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EXTENSIONS OF THE HEISENBERG GROUP BY d-PARAMETER GROUPS OF DILATIONS

Adisak Seesanea

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EXTENSIONS OF THE HEISENBERG GROUP BY d-PARAMETER GROUPS OF DILATIONS

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ศึกษาภาคขยายของกรุปไฮเซนเบิร์กแบบหลายมิติซึ่งขยายโดยกรุป d-พารามิเตอร์ของเมทริกซ์เปลี่ยนขนาด โดยทำการจำแนกกรุปภาคขยายบางส่วนขึ้นอยู่กับสมสัณฐานของกรุปด้วยวิธีการพีชคณิตลี และจำแนกกรุปภาคขยายทั้งหมดในกรณี $d = 2$ นอกจากนี้ได้แสดงว่ากรุปเหล่านั้นสมสัณฐานกันกับทั้งกรุปย่อยของกรุปซิมเพลกติกและกรุปย่อยของกรุปสัมพรรคทำให้ได้ว่ากรุปเหล่านั้นมีทั้งตัวแทนเมตาเพลกติกและตัวแทนเวฟเลท ยิ่งไปกว่านั้นยังแสดงให้เห็นว่าตัวแทนเมตาเพลกติกดังกล่าวเป็นผลรวมของตัวแทนย่อยที่เหมือนกันสองตัวของตัวแทนเวฟเลทดังกล่าวด้วย

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HEISENBERG GROUP / LIE ALGEBRA / METAPLECTIC REPRESENTATION / WAVELET REPRESENTATION

Group extensions of the multidimensional Heisenberg group by d-parameter groups are investigated. The extended groups are partially classified, up to isomorphism, by employing Lie algebra methods, and in case $d = 2$ a complete classification is given. It is shown that they are isomorphic to subgroups of both, the symplectic group and the affine group, and thus possess a metaplectic and a wavelet representation. It is further shown that the metaplectic representation is a sum of two copies of a subrepresentation of the wavelet representation.

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CONTENTS

CONTENTS (Continued)

CONTENTS (Continued)

LIST OF TABLES

Table **Page**

CHAPTER I

INTRODUCTION

The purpose of this thesis is to study extensions of the multidimensional Heisenberg group and their unitary representations. Before going into further details, we need to introduce the basic concepts, and review some of the literature on this topic.

1.1 Extensions of the Heisenberg group

1.1.1 Background

One way in which groups can naturally be enlarged originates with group actions. Let $(M, +)$ be a given group on which a second group $(H, \cdot)$ acts by automorphisms α_h. The Cartesian product

$$M \times H = \{ (m, h) : m \in M,\, h \in H \}$$

of the two groups can be given a group structure different from the usual product group operation by setting

$$(m, h)(\tilde{m}, \tilde{h}) = (m + \alpha_h(\tilde{m}), h\tilde{h}).$$

This new group is called the *semi-direct product* of the two groups and denoted by $M \rtimes_\alpha H$. One easily verifies that M is isomorphic to the normal subgroup $\{(m, e) : m \in M\}$ of $M \rtimes_\alpha H$, while H is isomorphic to both, the subgroup $\{(0, h) : h \in H)\}$ of $M \rtimes_\alpha H$, and to the quotient $(M \rtimes_\alpha H)/M$. When the two component groups are topological groups and the action α is continuous, then the semi-direct product will again be a topological group in the product topology.

A simple example of this construction is the *affine group*. The general linear group $GL_n(\mathbb{R})$ naturally acts on $\mathbb{R}^n$ by matrix multiplication,

$$\alpha_h(m) = hm \qquad (m \in \mathbb{R}^n, h \in GL_n(\mathbb{R})),$$

and the semi-direct product $\mathbb{R}^n \rtimes_\alpha GL_n(\mathbb{R})$ is then isomorphic to the group formed by translations and linear transformations in Euclidean space, namely the affine group $\mathit{Aff}(n, \mathbb{R})$.

In this thesis, we consider semi-direct products involving the *Heisenberg group*. Recall that the matrix $\mathcal{J} = \left[\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right]$ determines a skew-symmetric and bilinear form on $\mathbb{R}^{2n}$, the *symplectic form*, by

$$[\![w, \tilde{w}]\!] = w^T \mathcal{J} \tilde{w} \qquad (w, \tilde{w} \in \mathbb{R}^{2n}). \tag{1.1}$$

The Heisenberg group is the set

$$\mathbb{H}^n = \big\{ (w, z) \ : \ w \in \mathbb{R}^{2n}, \ z \in \mathbb{R} \big\}$$

endowed with the topology of $\mathbb{R}^{2n+1}$ and the group operation

$$(w, z)(\tilde{w}, \tilde{z}) = \big(w + \tilde{w}, z + \tilde{z} + \frac{1}{2}[\![w, \tilde{w}]\!]\big).$$

It plays a fundamental role in quantum mechanics (Folland, 1989) and signal processing (Gröchenig, 2000).

Now it is known (Folland, 1989) that every automorphism of the Heisenberg group is composed of automorphisms of four basic types: an *inner automorphism*, *inversion*, a *dilation* and a *symplectic automorphism*. The first two types of automorphisms are not of interest here, because inner automorphisms keep the elements in the *phase space* $W = \{(w,0) \in \mathbb{H}^n : w \in \mathbb{R}^{2n}\}$ fixed, and inversion is of finite order two. Dilations of the Heisenberg group are automorphisms of the form

$$\alpha_\lambda(w,z) = (\lambda w, \lambda^2 z)$$

for some nonzero real number λ, while symplectic automorphisms are determined by symplectic matrices: Recall here that the *symplectic group* $Sp(n,\mathbb{R})$ is the set of all invertible matrices preserving the symplectic form (1.1),

$$Sp(n,\mathbb{R}) = \left\{ \mathcal{A} \in GL_{2n}(\mathbb{R}) \ : \ [\![\mathcal{A}w, \mathcal{A}\tilde{w}]\!] = [\![w, \tilde{w}]\!] \quad \forall\ w, \tilde{w} \in \mathbb{R}^{2n} \ \right\}.$$

Each of its elements naturally defines an automorphism $\alpha_{\mathcal{A}}$ of $\mathbb{H}^n$ by

$$\alpha_{\mathcal{A}}(w,z) = (\mathcal{A}w, z)$$

fixing the elements of the center $Z = \{(0,z) : z \in \mathbb{R}\}$ of $\mathbb{H}^n$. The corresponding semi-direct product $\mathbb{H}^n \rtimes_\alpha Sp(n,\mathbb{R})$ has been studied by Cordero et al. (2006).

Our interest centers around automorphisms which are composed of dilations and automorphisms of symplectic type, but which leave the two components

$$X = \{((x,0),0) : x \in \mathbb{R}^n\} \qquad \text{and} \qquad Y = \{((0,y),0) : y \in \mathbb{R}^n\}$$

of the phase space $W = \{(w,0) : w \in \mathbb{R}^{2n}\}$ invariant. This invariance condition restricts the symplectic automorphisms to those determined by symplectic matrices of the form

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & [A^{-1}]^T \end{bmatrix} \qquad (A \in GL_n(\mathbb{R})),$$

and makes it possible to work with the *polarized Heisenberg group* $\mathbb{H}^n_{pol}$, which has a representation as the matrix group

$$\mathbb{H}^n_{pol} = \left\{ h(x,y,z) \ = \ \begin{bmatrix} 1 & y^T & z \\ 0 & I_n & x \\ 0 & 0 & 1 \end{bmatrix} \ : \ x, y \in \mathbb{R}^n,\ z \in \mathbb{R} \right\} \subset GL_{n+2}(\mathbb{R}).$$

In fact, the two Heisenberg groups are isomorphic via the map

$$\Psi : (w, z) \in \mathbb{H}^n \mapsto h(x, y, z + \tfrac{1}{2} y^T x) \in \mathbb{H}^n_{pol} \qquad (w = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right],\ x, y \in \mathbb{R}^n).$$

Now consider the closed subgroup of $GL_{n+2}(\mathbb{R})$ of the form

$$D_0 = \left\{ d(\lambda, A) := \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda A & 0 \\ 0 & 0 & 1 \end{bmatrix} : \lambda \in \mathbb{R}\backslash\{0\},\ A \in GL_n(\mathbb{R}) \right\}.$$

Direct computation shows that an automorphism $\alpha_{\mathcal{A}} \circ \alpha_\lambda$ of $\mathbb{H}^n$ is carried by Ψ to the automorphism of $\mathbb{H}^n_{pol}$ determined by conjugation with $d(\lambda, A)$,

$$\Psi((\alpha_{\mathcal{A}} \circ \alpha_\lambda)(w, z)) = d(\lambda, A)\Psi(w, z)d(\lambda, A)^{-1}.$$

It is thus natural to consider semidirect products of the form

$$\mathbb{H}^n_{pol} \rtimes_\alpha D$$

for closed subgroups D of D_0. It is more convenient to reparametrize elements of D_0 in the form

$$D_0 = \left\{ d(a, A) := \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix} : a > 0,\ A \in GL_n(\mathbb{R}) \right\};$$

then the semi-direct products can be represented as matrix groups

$$\mathbb{H}^n_{pol} \rtimes D \cong \left\{ h(x, y, z)d(a, A) \ :\ h(x, y, z) \in \mathbb{H}^n_{pol},\ d(a, A) \in D \right\} \subset GL_{n+2}(\mathbb{R}). \tag{1.2}$$

When $D = \{ d(1, A) : A \in GL_n(\mathbb{R}) \} \cong GL_n(\mathbb{R})$, this semi-direct product is called the *affine-Weyl-Heisenberg group* which has been extensively studied by several authors (Ali, Antoine and Gazeau, 2000; Hogan and Lakey, 1995; Kalisa and Torrésani, 1993; and Torrésani, 1991).

The case $n = 1$ (so that A is a scalar) and $D = \{d(A^p, A) : A > 0\} \simeq \mathbb{R}^+$ for fixed p has already been studied in Schulz and Taylor (1999), where the semi-direct products were classified up to isomorphism. It was further noticed that they are isomorphic to subgroups of the affine group $Aff(2, \mathbb{R})$,

$$\mathbb{H}^1_{pol} \rtimes D \cong \mathbb{R}^2 \rtimes H \tag{1.3}$$

where H is a closed subgroup of $GL_2(\mathbb{R})$, and $\mathbb{R}^2$ and H are identified with the groups of matrices,

$$\mathbb{R}^2 \cong \{\, h(x, 0, z) \;:\; x, z \in \mathbb{R}\,\} \subset GL_3(\mathbb{R})$$

and

$$H \cong \{\, h(0, y, 0)d(A^p, A) \;:\; A > 0\,\} \subset GL_3(\mathbb{R}),$$

respectively. Namngam (2010) has considered the case of arbitrary n, where the groups D are one-parameter groups

$$D = \{d(e^{pt}, e^{Bt}) : t \in \mathbb{R}\} \tag{1.4}$$

for some fixed number p and $B \in M_n(\mathbb{R})$, and has classified the semi-direct products up to isomorphism with regards to the choice of p and B.

1.1.2 The first objective

The main purpose of this thesis is to continue the study and classification of semi-direct products $\mathbb{H}^n_{pol} \rtimes_\alpha D$ in higher dimensions. We will consider groups D which are d-parameter groups, that is, groups which are of the form

$$D = \left\{ d(e^{p_1 t_1 + \dots p_d t_d}, e^{t_1 B_1 + \dots + t_d B_d}) : (t_1, \dots, t_d) \in \mathbb{R}^d \right\}. \tag{1.5}$$

for fixed real numbers p_k and commuting matrices B_k, and which are isomorphic to $\mathbb{R}^d$. Using Lie algebra techniques, we will prove some results towards their classification, and in case $d = 2$ we can give a complete theorem on classification.

1.2 Affine subgroups of the symplectic group

1.2.1 Background

In Cordero et al. (2006) and also in Czaja and King (2012, 2013), two subgroups of the symplectic group $Sp(n+1, \mathbb{R})$, denoted $(CDS)_{n+1}$ and $(TDS)_{n+1}$ were shown to be isomorphic to subgroups of the affine group $Aff(n+1, \mathbb{R})$, and it was shown that their metaplectic representations and wavelet representations have equivalent subrepresentations. Later it was demonstrated by Namngam (2010) that these two groups belong to the class of groups of the form (1.2), where D is a one-parameter group of form (1.4) with B of particularly simple form. De Mari and De Vito (2013) and Namngam (2010) generalized the ad-hoc techniques of Cordero et al. (2006) to identify general classes of subgroups of the symplectic group which are isomorphic to subgroups of the affine group, and to obtain connections between their metaplectic and wavelet representations.

1.2.2 The second objective

The secondary purpose of this thesis is to apply the techniques of De Mari and De Vito (2013) and Namngan (2010) to show that the semi-direct products $\mathbb{H}^n_{pol} \rtimes_\alpha D$ considered here, with D as in (1.5), follow this pattern. We show that they are isomorphic to subgroups of the symplectic group $Sp(n+1, \mathbb{R})$ as well as the affine group $Aff(n+1, \mathbb{R})$, and study connections between their metaplectic and wavelet representations.

1.3 Organization

This thesis is organized as follows. In Chapter II, we introduce the notation and review the main concepts and theorems used throughout, mainly covering topics from the theory of locally compact groups and Lie algebras. In Chapter III, we apply Lie algebra techniques to prove some theorems on the classification of the extended groups. In the special case of 2-paramter groups, we provide an explicit list of all equivalence classes of extended groups in dimensions $n = 1, 2, 3$. Chapter IV is used to showing that the extended groups are isomorphic to subgroups of both, the symplectic group and the affine group, and to studying their metaplectic and affine representations. Chapter V concludes by summarizing the results achieved.

CHAPTER II

BASIC BACKGROUND

Throughout this thesis, we assume that the reader is familiar with the fundamental concepts from topology, algebra, measure theory, and Hilbert spaces. In this chapter, we document definitions and facts of the lesser known concepts used, mainly covering topics related to topological groups and their representations, semi-direct products of such groups, and matrix groups and their Lie algebras. Details and proofs can be found in standard textbooks such as Baker (2001), Folland (1989, 1995, 1999), and Jacobson (1962).

2.1 Topological concepts

2.1.1 Simply connected spaces

Throughout this section, we let X and Y be topological spaces.

Definition 2.1. Let $f_0, f_1 : X \to Y$ be continuous functions. Then f_0 is *homotopic* to f_1 if there exists a continuous function $F : [0,1] \times X \to Y$ such that

$$F(0,x) = f_0(x) \qquad \text{and} \qquad F(1,x) = f_1(x) \qquad \forall x \in X.$$

The function F is called a *homotopy* from f_0 to f_1.

Remark 2.2. For $(t,x) \in [0,1] \times X$, we may regard t as measuring time. Then $f_t(x) := F(t,x)$ is a 1-parameter family of functions maps $X \to Y$. At time $t = 0$, we have the function f_0. At time $t = 1$, we have function f_1. As time increase from 0 to 1, the function f_0 is deformed continuously to the function f_1.

Definition 2.3. Let $x_0, x_1 \in X$. A *path* from x_0 to x_1 (with origin at x_0 and end at x_1) is a continuous function $\rho : [0,1] \to X$ such that

$$\rho(0) = x_0 \qquad \text{and} \qquad \rho(1) = x_1.$$

In case $x_0 = x_1$ the origin and end points coincide, and the path ρ is called a *loop* with basepoint x_0.

Definition 2.4. A space X is *path connected* if, for any given pair of points $x_0, x_1 \in X$, there exists a path with origin at x_0 and end at x_1.

Example 2.5. We provide a couple of simple examples.

(1) The Euclidean space $\mathbb{R}^n$ is path connected. In fact, every pair x_0, x_1 of points in $\mathbb{R}^n$ can be connected by a path which constitutes a line segment,

$$\rho(t) = (1-t)x_0 + tx_1.$$

(2) The unit circle

$$S^1 = \{z = e^{i\theta} \in \mathbb{C} : 0 \leq \theta < 2\pi\}$$

and endowed with the relative topology is path connected. In fact, let $x_0 = e^{i\theta_0}$ and $x_1 = e^{i\theta_1}$ be any pair of points in S^1. Then

$$\rho(t) = e^{i[(1-t)\theta_0 + t\theta_1]}$$

will be a path in S^1 from x_0 to x_1.

Definition 2.6. Two paths in X, ρ and σ from x_0 to x_1 are *homotopic* if there exists a continuous function $F : [0,1] \times [0,1] \to X$ such that

$$F(0,w) = \rho(w) \qquad \text{and} \qquad F(1,w) = \sigma(w) \qquad \forall w \in [0,1]$$

$$F(t,0) = x_0 \qquad \text{and} \qquad F(t,1) = x_1 \qquad \forall t \in [0,1]$$

Remark 2.7. The homotopy of paths in definition 2.6 is really a homotopy in the usual sense equipped with the additional requirement that the origin and end points are fixed throughout the homotopy.

Example 2.8. Let x_0, x_1 be two points in $\mathbb{R}^n$. Then any two paths ρ and σ from x_0 to x_1 are homotopic. In fact, the map

$$F(t, w) = (1 - t)\rho(w) + t\sigma(w)$$

satisfies all the conditions of Definition 2.6.

Definition 2.9. A space X is called *simply connected* if it is path connected and every loop in X is homotopic to a constant path.

Remark 2.10. Intuitively, any loop in a simply connected space can be shrunk continuously to a point contained in that loop.

Example 2.11. By example 2.8, $\mathbb{R}^n$ is simply connected. On the other hand, the unit circle S^1 is not simply connected: Intuitively, the loop $\rho(t) = e^{2i\pi t}$ with basepoint 1 can not be shrunk continuously in S^1 to the point 1.

2.2 Locally compact groups

Definition 2.12. A *topological group* G is a group endowed with a topology so that the group operations are continuous, that is, (when the group is written multiplicatively)

(1) the multiplication map $(x, y) \mapsto xy$ is continuous from $G \times G$ to G, and

(2) the inversion map $x \mapsto x^{-1}$ is continuous from G to G.

Definition 2.13. A topological group G is called *locally compact* if it is a locally compact Hausdorff space in its topology.

Example 2.14. The groups $(\mathbb{R}^n, +)$ and $(\mathbb{R}^+, \cdot)$ are locally compact, but non-compact groups in the usual topology. Here, $\mathbb{R}^+$ denotes the set of positive real numbers. The unit circle S^1 is a compact group under the multiplication of complex numbers. When S^1 is considered as a topological group, it is usually denoted by Π.

Remark 2.15. In the realm of topological groups, the word *isomorphism* means an algebraic isomorphism which is also a homeomorphism.

Example 2.16. Let $\Bbbk = \mathbb{C}$ or $\mathbb{R}$. The set of all $n \times n$ matrices whose entries are in $\Bbbk$ is denoted by $M_n(\Bbbk)$. It is a vector space under the operations of matrix addition and scalar multiplication. Moreover, $M_n(\Bbbk)$ is a topological space as $M_n(\Bbbk)$ is isomorphic to $\Bbbk^{n^2}$ and thus inherits the topology of $\Bbbk^{n^2}$. In fact, this isomorphism makes $M_n(\Bbbk)$ a locally compact group under matrix addition, which is isomorphic to the group $\Bbbk^{n^2}$.

Next, we consider the subset $GL_n(\Bbbk) = \{a \in M_n(\Bbbk) : \det a \neq 0\}$ which is a group under matrix multiplication, called the *general linear group*. As the determinant $\det : M_n(\Bbbk) \to \Bbbk$ is a continuous function, then $GL_n(\Bbbk) = \det^{-1}(\Bbbk \backslash \{0\})$ is open in $M_n(\Bbbk)$, hence $GL_n(\Bbbk)$ is a locally compact group.

2.2.1 Continuous group actions

Definition 2.17. Let G be a group with identity element e, and X a set. A *(left) group action* of G on X is a binary operator $\alpha : G \times X \to X$ satisfying

(1) $\alpha_e(x) = x \qquad \forall x \in X$, and "Identity"

(2) $\alpha_g(\alpha_{\tilde{g}}(x)) = \alpha_{g\tilde{g}}(x) \qquad \forall g, \tilde{g} \in G,\ \forall x \in X.$ "Associativity"

where $\alpha_g(x) := \alpha(g, x)$ for $(g, x) \in G \times X$. The triplet (X, G, α) is called a *transformation group*, X is called a *(left)* G-set, and we say G *acts on* X *(on the* *left) by* α .

When G is a topological group and X a topological space, then one requires in addition that the map α be continuous. In this case, X is called a *G-space*.

When both G and X are topological groups, one also requires that G acts on X by automorphisms of X, that is, for each $g \in G$, the map $\alpha_g : X \mapsto X$ is an automorphism of X.

Example 2.18. Here are a few of simple examples of topological groups acting on topological groups.

(1) Let G be any topological group, and $X = G$. Then G acts on itself by *left translation*,

$$\alpha_g(x) = gx \qquad \forall (g, x) \in G \times G.$$

(2) Let G be any topological group, and N a closed normal subgroup. Then G acts on $X = N$ by *conjugation*,

$$\alpha_g(n) = gng^{-1} \qquad \forall (g, n) \in G \times N.$$

(3) Let $X = (\mathbb{R}, +)$ and $G = (\mathbb{R}^+, \cdot)$. Then $\mathbb{R}^+$ acts on $\mathbb{R}$ by multiplication,

$$\alpha_a(x) = ax \qquad \forall (a, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

(4) More generally, let $X = \mathbb{k}^n$ and $G = GL_n(\mathbb{k})$. Then $GL_n(\mathbb{k})$ acts (continuously) on $\mathbb{k}^n$ by multiplication,

$$\alpha_a(x) = ax \qquad \forall (a, x) \in GL_n(\mathbb{k}) \times \mathbb{k}^n.$$

(5) Let $Sym(n, \mathbb{R}) = \{m \in M_n(\mathbb{R}) : m = m^T\}$ denote the set of all symmetric $n \times n$ matrices. Clearly, $Sym(n, \mathbb{R})$ is a closed linear subspace of $M_n(\mathbb{R})$ and

hence a locally compact group. We observe that $GL_n(\mathbb{R})$ acts continuously on $X = Sym(n, \mathbb{R})$ by

$$\alpha_a(m) = [a^{-1}]^T m a^{-1} \qquad \forall (a, m) \in GL_n(\mathbb{R}) \times Sym(n, \mathbb{R}).$$

We note that in the last two examples, the space X is not only a group, but a vector space, and the action α of G is by vector space automorphisms of X. We will refer to these two examples in later chapters.

2.2.2 Group extensions and semi-direct products

In mathematics, the word *extension* usually means that we enlarge a given object to a larger object in the same category. For example, extending a map usually means enlarging its domain.

In the theory of groups, however, the meaning is usually different:

Definition 2.19. Let G, N and D be groups. Then G is called an *extension* of D by N, if the following conditions are satisfied:

(1) N is a normal subgroup of G, and

(2) D is isomorphic to the quotient group G/N.

Note that D need not be a subgroup of G.

Remark 2.20. One may define a group extensions in a slightly more general way, in the sense that N need only be isomorphic to a normal subgroup of G.

Remark 2.21. When dealing with topological groups, we of course require N to be a closed subgroup of G, and D to be homeomorphic to the quotient space D/N.

Next, we wish to define the external semi-direct product of two groups and show that this is one particular way of obtaining group extensions. However, we first recall the concept of an inner semi-direct product.

Definition 2.22. Let G be a group with identity element e, and N and D subgroups of G. Then G is called the *inner semi-direct product of N by D*, denoted by $G = N \rtimes D$, if

(1) N is a normal subgroup of G,

(2) $G = ND$, and

(3) $N \cap D = \{e\}$.

Remark 2.23. From (2) and (3) it follows that every element in G has a unique representation as $g = nd$ with $n \in N$ and $d \in D$. In addition, D is isomorphic to the quotient G/N. Thus, G is an extension of D by N, with the added property that D is a subgroup of G.

Remark 2.24. Given an inner semi-direct product G of N by D, let α denote the action of D on N by conjugation,

$$\alpha_d(n) = dnd^{-1}, \qquad n \in N,\ d \in D.$$

Then for all $g = nd$ and $\tilde{g} = \tilde{n}\tilde{d}$ in G we have

$$g\tilde{g} = (nd)(\tilde{n}\tilde{d}) = nd\tilde{n}d^{-1}d\tilde{d} = (n\alpha_d(\tilde{n}))\,(d\tilde{d}). \tag{2.1}$$

The following well-known theorem shows a converse of this: If D is a group acting on another group N, then both groups can be embedded in a larger group G which is the semi-direct product of N by D, with group operation satisfying (2.1). Its proof uses only standard tools of the algebra of groups.

Theorem 2.25. Let N and D be multiplicative groups with identity elements e_N and e_D, respectively, and let α be an action of D on N by automorphisms. Set

$$G := N \times D.$$

Then G is a group under group operation defined by

$$(n,d)(\tilde{n},\tilde{d}) = (n\alpha_d(\tilde{n}), d\tilde{d}).$$

The identity element of G is

$$(e_N, e_D),$$

and the inverse of each element (n,d) is

$$\left(\alpha_{d^{-1}}(n^{-1}), d^{-1}\right).$$

Furthermore,

(i) $N' := N \times \{e_D\}$ is a normal subgroup of G, and isomorphic to N

(ii) $D' := \{e_N\} \times D$ is a subgroup of G, and isomorphic to D

(iii) $G = N'D'$

(iv) $N' \cap D' = \{(e_N, e_D)\}$

That is, the group G can be represented as the inner semi-direct product of H' by D'. We therefore denote this new group by

$$G = N \rtimes_\alpha D,$$

called the *semi-direct product of N by D with respect to α*.

Remark 2.26. We make the following comments.

(1) If N and D are locally compact groups, and α is a continuous action of D on N by automorphisms, then the semi-direct product $N \rtimes_\alpha D$ will again be a locally compact group with respect to the product topology on $N \times D$. In addition, N and D will be (isomorphic to) closed subgroups of $N \rtimes_\alpha D$.

(2) If N and D are closed subgroups of $GL_n(\mathbb{R})$, and D acts on N by conjugation,

$$\alpha_d(n) = dnd^{-1} \qquad (n \in N,\ d \in D),$$

then the semi-direct product $N \rtimes_\alpha D$ is isomorphic to the subgroup of $GL_n(\mathbb{R})$,

$$\{nd : n \in N,\, d \in D\}.$$

(3) In this thesis, the group N is usually abelian, with group operation written as addition. Then the group operations in the semi-direct product become

$$(n, d)(\tilde{n}, \tilde{d}) = (n + \alpha_d(\tilde{n}), d\tilde{d}).$$

and

$$(n, d)^{-1} = \left(-\alpha_{d^{-1}}(n), d^{-1}\right).$$

Remark 2.27. Suppose $G = N \rtimes_\alpha D$ is the semi-direct product of N by D. Then G is an extension of D by N. In fact, by the above construction of the semi-direct product, it is left to show that D' is isomorphic to the quotient group G/N'. By the second isomorphism theorem, we have

$$D' \cong D'/(e_N, e_D) = D'/(D' \cap N') \cong D'N'/N' = N'D'/N' = G/N'$$

where N', D' are as defined in Theorem 2.25.

We now have two notions: *extension of D by N* and *semi-direct product of N by D*. Since the group extensions in this thesis arise from semi-direct products, we will use the terminology *extension of N by D* to mean the semi-direct product of N by D, which is different from the meaning of *group extension* as given in Definition 2.19, but in line with the common understanding of the word extension.

2.2.3 Group representations

Since the structure of operators on Hilbert spaces is well understood, it is often useful to represent a given topological group in the form of a group of operators on a Hilbert space, in order to study its properties.

Definition 2.28. A *(unitary) representation* of a locally compact group G is a map π from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on some non-zero Hilbert space $\mathcal{H}_\pi$ satisfying

(1) π is a homomorphism, that is,

$$\pi(xy) = \pi(x)\pi(y) \qquad \text{and} \qquad \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$$

for all $x, y \in G$, and where $\pi(x)^*$ denotes the adjoint operator of $\pi(x)$.

(2) π is continuous when $\mathcal{U}(\mathcal{H}_\pi)$ carries the strong operator topology, that is, the map $x \mapsto \pi(x)u$ is continuous from G to $\mathcal{H}_\pi$, for all vectors $u \in \mathcal{H}_\pi$.

Definition 2.29. Let π be a representation of a locally compact group G on a Hilbert space $\mathcal{H}_\pi$. A closed subspace $\mathcal{K}$ of $\mathcal{H}_\pi$ is called π-*invariant*, if

$$\pi(x)\mathcal{K} \subset \mathcal{K} \qquad \forall\, x \in G.$$

Remark 2.30. Using the above definition, it is not difficult to show that if $\mathcal{K}$ is a closed π-invariant subspace, then its orthogonal complement,

$$\mathcal{K}^\perp := \{u \in \mathcal{H}_\pi : u \perp \mathcal{K}\},$$

is also π-invariant.

Then the restrictions of π to $\mathcal{K}$ and $\mathcal{K}^\perp$, respectively, are again representations of G; we have split π into *subrepresentations*.

Definition 2.31. Two representations π_1, π_2 of a locally compact group G are *(unitarily) equivalent* if there exists a unitary operator $\mathcal{U} : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ with

$$\pi_2(x) = \mathcal{U}\pi_1(x)\mathcal{U}^{-1}$$

for all $x \in G$. We write $\pi_1 \simeq \pi_2$.

Unitary equivalence essentially means that the Hilbert spaces $\mathcal{H}_{\pi_1}$ and $\mathcal{H}_{\pi_2}$ are isomorphic and, up to this isomorphism, $\pi_1(x)$ and $\pi_2(x)$ are the same unitary operators.

Instead of splitting representations, one can also combine them to form new representations:

Definition 2.32. Let $\{\pi_i\}_{i\in I}$ be a family of representations of a locally compact group G, and $\mathcal{H} = \bigoplus\limits_{i\in I} \mathcal{H}_{\pi_i}$ denote the direct sum of the underlying Hilbert spaces. The *direct sum* of the representations $\{\pi_i\}_{i\in I}$ is the representation π of G on $\mathcal{H}$ defined by

$$\pi(x)(v) = \sum_{i\in I} \pi_i(x)v_i \qquad (v = \sum_{i\in I} v_i,\ v_i \in \mathcal{H}_{\pi_i}, i \in I).$$

We write $\pi = \bigoplus\limits_{i\in I} \pi_i$.

2.3 Lie algebras

Definition 2.33. An *algebra* $\mathfrak{g}$ is a vector space over a field $\mathbb{k}$ endowed with a bilinear map $(a, b) \mapsto ab$ on $\mathfrak{g}$, that is,

$$a(\alpha b + \beta c) = \alpha ab + \beta ac \qquad \text{and} \qquad (\alpha b + \beta c)a = \alpha ba + \beta ca$$

for all $\alpha, \beta \in \mathbb{k}, a, b, c \in \mathfrak{g}$. An algebra $\mathfrak{g}$ is called *associative* if

$$a(bc) = (ab)c$$

for all $a, b, c \in \mathfrak{g}$.

Example 2.34. We provide some well-known associative algebras.

(1) The space $M_n(\Bbbk)$ with matrix multiplication as bilinear map.

(2) More generally, the space of all endomorphism of a vector space V, $End(V)$, with the composition of operators as bilinear map.

Definition 2.35. A *subalgebra* $\mathfrak{h}$ of an algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ that closed under the binary operation, that is, $ab \in \mathfrak{h}$ for all $a, b \in \mathfrak{h}$.

Definition 2.36. A *Lie algebra* $\mathfrak{g}$ is an algebra whose bilinear map (usually denoted $(a, b) \mapsto [a, b]$ and called the *Lie bracket*) satisfies the additional conditions

(L1) $[a, a] = 0$, and "Alternating property"

(L2) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ "Jacobi identity"

for all $a, b, c \in \mathfrak{g}$.

Remark 2.37. Applying (L1) and bilinearity to $[a+b, a+b]$, we immediately obtain that (L1) implies "anticommutativity" of the Lie bracket; (L1′): $[a, b] = -[b, a]$. Conversely, (L1′) will imply (L1) when the characteristic of the field $\Bbbk$ is not 2. Hence, if we consider Lie algebras over the fields $\mathbb{R}$ or $\mathbb{C}$, then conditions (L1) and (L1′) are equivalent.

Example 2.38. The following examples are Lie algebras.

(1) Any vector space $\mathfrak{g}$ with trivial bracket $[a, b] = 0$. This is called an *abelian Lie algebra*.

(2) The three-dimensional Euclidean space $\mathbb{R}^3$ with the Lie bracket given by the cross product of vectors.

(3) Any associative algebra with the Lie bracket given by

$$[a,b] := ab - ba$$

becomes a Lie algebra. In fact, property (L1) is obvious. As for (L2) we calculate

$$[a,bc] = abc - bca = (ab - ba)c + b(ac - ca) = [a,b]c + b[a,c],$$

and this implies

$$\begin{aligned}[a,[b,c]] = [a, bc - cb] &= [a,bc] - [a,cb] \\ &= ([a,b]c + b[a,c]) - ([a,c]b + c[a,b]) \\ &= [[a,b],c] + [b,[a,c]].\end{aligned}$$

Applying (L1′), the Jacobi identity follows.

In particular, $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ are Lie algebras.

Lie algebras have a rich structure and have been studied in great detail. We only introduce the few structural properties needed in the next chapter.

Definition 2.39. Let A, B be subsets of a Lie algebra $\mathfrak{g}$. Then $[A,B]$ denotes the vector subspace of $\mathfrak{g}$,

$$[A,B] := span\big\{[a,b] \,:\, a \in A,\, b \in B\big\}.$$

It is clear that if A and B are vector subspaces of $\mathfrak{g}$, then

$$[A,B] = \big\{[a,b] \,:\, a \in A,\, b \in B\big\}.$$

Definition 2.40. Let $\mathfrak{h}$ be a vector subspace of a Lie algebra $\mathfrak{g}$. Then

(1) $\mathfrak{h}$ is called a *(Lie) subalgebra* of $\mathfrak{g}$, if $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$.

(2) $\mathfrak{h}$ is called an *ideal* of $\mathfrak{g}$, if $[\mathfrak{h},\mathfrak{g}] \subseteq \mathfrak{h}$.

Remark 2.41. Any subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a Lie algebra with the induced Lie bracket.

Definition 2.42. Let $\mathfrak{g}$ be a Lie algebra.

(1) Then $[\mathfrak{g},\mathfrak{g}]$ is a subalgebra of $\mathfrak{g}$, called the *derived algebra.*

(2) The *center* of $\mathfrak{g}$ is the subalgebra

$$Z(\mathfrak{g}) = \{z \in \mathfrak{g} : [z,x] = 0 \text{ for all } x \in \mathfrak{g}\}.$$

Remark 2.43. Obviously, the derived algebra $[\mathfrak{g},\mathfrak{g}]$ and the center $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ are ideals of $\mathfrak{g}$.

Definition 2.44. Let $\mathfrak{g}$ be a Lie algebra. Then the sequence $\{\mathfrak{g}_k\}_{k=0}^{\infty}$ of subalgebras

$$\mathfrak{g}_0 = \mathfrak{g}, \qquad \mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}_0], \qquad \mathfrak{g}_2 = [\mathfrak{g},\mathfrak{g}_1], \quad \ldots\ldots \quad \mathfrak{g}_{k+1} = [\mathfrak{g},\mathfrak{g}_k], \quad \ldots\ldots$$

is called its *lower central series.* The Lie algebra $\mathfrak{g}$ is called *(k-step) nilpotent* if there exists a k so that $\mathfrak{g}_k = \{0\}$, while $\mathfrak{g}_{k-1} \neq \{0\}$.

For example, a nontrivial Lie algebra $\mathfrak{g}$ is 1-step nilpotent if and only if it is abelian.

Remark 2.45. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called nilpotent, if it is nilpotent as a Lie algebra in its own right. Since the sum of two ideals in $\mathfrak{g}$ is again an ideal, and the sum of nilpotent ideals is again nilpotent, every Lie algebra contains a largest nilpotent ideal $\mathfrak{n}$ called the *nilradical* of $\mathfrak{g}$, which is unique.

Definition 2.46. A linear map $\Phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ between Lie algebras over the same field $\Bbbk$ is called a *Lie algebra homomorphism*, if Φ preserves the Lie bracket, i.e.,

$$\Phi([a,b]) = [\Phi(a),\Phi(b)] \qquad \forall a,b \in \mathfrak{g}.$$

If in addition, Φ is a bijection, then it is called a *Lie algebra isomorphism*, and $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are said to be *isomorphic* Lie algebras.

Let $\mathfrak{n}$ be in ideal of the Lie algebra $\mathfrak{g}$. Then in particular, $\mathfrak{n}$ is a linear subspace, hence the collection of cosets $\mathfrak{h} = \{a + \mathfrak{n} : a \in \mathfrak{g}\}$ is again a vector space, usually denoted $\mathfrak{h} = \mathfrak{g}/\mathfrak{n}$. We now have:

Proposition 2.47. $\qquad [a + \mathfrak{n}, b + \mathfrak{n}] := [a, b] + \mathfrak{n} \qquad (a, b \in \mathfrak{g})$

well defines a Lie bracket on $\mathfrak{h}$, and the quotient map $\Phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

The usual isomorphism theorems apply. For example:

Proposition 2.48. Let $\Phi : \mathfrak{g} \mapsto \tilde{\mathfrak{g}}$ be an isomorphism of Lie algebras, and $\mathfrak{n}$ an ideal of $\mathfrak{g}$. Then $\tilde{\mathfrak{n}} = \Phi(\mathfrak{n})$ is an ideal of $\tilde{\mathfrak{g}}$, and $\hat{\Phi} : a + \mathfrak{n} \mapsto \Phi(a) + \tilde{\mathfrak{n}}$ defines an isomorphism between the quotient algebras $\mathfrak{h} = \mathfrak{g}/\mathfrak{n}$ and $\tilde{\mathfrak{h}} = \tilde{\mathfrak{g}}/\tilde{\mathfrak{n}}$.

2.4 Matrix groups

A closed subgroup G of $GL_n(\mathbb{k})$ is called a *matrix group*. Note that every closed subgroup of a matrix group is also a matrix group.

2.4.1 The tangent space of a matrix group as a Lie algebra

Every matrix group has an associated Lie algebra, its tangent space at the identity:

Definition 2.49. Let G be a matrix group. A *differentiable curve* in G is a function $\gamma : (a, b) \subseteq \mathbb{R} \to G$ such that the derivative $\gamma'(t)$ exists for each $t \in (a, b)$. Here $\gamma'(t)$ is defined as an element of $M_n(\mathbb{k})$ by

$$\gamma'(t) = \lim_{s \to t} \frac{1}{s - t} \left(\gamma(t) - \gamma(s) \right),$$

provided this limit exists.

A related notion is the following:

Definition 2.50. A *d-parameter group* in a matrix group G is a continuous homomorphism $\gamma : \mathbb{R}^d \to G$.

Definition 2.51. Let G be a matrix group, The *tangent space* of G at $A \in G$ is defined by

$$T_A G = \{\gamma'(0) \in M_n(\Bbbk) \,:\, \gamma \,\text{is a differentiable curve in}\, G \,\text{with}\, \gamma(0) = A\}\,.$$

Theorem 2.52. Let G be a matrix group and $I \in G$ the identity matrix. Then $T_I G$ is a subalgebra of the Lie algebra $M_n(\Bbbk)$. This Lie algebra is called the *Lie algebra of* G, denoted by $\mathfrak{g}$.

In general, nonisomorphic matrix groups may have isomorphic Lie algebras. This is, however, not the case for simply connected groups:

Theorem 2.53. Let G and H be simply connected matrix groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then G and H are isomorphic if and only if $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.

2.4.2 The matrix exponential

Definition 2.54. Given $A \in M_n(\Bbbk)$, the matrix *exponential* of A is defined by the matrix-valued series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

which converges for all $A \in M_n(\Bbbk)$.

Definition 2.55. The *logarithm* of A is defined by the matrix-valued series

$$\log(A) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (A - I)^k$$

which converges and hence is defined for $\|A - I\| < 1$.

Theorem 2.56. Let $A, B \in M_n(\mathbb{k})$.

(1) If A and B commute, then $e^{A+B} = e^A e^B$.

(2) In particular, $e^A \in GL_n(\mathbb{k})$, and $\left(e^A\right)^{-1} = e^{-A}$.

(3) $\det\left(e^A\right) = e^{tr(A)}$.

(4) The maps $A \mapsto e^A$ and $A \mapsto \log(A)$ are continuous.

Proposition 2.57. Let $A \in M_n(\mathbb{k})$. Then $\gamma(t) = e^{tA}$ $(t \in \mathbb{R})$ is a differentiable curve in $GL_n(\mathbb{R})$, and

$$\gamma'(t) = \gamma(t)A \qquad (t \in \mathbb{R}).$$

Proof. First let $t = 0$. By definition of the derivative and the exponential,

$$\lim_{s\to 0} \frac{\gamma(s) - \gamma(0)}{s - 0} = \lim_{s\to 0} \frac{1}{s}\left[\sum_{k=0}^{\infty} \frac{1}{k!}(sA)^k - I_n\right] = \lim_{s\to 0} \sum_{k=1}^{\infty} \frac{1}{k!} s^{k-1} A^k = A$$

as for $s \neq 0$,

$$\left\|\sum_{k=1}^{\infty} \frac{1}{k!} s^{k-1} A^k - A\right\| = \left\|\sum_{k=2}^{\infty} \frac{1}{k!} s^{k-1} A^k\right\| \leq \sum_{k=2}^{\infty} \frac{1}{k!} |s|^{k-1} \|A\|^k = \frac{1}{|s|} \sum_{k=2}^{\infty} \frac{1}{k!} |s|^k \|A\|^k$$
$$= \frac{e^{|s|\,\|A\|} - 1}{|s|} - \|A\| \quad \to 0 \qquad \text{as } s \to 0$$

in the operator norm. That is, γ is differentiable at 0, and $\gamma'(0) = A = \gamma(0)A$.

Next let t be arbitrary. As γ is a group homomorphism, then by the above,

$$\lim_{s\to t} \frac{\gamma(s) - \gamma(t)}{s - t} = \lim_{s\to t} \frac{e^{sA} - e^{tA}}{s - t} = e^{tA} \lim_{s\to t} \frac{e^{(s-t)A} - I_n}{s - t}$$
$$= e^{tA} \lim_{u\to 0} \frac{\gamma(u) - \gamma(0)}{u - 0} = \gamma(t)A$$

which proves the assertion. □

The next theorem generalizes Proposition 2.57 and is fundamental in the connection between Lie algebras and Lie groups. Although it can be stated in more general terms, the following simplified version will suffice.

Theorem 2.58. There exists an open neighborhood U of 0 in $M_n(\mathbb{k})$, $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$, which is being mapped homeomorphically by the exponential map $exp : A \mapsto e^A$ onto an open neighborhood V of the identity I in $GL_n(\mathbb{k})$. Its inverse map is given by the logarithm log. Furthermore, if $\eta : (a, b) \to U$ is a differentiable curve in U, then $\gamma : (a, b) \to V$ given by $\gamma(t) = e^{\eta(t)}$ is a differentiable curve in V, with $\gamma'(t) = e^{\eta(t)}\eta'(t)$ for $t \in (a, b)$. Conversely, every differentiable curve in V is of such a form.

CHAPTER III

CLASSIFICATION OF EXTENSIONS OF THE HEISENBERG GROUP

In this chapter, we first extend the multidimensional polarized Heisenberg group by d-parameter groups of dilations. We then strive to classify the extended groups, up to isomorphism, by using Lie algebra techniques.

3.1 Preliminaries

We begin by reviewing the Heisenberg groups and their Lie algebras in greater detail than was done in the Introduction.

3.1.1 The Heisenberg group

Let I_n denote the $n \times n$ identity matrix, and let $\mathcal{J}$ denote the $2n \times 2n$ skew-symmetric matrix

$$\mathcal{J} := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}. \tag{3.1}$$

As any matrix does, $\mathcal{J}$ determines a bilinear form $[\![\cdot\,,\cdot]\!]$ on $\mathbb{R}^{2n}$ by

$$[\![w, \tilde{w}]\!] = w^T \mathcal{J} \tilde{w}$$

for $w, \tilde{w} \in \mathbb{R}^{2n}$. This form is, however, not an inner product: In fact it is

(1) *skew-symmetric:* Since $\mathcal{J}$ is a skew-symmetric matrix, $\mathcal{J}^T = -\mathcal{J}$, then for all $w, \tilde{w} \in \mathbb{R}^{2n}$,

$$[\![w, \tilde{w}]\!] = w^T \mathcal{J} \tilde{w} = \tilde{w}^T \mathcal{J}^T w = -\tilde{w}^T \mathcal{J} w = -[\![\tilde{w}, w]\!].$$

(2) *totally isotropic.* By skew-symmetry, $[\![w,\tilde{w}]\!] = -[\![w,\tilde{w}]\!]$, we have

$$[\![w,w]\!] = 0$$

for all $w \in \mathbb{R}^{2n}$.

(3) *non-degenerate.* For every nonzero $w \in \mathbb{R}^{2n}$ there clearly exists $\tilde{w} \in \mathbb{R}^{2n}$ so that

$$[\![w,\tilde{w}]\!] \neq 0.$$

The *Heisenberg group* is the set

$$\mathbb{H}^n = \big\{(w,z) \ : \ w \in \mathbb{R}^{2n}, \ z \in \mathbb{R}\,\big\}$$

endowed with the topology of $\mathbb{R}^{2n+1}$ and the group operation

$$(w,z)(\tilde{w},\tilde{z}) = \big(w+\tilde{w}, z+\tilde{z}+\frac{1}{2}[\![w,\tilde{w}]\!]\big).$$

The Heisenberg group has three main components. Decomposing the underlying set $\mathbb{R}^{2n+1}$ as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, then the three components are

$$X = \{(x,0,0) : x \in \mathbb{R}^n\}, \quad Y = \{(0,y,0) : y \in \mathbb{R}^n\}, \quad Z = \{(0,0,z) : z \in \mathbb{R}\}.$$

They are closed abelian subgroups of $\mathbb{H}^n$, and Z is its center. Furthermore, the closed abelian subgroup

$$W = \{(w,0) : w \in \mathbb{R}^{2n}\} = \{(x,y,0) : x,y \in \mathbb{R}^n\},$$

is called the *phase space*.

It is not difficult to verify that $\mathbb{H}^n$, as a topological group, is isomorphic to a matrix group,

$$\mathbb{H}^n \cong \left\{ h_o(w,z) = \begin{bmatrix} 1 & w^T\mathcal{J} & 2z \\ 0 & I_{2n} & w \\ 0 & 0 & 1 \end{bmatrix} \ : \ w \in \mathbb{R}^{2n}, \ z \in \mathbb{R} \right\} \subset GL_{2n+2}. \qquad (3.2)$$

As outlined in the Introduction, since we will consider automorphisms leaving the subgroups X and Y invariant, it is better to split elements w of the phase space as $w = (x, y)$ and work with the *polarized Heisenberg group* $\mathbb{H}^n_{pol}$,

$$\mathbb{H}^n_{pol} = \left\{\, (x, y, z) \;:\; x, y \in \mathbb{R}^n,\, z \in \mathbb{R} \,\right\}$$

which has the group operation

$$(x, y, z)(\tilde{x}, \tilde{y}, \tilde{z}) = \big(x + \tilde{x}, y + \tilde{y}, z + \tilde{z} + y^T \tilde{x}\big)$$

and the simpler representation as a matrix group

$$\mathbb{H}^n_{pol} = \left\{ h(x, y, z) = \begin{bmatrix} 1 & y^T & z \\ 0 & I_n & x \\ 0 & 0 & 1 \end{bmatrix} : x, y \in \mathbb{R}^n,\; z \in \mathbb{R} \right\} \subset GL_{n+2}(\mathbb{R}). \quad (3.3)$$

The two Heisenberg groups are isomorphic via the map

$$\Psi : (w, z) \in \mathbb{H}^n \mapsto h(x, y, z + \tfrac{1}{2} y^T x) \in \mathbb{H}^n_{pol} \qquad (w = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right],\; x, y \in \mathbb{R}^n).$$

3.1.2 The Heisenberg algebra

Recall that the Lie algebra of a matrix group G is defined as the tangent space of G at the identity. By standard computation (see for example the computations in Section 3.3.1 below), the Lie algebra $\mathfrak{h}^n$ of the Heisenberg group $\mathbb{H}^n$ in (3.2) is the $2n + 1$ dimensional matrix subalgebra of $M_{2n+2}(\mathbb{R})$,

$$\mathfrak{h}^n = \left\{ \begin{bmatrix} 0 & w^T \mathcal{J} & 2z \\ 0 & 0 & w \\ 0 & 0 & 0 \end{bmatrix} : w \in \mathbb{R}^{2n},\; z \in \mathbb{R} \right\} = U_W \oplus U_Z$$

where

$$U_W = \{W_w : w \in \mathbb{R}^{2n}\}, \qquad U_Z = \{Z_z : z \in \mathbb{R}\}$$

with

$$W_w = \begin{bmatrix} 0 & w^T\mathcal{J} & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{bmatrix}, \qquad Z_z = \begin{bmatrix} 0 & 0 & 2z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Lie brackets are given by

$$[W_w, Z_z] = 0 \qquad \text{and} \qquad [W_w, W_{\tilde{w}}] = Z_{[\![w,\tilde{w}]\!]}. \tag{3.4}$$

Similarly, the Lie algebra $\mathfrak{h}^n_{pol}$ of the polarized Heisenberg group $\mathbb{H}^n_{pol}$ in (3.3) is the $2n+1$ dimensional matrix subalgebra of $M_{n+2}(\mathbb{R})$,

$$\mathfrak{h}^n_{pol} = \left\{ \begin{bmatrix} 0 & y^T & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix} \ : \ x, y \in \mathbb{R}^n, \ z \in \mathbb{R} \right\} = V_X \oplus V_Y \oplus V_Z$$

where

$$V_X = \{X_x : x \in \mathbb{R}^n\}, \qquad V_Y = \{Y_y : y \in \mathbb{R}^n\}, \qquad V_Z = \{Z_z : z \in \mathbb{R}\}$$

with

$$X_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y_y = \begin{bmatrix} 0 & y^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Z_z = \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Lie brackets are given by

$$[X_x, X_{\tilde{x}}] = [Y_y, Y_{\tilde{y}}] = [X_x, Z_z] = [Y_y, Z_z] = 0 \qquad \text{and} \qquad [Y_y, X_x] = Z_{y^T x}.$$

Let us combine the two subspaces V_X and V_Y to

$$V_W = V_X \oplus V_Y = \left\{ W_w = \begin{bmatrix} 0 & y^T & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix} \ : \ w = (x, y), \ x, y \in \mathbb{R}^n \right\}.$$

Then the above Lie brackets become

$$\begin{aligned}[W_w, W_{\tilde{w}}] &= [X_x + Y_y, X_{\tilde{x}} + Y_{\tilde{y}}] = [Y_y, X_{\tilde{x}}] - [Y_{\tilde{y}}, X_x] \\ &= Z_{y^T \tilde{x}} - Z_{\tilde{y}^T x} = Z_{y^T \tilde{x} - \tilde{y}^T x} = Z_{[\![w, \tilde{w}]\!]} \\ [W_w, Z_z] &= [X_x + Y_y, Z_z] = [X_x, Z_z] + [Y_y, Z_z] = 0. \end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5) we observe that the linear isomorphism $\Phi : \mathfrak{h}^n \to \mathfrak{h}^n_{pol}$ given by $W_w \in U_W \mapsto W_w \in V_W$ and $Z_z \in U_Z \mapsto Z_z \in V_Z$ preserves Lie brackets, that is, is an isomorphism of Lie algebras. This was of course expected, as isomorphic matrix groups have isomorphic Lie algebras. For this reason, we need not distinguish between the two Lie algebras, and simply denote them by $\mathfrak{h}^n$.

The Lie algebra $\mathfrak{h}^n$ is easily seen to be isomorphic to the Lie algebra $\mathbb{R}^{2n+1}$ generated by the basis elements $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ with Lie brackets

$$[x_i, z] = [y_i, z] = 0 \quad \text{and} \quad [y_i, x_j] = \delta_{i,j} z,$$

for all $i, j = 1, \ldots, n$. This algebra is called the *Heisenberg algebra*. Thus, $\mathfrak{h}^n$ and $\mathfrak{h}^n_{pol}$ are merely two realizations of the Heisenberg algebra.

We note that the Heisenberg algebra is 2-step nilpotent.

Remark 3.1. The construction of the Heisenberg group and Heisenberg algebra can be done in reverse: Some authors, for example, Folland (1989), first define the Heisenberg algebra as above, and then exponentiate the Lie algebra using the Baker-Campbell-Hausdorff formula to get a Lie group called the Heisenberg group.

3.2 Extensions of the Heisenberg group

3.2.1 The groups $G_{p,B}$

Fix $d \in \mathbb{N}$. For given fixed numbers $p_1, \ldots, p_d \in \mathbb{R}$ and fixed commuting matrices $B_1, \ldots, B_d \in M_n(\mathbb{R})$, let us set

$$V_B := \operatorname{span}(B_1, \ldots, B_d) \subset M_n(\mathbb{R})$$

and

$$p := (p_1, \ldots, p_d) \qquad \text{and} \qquad B := (B_1, \ldots, B_d).$$

We also set

$$D_{p,B} = \left\{ d(t) := \begin{bmatrix} e^{pt} & 0 & 0 \\ 0 & e^{Bt} & 0 \\ 0 & 0 & 1 \end{bmatrix} \ : \ t \in \mathbb{R}^d \right\}$$

where pt and Bt denote "scalar" products

$$pt = p_1 t_1 + \cdots + p_d t_d \in \mathbb{R} \qquad \text{and} \qquad Bt = B_1 t_1 + \cdots + B_d t_d \in V_B$$

for $t = (t_1, \ldots, t_d)^T \in \mathbb{R}^d$. Then $D_{p,B}$ is an abelian (not necessarily closed) subgroup of $GL_{n+2}(\mathbb{R})$. Conjugation by elements of $D_{p,B}$ naturally defines a continuous action α of $\mathbb{R}^d$ on $\mathbb{H}^n_{pol}$ by

$$\begin{aligned} \alpha_t\big(\, h(x,y,z)\,\big) &:= d(t)\ h(x,y,z)\ d(t)^{-1} \\ &= \begin{bmatrix} e^{pt} & 0 & 0 \\ 0 & e^{Bt} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y^T & z \\ 0 & I_n & x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-pt} & 0 & 0 \\ 0 & e^{-Bt} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & e^{pt} y^T e^{-Bt} & e^{pt} z \\ 0 & I_n & e^{Bt} x \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{3.6}$$

that is,

$$\alpha_t\big(h(x,y,z)\big) = h\big(e^{Bt}x, e^{pt}[e^{-Bt}]^T y, e^{pt} z\big). \tag{3.7}$$

We can thus form the semidirect product

$$G_{p,B} := \mathbb{H}^n_{pol} \rtimes_\alpha \mathbb{R}^d.$$

The group operation is given by

$$\begin{aligned}(h(x,y,z),t)\left(h(\tilde{x},\tilde{y},\tilde{z}),\tilde{t}\right) &= \left(h(x,y,z)\,\alpha_t\left(h(\tilde{x},\tilde{y},\tilde{z})\right), t+\tilde{t}\right)\\ &= \left(h(x+e^{Bt}\tilde{x}, y+e^{pt}[e^{-Bt}]^T\tilde{y}, z+e^{pt}\tilde{z}+y^Te^{Bt}x), t+\tilde{t}\right).\end{aligned}$$

Alternatively, we may represent elements of $G_{p,B}$ as quadruples $g(t,x,y,z)$; in this case the group operation becomes

$$g(t,x,y,z)g(\tilde{t},\tilde{x},\tilde{y},\tilde{z}) = g(t+\tilde{t}, x+e^{Bt}\tilde{x}, y+e^{pt}[e^{-Bt}]^T\tilde{y}, z+e^{pt}\tilde{z}+y^Te^{Bt}\tilde{x}). \quad (3.8)$$

3.2.2 The groups $G_{p,B}$ as closed subgroups of $GL_{n+2}(\mathbb{R})$

It is not difficult to verify that each group $G_{p,B}$ is isomorphic as a topological group to the closed subgroup of $GL_{d+n+2}(\mathbb{R})$,

$$G^{(1)}_{p,B} := \left\{ \tilde{g}(t,x,y,z) = \begin{bmatrix} \begin{bmatrix} e^{pt} & y^Te^{Bt} & z \\ 0 & e^{Bt} & x \\ 0 & 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} e^{t_1} & 0 & \dots & 0 \\ 0 & e^{t_2} & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & e^{t_d} \end{bmatrix} \end{bmatrix} : \begin{array}{l} t \in \mathbb{R}^d \\ t = (t_1,\dots,t_d)^T \\ x,y \in \mathbb{R}^n \\ z \in \mathbb{R} \end{array} \right\}. \quad (3.9)$$

This representation of $G_{p,B}$ is not very useful. Since $\mathbb{H}^n_{pol}$ is a subgroup of $GL_{n+2}(\mathbb{R})$, we wish to represent $G_{p,B}$ as a subgroup of the same. For this we need to identify $D_{p,B}$ with $\mathbb{R}^d$, and make sure that $D_{p,B}$ is a matrix group, that is, is closed in $GL_{n+2}(\mathbb{R})$. This is possible under some mild assumptions on the matrices $B_1,\dots,B_d$. The main ingredient is the proof of Lemma 11 in Bruna et al. (2011) which shows the following:

Lemma 3.2. Let $A_1,\dots,A_d$ be $m \times m$ commuting matrices such that

(A1) $A_1, \ldots, A_d$ are linearly independent, and

(A2) no nonzero element of $V_A := \operatorname{span}(A_1, \ldots, A_d)$ is similar to a skew-symmetric matrix.

Then the exponential map $exp : A \mapsto e^A$ is an isomorphism and homeomorphism of the additive group V_A onto a closed subgroup of $GL_m(\mathbb{R})$.

Let us now set

$$M_k = \begin{bmatrix} p_k & 0 & 0 \\ 0 & B_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (k = 1, \ldots, d), \tag{3.10}$$

and also

$$V_M = \operatorname{span}(M_1, \ldots, M_d)$$

so that

$$D_{p,B} = \left\{ e^{Mt} \ : \ M = (M_1, \ldots M_d),\ t \in \mathbb{R}^d \right\}.$$

Now assume that the matrices $M_1, \ldots, M_d$ satisfy the conditions (A1)–(A2). (This certainly is the case if $B_1, \ldots, B_d$ themselves satisfy (A1)–(A2). When $p_k = \delta_{j,k}$, this is the case if and only if $B_1, \ldots, B_{j-1}, B_{j+1}, \ldots, B_d$ satisfy (A1)–(A2).) In particular, $\dim(V_M) = d$. Applying Lemma 3.2 to the matrices $M_1, \ldots, M_d$ shows that the map $t \mapsto d(t)$ is an isomorphism and homeomorphism of $\mathbb{R}^d$ onto $D_{p,B}$ and that $D_{p,B}$ is closed in $GL_{n+2}(\mathbb{R})$, and hence by (3.6) and Remark 2.26,

$$G_{p,B} \cong G^{(2)}_{p,B} := \left\{ g(t,x,y,z) = \begin{bmatrix} e^{pt} & y^T e^{Bt} & z \\ 0 & e^{Bt} & x \\ 0 & 0 & 1 \end{bmatrix} \ : \ \begin{matrix} t \in \mathbb{R}^d \\ x, y \in \mathbb{R}^n \\ z \in \mathbb{R} \end{matrix} \right\}. \tag{3.11}$$

3.3 Classification of the groups $G_{p,B}$

Observe that each group $G_{p,B} = \mathbb{H}^n_{pol} \rtimes \mathbb{R}^d$ is the topological product of two simply connected groups, and hence is simply connected. It follows that the isomorphic matrix groups $G^{(1)}_{p,B} \cong G_{p,B}$ are simply connected. Thus, in order to classify the groups $G_{p,B}$, by Theorem 2.53, it suffices to classify their Lie algebras $\mathfrak{g}^{(1)}_{p,B}$.

On the other hand, under the additional assumptions (A1)–(A2), the matrix groups $G_{p,B}$ are isomorphic to the groups $G^{(2)}_{p,B}$, so in order to classify the groups $G_{p,B}$ satisfying (A1)–(A2), it suffices to classify the Lie algebras $\mathfrak{g}^{(2)}_{p,B}$ of $G^{(2)}_{p,B}$.

Suppose first that (A1)–(A2) hold. Since isomorphic matrix groups have isomorphic Lie algebras, it follows that $\mathfrak{g}^{(1)}_{p,B} \cong \mathfrak{g}^{(2)}_{p,B}$. This fact can easily be established directly, by computing the two Lie algebras. Of course we prefer to work with $\mathfrak{g}^{(2)}_{p,B}$, as it is a subalgebra of $M_{n+2}(\mathbb{R})$, and its elements have a simpler matrix representation than those of $\mathfrak{g}^{(1)}_{p,B} \subset M_{d+n+2}(\mathbb{R})$.

Since the motivation of this work was to study the groups $G^{(2)}_{p,B}$, we will work with the Lie algebras $\mathfrak{g}^{(2)}_{p,B}$ in what follows. This is not a restriction, however. First of all, the Lie algebras $\mathfrak{g}^{(1)}_{p,B}$ can be computed and analyzed in a similar way as we will do below with $\mathfrak{g}^{(2)}_{p,B}$. Secondly, one can show that even when condition (A2) is not satisfied, the Lie algebras $\mathfrak{g}^{(1)}_{p,B}$ are isomorphic to Lie subalgebras of $M_{n+2}(\mathbb{R})$ of the form $\mathfrak{g}^{(2)}_{p,B}$ discussed below. The computations are not difficult, but we omit them for brevity. We therefore will simply use the symbol $\mathfrak{g}_{p,B}$ to denote $\mathfrak{g}^{(2)}_{p,B}$.

In fact, since $G_{p,B} = \mathbb{H}^n_{pol} \rtimes \mathbb{R}^d$ is the semi-direct product of two Lie groups, it is a Lie group in it own right, and its Lie algebra is the semi-direct product of their Lie algebras. We therefore could avoid working with Lie algebras in the form of concrete matrix algebras altogether. As in this thesis we have introduced Lie groups and their Lie algebras in the context of matrix groups only, we prefer to go the route of computing the Lie algebras as tangent spaces of matrix groups.

3.3.1 The Lie algebras $\mathfrak{g}_{p,B}$

We now compute the Lie algebras $\mathfrak{g}_{p,B} = \mathfrak{g}^{(2)}_{p,B}$ of $G^{(2)}_{p,B}$. As noted above, one can show that the Lie algebra of every group $G_{p,B}$ is of this form, by applying similar computations to $G^{(1)}_{p,B}$, provided that condition (A1) of Lemma 3.2 applies to the family of matrices $M_1, \ldots, M_d$.

Proposition 3.3. The Lie algebra $\mathfrak{g}_{p,B}$ of $G^{(2)}_{p,B}$ coincides with the set of matrices

$$L := \left\{ \begin{bmatrix} pt & y^T & z \\ 0 & Bt & x \\ 0 & 0 & 0 \end{bmatrix} : \ t \in \mathbb{R}^d, \ x, y \in \mathbb{R}^n, \ z \in \mathbb{R} \right\}. \tag{3.12}$$

Proof. First we note that, since for fixed t, the map $y^T \mapsto y^T e^{Bt}$ is one-to-one, the group $G^{(2)}_{p,B}$ can be identified, as a subset of $M_{n+2}(\mathbb{R})$, with the set of matrices

$$M_{p,B} = \left\{ \begin{bmatrix} e^{pt} & y^T & z \\ 0 & e^{Bt} & x \\ 0 & 0 & 1 \end{bmatrix} : \ t \in \mathbb{R}^d, \ x, y \in \mathbb{R}^n, \ z \in \mathbb{R} \right\}.$$

Since the topological structure of $G_{p,B}$ is that of a topological subspace of $M_{n+2}(\mathbb{R})$, we may choose this representation of $G^{(2)}_{p,B}$ to obtain the tangent space at the identity matrix.

First we show that $L \subset \mathfrak{g}_{p,B}$. For this, let

$$l_0 = \begin{bmatrix} pt_0 & y_0^T & z_0 \\ 0 & Bt_0 & x_0 \\ 0 & 0 & 0 \end{bmatrix} \in L$$

be given. Define a map $\gamma : \mathbb{R} \to M_{p,B}$ by

$$\gamma(s) = \begin{bmatrix} e^{s(pt_0)} & sy_0^T & sz_0 \\ 0 & e^{s(Bt_0)} & sx_0 \\ 0 & 0 & 1 \end{bmatrix}$$

for each $s \in \mathbb{R}$. Using Proposition 2.57 one quickly verifies that each entry is differentiable with respect to s, and in fact,

$$\begin{aligned}
\frac{d}{ds}e^{s(pt_0)}\Big|_{s=0} &= (pt_0)e^{s(pt_0)}\Big|_{s=0} = pt_0 \\
\frac{d}{ds}sx_0\Big|_{s=0} &= x_0\Big|_{s=0} = x_0 \\
\frac{d}{ds}sy_0^T\Big|_{s=0} &= y_0^T\Big|_{s=0} = y_0^T \\
\frac{d}{ds}sz_0\Big|_{s=0} &= z_0\Big|_{s=0} = z_0 \\
\frac{d}{ds}e^{s(Bt_0)}\Big|_{s=0} &= e^{s(Bt_0)}(Bt_0)\Big|_{s=0} = Bt_0.
\end{aligned}$$

Hence, $\gamma(s)$ is differentiable, and $\gamma'(0) = l_0$. Since $\gamma(0) = I_{n+2}$, it follows that $l_o \in \mathfrak{g}_{p,B}$. This shows that $L \subset \mathfrak{g}_{p,B}$.

Conversely, Let $X \in \mathfrak{g}_{p,B}$ be given. Then

$$X = \gamma'(0) \qquad \exists \text{ a differentiable curve } \gamma : (-\epsilon, \epsilon) \subset \mathbb{R} \to M_{p,B} \text{ with } \gamma(0) = I_{n+2}.$$

We can decompose γ into its components,

$$\gamma(s) = \begin{bmatrix} \gamma_1(s) & \gamma_3(s)^T & \gamma_5(s) \\ 0 & \gamma_2(s) & \gamma_4(s) \\ 0 & 0 & 1 \end{bmatrix}, \qquad (s \in (-\epsilon, \epsilon))$$

for some differentiable maps

$$\gamma_1 : (-\epsilon, \epsilon) \to \mathbb{R}^+, \qquad \gamma_2 : (-\epsilon, \epsilon) \to \{A \in GL_n : A = e^{B_o}\ \exists B_o \in V_B\},$$

$$\gamma_3, \gamma_4 : (-\epsilon, \epsilon) \to \mathbb{R}^n, \qquad \gamma_5 : (-\epsilon, \epsilon) \to \mathbb{R},$$

with $\gamma_1(0) = 1$, $\gamma_2(0) = I_n$ and $\gamma_i(0) = 0$ for $i = 3, \ldots, 5$. Now as γ is differentiable on $(-\epsilon, \epsilon)$, then so are all of its components γ_i. The important observation is that γ_1 and γ_2 can be expressed as exponentials of differentiable paths. In fact, since γ_1 and γ_2 are differentiable, then

$$\tilde{\gamma}(s) = \begin{bmatrix} \gamma_1(s) & 0 & 0 \\ 0 & \gamma_2(s) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

defines a differentiable curve $(-\epsilon, \epsilon) \to D_{p,B}$ with $\tilde{\gamma}(0) = I_{n+2}$. Reducing ϵ if necessary, by Theorem 2.58 we may assume that there exists a differentiable curve $\hat{\gamma}(s) : (-\epsilon, \epsilon) \to V_M \subset M_{n+2}(\mathbb{R})$, with $e^{\hat{\gamma}(s)} = \gamma(s)$ and $\hat{\gamma}(0) = 0$. Expressed in component form,

$$\hat{\gamma}(s) = \begin{bmatrix} \hat{\gamma}_1(s) & 0 & 0 \\ 0 & \hat{\gamma}_2(s) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $e^{\hat{\gamma}_i(s)} = \gamma_i(s)$, $i = 1, 2$. Since V_M is a finite dimensional vector space, derivatives of curves in V_M are again elements of V_M, so that there exists $t_0 \in \mathbb{R}^d$ with

$$\hat{\gamma}'(0) = Mt_0 = \begin{bmatrix} pt_0 & 0 & 0 \\ 0 & Bt_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that since $\hat{\gamma}$ is differentiable, so are its components $\hat{\gamma}_1$ and $\hat{\gamma}_2$, and $\hat{\gamma}_1'(0) = pt_0$ and $\hat{\gamma}_2'(0) = Bt_0$. Differentiating componentwise and applying the chain rule in Theorem 2.58, we obtain

$$X = \gamma'(0) = \begin{bmatrix} \gamma_1'(0) & \left[\gamma_3^T\right]'(0) & \gamma_5'(0) \\ 0 & \gamma_2'(0) & \gamma_4'(0) \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \left(e^{\hat{\gamma}_1}\right)'(0) & [\gamma_3^T]'(0) & \gamma_5'(0) \\ 0 & \left(e^{\hat{\gamma}_2}\right)'(0) & \gamma_4'(0) \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\hat{\gamma}_1(0)}\hat{\gamma}_1'(0) & [\gamma_3'(0)]^T & \gamma_5'(0) \\ 0 & e^{\hat{\gamma}_2(0)}\hat{\gamma}_2'(0) & \gamma_4'(0) \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} pt_0 & [\gamma_3'(0)]^T & \gamma_5'(0) \\ 0 & Bt_0 & \gamma_4'(0) \\ 0 & 0 & 0 \end{bmatrix} \in L.$$

This shows that $\mathfrak{g}_{p,B} \subset L$, and thus proves the assertion. □

We see immediately that each Lie algebra $\mathfrak{g}_{p,B}$ decomposes into the direct sum $V_M \oplus V_H$ where $V_H = \mathfrak{h}^n$ is the Heisenberg Lie algebra, and V_M the abelian Lie algebra spanned by the matrices M_k. Thus, under assumption that $M_1, \ldots, M_d$ be linearly independent (which holds as (A1) is satisfied) we have the decomposition

$$\mathfrak{g}_{p,B} = V_M \oplus V_H = \underbrace{V_{M_1} \oplus \cdots \oplus V_{M_d}}_{V_M} \oplus \underbrace{\overbrace{V_X \oplus V_Y}^{V_W} \oplus V_Z}_{V_H},$$

where

$$V_{M_k} = \{t_k M_k : t_k \in \mathbb{R}\} \qquad (k = 1, \dots, d),$$

$$V_X = \{X_x : x \in \mathbb{R}^n\}, \qquad V_Y = \{Y_y : y \in \mathbb{R}^n\}, \qquad V_Z = \{Z_z : z \in \mathbb{R}\},$$

with

$$X_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y_y = \begin{bmatrix} 0 & y^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Z_z = \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\mathfrak{g}_{p,B}$ has dimension $d + 2n + 1$. The only possibly nonzero Lie brackets are determined by

$$\begin{aligned} &[M_k, X_x] = X_{B_k x}, \quad [M_k, Y_y] = Y_{(p_k I - B_k^T) y}, \\ &[M_k, Z_z] = Z_{p_k z}, \quad [Y_y, X_x] = Z_{y^T x}, \end{aligned} \tag{3.13}$$

$k = 1, \dots, d$. For the purpose of classifying this type of Lie algebras, the matrices M_k need not satisfy condition (A2). Observe that V_H is an ideal of the nilradical.

As shown with the Heisenberg algebra, it will be convenient to denote elements $X_x + Y_y$ of V_W by W_w, where $w = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. In this notation, some of the Lie brackets in (3.13) become

$$[W_w, W_{\tilde{w}}] = [X_x + Y_y, X_{\tilde{x}} + Y_{\tilde{y}}] = Z_{y^T \tilde{x} + \tilde{y}^T x} = Z_{[\![w, \tilde{w}]\!]},$$

and also

$$[M_k, W_w] = [M_k, X_x] + [M_k, Y_y] = X_{B_k x} + Y_{(p_k I_n - B_k^T) y} = W_{C_k w} \tag{3.14}$$

with

$$C_k = \begin{bmatrix} B_k & 0 \\ 0 & p_k I_n - B_k^T \end{bmatrix} \in M_{2n}(\mathbb{R}). \tag{3.15}$$

The following lemma characterizes the automorphisms of the Heisenberg subalgebra $V_H = \mathfrak{h}^n$. We note that a similar characterization can be found in Folland (1989).

Lemma 3.4. Let a triple (λ, u, S) be given, where $\lambda > 0$, $u \in \mathbb{R}^{2n}$, and $S \in GL_{2n}(\mathbb{R})$ satisfies $S^T \mathcal{J} S = \pm\mathcal{J}$. Then

$$\begin{aligned} &\Phi(W_w) = W_{\lambda S w} + Z_{u^T w} \quad \text{and} \\ &\Phi(Z_z) = Z_{\pm\lambda^2 z} \qquad\qquad (W_w \in V_W,\, Z_z \in V_Z) \end{aligned} \tag{3.16}$$

defines an automorphism of the Heisenberg algebra $\mathfrak{h}^n$. Conversely, every automorphism of $\mathfrak{h}^n$ is of this form.

Proof. It is clear that the linear map Φ defined by (3.16) constitutes a linear automorphism of $\mathfrak{h}^n$. Moreover, by assumption on S, we have for all $w, \tilde{w} \in \mathbb{R}^{2n}$,

$$\begin{aligned} [\Phi(W_w), \Phi(W_{\tilde{w}})] &= [W_{\lambda S w} + Z_{u^T w}, W_{\lambda S \tilde{w}} + Z_{u^T \tilde{w}}] = Z_{[\![\lambda S w, \lambda S \tilde{w}]\!]} \\ &= Z_{\pm\lambda^2 [\![w,\tilde{w}]\!]} = \Phi\left(Z_{[\![w,\tilde{w}]\!]}\right) = \Phi\left([W_w, W_{\tilde{w}}]\right), \end{aligned} \tag{3.17}$$

and it follows that Φ preserves the Lie brackets.

Conversely, let Φ be a Lie algebra automorphism of $\mathfrak{h}^n$. In light of the decomposition $\mathfrak{h}^n = V_W \oplus V_Z$ and since Φ leaves the center V_Z invariant, Φ has a matrix representation

$$\Phi \leftrightarrow \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_{11} \in GL_{2n}(\mathbb{R})$ and $a_{22} \neq 0$. Computing as in (3.17) we have for all $w, \tilde{w} \in \mathbb{R}^{2n}$,

$$\begin{aligned} Z_{a_{22}[\![w,\tilde{w}]\!]} &= \Phi\left(Z_{[\![w,\tilde{w}]\!]}\right) = \Phi\left([W_w, W_{\tilde{w}}]\right) = [\Phi(W_w), \Phi(W_{\tilde{w}})] \\ &= [W_{a_{11}w} + Z_{a_{21}w}, W_{a_{11}\tilde{w}} + Z_{a_{21}\tilde{w}}] = Z_{[\![a_{11}w, a_{11}\tilde{w}]\!]} \end{aligned}$$

Set $\lambda = \sqrt{|a_{22}|}$, $S = \frac{1}{\lambda} a_{11}$ and $u = a_{21}^T$. Then $[\![Sw, S\tilde{w}]\!] = \operatorname{sgn}(a_{22})[\![w, \tilde{w}]\!]$, that is $S^T \mathcal{J} S = \operatorname{sgn}(a_{22})\mathcal{J}$, and the assertion follows. □

3.3.2 Classification of the Lie algebras $\mathfrak{g}_{p,B}$

Let us first introduce some normalization to the class of Lie algebras $\mathfrak{g}_{p,B}$. Given two algebras $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{\tilde{p},\tilde{B}}$, their Heisenberg parts are identical, so we will use the same symbol V_H to denote the two. The remaining component spaces will be denoted by V_M and $V_{\tilde{M}}$, respectively. V_M has a basis $\{M_1, \ldots, M_d\}$ while $V_{\tilde{M}}$ has a basis $\{\tilde{M}_1, \ldots, \tilde{M}_d\}$ as determined in (3.10).

Theorem 3.5. If any of the following properties hold, then two Lie algebras $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{\tilde{p},\tilde{B}}$ are isomorphic:

1. $\tilde{p} = p$ and there exists $S \in Sp(n, \mathbb{R})$ so that

$$\tilde{C}_k = S C_k S^{-1} \qquad (k = 1, \ldots, d),$$

   with C_k and $\tilde{C}_k$ given as in (3.15).

2. $\tilde{p} = p$ and there exists $V \in GL_n(\mathbb{R})$ so that

$$\tilde{B}_k = V B_k V^{-1} \qquad (k = 1, \ldots, d).$$

3. Each $\tilde{M}_i$ is a linear combination of $M_1, \ldots, M_d$,

$$\tilde{M}_i = \sum_{k=1}^{d} a_{ik} M_k$$

   with $\det(A) \neq 0$ where $A = [a_{ik}]$.

4. There exists $\alpha \neq 0$ so that $\tilde{M}_k = \alpha M_k$ for all $k = 1, \ldots, d$.

Proof. 1. Define a linear isomorphism $\Phi : \mathfrak{g}_{p,B} \to \mathfrak{g}_{\tilde{p},\tilde{B}}$ by

$$\Phi(M_k) = \tilde{M}_k, \qquad \Phi(W_w) = W_{Sw}, \qquad \Phi(Z_z) = Z_z.$$

In light of Lemma 3.4 one only needs to verify that Lie brackets involving the matrices M_k are preserved. This is indeed the case, as by (3.14),

$$\begin{aligned}[\Phi(M_k), \Phi(W_w)] = &\ [\tilde{M}_k, W_{Sw}] = W_{\tilde{C}_k Sw} \\ &= W_{SC_k w} = \Phi\left(W_{C_k w}\right) = \Phi\left([M_k, W_w]\right)\end{aligned}$$

and, by (3.13),

$$[\Phi(M_k), \Phi(Z_z)] = \ [\tilde{M}_k, Z_z] = Z_{\tilde{p}_k z} = Z_{p_k z} = \Phi\left([M_k, Z_z]\right).$$

for all $k = 1, \ldots, d$.

2. Simply apply the above to

$$S = \begin{bmatrix} V & 0 \\ 0 & (V^{-1})^T \end{bmatrix}.$$

3. This is merely a change of basis of the subalgebra V_M, and hence both Lie algebras coincide.

4. This is a particular change of basis, choosing $a_{ik} = \alpha\delta_{i,k}$.

□

Replacing the matrices $M_1, \ldots, M_d$ (and consequently $B_1, \ldots, B_d$) with appropriate linear combinations, by Theorem 3.5, we may from here on assume that $p_1 \in \{0, 1\}$ and $p_k = 0$ for $k \geqslant 2$. After this normalization of the basis of V_M, we aim to give a partial converse of Theorem 3.5.

Remark 3.6. If two normalized Lie algebras $\mathfrak{g}_{p,B}$, and $\mathfrak{g}_{\tilde{p},\tilde{B}}$ are isomorphic, then $p_1 = \tilde{p}_1$ (i.e. $p = \tilde{p}$). In fact, if $\Phi : \mathfrak{g}_{p,B} \to \mathfrak{g}_{\tilde{p},\tilde{B}}$ is a Lie algebra isomorphism, then Φ maps center onto center. Since every Lie algebra $\mathfrak{g}_{p,B}$ has trivial center when $p_1 = 1$, and center V_Z when $p_1 = 0$, it immediately follows that $p_1 = \tilde{p}_1$.

This remark shows that the normalized Lie algebras $\mathfrak{g}_{p,B}$ need only be classified with respect to the various choices of B.

Theorem 3.7. Let $\Phi : \mathfrak{g}_{p,B} \to \mathfrak{g}_{p,\tilde{B}}$ be an isomorphism of normalized Lie algebras mapping V_H onto V_H. Then there exists $S \in Sp(n, \mathbb{R})$ so that, after replacing the matrices $\tilde{M}_1, \ldots, \tilde{M}_d$ with suitable linear combinations,

$$\tilde{C}_k = SC_kS^{-1}, \qquad k = 1, \ldots d, \tag{3.18}$$

with C_k and $\tilde{C}_k$ given as in (3.15).

Proof. Suppose that $\Phi : V_H \to V_H$. Then in light of Lemma 3.4, Φ has the matrix representation

$$\Phi \leftrightarrow \begin{bmatrix} E_{11} & 0 & 0 \\ E_{21} & E_{22} & 0 \\ E_{31} & E_{32} & E_{33} \end{bmatrix}, \tag{3.19}$$

corresponding to the decomposition $\mathfrak{g}_{p,B} = V_M \oplus V_W \oplus V_Z$. Note that composing Φ with the automorphism Ψ of $\mathfrak{g}_{p,\tilde{B}}$ given by the matrix

$$\Psi \leftrightarrow \begin{bmatrix} I_d & 0 & 0 \\ 0 & \lambda\mathcal{J} & 0 \\ 0 & 0 & -\lambda^2 \end{bmatrix}, \qquad \text{resp.} \qquad \Psi \leftrightarrow \begin{bmatrix} I_d & 0 & 0 \\ 0 & \lambda I_{2n} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix},$$

depending on the sign of E_{33}, where $\lambda = |E_{33}|^{-1/2}$, we may assume that $E_{33} = 1$.

After a suitable change of basis in $V_{\tilde{M}}$, which affects the first column of matrix (3.19) only, we may assume that $E_{11} = I_d$. It is important to observe that this change of basis can be done without changing the values of p_k. This is clear when $p_1 = 0$. On the other hand, suppose that $p_1 = 1$. Now if $E_{11} = [e_{ik}]$, then $\Phi(M_k) = \sum_i e_{ik}\tilde{M}_i + H_k$ for some $H_k \in V_H$ and it follows that for all $z \in \mathbb{R}$,

$$\begin{aligned} [\Phi(M_k), \Phi(Z_z)] &= \left[\left(\textstyle\sum_i e_{ik}\tilde{M}_i\right) + H_k, Z_{E_{33}z}\right] \\ &= \sum_i e_{ik}\left[\tilde{M}_i, Z_{E_{33}z}\right] = \sum_i e_{ik}Z_{p_iE_{33}z} = e_{1k}Z_{E_{33}z} \end{aligned} \tag{3.20}$$

while also

$$[\Phi(M_k), \Phi(Z_z)] = \Phi([M_k, Z_z]) = \Phi(Z_{p_k z}) = \Phi(\delta_{1,k} Z_z) = \delta_{1,k} Z_{E_{33} z}. \tag{3.21}$$

Comparing these two equations we obtain that

$$e_{1k} = \delta_{1,k} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1. \end{cases}$$

This shows that the change of basis can be done by replacing each $\tilde{M}_k$ with $\tilde{M}_k + \tilde{N}_k$ for some $\tilde{N}_k \in \text{span}(\tilde{M}_2, \dots \tilde{M}_m)$, and thus preserving the values of p_k. The isomorphism Φ now has the form

$$\Phi \leftrightarrow \begin{bmatrix} I_m & 0 & 0 \\ E_{21} & E_{22} & 0 \\ E_{31} & E_{32} & 1 \end{bmatrix}$$

with $E_{22} \in GL_{2n}(\mathbb{R})$.

It is easy to verify that a linear isomorphism determined by such a matrix preserves the Lie brackets if and only if

$$[\![E_{22} w, E_{22} \tilde{w}]\!] = [\![w, \tilde{w}]\!] \tag{3.22}$$

$$\tilde{C}_k = E_{22} C_k E_{22}^{-1} \tag{3.23}$$

$$\tilde{C}_k E_{21}^{(j)} = \tilde{C}_j E_{21}^{(k)}$$

$$p_k E_{31}^{(j)} + [\![E_{21}^{(k)}, E_{21}^{(j)}]\!] = p_j E_{31}^{(k)}$$

$$[\![E_{21}^{(k)}, w]\!] = E_{32} E_{22}^{-1} w$$

for all $j, k = 1, \dots, d$ and $w, \tilde{w} \in \mathbb{R}^{2n}$, with $E_{21}^{(k)}$ and $E_{31}^{(k)}$ denoting the k-th columns of the matrices E_{21} and E_{31}, respectively. These identities remain valid if we modify Φ so that $E_{21} = E_{31} = E_{32} = 0$, that is

$$\Phi \leftrightarrow \begin{bmatrix} I_d & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.24}$$

Choosing $S = E_{22}$, the identities (3.22) and (3.23) now yield the assertion. $\square$

Definition 3.8. Matrices $A_1, \ldots, A_d \in M_n(\mathbb{R})$ are said to be *linearly nilindependent*, if no nontrivial linear combination is nilpotent.

Clearly, linear nilindependence implies linear independence. In case of a normalized Lie algebra $\mathfrak{g}_{p,B}$ we thus have:

1. When $p_1 = 0$ then $M_1, \ldots, M_d$ are linearly nilindependent iff $B_1, \ldots, B_m$ are linearly nilindependent.

2. When $p_1 = 1$ then $M_1, \ldots, M_d$ are linearly nilindependent iff $B_2, \ldots, B_m$ are linearly nilindependent.

The next result shows that nilindependence guarantees that V_H is mapped to V_H. It can also be obtained from the classification of the Lie algebras whose nilradical is the Heisenberg algebra, given by Rubin and Winternitz (1993).

Corollary 3.9. Let $\Phi : \mathfrak{g}_{p,B} \to \mathfrak{g}_{p,\tilde{B}}$ be an isomorphism of normalized Lie algebras. If $M_1, \ldots, M_d$ are linearly nilindependent, then there exists $S \in Sp(n, \mathbb{R})$ so that, after replacing the matrices $\tilde{M}_1, \ldots, \tilde{M}_d$ with a suitable basis of $V_{\tilde{M}}$,

$$\tilde{C}_k = S C_k S^{-1}, \qquad k = 1, \ldots, d.$$

Proof. Since V_H is a nilpotent ideal, it is contained in the nilradical of the algebras $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{p,\tilde{B}}$. Now by non-nilpotency of all nonzero elements of V_M, the nilradical of $\mathfrak{g}_{p,B}$ coincides with V_H, and hence has dimension $2n + 1$. Since Φ is an isomorphism between the nilradicals, then the nilradical of $\mathfrak{g}_{p,\tilde{B}}$ has the same dimension, and hence must coincide with V_H as well. Thus, Theorem 3.7 applies. $\square$

Next, we make the relationship between the matrices B_k and $\tilde{B}_k$ more precise. First some remarks and observations:

We will make use of a result by Bruna et al. (2011), which generalizes the well known theorem on the real Jordan normal form of a single matrix to collections of commuting matrices.

Theorem 3.10. Let $l \in \mathbb{N}$ and $B_1, \ldots, B_d \in M_n(\mathbb{R})$ be commuting matrices. Then there exist $S \in GL_n(\mathbb{R})$, $m_r \in \mathbb{N}$ and $\mathbb{K}_r \in \{\mathbb{R}, \mathbb{C}\}$ (for $r = 1, \ldots \ell$) so that

$$\sum_{r=1}^{\ell} m_r \cdot \dim_{\mathbb{R}} \mathbb{K}_r = n$$

and, for $k = 1, \ldots, d$,

$$SB_kS^{-1} = \begin{bmatrix} B_{k,1} & 0 & 0 & \ldots & 0 \\ 0 & B_{k,2} & 0 & \ldots & 0 \\ 0 & 0 & B_{k,3} & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B_{k,\ell} \end{bmatrix}, \tag{3.25}$$

with blocks

$$B_{k,r} \in \mathbb{K}_r \cdot I_{m_r} + \mathcal{N}(m_r, \mathbb{K}_r). \tag{3.26}$$

Here, $\mathcal{N}(m, \mathbb{K})$ denotes the set of all properly upper triangular $m \times m$ matrices, and I_m the identity matrix in $M_m(\mathbb{K})$. The entries of a block $B_{k,r}$ are real numbers in case $\mathbb{K}_r = \mathbb{R}$, and else are 2×2-blocks of the form $\left[\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right]$ corresponding to the natural embedding $\alpha + i\beta \mapsto \left[\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right]$ of $\mathbb{C}$ in $\mathbb{R}^2$.

Remark 3.11. The proof of this theorem in Bruna et al. (2011) shows the following: For each k, let Λ_k denote the set of eigenvalues of B_k. Here a conjugate pair of complex eigenvalues is considered as one single eigenvalue with imaginary part $\Im(\lambda) > 0$. Set

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_m.$$

Then for each r, $1 \leq r \leq m$, there exists a unique

$$\lambda(r) = (\lambda_1^{(r)}, \lambda_2^{(r)}, \ldots, \lambda_m^{(r)}) \in \Lambda$$

so that

$$B_{k,r} = \lambda_k^{(r)} \cdot I_{m_r} + N_{k,r}, \qquad N_{k,r} \in \mathcal{N}(d_r, \mathbb{K}).$$

We set

$$\Lambda_B = \{\lambda(r) : 1 \leq r \leq \ell\} \subseteq \Lambda,$$

the *joint spectrum* of the matrices B_k. Note that the mapping $r \mapsto \lambda(r)$ need not be one-to-one, as different blocks may have the same joint eigenvalues.

Remark 3.12. Suppose matrices $B_1, \ldots, B_d$ in blockdiagonal form as in Theorem 3.10 generate a Lie algebra $\mathfrak{g}_{p,B}$. Fix r, $1 \leq r \leq \ell$, and replace the r-th block $B_{k,r}$ of each matrix with $p_k I_s - B_{k,r}^T$ where $s = \dim_{\mathbb{R}} \mathbb{K} \cdot m_r$ is the size of the block. (For ease of notation, we will simply write $p_k - B_{k,r}^T$.) The resulting matrices will possess the same block structure, and after a suitable change of the basis vectors of the r-th block, will again have upper triangular blocks as in the Theorem. We will call this process of replacing each $B_{k,r}$ $(k = 1, \ldots, d)$ with $p_k - B_{k,r}^T$ a *flip of the r-th blocks*. Obviously, such a flip will replace the eigenvalue $\lambda_k^{(r)}$ of B_k belonging to the r-th block with $p_k - \overline{\lambda_k^{(r)}}$, for $1 \leq k \leq d$. (The complex conjugate is required here by our agreement that $\Im(\lambda_k^{(r)}) > 0$ for all complex eigenvalues $\lambda_k^{(r)}$.)

We next introduce a particular class of symplectic matrices.

Remark 3.13. Let K, M be matrices of sizes $r \times r$, L and N be matrices of sizes

$s \times s$, and O and P of sizes $q \times q$. Set

$$C = \left[\begin{array}{ccc|ccc} K & 0 & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & O & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & N & 0 \\ 0 & 0 & 0 & 0 & 0 & P \end{array}\right] \quad \text{and} \quad \mathcal{J}_{r,s,q} = \left[\begin{array}{ccc|ccc} I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_s & 0 \\ 0 & 0 & 0 & 0 & 0 & I_q \\ \hline 0 & 0 & 0 & I_r & 0 & 0 \\ 0 & I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_q \end{array}\right].$$

Then $\mathcal{J}_{r,s,q}$ is a symplectic matrix, $\mathcal{J}_{r,s,q} \in Sp(r+s+q,\mathbb{R})$, and

$$\mathcal{J}_{r,s,q} C \mathcal{J}_{r,s,q}^{-1} = \left[\begin{array}{ccc|ccc} K & 0 & 0 & 0 & 0 & 0 \\ 0 & N & 0 & 0 & 0 & 0 \\ 0 & 0 & O & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & P \end{array}\right].$$

That is, conjugation by $\mathcal{J}_{r,s,q}$ exchanges the blocks N and L.

Theorem 3.14. Consider two isomorphic Lie algebras $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{p,\tilde{B}}$, where a basis of $V_{\tilde{M}}$ has been chosen so that (3.18) holds.

1. If every joint eigenvalue $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda_B$ has at least one component λ_k whose real part satisfies $\Re(\lambda_k) \neq p_k/2$, then there exist matrices

$$D_k = \begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix}, \qquad \tilde{D}_k = \begin{bmatrix} \tilde{E}_k & 0 \\ 0 & \tilde{F}_k \end{bmatrix} \quad \text{and} \quad A_o, U, V \in GL_n(\mathbb{R})$$

so that for each $k = 1, \ldots, d$,

(a) $\tilde{D}_k \simeq D_k$ by means of $Ad(A_o)$,

(b) $B_k \simeq \begin{bmatrix} E_k & 0 \\ 0 & p_k - F_k^T \end{bmatrix}$ by means of $Ad(U)$, and

(c) $\tilde{B}_k \simeq \begin{bmatrix} \tilde{E}_k & 0 \\ 0 & p_k - \tilde{F}_k^T \end{bmatrix}$ by means of $Ad(V)$.

(The blocks E_k and $\tilde{E}_k$ need not be of same size.)

2. If in addition, for all joint eigenvalues $\lambda \in \Lambda_B$ the "conjugate" $\lambda^c = p - \overline{\lambda}$ is not contained in Λ_B, and the same is true for the joint eigenvalues $\lambda \in \Lambda_{\tilde{B}}$, then there exist matrices

$$D_k = \begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix} \quad \text{and} \quad U, W \in GL_n(\mathbb{R})$$

so that for each $k = 1, \ldots, d$,

(a) $B_k \simeq D_k$ by means of $Ad(U)$, and

(b) $\tilde{B}_k \simeq \begin{bmatrix} E_k & 0 \\ 0 & p_k - F_k^T \end{bmatrix}$ by means of $Ad(W)$.

Proof. By Theorem 3.10, there exist invertible matrices U and V so that

$$U^{-1}B_kU = \begin{bmatrix} B_{k,1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_{k,\ell} \end{bmatrix} \quad \text{and} \quad V^{-1}\tilde{B}_kV = \begin{bmatrix} \tilde{B}_{k,1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \tilde{B}_{k,\tilde{\ell}} \end{bmatrix} \tag{3.27}$$

for $k = 1, \ldots, d$, and the blocks $B_{k,r}$ and $\tilde{B}_{k,\tilde{r}}$ have the upper-triangular form (3.26). Merging blocks belonging to the same joint eigenvalue (which effectively is first a switching of blocks followed by a merging of some adjacent blocks, and affects the matrices U and V), we may assume that the maps $r \in \{1, \ldots, \ell\} \mapsto \lambda \in \Lambda_B$ and $r \in \{1, \ldots, \tilde{\ell}\} \mapsto \lambda \in \Lambda_{\tilde{B}}$ are one-to-one. Applying Theorem 3.5, part 2 and its proof with the matrices U and V, we may thus assume from here on that B_k and $\tilde{B}_k$ are in this block-diagonal form.

We split the joint spectrum Λ_B into three subsets as follows. First let

$$\Lambda_B^0 = \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda_B : \Re(\lambda_k) = p_k/2 \text{ for all } k = 1, \ldots, d\}.$$

The remaining elements of λ_B have the property that there exists a first $k = k_o$ so that $\Re(\lambda_{k_o}) \neq p_{k_o}/2$. We set

$$\Lambda_B^+ = \{\lambda \in \Lambda_B : \Re(\lambda_{k_o}) > p_{k_o}/2\}$$
$$\Lambda_B^- = \{\lambda \in \Lambda_B : \Re(\lambda_{k_o}) < p_{k_o}/2\}\,.$$

Note that any of these subsets may be empty. By exchanging the blocks in (3.27) (which can be effected by changing the diagonalizing matrices U and V used earlier) we may assume that the surjection $r \in \{1, \ldots \ell\} \mapsto \lambda(r) \in \Lambda_B$ has the following property: There exist ℓ_1, ℓ_2 so that

$$\begin{aligned} \lambda(r) &\in \Lambda_B^+ \quad \text{when} \quad 1 \leq r \leq \ell_1 \\ \lambda(r) &\in \Lambda_B^- \quad \text{when} \quad \ell_1 < r \leq \ell_2 \\ \lambda(r) &\in \lambda_B^0 \quad \text{when} \quad \ell_2 < r \leq \ell \end{aligned}$$

Thus,

$$B_k = \begin{bmatrix} E_k & 0 & 0 \\ 0 & L_k & 0 \\ 0 & 0 & O_k \end{bmatrix}$$

where

$$E_k = \begin{bmatrix} B_{k,1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_{k,\ell_1} \end{bmatrix}, \quad L_k = \begin{bmatrix} B_{k,\ell_1+1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_{k,\ell_2} \end{bmatrix}, \quad O_k = \begin{bmatrix} B_{k,\ell_2+1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_{k,\ell} \end{bmatrix}.$$

The matrices E_k, L_k and O_k are of sizes

$$n_E = \sum_{r=1}^{\ell_1} m_r \cdot \dim_{\mathbb{R}} \mathbb{K}_r, \quad n_L = \sum_{r=\ell_1+1}^{\ell_2} m_r \cdot \dim_{\mathbb{R}} \mathbb{K}_r, \quad n_O = \sum_{r=\ell_2+1}^{\ell} m_r \cdot \dim_{\mathbb{R}} \mathbb{K}_r,$$

respectively, where zero is a permitted value.

Next we flip the blocks L_k and replace the matrices B_k by

$$\bar{B}_k = \begin{bmatrix} E_k & 0 & 0 \\ 0 & p_k - L_k^T & 0 \\ 0 & 0 & O_k \end{bmatrix} = \begin{bmatrix} H_k & 0 \\ 0 & O_k \end{bmatrix}, \qquad (k = 1, \ldots, d).$$

Each $\bar{B}_k$ is a block-diagonal matrix with either upper or lower triangular blocks, which have the same sizes and positions as the blocks of B_k. If $\Lambda_{\bar{B}}$ denotes the joint spectrum of the matrices $\bar{B}_k$, then,

$$\Lambda_{\bar{B}}^- = \emptyset, \qquad \Lambda_{\bar{B}}^+ = \Lambda_B^+ \cup \{\lambda^c = p - \overline{\lambda} : \lambda \in \Lambda_B^-\}, \qquad \Lambda_{\bar{B}}^0 = \Lambda_B^0. \tag{3.28}$$

Thus, the blocks inside H_k have joint eigenvalues $\lambda \in \Lambda_{\bar{B}}^+$, while those in O_k have joint eigenvalues $\lambda \in \Lambda_{\bar{B}}^0$. It may happen that a "conjugate" pair of joint eigenvalues $\lambda(r_1) \in \Lambda_B^+$ and $\lambda(r_2) = \lambda(r_1)^c \in \Lambda_B^-$ combine to a single joint eigenvalue of $\bar{B}_1, \ldots, \bar{B}_d$; in this case, we will merge all blocks in the $\bar{B}_k$ belonging to this new joint eigenvalue to one single block. This does, however, not happen under the additional assumption in part 2 of this theorem, that only one of the two is a joint eigenvalue, as then the union in (3.28) is disjoint. Furthermore, this joining of blocks can be achieved by first conjugating each $\bar{B}_k$ with an invertible matrix of the form

$$Q = \begin{bmatrix} Q_o & 0 \\ 0 & I_{n_O} \end{bmatrix}$$

with $Q_o \in GL_{n_E+n_L}(\mathbb{R})$, which results in making blocks to be joint adjacent, and then merging these adjacent blocks to a single block. After the joining of blocks, the matrices $\bar{B}_k$ (we still use the same symbol to avoid symbol overload), then

$$\bar{B}_k = Q \begin{bmatrix} H_k & 0 \\ 0 & O_k \end{bmatrix} Q^{-1}$$

will consist of fewer but larger blocks than the matrices B_k; if $\bar{\ell}$ denotes the number of blocks of the $\bar{B}_k$ after joining, then $\bar{\ell} \leq \ell$. This merging of blocks does not modify the spectral sets (3.28), but ensures that the map $r \in \{1, \ldots, \bar{\ell}\} \mapsto \Lambda_{\bar{B}}$ is still one-to-one.

We now show that $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{p,\bar{B}}$ are isomorphic. Observe that

$$\bar{C}_k = \begin{bmatrix} \bar{B}_k & 0 \\ 0 & p_k - \bar{B}_k{}^T \end{bmatrix}$$

$$= \begin{bmatrix} Q & 0 \\ 0 & [Q^{-1}]^T \end{bmatrix} \begin{bmatrix} H_k & 0 & 0 & 0 \\ 0 & O_k & 0 & 0 \\ 0 & 0 & p_k - H_k^T & 0 \\ 0 & 0 & 0 & p_k - O_k^T \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^T \end{bmatrix}$$

$$= \begin{bmatrix} Q & 0 \\ 0 & [Q^{-1}]^T \end{bmatrix} \begin{bmatrix} E_k & 0 & 0 & 0 & 0 & 0 \\ 0 & p_k - L_k^T & 0 & 0 & 0 & 0 \\ 0 & 0 & O_k & 0 & 0 & 0 \\ 0 & 0 & 0 & p_k - E_k^T & 0 & 0 \\ 0 & 0 & 0 & 0 & L_k & 0 \\ 0 & 0 & 0 & 0 & 0 & p_k - O_k^T \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & Q^T \end{bmatrix}$$

$$= T \begin{bmatrix} E_k & 0 & 0 & 0 & 0 & 0 \\ 0 & L_k & 0 & 0 & 0 & 0 \\ 0 & 0 & O_k & 0 & 0 & 0 \\ 0 & 0 & 0 & p_k - E_k^T & 0 & 0 \\ 0 & 0 & 0 & 0 & p_k - L_k^T & 0 \\ 0 & 0 & 0 & 0 & 0 & p_k - O_k^T \end{bmatrix} T^{-1}$$

$$= T \begin{bmatrix} B_k & 0 \\ 0 & p_k - B_k^T \end{bmatrix} T^{-1} = T C_k T^{-1}$$

where

$$T = \begin{bmatrix} Q & 0 \\ 0 & [Q^{-1}]^T \end{bmatrix} \mathcal{J}_{n_E,n_L,n_O} \in Sp(n,\mathbb{R}).$$

Hence by Theorem 3.5, part 1, $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{p,\bar{B}}$ are isomorphic.

We do the same construction with the matrices $\tilde{B}_k$ to obtain matrices $\bar{\tilde{B}}_k$ and symplectic matrices $\tilde{T}$ so that

$$\bar{\tilde{C}}_k = \begin{bmatrix} \bar{\tilde{B}}_k & 0 \\ 0 & p_k - \bar{\tilde{B}}_k^T \end{bmatrix} = \tilde{T} \begin{bmatrix} \tilde{B}_k & 0 \\ 0 & p_k - \tilde{B}_k^T \end{bmatrix} \tilde{T}^{-1} = \tilde{T}\tilde{C}_k\tilde{T}^{-1}$$

resulting in isomorphic algebras $\mathfrak{g}_{p,\tilde{B}}$ and $\mathfrak{g}_{p,\bar{\tilde{B}}}$. Composing the isomorphism $\mathfrak{g}_{p,B} \mapsto \mathfrak{g}_{p,\tilde{B}}$ determined by $S \in Sp(n,\mathbb{R})$ in (3.18) with the isomorphisms determined by T and $\tilde{T}$ we now obtain a Lie algebra isomorphism $\Phi : \mathfrak{g}_{p,\bar{B}} \to \mathfrak{g}_{p,\bar{\tilde{B}}}$ implemented by $G = \tilde{T}ST^{-1} \in Sp(n,\mathbb{R})$:

$$\bar{\tilde{C}}_k = \tilde{T}\tilde{C}_k\tilde{T}^{-1} = \tilde{T}SC_kS^{-1}\tilde{T}^{-1} = \tilde{T}ST^{-1}\bar{C}_kTS^{-1}\tilde{T}^{-1} = G\bar{C}_kG^{-1} \tag{3.29}$$

for $k = 1, \ldots, d$.

Now the matrices $\bar{C}_k$ are block-diagonal with triangular blocks which arise as follows: Each block $\bar{B}_{k,r}$ belonging to an eigenvalue $\lambda_k^{(r)}$ of $\bar{B}_k$ gives rise to two blocks, one in the top-left corner of $\bar{C}_k$ with eigenvalue $\lambda_k^{(r)}$, and one in the bottom right corner of $\bar{C}_k$ with eigenvalue $p_k - \overline{\lambda_k^{(r)}}$. It follows that the joint spectrum $\Lambda_{\bar{C}}$ of the matrices $\bar{C}_k$ has the following form: $\Lambda^+_{\bar{C}} = \Lambda^+_{\bar{B}}$, $\Lambda^-_{\bar{C}} = \{\lambda^c = p - \bar{\lambda} : \lambda \in \Lambda^+_{\bar{B}}\}$, and $\Lambda^0_{\bar{C}} = \Lambda^0_{\bar{B}}$. The last two identities hold as every complex conjugate pair of eigenvalues has been identified to a single eigenvalue with positive imaginary part. Now blocks of $\bar{C}_k$ belonging to $\lambda \in \Lambda^+_{\bar{C}}$ lie in the upper-left corner of $\bar{C}_k$, while those belonging to $\lambda \in \Lambda^-_{\bar{C}}$ lie in the lower-right corner of $\bar{C}_k$. Each $\lambda \in \Lambda^0_{\bar{C}}$ gives rise to two blocks of $\bar{C}_k$, one in each corner.

The same arguments apply similarly to the matrix $\bar{\tilde{C}}_k$. Since the families of block diagonal matrices $\{\bar{C}_k\}_{k=1}^d$ and $\{\bar{\tilde{C}}_k\}_{k=1}^d$ are similar via the map $\mathrm{Ad}(G)$

in (3.29), they have the same joint spectrum, $\Lambda_{\bar{C}} = \Lambda_{\bar{\bar{C}}}$, and the same block sizes belonging to each joint eigenvalue. It follows that $\Lambda_{\bar{B}} = \Lambda_{\bar{\bar{B}}}$, and that this similarity carries a block in $\bar{C}_k$, corresponding to a joint eigenvalue $\lambda(r) \in \Lambda^+_{\bar{C}} = \Lambda^+_{\bar{B}}$, to a block in $\bar{\bar{C}}_k$ corresponding to the same joint eigenvalue $\lambda(\tilde{r}) \in \Lambda^+_{\bar{\bar{C}}} = \Lambda^+_{\bar{\bar{B}}}$, and hence must carry H_k onto $\tilde{H}_k$ (and similarly $p_k - H_k^T$ onto $p_k - \tilde{H}_k^T$), and each block in H_k onto a block in $\tilde{H}_k$. In addition, $\Lambda^+_{\bar{C}} = \Lambda^+_{\bar{\bar{C}}}$ so that $\Lambda^+_{\bar{B}} = \Lambda^+_{\bar{\bar{B}}}$. Thus, G is of the form

$$G = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & C \\ 0 & 0 & [A^{-1}]^T & 0 \\ 0 & D & 0 & E \end{bmatrix}, \qquad A \in GL_{n_H}(\mathbb{R}), \qquad \begin{bmatrix} B & C \\ D & E \end{bmatrix} \in Sp(n_O, \mathbb{R}),$$

where $n_H = n_E + n_L$.

Now suppose as stated in the assumption of the theorem, that $\Lambda^0_B = \emptyset$, so that $\Lambda^0_{\bar{B}} = \emptyset$, $\bar{B}_k = QH_kQ^{-1}$, $\bar{\bar{B}}_k = \tilde{Q}\tilde{H}_k\tilde{Q}^{-1}$ and G is of the form

$$G = \begin{bmatrix} A & 0 \\ 0 & [A^{-1}]^T \end{bmatrix}.$$

Then by (3.29),

$$\bar{\bar{C}}_k = \begin{bmatrix} \bar{\bar{B}}_k & 0 \\ 0 & p_k - \bar{\bar{B}}_k \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & [A^{-1}]^T \end{bmatrix} \begin{bmatrix} \bar{B}_k & 0 \\ 0 & p_k - \bar{B}_k \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & A^T \end{bmatrix}$$

from which we obtain

$$\bar{\bar{B}}_k = A\bar{B}_kA^{-1},$$

that is,

$$\tilde{Q} \begin{bmatrix} \tilde{E}_k & 0 \\ 0 & p_k - \tilde{L}_k^T \end{bmatrix} \tilde{Q}^{-1} = AQ \begin{bmatrix} E_k & 0 \\ 0 & p_k - L_k^T \end{bmatrix} Q^{-1}A^{-1}.$$

Setting $F_k = p_k - L_k^T$ and $\tilde{F}_k = p_k - \tilde{L}_k^T$ and $A_o = \tilde{Q}^{-1}AQ$, the assertion (a) follows. In addition, as

$$B_k = \begin{bmatrix} E_k & 0 \\ 0 & L_k \end{bmatrix} = \begin{bmatrix} E_k & 0 \\ 0 & p_k - F_k^T \end{bmatrix},$$

and similarly for $\tilde{B}_k$. then assertions (b) and (c) follow.

We observe that $\mathrm{Ad}(A)$ switches blocks. In fact, let

$$\phi : r \mapsto \tilde{r}$$

be the bijection which identifies the joint spectra $\Lambda^+_{\bar{C}} = \Lambda^+_{\bar{B}} = \Lambda_{\bar{B}}$ and $\Lambda^+_{\bar{\tilde{C}}} = \Lambda^+_{\bar{\tilde{B}}} = \Lambda_{\bar{\tilde{B}}}$, that is, $\tilde{\lambda}(\phi(r)) = \lambda(r)$ for $\lambda(r) \in \Lambda_{\bar{B}}$ and $\tilde{\lambda}(\tilde{r}) \in \Lambda_{\bar{\tilde{B}}}$. Correspondingly, $\mathrm{Ad}(A)$ maps the r-th block of $\bar{B}_k$ to the $\tilde{r} = \phi(r)$-th block of $\bar{\tilde{B}}_k$. The corresponding decompositions $\mathbb{R}^n = \oplus_r V_r$ and $\mathbb{R}^n = \oplus_{\tilde{r}} V_{\tilde{r}}$ give a decomposition of A into block form, $A = [a_{\tilde{r}r}]$, where $a_{\tilde{r}r} \neq 0 \Leftrightarrow \tilde{r} = \phi(r)$. Each row and each column of A contains exactly one nonzero entry which is a square matrix. In addition, $A^{-1} = [b_{r\tilde{r}}]$ where $b_{r\tilde{r}} \neq 0 \Leftrightarrow \tilde{r} = \phi(r)$, in which case $b_{r\tilde{r}} = a_{\tilde{r}r}^{-1}$. Furthermore, $(A^{-1})^T = [c_{\tilde{r}r}]$ where $c_{\tilde{r}r} \neq 0 \Leftrightarrow \tilde{r} = \phi(r)$, in which case $c_{\tilde{r}r} = b_{r\tilde{r}}^T = (a_{\tilde{r}r}^{-1})^T$.

Finally, suppose in addition that the "conjugate" λ^c of any joint eigenvalue $\lambda \in \Lambda_B$, respectively $\lambda \in \Lambda_{\tilde{B}}$, is not a joint eigenvalue. Then there is an exact one-to-one correspondence between the (merged) blocks of B_k and those of $\bar{B}_k$, and similarly between those of $\tilde{B}_k$ and $\bar{\tilde{B}}_k$, and in particular, $Q = \tilde{Q} = I_n$. Since $\bar{B}_k$ and $\bar{\tilde{B}}_k$ have the same number of blocks of equal size, it follows that so do B_k and $\tilde{B}_k$; in particular, $\ell = \tilde{\ell}$. Now (3.29) gives

$$\tilde{C}_k = \tilde{T}^{-1}GTC_kT^{-1}G^{-1}\tilde{T} = (\tilde{T}^{-1}GT)C_k(\tilde{T}^{-1}GT)^{-1}.$$

where $T = \mathcal{J}_{n_E,n_L,0}$ and $\tilde{T} = \mathcal{J}_{n_{\tilde{E}},n_{\tilde{L}},0}$ We analyze the action of $\mathrm{Ad}(\tilde{T}^{-1}GT)$. Begin with an arbitrary block $B_{k,r}$ of any B_k. This block results in precisely two blocks of C_k,

$$E^{(u)}_{k,r} = B_{k,r} \qquad \text{and} \qquad E^{(l)}_{k,r} = p_k - B^T_{k,r}$$

located in the upper-left and lower-right corners of C_k, respectively, both at position r in each corner. Applying $\mathrm{Ad}(T)$ either keeps these two blocks in place, or exchanges them. The two resulting blocks $F^{(u)}_{k,r}$ and $F^{(l)}_{k,r}$ in the r-th position of each corner are thus either

$$F^{(u)}_{k,r} = E^{(u)}_{k,r}, \quad F^{(l)}_{k,r} = E^{(l)}_{k,r}, \qquad \text{or} \qquad F^{(u)}_{k,r} = E^{(l)}_{k,r}, \quad F^{(l)}_{k,r} = E^{(u)}_{k,r}.$$

Next applying $\mathrm{Ad}(G)$ switches positions of these blocks within the same corners, moving them to position $\tilde{r} = \phi(r)$,

$$G^{(u)}_{k,\tilde{r}} = a_{\tilde{r}r} F^{(u)}_{k,r} a^{-1}_{\tilde{r}r}, \qquad G^{(l)}_{k,\tilde{r}} = (a^{-1}_{\tilde{r}r})^T F^{(l)}_{k,r} a^T_{\tilde{r}r}.$$

Applying $\mathrm{Ad}(\tilde{T})$ at last again either keeps these blocks in place, or switches them between corners, to obtain blocks of $\tilde{C}_k$ of the form

$$H^{(u)}_{k,\tilde{r}} = G^{(u)}_{k,\tilde{r}}, \quad H^{(l)}_{k,\tilde{r}} = G^{(l)}_{k,\tilde{r}}, \qquad \text{or} \qquad H^{(u)}_{k,\tilde{r}} = G^{(l)}_{k,\tilde{r}}, \quad H^{(l)}_{k,\tilde{r}} = G^{(u)}_{k,\tilde{r}}.$$

Analyzing the left-upper blocks $\tilde{B}_{k,\tilde{r}} = H^{(u)}_{k,\tilde{r}}$ of $\tilde{C}_k$, there are now four possibilities:

(1) (never switched between corners)

$$\tilde{B}_{k,\tilde{r}} = a_{\tilde{r}r} E^{(u)}_{k,r} a^{-1}_{\tilde{r}r} = a_{\tilde{r}r} B_{k,r} a^{-1}_{\tilde{r}r},$$

(2) ($\mathrm{Ad}(T)$ and $\mathrm{Ad}(\tilde{T})$ switched between corners)

$$\tilde{B}_{k,\tilde{r}} = (a^{-1}_{\tilde{r}r})^T E^{(u)}_{k,r} a^T_{\tilde{r}r} = (a^{-1}_{\tilde{r}r})^T B_{k,r} a^T_{\tilde{r}r},$$

(3) (only $\mathrm{Ad}(T)$ switched between corners)

$$\tilde{B}_{k,\tilde{r}} = a_{\tilde{r}r} E^{(l)}_{k,r} a^{-1}_{\tilde{r}r} = a_{\tilde{r}r} \left(p_k - B^T_{k,r}\right) a^{-1}_{\tilde{r}r},$$

(4) (only $\mathrm{Ad}(\tilde{T})$ switched between corners)

$$\tilde{B}_{k,\tilde{r}} = (a^{-1}_{\tilde{r}r})^T E^{(l)}_{k,r} a^T_{\tilde{r}r} = (a^{-1}_{\tilde{r}r})^T \left(p_k - B^T_{k,r}\right) a^T_{\tilde{r}r},$$

where $\tilde{r} = \phi(r)$ throughout. After reordering the indices r and $\tilde{r}$ we may assume that $\tilde{r} = \phi(r) = r$, and (1) and (2) occur for $r \leq \ell_1$ while (3) and (4) occur for $\ell_1 < r \leq \ell$. (this reordering affects the matrices U and V only.) Setting $E_k = \text{diag}(B_{k,1}, \ldots, B_{k,\ell_1})$ and $F_k = \text{diag}(B_{k,\ell_1+1}, \ldots, B_{k,\ell})$, it follows that

$$B_k = \begin{bmatrix} E_k & 0 \\ 0 & F_k \end{bmatrix}$$

and there exists a block-diagonal matrix A_1 so that

$$\tilde{B}_k = \text{diag}(\tilde{B}_{k,1}, \ldots \tilde{B}_{k,\ell}) = A_1 \begin{bmatrix} E_k & 0 \\ 0 & p_k - F_k^T \end{bmatrix} A_1^{-1}.$$

Setting $W = VA_1$, the second assertion follows. □

3.3.3 Classification of the Lie algebras $\mathfrak{g}_{p,B}$ generated by pairs of commuting matrices

We now show that when $d = 2$, the nilindependence requirement of Corollary 3.9 may be removed. Given two commuting nonzero matrices $B_1, B_2 \in M_n(\mathbb{R})$, let M_1, M_2 be as above, with $p_1 \in \{0, 1\}$ and $p_2 = 0$. When $p_1 = 0$ we need to impose the requirement that B_1 and B_2 be linearly independent, in order for (A1) to hold.

We next investigate properties of Lie algebra isomorphisms between two normalized Lie algebras $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{p,\tilde{B}}$. Every isomorphism $\Phi : \mathfrak{g}_{p,B} \to \mathfrak{g}_{p,\tilde{B}}$ can be represented in matrix form as

$$\Phi \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \qquad (3.30)$$

by using the decomposition $\mathfrak{g}_{p,B} = V_{M_1} \oplus V_{M_2} \oplus V_W \oplus V_Z$. Our goal is to show that $a_{13} = a_{23} = a_{14} = a_{24} = 0$, which guarantees that Φ maps V_H onto V_H. We begin with the following observation.

Lemma 3.15. Let $\Phi : \mathfrak{g}_{p,B} \to \mathfrak{g}_{p,\tilde{B}}$ be a Lie algebra isomorphism which has the matrix representation

$$\Phi \leftrightarrow \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}. \tag{3.31}$$

Then $a_{23} = 0$.

Proof. Since Φ maps the ideal V_Z onto V_Z, it factors to a Lie algebra isomorphism $\hat{\Phi} : \mathfrak{h} = \mathfrak{g}_{p,B}/V_Z \simeq V_{M_1} \oplus V_{M_2} \oplus V_W \to \tilde{\mathfrak{h}} = \mathfrak{g}_{p,\tilde{B}}/V_Z \simeq V_{\tilde{M}_1} \oplus V_{\tilde{M}_2} \oplus V_W$ whose matrix representation is

$$\hat{\Phi} \leftrightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Let us set $\mathfrak{k} = V_{M_2} \oplus V_W$, an ideal in $\mathfrak{h}$ of dimension $2n+1$, and similarly, $\tilde{\mathfrak{k}} = V_{\tilde{M}_2} \oplus V_W$. Then $\hat{\Phi}$ maps $\mathfrak{k}$ onto $\tilde{\mathfrak{k}}$, and V_W is an abelian ideal of codimension one in $\mathfrak{k}$, respectively $\tilde{\mathfrak{k}}$.

We claim that V_W is the unique such ideal. For suppose that J is another abelian ideal of codimension one in $\mathfrak{k}$. Let $U_1, \dots, U_{2n}$ be a basis of J. Then each U_i is of the form

$$U_i = \alpha_i M_2 + W_{w_i}, \quad \alpha_i \in \mathbb{R},\ i = 1, \dots, 2n.$$

If $\alpha_i = 0$ for all i, the claim is proved, otherwise we may assume, without loss of generality, that $\alpha_1 = 1$ and $\alpha_i = 0$ for all $i \geq 2$. Now since $B_2 \neq 0$, there exist

$x_o, y_o \in \mathbb{R}^n$ so that

$$[U_1, X_{x_o}] = [M_2, X_{x_o}] = X_{B_2 x_o} \neq 0$$

$$[U_1, Y_{y_o}] = [M_2, Y_{y_o}] = Y_{-B_2^T y_o} \neq 0.$$

Since J is abelian, it follows that $X_{x_o}, Y_{y_o} \notin J$, contradicting the assumption that $\operatorname{codim}(J) = 1$. This proves the claim.

From the claim it follows immediately that $\hat{\Phi}$ maps V_W onto V_W, and hence that $a_{23} = 0$. □

Theorem 3.16. Let $\Phi : \mathfrak{g}_{p,B} \to \mathfrak{g}_{p,\tilde{B}}$ be a Lie algebra isomorphism of normalized Lie algebras. Then Φ maps V_H onto V_H.

Proof. We consider five distinct possibilities: $p_1 = 1$ and B_2 is nilpotent, $p_1 = 1$ and B_2 is not nilpotent, $p_1 = 0$ and none of B_1 and B_2 is nilpotent, $p_1 = 0$ and exactly one of B_1 and B_2 is nilpotent, and $p_1 = 0$ and both, B_1 and B_2 are nilpotent.

As will be seen below, in each of the five cases, $\mathfrak{g}_{p,B}$ will have a different algebraic structure. Thus, two Lie algebras which are isomorphic via some isomorphism Φ must both belong to the same of the five cases.

- *Case 1: $p_1 = 1$ and B_2 is not nilpotent*

  Here, $\mathfrak{g}_{p,B}$ has nilradical V_H which is of dimension $2n+1$. Since Φ maps nilradical to nilradical, it follows that $\mathfrak{g}_{p,\tilde{B}}$ has nilradical of dimension $2n+1$ as well, which thus must coincide with V_H. That is, Φ maps V_H onto V_H.

- *Case 2: $p_1 = 1$ and B_2 is nilpotent*

  Here, $\mathfrak{g}_{p,B}$ has nilradical $V_{M_2} \oplus V_H$ of dimension $2n + 2$. Since $p_1 = 1$, and the nilradical of $\mathfrak{g}_{p,\tilde{B}}$ has dimension $2n + 2$ as well, $\mathfrak{g}_{p,\tilde{B}}$ must belong to case 2. It follows that $\tilde{B}_2$ is nilpotent and the nilradical of $\mathfrak{g}_{p,\tilde{B}}$ is $V_{\tilde{M}_2} \oplus V_H$. In addition, as Φ maps the center V_Z of the nilradical onto the center V_Z of the nilradical, it follows that Φ has matrix form

$$\Phi \leftrightarrow \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}. \tag{3.32}$$

Replacing $\tilde{M}_1$ with a suitable linear combination $\tilde{M}_1 + \beta\tilde{M}_2$, we may assume that $a_{21} = 0$. Applying Lemma 3.15 it follows that $a_{23} = 0$, that is, Φ maps V_H onto V_H.

- *Case 3: $p_1 = 0$ and none of B_1, B_2 is nilpotent*

  Simply apply the argument of case 1.

- *Case 4: $p_1 = 0$ and one of B_1, B_2 is nilpotent*

  Without loss of generality, we may assume that B_2 is nilpotent, but B_1 is not. Then $\mathfrak{g}_{p,B}$ has nilradical $V_{M_2} \oplus V_H$ of dimension $2n+2$. Since $p_1 = 0$ and the nilradical of $\mathfrak{g}_{p,\tilde{B}}$ has dimensions $2n+2$, the latter algebra must again belong to case 4, so that replacing $\tilde{B}_1$ and $\tilde{B}_2$ by suitable linear combinations,$\mathfrak{g}_{p,\tilde{B}}$ will have nilradical $V_{\tilde{M}_2} \oplus V_H$. The remainder of the argument follows that of case 2.

- *Case 5: $p_1 = 0$ and both, B_1 and B_2 are nilpotent*

  Here, $\mathfrak{g}_{p,B}$ is itself nilpotent with center V_Z. Hence $\mathfrak{g}_{p,\tilde{B}}$ is also nilpotent and belongs to case 5. Since Φ maps center to center, it has the form (3.30) with $a_{14} = a_{24} = a_{34} = 0$. We begin by considering the induced isomorphism

$\hat{\Phi} : \mathfrak{h} = \mathfrak{g}_{p,B}/V_Z \to \tilde{\mathfrak{h}} = \mathfrak{g}_{p,\tilde{B}}/V_Z,$

$$\hat{\Phi} \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Since V_W is an ideal of codimension two in $\mathfrak{h}$, then $\tilde{I} := \hat{\Phi}(V_W)$ will be an abelian ideal of codimension two in $\tilde{\mathfrak{h}}$, that is, of dimension $2n$.

We claim that $\tilde{I} = V_W$. Suppose to the contrary that $\tilde{I} \neq V_W$. Denoting by P_o the projection of $\tilde{\mathfrak{h}}$ onto $V_{\tilde{M}} = V_{\tilde{M}_1} \oplus V_{\tilde{M}_2}$ and setting $V_o = P_o(\tilde{I})$, we then obtain that $\dim(V_o) \in \{1, 2\}$.

- *Subcase 5a:* $\dim(V_o) = 1$. Then elements of $\tilde{I}$ are of the form

$$A = \alpha \tilde{M}_o + W_w, \qquad \alpha \in \mathbb{R},\ W_w \in V_W$$

for some fixed nonzero $\tilde{M}_o = \operatorname{diag}(0, \tilde{B}_o, 0) \in V_{\tilde{M}}$. Fix one such A with $\alpha = 1$. Then there exist $x_o, y_o \in \mathbb{R}^n$ so that

$$[A, X_{x_o}] = [\tilde{M}_o, X_{x_o}] = X_{\tilde{B}_o x_o} \neq 0 \qquad \text{and}$$
$$[A, Y_{y_o}] = [\tilde{M}_o, Y_{y_o}] = Y_{-\tilde{B}_o^T y_o} \neq 0.$$

Since $\tilde{I}$ has codimension two in $\tilde{\mathfrak{h}}$, it follows that $\tilde{\mathfrak{h}} = \tilde{I} \oplus < X_{x_o}, Y_{y_o} >$ where $< X_{x_o}, Y_{y_o} >$ denotes $\operatorname{span}(X_{x_o}, Y_{y_o})$. In fact, suppose $\alpha X_{x_o} + \beta Y_{y_o} \in \tilde{I}$ for some scalars α, β . Then

$$0 = [A, \alpha X_{x_o} + \beta Y_{y_o}] = \alpha [A, X_{x_o}] + \beta [A, Y_{y_o}] = \alpha X_{\tilde{B}_o x_o} + \beta Y_{-\tilde{B}_o^T y_o},$$

which implies that $\alpha = \beta = 0$. Now as $< X_{x_o}, Y_{y_o} > \subseteq V_W$ we have

$$V_o = P_o(\tilde{I}) = P_o(\tilde{I} \oplus < X_{x_o}, Y_{y_o} >) = P_o(\mathfrak{h}) = V_{\tilde{M}}$$

contradicting the fact that V_o has dimension one.

∘ *Subcase 5b:* $\dim(V_o) = 2$.

Then $V_o = V_{\tilde{M}}$. Note that by nilpotency of $\tilde{B}_1$ and $\tilde{B}_2$, all linear combinations $\alpha\tilde{B}_1 + \beta\tilde{B}_2$ are again nilpotent and thus have nontrivial null spaces.

⋄ *Subcase 5b-1: there exists $\tilde{B}_o = \alpha\tilde{B}_1 + \beta\tilde{B}_2$ whose null space has dimension $\leq n-2$.*

Set $\tilde{M}_o = \alpha\tilde{M}_1 + \beta\tilde{M}_2$ and pick any $A \in \tilde{I}$ with $P_o(A) = \tilde{M}_o$. By choice of $\tilde{B}_o$, there exist two elements $x_1, x_2 \in \mathbb{R}^n$ with $[\tilde{M}_o, X_{x_1}] = X_{\tilde{B}_o x_1}$ and $[\tilde{M}_o, X_{x_2}] = X_{\tilde{B}_o x_2}$ linearly independent. Also, pick $y_1 \in \mathbb{R}^n$ with $[\tilde{M}_o, Y_{y_1}] = Y_{-\tilde{B}_o^T y_1} \neq 0$. We observe that $\tilde{I} + < X_{x_1}, X_{x_2}, Y_{y_1} >$ is a $2n+3$ dimensional subspace of $\tilde{\mathfrak{h}}$. In fact, suppose $\alpha X_{x_1} + \beta X_{x_2} + \gamma Y_{y_1} \in \tilde{I}$ for some scalars α, β, γ. Then

$$\begin{aligned} 0 &= \left[\tilde{M}_o, \alpha X_{x_1} + \beta X_{x_2} + \gamma Y_{y_1}\right] \\ &= \alpha[\tilde{M}_o, X_{x_1}] + \beta[\tilde{M}_o, X_{x_2}] + \gamma[\tilde{M}_o, Y_{y_1}] \\ &= \alpha X_{\tilde{B}_o x_1} + \beta X_{\tilde{B}_o x_2} + \gamma Y_{-\tilde{B}_o^T y_1} \end{aligned}$$

from which it follows that $\alpha = \beta = \gamma = 0$. This, however, contradicts the fact that $\tilde{\mathfrak{h}}$ has dimension $2n+2$.

⋄ *Subcase 5b-2: the null spaces of all nonzero $\alpha\tilde{B}_1 + \beta\tilde{B}_2$ have dimensions $n-1$.*

Pick elements $A_1 = \tilde{M}_1 + W_{w_1}$ and $A_2 = \tilde{M}_2 + W_{w_2}$ $(W_{w_1}, W_{w_2} \in V_W)$ of $\tilde{I}$. Since

$$\mathrm{ad}(A_i)(X_x) = X_{\tilde{B}_i x} \qquad \text{and} \qquad \mathrm{ad}(A_i)(Y_y) = Y_{-\tilde{B}_i^T y}, \qquad i = 1, 2, \quad (3.33)$$

it follows that $\ker(\mathrm{ad}(A_1))$ and $\ker(\mathrm{ad}(A_2))$ both have codimensions of at least 2 in $\tilde{\mathfrak{h}}$. In addition, since $\tilde{I}$ is abelian, then $\tilde{I} \subseteq \ker(\mathrm{ad}(A_1)) \cap \ker(\mathrm{ad}(A_2))$. Comparing dimensions, it follows that $\tilde{I} = \ker(\mathrm{ad}(A_1)) =$

$\ker(\mathrm{ad}(A_2))$. Now (3.33) shows that $\ker(\mathrm{ad}(A_i)_{|V_W})$ splits into subspaces V_{X_o} and V_{Y_o} of V_X, respectively V_Y, of codimensions one. Hence we can decompose V_X and V_Y as direct sums

$$V_X = V_{X_o} \oplus < X_{x_o} >, \qquad V_Y = V_{Y_o} \oplus < Y_{y_o} > \tag{3.34}$$

of subspaces. Here we have chosen the vectors x_o and y_o so that $x_o \perp X_o$ and $y_o \perp Y_o$ in $\mathbb{R}^n$ with respect to the usual inner product. Now since $X_o = \ker(\tilde{B}_i)$ and also $Y_o = \ker(\tilde{B}_i^T) = \mathrm{range}(\tilde{B}_i)^\perp$ $(i = 1, 2)$, it follows that, after expressing the common domain space as $X_o \oplus < x_o >$ and the common range space as $Y_o \oplus < y_o >$, the matrices $\tilde{B}_i$ take the form

$$\tilde{B}_i = \begin{bmatrix} 0 & 0 \\ 0 & b_i \end{bmatrix}$$

for scalars b_1 and b_2, contradicting the linear independence of the two matrices.

Thus, the claim is proved. It follows immediately that $a_{13} = a_{23} = 0$. Since Φ maps center V_Z onto center V_Z, then also $a_{14} = a_{24} = a_{34} = 0$. That is, Φ maps V_H onto V_H.

This completes the proof. □

Combining Theorems 3.5, 3.7, 3.16, and Remark 3.6, we arrive at:

Corollary 3.17. Let $d = 2$. Then two normalized Lie algebra $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{\tilde{p},\tilde{B}}$ are isomorphic iff

1. $p = \tilde{p}$, and

2. there exists $S \in Sp(n, \mathbb{R})$ so that, after replacing the matrices $\tilde{M}_1, \tilde{M}_2$ with a suitable basis of $V_{\tilde{M}}$,

$$\tilde{C}_k = SC_kS^{-1}, \qquad k = 1, 2,$$

with C_k and $\tilde{C}_k$ given as in (3.15).

Table 3.1 lists the equivalence classes of all Lie algebras $\mathfrak{g}_{p,B}$ generated by two commuting matrices B_1 and B_2 in the lowest dimensions, namely for $n = 1, 2, 3$. Detailed explanations of this procedure for the case $n = 3$ are given in the Appendix. As the cases $n = 1, 2$ are less difficult and are special cases of $n = 3$, detailed explanations are omitted for $n = 1, 2$. Note that when $n = 2$, the non-nilpotent cases can also be obtained from the list in Rubin and Winternitz (1993).

Table 3.1: Equivalence classes of the Lie algebras $\mathfrak{g}_{p,B}$ for $n = 1, 2, 3$

	Name	B_1	B_2	Range of parameters	Remarks
$n=1$					
$p=0$		—	—		none exists
$p=1$	$\mathfrak{g}^1_{1,1}$	$\begin{bmatrix}\frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}1\end{bmatrix}$		
$n=2$					
$p=0$	$\mathfrak{g}^2_{0,1}$	$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$		
	$\mathfrak{g}^2_{0,2}$	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$		B_2 is nilpotent
	$\mathfrak{g}^2_{0,3}$	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$		
$p=1$	$\mathfrak{g}^2_{1,1}$	$\begin{bmatrix}\frac{1}{2} & 0\\ 0 & b\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & d\end{bmatrix}$	$b > \frac{1}{2},\ 0 \leq \lvert d\rvert \leq 1$ $b = \frac{1}{2},\ 0 \leq d \leq 1$	
	$\mathfrak{g}^2_{1,2}$	$\begin{bmatrix}\frac{1}{2} & 1\\ 0 & \frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}1 & d\\ 0 & 1\end{bmatrix}$	$d \geq 0$	
	$\mathfrak{g}^2_{1,3}$	$\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$		
	$\mathfrak{g}^2_{1,4}$	$\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$	$a \geq \frac{1}{2}$	B_2 is nilpotent

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Table 3.1 – *Continued*

	Name	B_1	B_2	Range of parameters	Remarks
	$\mathfrak{g}^2_{1,5}$	$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$	$\begin{bmatrix} c & 1 \\ -1 & c \end{bmatrix}$	$a \geq \frac{1}{2},\ c \geq 0$	
	$\mathfrak{g}^2_{1,6}$	$\begin{bmatrix} \frac{1}{2} & b \\ -b & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$b > 0$	case $b = 0$ is $\mathfrak{g}^2_{1,1}$
$n = 3$					
$p = 0$	$\mathfrak{g}^3_{0,1}$	$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$1 \leq a \leq b$ $a = 0,\ 1 \leq b$ $a = b = 0$	
	$\mathfrak{g}^3_{0,2}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$a \geq 0$	
	$\mathfrak{g}^3_{0,3}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
	$\mathfrak{g}^3_{0,4}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$a \geq 0$	B_2 is nilpotent
	$\mathfrak{g}^3_{0,5}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		B_2 is nilpotent
	$\mathfrak{g}^3_{0,6}$	$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$a \in \mathbb{R}$	B_2 is nilpotent
	$\mathfrak{g}^3_{0,7}$	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		B_2 is nilpotent
	$\mathfrak{g}^3_{0,8}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		B_2 is nilpotent
	$\mathfrak{g}^3_{0,9}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		B_1, B_2 are nilpotent

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Table 3.1 – *Continued*

	Name	B_1	B_2	Range of parameters	Remarks
	$\mathfrak{g}^3_{0,10}$	$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$		B_1, B_2 are nilpotent
	$\mathfrak{g}^3_{0,11}$	$\begin{bmatrix}a&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}b&0&0\\0&0&1\\0&-1&0\end{bmatrix}$	$a \geq 0,\ b \geq 0$	
	$\mathfrak{g}^3_{0,12}$	$\begin{bmatrix}0&0&0\\0&a&1\\0&-1&a\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$a \geq 0$	
$p=1$	$\mathfrak{g}^3_{1,1}$	$\begin{bmatrix}a&0&0\\0&b&0\\0&0&\frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}c&0&0\\0&d&0\\0&0&1\end{bmatrix}$	$a \geq b > \frac{1}{2},\ c \geq 0,\ d \in \mathbb{R}$ $a > b = \frac{1}{2},\ c \geq 0,\ d \geq 0$ $a = b = \frac{1}{2},\ c \geq d \geq 0$	
	$\mathfrak{g}^3_{1,2}$	$\begin{bmatrix}a&0&0\\0&\frac{1}{2}&1\\0&0&\frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}b&0&0\\0&1&c\\0&0&1\end{bmatrix}$	$a > \frac{1}{2},\ b > 0,\ c \in \mathbb{R}$ $a > \frac{1}{2},\ b = 0,\ c \geq 0$ $a = \frac{1}{2},\ b \geq 0,\ c \geq 0$	
	$\mathfrak{g}^3_{1,3}$	$\begin{bmatrix}a&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}b&0&0\\0&1&1\\0&0&1\end{bmatrix}$	$a \geq \frac{1}{2},\ b \geq 0$	
	$\mathfrak{g}^3_{1,4}$	$\begin{bmatrix}a&0&0\\0&b&0\\0&0&b\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$	$a \geq \frac{1}{2},\ b \geq \frac{1}{2}$	
	$\mathfrak{g}^3_{1,5}$	$\begin{bmatrix}a&0&0\\0&b&0\\0&0&b\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	$a \geq \frac{1}{2},\ b \geq \frac{1}{2}$	B_2 is nilpotent
	$\mathfrak{g}^3_{1,6}$	$\begin{bmatrix}\frac{1}{2}&0&0\\0&a&1\\0&0&a\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$a \geq \frac{1}{2}$	
	$\mathfrak{g}^3_{1,7}$	$\begin{bmatrix}\frac{1}{2}&a&b\\0&\frac{1}{2}&a\\0&0&\frac{1}{2}\end{bmatrix}$	$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$	$a \geq 0,\ b \in \mathbb{R}$	

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Table 3.1 – *Continued*

Name	B_1	B_2	Range of parameters	Remarks
$\mathfrak{g}^3_{1,8}$	$\begin{bmatrix} \frac{1}{2} & a & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a \in \mathbb{R}$	
$\mathfrak{g}^3_{1,9}$	$\begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
$\mathfrak{g}^3_{1,10}$	$\begin{bmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$a \geq \frac{1}{2}$	B_2 is nilpotent
$\mathfrak{g}^3_{1,11}$	$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$a \geq \frac{1}{2}$	B_2 is nilpotent
$\mathfrak{g}^3_{1,12}$	$\begin{bmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$a \in \mathbb{R}$	B_2 is nilpotent
$\mathfrak{g}^3_{1,13}$	$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$a \geq \frac{1}{2}$	B_2 is nilpotent
$\mathfrak{g}^3_{1,14}$	$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$	$\begin{bmatrix} c & 0 & 0 \\ 0 & d & 1 \\ 0 & -1 & d \end{bmatrix}$	$a > \frac{1}{2},\ b > \frac{1}{2},\ c \in \mathbb{R},\ d \geq 0$ $a = \frac{1}{2},\ b \geq \frac{1}{2},\ c \geq 0,\ d \geq 0$ $a > \frac{1}{2},\ b = \frac{1}{2},\ c \geq 0,\ d \geq 0$	
$\mathfrak{g}^3_{1,15}$	$\begin{bmatrix} a & 0 & 0 \\ 0 & \frac{1}{2} & b \\ 0 & -b & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a \geq \frac{1}{2},\ b > 0,\ c \geq 0$	
$\mathfrak{g}^3_{1,16}$	$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$a \geq \frac{1}{2},\ b > 0$	

CHAPTER IV

REPRESENTATIONS OF THE GROUPS $G_{p,B}$

In this chapter, we show that the groups $G_{p,B}$ can be represented as subgroups of both, the symplectic group $Sp(n+1,\mathbb{R})$, as well as the affine group $\mathit{Aff}(n+1)$. Thus, they possess both, a metaplectic and a wavelet representation. We also show that the metaplectic representation is equivalent to a sum of two copies of a subrepresentation of the wavelet representation.

4.1 Preliminaries

Throughout, symbols x, y will denote vectors in Euclidean space $\mathbb{R}^n$ written as column vectors, while Greek symbols ξ, η will denote elements in the Euclidean space written as row vectors. For ease of distinction, we denote the space of row vectors by $\widehat{\mathbb{R}^n}$. The transpose of a vector or matrix x is denoted by x^T, hence the inner product in $\mathbb{R}^n$ is $x \cdot y = y^T x$.

4.1.1 The Fourier transform

The *Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2i\pi\xi x}\, dx \qquad (\xi \in \widehat{\mathbb{R}^n}).$$

By the Plancherel Theorem, the restriction of the map $f \mapsto \hat{f}$ to $(L^1 \cap L^2)(\mathbb{R}^n)$ extends uniquely to a unitary operator $\mathcal{F} : f \in L^2(\mathbb{R}^n) \mapsto \hat{f} \in L^2(\widehat{\mathbb{R}^n})$ which is also called the Fourier transform.

4.1.2 Translation, modulation, dilation, chirp operators

The two standard unitary representations of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ are *translation* and *modulation*, defined by

$$(T_x f)(y) = f(y - x) \qquad \text{and} \qquad (E_x f)(y) = e^{2i\pi x^T y} f(y),$$

and the corresponding operators on $L^2(\widehat{\mathbb{R}^n})$ are defined similarly,

$$(\hat{T}_x g)(\xi) = g(\xi - x^T) \qquad \text{and} \qquad (\hat{E}_x g)(\xi) = e^{2i\pi \xi x} g(\xi),$$

for $x, y \in \mathbb{R}^n$, $\xi \in \widehat{\mathbb{R}^n}$, $f \in L^2(\mathbb{R}^n)$ and $g \in L^2(\widehat{\mathbb{R}^n})$.

The natural representations of $GL_n(\mathbb{R})$ on the spaces $L^2(\mathbb{R}^n)$ and $L^2(\widehat{\mathbb{R}^n})$ are given by left and right *dilation*, respectively,

$$(S_a f)(y) = |\det a|^{-1/2} f(a^{-1} y) \qquad \text{and} \qquad (\hat{S}_a g)(\xi) = |\det a|^{1/2} g(\xi a),$$

for $a \in GL_n(\mathbb{R})$, $y \in \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n)$ and $g \in L^2(\widehat{\mathbb{R}^n})$. The Fourier transform intertwines some of these representations,

$$\hat{E}_{-x} = \mathcal{F} T_x \mathcal{F}^{-1}, \quad \hat{T}_x = \mathcal{F} E_x \mathcal{F}^{-1} \quad \text{and} \quad \hat{S}_a = \mathcal{F} S_a \mathcal{F}^{-1}. \tag{4.1}$$

The additive group $Sym_n(\mathbb{R})$ of $n \times n$ symmetric matrices also possess a representation on $L^2(\mathbb{R}^n)$ by *chirps*, and defined by

$$(U_m f)(q) = e^{i\pi q^T m q} f(q)$$

for $m \in Sym_n(\mathbb{R})$, $f \in L^2(\mathbb{R}^n)$ and $q \in \mathbb{R}^n$.

4.1.3 The affine group and the wavelet representation

The *affine group* $Aff(n, \mathbb{R})$ is the group formed by the invertible linear transformations and translations in Euclidean space. It takes the form of a semi-direct product $\mathbb{R}^n \rtimes_\alpha GL_n(\mathbb{R})$, where the action α is simply matrix multiplication,

$$\alpha_a(x) = ax$$

for $x \in \mathbb{R}^n$, $a \in GL_n(\mathbb{R})$. Thus the group operation is

$$(x, a)(\tilde{x}, \tilde{a}) = (x + a\tilde{x}, a\tilde{a})$$

for $(x, a), (\tilde{x}, \tilde{a}) \in \mathit{Aff}(n, \mathbb{R})$.

If H is a closed subgroup of $GL_n(\mathbb{R})$, then the corresponding subgroup of the affine group can be represented as the matrix group

$$\mathbb{R}^n \rtimes_\alpha H \cong \left\{ \begin{bmatrix} a & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R}^n,\ a \in H \right\} \subset GL_{n+1}(\mathbb{R}).$$

Since $S_a T_x S_{a^{-1}} = T_{ax}$, translation and left dilation compose to a unitary representation π of such subgroups on $L^2(\mathbb{R}^n)$, called the *wavelet representation* or *affine representation*, by

$$\pi(x, a) = T_x S_a, \qquad (x, a) \in \mathbb{R}^n \rtimes_\alpha H.$$

Conjugating by the Fourier transform, (4.1) yields an equivalent representation $\hat{\pi}$ on $L^2(\widehat{\mathbb{R}^n})$ given by

$$\hat{\pi}(x, a) = \hat{E}_{-x} \hat{S}_a, \qquad (x, a) \in \mathbb{R}^n \rtimes_\alpha H. \tag{4.2}$$

We call this the *wavelet representation in Fourier space.*

4.1.4 The symplectic group and the metaplectic representation

Recall from the Introduction that the *symplectic group* $Sp(n, \mathbb{R})$ is the set of all $2n \times 2n$ invertible matrices preserving the symplectic form,

$$Sp(n, \mathbb{R}) = \left\{ \mathcal{A} \in GL_{2n}(\mathbb{R}) \ : \ [\![\mathcal{A}w, \mathcal{A}\tilde{w}]\!] = [\![w, \tilde{w}]\!] \qquad \forall\ w, \tilde{w} \in \mathbb{R}^{2n} \right\}.$$

(Some authors denote this group by $Sp(2n, \mathbb{R})$.) For details on its structure, see Folland (1989). The matrix $\mathcal{J}$ is one of its elements, and the groups $Sym(n, \mathbb{R})$

and $GL_n(\mathbb{R})$ are naturally embedded in $Sp(n,\mathbb{R})$ in form of the closed subgroups

$$\begin{aligned} N &= \left\{ \mathcal{N}_m := \begin{bmatrix} I_n & 0 \\ m & I_n \end{bmatrix} : m \in Sym(n,\mathbb{R}) \right\}, \quad \text{and} \\ L &= \left\{ \mathcal{L}_a := \begin{bmatrix} a & 0 \\ 0 & (a^{-1})^T \end{bmatrix} : a \in GL_n(\mathbb{R}) \right\} \quad \text{respectively.} \end{aligned} \tag{4.3}$$

One can show that the group $Sp(n,\mathbb{R})$ is generated by $L \cup N \cup \{\mathcal{J}\}$.

There is a projective representation μ of $Sp(n,\mathbb{R})$ on $L^2(\mathbb{R}^n)$ called the *metaplectic representation* which, for the three types of generating matrices, is given by

$$\mu(\mathcal{L}_a) = S_a, \qquad \mu(\mathcal{N}_m) = U_m, \qquad \mu(-\mathcal{J}) = (-i)^{n/2}\,\mathcal{F}.$$

The word projective here means that μ is a homomorphism of the group $Sp(n,\mathbb{R})$ into the unitary group of $L^2(\mathbb{R}^n)$ only up to a factor of ± 1:

$$\mu(\mathcal{A}\mathcal{B}) = \pm\mu(\mathcal{A})\mu(\mathcal{B}) \qquad (\mathcal{A},\mathcal{B} \in Sp(n,\mathbb{R})).$$

The problem here is the matrix $\mathcal{J}$. However, when restricted to the subgroup generated by $L \cup N$, μ is a group homomorphism.

4.1.5 Subgroups of the symplectic group which possess a wavelet representation

We next consider a class of subgroups of $Sp(n,\mathbb{R})$ which arise as semidirect products of a vector group with a group of dilations. We begin with the linear action α of $GL_n(\mathbb{R})$ on the vector space $Sym(n,\mathbb{R})$ of Example 2.18,

$$\alpha_a(m) = (a^{-1})^T m a^{-1} \qquad \big(a \in GL_n(\mathbb{R}),\ m \in Sym(n,\mathbb{R})\big). \tag{4.4}$$

Let E be a closed subgroup $GL_n(\mathbb{R})$ and M an $l-$dimensional and E-invariant linear subspace of $Sym(n,\mathbb{R})$. Invariant means that $\alpha_a(m) \in M$ for all $m \in M$ and

$a \in E$. As can be seen from (4.3), M and E are isomorphic to closed subgroups of $Sp(n, \mathbb{R})$, and the action α is implemented by conjugation under this isomorphism,

$$\mathcal{L}_a \mathcal{N}_m \mathcal{L}_a^{-1} = \mathcal{N}_{(a^{-1})^T m a^{-1}}.$$

Consequently, by Remark 2.26, the semidirect product $M \rtimes_\alpha E$ is isomorphic to a closed subgroup of $Sp(n, \mathbb{R})$,

$$M \rtimes_\alpha E \cong K := \left\{ \mathcal{N}_m \mathcal{L}_a = \begin{bmatrix} a & 0 \\ ma & (a^{-1})^T \end{bmatrix} \;:\; m \in M,\; a \in E \right\}. \tag{4.5}$$

The restriction of the metaplectic representation to K, which we simply call the metaplectic representation of $K = M \rtimes_\alpha E$, is given by

$$\mu(m, a) := \mu(\mathcal{N}_m \mathcal{L}_a) = U_m S_a, \qquad ((m, a) \in M \rtimes_\alpha E), \tag{4.6}$$

and it is a proper representation, that is, a group homomorphism.

Next we show that the groups $M \rtimes_\alpha E$ also have a wavelet representation. To do so, identify the vector space M with Euclidean space $\mathbb{R}^l$ by fixing a basis. Since the action α is by invertible linear transformations, there exists a continuous homomorphism $\varphi : a \mapsto h_a$ of E onto a (not necessarily closed) subgroup H of $GL_l(\mathbb{R})$ satisfying

$$\alpha_a(m) = h_a m \qquad (m \in \mathbb{R}^l,\; a \in E),$$

which, as one easilly verifies, naturally extends to a group homomorphism φ of $M \rtimes_\alpha E$ onto the subgroup $\mathbb{R}^l \rtimes_\alpha H$ of $Aff(l, \mathbb{R})$ by

$$\varphi(m, a) = (m, h_a). \tag{4.7}$$

(For ease of notation, we will denote these semi-direct products by $M \rtimes E$ and $\mathbb{R}^l \rtimes H$.) Now composition of the homomorphism φ with the wavelet representation

(4.2) in Fourier space yields a wavelet representation of $M \rtimes E$ on $L^2(\widehat{\mathbb{R}^l})$, also denoted by $\hat{\pi}$, and given by

$$\hat{\pi}(m,a) = \hat{E}_{-m}\hat{S}_{h_a}. \tag{4.8}$$

4.2 The groups $G_{p,B}$ are subgroups of $Sp(n+1,\mathbb{R})$ and $Aff(n+1,\mathbb{R})$

We apply the discussion in the previous section to show that the each group $G_{p,B}$ can be represented as a subgroup of the form $M \rtimes E$ of the symplectic group, and as a subgroup of the form $\mathbb{R}^{n+1} \rtimes H$ of the affine group. We will impose the assumptions (A1) and (A2) of Chapter 3, which ensures that each group $G_{p,B}$ can be represented as a matrix group of the form (3.11).

From now on, M will denote the $l = n+1$ dimensional vector subspace of $Sym(n+1,\mathbb{R})$,

$$M = \left\{ m(z,x) := \begin{bmatrix} -z & -x^T \\ -x & 0 \end{bmatrix} \;:\; x \in \mathbb{R}^n,\; z \in \mathbb{R} \right\}. \tag{4.9}$$

This parametrization reflects the identification of M with $\mathbb{R}^l = \mathbb{R}^{n+1}$ chosen,

$$m(z,x) \mapsto \begin{pmatrix} z \\ x \end{pmatrix}. \tag{4.10}$$

Furthermore, $E = E_{p,B}$ will be the closed subgroup of $GL_{n+1}(\mathbb{R})$,

$$E_{p,B} = \left\{ a(t,y) := \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}y & I_n \end{bmatrix} \begin{bmatrix} e^{-pt/2} & 0 \\ 0 & e^{pt/2}\left[e^{-Bt}\right]^T \end{bmatrix} \;:\; t \in \mathbb{R}^d,\; y \in \mathbb{R}^n \right\}.$$

The group law in $E_{p,B}$ is

$$a(t,y)a(\tilde{t},\tilde{y}) = a(t+\tilde{t}, y + e^{pt/2}\left[e^{-Bt}\right]^T \tilde{y}). \tag{4.11}$$

Now M is invariant under the $E_{p,B}$-action (4.4), in fact

$$\alpha_{a(t,y)}(m(z,x)) = (a(t,y)^{-1})^T m(z,x) a(t,y)^{-1} = m(e^{pt}z + y^T e^{Bt}x, e^{Bt}x). \tag{4.12}$$

By (4.5), the semi-direct product $M \rtimes E_{p,B}$ can be identified with a closed subgroup of $Sp(n+1,\mathbb{R})$,

$$M \rtimes E_{p,B} \cong K_{p,B} := \left\{ k(t,x,y,z) = \begin{bmatrix} a(t,y) & 0 \\ m(z,x)a(t,y) & [a(t,y)^{-1}]^T \end{bmatrix} : \begin{array}{c} z \in \mathbb{R}, \\ t \in \mathbb{R}^d, \\ x,y \in \mathbb{R}^n \end{array} \right\}$$

having the group law

$$k(t,x,y,z)\, k(\tilde{t},\tilde{x},\tilde{y},\tilde{z}) = k(t+\tilde{t}, x+e^{Bt}\tilde{x}, y+e^{pt}\left[e^{-Bt}\right]^T \tilde{y}, z+e^{pt}\tilde{z}+y^T e^{Bt}\tilde{x}),$$

which is precisely the law (3.8) of $G_{p,B}$. It is now easy to see that the matrix groups $G_{p,B}$ and $K_{p,B}$ are isomorphic.

Next we compute the homomorphism $\varphi : M \rtimes E_{p,B} \to \mathbb{R}^{n+1} \rtimes H$ of (4.7). Using the identification (4.10) of M with $\mathbb{R}^{n+1}$ and equation (4.12) we obtain that

$$h_{a(t,y)} \begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} e^{pt}z + y^T e^{Bt}x \\ e^{Bt}x \end{pmatrix},$$

so that

$$H = H_{p,B} = \left\{ h_{a(t,y)} = \begin{bmatrix} e^{pt} & y^T e^{Bt} \\ 0 & e^{Bt} \end{bmatrix} : t \in \mathbb{R}^d,\ y \in \mathbb{R}^n \right\} \subset GL_{n+1}(\mathbb{R}).$$

We observe that by assumptions (A1)–(A2), this group is closed in $GL_{n+1}(\mathbb{R})$, and the map $\varphi : E_{p,B} \to H_{p,B}$ is an isomorphism of matrix groups. Hence,

$$G_{p,B} \cong M \rtimes E_{p,B} \cong \mathbb{R}^{n+1} \rtimes H_{p,B} = \left\{(m, h_a) : m \in \mathbb{R}^{n+1},\ h_a \in H_{p,B}\right\}$$

$$\cong \left\{ \begin{bmatrix} h_{a(t,y)} & \binom{z}{x} \\ 0 & 1 \end{bmatrix} : z \in \mathbb{R},\ t \in \mathbb{R}^d,\ x,y \in \mathbb{R}^n \right\}$$

which is a closed subgroup of $\mathit{Aff}(n+1,\mathbb{R})$.

4.3 The symplectic and wavelet representations of the groups $G_{p,B}$

By (4.8), the wavelet representation of $G_{p,B} \cong \mathbb{R}^{n+1} \rtimes H_{p,B}$ in Fourier space is given by

$$\hat{\pi}\big(\, g(t,x,y,z)\,\big) = \hat{E}_{-\left(\begin{smallmatrix} z \\ x \end{smallmatrix}\right)} \hat{S}_{h_{a(t,y)}},$$

that is,

$$\left[\hat{\pi}\big(g(t,x,y,z)\big)f\right](r,\xi) = \delta(t)^{1/2}\, e^{pt/2} e^{-2i\pi(rz+\xi x)}\, f\big(re^{pt}, (ry^T+\xi)e^{Bt}\big) \qquad (4.13)$$

for $f \in L^2(\widehat{\mathbb{R}^{n+1}})$, $r \in \mathbb{R}$, $\xi \in \widehat{\mathbb{R}^n}$ and $\delta(t) = \det\left(e^{Bt}\right) = e^{tr(Bt)}$. Clearly, $\widehat{\mathbb{R}^{n+1}}$ decomposes measurably into the two $H_{p,B}$-invariant open half spaces

$$\mathcal{O}_+ = \{(r,\xi) : r > 0\} \qquad \text{and} \qquad \mathcal{O}_- = \{(r,\xi) : r < 0\}.$$

It thus can be seen from (4.13) that $L^2(\mathcal{O}_+)$ and $L^2(\mathcal{O}_-)$ are both $\hat{\pi}$-invariant subspaces of $L^2(\widehat{\mathbb{R}^{n+1}})$ and consequently, the wavelet representation $\hat{\pi}$ splits into the direct sum $\hat{\pi} = \hat{\pi}_+ \oplus \hat{\pi}_-$ of the two subrepresentations $\hat{\pi}_\pm$ obtained by restricting $\hat{\pi}$ to these two invariant subspaces.

Similarly, by (4.6), the metaplectic representation of the group $G_{p,B} \cong M \rtimes E_{p,B}$ is given by

$$\mu\,(\, g(t,x,y,z)\,) = U_{m(z,x)} S_{a(t,y)}.$$

Since for each vector $q = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathbb{R}^{n+1}$, $u \in \mathbb{R}$, $v \in \mathbb{R}^n$ we have

$$q^T m(z,x) q = (u, v^T) m(z,x) \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = -(u^2 z + 2uv^T x), \qquad (4.14)$$

it follows that

$$\left[\mu\big(\, g(t,x,y,z)\,\big) f\right] \begin{pmatrix} u \\ v \end{pmatrix} = \delta(t)^{1/2} e^{pt(1-n)/4} e^{-i\pi(u^2 z + 2uv^T x)} f \begin{pmatrix} e^{pt/2} u \\ e^{-pt/2}\left[e^{Bt}\right]^T \left(\frac{u}{2} y + v\right) \end{pmatrix} \qquad (4.15)$$

for $f \in L^2(\mathbb{R}^{n+1})$, with $\delta(t) = \det\left(e^{Bt}\right)$. Clearly, $\mathbb{R}^{n+1}$ splits measurably into two $E_{p,B}$-invariant open half spaces

$$\mathcal{U}_+ = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \ : \ u > 0 \right\} \quad \text{and} \quad \mathcal{U}_- = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \ : \ u < 0 \right\}.$$

It can be seen from (4.15) that $L^2(\mathcal{U}_+)$ and $L^2(\mathcal{U}_-)$ are both μ-invariant subspaces of $L^2(\mathbb{R}^{n+1})$. Hence, μ splits into the direct sum $\mu = \mu_+ \oplus \mu_-$ of the two subrepresentations $\mu_\pm$ obtained by restricting μ to each of the two invariant subspaces $L^2(\mathcal{U}_\pm)$.

We next obtain a connection between these representations, by employing the techniques developed in Cordero et. al (2006), De Mari and De Vito (2013), and Namngam and Schulz (2013).

Proposition 4.1. The subrepresentations μ_+ and μ_- are both equivalent to $\hat{\pi}_+$.

Proof. Observe that for each $q \in \mathbb{R}^{n+1}$, the map

$$\begin{pmatrix} z \\ x \end{pmatrix} \mapsto q^T m(z,x) q$$

defines a linear functional on $\mathbb{R}^{n+1}$. Hence there exists a unique $\Psi(q) \in \widehat{\mathbb{R}^{n+1}}$ so that

$$q^T m(z,x) q = -2\Psi(q) \begin{pmatrix} z \\ x \end{pmatrix}$$

for all $z \in \mathbb{R}$, $x \in \mathbb{R}^n$. In fact, equation (4.14) shows that

$$\Psi(q) = \Psi \begin{pmatrix} u \\ v \end{pmatrix} = \left(\tfrac{1}{2} u^2, u v^T \right).$$

We observe that Ψ is smooth with Jacobian determinant

$$J_\Psi \begin{pmatrix} u \\ v \end{pmatrix} = u^{n+1}$$

which does not vanish on the open half planes $\mathcal{U}_+$ and $\mathcal{U}_-$. Thus, the restrictions of Ψ to these sets constitute diffeomorphisms

$$\Psi_+ : \mathcal{U}_+ \to \mathcal{O}_+ \qquad \text{and} \qquad \Psi_- : \mathcal{U}_- \to \mathcal{O}_+,$$

respectively. Furthermore, for $(r,\xi) \in \mathcal{O}_+ \subset \widehat{\mathbb{R}^{n+1}}$ with $r \in \mathbb{R}$, $\xi \in \widehat{\mathbb{R}^n}$ we have

$$\Psi_\pm^{-1}(r,\xi) = \begin{pmatrix} \pm\sqrt{2r} \\ \pm\frac{1}{\sqrt{2r}}\,\xi^T \end{pmatrix} \qquad \text{and} \qquad J_{\Psi_\pm^{-1}}(r,\xi) = \pm\,(2r)^{-(n+1)/2}.$$

It follows that the operators

$$Q_+ : L^2(\mathcal{O}_+) \to L^2(\mathcal{U}_+) \qquad \text{and} \qquad Q_- : L^2(\mathcal{O}_+) \to L^2(\mathcal{U}_-)$$

defined by

$$(Q_\pm f)(q) = |J_\Psi(q)|^{1/2}\, f\left(\Psi(q)\right) \qquad\qquad (f \in L^2(\mathcal{O}_+), q \in \mathcal{U}_\pm)$$

constitute Hilbert space isomorphism, whose inverses are given by

$$(Q_\pm^{-1} f)(\eta) = \left|J_{\Psi_\pm^{-1}}(\eta)\right|^{1/2} f\left(\Psi_\pm^{-1}(\eta)\right) \qquad\qquad (f \in L^2(\mathcal{U}_\pm), \eta \in \mathcal{O}_+).$$

We complete the proof by showing that

$$\mu_\pm = Q_\pm \hat{\pi}_+ Q_\pm^{-1}.$$

In fact, for all $f \in L^2(\mathcal{U}_\pm)$ and $q = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathbb{R}^{n+1}$ we have

$$\begin{aligned}
&\left[Q_\pm \hat{\pi}_+(t,x,y,z) Q_\pm^{-1} f\right](q) = \left|J_\Psi\begin{pmatrix} u \\ v \end{pmatrix}\right|^{1/2} \left[\hat{\pi}_+(t,x,y,z) Q_\pm^{-1} f\right]\left(\Psi\begin{pmatrix} u \\ v \end{pmatrix}\right) \\
&= |u|^{(n+1)/2} \delta(t)^{1/2} e^{pt/2} e^{-2i\pi\left((u^2/2)z + uv^T x\right)} \left[Q_\pm^{-1} f\right]\left(\tfrac{u^2}{2} e^{pt}, \left(\tfrac{u^2}{2} y^T + uv^T\right) e^{Bt}\right) \\
&= \delta(t)^{1/2} |u|^{(n+1)/2} e^{pt/2} e^{-i\pi\left(u^2 z + 2uv^T x\right)} \left|\pm u^2 e^{pt}\right|^{-(n+1)/4} f\begin{pmatrix} \pm\sqrt{u^2 e^{pt}} \\ \pm\frac{1}{\sqrt{u^2 e^{pt}}} [e^{Bt}]^T \left(\frac{u^2}{2} y + uv\right) \end{pmatrix} \\
&= \delta(t)^{1/2} e^{pt(1-n)/4} e^{-i\pi\left(u^2 z + 2uv^T x\right)} f\begin{pmatrix} e^{pt/2} u \\ e^{-pt/2} [e^{Bt}]^T \left(\frac{u}{2} y + v\right) \end{pmatrix}
\end{aligned}$$

which is precisely (4.15). □

It now follows immediately that the metaplectic representation μ of $G_{p,B}$ is equivalent to the sum of two copies of $\hat{\pi}_+$,

$$\mu = \mu_+ \oplus \mu_- \simeq \hat{\pi}_+ \oplus \hat{\pi}_+.$$

CHAPTER V

CONCLUSION

In this thesis, we have studied extensions of the multidimensional Heisenberg group $\mathbb{H}^n$ by groups of automorphisms. The particular feature of the automorphisms chosen is that, when the Heisenberg group is represented in matrix form as the polarized Heisenberg group $\mathbb{H}^n_{pol}$, they can be implemented by conjugation with invertible matrices. We only considered d-parameter groups of automorphisms, as they render the extended groups $G_{p,B}$ simply connected and hence uniquely determined by their Lie algebras. The objectives of the thesis were first, to at least partially classify the extended groups up to isomorphism, and second, to show that they can be embedded in both, the symplectic and affine groups, and to compare their metaplectic and wavelet representations. The results achieved are summarized below.

5.1 Classification

In order to classify the extended groups $G_{p,B}$ with respect to choices of $p = (p_1, \ldots, p_d)$ and $B = (B_1, \ldots, B_d)$, we considered the equivalent and simpler task of classifying their Lie algebras $\mathfrak{g}_{p,B}$ with the following outcome:

1. Theorem 3.5 gives sufficient conditions for two Lie algebras $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{\tilde{p},\tilde{B}}$ to be isomorphic. An immediate consequence of this theorem is that every Lie algebra $\mathfrak{g}_{p,B}$ is isomorphic to a Lie algebra with normalized parameters. That is, one may assume that $p_1 \in \{0, 1\}$ and $p_k = 0$ for $k = 2, \ldots, d$.

2. Theorem 3.7 gives necessary conditions for two Lie algebras to be isomorphic, under condition that the isomorphism between the two restricts to an isomorphism between their Heisenberg subalgebras. This is always the case when the matrices $B_1, \ldots, B_d$ are nilindependent, as shown in Corollary 3.9.

3. Given two isomorphic normalized Lie algebras $\mathfrak{g}_{p,B}$ and $\mathfrak{g}_{p,\tilde{B}}$, Theorem 3.14 compares the block structures of each of the pairs of matrices B_k and $\tilde{B}_k$ ($k = 1 \ldots d$) under some mild assumptions on the joint spectrum of the matrices.

4. In the case of two-parameter groups of automorphisms ($d = 2$), we showed in Theorem 3.16 that every isomorphism between two Lie algebras carries Heisenberg subalgebra to Heisenberg subalgebra, so that Theorem 3.7 always applies. This together with Theorem 3.5 led to the characterization of isomorphic Lie algebras in Corollary 3.17.

5. In the case of two-parameter groups of automorphisms ($d = 2$), the equivalence classes of isomorphic Lie algebras were explicitly computed for the low dimensional cases $n = 1, 2, 3$ as presented in Table 3.1.

5.2 Embedding in the symplectic and affine groups

We first showed that the extended groups $G_{p,B}$ are isomorphic to subgroups of the symplectic group $Sp(n + 1, \mathbb{R})$. In fact, they take the form of semidirect product subgroups of the form $M \rtimes E_{p,B}$, where M is a vector group and $E_{p,B}$ acts linearly on M. This makes it possible to represent them as semidirect products $\mathbb{R}^{n+1} \rtimes H_{p.B}$, with $H_{p,B}$ a closed subgroup of $GL_{n+1}(\mathbb{R})$, that is, as a subgroup of the affine group $Aff(n + 1, \mathbb{R})$.

We then computed the metaplectic representation μ and wavelet representation $\hat{\pi}$ of the groups $G_{p,B}$. We showed that μ and $\hat{\pi}$ split into sums of two subrepresentations, $\mu = \mu_+ \oplus \mu_-$ and $\hat{\pi} = \hat{\pi}_+ \oplus \hat{\pi}_-$, and that μ is equivalent to $\hat{\pi}_+ \oplus \hat{\pi}_+$.

5.3 Further work

Both topics covered in this thesis lead to opportunities for future research.

The immediate task would be to complete the classification of the Lie algebras $\mathfrak{g}_{p,B}$ when $d > 2$. This would begin with the investigation whether or under what conditions an isomorphism between two Lie algebras will map the Heisenberg algebra to the Heisenberg algebra, in case that the matrices B_k are not nilindependent. Once this classification has been achieved, it will be natural to look at general groups of automorphisms, beyond d-parameter groups.

The original motivation for studying the metaplectic and wavelet representations of these groups comes from the particular example of a group $M \rtimes E$ by Cordero et al. (2006) who implicitly used the equivalence of the subrepresentations μ_- and μ_+ with $\hat{\pi}^+$ to show that the $M \rtimes E$ is admissible for the metaplectic representation, by employing the well known results for admissible groups for the wavelet representation. We recall here that a group G is admissible for a representation $(\pi, \mathcal{H})$ if there exists $h \in \mathcal{H}$ so that

$$\|f\|_2 = \int_G |< f, \pi(g)h >|^2 \, dg$$

for all $f \in \mathcal{H}$. In Namngam (2010), it was shown that the groups $G_{p,B} \simeq M \rtimes E_{p,B}$ are admissible for the metaplectic representation when $d = 1$. In the case $d \geq 2$ considered in this thesis, the groups $M \rtimes E_{p,B}$ can be shown to not be admissible for the metaplectic representation, because their Lie algebras are too large in dimension. It will therefore be of interest to find and characterize subgroups of $M \rtimes E_{p,B}$ which are admissible instead.

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APPENDIX

APPENDIX

CLASSIFICATION OF THE LIE ALGEBRAS $\mathfrak{g}_{p,B}$ GENERATED BY PAIRS OF 3×3 COMMUTING MATRICES

In this appendix, we describe the procedure of classifiying all normalized Lie algebras $\mathfrak{g}_{p,B}$ generated by two commuting 3×3 matrices, the list of which has already been presented in the Table 3.1.

Let two commuting 3×3 matrices B_1 and B_2 be given. When $p_1 = 0$ we assume that the two are linearly independent, to ensure that the matrices M_1 and M_2 are linearly independent. Applying Theorem 3.10, it is not difficult to see that in some appropriate basis, the pair B_1, B_2 takes one of the following four forms:

Type I: Both matrices are diagonal,

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & 0 \\ 0 & 0 & \tilde{c} \end{bmatrix}.$$

Type II: Each matrix has one 2×2 upper triangular block,

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & \tilde{c} \\ 0 & 0 & \tilde{b} \end{bmatrix},$$

but the pair is not of type I.

Type III: Each matrix is a 3×3 upper triangular block,

$$B_1 = \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & \tilde{b} & \tilde{c} \\ 0 & \tilde{a} & \tilde{d} \\ 0 & 0 & \tilde{a} \end{bmatrix},$$

with $b\tilde{d} = \tilde{b}d$, and the pair is neither of type I nor type II.

Type IV: Each matrix has a 2×2 block with complex eigenvalues,

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & \tilde{c} \\ 0 & -\tilde{c} & \tilde{b} \end{bmatrix},$$

but the pair is not of type I.

In addition, by normalization we may assume that $p_1 \in \{0, 1\}$ and $p_2 = 0$.

By the various Theorems presented in Section 3.3.2, the following operations do not change the equivalence class of the Lie algebra, and will be used throughout:

1. Change the basis of the underlying vector space $\mathbb{R}^3$. We will denote the basis vectors before a change of basis by e_1, e_2, e_3, and those after the change of basis by f_1, f_2, f_3.
2. Multiply a matrix B_k by a scalar. However, when $p_1 = 1$ then only B_2 can be scaled in this manner.
3. Replace any B_k by a nontrivial linear combinations $\alpha B_1 + \beta B_2$, preserving linear independence of B_1 and B_2, when $p_1 = 0$. When $p_1 = 1$, then the only linear combination allowed is replacing B_1 with $B_1 + \alpha B_2$ for some scalar α.
4. Replace the r-th blocks $B_{k,r}$ of B_k with $p_k I - B_{k,r}^T$, for both $k = 1, 2$. (Recall that this is called a *flip* of the r-th blocks).

- **Case** $p_1 = 0$

Type I matrices

Without loss of generality, we may assume that $c \neq 0$. This allows scaling B_1 so that $c = 1$. Then replacing B_2 by $B_2 - \tilde{c}B_1$ we obtain $\tilde{c} = 0$.

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next we may assume that $\tilde{b} \neq 0$, by exchanging the basis vectors e_1 and e_2 if necessary. This permits scaling the matrix B_2 to $\tilde{b} = 1$, followed by replacing B_1 with $B_1 - bB_1$ so that $b = 0$,

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

We have obtained the algebra $\mathfrak{g}^3_{0,1}$.

Now we consider the range of parameters a and $\tilde{a}$. First we make $a, \tilde{a}$ non-negative.

- If $a, \tilde{a} \geq 0$, there is nothing to do.
- When $a, \tilde{a} < 0$, then we flip the e_1-block, to obtain $a > 0, \tilde{a} > 0$.
- When $a \geq 0$ but $\tilde{a} < 0$ we flip the e_1, e_3-block. This will render the entries of B_1 non-positive, which we remedy by scaling B_1 by -1. In addition this process replaces $\tilde{a}$ by $-\tilde{a} > 0$ so that now the entries of both matrices are non-negative.
- When $a < 0$ but $\tilde{a} \geq 0$ we proceed similarly, flipping the e_1, e_2-block.

Thus we may assume that $a, \tilde{a} \geq 0$.

Next, when $\tilde{a} < a$ we first exchange B_1 and B_2 and then exchange the two basis vectors e_2 and e_3, which allows us to assume that $a \leq \tilde{a}$.

Finally, we attempt to make $a, \tilde{a} \geq 1$.

1. If $0 < a < 1$ then we divide B_1 by a and subtract $\tilde{a}B_1$ from B_2 to obtain

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{\tilde{a}}{a} \end{bmatrix}.$$

   Next we flip the e_2-block and multiply B_2 by -1 to obtain matrices of the same form, but without the minus sign in B_2. Thus, all nonzero entries are ≥ 1. Switching the basis vectors by $f_1 = e_3$ and $f_3 = e_1$ (if $\frac{1}{a} \leq \frac{\tilde{a}}{a}$), or exchanging B_1 and B_2 and shifting the basis vectors to $f_1 = e_3$ and $f_2 = e_1$, $f_3 = e_2$ (if $\frac{1}{a} > \frac{\tilde{a}}{a}$) and then relabeling the variables, we obtain matrices of form (1), but with $1 \leq a \leq \tilde{a}$.

2. If $a = 0$ but $0 < \tilde{a} < 1$ then we divide B_2 by $\tilde{a}$ and relabel $\frac{1}{\tilde{a}}$ to $\tilde{a}$, and exchange the basis vectors e_1 and e_2 to obtain matrices as in (1) with $a = 0$ and $\tilde{a} \geq 1$.

We are thus left with three possibilitites: $1 \leq a \leq \tilde{a}$, or $a = 0$, $\tilde{a} \geq 1$, or $a = \tilde{a} = 0$.

Type II matrices

Here we must consider various possibilities:

1. $a \neq 0$ or $\tilde{a} \neq 0$. Without loss of generality, we may assume that $a \neq 0$. This permits scaling B_1 so that $a = 1$, followed by replacing B_2 with $B_2 - \tilde{a}B_1$, to obtain $\tilde{a} = 0$,

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{b} & \tilde{c} \\ 0 & 0 & \tilde{b} \end{bmatrix}.$$

   Next we consider several cases.

(a) $\tilde{b} \neq 0$. Here we can first scale B_2 to obtain $\tilde{b} = 1$, and then replace B_1 by $B_1 - bB_2$ to obtain $b = 0$,

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \tilde{c} \\ 0 & 0 & 1 \end{bmatrix}.$$

i. $\tilde{c} \neq 0$. We can then scale the basis vector e_3 to obtain $\tilde{c} = 1$. Now in case $c < 0$ we flip the (e_2, e_3)-block, then exchange the vectors e_2 and e_3 and finally multiply B_2 by -1 to obtain matrices of the same form, but with $c \geq 0$. We have obtained the algebra $\mathfrak{g}^3_{0,2}$.

ii. $\tilde{c} = 0$.

A. $c \neq 0$. We scale the basis vector e_3 to obtain $c = 1$. We now have

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

which is $\mathfrak{g}^3_{0,3}$.

B. $c = 0$. We now have matrices of type I; this case has already been covered above.

(b) $\tilde{b} = 0$. We then have

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{c} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\tilde{c} \neq 0$, we can scale B_2 to arrive at $\tilde{c} = 1$, and then replace B_1

with $B_1 - cB_2$ to obtain $c = 0$.

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In case $b < 0$ we flip the (e_2, e_3)-blocks of both matrices, exchange the basis vectors e_2 and e_3 and then multiply B_2 by -1 to obtain matrices of the above form, with $b \geq 0$ always. We have thus obtained $\mathfrak{g}^3_{0,4}$.

2. $a = \tilde{a} = 0$. Thus,

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{b} & \tilde{c} \\ 0 & 0 & \tilde{b} \end{bmatrix}.$$

(a) $b \neq 0$ or $\tilde{b} \neq 0$. Without loss of generality, $b \neq 0$. Here we scale B_1 to obtain $b = 1$, and then replace B_2 with $B_2 - \tilde{b}B_1$ to obtain $\tilde{b} = 0$. Since $\tilde{c} \neq 0$, we can then scale the matrix B_2 so that $\tilde{c} = 1$.

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Replacing B_1 with $B_1 - cB_2$ we may assume that $c = 0$. This is $\mathfrak{g}^3_{0,5}$.

(b) $\tilde{b} = \tilde{b} = 0$. This is not possible as B_1 and B_2 must be linearly independent.

Type III matrices

Here we must consider many possibilities.

1. $a \neq 0$ or $\tilde{a} \neq 0$. Without loss of generality, we may assume that $a \neq 0$. This permits scaling B_1 so that $a = 1$, and then replacing B_2 with $B_2 - \tilde{a}B_1$, to

obtain $\tilde{a} = 0$,

$$B_1 = \begin{bmatrix} 1 & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & \tilde{b} & \tilde{c} \\ 0 & 0 & \tilde{d} \\ 0 & 0 & 0 \end{bmatrix},$$

with $b\tilde{d} = \tilde{b}d$. Next we consider several cases:

(a) $\tilde{b} \neq 0$. Here we first scale B_2 to obtain $\tilde{b} = 1$, and then replace B_1 with $B_1 - bB_2$ to obtain $b = 0$, The condition $b\tilde{d} = \tilde{b}d$ now gives $d = 0$.

$$B_1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 1 & \tilde{c} \\ 0 & 0 & \tilde{d} \\ 0 & 0 & 0 \end{bmatrix}.$$

i. $\tilde{d} \neq 0$. We first scale e_3 so that $\tilde{d} = 1$. Then we change basis to $f_1 = e_1$, $f_2 = \tilde{c}e_1 + e_2$, $f_3 = e_3$ which changes $\tilde{c}$ to 0, but leaves all other entries of B_1 and B_2 unchanged. We have obtained $\mathfrak{g}^3_{0,6}$.

ii. $\tilde{d} = 0$. We change basis to $f_1 = e_1$, $f_2 = e_2$, $f_3 = e_3 - \tilde{c}e_2$ which gives

$$B_1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next we flip both matrices, multiply each by -1 and exchange basis vectors e_1 and e_3 to obtain

$$B_1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

When $c = 0$ we have obtained $\mathfrak{g}^3_{0,4}$ with $a = 1$. On the other hand, when $c \neq 0$ we scale the vector e_1 so that $c = 1$, and we have obtained $\mathfrak{g}^3_{0,7}$.

(b) $\tilde{b} = 0$. The condition $b\tilde{d} = \tilde{b}d$ gives $b = 0$ or $\tilde{d} = 0$.

i. $\tilde{d} = 0$. We thus have

$$B_1 = \begin{bmatrix} 1 & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & \tilde{c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $\tilde{c} \neq 0$. We thus scale B_2 to obtain $\tilde{c} = 1$, and replace B_1 with $B_1 - cB_2$ to obtain $c = 0$.

A. $b \neq 0$. Scaling the vector e_2 we may assume that $b = 1$. That is,

$$B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that when $d = 0$, then after flipping, multiplying both matrices by one, and changing basis to vectors $f_1 = e_2$, $f_2 = e_3$ and $f_3 = e_1$, we have obtained $\mathfrak{g}^3_{0,7}$. On the other hand, when $d \neq 0$ we scale e_3 to obtain $d = 1$ (which also scales B_2), and then rescale B_2 to keep it of the above form. We have obtained $\mathfrak{g}^3_{0,8}$.

B. $b = 0$. That is,

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

When $d = 0$, after exchanging basis vectors e_1 and e_2 we have obtained $\mathfrak{g}^3_{0,4}$ with $a = 1$. On the other hand, when $d \neq 0$ we scale e_3 to obtain $d = 1$ (which also scales B_2), and then rescale

B_2 to keep it of the above form. Exchanging basis vectors e_1 and e_2 we have obtained $\mathfrak{g}^3_{0,7}$.

ii. $b = 0$ but $\tilde{d} \neq 0$. We can scale B_2 so that $\tilde{d} = 1$ and replace B_1 with $B_1 - dB_2$ to obtain $d = 0$,

$$B_1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & \tilde{c} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Changing basis $f_1 = e_1$, $f_2 = \tilde{c}_1 e_1 + e_2$, $f_3 = e_3$ we obtain matrices of the above form, with $\tilde{c} = 0$. When $c = 0$, we have obtained $\mathfrak{g}^3_{0,4}$ with $a = 1$. On the other hand, when $c \neq 0$ we scale the vector e_3 to obtain $c = 1$ and rescale the matrix B_2 to recover $\tilde{d} = 1$. That is, we have obtained $\mathfrak{g}^3_{0,7}$.

2. $a = \tilde{a} = 0$. We have

$$B_1 = \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{0} & \tilde{b} & \tilde{c} \\ 0 & \tilde{0} & \tilde{d} \\ 0 & 0 & 0 \end{bmatrix},$$

with $b\tilde{d} = \tilde{b}d$.

(a) $b \neq 0$ or $\tilde{b} \neq 0$. Without loss of generality, we may assume that $b \neq 0$. Scaling B_1 to obtain $b = 1$ and replacing B_2 with $B_2 - \tilde{b}B_1$ so that $\tilde{b} = 0$, then the condition $b\tilde{d} = \tilde{b}d$ gives $\tilde{d} = 0$. Scaling B_2 we now obtain $\tilde{c} = 1$:

$$B_1 = \begin{bmatrix} 0 & 1 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now replacing B_1 with $B_1 - cB_2$ we obtain $c = 0$.

i. $d \neq 0$. Scaling the basis vector e_3 we obtain $d = 1$. We then rescale B_2 so that it recovers the above form, and we have obtained $\mathfrak{g}^3_{0,9}$.

ii. $d = 0$. This is $\mathfrak{g}^3_{0,10}$.

(b) $b = \tilde{b} = 0$. We have

$$B_1 = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & \tilde{c} \\ 0 & 0 & \tilde{d} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since B_1 and B_2 are linearly independent, we can replace them with appropriate linear combinations to change them to form

$$B_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Flipping both matrices, i.e. replacing B_k with $-B_k^T$, followed with multiplication by -1, we obtain

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Finally, exchanging the pair of basis vectors e_1 and e_3, we arrive at $\mathfrak{g}^3_{0,10}$.

Type IV matrices

Without loss of generality, $\tilde{c} \neq 0$. Thus, we may scale B_2 so that $\tilde{c} = 1$ and then replace B_1 with $B_1 - cB_2$ to obtain $c = 0$,

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & 1 \\ 0 & -1 & \tilde{b} \end{bmatrix},$$

1. $b \neq 0$. We scale B_1 to obtain $b = 1$ and replace B_2 with $B_2 - \tilde{b}B_1$ to obtain $\tilde{b} = 0$.

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

   - When $a, \tilde{a} \geq 0$, we are done.
   - When $a, \tilde{a} \leq 0$, then we flip the e_1-block, to obtain $a, \tilde{a} > 0$ in the first entries.
   - When $a \geq 0$ but $\tilde{a} < 0$ we flip both matrices. This will render the entries of B_1 non-positive, which we remedy by scaling B_1 by -1. In addition this process replaces $\tilde{a}$ with $-\tilde{a} > 0$ while leaving the remaining entries of B_2 unchanged, so that we may assume that $a, \tilde{a} \geq 0$.
   - When $a < 0$ but $\tilde{a} \geq 0$ we first flip the e_1-block to obtain one of the previous scenarios, and continue as above.

   We have thus obtained the algebra $\mathfrak{g}^3_{0,11}$.

2. $b = 0$. Since $a \neq 0$, we can scale B_1 to $a = 1$, and then replace B_2 with $B_2 - \tilde{a}B_1$ to obtain $\tilde{a} = 0$. Finally, when $\tilde{b} < 0$ we flip the (e_2, e_3)-block to ensure $\tilde{b} \geq 0$. We have thus obtained the algebra $\mathfrak{g}^3_{0,12}$.

- **Case** $p_1 = 1$

The only linear combinations permitted which involve the matrix B_1 consist of adding a multiple of B_2 to B_1.

Type I matrices

Without loss of generality, we may first assume that $\tilde{c} \neq 0$, which allows us to scale B_2 so that $\tilde{c} = 1$. Next we add an appropriate multiple of B_2 to B_1 to obtain $c = \frac{1}{2}$.

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exchanging the basis vectors e_1 and e_2 if necessary, we may assume that $|b - \frac{1}{2}| \leq |a - \frac{1}{2}|$. Next flipping the e_1 and/or e_2-blocks, we can obtain that $\frac{1}{2} \leq b \leq a$. When $\tilde{a} < 0$ then we flip the e_3-block and multiply B_2 by -1 to obtain B_2 of the above form with $\tilde{a} > 0$, while B_1 remains unchanged. We thus have obtained $\mathfrak{g}^3_{1,1}$.

In the special case where $b = \frac{1}{2}$ we can also render $\tilde{b} \geq 0$, by simply flipping the e_2-block. In addition, when $a = b = \frac{1}{2}$ we may exchange the vectors e_1 and e_2 so that $\tilde{a} \geq \tilde{b} \geq 0$.

Type II matrices

Here we must consider various possibilities:

1. $\tilde{b} \neq 0$. Here we scale B_2 to obtain $\tilde{b} = 1$, and subtract a multiple of B_2 from B_1 to obtain $b = \frac{1}{2}$.

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{1}{2} & c \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & 1 & \tilde{c} \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Flipping the e_1-block if necessary, we may assume that $a \geq \frac{1}{2}$.

In case $\tilde{a} < 0$, then we first flip the (e_2, e_3)-blocks, to obtain

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -c & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\tilde{c} & -1 \end{bmatrix}.$$

After exchanging the vectors e_2 and e_3 we obtain

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{1}{2} & -c \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & -1 & -\tilde{c} \\ 0 & 0 & -1 \end{bmatrix}.$$

Multiplying B_2 by -1, we arrive at a pair of matrices of form (2) with $a \geq \frac{1}{2}$, $\tilde{a} \geq 0$.

(a) $c \neq 0$. We scale the vector e_3 to render $c = 1$ in (2), and have obtained $\mathfrak{g}^3_{1,2}$.

In the special where $\tilde{a} = 0$ we can render the values of $\tilde{c}$ non-negative: Flip the (e_2, e_3)-blocks, then exchange the vectors e_2 and e_3 and scale e_3 by -1. This gives

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & \tilde{c} \\ 0 & 0 & -1 \end{bmatrix}.$$

Now multiplying B_2 by -1 will return B_2 to its original form, except that $\tilde{c} < 0$ has been replaced by $-\tilde{c} > 0$.

Similarly, in the special case where $a = \frac{1}{2}$, we may also render $\tilde{c}$ non-negative: Flip both matrices, then exchange the vectors e_2 and e_3, multiply e_3 by -1 and finally multiply B_2 by -1.

(b) $c = 0$ and $\tilde{c} \neq 0$. We scale the vector e_3 to render $\tilde{c} = 1$ in (2), and have obtained $\mathfrak{g}^3_{1,3}$.

(c) $c = \tilde{c} = 0$. This is the type I case which has already been treated.

2. $\tilde{b} = 0$. We have

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & 0 & \tilde{c} \\ 0 & 0 & 0 \end{bmatrix}.$$

If necessary, flipping the (e_2, e_3)-block followed by an exchange of e_2 and e_3 we may assume that $b \geq \frac{1}{2}$.

(a) $\tilde{c} \neq 0$. Subtracting $\frac{c}{\tilde{c}} B_2$ from B_1 we obtain that $c = 0$. Flipping the e_1-block if necessary, we may assume that $a \geq \frac{1}{2}$.

i. $\tilde{a} \neq 0$. We scale B_2 so that $\tilde{a} = 1$. Scaling the vector e_3 we obtain that $\tilde{c} = 1$. We thus have obtained $\mathfrak{g}^3_{1,4}$.

ii. $\tilde{a} = 0$. We scale B_2 so that $\tilde{c} = 1$ and have obtained $\mathfrak{g}^3_{1,5}$.

(b) $\tilde{c} = 0$ and $c \neq 0$. Then $\tilde{a} \neq 0$, so we can scale B_2 so that $\tilde{a} = 1$. We subtract a multiple of B_2 from B_1 to obtain $a = \frac{1}{2}$. Finally, we scale the vector e_3 so that $c = 1$. We have obtained $\mathfrak{g}^3_{1,6}$.

(c) $\tilde{c} = c = 0$. This is the type I case, which has already been covered.

Type III matrices

As a reminder, here

$$B_1 = \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & \tilde{b} & \tilde{c} \\ 0 & \tilde{a} & \tilde{d} \\ 0 & 0 & \tilde{a} \end{bmatrix} \tag{3}$$

with $b\tilde{d} = \tilde{b}d$.

1. $\tilde{a} \neq 0$. Here we scale B_2 to obtain $\tilde{a} = 1$, and subtract an appropriate

multiple of B_2 from B_1 to obtain $a = \frac{1}{2}$:

$$B_1 = \begin{bmatrix} \frac{1}{2} & b & c \\ 0 & \frac{1}{2} & d \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & \tilde{b} & \tilde{c} \\ 0 & 1 & \tilde{d} \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) $\tilde{b} \neq 0$. We scale the basis vector e_2 so that $\tilde{b} = 1$. Now setting $f_1 = e_1$, $f_2 = e_2$ and $f_3 = e_3 - \tilde{c}e_2$ we obtain that $\tilde{c} = 0$.

i. $\tilde{d} \neq 0$. Next we scale the basis vector e_3 so that $\tilde{d} = 1$. Now the condition $b\tilde{d} = \tilde{b}d$ gives us $b = d$:

$$B_1 = \begin{bmatrix} \frac{1}{2} & b & c \\ 0 & \frac{1}{2} & b \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

When $b < 0$ we flip both matrices and exchange the vectors e_1 and e_3, followed by multiplying B_2 by -1. This will return B_1, B_2 to the above form, now with $b \geq 0$. We have obtained $\mathfrak{g}^3_{1,7}$.

ii. $\tilde{d} = 0$. The condition $b\tilde{d} = \tilde{b}d$ gives $d = 0$.

$$B_1 = \begin{bmatrix} \frac{1}{2} & b & c \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When $c = 0$ this is the type II case. We may thus suppose that $c \neq 0$. Scaling the vector e_3 we obtain $c = 1$. We have obtained $\mathfrak{g}^3_{1,8}$.

(b) $\tilde{b} = 0$. The condition $b\tilde{d} = \tilde{b}d$ gives $b = 0$ or $\tilde{d} = 0$.

i. $\tilde{d} = 0$. Then

$$B_1 = \begin{bmatrix} \frac{1}{2} & b & c \\ 0 & \frac{1}{2} & d \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & \tilde{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A. $\tilde{c} = 0$. Then $B_2 = I_3$.

– If $b = c = d = 0$, this is the type I case.

– If $b = c = 0$ but $d \neq 0$, this is the type II case.

– If $c = d = 0$ but $b \neq 0$, this is the type II case.

– If $b = d = 0$ but $c \neq 0$, this is the type II case.

We may thus assume that at most one of b, c, d equals zero, and must consider three possibilities:

– When $b \neq 0$ and $d \neq 0$, we scale the basis vectors e_2 and e_3 to obtain $b = d = 1$. We then replace e_3 with $e_3 - ce_2$ to obtain $c = 0$. We have obtained $\mathfrak{g}^3_{1,9}$.

– When $b \neq 0$ but $d = 0$, then $c \neq 0$. Scaling the basis vectors e_2 and e_3 we obtain $b = c = 1$.

$$B_1 = \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Now changing basis by $f_1 = e_1$, $f_2 = \frac{1}{2}(e_2 + e_3)$ and $f_3 = \frac{1}{2}(e_2 - e_3)$ we arrive at

$$B_1 = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

which is a type II scenario.

– When $d \neq 0$ but $b = 0$, then $c \neq 0$.

$$B_1 = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Flipping both matrices, exchanging the basis vectors e_1 and e_3, multiplying B_2 as well as e_2 and e_3 by -1, we arrive at (4) which is a type II scenario.

ii. $b = 0$, but $\tilde{d} \neq 0$.

$$B_1 = \begin{bmatrix} \frac{1}{2} & 0 & c \\ 0 & \frac{1}{2} & d \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & \tilde{c} \\ 0 & 1 & \tilde{d} \\ 0 & 0 & 1 \end{bmatrix}.$$

We flip both matrices and then exchange vectors e_1 and e_3 to obtain matrices of the form

$$B_1 = \begin{bmatrix} \frac{1}{2} & b & c \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & \tilde{b} & \tilde{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

with $\tilde{b} \neq 0$. This is the scenario $\tilde{b} \neq 0$, $\tilde{d} = 0$ already discussed above.

2. $\tilde{a} = 0$ in (3). When $a < \frac{1}{2}$ we flip both matrices and exchange basis vectors e_1 and e_3 to obtain matrices of the form (3) with $a \geq \frac{1}{2}$ and $\tilde{a} = 0$.

(a) $\tilde{b} \neq 0$ and $\tilde{d} \neq 0$. Here we can scale the vectors e_2 and e_3 to obtain $\tilde{b} = \tilde{d} = 1$. Replacing e_3 with $e_3 - \tilde{c}e_2$ we then have $\tilde{c} = 0$:

$$B_1 = \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Subtracting dB_2 from B_1 we obtain $d = 0$. The condition $b\tilde{d} = \tilde{b}d$ now gives $b = 0$.

$$B_1 = \begin{bmatrix} a & 0 & c \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

i. $c \neq 0$. We scale the basis vectors e_2 and e_3 by $\frac{1}{c}$ and then multiply B_2 by c to obtain the same matrices, but with $c = 1$. We thus have obtained $\mathfrak{g}^3_{1,10}$.

ii. $c = 0$. We have obtained $\mathfrak{g}^3_{1,11}$.

(b) $\tilde{b} = 0$ but $\tilde{d} \neq 0$. We scale e_3 to obtain $\tilde{d} = 1$.

$$B_1 = \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & \tilde{c} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Changing basis to $f_1 = e_1$, $f_2 = \tilde{c}e_1 + e_2$, $f_3 = e_3$ we obtain matrices of the same form, with $\tilde{c} = 0$. We can now subtract dB_2 from B_1 to obtain $d = 0$. The condition $b\tilde{d} = \tilde{b}d$ now gives $b = 0$.

$$B_1 = \begin{bmatrix} a & 0 & c \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

i. $c \neq 0$. Replacing the vector e_3 with $\frac{1}{c}e_3$ and then multiplying B_2 by c we arrive at matrices of the same form, with $c = 1$. We have obtained $\mathfrak{g}^3_{1,12}$, subcase $a \geq \frac{1}{2}$.

ii. $c = 0$. This is a type II scenario.

(c) $\tilde{b} \neq 0$ but $\tilde{d} = 0$. We scale e_2 to obtain $\tilde{b} = 1$.

$$B_1 = \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 1 & \tilde{c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We now flip both matrices, followed by exchanging the vectors e_1 and e_3 and multiplication of B_2 by -1 to obtain the scenario (5), but with $a \leq \frac{1}{2}$. Proceeding similarly, we either obtain $\mathfrak{g}^3_{1,12}$, subcase $a \leq \frac{1}{2}$, or a type II scenario.

(d) $\tilde{b} = \tilde{d} = 0$. Since $\tilde{c} \neq 0$ we can scale B_2 to obtain $\tilde{c} = 1$. Now subtracting cB_2 from B_2 we arrive at $c = 0$,

$$B_1 = \begin{bmatrix} a & b & 0 \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

i. $b \neq 0$ and $d \neq 0$. Scaling the vectors e_2 and e_3 and scaling B_2 with appropriate factors, we arrive at the above matrices with $b = d = 1$. We have obtained $\mathfrak{g}^3_{1,13}$.

ii. $b = 0$ but $d \neq 0$. Scaling the vector e_3 and also the matrix B_2 we obtain $d = 1$. Exchanging vectors e_1 and e_2 we arrive at $\mathfrak{g}^3_{1,12}$, subcase $a \geq \frac{1}{2}$.

iii. $b \neq 0$ but $d = 0$. Flipping the two matrices, exchanging vectors e_1 and e_3, and multiplying B_2 by -1 we arrive at the scenario $b = 0, d \neq 0$ but with $a \leq \frac{1}{2}$. This is $\mathfrak{g}^3_{1,12}$, subcase $a \leq \frac{1}{2}$.

iv. $b = d = 0$. This is a type II scenario.

Type IV matrices

As a reminder,

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & \tilde{c} \\ 0 & -\tilde{c} & \tilde{b} \end{bmatrix},$$

with either $c \neq 0$ or $\tilde{c} \neq 0$.

1. $\tilde{c} \neq 0$. We scale B_2 so that $\tilde{c} = 1$. Then we subtract cB_2 from B_1 so that $c = 0$.

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & 1 \\ 0 & -1 & \tilde{b} \end{bmatrix}.$$

Flipping the first blocks if necessary, we may always assume that $a \geq \frac{1}{2}$. Similarly, flipping the (e_2, e_3)-blocks if necessary, we may assume that $b \geq \frac{1}{2}$. When $\tilde{b} < 0$ we exchange the vectors e_2 and e_3 and then multiply B_2 by -1 to obtain matrices of the same form, with $\tilde{b} \geq 0$. This is $\mathfrak{g}^3_{1,14}$.

In the special case $a = \frac{1}{2}$ we can render $\tilde{a} \geq 0$ by flipping the e_1-block. Similarly, in the special case $b = \frac{1}{2}$ we can render $\tilde{a} \geq 0$ as follows: When $\tilde{a} < 0$, multiply B_2 by -1 and correct the change of signs of the other entries of B_2 by flipping the (e_2, e_3)-blocks and then exchanging the vectors e_2 and e_3.

2. $\tilde{c} = 0$. Then $c \neq 0$.

   (a) $\tilde{b} \neq 0$. We scale B_2 so that $\tilde{b} = 1$.

$$B_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \tilde{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Subtracting a multiple of B_2 from B_1 we obtain $b = \frac{1}{2}$. Flipping the first block if necessary, we may assume that $a \geq \frac{1}{2}$. When $\tilde{a} < 0$ we may multiply B_2 by -1 and flip the (e_2, e_3)-blocks to obtain the same type of matrices, but with $\tilde{a} \geq 0$. Finally, if $c < 0$ we exchange the vectors e_2 and e_3 to obtain that $c > 0$ always. This is $\mathfrak{g}^3_{1,15}$.

(b) $\tilde{b} = 0$. We scale B_2 so that $\tilde{a} = 1$. Subtracting a multiple of B_2 from B_1 we obtain $a = \frac{1}{2}$.

$$B_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Flipping the (e_2, e_3)-blocks if necessary, we obtain $b \geq \frac{1}{2}$. Finally, if $c < 0$ we exchange the vectors e_2 and e_3 to obtain that $c > 0$ always. This is $\mathfrak{g}^3_{1,16}$.

It is left to verify that no two Lie algebras in Table 3.1 are isomorphic. This can be done mainly by studying the eigenvalues and properties of the operator $Ad(M)$ where $M \in V_M$. Because it is a tedious process, we do not present the details here.

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- Extensions of the Heisenberg Group by Two-Parameter Groups of Dilations (with E. Schulz), **Proceedings of the Annual Pure and Applied Mathematics Conference 2013**, 147-162.