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Generalization of the Equivalence Transformations

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This report is devoted to generalization of the equivalence transformations. Let a system of differential equations be given. Almost all systems of differential equations have arbitrary elements: arbitrary functions or arbitrary constants.

The notion of an arbitrary element is related to the fact that a lot of particular problems of mathematical physics contain experimentally determined parameters or functions. And these parameters and functions play the role of arbitrary elements. For example, in gas dynamic equations such an arbitrary element is the state equation. The problem of finding these transformations is one of the stages in group classification.

The nondegenerate change of dependent and independent variables, which transfers any systems of differential equations of a given class to systems of equations of the same class is called the transformation of equivalence of the given class of equations. We shall follow the infinitesimal approach to calculation of the transformations of equivalence which was used for the first time by L.V. Ovsyannikov.

1 Arbitrary elements

The systems of equations with arbitrary elements $\phi = (\phi^1, \ldots, \phi^r)$ can be written in the following way:

$$F_k(x, u, p, \phi) = 0, (k = 1, 2, \ldots, s). \tag{1}$$

Here $x = (x_1, \ldots, x_n) \in R^n$ are independent variables, $u = (u^1, \ldots, u^m) \in R^m$ are dependent variables, p are derivatives of dependent variables u with respect to independent x until some order r. For the sake of simplicity below we consider the first-order systems $(r = 1)$. Without the loss of generality we take that in the system arbitrary elements depend only from dependent and independent variables. The system (1) is called the system with arbitrary elements and determines the class of differential equations, a concrete representative of which is determined by setting the arbitrary elements $\phi(x, u)$.

The problem of finding the equivalent transformations consists in construction of such a transformation of space $R^{n+m+r}(x, u, \phi)$ that preserves the equations only changing their representative $\phi = \phi(x, u)$. For that purpose we search for a one-parametrical group of transformations of the space R^{n+m+r}

$$x' = f^x(x, u, \phi; a), \quad u' = f^u(x, u, \phi; a), \quad \phi' = f^\phi(x, u, \phi; a) \tag{2}$$

with the group parameter a. Operators of this group have the form:

$$X^e = \xi^x\partial_x + \zeta^u\partial_u + \zeta^\phi\partial_\phi \tag{3}$$

where the coordinates are:

$$\xi^i = \xi^i(x,u,\phi), \quad \zeta^{u^j} = \zeta^{u^j}(x,u,\phi), \quad \zeta^{\phi^k} = \zeta^{\phi^k}(x,u,\phi)$$

$$(i = 1,\dots,n; \quad j = 1,\dots,m; \quad k = 1,\dots,r).$$

A note. Earlier [1] it was supposed that

$$\frac{\partial \xi^i}{\partial \phi^k} = 0, \qquad \frac{\partial \zeta^{u^i}}{\partial \phi^k} = 0 \quad (i = 1,\dots,n; \ \ j = 1,\dots,m; \ \ k = 1,\dots,r).$$

Since there are the derivatives p in (1), it is necessary to determine how they are transformed or how the coordinates of the operator $\bar{X}^e$, expended to the operator X^e, are determined.

As the functions $\phi(x,u)$ and $u(x)$ act in different spaces, first we must understand how these functions are transformed under action of the group (2), on which the following constraint is imposed. Any solution $u_0(x)$ of the system (1) with the functions $\phi(x,u)$ when acted upon by (2) transforms once more into the solution of equations of the form (1) but with another (transformed) functions $\phi_a(x,u)$ which is defined in the usual way. By solving the following relations for (x,u)

$$x' = f^x(x,u,\phi(x,u);a), \qquad u' = f^u(x,u,\phi(x,u);a),$$

we obtain

$$x = g^x(x',u';a), \qquad u = g^u(x',u';a) \tag{4}$$

after which the transformed functions $T_a(\phi)$ are determined

$$\phi_a(x',u') = f^\phi(x,u,\phi(x,u);a),$$

where instead of (x,u) we have substituted their expressions (4). The transformed solution $T_a(u) = u_a(x)$ is obtained by solving the relations

$$x' = f^x(x,u_0(x),\phi(x,u_0(x));a)$$

for $x = \psi^x(x';a)$ and substituting these solutions into

$$u_a(x') = f^u(x,u_0(x),\phi_a(x,u_0(x));a).$$

Lemma. *The transformations $T_a(u)$ constructed in this way form a group.*

Proof. Since (2) forms a one-parameter group of continuous transformations, then, by the method of construction, the equality $T_b(T_a(\phi)) = T_{a+b}(\phi)$ is satisfied. Taking this property into account, equating $T_b(T_a(u))$ and $T_{a+b}(u)$, we complete the proof of the lemma.

In agreement with the construction, the extended operator

$$\bar{X}^e = X^e + \zeta^{u_x}\partial_{u_x} + \zeta^{\phi_x}\partial_{\phi_x} + \zeta^{\phi_u}\partial_{\phi_u} + \cdots$$

has the following coordinates. The coordinates of the extended operator $\bar{X}^e$ connected with the dependent functions are defined by the formulas

$$\zeta^{u_\lambda} = D^e_\lambda\zeta^u - u_x D^e_\lambda\xi^x, \qquad D^e_\lambda = \partial_\lambda + u_\lambda\partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi.$$

Here λ takes the values x_i. The coordinates of the extended operator, connected with arbitrary elements, are defined by the formulas

$$\zeta^{\phi_\lambda} = \tilde{D}^e_\lambda \zeta^\phi - \phi_x \tilde{D}^e_\lambda \xi^x - \phi_u \tilde{D}^e_\lambda \zeta^u, \qquad \tilde{D}^e_\lambda = \partial_\lambda + \phi_\lambda \partial_\phi \quad (\lambda = u^j, x_i).$$

To construct the equivalence group (2), we need to obtain the group admissible by it with the extended operator $\bar{X}^e$ constructed above. Also here we must take into account possible, special, previously known properties of arbitrary elements (for example, $\phi_{x_i} = 0$).

Thus, the finding of equivalence group is executed with help of the usual algorithm of a finding of the admissible group of continuous transformations but with a more general kind of coordinates of the operator X^e. The assumption of dependence of all its coordinates on arbitrary elements, in general case, expand the equivalence group in comparison with the earlier used algorithm [1].

2 Examples

We can take as one of the examples of extension of the equivalence group in this approach the group for the system of two equations with two independent variables [2]

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} p(u, v) = 0, \qquad \frac{\partial v}{\partial t} = g(u, v). \tag{5}$$

If we search for the equivalence group of this system such that all the coefficients of the operator can depend on arbitrary elements, we must add one more $p\partial_u + x\partial_t$, corresponding to the transformation to the operators from [2]

$$u' = ap + u, \quad v' = v, \quad x' = x, \quad t' = t + ax, \quad g' = g, \quad p' = p.$$

Another example of extension of the equivalence group is the group of equivalence for the system of two quasilinear differential equations

$$u_y \phi_u + \phi_u(\phi^2 - 2u)u_x + v_x e^{-1/\phi_u} = 0, \qquad v_y - v_x v e^{-\phi} = 0,$$

where $\phi = \phi(u)$. If we search for the equivalence group with the infinitesimal operator $X^e = \xi^x \partial_x + \xi^y \partial_y + \zeta^u \partial_u + \zeta^v \partial_v + \zeta^\phi \partial_\phi$, so that

$$\xi^x_\phi = \xi^y_\phi = \zeta^u_\phi = \zeta^v_\phi = 0,$$

then the group is three-parametric with the operators

$$x\partial_x + y\partial_y, \quad \partial_x, \quad \partial_y.$$

Using the approach developed here, when the dependence of all coordinates of the infinitesimal operator from the arbitrary element ϕ is proposed, one more operator is added:

$$\phi\partial_u + v\partial_v + \partial_\phi.$$

3 x-Autonomy of a quasilinear system of equations

A Lie-group with infinitesimal operators having a form

$$X = \xi^i(x,u,\phi)\partial_{x_i} + \zeta^k(x,u,\phi)\partial_{u^k} + \zeta^{\phi^j}(x,u,\phi)\partial_{\phi^j}$$

is called x-autonomous if all coordinates ξ^i do not depend from variables u, ϕ. While calculating groups of equivalence of differential equations as well as basic admissible groups, the necessity in finding out whether a group is x-autonomous arises.

The practice shows that many systems of differential equations have this property. In a general case getting similar results requires an investigation for compatibility of determining equations for coordinates of the infinitesimal operator.

In [3] for the systems of quasilinear differential equations

$$\frac{\partial u^k}{\partial t} = a^{\alpha k}_{\beta}\frac{\partial u^\beta}{\partial x_\alpha} + v^k \quad (k = 1,\ldots,m, \quad m > 1, \quad n > 0), \tag{6}$$

sufficient conditions for x-autonomy of admissible Lie-groups were obtained on the basis of analysis of determining 2-equations. A similar result takes place in finding an equivalence group.

Really, let us take

$$a^{ik}_l = a^{ik}_l(x,u,\phi), \qquad v^k = v^k(x,u,\phi).$$

The determining equations for coordinates of infinitesimal operators of the equivalence group (as well as for basic groups admissible by (6) [3]) have the structure of inhomogeneous square forms from parametric highest-order derivatives $\partial u^i/\partial x_k$ and hence are split into three subsystems of equations: square (2-equations), linear (1-equations) and "zero" (0-equations) in relation to these derivatives.

The 2-equations are the following

$$\begin{gathered} a^{ik}_q a^{js}_p h_s + a^{jk}_p a^{is}_q h_s - a^{jk}_s a^{is}_q h_p - a^{ik}_s a^{js}_p h_q + \\ \delta^k_q a^{js}_p b^i_s + \delta^k_p a^{is}_q b^j_s - a^{jk}_q b^i_p - a^{ik}_p b^j_q = 0 \end{gathered} \tag{7}$$

$$(i,j = 1,\ldots,n; k,p,q = 1,\ldots,m).$$

Here

$$b^\gamma_\lambda = \frac{\partial \xi^\gamma}{\partial u^\lambda} + \phi^\mu_{u^\lambda}\frac{\partial \xi^\gamma}{\partial \phi^\mu}, \qquad h_\lambda = \frac{\partial \xi^0}{\partial u^\lambda} + \phi^\mu_{u^\lambda}\frac{\partial \xi^0}{\partial \phi^\mu}.$$

The 1-equations are the following

$$\begin{gathered} a^{\alpha\lambda}_\beta\Big(\frac{\partial \zeta^{u^k}}{\partial u^\lambda} + \phi^\mu_{u^\lambda}\frac{\partial \zeta^{u^k}}{\partial \phi^\mu}\Big) - a^{\alpha k}_\gamma\Big(\frac{\partial \zeta^{u^\gamma}}{\partial u^\beta} + \phi^\mu_{u^\beta}\frac{\partial \zeta^{u^\gamma}}{\partial \phi^\mu}\Big) + \\ a^{\gamma k}_\beta\Big(\frac{\partial \xi^\alpha}{\partial x_\gamma} + \phi^\mu_{x_\gamma}\frac{\partial \xi^\alpha}{\partial \phi^\mu}\Big) + a^{\gamma k}_\nu\Big(\frac{\partial \xi^0}{\partial x_\gamma} + \phi^\mu_{x_\gamma}\frac{\partial \xi^0}{\partial \phi^\mu}\Big)a^{\alpha\nu}_\beta - \\ a^{\alpha k}_\beta\Big(\frac{\partial \xi^0}{\partial t} + \phi^\mu_t\frac{\partial \xi^0}{\partial \phi^\mu}\Big) - \delta^k_\beta\Big(\frac{\partial \xi^\alpha}{\partial t} + \phi^\mu_t\frac{\partial \xi^\alpha}{\partial \phi^\mu}\Big) - \\ \delta^k_\beta v^\lambda b^\alpha_\lambda - a^{\alpha k}_\beta v^\lambda h_\lambda - v^k a^{\alpha\lambda}_\beta h_\lambda + a^{\alpha k}_\gamma v^\gamma h_\beta - X(a^{\alpha k}_\beta) = 0. \end{gathered} \tag{8}$$

The equations (7) compose the system of linear homogeneous equations with respect to $m(n+1)$ unknowns h_p, b_p^i. It coincides with the system studied in the [3] while investigating x-autonomy. Hence, if it has a nonzero solution, the coefficients a_l^{ik} generate a specific algebraic structure, the description of which is given in [3]. For example, if the system (7) has a nonzero solution, each of the matrices A^i with elements a_l^{ik} satisfies the square equations

$$(A^i)^2 = c_i A^i + d_i I \tag{9}$$

with scalar coefficients c_i, d_i.

Let just only one of the conditions (9) not be fulfilled. Then from (7) we get

$$h_\lambda = 0, \quad b_\lambda^\gamma = 0 \quad (\lambda = 1, \ldots, m; \quad \gamma = 1, \ldots, n). \tag{10}$$

In the equations (7), (8) it is also necessary to take into account additional constraints on the arbitrary elements (if they exist).

If there are no constraints on the derivatives from arbitrary elements $\partial\phi^\mu/\partial u^\lambda$ (that is they are parametric in finding the equivalence transformations), then after the splitting of equations (10) with respect to them the x-autonomy of equivalence group is got. And also from the (8) we have

$$\frac{\partial \zeta^{u^k}}{\partial \phi^\mu} = 0, \qquad \frac{\partial \xi^i}{\partial \phi^\mu} = 0 \quad (i = 1, \ldots, n; \quad k = 1, \ldots, m; \quad \mu = 1, \ldots, r).$$

It means in this case there is no expansion of the group of equivalence transformations in comparison with the earlier used way [1].

In mathematical modelling the arbitrary elements ϕ^μ are supposed to be independent from independent variables $\phi = \phi(u)$. This leads to the additional determining equations

$$\frac{\partial \zeta^{\phi^\mu}}{\partial x_\gamma} - \phi^\mu_{u^\beta} \frac{\partial \zeta^{u^\beta}}{\partial x_\gamma} = 0.$$

From these relations we also obtain the (u, ϕ)-autonomy and u-autonomy of the group of equivalence.

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