Shape Preserving Spline Approximation via Local Algorithms

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Abstract. The main goal of this paper is to construct an algorithm for shape preserving spline approximation of complex multivalued surfaces that appear in some applications. We consider firstly two different local 1-D algorithms of shape preserving interpolation and approximation. Then we use the direct product of 1-D splines and special monotonizing parametrization to obtain the surface satisfying given tolerances that inherits some shape preserving properties of 1-D splines.

§1. Introduction

In many practical problems we deal with approximation of discrete data when geometric properties of the data such as positivity, monotonicity, convexity, presence of linear sections, the angles and the bends should be retained. Standard approaches such as spline interpolation, NURBS and other are usually fail in the treatment of this problem. To obtain the necessary solution many authors [1,2,5,7] introduce some parameters in the structure of the spline. Then they choose these factors in such a way to satisfy the geometric constraints. The key idea here is to develop algorithms for automatic selection of these parameters. We consider two such local 1-D algorithms based on generalized cubic splines. Then the direct product of 1-D splines and special monotonizing parametrization are used to obtain the shape preserving approximation of complex multivalued surfaces. We formulate the shape preserving properties of 2-D splines and give some numerical examples.

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§2. The Class of Shape Preserving Interpolants

Let the sequence of points $V = \{P_i \,|\, i = 0, 1, \ldots, N\}$, $P_i = (x_i, f_i)$, on the plane $\mathbb{R}^2$ be fixed, where $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ forms a partition of the interval $[a, b]$. We introduce the notation for the first two devided differences $\Delta_i f = (f_{i+1} - f_i)/h_i$, $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, N-1$; $\delta_i f = \Delta_i f - \Delta_{i-1} f$, $i = 1, 2, \ldots, N-1$. As usual, we shall say that the initial data increases monotonically (decreases monotonically) on the subinterval $[x_n, x_k]$, $n > k$, if $\Delta_i f > 0$ ($\Delta_i f < 0$), $i = n, \ldots, k-1$. We say it is convex down (up) on $[x_n, x_k]$, $k > n+1$ if $\delta_i f > 0$ ($\delta_i f < 0$), $i = n, \ldots, k-2$.

We call the problem of searching for a sufficiently smooth function $S(x)$ such that $S(x_i) = f_i$, $i = 0, 1, \ldots, N$, and $S(x)$ preserves the form of the initial data, a *shape preserving interpolation problem.* It means that $S(x)$ should monotonically increase or decrease if the data has the same behaviour. Analogously, $S(x)$ should also be convex (concave) in data convexity (concavity) intervals.

Evidently the solution of the shape preserving interpolation problem is not unique. We formalize the class of functions in which we search for the solution.

Definition 2.1. *The set of functions $I(V)$ is called the class of shape preserving functions if for any $S(x) \in I(V)$ the following conditions are met:*

(1) $S(x) \in C^2[a, b]$;

(2) $S(x_i) = f_i$, $i = 0, 1, \ldots, N$;

(3) $S'(x)\Delta_i f \geq 0$ *if* $\Delta_i f \neq 0$ *and* $S'(x) = 0$ *if* $\Delta_i f = 0$ *for all* $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$; *and*

(4) $S''(x_i)\delta_i f \geq 0$, $i = 1, 2, \ldots, N-1$; $S''(x)\delta_j f \geq 0$, $x \in [x_i, x_{i+1}]$, $j = i, i+1$ *if* $\delta_i f \delta_{i+1} f \geq 0$; $S(x)$ *has no more than one inflection point* $\overline{x}$ *in the interval* (x_i, x_{i+1}) *if* $\delta_i f \delta_{i+1} f < 0$ *and also* $S''(x)\delta_i f \geq 0$ *for* $x \in [x_i, \overline{x}]$ *and the number of inflection points in the interval* (x_{i-1}, x_{i+1}) *does not exceed the number of sign changes in the sequence* $\delta_{i-1} f, \delta_i f, \delta_{i+1} f$.

Remark. When counting the number of sign changes in the sequence $\delta_{i-1} f, \delta_i f$, $\delta_{i+1} f$, the zeros are omitted.

The following propositions, characterizing the properties of shape preserving interpolants, are proved by using simple geometric considerations.

Lemma 2.1. *If $\Delta_{i-1} f \Delta_i f \leq 0$, then for the function $S(x)$ to be shape preserving, it is necessary that $S'(x_i) = 0$.*

Lemma 2.2. *If $\delta_i f = 0$ and $\delta_{i-1} f \delta_{i+1} f \geq 0$, then the unique shape preserving function on the interval $[x_{i-1}, x_{i+1}]$ is the straight line passing through the points P_{i-1}, P_i, P_{i+1}.*

Corollary 2.1. *If $\delta_i f = \delta_{i+1} f = 0$, then the unique shape preserving function in the interval $[x_{i-1}, x_{i+1}]$ is the straight line passing through the points P_j, $j = i-1, i, i+1, i+2$.*

Lemma 2.3. *If $\delta_i f = 0$ and $\delta_{i-1} f \delta_{i+1} f < 0$, then for $S(x) \in I(V)$ it is necessary that one of the following conditions be met:*
(1) $S'(x_i)\delta_{i-1} f > \Delta_i f \delta_{i-1} f$, $S''(x_i) = 0$;
(2) $S'(x) = \Delta_i f$, $S''(x) = 0$ for all $x \in [x_{i-1}, x_{i+1}]$.

Lemma 2.4. *Let $\delta_i f \neq 0$ and $S''(x_i)S''(x) \geq 0$ for all $x \in [z_1, z_2]$, $z_1, z_2 \in [x_i, x_{i+1}]$. Then for $S(x) \in I(V)$ it is necessary that one of the following conditions be met:*
(1) $S'(z_1) < \Delta_z S < S'(z_2)$ for $\delta_i f > 0$,
(2) $S'(z_1) > \Delta_z S > S'(z_2)$ for $\delta_i f < 0$,
(3) $S'(x) = \Delta_z S$, $S''(x) = 0$ for all $x \in [z_1, z_2]$,
where $\Delta_z S = (S(z_2) - S(z_1))/(z_2 - z_1)$.

Lemma 2.4 immediately implies

Corollary 2.2. *If $\delta_i f \delta_{i+1} f > 0$ and $S'(x_j) \neq \Delta_i f$, $j = i, i+1$, then for $S(x) \in I(V)$ it is necessary that the condition*

$$S'(x_i)\delta_i f < \Delta_i f \delta_i f < S'(x_{i+1})\delta_i f$$

holds.

Corollary 2.3. *If $\delta_{i-1} f \delta_i f > 0$ and $\delta_i f \delta_{i+1} f > 0$, then for $S(x)$ to be shape preserving it is necessary that the inequalities hold:*

$$\min(\Delta_{i-1} f, \Delta_i f) \leq S'(x_i) \leq \max(\Delta_{i-1} f, \Delta_i f).$$

Lemma 2.5. *If $S'(x_i) = 0$, then for $S(x)$ to be shape preserving it is necessary that $S''(x_i)\Delta_i f \geq 0$, $S''(x_i)\Delta_{i-1} f \leq 0$.*

Theorem 2.1. *For the existence of a shape preserving function it is necessary and sufficient that none of the following conditions hold:*
(1) $\Delta_{i-1} f \Delta_i f \leq 0$, $\Delta_{i-1} f \neq 0$, $\delta_{i-2} f \delta_i f \geq 0$, $\delta_{i-1} f = 0$, $i = 3, \dots, N-1$,
(2) $\Delta_{i-1} f \Delta_i f \leq 0$, $\Delta_i f \neq 0$, $\delta_i f \delta_{i+2} f \geq 0$, $\delta_{i+1} f = 0$, $i = 1, \dots, N-3$,
(3) $\delta_i f \neq 0$, $\delta_{i-1} f = \delta_{i+1} f = 0$, $\delta_i f \delta_k f \geq 0$, $k = i-2, i+2$, $i = 3, \dots, N-3$.

Necessity of this assertion is proved directly by using Lemmas 2.1–2.5. The proof of the sufficiency consists in local constructing the shape preserving function $S(x)$ which interpolates arbitrary data and for which the conditions (1)–(3) of the Theorem 2.1 are not satisfied.

We define now the admissible values $S_i^{(r)} = S^{(r)}(x_i)$, $r = 1, 2$, in the knots of the mesh Δ. The choice of these values should be subjected to the following constraints:

$$\begin{gathered} \min(\Delta_{i-1} f, \Delta_i f) < S_i' < \max(\Delta_{i-1} f, \Delta_i f) \quad \text{and} \quad \delta_i f S_i'' \geq 0 \\ \text{if} \quad \delta_i f \neq 0, \quad 1 \leq i \leq N-1; \end{gathered} \tag{2.1}$$

$$\begin{aligned}&(S'_i - \Delta_i f)\delta_{i-1} f > 0,\ S'_i \Delta_i f \geq 0,\ S''_i = 0 \\ &\qquad \text{if} \quad \delta_i f = 0,\ \delta_{i-1} f \delta_{i+1} f < 0, \quad 2 \leq i \leq N-2;\end{aligned} \tag{2.2}$$

$$\begin{aligned}&(S'_1 - \Delta_1 f)\delta_2 f < 0,\ S'_1 \Delta_1 f \geq 0,\ S''_1 = 0 \quad \text{if} \quad \delta_1 f = 0, \\ &(S'_{N-1} - \Delta_{N-1} f)\delta_{N-2} f > 0,\ S'_{N-1} \Delta_{N-1} f \geq 0,\ S''_{N-1} = 0 \\ &\qquad \text{if} \ \ \delta_{N-1} f = 0;\end{aligned} \tag{2.3}$$

$$\begin{aligned}&(\Delta_0 f - S'_0)\delta_1 f > 0,\ S'_0 \Delta_0 f \geq 0\,(\Delta_0 f \neq 0),\ S''_0 \delta_1 f \geq 0 \ \ \text{if} \ \ \delta_1 f \neq 0, \\ &(\Delta_0 f - S'_0)\delta_2 f < 0,\ S'_0 \Delta_0 f \geq 0\,(\Delta_0 f \neq 0),\ S''_0 \delta_2 f \leq 0 \\ &\qquad \text{if} \quad \delta_1 f = 0,\ \delta_2 f \neq 0;\end{aligned} \tag{2.4}$$

$$\begin{aligned}&(S'_N - \Delta_{N-1} f)\delta_{N-1} f > 0,\ S'_N \Delta_{N-1} f \geq 0\,(\Delta_{N-1} f \neq 0),\ S''_N \delta_{N-1} f \geq 0 \\ &\qquad \text{if} \quad \delta_{N-1} f \neq 0, \\ &(S'_N - \Delta_{N-1} f)\delta_{N-2} f < 0,\ S'_N \Delta_{N-1} f \geq 0\,(\Delta_{N-1} f \neq 0),\ S''_N \delta_{N-2} f \leq 0 \\ &\qquad \text{if} \quad \delta_{N-1} f = 0, \delta_{N-2} f \neq 0.\end{aligned} \tag{2.5}$$

For the constructing of the shape preserving function $S(x)$ it is sufficient to eliminate from the consideration the intervals of the $S(x)$ linearity and to define $S(x)$ in arbitrary subinterval $[x_i, x_{i+1}]$ for the following possible configurations of the data:
(A) $\delta_i f \delta_{i+1} f > 0,\ \ 0 \leq i \leq N-1;$
(B) $\delta_i f = 0,\ \ \delta_{i-1} f \delta_{i+1} f < 0,\ 1 \leq i \leq N-1;$
(C) $\delta_i f \delta_{i+1} f < 0,\ \ 1 \leq i \leq N-2$
(if $i = 0, N$, then we formally set $\delta_i f = S''_i$).

By introducing on the straight line, joining the points P_i, P_{i+1}, an additional inflection point extending the mesh Δ the case (C) is reduced to the case (B). In cases (A) and (B) the problem of the shape preserving function construction can be reduced [3] to the solution in $[x_i, x_{i+1}]$ of the Hermite interpolation problem by the given values $S_j^{(r)}$, $r = 0, 1, 2$; $j = i, i+1$ with the function monotonicity and convexity requirement in this interval and additional restrictions

$$\min(S'_i, S'_{i+1}) < \Delta_i f < \max(S'_i, S'_{i+1}). \tag{2.6}$$

$$\Delta_i f S'_j \geq 0, \quad j = i, i+1. \tag{2.7}$$

$$S''_j/(S'_{i+1} - S'_i) \geq 0, \quad j = i, i+1. \tag{2.8}$$

According to the Definition 2.1 the following relations should be satisfied too:

$$S''(x)S''(x_j) \geq 0, \quad j = i, i+1; \quad x \in [x_i, x_{i+1}]. \tag{2.9}$$

§3. The Solution of the Hermite Interpolation Problem with Constraints

The question of local construction of the shape preserving function $S(x)$ can be solved by using generalized cubic splines [8,9].

Definition 3.1. *Our generalized cubic spline on the mesh Δ will be a function $S(x) \in C^2[a,b]$ such that in any subinterval $[x_j, x_{j+1}]$ it has the form*

$$S(x) = [S_j - \varphi_j(0)h_j^2 S_j''](1-t) + [S_{j+1} - \psi_j(1)h_j^2 S_{j+1}'']t \\ +\varphi_j(t)h_j^2 S_j'' + \psi_j(t)h_j^2 S_{j+1}'', \tag{3.1}$$

where $t = (x - x_j)/h_j$ and the functions $\varphi_j(t)$, $\psi_j(t)$ satisfy the conditions

$$\varphi_j^{(r)}(1) = \psi_j^{(r)}(0) = 0, \quad r = 0,1,2; \quad \varphi_j''(0) = \psi_j''(1) = 1.$$

We assume that $\varphi_j''(t)$, $\psi_j''(t)$ are continuous monotonic functions of the variable $t \in [0,1]$ values and

$$\varphi_j(t) = \varphi(p_j, t), \quad \psi_j(t) = \varphi(q_j, 1-t), \quad p_j, q_j \geq 0. \tag{3.2}$$

To solve the Hermite interpolation problem with constraints on the interval $[x_i, x_{i+1}]$ let us define a function

$$S(x) = \begin{cases} S(x, x_i, x_{i1}) & \text{if} \quad x \in [x_i, x_{i1}]; \\ S(x, x_{i1}, x_{i+1}) & \text{if} \quad x \in [x_{i1}, x_{i+1}], \end{cases}$$

which has the form (3.1) on the intervals $[x_i, x_{i1}]$, $[x_{i1}, x_{i+1}]$ and satisfies the interpolation and smoothness conditions

$$S^{(r)}(x_j) = f_j^{(r)}, \quad S^{(r)}(x_{i1} - 0) = S^{(r)}(x_{i1} + 0), \quad r = 0,1,2; \quad j = i, i+1.$$

We assume that the inequalities (2.6)–(2.8) are fulfilled and according to (2.7) we have $S_i' S_{i+1}' \geq 0$.

Let us introduce the notations

$$h_{i1} = x_{i1} - x_i, \quad \mu_{i1} = 1 - \lambda_{i1} = h_{i1}/h_i, \quad \tau_i = \frac{S_{i+1}' - \Delta_i f}{S_{i+1}' - S_i'},$$

$$\alpha_i = \frac{S_{i1} - f_i}{h_{i1}}, \quad \beta_i = \frac{f_{i+1} - S_{i1}}{h_i - h_{i1}}, \quad \sigma_j = \frac{h_i S_j''}{S_{i+1}' - S_i'}, \quad j = i, i+1.$$

According to these notations and by the inequalities (2.6) we have

$$\Delta_i f = \tau_i S_i' + (1 - \tau_i) S_{i+1}', \quad 0 < \tau_i < 1. \tag{3.3}$$

Using the formula (3.1) we obtain

$$\alpha_i = \frac{1}{\psi_i'(1)}\Big\{h_{i1}T_i^{-1}S_i'' - [\psi_i(1)-\psi_i'(1)]S_i' + \psi_i(1)S_{i1}'\Big\},$$
$$\beta_i = \frac{1}{\varphi_{i1}'(0)}\Big\{(h_i-h_{i1})T_{i1}^{-1}S_{i+1}'' - \varphi_{i1}(0)S_{i1}' + [\varphi_{i1}(0)+\varphi_{i1}'(0)]S_{i+1}'\Big\}, \tag{3.4}$$
$$T_j^{-1} = [\varphi_j(0)+\varphi_j'(0)][\psi_j(1)-\psi_j'(1)] - \varphi_j(0)\psi_j(1), \quad j=i,i1.$$

By the continuity of the spline second derivative in the knot x_{i1} we have the equation

$$S_{i1}' = \left[\frac{\lambda_{i1}}{\psi_i'(1)} - \frac{\mu_{i1}}{\varphi_{i1}'(0)}\right]^{-1}\left[\frac{\lambda_{i1}}{\psi_i'(1)}S_i' - \frac{\mu_{i1}}{\varphi_{i1}'(0)}S_{i+1}' - \frac{\varphi_i'(0)}{\psi_i'(1)}\lambda_{i1}h_{i1}S_i'' + \frac{\psi_{i1}'(1)}{\varphi_{i1}'(0)}\mu_{i1}(h_i-h_{i1})S_{i+1}''\right]. \tag{3.5}$$

Now taking into account the identity $\mu_{i1}\alpha_i + \lambda_{i1}\beta_i = \Delta_i f$ and substituting here the expressions for α_i, β_i, S_{i1}' from (3.4) and (3.5) we arrive at the equation with respect to μ_{i1}

$$\Phi_i(\mu_{i1}) = A_i\mu_{i1}^3 + B_i\mu_{i1}^2 + C_i\mu_{i1} + D_i = 0. \tag{3.6}$$

If now to set $p_{i1} = q_i$ then according to (3.2) we have in (3.6) the coefficient $A_i = 0$ and to define μ_{i1} we obtain the quadratic equation

$$\Phi_i(\mu_{i1}) = \frac{1}{\psi_i'(1)}\left[\hat B_i\mu_{i1}^2 + \hat C_i\mu_{i1} + \hat D_i\right] = 0, \tag{3.7}$$

where

$$\hat B_i = [\psi_i(1)\varphi_i'(0) + T_i^{-1}]\sigma_i + [\psi_i(1)\psi_{i1}'(1) - T_{i1}^{-1}]\sigma_{i+1},$$
$$\hat C_i = -\psi_i(1)\varphi_i'(0)\sigma_i + [-\psi_i(1)\psi_{i1}(1) + 2T_{i1}^{-1}]\sigma_{i+1} + 2\psi_i(1) - \psi_i'(1),$$
$$\hat D_i = -T_{i1}^{-1}\sigma_{i+1} - \psi_i(1) + \tau_i\psi_i'(1).$$

Since

$$\Phi_i(0) = \tau_i - \frac{1}{\psi_i'(1)}\big[T_{i1}^{-1}\sigma_{i+1} + \psi_i(1)\big],$$
$$\Phi_i(1) = -(1-\tau_i) + \frac{1}{\psi_i'(1)}\big[T_i^{-1}\sigma_i + \psi_i(1)\big],$$

we can find such $\overline{p}_i$, $\overline{q}_i$, $\overline{q}_{i1}$, that for all $p_i \ge \overline{p}_i$, $q_i \ge \overline{q}_i$, $q_{i1} \ge \overline{q}_{i1}$ according to (3.3) we have $\Phi_i(0) > 0$, $\Phi_i(1) < 0$. Thus the equation (3.7) has a unique root $\mu_{i1} \in (0,1)$.

As $p_{i1} = q_i$ we can rewrite the equation (3.5) in the form

$$S'_{i1} = S'_i + \mu_{i1}(S'_{i+1} - S'_i) - \lambda_{i1}\mu_{i1}h_i[\varphi'_i(0)S''_i + \psi'_{i1}(1)S''_{i+1}]. \qquad (3.8)$$

Considering $f(x)$ as a sufficiently smooth function we assume that $S_j^{(r)} - f_j^{(r)} = O(h_i^{k+1-r})$, $r = 1, 2$; $j = i, i+1$; $k = 2$ or $k = 3$. Then using (3.8) we obtain

$$\begin{aligned} S'_{i1} - f'(x_{i1}) =& S'_i - f'_i - [\varphi'_i(0) + \psi'_{i1}(1)]\lambda_{i1}\mu_{i1}h_i f''_i \\ &+ h_{i1}(h_i - h_{i1})[1/2 - \psi'_{i1}(1)]f'''_i + O(h_i^k). \end{aligned}$$

It implies that the approximation error order will increase for the derivative of the spline in the point x_{i1} if according to (3.2) we set $q_{i1} = p_i$.

We consider now the question of the shape preserving properties for the generalized spline $S(x)$ in the interval $[x_i, x_{i+1}]$. The following criterion is valid.

Theorem 3.1. *By the fulfillment the restrictions*

$$\frac{\psi_i(1)}{\psi'_i(1)} - \varphi'_i(0)\sigma_i < 1 - \tau_i, \quad \frac{\psi_i(1)}{\psi'_i(1)} - \varphi'_i(0)\sigma_{i+1} < \tau_i, \qquad (3.9)$$

the unique shape preserving generalized cubic spline $S(x)$ exists solving the Hermite interpolation problem with restrictions (2.6)–(2.9).

Proof: The conditions (2.6) – (2.8) are fulfilled by the construction. The requirement (2.9) means the absence on $[x_i, x_{i+1}]$ of inflection points for the shape preserving function $S_f(x)$. Let us show that for the spline $S(x)$ this condition will be fulfilled if the inequalities are valid

$$\begin{aligned} \min(\alpha_i, \beta_i) < S'_{i1} < \max(\alpha_i, \beta_i), \\ \min(S'_i, \Delta_i f) < \alpha_i < \max(S'_i, \Delta_i f), \\ \min(S'_{i+1}, \Delta_i f) < \beta_i < \max(S'_{i+1}, \Delta_i f). \end{aligned}$$

It is convenient to rewrite these inequalities in the form

$$\begin{aligned} \alpha_i(S'_{i+1} - S'_i)^{-1} < S'_{i1}(S'_{i+1} - S'_i)^{-1} < \beta_i(S'_{i+1} - S'_i)^{-1}, \\ S'_i(S'_{i+1} - S'_i)^{-1} < \alpha_i(S'_{i+1} - S'_i)^{-1} < \Delta_i f(S'_{i+1} - S'_i)^{-1}, \\ \Delta_i f(S'_{i+1} - S'_i)^{-1} < \beta_i(S'_{i+1} - S'_i)^{-1} < S'_{i+1}(S'_{i+1} - S'_i)^{-1}. \end{aligned} \qquad (3.10)$$

From (3.4) and (3.8) we find

$$\alpha_i = S'_i + \mu_{i1}(S'_{i+1} - S'_i)\frac{\psi_i(1)}{\psi'_i(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi'_i(0)(\sigma_{i+1} - \sigma_i)\right], \qquad (3.11)$$

$$\beta_i = S'_{i+1} - \lambda_{i1}(S'_{i+1} - S'_i)\frac{\psi_i(1)}{\psi'_i(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1} - \mu_{i1}\varphi'_i(0)(\sigma_{i+1} - \sigma_i)\right].$$

It enables us to write the conditions (3.10) in the form

$$\frac{\psi_i(1)}{\psi_i'(1)}\big[1+\frac{T_i^{-1}}{\psi_i(1)}\sigma_i+\lambda_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i)\big]<1+\lambda_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i),$$
$$\frac{\psi_i(1)}{\psi_i'(1)}\big[1+\frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1}-\mu_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i)\big]<1-\mu_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i),$$
$$0<\mu_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\big[1+\frac{T_i^{-1}}{\psi_i(1)}\sigma_i+\lambda_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i)\big]<1-\tau_i,$$
$$0<\lambda_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\big[1+\frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1}-\mu_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i)\big]<\tau_i.$$

To fulfill these inequalities and the conditions $\Phi_i(0)>0$, $\Phi_i(1)<0$ it is sufficient to choose the parameters p_i, q_i in such a way that the restrictions (3.9) are satisfied.

According to (3.1)

$$S_i'=\alpha_i+[\varphi_i(0)+\varphi_i'(0)]h_{i1}S_i''-\psi_i(1)h_{i1}S_{i1}''.$$

Then by substituting here the expression for α_i from (3.11) we have

$$S_{i1}''=\frac{S_{i+1}'-S_i'}{h_i\psi_i'(1)}\big[1+\varphi_i'(0)(\mu_{i1}\sigma_i+\lambda_{i1}\sigma_{i+1})\big]. \tag{3.12}$$

If the inequalities (3.9) are fulfilled, the expression in square parentheses in (3.12) is positive and $S_{i1}''(S_{i+1}'-S_i')\geq 0$. As $S_j''(S_{i+1}'-S_i')\geq 0$, $j=i,i+1$, we conclude from here that $S_{i1}''S_j''\geq 0, \quad j=i,i+1$.

From (3.1) on the interval $[x_i,x_{i1}]$ we have

$$S''(x)=S_i''\varphi_i''(t)+S_{i1}''\psi_i''(t).$$

Since $\varphi_i''(t),\psi_i''(t)\geq 0$ for $t\in[0,1]$, then

$$S''(x)S_j''\geq 0,\quad j=i,i1 \quad\text{for}\quad x\in[x_i,x_{i1}].$$

We arrive at an analogous conclusion by considering the subinterval $[x_{i1},x_{i+1}]$. As a result the function $S''(x)$ is convex in the interval $[x_i,x_{i+1}]$ and $S'(x)$ is monotone. Because of the assumption $S_i'S_{i+1}'\geq 0$ the function $S(x)$ has the monotonicity property. The theorem is proved. ■

The given construction completes the proof of the sufficiency conditions of Theorem 2.1 from the previous section.

§4. The Problem of Shape Preserving Approximation

Suppose a set of intervals $F = \{F_i \,|\, i = 0, \ldots, N\}$, $F_i \equiv [f_i - \varepsilon_i, f_i + \varepsilon_i]$, $i = 0, \ldots, N$, with prescribed small $\varepsilon_i > 0$ on a grid $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ be given. We call the problem of searching for a sufficiently smooth function $S(x) \in C^2[a, b]$ such that $S(x_i) \in F_i$, $i = 0, \ldots, N$, and $S(x)$ preserves the shape of the initial data a *shape preserving approximation problem.*

To formalize this problem we introduce the interval differences [6]

$$\Delta_i F = h_i^{-1}(F_{i+1} - F_i) = [\Delta_i f - e_i, \Delta_i f + e_i], \quad e_i = h_i^{-1}(\varepsilon_i + \varepsilon_{i+1}), \quad i = 0, \ldots, N-1,$$

$$\delta_i F = \Delta_i F - \Delta_{i-1} F = [\delta_i f - E_i, \delta_i f + E_i], \quad E_i = e_{i-1} + e_i, \quad i = 1, \ldots, N-1,$$

$$[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1] > 0 \quad \text{if and only if} \quad a_1 > b_2.$$

The initial data are said to increase (decrease) monotonically on a subinterval $[x_R, x_K]$, $K > R$, if $\Delta_i F > 0$ ($\Delta_i F < 0$), $i = R, \ldots, K-1$. The data are called convex downward (upward) on $[x_R, x_K]$, $K > R + 1$, if $\delta_i F > 0$ ($\delta_i F < 0$), $i = R+1, \ldots, K-1$.

We assume that the intervals $\Delta_i F$, $\delta_i F$ for all i do not contain zeros, i.e. $(\Delta_i f)^2 > e_i^2$, $i = 0, \ldots, N-1$; $(\delta_i f)^2 > E_i^2$, $i = 1, \ldots, N-1$.

If the values of a function $S(x)$ are such that $S(x_i) \in F_i$, $i = 0, \ldots, N$, then we have $\Delta_i S \in \Delta_i F$, $i = 0, \ldots, N-1$, $\delta_i S \in \delta_i F$, $i = 1, \ldots, N-1$. Taking into account the inequalities for the initial data, we obtain $\Delta_i S\, \Delta_i f > 0$, $i = 0, \ldots, N-1$, $\delta_i S\, \delta_i f > 0$, $i = 1, \ldots, N-1$.

Definition 4.1. *The set of functions $I(\Delta, F)$ is called the class of shape preserving approximants if for any function $S(x) \in I(\Delta, F)$ the following conditions are met:*

1. $S(x) \in C^2[a, b]$;
2. $S(x_i) \in F_i$, $i = 0, \ldots, N$;
3. $S(x)$ *is monotonic in* $[x_i, x_{i+1}]$, $i = 1, \ldots, N-2$ *if* $\Delta_{i-1} f \Delta_i f > 0$, $\Delta_i f \Delta_{i+1} f > 0$; $S(x)$ *is monotonic in* $[x_0, x_1]$ *if* $\Delta_0 f \Delta_1 f > 0$ *and in* $[x_{N-1}, x_N]$ *if* $\Delta_{N-2} f \Delta_{N-1} f > 0$; $S'(x)$ *has one sign change in* $[x_{i-1}, x_{i+1}]$, $i = 1, \ldots, N-1$ *if* $\Delta_{i-1} f \Delta_i f < 0$; *the number of sign changes of the function* $S'(x)$ *in* $[a, b]$ *coincides with that in the sequence* $\Delta_0 f, \Delta_1 f, \ldots, \Delta_{N-1} f$; *and*
4. $S''(x_i)\delta_i f \geq 0$, $i = 1, \ldots, N-1$; *the number of sign changes of the function* $S''(x)$ *in* $x \in [a, b]$ *coincides with that in the sequence* $\delta_1 f, \delta_2 f, \ldots, \delta_{N-1} f$.

The shape preserving approximation problem is, by definition, the problem of searching for a function $S(x) \in I(\Delta, F)$. We seek a solution of the shape preserving approximation problem in the form of generalized cubic spline complying with the Definition 3.1.

§5. Algorithm of Shape Preserving Local Approximation

The set of splines complying with Definition 3.1 is denoted by S_3^G. As $dim(S_3^G) = 4N - 3(N-1) = N+3$, for the constructing in S_3^G a basis of B-splines, i. e. nonnegative functions with local minimum-length supports, we extend mesh Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$ such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+1} < x_{N+2} < x_{N+3}$.

Simple calculations permit us to obtain an explicit form of B-splines

$$B_i(x) = \begin{cases} B_i''(x_{i-1})\Psi_{i-2}(x), \quad x \in [x_{i-2}, x_{i-1}], \\ B_i''(x_{i-1})[v_{i-1} + v_{i-1}'(x - x_{i-1})] \\ \quad + B_i''(x_{i-1})\Phi_{i-1}(x) + B_i''(x_i)\Psi_{i-1}(x), \quad x \in [x_{i-1}, x_i], \\ -B_i''(x_{i+1})[v_{i+1} + v_{i+1}'(x - x_{i+1})] \\ \quad + B_i''(x_i)\Phi_i(x) + B_i''(x_{i+1})\Psi_i(x), \quad x \in [x_i, x_{i+1}], \\ B_i''(x_{i+1})\Phi_{i+1}(x), \quad x \in [x_{i+1}, x_{i+2}], \\ 0, \quad x \notin [x_{i-2}, x_{i+2}], \end{cases} \tag{5.1}$$

where

$$\Phi_j(x) = \varphi_j\left(\frac{x - x_j}{h_j}\right)h_j^2, \quad \Psi_j(x) = \psi_j\left(\frac{x - x_j}{h_j}\right)h_j^2,$$
$$v_j^{(r)} = \psi_{j-1}^{(r)}(1)h_{j-1}^{2-r} - \varphi_j^{(r)}(0)h_j^{2-r}, \quad r = 0, 1.$$
$$B_i''(x_j) = \frac{y_{i+1} - y_{i-1}}{v_j'\omega_{i-1}'(y_j)}, \quad y_j = x_j - v_j/v_j', \quad j = i-1, i, i+1,$$
$$\omega_{i-1}(x) = (x - y_{i-1})(x - y_i)(x - y_{i+1}).$$

Further we consider the case when 'averaged nodes' of B-splines $y_i = x_i - v_i/v_i'$, $i = 0, \ldots, N$ coincide with the nodes of main mesh Δ, i.e. $v_i = \psi_{i-1}(1)h_{i-1}^2 - \varphi_i(0)h_i^2 = 0$, $i = 0, \ldots, N$ and $x_{-i} = x_0 - ih_0$, $x_{N+i} = x_N + ih_{N-1}$, $i = 1, 2, 3$.

Basis splines $B_i(x)$, $i = -1, \ldots, N+1$ have the following properties: $B_i(x) > 0$ if $x \in (x_{i-2}, x_{i+2})$ and $B_i(x) \equiv 0$ otherwise,

$$\sum_{j=-1}^{N+1} B_j(x) \equiv 1 \quad \text{for} \quad x \in [a, b].$$

Any spline $S(x) \in S_3^G$ can be uniquely represented in the form

$$S(x) = \sum_{j=-1}^{N+1} b_j B_j(x) \quad \text{for} \quad x \in [a, b] \tag{5.2}$$

with some constant coefficients b_j.

From (5.1) the expression (5.2) for spline $S(x)$ in the subinterval $[x_i, x_{i+1}]$ is transformed to the form

$$S(x) = b_i + \Delta_i b(x - x_i) + \varphi_i(t) h_i^2 \delta_i b / v_i' + \psi_i(t) h_i^2 \delta_{i+1} b / v_{i+1}', \tag{5.3}$$

where $\delta_j b = \Delta_j b - \Delta_{j-1} b$, $j = i, i+1$, $\Delta_j b = (b_{j+1} - b_j)/h_j$.

Whence the formulae

$$S(x_i) = b_i + \delta_i b \left(\frac{\psi_{i-1}'(1)}{\psi_{i-1}(1)} \frac{1}{h_{i-1}} - \frac{\varphi_i'(0)}{\varphi_i(0)} \frac{1}{h_i} \right)^{-1}, \tag{5.4}$$

$$S'(x_i) = \frac{1}{v_i'} [\psi_{i-1}'(1) h_{i-1} \Delta_i b - \varphi_i'(0) h_i \Delta_{i-1} b], \tag{5.5}$$

$$S''(x_i) = \delta_i b / v_i' \tag{5.6}$$

follow, and vice versa

$$\begin{aligned} b_{i-1} &= S(x_i) - h_{i-1} S'(x_i) + h_{i-1}^2 [-\psi_{i-1}(1) + \psi_{i-1}'(1)] S''(x_i), \\ b_i &= S(x_i) - h_i^2 \varphi_i(0) S''(x_i), \\ b_{i+1} &= S(x_i) + h_i S'(x_i) - h_i^2 [\varphi_i(0) + \varphi_i'(0)] S''(x_i), \\ & \qquad i = 0, \dots, N. \end{aligned} \tag{5.7}$$

Algorithm 5.1. We compute the coefficients in (5.2) from formulae (5.7), with $S''(x_i)$ being approximated using the second divided difference

$$b_i = f_i - 2h_i^2 (h_{i-1} + h_i)^{-1} \varphi_i(0) \delta_i f, \quad i = 1, \dots, N-1. \tag{5.8}$$

To determine the coefficients b_i, for $i = -1, 0, N, N+1$, we use the boundary conditions: $S^{(k)}(x_i) = f_i^{(k)}$, $i = 0, N$, $k = 0, 1$. Using the formulae (5.4)–(5.7), we write out

$$\begin{aligned} b_{-1} &= b_1 - 2h_0 f_0', \\ b_0 &= f_0 - (f_0 + h_0 f_0' - b_1)[1 + \varphi_0'(0)/\varphi_0(0)]^{-1}, \\ b_N &= f_N - (f_N - h_{N-1} f_N' - b_{N-1}) \\ & \qquad \times [1 - \psi_{N-1}'(1)/\psi_{N-1}(1)]^{-1}, \\ b_{N+1} &= b_{N-1} + 2h_{N-1} f_N'. \end{aligned} \tag{5.9}$$

We find the parameters p_i, q_i, $i = 0, \dots, N-1$, from the shape preservity conditions formulated in Definition 4.1 in two steps. Using the constraints $|b_i - f_i| \leq \varepsilon_i$, $i = 1, \dots, N-1$, which in view of (6.1) are equivalent to

$$2h_i^2 (h_{i-1} + h_i)^{-1} \varphi_i(0) |\delta_i f| \leq \varepsilon_i, \quad i = 1, \dots, N-1 \tag{5.10}$$

we first find p_i and obtain q_{i-1} from the condition $v_i = 0$ or $\varphi(q_{i-1}, 0)h_{i-1}^2 = \varphi(p_i, 0)h_i^2$.

According to (5.9) the quantities p_0, q_{N-1} are selected so as to satisfy the inequalities

$$\begin{aligned} |b_0 - f_0| &= |f_0 + h_0 f_0' - b_1||1 + \varphi_0'(0)/\varphi_0(0)|^{-1} \le \varepsilon_0, \\ |b_N - f_N| &= |f_N - h_{N-1} f_N' - b_{N-1}| \\ &\qquad \times |1 - \psi_{N-1}'(1)/\psi_{N-1}(1)|^{-1} \le \varepsilon_N. \end{aligned} \tag{5.11}$$

Finally p_i, q_i we find from the constraints $|S(x_i) - f_i| \le \varepsilon_i$, $i = 0, \dots, N$. From (5.4) and (5.8)

$$\begin{aligned} S(x_i) = f_i + H_i^{-1} \Big\{ & - \frac{2h_{i-1}^2 \varphi_{i-1}(0)}{h_{i-2} + h_{i-1}} \delta_{i-1} f - \frac{2h_{i+1}^2 \varphi_{i+1}(0)}{h_i(h_i + h_{i+1})} \delta_{i+1} f \\ & + [1 + 2h_{i-1}^{-1} h_i \varphi_i(0) - 2(h_{i-1} + h_i)^{-1} (\psi_{i-1}'(1) h_{i-1} - \varphi_i'(0) h_i)] \delta_i f \Big\}, \end{aligned} \tag{5.12}$$

where

$$H_i = \frac{\psi_{i-1}'(1)}{\psi_{i-1}(1)} \frac{1}{h_{i-1}} - \frac{\varphi_i'(0)}{\varphi_i(0)} \frac{1}{h_i}.$$

Therefore according to the estimate (5.10) by (5.12) we obtain

$$|S(x_i) - f_i| \le H_i^{-1} \theta_i \le \varepsilon_i, \tag{5.13}$$

where $\theta_i = \varepsilon_{i-1} h_{i-1}^{-1} + \frac{4}{3} |\delta_i f| + \varepsilon_{i+1} h_i^{-1}$.

If $h_{i-1} \le h_i$ from (5.13) we have

$$|S(x_i) - f_i| \le \theta_i \left[\left(\frac{1}{h_{i-1}} + \frac{1}{h_i} \right) \frac{\psi_{i-1}'(1)}{\psi_{i-1}(1)} \right]^{-1} \le \varepsilon_i.$$

Taking into account the constraint for generalized cubic splines $\psi_{i-1}'(1)/\psi_{i-1}(1) \ge 3$, we can set

$$\frac{\psi_{i-1}'(1)}{\psi_{i-1}(1)} - 3 = \max \left(\frac{h_{i-1} h_i}{h_{i-1} + h_i} \frac{\theta_i}{\varepsilon_i} - 3, 0 \right), \quad i = 2, \dots, N-2 \tag{5.14}$$

to define q_{i-1}. The value p_i is calculated from condition $v_i = 0$. The case in which $h_i < h_{i-1}$ and $p_i < q_{i-1}$ is considered in a similar manner.

For $i = 1, N-1$, according to (5.9) and (5.11) we come again to the formula (5.14) that permits us to choose the parameters q_0, p_{N-1}. We fulfill the conditions of the definition 4.1 by the final choice of the parameters p_i, q_i, $i = 0, \dots, N-1$. This gives us the following result.

Theorem 5.1. *If the inequalities*

$$\delta_1 f(\Delta_0 f - f'_0) > |\delta_1 f|\varepsilon_1 h_0^{-1}, \quad f'_0 \Delta_0 f \ge 0,$$
$$\delta_{N-1} f(f'_N - \Delta_{N-1} f) > |\delta_{N-1} f|\varepsilon_{N-1} h_{N-1}^{-1}, \quad f'_N \Delta_{N-1} f \ge 0,$$

are valid, the generalized cubic spline $S(x)$ constructed by Algorithm 5.1 of three-point local approximation is a shape preserving approximant.

Remark 5.1. For $f(x) \equiv 1$ and $f(x) \equiv x$ by immediate checking we have $b_i = 1$ and $b_i = x_i$, $i = -1, \dots, N+1$, respectively and therefore according to (5.3) the shape preserving spline $S(x)$ reproduces the straight lines.

§6. Shape Preserving Approximation of Surfaces

Let the domain $G : [c, d] \times [0, 1] \subset WU$ be partitioned by straight lines $w = w_i$, $i = 0, \dots, N$, of the grid $\Delta_w : c = w_0 < w_1 < \cdots < w_N = d$ into N rectangular subdomains. Assume that a grid $\Delta_u^i : 0 = u_0^i < u_1^i < \cdots < u_{M_i}^i = 1$, $i = 0, 1, \dots, N$, is given on every straight line $w = w_i$. The number of grid nodes and their positions on grids Δ_u^i, $i = 0, \dots, N$, are independent of one another. The values f_{ij} of some function $f(w, u)$ are given with tolerances ε_{ij} at the nodes u_j^i, $j = 0, \dots, M_i$, $i = 0, \dots, N$.

A surface of the class $C^{2,2}(G)$, passing through the points $P_{ij} = (w_i, u_j^i, \tilde{f}_{ij})$, where $\tilde{f}_{ij} \in [f_{ij} - \varepsilon_{ij}, f_{ij} + \varepsilon_{ij}]$, $j = 0, \dots, M_i$, $i = 0, \dots, N$, can be constructed by generalizing the algorithm of local approximation by splines from Section 5. In addition to being efficient at constructing the surface, these algorithm also preserve the shape of input data.

The surface is sought in the form of a function:

$$S(w, u) = \sum_{i=-1}^{N+1} b_i(u) B_i(w). \tag{6.1}$$

where the generalized basis splines $B_i(w)$ are the same as in (5.2). The functions $b_i(u)$, $i = -1, \dots, N+1$, generalize local approximation formulae from Sections 5 (Algorithms 5.1) and are linear combinations of the one-dimensional interpolation shape preserving splines $S_i(u)$, $i = 0, \dots, N$ described in Sections 2 and 3. These splines define curves along sections $w = w_i$, $i = 0, \dots, N$, and pass through the points (u_j^i, f_{ij}), $j = 0, \dots, M_i$.

Formally, necessary formulae (Algorithm 6.1) can be obtained by replacing the values $f_j^{(k)}$ in Algorithm 5.1 by the functions $S_j^{(k)}(u)$, $k = 0, 1, 2$, respectively. Similar changes are made to boundary conditions. For the scheme given below we use the boundary conditions: $S(w_i, u) = S_i(u)$, $\frac{\partial}{\partial w} S(w_i, u) = g_i(u)$, $i = 0, N$, with $g_i(u) = \frac{\partial}{\partial w} f(w_i, u)$.

Algorithm 6.1. We compute the coefficients in (6.1) by the formulae:

$$
\begin{aligned}
b_{-1}(u) &= b_1(u) - 2h_0 g_0(u), \\
b_0(u) &= S_0(u) - [S_0(u) + h_0 g_0(u) - b_1(u)] \\
&\qquad \times [1 + \varphi_0'(0)/\varphi_0(0)]^{-1}, \\
b_i(u) &= S_i(u) - 2h_i^2 (h_{i-1} + h_i)^{-1} \varphi_i(0) \delta_i S(u), \\
&\qquad i = 1, \ldots, N-1, \\
b_N(u) &= S_N(u) - [S_N(u) - h_{N-1} g_N(u) - b_{N-1}(u)] \\
&\qquad \times [1 - \psi_{N-1}'(1)/\psi_{N-1}(1)]^{-1}, \\
b_{N+1}(u) &= b_{N-1}(u) + 2h_{N-1} g_N(u),
\end{aligned}
\tag{6.2}
$$

where

$$\delta_i S(u) = \Delta_i S(u) - \Delta_{i-1} S(u), \quad \Delta_j S(u) = [S_{j+1}(u) - S_j(u)]/h_j, \ j = i-1, i.$$

The approximating spline $S(w, u)$ possesses the following properties of preserving the shape of the initial data.

Property 6.1. *Let functions $S_j(u)$, $j = i-1, \ldots, i+2$, $1 \le i \le N-1$, be monotonic and/or convex on the interval $[\tilde{u}_m, \tilde{u}_{m+1}]$ and satisfy the conditions*

$$S_j^{(k)}(u) \delta_j f^{(k)}(u) < 0, \quad j \ne 0, N, \quad S_j^{(k)}(u) g_j^{(k)}(u) < 0, \quad j = 0, N,$$

where $k = 1$ and/or $k = 2$, respectively. Then for any fixed $\tilde{w} \in [w_i, w_{i+1}]$, $2 \le i \le N-3$, the generalized spline $S(w, u)$ constructed by Algorithm 6.1 will be monotonic and/or convex in $[\tilde{u}_m, \tilde{u}_{m+1}]$.

Property 6.2. *Let the choice of parameters p_i, q_i, $i = -2, \ldots, N+2$, of a generalized spline $S(w, u)$ ensures the following estimate for any $\tilde{S}_j(u)$ such that $\Delta_i \tilde{S}(u)$, $\delta_i \tilde{S}(u)$ do not change sign for all $u \in [0, 1]$,*

$$|\tilde{S}_j(u) - S_j(u)| \le E_j(u), \quad j = 0, \ldots, N,$$

where $E_j(u)$ are given functions. Then for any fixed u the spline $S_u(w) = S(w, u)$ is a shape preserving approximant.

To prove these assertions, it is sufficient to take advantage of the relations

$$\frac{\partial^k}{\partial u^k} S(w, u) = \sum_{i=-1}^{N+1} b_i^{(k)}(u) B_i(w), \quad k = 1, 2,$$

use the expressions (6.2) for the coefficients $b_i(u)$, and take into account the finiteness of the B-splines: $B_i(w) > 0$ at $w \in (w_{i-2}, w_{i+2})$ and $B_i(w) \equiv 0$ at $w \notin (w_{i-2}, w_{i+2})$.

A nonunique shape preserving surface given point by point as a family of curvilinear nonintersecting sections can be constructed by introducing the standard parametrization

$$x = S^x(w,u), \quad y = S^y(w,u), \quad z = S^z(w,u). \tag{6.3}$$

We choose the nodes of nonuniform meshes in the directions w and u according to the results [4]. In our case the original points $T_{ij} = (x_{ij}, y_{ij}, z_{ij})$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$, are considered to belong to the parallelepiped $\prod_{ij} = \{\tilde{\chi}_{ij} \| \tilde{\chi}_{ij} - \chi_{ij}| \le \varepsilon_{ij}^{\chi}\}$, where we put $\chi_{ij} = \chi(w_i, u_j)$ for every coordinate function in (6.3) and ε_{ij}^{χ} is the admissible deviation for the appropriate variable. The resulting surface is obtained as a triple of shape preserving splines constructed by the above algorithm.

§7. Numerical Examples

In the numerical tests the initial data were given point by point as a collection of nonintersecting, in general, curvilinear sections of a 3-D body. At the beginning via the 1-D algorithm of shape preserving interpolation from sections 2 and 3 the system of curves along the initial cross-sections was constructed. Along the orthogonal direction the set of generalized local approximation splines was generated. On the resulting surface the system of curvilinear coordinate lines forming the regular mesh was constructed. Along these lines shape properties of the initial data such as convexity, monotonicity, presence of linear sections and other were retained. The defining functions $\varphi_i(t)$, $\psi_i(t)$ for the generalized cubic spline were taken in the form

$$\varphi_i(t) = \varphi(p_i, t) = P_i(1-t)^3/[1 + p_i t(1-t)],$$
$$\psi_i(t) = \varphi(q_i, 1-t), \quad P_i^{-1} = 2(1+p_i)(3+p_i),$$

that correspond to rational cubic splines with quadratic denominator. The initial data and the resulting shape preserving surface are given in figures 1 and 2, correspondently.

Fig. 1. The initial aircraft data.

Fig. 2. The resulting shape preserving aircraft surface.

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