GB-splines and algorithms of shape preserving approximation

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Abstract. This paper defines a class of functions with shape preserving properties ("isogeometry") determined by a given set of intervals on the plane $\mathbb{R}^2$. Based on the definition, we provide very simple one- and three-point local algorithms for convex and monotone approximation of curves and surfaces by C^2 generalized cubic splines. The generalized splines are represented as a linear combination of GB-splines. In 2-D case tensor-product splines are used. We give the results of some numerical tests.

Key words: GB-splines, shape preserving, isogeometric approximation, tensor product surfaces.

1. Introduction

For the shape-preserving interpolation problem solving various types of generalized cubic splines were successfully applied [Beatson & Wolkowitz '89; Boor '78; Gregory '86; Sakai & Silanes '86; Schaback '90]. By inserting some parameters in the structure of a spline we can ensure the desired geometric properties of the spline curve and, in particular, preserve monotonicity and convexity of initial data by choosing the parameters properly. The key moment is here the developing of algorithms for authomatic choice of parameters. The known algorithms are based mainly on the piecewise representation of the splines [McCartin '90; Miroshnichenko '84].

In many practical problems initial data are known approximately. Therefore of interest is developing smoothers of initial data that would work within a prescribed error and at the same time inherit geometric peculiarities of the data. The statement of the problem was considered in [Grebennikov '83;

† Supported in part by the Russian Fundamental Research fund (93-01-00495)

Schmidt & Scholz '90]. In [Grebennikov '85] a sequential smoother was developed for this purpose. In [Kvasov & Vanin '93; Kvasov & Yatsenko '90] direct isogeometric approximation algorithms were suggested based on the tool of rational cubic and parabolic B-splines.

In the present paper the problem of isogeometric approximation for interval data is formalized by introducing the notion of a class of functions with the isogeometry. The algorithms of isogeometric local approximation are developed on the base of the generalized B-spline tool. The data are smoothed using one- and three-point local approximation formulae with automatic choice of the parameters, with convexity and monotonicity of initial data being retained.

By the surface approximation we suppose the initial data be given point by point as a collection of nonintersecting and, in general, curvilinear sections of a 3-D body. At the beginning via the algorithm of isogeometric interpolation [Kvasov & Yatsenko '87] the system of curves along the initial cross-sections is constructed. The 2-D spline is defined as the tensor product of one-dimensional splines. To this end along the orthogonal direction the set of generalized local approximation splines is generated. On the resulting surface the system of curvilinear coordinate lines forming the regular meth is constructed. Along these lines such shape properties of the initial data as convexity, monotonicity, the presence of linear sections, the angles and the bends are retained. The possibilities of the algorithm are illustrated by test examples.

2. The Problem of Isogeometric Approximation

Suppose a set of intervals $F = \{F_i \,|\, i = 0, \ldots, N\}$, $F_i \equiv [f_i - \varepsilon_i, f_i + \varepsilon_i]$, $i = 0, \ldots, N$, with prescribed small $\varepsilon_i > 0$ on a grid $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ be given. We call the problem of searching for a sufficiently smooth function $S(x) \in C^2[a, b]$ such that $S(x_i) \in F_i$, $i = 0, \ldots, N$, and $S(x)$ preserves the shape of the initial data an *isogeometric approximation problem.*

To formalize this problem we introduce the notations:

$$\begin{aligned} &\Delta_i S = (S(x_{i+1}) - S(x_i))/h_i, \quad h_i = x_{i+1} - x_i, \quad i = 0, \ldots, N-1; \\ &\delta_i S = \Delta_i S - \Delta_{i-1} S, \quad i = 1, \ldots, N-1 \end{aligned}$$

and use the interval differences [Shokin '81]

$$\Delta_i F = h_i^{-1}(F_{i+1} - F_i) = [\Delta_i f - e_i, \Delta_i f + e_i], \quad e_i = h_i^{-1}(\varepsilon_i + \varepsilon_{i+1}), \\ i = 0, \ldots, N-1,$$

$$\delta_i F = \Delta_i F - \Delta_{i-1} F = [\delta_i f - E_i, \delta_i f + E_i], \quad E_i = e_{i-1} + e_i, \quad i = 1, \ldots, N-1,$$

$$[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1] > 0 \quad \text{if and only if} \quad a_1 > b_2.$$

The initial data are said to increase (decrease) monotonically on a subinterval $[x_R, x_K]$, $K > R$, if $\Delta_i F > 0$ ($\Delta_i F < 0$), $i = R, \ldots, K-1$. The data are called convex downwards (upwards) on $[x_R, x_K]$, $K > R+1$, if $\delta_i F > 0$ ($\delta_i F < 0$), $i = R+1, \ldots, K-1$.

We assume that the intervals $\Delta_i F$, $\delta_i F$ for all i do not contain zeros, i.e. the initial data satisfy the conditions

$$(\Delta_i f)^2 > e_i^2, \quad i = 0, \ldots, N-1; \quad (\delta_i f)^2 > E_i^2, \quad i = 1, \ldots, N-1. \tag{2.1}$$

If these inequalities are valid all over the interval $[a, b]$, the initial data uniquely define monotonicity and convexity of an approximation function $S(x)$. If the values of a function $S(x)$ are such that $S(x_i) \in F_i$, $i = 0, \ldots, N$, then we have $\Delta_i S \in \Delta_i F$, $i = 0, \ldots, N-1$, $\delta_i S \in \delta_i F$, $i = 1, \ldots, N-1$. Taking into account the inequalities for the initial data, we obtain

$$\Delta_i S \, \Delta_i f > 0, \quad i = 0, \ldots, N-1, \quad \delta_i S \, \delta_i f > 0, \quad i = 1, \ldots, N-1. \tag{2.2}$$

Definition 2.1. *The set of functions $I(\Delta, F)$ is called the class of isogeometric approximants if for any function $S(x) \in I(\Delta, F)$ the following conditions are met:*

1. $S(x) \in C^2[a, b]$;
2. $S(x_i) \in F_i$, $i = 0, \ldots, N$;
3. *$S(x)$ is monotonic in $[x_i, x_{i+1}]$, $i = 1, \ldots, N-2$ if $\Delta_{i-1} f \Delta_i f > 0$, $\Delta_i f \Delta_{i+1} f > 0$; $S(x)$ is monotonic in $[x_0, x_1]$ if $\Delta_0 f \Delta_1 f > 0$ and in $[x_{N-1}, x_N]$ if $\Delta_{N-2} f \Delta_{N-1} f > 0$;*
   $S'(x)$ has one sign change in $[x_{i-1}, x_{i+1}]$, $i = 1, \ldots, N-1$ if $\Delta_{i-1} f \Delta_i f < 0$; the number of sign changes of the function $S'(x)$ in $[a, b]$ coincides with that in the sequence $\Delta_0 f, \Delta_1 f, \ldots, \Delta_{N-1} f$; and
4. *$S''(x_i) \delta_i f \geq 0$, $i = 1, \ldots, N-1$; the number of sign changes of the function $S''(x)$ in $x \in [a, b]$ coincides with that in the sequence $\delta_1 f, \delta_2 f, \ldots, \delta_{N-1} f$.*

The isogeometric approximation problem is, by definition, the problem of searching for a function $S(x) \in I(\Delta, F)$. The solution of isogeometric approximation problem we seek in the form of generalized cubic spline.

3. A Basis of Generalized B-splines

Let the partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ on the interval $[a, b]$ be given. For a sufficiently smooth function $S(x)$ we set $S_i^r = S^{(r)}(x_i)$, $r = 0, 1, 2,\ i = 0, \dots, N$.

Definition 3.1. *Our generalized cubic spline on the mesh Δ is a function $S(x) \in C^2[a, b]$ such that in any subinterval $[x_i, x_{i+1}]$ it has the form*

$$\begin{aligned} S(x) = [S_i - \varphi_i(0)h_i^2 S_i''](1-t) + [S_{i+1} - \psi_i(1)h_i^2 S_{i+1}'']t \\ +\varphi_i(t)h_i^2 S_i'' + \psi_i(t)h_i^2 S_{i+1}'', \end{aligned} \tag{3.1}$$

where $t = (x - x_i)/h_i$ and the functions $\varphi_i(t)$, $\psi_i(t)$ satisfy the conditions

$$\varphi_i^{(r)}(1) = \psi_i^{(r)}(0) = 0, \quad r = 0, 1, 2; \quad \varphi_i''(0) = \psi_i''(1) = 1.$$

The set of splines complying with Definition 3.1 is denoted by S_3^G. The choice of functions $\varphi_i(t)$, $\psi_i(t)$, depending on the parameters is essential for the properties of the spline. By this reason follow to [Zav'yalov '90] we call these ones the defining functions.

Let us construct a system of basis functions with minimum-length supports for the set of generalized cubic splines S_3^G. As $dim(S_3^G) = 4N - 3(N-1) = N + 3$, we extend mesh Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$ such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+1} < x_{N+2} < x_{N+3}$.

We demand that the basis splines $B_i(x)$, $i = -1, \dots, N+1$, have the properties

$$B_i(x) \begin{cases} > 0, & \text{if} \quad x \in (x_{i-2}, x_{i+2}), \\ \equiv 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

$$\sum_{j=-1}^{N+1} B_j(x) \equiv 1 \quad for \quad x \in [a, b]. \tag{3.3}$$

To provide property (3.2) we suppose that in (3.1) the second derivatives of the defining functions $\varphi_i''(t)$, $\psi_i''(t)$, $t \in [0, 1]$ for the generalized cubic spline are nonnegative monotonic functions.

According to Definition 3.1, the basis spline $B_i(x)$ – different from zero only in the interval (x_{i-2}, x_{i+2}) – should have the form

$$B_i(x) = \begin{cases} B_i''(x_{i-1})\Psi_{i-2}(x), \quad x \in [x_{i-2}, x_{i-1}], \\ B_i''(x_{i-1})[v_{i-1} + v_{i-1}'(x - x_{i-1})] \\ \quad + B_i''(x_{i-1})\Phi_{i-1}(x) + B_i''(x_i)\Psi_{i-1}(x), \quad x \in [x_{i-1}, x_i], \\ -B_i''(x_{i+1})[v_{i+1} + v_{i+1}'(x - x_{i+1})] \\ \quad + B_i''(x_i)\Phi_i(x) + B_i''(x_{i+1})\Psi_i(x), \quad x \in [x_i, x_{i+1}], \\ B_i''(x_{i+1})\Phi_{i+1}(x), \quad x \in [x_{i+1}, x_{i+2}], \\ 0, \quad x \notin [x_{i-2}, x_{i+2}], \end{cases} \tag{3.4}$$

where

$$\Phi_j(x) = \varphi_j\left(\frac{x - x_j}{h_j}\right)h_j^2, \quad \Psi_j(x) = \psi_j\left(\frac{x - x_j}{h_j}\right)h_j^2,$$

$$v_j^{(r)} = \psi_{j-1}^{(r)}(1)h_{j-1}^{2-r} - \varphi_j^{(r)}(0)h_j^{2-r}, \quad r = 0, 1.$$

As

$$\sum_{j=i-1}^{i+1} B_i''(x_j)[v_j + v_j'(x - x_j)] \equiv 0,$$

we have

$$\sum_{j=i-1}^{i+1} B_i''(x_j)v_j'y_j^r = 0, \quad y_j = x_j - v_j/v_j', \quad r = 0, 1.$$

From (3.3) and (3.4) for $x \in [x_i, x_{i+1}]$ we get

$$\sum_{j=i-1}^{i+2} B_j(x) = \Phi_i(x)\sum_{j=i-1}^{i+1} B_j''(x_i) + \Psi_i(x)\sum_{j=i}^{i+2} B_i''(x_{i+1})$$
$$- B_i''(x_{i+1})v_{i+1}'(x - y_{i+1}) + B_{i+1}''(x_i)v_i'(x - y_i) \equiv 1.$$

Because of the linear independance of 1, x, $\Phi_i(x)$ and $\Psi_i(x)$ we obtain the equations

$$\sum_{j=i-1}^{i+1} B_j''(x_i) = \sum_{j=i}^{i+2} B_j''(x_{i+1}) = 0,$$

$$B_i''(x_{i+1})v_{i+1}'y_{i+1}^r - B_{i+1}''(x_i)v_i'y_i^r = \delta_{1r}, \quad r = 0, 1.$$

From this system we find by elimination

$$B_i''(x_j) = \frac{y_{i+1} - y_{i-1}}{v_j'\omega_{i-1}'(y_j)}, \quad j = i - 1, i, i + 1,$$

where $\omega_{i-1}(x) = (x - y_{i-1})(x - y_i)(x - y_{i+1})$.

Using methods in [Zav'yalov et al. '80], it is possible to show that the splines $B_i(x)$, $i = -1, \ldots, N+1$ are the nonnegative functions with minimum-length supports such that the identities

$$\sum_{j=-1}^{N+1} y_j^r B_j(x) \equiv x^r, \quad r = 0, 1 \quad \text{for} \quad x \in [a, b]$$

are valid. These functions are linearly independent and form a basis in the space of generalized cubic splines S_3^G. Therefore any spline $S(x) \in S_3^G$ can be uniquely represented in the form

$$S(x) = \sum_{j=-1}^{N+1} b_j B_j(x) \quad \text{for} \quad x \in [a, b] \tag{3.5}$$

with some constant coefficients b_j.

Further we consider the case when 'averaged nodes' of B-splines $y_i = x_i - v_i/v_i'$, $i = 0, \ldots, N$ coincide with the nodes of main mesh Δ, i.e. $v_i = \psi_{i-1}(1)h_{i-1}^2 - \varphi_i(0)h_i^2 = 0$, $i = 0, \ldots, N$ and $x_{-i} = x_0 - ih_0$, $x_{N+i} = x_N + ih_{N-1}$, $i = 1, 2, 3$.

From (3.4) the expression (3.5) for spline $S(x)$ in the subinterval $[x_i, x_{i+1}]$ is transformed to the form

$$S(x) = b_i + \Delta_i b(x - x_i) + \varphi_i(t)h_i^2 \delta_i b/v_i' + \psi_i(t)h_i^2 \delta_{i+1} b/v_{i+1}', \tag{3.6}$$

where $\delta_j b = \Delta_j b - \Delta_{j-1} b$, $j = i, i+1$, $\Delta_j b = (b_{j+1} - b_j)/h_j$.

Whence the formulae are followed

$$S(x_i) = b_i + \delta_i b \left(\frac{\psi_{i-1}'(1)}{\psi_{i-1}(1)} \frac{1}{h_{i-1}} - \frac{\varphi_i'(0)}{\varphi_i(0)} \frac{1}{h_i} \right)^{-1}, \tag{3.7}$$

$$S'(x_i) = \frac{1}{v_i'} [\psi_{i-1}'(1) h_{i-1} \Delta_i b - \varphi_i'(0) h_i \Delta_{i-1} b], \tag{3.8}$$

$$S''(x_i) = \delta_i b / v_i' \tag{3.9}$$

and vice versa

$$\begin{aligned} b_{i-1} &= S(x_i) - h_{i-1} S'(x_i) + h_{i-1}^2 [-\psi_{i-1}(1) + \psi_{i-1}'(1)] S''(x_i), \\ b_i &= S(x_i) - h_i^2 \varphi_i(0) S''(x_i), \\ b_{i+1} &= S(x_i) + h_i S'(x_i) - h_i^2 [\varphi_i(0) + \varphi_i'(0)] S''(x_i), \\ & \qquad i = 0, \ldots, N. \end{aligned} \tag{3.10}$$

The choice of defining functions $\varphi_i(t)$, $\psi_i(t)$ we subject to conditions: $\varphi_i(t) = \varphi(p_i, t)$, $\psi_i(t) = \varphi(q_i, 1-t)$, where parameters $p_i, q_i \geq 0$ for all i and $\varphi(p, 0)$ is a strictly monotonic decreasing function.

We display some commonly used examples of defining functions $\varphi_i(t)$, $\psi_i(t)$ in (3.1).

a) Rational splines with linear denominator [Späth '90]:

$$\varphi_i(t) = P_i(1-t)^3/(1+p_i t), \quad P_i^{-1} = 2(3 + 3p_i + p_i^2).$$

b) Rational splines with quadratic denominator:

$$\varphi_i(t) = P_i(1-t)^3/[1 + p_i t(1-t)], \quad P_i^{-1} = 2(1+p_i)(3+p_i).$$

c) Exponential splines [Koch & Lyche '93; Späth '90]:

$$\varphi_i(t) = (1-t)^3 e^{-p_i t}/(6 + 6p_i + p_i^2).$$

d) Hyperbolic splines (see [McCartin '90] and the numerous references in this paper):

$$\varphi_i(t) = \frac{\sinh p_i(1-t) - p_i(1-t)}{p_i^2 \sinh(p_i)}.$$

e) Splines with additional nodes [Pruess '79]:

$$\varphi_i(t) = \frac{1}{6(1+p_i)^2}[1 - (1+p_i)t]_+^3, \quad x_+ = \max(0, x).$$

Different generalizations of the parabolic splines in [Stechkin & Subbotin '76] very useful for practical calculations can be easily included in our scheme. To define such splines we construct the additional mesh $\overline{\Delta} = \{\overline{x}_i \mid i = -2, \ldots, N+3\}$, where $x_{i-1} < \overline{x}_i < x_i$, $\overline{x}_{i+1} = x_i + \alpha_i h_i = x_{i+1} - \beta_i h_i$. Based on the representation (3.1) we can suggest the following examples of the defining functions $\varphi_i(t) = \varphi(\alpha_i, p_i, t)$, $\psi_i(t) = \varphi(\beta_i, q_i, 1-t)$:

a) $\varphi_i(t) = P_i(\alpha_i - t)_+^2/(1 + p_i t)$, $P_i^{-1} = 2(1 + \alpha_i p_i)^2$;

b) $\varphi_i(t) = P_i(\alpha_i - t)_+^2/[1 + p_i t(1-t)]$, $P_i^{-1} = 2[(1+\alpha_i p_i)^2 + \alpha_i^2 p_i]$;

c) $\varphi_i(t) = P_i e^{-p_i t}(\alpha_i - t)_+^2$, $P_i^{-1} = (2 + \alpha_i p_i)^2 - 2$;

d) $\varphi_i(t) = [\alpha_i/(1+p_i) - t]_+^2/2$.

4. One-point Algorithm of Isogeometric Local Approximation

In (3.5) we set $b_i = f_i$, $i = 0, \ldots, N$. The coefficients b_{-1} and b_{N+1} can be computed by different methods depending the problem solved. For example these coefficients can be find from the one of the following boundary conditions [Zav'yalov et al. '80]:

I. $S'(x_i) = f'_i$, $i = 0, N$;
II. $S''(x_i) = f''_i$, $i = 0, N$;
III. The periodic problem: $h_{N+i} = h_i$, $f_{N+i} = f_i$;
IV. $S(x_i) = f_i$, $i = 0, N$.

From the condition $v_i = 0$ or $\varphi(q_{i-1}, 0)h_{i-1}^2 = \varphi(p_i, 0)h_i^2$, $i = 0, N$ using the strict monotonicity of function $\varphi(p, 0)$ and the equalities $h_{-1} = h_0$, $h_N = h_{N-1}$ we have $q_{-1} = p_0$, $q_{N-1} = p_N$. Therefore by adding the first and third equations in (3.10) we obtain for the type I boundary conditions at $i = 0, N$

$$b_{-1} = f_1 - 2h_0 f'_0, \quad b_{N+1} = f_{N-1} + 2h_{N-1} f'_N. \tag{4.1}$$

For the type II boundary conditions by virtue of (3.10) we have

$$\begin{aligned} b_{-1} &= 2f_0 - f_1 - 2h_0^2 \varphi'_0(0) f''_0, \\ b_{N+1} &= 2f_N - f_{N-1} + 2h_{N-1}^2 \psi'_{N-1}(1) f''_N. \end{aligned} \tag{4.2}$$

In the periodic case $b_{-1} = b_{N-1}$, $b_{N+1} = b_1$. According to (3.10) the case of type IV boundary conditions is equivalent to prescribing $S''(x_i) = 0$, $i = 0, N$ or by virtue of (3.6) $\delta_i b = 0$, $i = 0, N$. Of course, in left and right boundary points different types of conditions can be considered.

The parameters q_{i-1}, p_i, $i = 1, \ldots, N-1$ (in the periodic case $i = 0, \ldots, N$) are chosen to guarantee $|S(x_i) - f_i| \le \varepsilon_i$.

By the formula (3.7) we have

$$S(x_i) - f_i = \left(\frac{\psi'_{i-1}(1)}{\psi_{i-1}(1)} \frac{1}{h_{i-1}} - \frac{\varphi'_i(0)}{\varphi_i(0)} \frac{1}{h_i} \right)^{-1} \delta_i f, \quad i = 1, \ldots, N-1. \tag{4.3}$$

As for $h_{i-1} \le h_i$ the equation $v_i = 0$ yield the inequality $q_{i-1} \le p_i$ and the relation (4.3) permits a simple choice of the parameters q_{i-1}, p_i. If $h_{i-1} \le h_i$ we use

$$|S(x_i) - f_i| \le |\delta_i f| \left[\left(\frac{1}{h_{i-1}} + \frac{1}{h_i} \right) \frac{\psi'_{i-1}(1)}{\psi_{i-1}(1)} \right]^{-1} \le \varepsilon_i.$$

Taking into account the constraint for generalized cubic splines $\psi'_{i-1}(1)/\psi_{i-1}(1) \ge 3$, we can set

$$\frac{\psi'_{i-1}(1)}{\psi_{i-1}(1)} - 3 = \max \left(\frac{|\delta_i f| h_{i-1} h_i}{h_{i-1} + h_i} \frac{1}{\varepsilon_i} - 3, 0 \right)$$

to define q_{i-1} and the value p_i is calculated from condition $v_i = 0$ or

$$\varphi(q_{i-1}, 0)h_{i-1}^2 = \varphi(p_i, 0)h_i^2, \quad i = 1, \ldots, N-1.$$

The case in which $h_i < h_{i-1}$ and $p_i < q_{i-1}$ is considered in a similar manner.

In case of type I boundary conditions using the formulae (3.7) and (4.1) to find p_0, q_{N-1} we analogously have

$$\frac{-\varphi_0'(0)}{\varphi_0(0)} - 3 = \max\left(\frac{h_0}{\varepsilon_0}|\Delta_0 f - f_0'| - 3, 0\right),$$
$$\frac{\psi_{N-1}'(1)}{\psi_{N-1}(1)} - 3 = \max\left(\frac{h_{N-1}}{\varepsilon_N}|f_N' - \Delta_{N-1}| - 3, 0\right).$$

For type II boundary conditions by virtue of (3.10) we obtain

$$S(x_i) = f_i + h_i^2\varphi_i(0)f_i'', \quad i = 0, N,$$

whence the restrictions

$$[\varphi_i(0)]^{-1} \geq \frac{h_i^2|f_i''|}{\varepsilon_i}, \quad i = 0, N,$$

are satisfied.

For generalized cubic splines the inequality $1/\varphi_i(0) \geq 6$ is valid and the p_0, p_N values are chosen by the rule

$$[\varphi_i(0)]^{-1} - 6 = \max\left(\frac{h_i^2|f_i''|}{\varepsilon_i} - 6, 0\right), \quad i = 0, N.$$

After the calculation of the parameters p_0, q_{N-1} the coefficients b_j, $j = -1, N+1$ are found from (4.2).

Lemma 4.1. *The function $S''(x)$ changes sign on the interval $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, only once if the condition $S''(x_i)S''(x_{i+1}) < 0$ is fulfilled.*

Proof: According to (3.6) in $x \in [x_i, x_{i+1}]$, we have

$$S''(x) = \varphi_i''(t)S''(x_i) + \psi_i''(t)S''(x_{i+1}). \tag{4.4}$$

By assumption for $t \in [0, 1]$ the function $\varphi_i''(t) \geq 0$ monotonically decreases ($\psi_i''(t) \geq 0$ monotonically increases). Therefore, if $S''(x_i)S''(x_{i+1}) > 0$ by virtue of (4.4) in $x \in [x_i, x_{i+1}]$ the function $S'''(x)$ is of constant sign. If $S''(x_i)S''(x_{i+1}) < 0$ then by virtue of constant sign

$$\frac{d}{dx}S''(x) = h_i^{-1}[\varphi_i'''(t)S''(x_i) + \psi_i'''(t)S''(x_{i+1})] \quad \text{in} \quad [x_i, x_{i+1}]$$

the function $S''(x)$ is monotonic and has precisely one sign change. □

Theorem 4.1. *If the inequalities*

$$\begin{gathered}(\Delta_0 f - f_0')\delta_1 f > 0, \quad f_0'\Delta_0 f \geq 0, \\ (f_N' - \Delta_{N-1} f)\delta_{N-1} f > 0, \quad f_N'\Delta_{N-1} f \geq 0,\end{gathered} \tag{4.5}$$

$$f_0''\delta_1 f \geq 0, \quad f_N''\delta_{N-1} f \geq 0 \tag{4.6}$$

(by assumption $\delta_1 f \neq 0$ and $\delta_{N-1} f \neq 0$) are valid, the generalized cubic spline $S(x)$ constructed by Algorithm of one-point local approximation is an isogeometric approximant.

Proof: In fact by (3.9) we have $S''(x_i) = \delta_i f / v_i'$, $i = 1, \ldots, N-1$. Since $v_i' > 0$ and with allowance made for the restrictions on the initial data (2.1), we derive $S''(x_i)\delta_i f > 0$, $i = 1, \ldots, N-1$.

For type I boundary conditions it follows from (3.9) and (4.1) that $S''(x_0) = \delta_0 f / v_0' = 2(v_0')^{-1}(\Delta_0 f - f_0')$. Therefore, by virtue of (4.5) and (4.6) in the case of type I and II boundary conditions the inequality $S''(x_0)S''(x_1) > 0$ is fulfilled. Analogously the inequality $S''(x_{N-1})S''(x_N) > 0$ is established. Taking into account Lemma 4.1 we conclude that the number of sign changes of the function $S''(x)$ for $[a, b]$ coincides with the one in the sequence $\delta_i f$, $i = 1, \ldots, N-1$. Thus the conditions 4. from Definition 2.1 are met.

Let us now introduce the mesh $\gamma : a = z_0 < z_1 < \cdots < z_N < z_{N+1} = b$, where if $S''(x_{i-1})S''(x_i) \geq 0$, $i = 1, \ldots, N$ we set $z_i = \xi_i \in (x_i, x_{i+1})$ according to the equality $S(x_i) - S(x_{i-1}) = S'(\xi_i)(x_i - x_{i-1})$, and if $S''(x_{i-1}) \times S''(x_i) < 0$ we choose $z_i = x^*$ from the condition $S''(x^*) = 0$, $x^* \in (x_{i-1}, x_i)$.

By construction $S''(z_j)S''(x_i) \geq 0$, $j = i, i \pm 1$. By virtue of the restrictions on the initial data (2.1) and by (3.9) we have $S''(x_i) \neq 0$. Therefore according to Lemma 4.1 $S''(x)$ is of constant sign and $S'(x)$ is monotonic in $[z_i, z_{i+1}]$, $i = 1, \ldots, N-1$, respectively. In $[z_0, z_1]$, $[z_N, z_{N+1}]$ the monotonicity of $S'(x)$ follows from the inequalities $S''(x_i)S''(x_{i+1}) > 0$, $i = 0, N-1$, and from Lemma 4.1.

Let us show that at any inflection point $x^* \in [x_i, x_{i+1}]$, $i = 1, \ldots, N-2$, we have $S'(x^*)\Delta_i f > 0$. From the above it follows that $\delta_i f \delta_{i+1} f < 0$ and one of the two possible cases $\delta_i f \Delta_i f < 0$ or $\delta_i f \Delta_i f > 0$ takes place.

From (3.6) for $x \in [x_i, x_{i+1}]$, we have

$$S'(x) = \Delta_i f + \varphi_i'(t) h_i \delta_i f / v_i' + \psi_i'(t) h_i \delta_{i+1} f / v_{i+1}',$$

where $\varphi_i'(t) \le 0$, $\psi_i'(t) \ge 0$.

Taking into account the signs of the functions $\varphi_i'(t)$, $\psi_i'(t)$, we have $S'(x)\Delta_i f > 0$ at $\delta_i f \Delta_i f < 0$ and therefore $S'(z_{i+1})\Delta_i f = S'(x^*)\Delta_i f > 0$.

Now let $\delta_i f \Delta_i f > 0$. We suppose $\delta_i f > 0$ without loss of generality. The derivative is extremal at an inflection point, i.e. $\Delta_i S < S'(x^*)$. According to (2.2) we have $\Delta_i S \Delta_i f > 0$, and we again come to the inequality $S'(x^*) \times \Delta_i f > 0$. The case in which $\delta_i f < 0$ is considered in a similar manner.

By the construction, we evidently have $S'(z_{i+1})\Delta_i f > 0$ for $S''(x_i) \times S''(x_{i+1}) \ge 0$, $i = 0, \ldots, N-1$. Therefore we have $S'(z_{i+1})\Delta_i f > 0$, $i = 0, \ldots, N-1$, at the nodes of the grid γ.

It was proved above that $S'(x)$ is monotonic in $[z_j, z_{j+1}]$, $j = i, i+1$ and $S'(z_{i+1})\Delta_i f > 0$. Hence $S'(x)$ is monotonic in $[z_i, z_{i+2}]$. If now $\Delta_i f \Delta_j f > 0$, $j = i \pm 1$, we have $S'(z_i)S'(z_{i+2}) > 0$. Hence $S'(x)$ is of constant signs in $[z_i, z_{i+1}]$ and, in particular, in $[x_i, x_{i+1}]$ and $S(x)$ is monotonic in $[x_i, x_{i+1}]$, $i = 1, \ldots, N-1$.

Under the assumption that $\Delta_0 f \Delta_1 f > 0$, a function $S'(x)$ monotonic in $[z_1, z_2]$ will be of constant signs in this interval. Therefore $S'(x_1)\Delta_0 f > 0$. It has been shown above that $S'(x)$ is monotonic in $[x_0, x_1]$. According to (4.5), $f_0' \Delta_0 f \ge 0$ and since $S'(x_1)\Delta_0 f > 0$ the function $S(x)$ is monotonic in $[x_0, x_1]$.

For type II boundary conditions by (3.8) and (4.2) we have $S'(x_0) = \Delta_0 f + h_0 \varphi_0'(0) f_0''$. From here it follows that for a suitable choice of the parameter p_0 we have $S'(x_0)\Delta_0 f > 0$. As $S'(x_1)\Delta_0 f > 0$ the function $S(x)$ is again monotonic in $[x_0, x_1]$. The case of the interval $[x_{N-1}, x_N]$ is considered analogously.

Because the inequalities $S'(x_0)\Delta_0 f > 0$, $S'(z_{i+1})\Delta_i f > 0$, $i = 0, \ldots, N-1$, $S'(x_N)\Delta_{N-1} f > 0$ are valid and $S'(x)$ is monotonic in $[z_i, z_{i+1}]$, $i = 0, \ldots, N$, the function $S'(x)$ changes sign in $[z_i, z_{i+1}]$ and hence in $[x_{i-1}, x_{i+1}]$, $i = 1, \ldots, N-1$, provided $\Delta_{i-1} f \Delta_i f < 0$. The number of sign inversions of the function $S'(x)$ in $[a, b]$ coincides with the one in the sequence $\Delta_0 f, \Delta_1 f, \ldots, \Delta_{N-1} f$. $\square$

5. Three-point Algorithm of Isogeometric Local Approximation

We compute the coefficients in (3.5) from formulae (3.10), with $S''(x_i)$ being approximated using the second divided difference

$$b_i = f_i - 2h_i^2 (h_{i-1} + h_i)^{-1} \varphi_i(0) \delta_i f, \quad i = 1, \ldots, N-1. \tag{5.1}$$

To determine the coefficients b_i, for $i=-1,0,N,N+1$, we use one of the extended boundary conditions of the same types as in Algorithm of one-point approximation.

I. $S^{(k)}(x_i) = f_i^{(k)}, \;\; i=0,N, \;\; k=0,1;$

II. $S^{(k)}(x_i) = f_i^{(k)}, \;\; i=0,N, \;\; k=0,2;$

III. The periodic problem: $h_{N+i}=h_i$, $f_{N+i}=f_i$;

IV. $S(x_i)=f_i$, $i=0,1,N-1,N$.

Using the formulae (3.7)–(3.10), we write out the explicit form of the coefficients b_i, $i=-1,0,N,N+1$ for interesting us boundary conditions. To reduce the exposition the type IV boundary conditions are not considered.

Type I.

$$\begin{aligned} b_{-1} &= b_1 - 2h_0 f_0', \\ b_0 &= f_0 - (f_0 + h_0 f_0' - b_1)[1+\varphi_0'(0)/\varphi_0(0)]^{-1}, \\ b_N &= f_N - (f_N - h_{N-1}f_N' - b_{N-1})[1-\psi_{N-1}'(1)/\psi_{N-1}(1)]^{-1}, \\ b_{N+1} &= b_{N-1} + 2h_{N-1}f_N'. \end{aligned} \tag{5.2}$$

Type II.

$$\begin{aligned} b_{-1} &= 2f_0 - b_1 - 2h_0^2[\varphi_0(0)+\varphi_0'(0)]f_0'', \\ b_0 &= f_0 - h_0^2\varphi_0(0)f_0'', \\ b_N &= f_N - h_{N-1}^2\psi_{N-1}(1)f_N'', \\ b_{N+1} &= 2f_N - b_{N-1} - 2h_{N-1}^2[\psi_{N-1}(1)+\psi_{N-1}'(1)]f_N''. \end{aligned} \tag{5.3}$$

Type III. In the case of periodic boundary conditions formula (5.1) is valid for $i=1,\dots,N$ and $b_{N+i}=b_i$ for all i.

We find the parameters p_i, q_i, $i=0,\dots,N-1$, from the isogeometric conditions formulated in Definition 2.1 in two steps. Using the constraints $|b_i - f_i| \le \varepsilon_i$, $i=1,\dots,N-1$, which in view of (5.1) are equivalent to

$$2h_i^2(h_{i-1}+h_i)^{-1}\varphi_i(0)|\delta_i f| \le \varepsilon_i, \quad i=1,\dots,N-1 \tag{5.4}$$

(in the periodic case $i=0,\dots,N$), we first find p_i and obtain q_{i-1} from the condition $v_i=0$.

For type I boundary conditions according to (5.2) the quantities p_0, q_{N-1} are selected so as to satisfy the inequalities

$$\begin{aligned} |b_0 - f_0| &= |f_0 + h_0 f_0' - b_1||1+\varphi_0'(0)/\varphi_0(0)|^{-1} \le \varepsilon_0, \\ |b_N - f_N| &= |f_N - h_{N-1}f_N' - b_{N-1}||1-\psi_{N-1}'(1)/\psi_{N-1}(1)|^{-1} \le \varepsilon_N. \end{aligned} \tag{5.5}$$

For type II boundary conditions by virtue of the formulae (5.3) the p_0, q_{N-1} values we choose from the estimates

$$|b_0 - f_0| = h_0^2\varphi_0(0)|f_0''| \le \varepsilon_0,$$
$$|b_N - f_N| = h_{N-1}^2\psi_{N-1}(1)|f_N''| \le \varepsilon_N. \tag{5.6}$$

Finally p_i, q_i we find from the constraints $|S(x_i) - f_i| \le \varepsilon_i$, $i = 0, \dots, N$. According to (3.7) and (5.1), we have

$$S(x_i) = f_i + H_i^{-1}\Big\{ -\frac{2h_{i-1}^2\varphi_{i-1}(0)}{h_{i-2}+h_{i-1}}\delta_{i-1}f - \frac{2h_{i+1}^2\varphi_{i+1}(0)}{h_i(h_i+h_{i+1})}\delta_{i+1}f$$
$$+ [1 + 2h_{i-1}^{-1}h_i\varphi_i(0) - 2(h_{i-1}+h_i)^{-1}(\psi_{i-1}'(1)h_{i-1} - \varphi_i'(0)h_i)]\delta_i f\Big\}, \tag{5.7}$$

where

$$H_i = \frac{\psi_{i-1}'(1)}{\psi_{i-1}(1)}\frac{1}{h_{i-1}} - \frac{\varphi_i'(0)}{\varphi_i(0)}\frac{1}{h_i}.$$

For generalized cubic splines at any relations between h_{i-1} and h_i by virtue of the condition $v_i = 0$ we have

$$h_i h_{i-1}^{-1}\varphi_i(0) = h_{i-1}h_i^{-1}\psi_{i-1}(1) \le 1/6.$$

Therefore according to the estimate (5.4) by (5.7) we obtain

$$|S(x_i) - f_i| \le H_i^{-1}\theta_i \le \varepsilon_i,$$

where $\theta_i = \varepsilon_{i-1}h_{i-1}^{-1} + \frac{4}{3}|\delta_i f| + \varepsilon_{i+1}h_i^{-1}$.

For $h_{i-1} \le h_i$, from here as in Algorithm of one-point approximation we find q_{i-1} from the relation

$$\frac{\psi_{i-1}'(1)}{\psi_{i-1}(1)} - 3 = \max\left(\frac{h_{i-1}h_i}{h_{i-1}+h_i}\frac{\theta_i}{\varepsilon_i} - 3, 0\right), \quad i = 2, \dots, N-2 \tag{5.8}$$

and the values p_i are found from the condition $v_i = 0$.

In the case of type I, II boundary conditions for $i = 1, N-1$, according to (5.2), (5.3) and using (5.5), (5.6), we come again to the formula (5.8) that permits us to choose the parameters q_0, p_{N-1}.

For type I boundary conditions and $h_0 \le h_1$, $h_{N-1} \le h_{N-2}$ we can find q_0, p_{N-1} using also the relations

$$\frac{\psi_0'(1)}{\psi_0(1)} - 3 = \max\left(\frac{h_0 h_1}{h_0+h_1}\frac{\tilde\theta_1}{\varepsilon_1} - 3, 0\right),$$
$$\frac{\varphi_{N-1}'(0)}{\varphi_{N-1}(0)} - 3 = \max\left(\frac{h_{N-2}h_{N-1}}{h_{N-2}+h_{N-1}}\frac{\tilde\theta_{N-1}}{\varepsilon_{N-1}} - 3, 0\right),$$

where

$$\tilde{\theta}_1 = \varepsilon_2 h_1^{-1} + \frac{4}{3}|\delta_1 f| + (|\Delta_0 f - f_0'| + \varepsilon_1 h_0^{-1})|1 + \varphi_0'(0)/\varphi_0(0)|^{-1},$$

$$\tilde{\theta}_{N-1} = \varepsilon_{N-2} h_{N-1}^{-1} + \frac{4}{3}|\delta_{N-1} f| + (|\Delta_{N-1} f - f_N'| + \varepsilon_{N-1} h_{N-1}^{-1}) \times |1 - \psi_{N-1}'(1)/\psi_{N-1}(1)|^{-1}.$$

Theorem 5.1. *If the inequalities*

$$\begin{aligned} \delta_1 f(\Delta_0 f - f_0') &> |\delta_1 f|\varepsilon_1 h_0^{-1}, & f_0'\Delta_0 f &\geq 0, \\ \delta_{N-1} f(f_N' - \Delta_{N-1} f) &> |\delta_{N-1} f|\varepsilon_{N-1} h_{N-1}^{-1}, & f_N'\Delta_{N-1} f &\geq 0, \end{aligned} \tag{5.9}$$

$$f_0''\delta_1 f \geq 0, \quad f_N''\delta_{N-1} f \geq 0, \tag{5.10}$$

are valid, the generalized cubic spline $S(x)$ constructed by Algorithm of three-point local approximation is an isogeometric approximant.

Proof: By virtue of the restrictions imposed on the initial data (2.1) and (2.2), it follows from (5.4)–(5.6) that

$$\begin{aligned} \Delta_i b\, \Delta_i f &> 0, \quad i = 0, \ldots, N-1, \\ (\Delta_i b - \Delta_{i-1} b)\delta_i f &> 0, \quad i = 1, \ldots, N-1. \end{aligned} \tag{5.11}$$

From here according to (3.9) $S''(x_i)\delta_i f > 0$, $i = 1, \ldots, N-1$. For the type I boundary conditions in the point x_0 we find $S''(x_0) = (v_0')^{-1}(\Delta_0 b - \Delta_{-1} b) = -h_0^{-2}[\varphi_0(0) + \varphi_0'(0)]^{-1}(b_1 - f_0 - h_0 f_0')$ from formulae (3.9) and (5.2). In the case $\delta_1 f > 0$ according to (5.9) we have $f_0' < \Delta_0 f - \varepsilon_1 h_0^{-1}$. Therefore, $b_1 - f_0 - h_0 f_0' > b_1 - f_0 - h_0 \Delta_0 f + \varepsilon_1 = b_1 - f_1 + \varepsilon_1$. Taking into account the estimate (5.4), we obtain $S''(x_0) \geq 0$, i.e. $S''(x_0)S''(x_1) \geq 0$. We have the same inequality for $\delta_1 f < 0$. In a similar way, we can arrive at the estimate $S''(x_{N-1})S''(x_N) \geq 0$. For the type II boundary conditions we get $S''(x_j)S''(x_{j+1}) \geq 0$, $j = 0, N-1$ from (5.10). Now, using Lemma 4.1, we conclude that conditions 4. from Definition 2.1 are met.

With regard to the fulfilment of inequalities (5.11) the procedure for testing condition 3. from Definition 2.1 does not differ from the proof of Theorem 4.1. The theorem is proved. □

Remark 5.1. For $f(x) \equiv 1$ and $f(x) \equiv x$ both in the case of one-point and three-point algorithms for type I, II, IV boundary conditions by immediate checking we have $b_i = 1$ and $b_i = x_i$, $i = -1, \ldots, N+1$, respectively and therefore according to (3.6) the spline with isogeometry $S(x)$ reproduces the straight lines. For type III boundary conditions the spline $S(x)$ will be exact for the constants.

Remark 5.2. For $p_i = q_i = 0$ the equality $v_i = 0$, $i = 1, \ldots, N-1$, is valid only for a uniform grid Δ. In this case, according to (5.1), we obtain the well-known three-point scheme of local approximation by cubic splines in [Zav'yalov et al. '80].

6. Isogeometric Approximation of Surfaces

Let a domain $G : [c, d] \times [0, 1]$ in a plane WU be partitioned by straight lines $w = w_i$, $i = 0, \ldots, N$, of the grid $\Delta_w : c = w_0 < w_1 < \cdots < w_N = d$ into N rectangular subdomains. Assume that a grid $\Delta_u^i : 0 = u_0^i < u_1^i < \cdots < u_{M_i}^i = 1$, $i = 0, 1, \ldots, N$, is given on every straight line $w = w_i$. The number of the grid nodes and their position on grids Δ_u^i, $i = 0, \ldots, N$, are independent of one another. The values f_{ij} of some function $f(w, u)$ are given with tolerances ε_{ij} at the nodes u_j^i, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$.

A surface of the class $C^{2,2}(G)$, passing through the points $P_{ij} = (w_i, u_j^i, \tilde{f}_{ij})$, where $\tilde{f}_{ij} \in [f_{ij} - \varepsilon_{ij}, f_{ij} + \varepsilon_{ij}]$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$, can be constructed by generalizing algorithms of local approximation by splines from Sections 4 and 5. In addition to being efficient at constructing the surface, these algorithms also preserve the shape of input data.

The surface is sought in the form of a function:

$$S(w, u) = \sum_{i=-1}^{N+1} b_i B_i(w),$$

where the generalized basis splines $B_i(w)$ are the same as in (3.5). The functions $b_i(u)$, $i = -1, \ldots, N + 1$, generalize local approximation formulae from Sections 4, 5 (Algorithms 1, 2) and are linear combinations of one-dimensional interpolation splines with isogeometry $S_i(u)$, $i = 0, \ldots, N$ in [Kvasov & Yatsenko '87]. These splines define curves along sections $w = w_i$, $i = 0, \ldots, N$, and pass through the points (u_j^i, f_{ij}), $j = 0, \ldots, M_i$.

Formally, necessary formulae (Algorithms 3, 4) can be obtained by replacing the values $f_j^{(k)}$ in Algorithms 1, 2 by the functions $S_j^{(k)}(u)$, $k = 0, 1, 2$, respectively. Similar changes are made in the boundary conditions, which can be specified for the boundary conditions.

As above we consider the four types of boundary conditions. For "one-point" scheme we have:

I. $\frac{\partial}{\partial w} S(w_i, u) = \frac{\partial}{\partial w} f(w_i, u)$, $i = 0, N$;

II. $\frac{\partial^2}{\partial w^2} S(w_i, u) = \frac{\partial^2}{\partial w^2} f(w_i, u)$, $i = 0, N$;

III. The periodic problem: $h_{N+i} = h_i$, $S_{N+i}(u) = S_i(u)$ for $h_i = w_{i+1} - w_i$;

IV. $S(w_i, u) = S_i(u)$, $i = 0, N$.

For "three-point" scheme to the type I, II boundary conditions we add the type IV conditions and the type IV boundary conditions are fulfilled for $i = 0, 1, N - 1, N$. As the formulae for functions $b_i(u)$, $i = -1, \ldots, N + 1$ are the direct generalization of local approximation formulae from Sections 4 and

5 we will consider only the type I boundary conditions. We use the notation $g_i(u) = \frac{\partial}{\partial w} f(w_i, u)$, $i = 0, N$.

Algorithm 3. One-point scheme:

$$\begin{aligned} b_{-1}(u) &= S_1(u) - 2h_0 g_0(u), \\ b_i(u) &= S_i(u), \quad i = 0, \dots, N, \\ b_{N+1}(u) &= S_{N-1}(u) + 2h_{N-1} g_N(u). \end{aligned} \tag{6.1}$$

Algorithm 4. Three-point scheme:

$$\begin{aligned} b_{-1}(u) &= b_1(u) - 2h_0 g_0(u), \\ b_0(u) &= S_0(u) - [S_0(u) + h_0 g_0(u) - b_1(u)][1 + \varphi_0'(0)/\varphi_0(0)]^{-1}, \\ b_i(u) &= S_i(u) - 2h_i^2 (h_{i-1} + h_i)^{-1} \varphi_i(0) \delta_i S(u), \quad i = 1, \dots, N-1, \\ b_N(u) &= S_N(u) - [S_N(u) - h_{N-1} g_N(u) - b_{N-1}(u)] \\ &\qquad \times [1 - \psi_{N-1}'(1)/\psi_{N-1}(1)]^{-1}, \\ b_{N+1}(u) &= b_{N-1}(u) + 2h_{N-1} g_N(u), \end{aligned} \tag{6.2}$$

where

$$\delta_i S(u) = \Delta_i S(u) - \Delta_{i-1} S(u), \ \Delta_j S(u) = [S_{j+1}(u) - S_j(u)]/h_j, \ j = i-1, i.$$

In order to calculate the boundary conditions we can use one-parameter interpolation Lagrange polynomials of the second and third degree. In accordance with isogeometry restrictions (4.5), (5.9), (5.10) in [Kvasov & Yatsenko '87,§ 4], we put

$$\begin{aligned} g_0(x) &= \begin{cases} \frac{\partial}{\partial w} L_{2,0}(w_0, u) & \text{if } \frac{\partial}{\partial w} L_{2,0}(w_0, u) \Delta_0 S(u) \geq 0, \ \delta_1 S(u) \neq 0, \\ \frac{\partial}{\partial w} L_{3,0}(w_0, u) & \text{if } \frac{\partial}{\partial w} L_{3,0}(w_0, u) \Delta_0 S(u) \geq 0, \ \delta_1 S(u) = 0, \\ 0, & \text{otherwise}, \end{cases} \\ g_N(u) &= \begin{cases} \frac{\partial}{\partial w} L_{2,N-2}(w_N, u) & \text{if } \frac{\partial}{\partial w} L_{2,N-2}(w_N, u) \Delta_{N-1} S(u) \geq 0, \\ & \qquad \delta_{N-1} S(u) \neq 0, \\ \frac{\partial}{\partial w} L_{3,N-3}(w_N, u) & \text{if } \frac{\partial}{\partial w} L_{3,N-3}(w_N, u) \Delta_{N-1} S(u) \geq 0, \\ & \qquad \delta_{N-1} S(u) = 0, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} L_{2,i}(w, u) &= S_i(u) + (w - w_i)[\Delta_i S(u) + (w - w_{i+1}) \delta_{i+1} S(u)/(w_{i+2} - w_i)], \\ L_{3,i}(w, u) &= [L_{2,i}(w, u)(w_{i+3} - w) + L_{2,i+1}(w, u)(w - w_i)]/(w_{i+3} - w_i). \end{aligned}$$

Instead of $g_i(u)$, $i = 0, N$, we can also consider interpolation isogeometric splines in [Kvasov & Yatsenko '87], which are constructed by the prescribed values $\partial f(w_j, u_j^i)/\partial w$, $j = 0, \ldots, M_i$, $i = 0, N$.

In practical calculations, every so often it is necessary to recalculate the prescribed values f_{ij} from the nodes of original irregular grid to the nodes of a regular grid on the domain G, i. e. to the points $(\tilde{w}_n, \tilde{u}_m)$, $m = 0, \ldots, \tilde{M}$, $n = 0, \ldots, \tilde{N}$. In this case, it is sufficient to know the values $g_j(\tilde{u}_m)$, $m = 0, \ldots, \tilde{M}$, $j = 0, N$, which can be found using (6.3).

The isogeometric spline $S(w, u)$ possesses the following properties of preserving the shape of the initial data.

Property 6.1. *Let functions $S_j(u)$, $j = i-1, \ldots, i+2$, $1 \le i \le N-2$ be monotonic and/or convex in the interval $[\tilde{u}_m, \tilde{u}_{m+1}]$. Then for any fixed $\tilde{w} \in [w_i, w_{i+1}]$, the generalized spline $S(w, u)$ constructed by Algorithm 3 will be monotonic and/or convex in the interval $[\tilde{u}_m, \tilde{u}_{m+1}]$.*

Property 6.2. *Let functions $S_j(u)$, $j = i-1, \ldots, i+2$, $1 \le i \le N-1$, be monotonic and/or convex on the interval $[\tilde{u}_m, \tilde{u}_{m+1}]$ and satisfy the conditions*

$$S_j^{(k)}(u)\delta_j^{(k)} f(u) < 0, \quad j \neq 0, N, \quad S_j^{(k)}(u) g_j^{(k)}(u) < 0, \quad j = 0, N,$$

where $k = 1$ and/or $k = 2$, respectively. Then for any fixed $\tilde{w} \in [w_i, w_{i+1}]$, $2 \le i \le N-3$, the generalized spline $S(w, u)$ constructed by Algorithm 4 will be monotonic and/or convex in $[\tilde{u}_m, \tilde{u}_{m+1}]$.

To prove these assertions, it is sufficient to take advantage of the relations

$$\frac{\partial^k}{\partial u^k} S(w, u) = \sum_{i=-1}^{N+1} b_i^{(k)}(u) B_i(w), \quad k = 1, 2,$$

use the expressions (6.1) and (6.2) for the coefficients $b_i(u)$, and take into account the finiteness of that the B-splines:$B_i(w) > 0$ at $w \in (w_{i-2}, w_{i+2})$ and $B_i(w) \equiv 0$ at $w \notin (w_{i-2}, w_{i+2})$.

Property 6.3. *Let the choice of parameters p_i, q_i, $i = -2, \ldots, N+2$, of a generalized spline $S(w, u)$ ensures the following estimate for any $\tilde{S}_j(u)$ such that $\Delta_i \tilde{S}(u)$, $\delta_i \tilde{S}(u)$ do not change sign for all $u \in [0, 1]$,*

$$|\tilde{S}_j(u) - S_j(u)| \le E_j(u), \quad j = 0, \ldots, N,$$

where $E_j(u)$ are given functions. Then for any fixed u the spline $S_u(w) = S(w, u)$ is an isogeometric approximant.

Proof: The proof of this statement follows from the above reasoning for one-dimensional local approximation splines. □

The calculation of spline $S(w,u)$ values can be realized most effectively to minimize the number of arithmetical operation executed for mentioned above regular resulting mesh. In this case at first the coefficients $b_i(\tilde{u}_m)$, $i=-1,\ldots,N+1$ are found and then by the formulae for the generalized B-splines the values $S(\tilde{w}_n,\tilde{u}_m)$, $n=0,\ldots,\tilde{N}$, $m=0,\ldots,\tilde{M}$, are computed.

A nonunique isogeometric surface given point by point as a family of curvilinear nonintersecting sections can be constructed by introducing the standard parametrization

$$x=S^x(w,u),\quad y=S^y(w,u),\quad z=S^z(w,u). \tag{6.4}$$

In this case the original points $T_{ij}=(x_{ij},y_{ij},z_{ij})$, $j=0,\ldots,M_i$, $i=0,\ldots,N$, are considered to belong to the parallelepiped $\prod_{ij}=\{\tilde{\chi}_{ij}|\ |\tilde{\chi}_{ij}-\chi_{ij}|\le\varepsilon_{ij}^{\chi}\}$, where we put $\chi_{ij}=\chi(w_i,u_j)$ for every coordinate function in (6.4) and ε_{ij}^{χ} is an admissible deviation with respect to appropriate variable. The resultant surface is obtained as a triple of isogeometric splines constructed by the above algorithm.

This algorithm can be assigned to a class of [Gordon '69] type algorithms, with the difference that the local approximation is used here instead of 'mixing' the functions $b_i(u)$, $i=0,\ldots,N$, by the use of fundamental splines and the surface is constructed in the space of isogeometric generalized splines.

7. Numerical Examples

That approximating generalized splines are isogeometric was proved under the restrictions (2.1) on original data that ensure the uniqueness of monotonicity and convexity conditions. Actually the algorithms we suggest work well for more general data, which is illustrated by the examples given below. A possibility to vary the parameters of admissible relative deviations from initial data ε_i gives us additional tool to control the 'smoothness' of the resultant curves while remaining within the prescribed approximation error.

The figures below illustrate the application of generalized cubic splines when prescribed curves and surfaces are approximated point by point. The defining functions were taken in the form

$$\varphi_i(t)=\varphi(p_i,t)=P_i(1-t)^3/[1+p_it(1-t)],$$
$$\psi_i(t)=\varphi(q_i,1-t),\quad P_i^{-1}=2(1+p_i)(3+p_i),$$

that corresponds to rational cubic splines with quadratic denominator. In all cases below, more precise three-point local approximation formulae (Algorithms 2, 4) were employed. For comparison, the same data were interpolated using a standard cubic spline in [Zav'yalov et al. '80] (in our case $p_i = q_i = 0$ for all i). The derivatives at end points were calculated using Lagrange polynomials of the second degree: $S'(x_0) = \mathbb{L}'_{2,0}(x_0)$, $S'(x_N) = \mathbb{L}'_{2,N-2}(x_N)$. These derivatives were then refined according to the isogeometry conditions (5.9) (for surfaces according to (6.3)). The limit admissible relative deviation from initial data was 10%. In the figures continuous and dashed lines denote, respectively, the graphs of the rational spline $S(x)$ and the cubic spline $S_3(x)$. Initial points are marked by crosses.

Figure 7.1a shows an example of approximating a unit-pulse function $f(x) = \max(0, 1 - 4|x - 1.75|)$ using the points $x_i = 1 + 0.25i$, $i = 0, 1, \ldots, 6$. In this case oscillations are typical of the cubic spline, while the rational spline is not sensitive to such outliers. By varying ε_i we can affect the 'corner radii'.

(a) (b)

Figure 7.1. Profiles of interpolation and isogeometric splines. (a) Approximation of a unit-pulse function; (b) data obtained by [Späth '69].

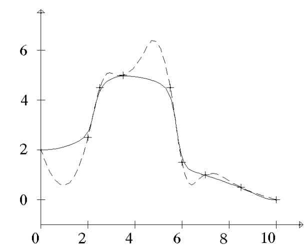
The data for Figs. 7.1b and 7.2 (Tables 7.1 and 7.2) were taken from [Späth '69] and [Späth '86], respectively. The cubic spline in Fig. 7.1b has extra inflection points on the first, third, fourth, and eighth intervals. The isogeometric spline reveals no such oscillations. Figure 7.2 reflects the same general tendencies in the behaviour of splines $S_3(x)$ and $S(x)$. By reducing ε_i at proper points, we can additionally 'press' the curve againts the data (see Fig. 7.2b) at an expense of making it more 'angular'.

Table 7.1. Data for Fig. 7.1b.

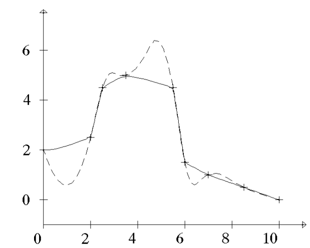
x_i	0.0	1.0	1.5	2.5	4.0	4.5	5.5	6.0	8.0	10.0
f_i	10.0	8.0	5.0	4.0	3.5	3.4	6.0	7.1	8.0	8.5

Table 7.2. Data for Fig. 7.2:

x_i	0	2.0	2.5	3.5	5.5	6.0	7	8.5	10
f_i	2	2.5	4.5	5.0	4.5	1.5	1	0.5	0



(a)



(b)

Figure 7.2. Data obtained by [Späth '86]. Variation of the isogeometric curve with decreasing the tolerance parameters ε_i.

As the data for Fig. 7.3a we took a function with discontinuous derivative, obtained by joining line segments and a semicircle $f(x) = 1 + [1 - (x-4)^2]^{1/2}$, $|x - 4| \leq 1$ and $f(x) = 1$ otherwise. From the geometric point of view the interpolation cubic spline curve is unsuitable here whereas the profile of the isogeometric spline is ideal (Fig. 7.3a).

The case of joining a quadrant of a circle with a line segment is presented in Fig. 7.3b. The resultant curve has here a discontinuous curvature at the joining point. The vertical tangent line on the left boundary was approximated by the value $S'(a) = 50$. From the viewpoint of geometric requirements, the behaviour of the cubic interpolant is once again far from being satisfactory while the rational spline once again produces no oscillations and automatically adjusts boundary conditions.

In many works devoted to isogeometric interpolation, algorithms are tested with [Akima '70] data (Table 7.3). The profiles of splines $S_3(x)$ and

(a) (b)

Figure 7.3. Joining of a part of the circle with line segments. (a) Semicircle; (b) one quadrant of a circle.

$S(x)$ obtained for these data are given in Fig. 7.4a. On a 'steep-gradient' interval a corridor of admissible deviation from initial data was a maximum: of $\varepsilon_7 = 10$, $\varepsilon_6 = \varepsilon_8 = \varepsilon_9 = 5$ and $\varepsilon_i = 1$ at the other points.

(a) (b)

Figure 7.4. Typical behaviour of interpolation and isogeometric splines, given fast- and slow-change sections of data. (a) Data obtained by [Akima '70]; (b) data obtained by [Fritsch & Carlson '80].

Table 7.3: Data for Fig. 7.4a.

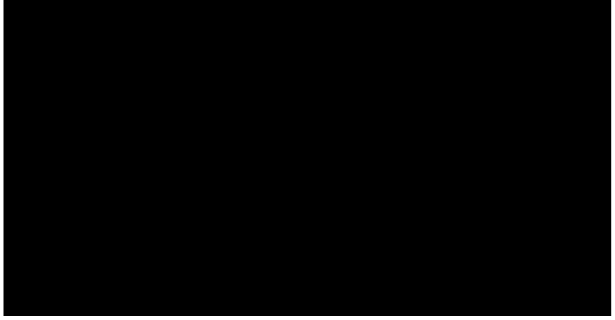
x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	56	60	85

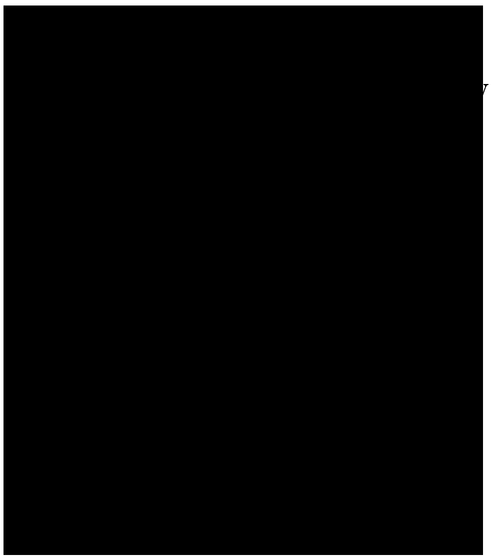
Figure 7.4b presents the results of approximating the data from [Fritsch & Carlson '80], namely $\{x_i\} = \{7.99, 8.09, 8.19, 8.7, 9.2, 10, 12, 15, 20\}$, $\{f_i\} = \{0, 2.76429E-5, 4.37498E-2,$ 0.169183, 0.469428, 0.943740, 0.998636, 0.999916, 0.999994$\}$. Here $\varepsilon_i = 0.1$ for all i.

A perspective plot of the data for the reconstruction of the 'viking ship' surface is shown in the figure 7.5a. The initial data were kindly allowed by prof. T. Lyche from the University of Oslo. The data is a collection of points in space, all in the same cartesian coordinate system. The original data have been split into three sets. The first set makes up the points along the top curve in the figure and the second set the bottom curve. Both of these are closed curves in the sense that the first and last points are identical. The third set of data points consists of the six ribs. Each rib is meant to be an open curve so the last point is not the same as the first point.

Using these data we constructed a system of nonintersecting curvilinear sections across the ship from side to side. As body cross-sections we have used the ribs and four-point sections in aft and bow parts of the ship extracted from the top and bottom curves. The cartesian coordinates of the nonunique isogeometric surface were calculated using the standard parametrization in (6.4). In figures 7.5b, 7.5c the resulting surface constructed by the algorithm 4 is shown in two different projections with 100×100 lines of a regular mesh.

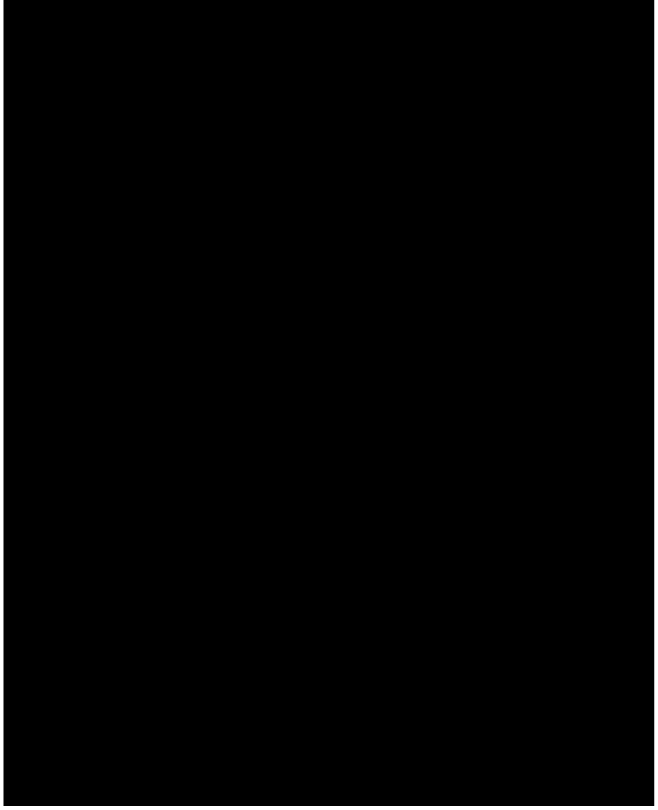
The main results of this paper were reported at the Third International Conference on Mathematical Methods in CAGD, Ulvik, Norway [Kvasov '94].





of the data.

(b) Isogeometric surface. First projection.



(c) Resulting isogeometric surface.

Figure 7.5. Approximation of a viking-ship surface.

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