

Applying the two-dimensional Dynkin's formula for the price dynamics (3.10) and volatility v_t in equation (3.4), we obtain that the value of a European-style option, as a function of the stock log return L_t denoted by

$$\tilde{C}(L_t, v_t, t; k, T) \equiv C(e^{L_t}, v_t, t; e^k, T)$$

$$= C(e^{\ln S_t}, v_t, t; e^{\ln K}, T)$$

$$= C(S_t, v_t, t; K, T),$$

i.e.,

$$\tilde{C}(l, v, t; k, T) = e^{-r(T-t)} E_{\mathcal{M}} \left[\max(e^{L_T} - K, 0) | L_t = l, v_t = v \right],$$

and satisfies the following PIDE:

$$
\begin{aligned}
0 &= \frac{\partial \tilde{C}}{\partial t} + \bar{\mathcal{A}}[\tilde{C}](l,v,t;k,T) \\
&\quad + \lambda^S \int_{\Re} [\tilde{C}(l+y,v,t;k,T) - \tilde{C}(l,v,t;k,T)]\phi_Y(y)dy \\
&\quad + \lambda^v \int_{\Re} [\tilde{C}(l,v+z,t;k,T) - \tilde{C}(l,v,t;k,T)]\phi_Z(z)dz.
\end{aligned}
\tag{3.11}
$$

Here the operator $\bar{\mathcal{A}}$ as in (3.8) is defined by

$$
\begin{aligned}
\bar{\mathcal{A}}[\tilde{C}](l,v,t;k,T) &= (r - \lambda^S m - \frac{1}{2}v)\frac{\partial \tilde{C}}{\partial l} + \kappa(\theta - v)\frac{\partial \tilde{C}}{\partial v} \\
&\quad + \frac{1}{2}v\frac{\partial^2 \tilde{C}}{\partial l^2} + \rho\sigma v\frac{\partial^2 \tilde{C}}{\partial l \partial v} \\
&\quad + \frac{1}{2}\sigma^2 v\frac{\partial^2 \tilde{C}}{\partial v^2} - r\tilde{C}.
\end{aligned}
$$

In the current state variable, the last line of equation (3.9) becomes

$$
\tilde{C}(l,v,t;k,T) = e^l \tilde{P}_1(l,v,t;k,T) - e^{k-r(T-t)}\tilde{P}_2(l,v,t;k,T) \tag{3.12}
$$

where $\tilde{P}_j(l,v,t;k,T) := P_j(e^l,v,t;e^k,T)$, $j = 1,2$.

The following lemma shows the relationship between $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of equation (3.12).

Lemma 3.1. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of the equation (3.12) satisfy the following PIDEs*

$$
\begin{aligned}
0 = \frac{\partial \tilde{P}_1}{\partial t} &+ \mathcal{A}[\tilde{P}_1](l,v,t;k,T) + v\frac{\partial \tilde{P}_1}{\partial l} + \rho\sigma v\frac{\partial \tilde{P}_1}{\partial v} + (r - \lambda^S m)\tilde{P}_1 \\
&+ \lambda^S \int_{\Re} [(e^y - 1)\tilde{P}_1(l+y,v,t:k,T)]\phi_Y(y)dy,
\end{aligned}
$$

and subject to the boundary condition at expiration time $t = T$;

$$
\tilde{P}_1(l,v,T;k,T) = 1_{l>k}. \tag{3.13}
$$

Moreover, $\tilde{P}_2$ satisfies the equation

$$0 = \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}[\tilde{P}_2](l, v, t; k, T) + r\tilde{P}_2,$$

and subject to the boundary condition at expiration time $t = T$;

$$\tilde{P}_2(l, v, T; k, T) = 1_{l>k}. \tag{3.14}$$

The operator $\mathcal{A}$ is defined by

$$\begin{aligned}\mathcal{A}[f](l, v, t; k, T) \quad := \quad & (r - \lambda^S m - \frac{1}{2}v)\frac{\partial f}{\partial l} + \kappa(\theta - v)\frac{\partial f}{\partial v} \\ & + \frac{1}{2}v\frac{\partial^2 f}{\partial l^2} + \rho\sigma v\frac{\partial^2 f}{\partial l \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f}{\partial v^2} - rf \\ & + \lambda^S \int\limits_{\Re} [f(l + y, v, t; k, T) - f(l, v, t; k, T)]\phi_Y(y)dy \\ & + \lambda^v \int\limits_{\Re} [f(l, v + z, t; k, T) - f(l, v, t; k, T)]\phi_Z(z)dz.\end{aligned} \tag{3.15}$$

Note that $1_{l>k} = 1$ if $l > k$ and otherwise $1_{1>k} = 0$.

Proof. We plan to substitute equation (3.12) into equation (3.11). Firstly, we compute

$$\begin{aligned}\frac{\partial \tilde{C}}{\partial t} &= e^l\frac{\partial \tilde{P}_1}{\partial t} - e^{k-r(T-t)}\frac{\partial \tilde{P}_2}{\partial t} - re^{k-r(T-t)}\tilde{P}_2 \\ \frac{\partial \tilde{C}}{\partial l} &= e^l\frac{\partial \tilde{P}_1}{\partial l} + e^l\tilde{P}_1 - e^{k-r(T-t)}\frac{\partial \tilde{P}_2}{\partial l} \\ \frac{\partial \tilde{C}}{\partial v} &= e^l\frac{\partial \tilde{P}_1}{\partial v} - e^{k-r(T-t)}\frac{\partial \tilde{P}_2}{\partial v} \\ \frac{\partial^2 \tilde{C}}{\partial l^2} &= e^l\frac{\partial^2 \tilde{P}_1}{\partial l^2} + 2e^l\frac{\partial \tilde{P}_1}{\partial l} + e^l\tilde{P}_1 - e^{k-r(T-t)}\frac{\partial^2 \tilde{P}_2}{\partial l^2} \\ \frac{\partial^2 \tilde{C}}{\partial l \partial v} &= e^l\frac{\partial^2 \tilde{P}_1}{\partial l \partial v} + e^l\frac{\partial \tilde{P}_1}{\partial v} - e^{k-r(T-t)}\frac{\partial^2 \tilde{P}_2}{\partial l \partial v} \\ \frac{\partial^2 \tilde{C}}{\partial v^2} &= e^l\frac{\partial^2 \tilde{P}_1}{\partial v^2} - e^{k-r(T-t)}\frac{\partial^2 \tilde{P}_2}{\partial v^2},\end{aligned}$$

$$
\begin{aligned}
&\tilde{C}(l+y,v,t;k,T)-\tilde{C}(l,v,t;k,T)\\
&=\left[e^{(l+y)}\tilde{P}_1(l+y,v,t;k,T)-e^{k-r(T-t)}\tilde{P}_2(l+y,v,t;k,T)\right]\\
&\quad-\left[e^{l}\tilde{P}_1(l,v,t;k,T)-e^{k-r(T-t)}\tilde{P}_2(l,v,t;k,T)\right]\\
&=e^{l}\left[e^{y}\tilde{P}_1(l+y,v,t;k,T)-\tilde{P}_1(l+y,v,t;k,T)\right]\\
&\quad+\left[e^{l}\tilde{P}_1(l+y,v,t;k,T)-e^{l}\tilde{P}_1(l,v,t;k,T)\right]\\
&\quad-e^{k-r(T-t)}\left[\tilde{P}_2(l+y,v,t;k,T)-\tilde{P}_2(l,v,t;k,T)\right]\\
&=e^{l}(e^{y}-1)\tilde{P}_1(l+y,v,t;k,T)+e^{l}\left[\tilde{P}_1(l+y,v,t;k,T)-\tilde{P}_1(l,v,t;k,T)\right]\\
&\quad-e^{k-r(T-t)}\left[\tilde{P}_2(l+y,v,t;k,T)-\tilde{P}_2(l,v,t;k,T)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{C}(l,v+z,t;k,T)-\tilde{C}(l,v,t;k,T)\\
&=\left[e^{l}\tilde{P}_1(l,v+z,t;k,T)-e^{k-r(T-t)}\tilde{P}_2(l,v+z,t;k,T)\right]\\
&\quad-\left[e^{l}\tilde{P}_1(l,v,t;k,T)-e^{k-r(T-t)}\tilde{P}_2(l,v,t;k,T)\right]\\
&=e^{l}\left[\tilde{P}_1(l,v+z,t;k,T)-\tilde{P}_1(l,v,t;k,T)\right]\\
&\quad-e^{k-r(T-t)}\left[\tilde{P}_2(l,v+z,t;k,T)-\tilde{P}_2(l,v,t;k,T)\right].
\end{aligned}
$$

We substitute all terms above into equation (3.11) and separate it by assumed independent terms of $\tilde{P}_1$ and $\tilde{P}_2$. This gives two PIDEs for the risk-neutralized probability for $\tilde{P}_j(l,v,t;k,T), j=1,2:$

$$
\begin{aligned}
0 \;=\;& \frac{\partial\tilde{P}_1}{\partial t}+\left(r-\lambda^{S}m-\frac{1}{2}v\right)\left(\frac{\partial\tilde{P}_1}{\partial l}+\tilde{P}_1\right)\\
&+\kappa(\theta-v)\frac{\partial\tilde{P}_1}{\partial v}+\frac{1}{2}v\left(\frac{\partial^2\tilde{P}_1}{\partial l^2}+2\frac{\partial\tilde{P}_1}{\partial l}+\tilde{P}_1\right)\\
&+\rho\sigma v\left(\frac{\partial^2\tilde{P}_1}{\partial l\partial v}+\frac{\partial\tilde{P}_1}{\partial v}\right)+\frac{1}{2}\sigma^2 v\frac{\partial^2\tilde{P}_1}{\partial v^2}-r\tilde{P}_1\\
&+\lambda^{S}\int_{\Re}[(e^{y}-1)\tilde{P}_1(l+y,v,t;k,T)+\tilde{P}_1(l+y,v,t;k,T)\\
&-\tilde{P}_1(l,v,t;k,T)]\phi_Y(y)dy\\
&+\lambda^{v}\int_{\Re}[\tilde{P}_1(l,v+z,t;k,T)-\tilde{P}_1(l,v,t;k,T)]\phi_Z(z)dz \qquad (3.16)
\end{aligned}
$$

subject to the boundary condition at the expiration time $t=T$ according to equation (3.13).

By using the notation in equation (3.15), PIDE (3.16) becomes

$$\begin{aligned} 0 &= \frac{\partial \tilde{P}_1}{\partial t} + \mathcal{A}[\tilde{P}_1](l,v,t;k,T) + v\frac{\partial \tilde{P}_1}{\partial l} + \rho\sigma v\frac{\partial \tilde{P}_1}{\partial v} + (r-\lambda^S m)\tilde{P}_1 \\ &\quad +\lambda^S \int_{\Re} [(e^y-1)\tilde{P}_1(l+y,v,t;k.T]\phi_Y(y)dy \\ &:= \frac{\partial \tilde{P}_1}{\partial t} + \mathcal{A}_1[\tilde{P}_1](l,v,t;k,T). \end{aligned}$$

For $\tilde{P}_2(l,v,t;k,T)$:

$$\begin{aligned} 0 &= \frac{\partial \tilde{P}_2}{\partial t} + r\tilde{P}_2 + \left(r-\lambda^S m - \frac{1}{2}v\right)\frac{\partial \tilde{P}_2}{\partial l} + \kappa(\theta-v)\frac{\partial \tilde{P}_2}{\partial v} \\ &\quad +\frac{1}{2}v\frac{\partial^2 \tilde{P}_2}{\partial l^2} + \rho\sigma v\frac{\partial^2 \tilde{P}_2}{\partial l\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \tilde{P}_2}{\partial v^2} - r\tilde{P}_2 \\ &\quad +\lambda^S \int_{\Re} [\tilde{P}_2(l+y,v,t;k,T) - \tilde{P}_2(l,v,t;k,T)]\phi_Y(y)dy \\ &\quad +\lambda^v \int_{\Re} [\tilde{P}_2(l,v+z,t;k,T) - \tilde{P}_2(l,v,t;k,T)]\phi_Z(z)dz \end{aligned} \tag{3.17}$$

subject to the boundary condition at the expiration time $t = T$ according to equation (3.14).

Again, by using the notation in equation (3.15), PIDE (3.17) becomes

$$\begin{aligned} 0 &= \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}[\tilde{P}_2](l,v,t;k,T) + r\tilde{P}_2 \\ &:= \frac{\partial \tilde{P}_2}{\partial t} + \mathcal{A}_2[\tilde{P}_2](l,v,t;k,T). \end{aligned}$$

The proof of Lemma is now completed. □

For $j = 1,2$ the characteristic functions for $\tilde{P}_j(l,v,t;k,T)$, with respect to the variable k are defined by

$$f_j(l,v,t;x,T) := -\int_{-\infty}^{\infty} e^{ixk} d\tilde{P}_j(l,v,t;k,T),$$

with a minus sign to account for the negativity of the measure $d\tilde{P}_j$.

Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t} + \mathcal{A}_j[f_j](l, v, t; k, T) = 0, \tag{3.18}$$

with the respective boundary conditions

$$\begin{aligned} f_j(l, v, T; x, T) &= -\int_{-\infty}^{\infty} e^{ixk} d\tilde{P}_j(l, v, T; k, T) \\ &= -\int_{-\infty}^{\infty} e^{ixk}(-\delta(k-l)dk) \\ &= e^{ixl} \end{aligned}$$

since

$$\begin{aligned} d\tilde{P}_j(l, v, T; k, T) &= d1_{l>k} \\ &= dH(l-k) \\ &= -\delta(k-l)dk. \end{aligned}$$

The following lemma shows how to calculate the functions $\tilde{P}_1$ and $\tilde{P}_2$ as they appeared in Lemma 3.1.

Lemma 3.2. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ can be calculated by the inverse Fourier transforms of the characteristic function, i.e.,*

$$P_j(l, v, t; k, T) = \frac{1}{2} + \frac{1}{\pi} \int_{0^+}^{+\infty} \text{Re}\left[\frac{e^{-ixk} f_j(l, v, t; x, T)}{ix}\right] dx$$

for $j = 1, 2$ with Re[·] *denoting the real component of a complex number.*

By letting $\tau = T - t$.

(i) The characteristic function f_1 is given by

$$f_1(l, v, t; x, t + \tau) = \exp(g_1(\tau) + v h_1(\tau) + ixl),$$

where

$$
\begin{aligned}
h_1(\tau) &= \frac{(\eta_1^2 - \Delta_1^2)(e^{\Delta_1 \tau} - 1)}{\sigma^2(\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1 \tau})} \\
g_1(\tau) &= ((r - \lambda^S m)ix - \lambda^S m)\tau \\
&\quad -\frac{\kappa\theta}{\sigma^2}\left(2\ln\left(1 - \frac{(\Delta_1 + \eta_1)(1 - e^{-\Delta_1 \tau})}{2\Delta_1}\right) + (\Delta_1 + \eta_1)\tau\right) \\
&\quad +\lambda^S \tau \int_{-\infty}^{\infty} (e^{(ix+1)y} - 1)\phi_Y(y)dy + \lambda^v \tau \int_{-\infty}^{\infty} (e^{zh_1(\tau)} - 1)\phi_Z(z)dz \\
\eta_1 &= \rho\sigma(ix + 1) - \kappa \\
\Delta_1 &= \sqrt{\eta_1^2 - \sigma^2 ix(ix + 1)}.
\end{aligned}
$$

(ii) The characteristic function f_2 is given by

$$
f_2(l, v, t; x, t + \tau) = \exp(g_2(\tau) + vh_2(\tau) + ixl + r\tau),
$$

where

$$
\begin{aligned}
h_2(\tau) &= \frac{(\eta_2^2 - \Delta_2^2)(e^{\Delta_2 \tau} - 1)}{\sigma^2(\eta_2 + \Delta_2 - (\eta_2 - \Delta_2)e^{\Delta_2 \tau})} \\
g_2(\tau) &= ((r - \lambda^S m)ix - r)\tau \\
&\quad -\frac{\kappa\theta}{\sigma^2}\left(2\ln\left(1 - \frac{(\Delta_2 + \eta_2)(1 - e^{-\Delta_2 \tau})}{2\Delta_2}\right) + (\Delta_2 + \eta_2)\tau\right) \\
&\quad +\lambda^S \tau \int_{-\infty}^{\infty} (e^{ixy} - 1)\phi_Y(y)dy + \lambda^v \tau \int_{-\infty}^{\infty} (e^{zh_2(\tau)} - 1)\phi_Z(z)dz \\
\eta_2 &= i\rho\sigma x - \kappa \\
\Delta_2 &= \sqrt{\eta_2^2 - \sigma^2 ix(ix - 1)}.
\end{aligned}
$$

Proof. *(i)* To solve for the characteristic function explicitly, letting $\tau = T - t$ be the time-to-go, we conjecture that the function f_1 is given by

$$
f_1(l, v, t; x, t + \tau) = \exp(g_1(\tau) + vh_1(\tau) + ixl), \qquad (3.19)
$$

and the boundary condition

$$
g_1(0) = 0 = h_1(0).
$$

This conjecture exploits the linearity of the coefficient in PIDE (3.18). Note that the characteristic function f_1 always exists.

In order to substitute (3.19) into (3.18), firstly, we compute

$$\begin{aligned}
\frac{\partial f_1}{\partial t} &= (-g_1'(\tau) - vh_1'(\tau))f_1 \\
\frac{\partial f_1}{\partial l} &= ixf_1 \\
\frac{\partial f_1}{\partial v} &= h_1(\tau)f_1 \\
\frac{\partial^2 f_1}{\partial l^2} &= -x^2 f_1 \\
\frac{\partial^2 f_1}{\partial l \partial v} &= ixh_1(\tau)f_1 \\
\frac{\partial^2 f_1}{\partial v^2} &= h_1^2(\tau)f_1
\end{aligned}$$

$$\begin{aligned}
f_1(l+y,v,t;x,t+\tau) - f_1(l,v,t;x,t+\tau) &= (e^{ixy}-1)f_1(l,v,t;x,t+\tau) \\
f_1(l,v+z,t;x,t+\tau) - f_1(l,v,t;x,t+\tau) &= (e^{zh_1(\tau)}-1)f_1(l,v,t;x,t+\tau)
\end{aligned}$$

and

$$\begin{aligned}
(e^y-1)f_1(l+y,v,t;x,t+\tau) &= (e^y-1)e^{g_1(\tau)+vh_1(\tau)+ix(l+y)} \\
&= (e^y-1)e^{ixy}f_1(l,v,t;x,t+\tau).
\end{aligned}$$

Substituting all the above terms into equation (3.18) and after canceling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned}
0 &= -g_1'(\tau) - vh_1'(\tau) + (r - \lambda^S m + \frac{1}{2}v)ix \\
&\quad +(\kappa(\theta - v) + \rho\sigma v)h_1(\tau) - \frac{1}{2}vx^2 + \rho\sigma vixh_1(\tau) + \frac{1}{2}\sigma^2 vh_1^2(\tau) - \lambda^S m \\
&\quad +\lambda^S \int_{\Re} (e^{(ix+1)y}-1)\phi_Y(y)dy + \lambda^v \int_{\Re} (e^{zh_1(\tau)}-1)\phi_Z(z)dz.
\end{aligned}$$

By separating the order v and ordering the remaining terms, we can reduce it to two ordinary differential equation (ODEs),

$$h_1'(\tau) = \frac{1}{2}\sigma^2 h_1^2(\tau) + (\rho\sigma(1+ix) - \kappa)h_1(\tau) + \frac{1}{2}ix - \frac{1}{2}x^2 \tag{3.20}$$

and

$$\begin{aligned} g_1'(\tau) &= \kappa\theta h_1(\tau) + (r - \lambda^S m)ix - \lambda^S m \\ &\quad + \lambda^S \int_{-\infty}^{\infty} (e^{(ix+1)y} - 1)\phi_Y(y)dy + \lambda^v \int_{-\infty}^{\infty} (e^{zh_1(\tau)} - 1)\phi_Z(z)dz. \end{aligned} \tag{3.21}$$

Let $\eta_1 = \rho\sigma(ix+1) - \kappa$ and substitute it into equation (3.20). We get

$$\begin{aligned} h_1'(\tau) &= \frac{1}{2}\sigma^2 h_1^2 + \eta_1 h_1 + \frac{1}{2}ix - \frac{1}{2}x^2 \\ &= \frac{1}{2}\sigma^2 \left(h_1^2 + \frac{2\eta_1}{\sigma^2}h_1 + \frac{1}{\sigma^2}ix(ix+1) \right) \\ &= \frac{1}{2}\sigma^2 \left(h_1 + \frac{2\eta_1 + \sqrt{4\eta_1^2 - 4\sigma^2 ix(ix+1)}}{2\sigma^2} \right) \\ &\quad \times \left(h_1 + \frac{2\eta_1 - \sqrt{4\eta_1^2 - 4\sigma^2 ix(ix+1)}}{2\sigma^2} \right) \\ &= \frac{1}{2}\sigma^2 \left(h_1 + \frac{\eta_1 + \Delta_1}{\sigma^2} \right) \left(h_1 + \frac{\eta_1 - \Delta_1}{\sigma^2} \right), \end{aligned}$$

where $\Delta_1 = \sqrt{\eta_1^2 - \sigma^2 ix(ix+1)}$.

By the method of variable separation, we have

$$\frac{2dh_1}{\left(h_1 + \frac{\eta_1+\Delta_1}{\sigma^2}\right)\left(h_1 + \frac{\eta_1-\Delta_1}{\sigma^2}\right)} = \sigma^2 d\tau.$$

Using partial fractions, we get

$$\frac{1}{\Delta_1}\left(\frac{1}{h_1 + \frac{\eta_1-\Delta_1}{\sigma^2}} - \frac{1}{h_1 + \frac{\eta_1+\Delta_1}{\sigma^2}} \right) dh_1 = d\tau.$$

Integrating both sides, we obtain

$$\ln\left(\frac{h_1 + \frac{\eta_1-\Delta_1}{\sigma^2}}{h_1 + \frac{\eta_1+\Delta_1}{\sigma^2}} \right) = \Delta_1\tau + C.$$

Using boundary condition $h_1(\tau = 0) = 0$, we get

$$C = \ln\left(\frac{\eta_1 - \Delta_1}{\eta_1 + \Delta_1} \right).$$

Solving for h_1, we obtain

$$h_1(\tau) = \frac{(\eta_1^2 - \Delta_1^2)(e^{\Delta_1\tau} - 1)}{\sigma^2(\eta_1 + \Delta_1 - (\eta_1 - \Delta_1)e^{\Delta_1\tau})}.$$

In order to solve $g_1(\tau)$ explicitly, we substitute $h_1(\tau)$ into equation (3.21) and integrate with respect to τ on both sides. Then we get

$$\begin{aligned} g_1(\tau) &= ((r-\lambda^S m)ix - \lambda^S m)\tau \\ &\quad -\frac{\kappa\theta}{\sigma^2}\left(2\ln\left(1-\frac{(\Delta_1+\eta_1)(1-e^{-\Delta_1\tau})}{2\Delta_1}\right)+(\Delta_1+\eta_1)\tau\right) \\ &\quad +\lambda^S\tau\int_{-\infty}^{\infty}(e^{(ix+1)y}-1)\phi_Y(y)dy+\lambda^v\tau\int_{-\infty}^{\infty}(e^{zh_1(\tau)}-1)\phi_Z(z)dz. \end{aligned}$$

(ii). The details of the proof are similar to case *(i).* Hence, we have

$$f_2(l,v,t;x,t+\tau) = \exp(g_2(\tau)+vh_2(\tau)+ixl+r\tau)$$

where $g_2(\tau),\ h_2(\tau),\ \eta_2$ and Δ_2 are as given in the Lemma.

We can thus evaluate the characteristic functions in explicit form. However, we are interested in the risk-neutral probabilities $\tilde{P}_j, j=1,2$. These can be inverted from the characteristic functions by performing the following integration

$$\tilde{P}_j(l,v,t;k,T) = \frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty}\mathrm{Re}\left[\frac{e^{-ixk}f_j(l,v,t;x,T)}{ix}\right]dx \tag{3.22}$$

for $j=1,2$.

To verify equation (3.22), firstly we note that

$$\begin{aligned} E_{\mathcal{M}}\left[e^{ix(\ln S_t-\ln K)}|\ln S_t=L_t, v_t=v\right] &= E_{\mathcal{M}}\left[e^{ix(L_t-k)}|L_t=l, v_t=v\right] \\ &= \int_{-\infty}^{\infty} e^{ix(l-k)}d\tilde{P}_j(l,v,t;k,T) \\ &= e^{-ixk}\int_{-\infty}^{\infty} e^{ixl}d\tilde{P}_j(l,v,t;k,T) \\ &= e^{-ixk}\int_{-\infty}^{\infty} e^{ixk}(-\delta(l-k)dk) \\ &= e^{-ixk}f_j(l,v,t;x,T). \end{aligned}$$

Then

$\frac{1}{2} + \frac{1}{\pi} \int\limits_{0^+}^{+\infty} \mathrm{Re}\left[\frac{e^{-ixk} f_j(l,v,t;x,T)}{ix}\right] dx$

$= \frac{1}{2} + \frac{1}{\pi} \int\limits_{0^+}^{+\infty} \mathrm{Re}\left[\frac{E_{\mathcal{M}}[e^{ix(\ln S_t - \ln K)} | \ln S_t = L_t, v_t = v]}{ix}\right] dx$

$= E_{\mathcal{M}}[\frac{1}{2} + \frac{1}{\pi} \int\limits_{0^+}^{+\infty} \mathrm{Re}\left[\frac{e^{ix(l-k)}}{ix}\right] dx |\, L_t = l, v_t = v]$

$= E_{\mathcal{M}}[\frac{1}{2} + \frac{1}{\pi} \int\limits_{0^+}^{+\infty} \frac{\sin(x(l-k))}{x}\, dx |\, L_t = l, v_t = v]$

$= E_{\mathcal{M}}[\frac{1}{2} + \mathrm{sgn}(l-k)\frac{1}{\pi} \int\limits_{0^+}^{+\infty} \frac{\sin x}{x}\, dx |\, L_t = l, v_t = v]$

$= E_{\mathcal{M}}[\frac{1}{2} + \frac{1}{2}\mathrm{sgn}(l-k) |\, L_t = l, v_t = v]$

$= E_{\mathcal{M}}[1_{l \geq k} |\, L_t = l, v_t = v]$,

where we have used the Dirichlet formula $\int\limits_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = 1$ and the function is defined as $\mathrm{sgn}(x) = 1$ if $x > 0, 0$ if $x = 0$ and -1 if $x < 0$.

□

In summary, we have just proved the following main theorem.

Theorem 3.3. *The value of a European call option of (3.3) is*

$$\tilde{C}(l, v, t; k, T) = e^l \tilde{P}_1(l, v, t; k, T) - e^{k - r(T-t)} \tilde{P}_2(l, v, t; k, T)$$

where $\tilde{P}_1$ and $\tilde{P}_2$ are given in Lemma 3.2.

Simulation Example

Let us consider the SET50 index. Figure 3.1 shows the daily prices of the data set consisting of the closing prices (Baht) of the SET50 index between January 4, 2011 and December 30, 2011. The empirical data set for these index prices were obtained from http://www.set.or.th

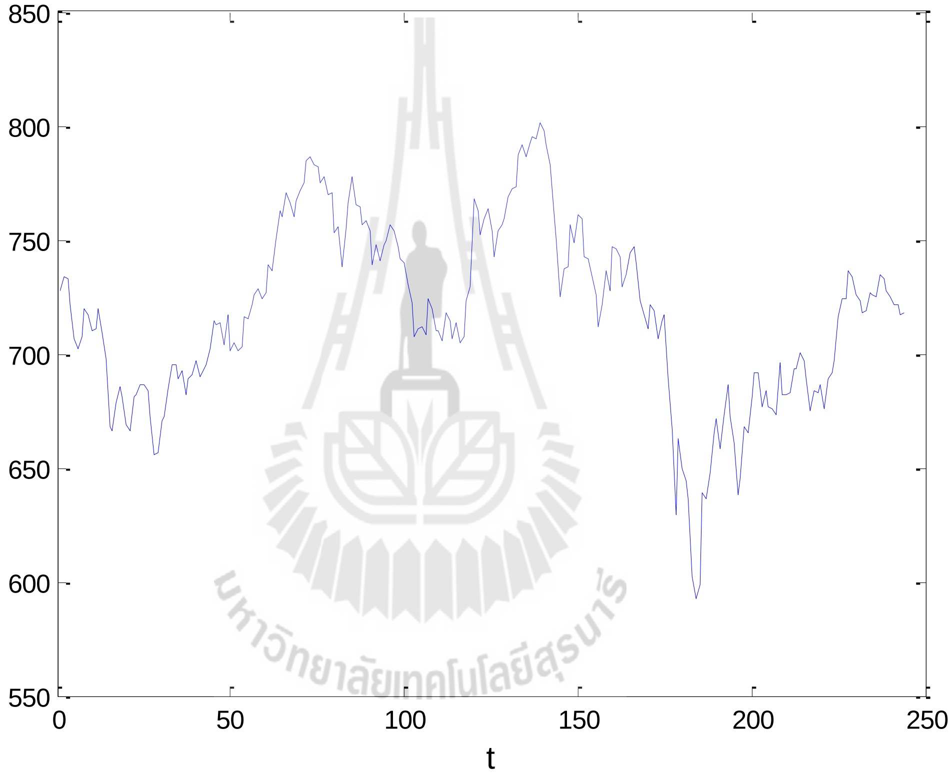

Figure 3.1 The daily price of SET50 index between January 4, 2011 and December 30, 2011.

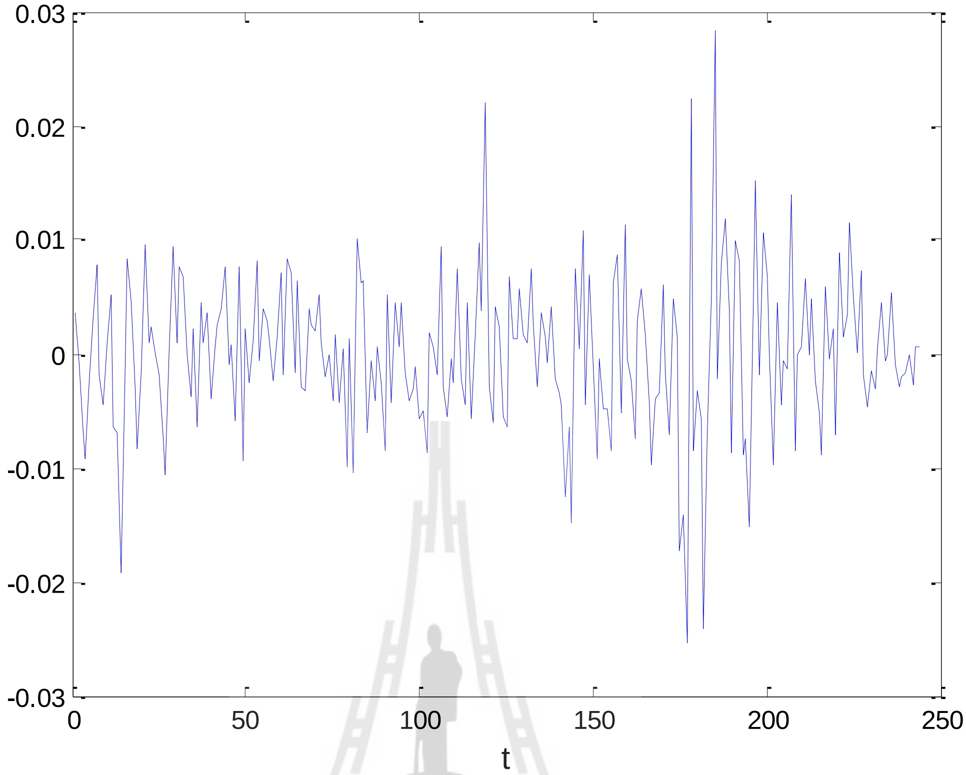

Figure 3.2 Log returns on the prices of SET50 index between January 4, 2011 and December 30, 2011.

The statistics of SET50 index and log returns are given in Table 3.1.

Table 3.1 Statistics of SET50 index data set.

	Asset prices	Log returns
Sample size	244	243
Mean	717.8971	-0.000023479
Standard deviation	39.38177	0.006808314
Skewness	-0.27213	0.039240447
Kurtosis	0.041693	2.428396942
Maximum	801.44	0.028318255
Minimum	592.57	-0.02539156

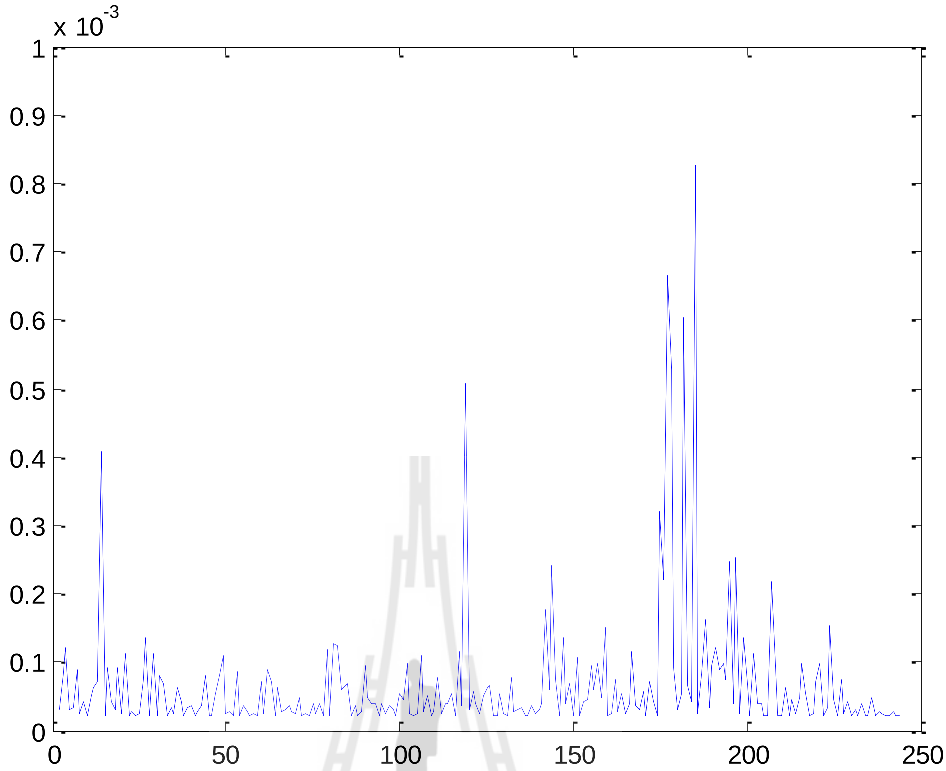

Figure 3.3 The historical volatility of SET50 index between January 4, 2011 and December 30, 2011.

This Figure show that historic volatility in not constant over time.

The model parameters are $S_0 = 727.9$, $\kappa = 1.5$, $r = -0.000023479$, $v_0 = 0.006808314$, $\theta = 0.04$, $\sigma = 0.2$, $\rho = 0.8$, $\lambda^S = 3$, $\lambda^v = 0.01$. After working 500 simulations and $N = 224$, we choose the smallest ARPE's sample path and shows the price simulation as compared to the empirical data of SET50 index close price in Figure 3.4.

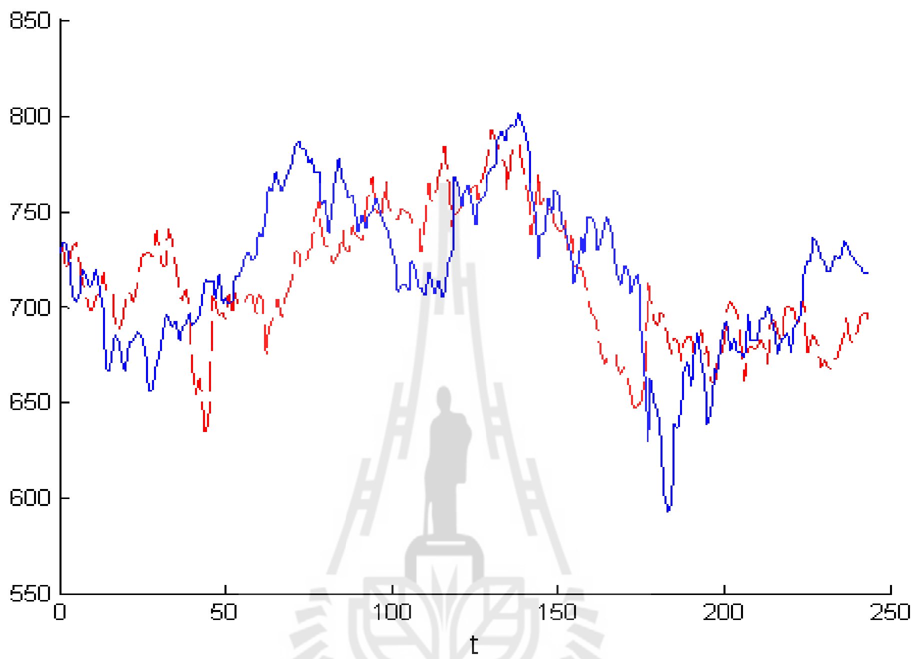

Figure 3.4 The price behavior of SET50 index between January 4, 2011 and December 30, 2011, as compared with a simulated from jump - diffusion and stochastic volatility with jump model. (solid line:=empirical data, dash line:=simulation data) with $N = 244$ and ARPE=0.0374.

For comparative purpose, we compute the Average Relative Percentage Error (ARPE). By definition

$$ARPE = \frac{1}{N}\sum_{k=1}^{N}\frac{|X_k - Y_k|}{X_k}\times 100$$

where N is the number of prices, $X = (X_k)_{k\geq 1}$ is the market price and $Y = (Y_k)_{k\geq 1}$ is the model price.

We present a numerical comparison between closed form solution and a Monte Carlo simulation option pricing. We apply the two techniques for the pricing of a European call option : $S_0 = 100$, $\kappa = 1.3$, $r = 0.05$, $v_0 = 0.034$, $\theta = 0.04$, $\sigma = 0.2$, $\rho = 0.7$, $\lambda^S = 3$, $\lambda^v = 0.02$.

Table 3.2 European call option price (with jump in volatility).

Strike price	Closed form solution	Monte Carlo
80	41.3645	40.7864
85	39.600	38.983
90	37.9592	38.569
95	36.4298	37.451
105	33.7072	32.890
110	32.4131	32.887
115	31.2371	31.678

CHAPTER IV

MEAN REVERTING PROCESS

4.1 Introduction

Empirical evidence on mean reversion in financial assets has been produced by Cecchetti et al. (1990) and Bessembinder et al. (1995), respectively. It has been documented that currency exchange rates also exhibit mean reversion. Jorion and Sweeney (1996) show how the real exchange rates revert to their mean levels and Sweeney (2006) provides empirical evidence of mean reversion in G-10 nominal exchange rates. Mean reversion also appears in some stock prices as evidenced by Poterba and Summers (1988).

In this chapter, we consider the problem of finding a closed-form formula for a European call option where the asset price follows mean reverting jump-diffusion and the stochastic volatility has jumps.

The rest of this chapter is organized as follows. In section 4.2, we briefly discuss model descriptions for option pricing. Deriving a formula for a characteristic function is presented in Section 4.3. Finally, a closed-form formula for a European call option in terms of characteristic functions is presented.

4.2 Model Descriptions

It is assumed that a risk-neutral probability measure $\mathcal{M}$ exists. The asset price S_t under this measure follows a mean reverting jump-diffusion process, and the volatility v_t follows mean reverting with jump, i.e. our models are governed by the following dynamics:

$$dS_t = b\left(a - \ln S_t - \frac{\lambda^S m}{b}\right) S_t dt + \sqrt{v_t} S_t dW_t^S + S_{t-} Y_t dN_t^S \qquad (4.1)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v + Z_t dN_t^v \qquad (4.2)$$

where $S_t, v_t, \kappa, \theta, \sigma, W_t^S$ and W_t^v are defined as Bates model, $a \in \Re$ is the mean of long-term asset price return, $b > 0$ is the rate at which the asset price return reverts toward its long-term mean, N_t^S and N_t^v are independent Poisson processes with constant intensities λ^S and λ^v respectively. Y_t and Z_t are proportional jump sizes of the asset price (4.1) and the jump size of the volatility process (4.2) respectively. Suppose that Y_t and Z_t are independent and identically distributed sequences with densities $\phi_{Y_t}(y) := \phi_Y(y)$, $\phi_{Z_t}(z) := \phi_Z(z)$ and $E[Y_t] := m < \infty$. Moreover, we assume that the jump processes N_t^S and N_t^v are independent of standard Brownian motions W_t^S and W_t^v.

Assume that the asset price S_t and the volatility v_t satisfy equations (4.1) and (4.2) respectively. Let $L_t = \ln S_t$, by the jump-diffusion chain rule, $\ln S_t$ satisfies the SDE

$$dL_t = b\left(a - L_t - \frac{\lambda^S m}{b} - \frac{v_t}{2b}\right) dt + \sqrt{v_t} dW_t^S + \ln(1 + Y_t) dN_t^S. \qquad (4.3)$$

4.3 Characteristic Function of Asset Price

We denote the characteristic function for $L_T = \ln S_T$ as

$$f(x : t, l, v) = E_{\mathcal{M}}[e^{ixL_T} | L_t = l, v_t = v] \qquad (4.4)$$

where $0 \leq t \leq T$ and $i = \sqrt{-1}$. Here L_t is the mean reverting asset price process with jumps specified by (4.3) and v_t is the volatility process specified by (4.2). The generalized Feynman-Kac theorem (Hanson (2007)) implies that $f(x : t, l, v)$

solves the following partial integro-differential equation (PIDE):

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} + b\left(a - l - \frac{\lambda^S m}{b} - \frac{v}{2b}\right)\frac{\partial f}{\partial l} \\ &+ \kappa(\theta - v)\frac{\partial f}{\partial v} + \frac{1}{2}v\frac{\partial^2 f}{\partial l^2} + \rho\sigma v\frac{\partial^2 f}{\partial l \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f}{\partial v^2} \\ &+ \lambda^S \int_{\Re} [f(x;t,l+y,v) - f(x;t,l,v)]\phi_Y(y)dy \\ &+ \lambda^v \int_{\Re} [f(x;t,l,v+z) - f(x;t,l,v)]\phi_Z(z)dz. \end{aligned} \tag{4.5}$$

Lemma 4.1. *Suppose that L_t follows the dynamics in (4.3). Then the characteristic function for L_T can be written in the form*

$$f(x:t,l,v) = \exp[B(t,T) + C(t,T)l + D(t,T)v + ixl], \tag{4.6}$$

where

$$\begin{aligned} B(t,T) &= (\frac{\lambda^S m}{b} - a)ix(e^{-b(T-t)} - 1) - \theta\kappa\int_{T-\tau}^{T} D(s,T)ds \\ &+ (T-t)\lambda^S \int_{\Re} \left[e^{ixy} - 1\right]\phi_Y(y)dy \\ &+ (T-t)\lambda^v \int_{\Re} \left[e^{zD(t,T)} - 1\right]\phi_Z(z)dz, \\ C(t,T) &= ix(e^{-b(T-t)} - 1), \\ D(t,T) &= U(e^{-b(T-t)}) + \frac{e^{-\kappa(T-t)}V(e^{-b(T-t)})}{-\frac{1}{U(1)} + \frac{\sigma^2}{2b}\int_1^{e^{-b(T-t)}} h^{\frac{\kappa}{b}-1}V(h)dh}, \end{aligned}$$

$$\begin{aligned} U(h) &= \frac{2bh}{\sigma^2}\frac{(\sqrt{1-p^2} - pi)\frac{\sigma x}{2b}\Phi(a^*, b^*, \frac{h}{\zeta}) + \frac{a^*}{b^*\zeta}\Phi(a^*+1, b^*+1, \frac{h}{\zeta})}{\Phi(a^*, b^*, \frac{h}{\zeta})}, \\ V(h) &= \frac{\Phi^2(a^*, b^*, \frac{1}{\zeta})e^{(\sqrt{1-\rho^2})\frac{\sigma x}{b}(1-h)}}{\Phi^2(a^*, b^*, \frac{h}{\zeta})}, \\ h &= e^{-b(T-t)}, \\ a^* &= \frac{\frac{b^*}{2}(\sqrt{\rho^2 - 1} + \rho) + \frac{\sigma}{4b}}{\sqrt{\rho^2 - 1}}, \\ b^* &= 1 - \frac{\kappa}{b}, \\ \zeta &= \frac{-b}{\sigma x\sqrt{1-\rho^2}} \end{aligned}$$

and $\Phi(\cdot,\cdot,\cdot)$ *is the degenerated hypergeometric function.*

Proof. From (4.4), it is clear that

$$f(x:T,l,v)=e^{ixl} \tag{4.7}$$

which is the boundary condition of PIDE (4.5). This implies that

$$B(T,T)=C(T,T)=D(T,T)=0. \tag{4.8}$$

First of all, we compute

$$\begin{aligned}
\frac{\partial f}{\partial t} &= (\frac{\partial B}{\partial t}+l\frac{\partial C}{\partial t}+v\frac{\partial D}{\partial t})f\\
\frac{\partial f}{\partial l} &= (C+ix)f\\
\frac{\partial f}{\partial v} &= Df\\
\frac{\partial^2 f}{\partial l^2} &= (C+ix)^2 f\\
\frac{\partial^2 f}{\partial v^2} &= D^2 f\\
\frac{\partial^2 f}{\partial l\partial v} &= (C+ix)Df
\end{aligned}$$

$$\lambda^S\int_{\Re}[f(x;t,l+y,v)-f(x;t,l,v)]\phi_Y(y)dy=(\lambda^S\int_{\Re}\left[e^{ixy}-1\right]\phi_Y(y)dy)f$$

$$\lambda^v\int_{\Re}[f(x;t,l,v+z)-f(x;t,l,v)]\phi_Z(z)dz=(\lambda^v\int_{\Re}\left[e^{zD(t,T)}-1\right]\phi_Z(z)dz)f.$$

Substituting all the above terms into equation (4.5) and using the fact that the function f is never zero, we obtain

$$\begin{aligned}
0 &= [B_t+C_tl+D_tv]+b[a-\frac{v}{2b}-\frac{\lambda^S m}{b}-l][C+ix]+\frac{1}{2}v(C+ix)^2\\
&\quad+\kappa(\theta-v)D+\frac{1}{2}v\sigma^2D^2+\rho\sigma v(ix+C)D\\
&\quad+\lambda^S\int_{\Re}[e^{ixy}-1]\phi_Y(y)dy+\lambda^v\int_{\Re}[e^{zD}-1]\phi_Z(z)dz,
\end{aligned}$$

where B_t, C_t and D_t are the partial derivatives with respect to t of functions B, C and D respectively. Rearranging the above equation, one obtains

$$\begin{aligned} 0 &= [B_t + (ba - \lambda^S m)(C + ix) + \kappa\theta D \\ &\quad + \lambda^S \int\limits_{\Re} [e^{ixy} - 1]\phi_Y(y)dy + \lambda^v \int\limits_{\Re} [e^{zD} - 1]\phi_Z(z)dz] \\ &\quad + [C_t - b(C + ix)]l \\ &\quad + [D_t - \frac{1}{2}(C + ix) + \frac{1}{2}(C + ix)^2 - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho\sigma(C + ix)D]v. \end{aligned} \tag{4.9}$$

This reduces the problem to one of solving three, much simpler, ordinary differential equations:

$$\begin{aligned} B_t &+ (ba - \lambda^S m)(C + ix) + \kappa\theta D + \lambda^S \int\limits_{\Re} [e^{ixy} - 1]\phi_Y(y)dy \\ &+ \lambda^v \int\limits_{\Re} [e^{zD} - 1]\phi_Z(z)dz = 0 \end{aligned} \tag{4.10}$$

$$C_t - b(C + ix) = 0 \tag{4.11}$$

$$\begin{aligned} D_t &+ \frac{1}{2}(C + ix)(C + ix - 1) \\ &- \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho\sigma(C + ix)D = 0 \end{aligned} \tag{4.12}$$

subject to boundary conditions (4.8).

The solution to equation (4.11) with the boundary condition $C(T, T) = 0$ is given by

$$C(t, T) = ix(e^{-b(T-t)} - 1). \tag{4.13}$$

We now consider equation (4.12). Substituting (4.13) in (4.12), one gets

$$D_t + \frac{1}{2}\left[ix + ix(e^{-b(T-t)} - 1)\right]\left[ix + ix(e^{-b(T-t)} - 1) - 1\right] - \kappa D + \frac{1}{2}\sigma^2 D^2$$

$$+\rho\sigma D\left[ix + ix(e^{-b(T-t)} - 1)\right] = 0$$

and, moreover,

$$D_t + \frac{1}{2}\left[ixe^{-b(T-t)}\right]\left[ixe^{-b(T-t)} - 1\right] - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho\sigma ix De^{-b(T-t)} = 0.$$

Hence,

$$D_t = -\frac{1}{2}\sigma^2 D^2 + \left[\kappa - \rho\sigma ixe^{-b(T-t)}\right] D + \frac{1}{2}\left[x^2 e^{-2b(T-t)} + ixe^{-b(T-t)}\right]. \tag{4.14}$$

Let $h = e^{-b(T-t)}$ and we define a new function $\hat{D}(h(t), T) := D(t, T)$. Then

$$\begin{aligned} \frac{\partial D(t,T)}{\partial t} &= \frac{\partial \hat{D}(h,T)}{\partial h}\frac{\partial h}{\partial t} \\ &= be^{-b(T-t)}\frac{\partial \hat{D}(h,T)}{\partial h}. \end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.14), we obtain the following Riccati equation

$$bh\frac{\partial \hat{D}}{\partial h} = -\frac{1}{2}\sigma^2 \hat{D}^2 + \left(\kappa - \rho\sigma ixe^{-b(T-t)}\right)\hat{D} + \frac{1}{2}\left(x^2 e^{-2b(T-t)} + ixe^{-b(T-t)}\right).$$

Multiplying the above equation by $\frac{1}{bh}$, one gets

$$\frac{\partial \hat{D}}{\partial h} = -\frac{1}{2bh}\sigma^2 \hat{D}^2 + \left(\frac{\kappa}{bh} - \frac{\rho\sigma ix}{b}\right)\hat{D} + \frac{1}{2b}\left(x^2 h + ix\right). \tag{4.16}$$

We shall solve the second order ODE (4.16) together with the initial condition $\hat{D}(1, T) = 0$. Let

$$\hat{D}(h,T) = \frac{2bhw'(h)}{\sigma^2 w(h)} \tag{4.17}$$

and taking the derivative of (4.17) with respect to h, one gets

$$\begin{aligned} \frac{\partial \hat{D}}{\partial h} &= \left[\sigma^2 w(h)\frac{\partial}{\partial h}(2bhw'(h)) - 2bhw'(h)\frac{\partial}{\partial h}(\sigma^2 w(h))\right]\frac{1}{\sigma^4 w^2(h)} \\ &= \left[\sigma^2 w(h)\left[2bw'(h) + 2bhw''(h)\right] - 2bh\sigma^2 (w'(h))^2\right]\frac{1}{\sigma^4 w^2(h)}. \end{aligned} \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.16), we have

$$\left[\sigma^2 w(h)\left[2bw'(h) + 2bhw''(h)\right] - 2bh\sigma^2 (w'(h))^2\right]\frac{1}{\sigma^4 w^2(h)}$$

$$= -\frac{\sigma^2}{2bh}\left(\frac{4b^2h^2(w'(h))^2}{\sigma^4 w^2(h)}\right) + \left(\frac{\kappa}{bh} - \frac{\rho\sigma x i}{b}\right)\left(\frac{2bhw'(h)}{\sigma^2 w(h)}\right) + \frac{1}{2b}(x^2h + ix).$$

Then

$$\sigma^2 w(h)\left[2bw'(h) + 2bhw''(h)\right]\frac{1}{\sigma^4 w^2(h)} = \left(\frac{\kappa}{bh} - \frac{\rho\sigma x i}{b}\right)\left(\frac{2bhw'(h)}{\sigma^2 w(h)}\right) + \frac{1}{2b}(x^2h + ix).$$

Multiplying the above equation by $\frac{\sigma^2 w(h)}{2b}$, one obtains

$$hw''(h) + w'(h) = \left(\frac{\kappa}{bh} - \frac{\rho\sigma x i}{b}\right)hw'(h) + \frac{\sigma^2 w(h)}{4b^2}(x^2h + ix)$$

or, equivalently,

$$hw''(h) - \left[(\frac{\kappa}{b} - 1) - h(\frac{\rho\sigma x i}{b})\right]w'(h) - \left[\frac{x^2\sigma^2 h}{4b^2} + \frac{ix\sigma^2}{4b^2}\right]w(h) = 0. \qquad (4.19)$$

The ODE (4.19) has a general solution of the form,

$$w(h) = e^{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma x}{2b}h}\left[C_1\Phi(a^*, b^*, \frac{h}{\zeta}) + C_2 h^{1-b^*}\Phi(a^* - b^* + 1, 2 - b^*, \frac{l}{\zeta})\right], \qquad (4.20)$$

where

$$a^* = \frac{(\sqrt{\rho^2-1}+\rho)\frac{b^*}{2} + \frac{\sigma}{4b}}{\sqrt{\rho^2-1}}$$

$$b^* = 1 - \frac{\kappa}{b}$$

and

$$\zeta = \frac{-b}{\sigma x\sqrt{1-\rho^2}}.$$

Here C_1 and C_2 are constants to be determined from the boundary conditions. $\Phi(a, b, z)$ is the degenerated hypergeometric function which has the following Kummer's series expansion

$$\Phi(a, b, z) = 1 + \sum_{k=1}^{\infty}\frac{(a)_k z^k}{(b)_k k!},$$

where

$$(a)_k = a(a+1)\cdots(a+k-1).$$

If we let $C_1 = 1$ and $C_2 = 0$ in (4.20) then a particular solution for (4.19) is

$$w(h) = e^{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma x}{2b}h}\left[\Phi(a^*, b^*, \frac{h}{\zeta})\right].$$

Using the transformation (4.17), Wong and Lo (2009) show that a particular solution for (4.16) is

$$U(h) = \frac{2bh}{\sigma^2}\frac{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma x}{2b}\Phi(a^*, b^*, \frac{h}{\zeta}) + \frac{a^*}{b^*\zeta}\Phi(a^*+1, b^*+1, \frac{h}{\zeta})}{\Phi(a^*, b^*, \frac{h}{\zeta})},$$

which can be used to obtain the general solution for (4.16) as follows

$$\hat{D}(h) = U(h) + \frac{\frac{\Phi^2(a^*,b^*,\frac{1}{\zeta})}{\Phi^2(a^*,b^*,\frac{h}{\zeta})}h^{\frac{\kappa}{b}}e^{-2(\sqrt{1-\rho^2})\frac{\sigma x}{2b}(h-1)}}{-\frac{1}{U(1)} + \frac{\sigma^2}{2b}\int_1^h \frac{\Phi^2(a^*,b^*,\frac{1}{\zeta})}{\Phi^2(a^*,b^*,\frac{h}{\zeta})}\eta^{\frac{\kappa}{b}-1}e^{-2(\sqrt{1-\rho^2})\frac{\sigma x}{2b}(\eta-1)}d\eta}. \quad (4.21)$$

We now consider the final ordinary differential equation (4.10). Substituting (4.21) and (4.13) in (4.10), we have

$$\begin{aligned} B_t(t,T) &= (\lambda^S m - ba)ixe^{-b(T-t)} - \kappa\theta D(t,T) \\ &\quad -\lambda^S \int_{\Re} [e^{ixy} - 1]\phi_Y(y)dy - \lambda^v \int_{\Re} [e^{zD} - 1]\phi_Z(z)dz. \end{aligned}$$

Integrating both sides of the above equation and invoking the condition $B(T,T) = 0$, we obtain

$$\begin{aligned} B(t,T) &= (\frac{\lambda^S m}{b} - a)ix(e^{-b(T-t)} - 1) - \kappa\theta\int_t^T D(s,T)ds \\ &\quad +(T-t)\lambda^S \int_{\Re} [e^{ixy} - 1]\phi_Y(y)dy \\ &\quad +(T-t)\lambda^v \int_{\Re} [e^{zD} - 1]\phi_Z(z)dz. \end{aligned} \quad (4.22)$$

□

We can conclude that the characteristic function of the mean reverting process (4.3) with stochastic volatility (4.2) is

$$f(x : t, l, v) = e^{B(t,T)+C(t,T)x+D(t,T)v+ixl},$$

where $B(t,T), C(t,T)$ and $D(t,T)$ are as given in the Lemma 4.1.

4.4 A Formula for European Option Pricing

The Fourier inversion technique illustrated in this section was first proposed by Carr & Madan (1999). It ensures that the Fourier transform of European option prices exists by the inclusion of an exponential damping factor. Moreover, singularities will be removed by this damping factor. Assume that $t = 0$ and we define $L_T = \ln S_T$ and $k = \ln K$. Moreover, we express the call price option $C(0, S_T)$ as a function of the log of the strike price K rather than the terminal log asset price S_T. The initial call value $C_T(K)$ is related to the risk-neutral density $q_T(l)$ by

$$\begin{aligned} C_T(K) &= e^{-rT}E[\max(S_T - K, 0)] \\ &= e^{-rT}E[\max(e^{\ln S_T} - e^{\ln K}, 0)] \\ &= e^{-rT}\int_{-\infty}^{\infty}\max(e^l - e^k, 0)q_T(l)dl \\ &= e^{-rT}\int_{k}^{\infty}(e^l - e^k)q_T(l)dl \end{aligned}$$

in which the expectation is taken with respect to some risk-neutral measure. Since

$$\lim_{K\to 0} C_T(K) = \lim_{k\to -\infty} C_T(e^k) = S_0,$$

we see that $C_T(e^k)$ is not in L^1, as $C_T(e^k)$ does not tend to zero for $k \to -\infty$. Now, consider the modified call price

$$c_T(k) = e^{\alpha k}C_T(e^k)$$

where $\alpha > 0$. Below we will show that under a certain assumption we have $c_T(k) \in L^1$, the space of integrable functions. For now, assume that the Fourier transform of $c_T(k)$ is well-defined:

$$\psi_T(u) = \int_{-\infty}^{\infty} e^{iuk}c_T(k)dk.$$

Inverting gives

$$c_T(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi_T(u)du.$$

or

$$\begin{aligned}
C_T(K) &= e^{-\alpha k}c_T(k) \\
&= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi_T(u)du \\
&= \frac{e^{-\alpha k}}{\pi}\mathrm{Re}\left\{\int_0^{\infty} e^{-iuk}\psi_T(u)du\right\}
\end{aligned}$$

where the last equality follows from the observation that

$$\int_{-\infty}^{\infty} e^{-iuk}\psi_T(u)du = \int_0^{\infty} e^{-iuk}\psi_T(u)du + \int_{-\infty}^{0} e^{-iuk}\psi_T(u)du,$$

and where the second term on the right-hand side can be rewritten as

$$\begin{aligned}
\int_{-\infty}^{0} e^{-iuk}\psi_T(u)du &= \int_0^{\infty} e^{iuk}\psi_T(-u)du \\
&= \int_0^{\infty} \overline{e^{-iuk}\psi_T(u)}du \\
&= \overline{\int_0^{\infty} e^{-iuk}\psi_T(-u)du}
\end{aligned}$$

yielding the claim. Note that we have a closed-form for the Fourier transform of $c_T(k)$:

$$\begin{aligned}
\psi_T(u) &= \int_{-\infty}^{\infty} e^{iuk} c_T(k)dk \\
&= \int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} C_T(k)dk \\
&= \int_{-\infty}^{\infty} e^{iuk} \left(\int_{k}^{\infty} e^{\alpha k} e^{-rT}(e^l - e^k) q_T(l)dl \right) dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(l) \left(\int_{-\infty}^{l} (e^{l+\alpha k} - e^{(1+\alpha)k}) e^{iuk} dk \right) dl \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(l) \left(\int_{-\infty}^{l} e^{(\alpha+iu)k} (e^l - e^k) dk \right) dl \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(l) \left(e^l \int_{-\infty}^{l} e^{(\alpha+iu)k} dk - \int_{-\infty}^{l} e^{(\alpha+iu+1)k} dk \right) dl \\
&= e^{-rT} \int_{-\infty}^{\infty} q_T(l) \left(e^l \left[\frac{1}{\alpha+iu} e^{(\alpha+iu)k} \right]_{-\infty}^{l} - \left[\frac{1}{\alpha+iu+1} e^{(\alpha+iu+1)k} \right]_{-\infty}^{l} \right) dl
\end{aligned} \tag{4.23}$$

Since for $\alpha > 0$

$$\lim_{k\to-\infty} \left| e^{(iu+\alpha)k} \right| = \lim_{k\to-\infty} \left| e^{(iu+\alpha+1)k} \right| = \lim_{k\to-\infty} \left| e^{(\alpha+1)k} \right| = 0,$$

equation (4.23) reduces to

$$\begin{aligned}
\psi_T(u) &= e^{-rT} \int_{-\infty}^{\infty} q_T(l) \left[\frac{e^{(\alpha+1+iu)l}}{\alpha+iu} - \frac{e^{(\alpha+1+iu)l}}{\alpha+iu+1} \right] dl \\
&= e^{-rT} \int_{-\infty}^{\infty} \left[\frac{(\alpha+iu+1)e^{(\alpha+1+iu)l} - (\alpha+iu)e^{(\alpha+1+iu)l}}{(\alpha+iu)(\alpha+iu+1)} \right] q_T(l)dl \\
&= e^{-rT} \int_{-\infty}^{\infty} \left[\frac{(\alpha+iu)e^{(\alpha+1+iu)l} + e^{(\alpha+1+iu)l} - (\alpha+iu)e^{(\alpha+1+iu)l}}{(\alpha+iu)(\alpha+iu+1)} \right] q_T(l)dl \\
&= e^{-rT} \int_{-\infty}^{\infty} \left[\frac{e^{(\alpha+1+iu)l}}{\alpha^2 + 2\alpha iu - u^2 + \alpha + iu} \right] q_T(l)dl \\
&= \frac{e^{-rT}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \int_{-\infty}^{\infty} e^{(\alpha+1+iu)l} q_T(l)dl \\
&= \frac{e^{-rT}}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} \int_{-\infty}^{\infty} e^{i(u-(\alpha+1)i)l} q_T(l)dl \\
&= \frac{e^{-rT} f(l, v, t; x = u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}
\end{aligned}$$

where f is the characteristic function defined in Lemma 4.1.

Lemma 4.2. *Let $\alpha > 0$. The Fourier transform of $c_T(k)$ exists (i.e. $c_T(k) \in L^1$) if $E[S_T^{\alpha+1}] < \infty$.*

Proof. Note that $E[S_T^{\alpha+1}] < \infty$ implies

$$\psi_T(0) < \infty, \tag{4.24}$$

since

$$\begin{aligned} |\psi_T(0)| &= \frac{e^{-rT}\,|f(-(\alpha+1)i)|}{\alpha^2+\alpha} \\ &= \frac{e^{-rT}E[S_T^{\alpha+1}]}{\alpha^2+\alpha}, \end{aligned}$$

where the last equality follows from

$$\begin{aligned} |f(-(\alpha+1)i)| &= \left|E[e^{(-(\alpha+1)i)i\log S_T}]\right| \\ &= \left|E[e^{(\alpha+1)\log S_T}]\right| \\ &= E[S_T^{\alpha+1}]. \end{aligned}$$

We have the equality

$$\psi_T(0) = \int_{-\infty}^{\infty} c_T(k)dk,$$

which follows from

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{iuk} c_T(k)dk.$$

Combining this with (4.24) completes the proof. □

Hence, the European call prices at time $t = 0$ with strike price $k = \ln K$ can then be numerically obtained by using the inverse transform:

$$\begin{aligned} C_T(k) &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi_T(u)du \\ &= \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-iuk}\frac{e^{-rT}f(l,v,t;x=u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}du. \end{aligned} \tag{4.25}$$

Integration (4.25) is a direct Fourier transform and lends itself to an application of the Fast Fourier Transform (FFT), which has also been done in Carr & Madan (1999).

Simulation Example

Let us consider the crude oil price. Figure 4.1 shows the daily prices of the data set consisting of daily closing prices (Dollars/Barrel) between January 2, 2008 and December 31, 2012. The empirical data set for these prices were obtained from http://www.eia.gov

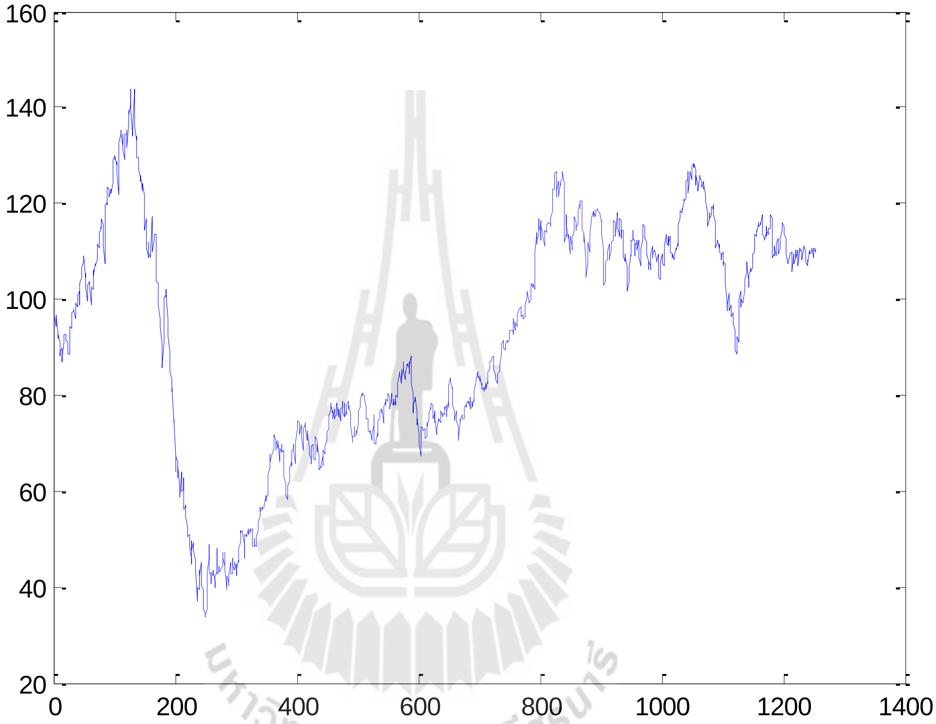

Figure 4.1 The daily price of crude oil between January 2, 2008 and December 31, 2012.

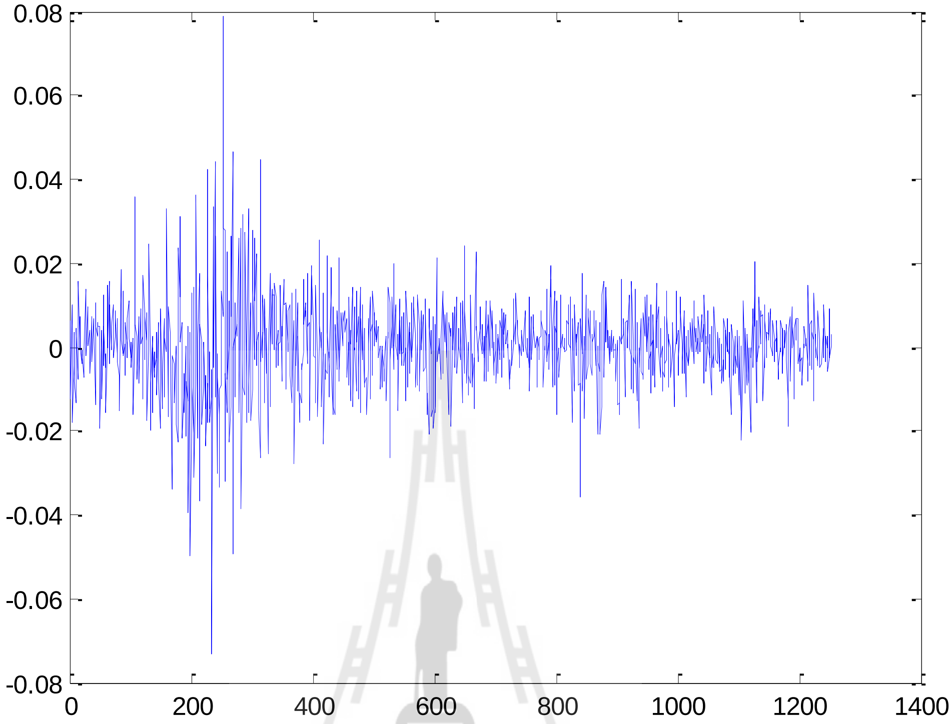

Figure 4.2 Log returns on the prices of crude oil between January 2, 2008 and December 31, 2012.

The statistics of crude oil price and log returns are given in Table 4.1.

Table 4.1 Statistics of crude oil price data set.

	Asset prices	Log returns
Sample size	1254	1253
Mean	92.12226	0.000046068
Standard deviation	24.44392	0.010455576
Skewness	-0.38537	0.018533709
Kurtosis	-0.77659	6.66512772
Maximum	143.95	0.078736459
Minimum	592.57	-0.073100493

We present a numerical comparison between FFT and a Monte Carlo simulation option pricing. We apply the two techniques for the pricing of a European call option: $S_0 = 1.3$, $b = 10$, $r = 0.12$, $v_0 = 0.18$, $a = 4.0399$, $\sigma = 0.04$, $\rho = 0.9$, $\lambda^S = 0.11$, $\lambda^v = 0.02$, $\kappa = 3.33$, $\theta = 0.5328$

Table 4.2 European call option price FFT vs. Monte Carlo

Strike price	FFT	Monte Carlo
0.375	0.9674	0.8712
0.456	0.8956	0.9710
0.5548	0.8079	0.8780
0.675	0.7014	0.7615
0.822	0.5720	0.6243
1	0.4239	0.4545
1.217	0.2740	0.2546
1.481	0.0532	0.0567
1.802	0.0014	0.0020

CHAPTER V

CONCLUSIONS

5.1 Conclusion

The aim of this thesis is to introduce an alternative model of stochastic volatility of jump-diffusion in which the asset prices follow a jump-diffusion with stochastic volatility. The following procedures were investigated:

1. We investigated the solution of the underlying asset by adding jumps in stochastic volatility.

2. The underlying asset was assumed to follow a mean reverting process.

3. The mathematical formula of the European option was formulated by inverting the characteristic function. In order to solve the characteristic function explicitly, we proved the lemma that established a relationship between stochastic volatility and partial differential equation in the general case. We got an explicit formula of the characteristic function. The formula of the European option can be expressed in terms of the probability function.

4. A simulation example shows the paths simulated by jump-diffusion with stochastic volatility and incorporating jump in stochastic volatility.

5.2 Research Possibility

1. A stochastic process with independent, stationary increments is called Lévy process. It represents the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals

of the same length. The most well known examples of Lévy processes are Brownian motion and the Poisson process. Thus the asset price can be extended to Lévy processes.

2. In this thesis, the interest rates are assumed to be constant over the period of analysis. In practice, interest rates are determined by monetary policy of a country according to its economic situation. Thus we may extent this thesis to the case where we have a stochastic volatility for the interest rate.

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APPENDIX

COMPUTER PROGRAM

MATLAB CODE

FFT for the jump-diffusion and stochastic volatility with jump Model

The following MATLAB routine computes the price of a European vanilla call option under the dynamics of the jump diffusion and stochastic volatility with jump model by means of the fast Fourier transform method of Carr and Madan (1999). The function, the jump diffusion and stochastic volatility with jump model fast Fourier transform takes the the jump diffusion and stochastic volatility with jump model parameters (KAPPA, THETA, SIGMAv, RHO, V0, LAMBDAJ, MUS, SIGMAS, MUV, r, T, S0, K, a1, b1) as inputs, as well as the risk-free rate of return (r), the maturity of the option (T), the spot price of the underlying (S0) and the strike price of the option (K). It outputs a single option price for the call option as well as the strike price on the FFT strike grid that is closest to K. It is simple to extend the code to output option prices for a range of strikes.

```
function [svjjfft strike] = SVJJFFT(KAPPA, THETA, SIGMAv, RHO, V0,
LAMBDAJ, MUS, SIGMAS, MUV, r, T, S0, K, a1, b1)
alpha = 0.75;
N = 2^12;
a = 600; %Upper limit of integration
eta = a/N; %Grid spacing for integration
lambda = (2*pi) / (N*eta); %Width of intervals btw successive strikes
b = N*lambda / 2;
if S0 >= K %For ITM and ATM options
u = (0:(N-1)) * eta; %Integration grid
```

```
v = u - (alpha + 1) * 1i;
%Characteristic Function Variables
%Diffusion Variables
ALPHA = -0.5*(v.^2 + 1i*v);
BETA = KAPPA - RHO*SIGMAv*1i*v;
GAMMA = (SIGMAv^2)/2;
d = sqrt(BETA.^2 - 4*ALPHA*GAMMA);
rpos = (BETA + d)/(SIGMAv^2);
rneg = (BETA - d)/(SIGMAv^2);
g = rneg./rpos;
D = rneg .* ((1 - exp(-d*T)) ./ (1 - g.*exp(-d*T)));
C = KAPPA * (rneg*T - (2/(SIGMAv^2)) * log((1 - g.*exp(-d*T)) ./ (1 - g)));
%Jump Variables
MUJ = exp(MUS + 0.5*SIGMAS^2) / (1 - RHOJ*MUV) - 1;
c = 1 - RHOJ*MUV*1i*v;
nu = ( (BETA + d) ./ ((BETA + d).*c - 2*MUV*ALPHA) ) * T + (
(4*MUV*ALPHA) ./ ((d.*c).^2 - (2*MUV*ALPHA - BETA.*c) ...
.^2) ) .* log( 1 - ( ((d-BETA).*c + 2*MUV*ALPHA) ./ (2*d.*c) ).*(1 - exp(-d*T)) );
P = -T*(1 + MUJ*1i*v) + exp( MUS*1i*v + 0.5*(SIGMAS^2)*(1i*v).^2 ).*nu;
%Characteristic Function and Fourier Transform
CharFun = exp(C*THETA + D*V0 + P*LAMBDAJ + 1i*v*(log(S0) + (r-q)*T));
FourierTrans = (exp(-r*T) * CharFun) ./ ((alpha + 1i*u) .* (alpha + 1i*u + 1));
SWeightings = (1/3) * (3 + (-1).^(1:N) - [1 zeros(1,N-1)]);
%Include Simpson's weightings
FFT = exp(1i*b*u) .* FourierTrans * eta .* SWeightings;
%FFT Routine
FFT = real(fft(FFT)); %Call MATLAB FFT routine
%Call Price Calculation
strikes = -b + lambda*(0:N-1); %Log-strike price grid
```

```
svjjfft = (exp(-strikes*alpha)/pi) .* FFT; %Include dampening factor
position = (log(K) + b) / lambda + 1; %Strike position on grid
svjjfft = (1-(position-floor(position))) * svjjfft(floor(position)) + (position-floor(position)) ...* svjjfft(floor(position)+1);
%Interpolated FFT call price
elseif S0 < K %For OTM options
u = (0:(N-1)) * eta; %Integration grid
v1 = u - 1i*alpha;
v2 = u + 1i*alpha;
w1 = u - 1i*alpha - 1i;
w2 = u + 1i*alpha - 1i;
%Characteristic Function 1 Variables
%Diffusion Variables
ALPHA1 = -0.5*(w1.^2 + 1i*w1);
BETA1 = KAPPA - RHO*SIGMAv*1i*w1;
GAMMA1 = (SIGMAv^2)/2;
d1 = sqrt(BETA1.^2 - 4*ALPHA1*GAMMA1);
rpos1 = (BETA1 + d1)/(SIGMAv^2);
rneg1 = (BETA1 - d1)/(SIGMAv^2);
g1 = rneg1./rpos1;
D1 = rneg1 .* ((1 - exp(-d1*T)) ./ (1 - g1.*exp(-d1*T)));
C1 = KAPPA * (rneg1*T - (2/(SIGMAv^2)) * log((1 - g1.*exp(-d1*T))./ (1 - g1)));
%Jump Variables
MUJ1 = exp(MUS + 0.5*SIGMAS^2) / (1 - RHOJ*MUV) - 1;
c1 = 1 - RHOJ*MUV*1i*w1;
nu1 = ( (BETA1 + d1) ./ ((BETA1 + d1).*c1 - 2*MUV*ALPHA1) ) * T + ...
( (4*MUV*ALPHA1) ./ ((d1.*c1).^2 - (2*MUV*ALPHA1 - BETA1.*c1) ...
.^2) ) .* log( 1 - ( ((d1-BETA1).*c1 + 2*MUV*ALPHA1) ./ (2*d1.*c1) ) .* (1 - exp(-d1*T)) );
```

```
P1 = -T*(1 + MUJ1*1i*w1) + exp( MUS*1i*w1 + 0.5*(SIGMAS^2)*(1i*w1).^2 ).*nu1;
CharFun1 = exp(C1*THETA + D1*V0 + P1*LAMBDAJ + 1i*w1*(log(S0) + (r-q)*T));
%Characteristic Function 2 Variables
ALPHA2 = -0.5*(w2.^2 + 1i*w2);
BETA2 = KAPPA - RHO*SIGMAv*1i*w2;
GAMMA2 = (SIGMAv^2)/2;
d2 = sqrt(BETA2.^2 - 4*ALPHA2*GAMMA2);
rpos2 = (BETA2 + d2)/(SIGMAv^2);
rneg2 = (BETA2 - d2)/(SIGMAv^2);
g2 = rneg2./rpos2;
D2 = rneg2 .* ((1 - exp(-d2*T)) ./ (1 - g2.*exp(-d2*T)));
C2 = KAPPA * (rneg2*T - (2/(SIGMAv^2)) * log((1 - g2.*exp(-d2*T)) ./ (1 - g2)));
%Jump Variables
MUJ2 = exp(MUS + 0.5*SIGMAS^2) / (1 - RHOJ*MUV) - 1;
c2 = 1 - RHOJ*MUV*1i*w2;
nu2 = ( (BETA2 + d2) ./ ((BETA2 + d2).*c2 - 2*MUV*ALPHA2) ) * T + ...
( (4*MUV*ALPHA2) ./ ((d2.*c2).^2 - (2*MUV*ALPHA2 - BETA2.*c2) ...
.^2) ) .* log( 1 - ( ((d2-BETA2).*c2 + 2*MUV*ALPHA2) ./ (2*d2.*c2) ) .* (1 - exp(-d2*T)) );
P2 = -T*(1 + MUJ2*1i*w2) + exp( MUS*1i*w2 + 0.5*(SIGMAS^2)*(1i*w2).^2 ).*nu2;
CharFun2 = exp(C2*THETA + D2*V0 + P2*LAMBDAJ + 1i*w2*(log(S0) + (r-q)*T));
%Characteristic Function and Fourier Transform
zeta1 = exp(-r*T) * ((1./(1 + 1i*v1)) - exp(r*T)./(1i*v1) - CharFun1./(v1.^2 - 1i*v1));
```

```
zeta2 = exp(-r*T) * ((1./(1 + 1i*v2)) - exp(r*T)./(1i*v2) - CharFun2./(v2.^2 -
1i*v2));
FourierTrans = (zeta1 - zeta2) / 2;
SWeightings = (1/3) * (3 + (-1).^(1:N) - [1 zeros(1,N-1)]);
%Include Simpson's weightings
FFT = exp(1i*b*u) .* FourierTrans * eta .* SWeightings;
%FFT Routine
FFT = real(fft(FFT)); %Call MATLAB FFT routine
%Call Price Calculation
strikes = -b + lambda*(0:N-1); %Log-strike price grid
svjjfft = (1 ./ (pi*sinh(alpha*strikes))) .* FFT;
%Include dampening factor
position = (log(K) + b) / lambda + 1; %Strike position on grid
svjjfft = (1-(position-floor(position))) * svjjfft(floor(position)) + (position-
floor(position)) ...* svjjfft(floor(position)+1);
%Interpolated FFT call price
end
strikes = -b + lambda*(0:N-1);
strike = exp(strikes(round(position))); %Strike price on strike grid
%closest to required strike
```

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- N. Nonthiya and P. Sattayatham (2011). Stochastic Volatility Jump-Diffusion Model for Option Pricing. **Journal of Mathematical Finance**. 1(3): 90-97.

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