Reduction Procedure and Generalized Simple Waves for Systems Written in the Riemann Variables.

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Abstract

In the manuscript the method of differential constraints is applied for systems written in the Riemann variables. We studied generalized simple waves. This class of solutions can be obtained by integrating a system of ordinary differential equations. Two models from continuum mechanics are studied: traffic flow and rate-type models.

1 Introduction

The method of differential constraints is one of the methods for constructing particular exact solutions of partial differential equations. The idea of the method was proposed by N.N.Yanenko [1]. A survey of the method can be found in the book [2]. The method is based on the following idea.

Consider a system of differential equations

$$S_i(x,u,p) = 0, \quad (i = 1,2,\ldots,s). \tag{1}$$

Here $x = (x_1, x_2, \ldots, x_n)$ are the independent variables, $u = (u^1, u^2, \ldots, u^m)$ are the dependent variables, $p = (p^j_\alpha)$ is the set of the derivatives $p^j_\alpha = \dfrac{\partial^{|}\alpha|u^j}{\partial x^\alpha}$, $(j = 1,2,\ldots,m; \|\alpha| \leq q)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Assume that a solution of system (1) satisfies the additional differential equations

$$\Phi_k(x,u,p) = 0, \quad (k = 1,2,\ldots,q). \tag{2}$$

The differential equations (2) are called differential constraints. A solution of system (1) satisfying (2) is called the solution characterized by the differential constraints (2).

The obtained system (1), (2) is an overdetermined system. The method of differential constraints requires for the overdetermined system (1), (2) to be compatible. The form of the differential constraints (the functions Φ_k) and a part of equations of the given system (the functions S_i) may not be known a priori.

The application of the method of differential constraints involves two stages. The first stage is to find the set of differential constraints (2) under which the overdetermined system is compatible. On this stage in the process of compatibility analysis (reducing the system to involutive) the overdetermined system (1), (2) can be supplemented by new equations. The second stage of the method is to construct solutions of the involutive overdetermined system. Because the solution has to satisfy the differential constraints, then it allows easier constructing particular solution of the given system (1).

The requirement of compatibility of system (1), (2) is very general. Therefore the method of differential constraints includes (almost) all known methods for constructing exact solutions of partial differential equations: group–invariant solutions, nonclassical and weak symmetries, partially invariant solutions, separation of variables, as well as many others.

Increasing the number of requirements on the differential constraints narrows the generality of the method and makes it more suitable for finding exact particular solutions. Shapeev [3] suggested that requiring the involutiveness of the overdetermined system (1), (2) would be one way in this direction. The classification of differential constraints and solutions characterized by them is carried out with respect to the functional arbitrariness of solutions of the overdetermined system (1), (2) and the order of highest derivatives, included in the differential constraints (2). With this refinement the method of differential constraints becomes a practical tool for obtaining exact particular solutions.

However the requirement of involutiveness for the overdetermined system involves a large arbitrariness in choice of differential constraints. Kaptsov [4] proposed an additional restriction: the conditions of involutiveness for the overdetermined system must coincide with the determining equations of the one–parameter group admitted by system (1).

The main object of our study in the manuscript is a system of quasilinear equations

$$\left\{ \begin{array}{l} u_t + a_{11}(u,v)u_x + a_{12}(u,v)v_x = f_1(u,v), \\ v_t + a_{21}(u,v)u_x + a_{22}(u,v)v_x = f_2(u,v), \end{array} \right. \tag{3}$$

with two independent variables (x,t) and two dependent variables (u,v). We assume that system (3) is strictly hyperbolic. Hence, the matrix A with entries a_{ij} has two distinctive eigenvalues λ and μ, $(\lambda \neq \mu)$. There are two linear independent left eigenvectors $l^{(\lambda)} = (l_{11}, l_{12})$, $l^{(\mu)} = (l_{21}, l_{22})$ such that the differential forms $l_{i1}du + l_{i2}dv$ are exact. This allows introducing the Riemann variables

$$\left(\begin{array}{c} dr \\ ds \end{array} \right) = L \left(\begin{array}{c} du \\ dv \end{array} \right),$$

where $L = (l_{ij})$. System (3) is rewritten in the Riemann variables

$$\left\{ \begin{array}{l} r_t + \lambda(r,s)r_x = f(r,s), \\ s_t + \mu(r,s)s_x = g(r,s), \end{array} \right. \tag{4}$$

where $f(r,s) = l_{11}f_1 + l_{12}f_2$, $g(r,s) = l_{21}f_1 + l_{22}f_2$. The derivatives with respect to r and s are rewritten as follows

$$\left\{ \begin{array}{l} \partial_r = \Delta^{-1}\left(l_{22}\partial_u - l_{21}\partial_v\right), \\ \partial_s = \Delta^{-1}\left(-l_{12}\partial_u + l_{11}\partial_v\right) \end{array} \right.$$

with $\Delta = \det(L)$.

The manuscript is devoted to study one class of solutions, which we call generalized simple waves. The main feature of these solutions is that these solutions are constructed by integrating a system of ordinary differential equations.

First we introduce some knowledge from the method of differential constraints that are necessary for our applications. The second part of the article is devoted to study generalized simple waves for systems of the form (3) written in Riemann variables. In the last part we apply the results of the previous parts to some models of interest in continuum mechanics.

2 Method of differential constraints

In this manuscript the method of differential constraints is applied to quasilinear systems of partial differential equations

$$\frac{\partial u}{\partial t} + Q\frac{\partial u}{\partial x} - f = 0. \tag{5}$$

Where $Q = Q(x, t, u)$ is a $m \times m$ matrix, $f = f(x, t, u)$ is a vector, E_r is a $r \times r$ unity matrix, $x_1 = x,\ x_2 = t$. For the simplicity, solutions characterized by the first order differential constraints

$$\Phi_k(x, t, u, u_x) = 0, \quad (k = 1, 2, \ldots, q). \tag{6}$$

are studied. It is assumed that the differential constraints satisfy the natural requirement

$$rank\left(\frac{\partial \Phi_k}{\partial u_x}\right) = q.$$

Remark 1. The study of differential constraints of higher order of the system (S) can be reduced to the study of differential constraints of the first order for the prolonged system.

2.1 Involutive conditions

Without loss of generality one can rewrite the system of differential equations and the differential constraints in the more suitable form

$$S \equiv Lu_t + ALu_x - Lf = 0, \tag{7}$$

$$\Phi = B_1 L u_x + \Psi = 0. \tag{8}$$

Here $L = L(x, t, u)$ is a nonsingular $m \times m$ matrix, $A = LQL^{-1}$, the function $\Psi = \Psi(x, t, u, y)$ depends on x, t, u and $y = B_2 L u_x$, B_1 and B_2 are rectangular $q \times m$ and $(q-1) \times m$ matrices with the elements

$$(B_1)_{ij} = \delta_{ij},\ (1 \le i \le q,\ 1 \le j \le m),$$
$$(B_2)_{kj} = \delta_{q+k,j},\ (1 \le k \le m-q,\ 1 \le j \le m),$$

δ_{ij} is the Kronecker's symbol. The matrices B_1 and B_2 have the following properties:

$$\begin{gathered} B_1 B_1' = E_q,\ B_2 B_2' = E_{m-q},\ B_1' B_1 + B_2' B_2 = E_m, \\ B_1 B_2' = 0,\ B_2 B_1' = 0. \end{gathered} \tag{9}$$

Note that if the matrix A is a diagonal matrix, then the matrix $B_j A B_j$ is diagonal and $B_i A B_j = 0$ $(i, j = 1, 2; i \ne j)$. For a hyperbolic system (5) the matrix A can be chosen diagonal.

Theorem 1. [5] Overdetermined system (7), (8) is involutive if and only if

$$(D_t \Phi + Z A B_1' D_x \Phi - Z D_x S)_{|(S\Phi)} = 0, \tag{10}$$

$$ZA - ZAB_1' Z = 0, \tag{11}$$

where $Z = B_1 + \Psi_y B_2$ and $(S\Phi)$ means the manifold

$$(S\Phi) \equiv \{(x, u, p) | S(x, u, p) = 0, \Phi(x, u, p) = 0\}.$$

Remark 2. Equations (11) mean that the symbol of the overdetermined system is involutive. In applications equations (11) are checked first, although they are contained in (10). Equations (11) mean that there are no new equations after prolongation the system.

Remark 3. Equations (11) are equivalent to

$$B_1 A B_2' - \Psi_y B_2 A B_1' \Psi_y + \Psi_y B_2 A B_2' - B_1 A B_1' \Psi_y = 0.$$

If the matrix A is a diagonal matrix with the diagonal entries λ_i $(i = 1, 2, \dots, m)$, then $B_1 A B_2' = 0$, $B_2 A B_1' = 0$, the matrices $B_1 A B_1'$, $B_2 A B_2'$ are diagonal and equations (11) become

$$(\lambda_i - \lambda_j)(\Psi_i)_{y_j} = 0,\ (i = 1, 2, ..., q;\ j = 1, 2, ..., m-q).$$

This means that Ψ_i can only depend on y_j such that $(\lambda_i - \lambda_j) = 0$. In particular, in the case of strictly hyperbolic systems $(\lambda_i - \lambda_j) \neq 0$ $(i \neq j)$, and equations (11) are reduced to [6]

$$\Psi_y = 0.$$

The last equations mean that for strictly hyperbolic systems the differential constraints are quasilinear.

If system (7), (8) is analytic, then its involutiveness provides an uniqueness and existence of the Cauchy problem. There are more weak requirements on the smoothness of system (7), (8) that are sufficient for the uniqueness and existence of the Cauchy problem. First proof for systems of the class C^2 was done in [7].

Assume that

$$L \in C^1(D),\ A \in C^1(D),\ f \in C^1(D),\ \Psi \in C^1(D) \tag{12}$$

in open domain $D \subset R^m \times R^2$.

Theorem 2. [5]. Let system (7) be a hyperbolic system with (12), and let equations (10), (11) be satisfied. Then there exists an unique solution $u(x,t) \in C^1$ of the Cauchy problem for system (7), (8) with the initial data $u(x,0) = \varphi(x) \in C^1$ satisfying the differential constraints (8) at $t = 0$.

The proof of the theorem is based on the theorem of existence for hyperbolic systems [8] that each coordinate of the vector–function $\mathcal{P}(x,t) = L(u(x,t),x,t)u_x(x,t)$ is continuously differentiable along its characteristic curve. The differential constraints in the strength of involutive conditions (10), (11) satisfies a linear homogeneous system of equations. By virtue of uniqueness of the solution of this system the proof is obtained.

There are also valid similar statements for other types of systems [5].

2.2 Generalized simple waves

One class of solutions, which generalizes class of simple waves is studied here. Assume that a system of quasilinear differential equations (S) admits $q = m - 1$ quasilinear differential constraints

$$\Phi = B_1 L u_x + \Psi_y B_2 L u_x + \phi = 0,$$

where $\phi = \phi(u,x,t)$ and $\Psi_y = \Psi_y(u,x,t)$ is a $(m-1) \times m$ matrix, $\lambda = B_2 A B_2'$, $y = B_2 L u_x$. Also assume that $B_2 A B_1' = 0$ and

$$L \in C^1(D),\ A \in C^1(D),\ f \in C^1(D),\ \Psi_y \in C^1(D),\ \phi \in C^1(D) \tag{13}$$

A solution satisfying these differential constraints we call a generalized simple wave[1]. We call it by generalized simple wave, because this class of solutions has similar properties as simple waves.

The conditions for the system $(S\Phi)$ to be involutive (11) are

$$\lambda\Psi_y = -B_1 A(B_2' - B_1'\Psi_y), \tag{14}$$

or it can be rewritten as

$$A(B_2' - B_1'\Psi_y) = \lambda(B_2' - B_1'\Psi_y). \tag{15}$$

Equations (10) become

$$\Omega_1 y^2 + \Omega_2 y + \Omega_3 = 0,$$

where $\Omega_1, \Omega_2, \Omega_3$ are functions, which depend on L, Ψ_y, A, ϕ and their derivatives [10]. Note also that in the strength of involutive conditions (14) the first function $\Omega_1 \equiv 0$. Because Ω_2 and Ω_3 do not depend on y, then the conditions of involutiveness require

$$\Omega_2 = 0,\ \Omega_3 = 0. \tag{16}$$

By virtue of the differential constraints and condition (15) for these solutions one can define all derivatives along the characteristic $\frac{dx}{dt} = \lambda$:

$$\begin{array}{c} B_1 L\frac{du}{dt} = B_1(A - \lambda E_m)B_1'\phi + B_1 Lf, \\ B_2 L\frac{du}{dt} = B_2 Lf,\ \frac{dx}{dt} = \lambda. \end{array} \tag{17}$$

Let $u_0(\xi) \in C^1$ satisfies the differential constraints

$$(B_1 + \Psi_y(u_0(\xi), \xi, 0)B_2)L(u_0(\xi), \xi, 0)u_0'(\xi) + \phi(u_0(\xi), \xi, 0) = 0.$$

There exists the unique solution $(v(\xi, t), x(\xi, t))$ of the Cauchy problem of the system of ordinary differential equations (17) with the initial data at $t = 0$:

$$v = u_0(\xi),\ x = a.$$

The dependence $x = x(\xi, t)$ can be solved with respect to $\xi = \xi(x, t)$ in some neighborhood of the point $(x_0, 0)$. We show that $u(x, t) = v(\xi(x, t), t)$ is a

[1]There is generalization of such class solutions for systems with more than two independent variables[9].

solution of the overdetermined system $(S\Phi)$. Exchanging the variables (x,t) onto (ξ,t) we have

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}+\lambda\frac{\partial}{\partial t},\ \frac{\partial}{\partial \xi}=x_\xi\frac{\partial}{\partial x},$$

where $x_\xi=\frac{\partial x}{\partial \xi}$. The left hand side of the differential constraints in the new independent variables is

$$B_1Lu_x+\Psi_yB_2Lu_x+\phi=\frac{1}{x_\xi}(B_1Lv_\xi+\Psi_yB_2Lv_\xi+x_\xi\phi).$$

Let $H=(B_1+\Psi_yB_2)Lv_\xi+x_\xi\phi$. By using the conditions of involutiveness (16), we obtain that $H=H(\xi,t)$ satisfies the linear differential equations

$$\frac{dH}{dt}=GH,$$

where $G=G(\xi,t)$ is some matrix, $\frac{d}{dt}$ is the partial derivative $\frac{\partial}{\partial t}$ in the variables (ξ,t). Because the initial values $H(\xi,0)=0$ and by virtue of the uniqueness of solution of the Cauchy problem of system of ordinary differential equations

$$H(\xi,t)=0.$$

It means that the differential constraints are satisfied. Rewriting equations (17) in the coordinates (x,t) one finds that

$$B_1L(u_t+\lambda u_x)=B_1(A-\lambda E_m)B_1'\phi+B_1Lf,$$
$$B_2L(u_t+\lambda u_x)=B_2Lf.$$

Substitution $\phi=-(B_1+\Psi_yB_2)Lu_x$ into the last equations gives

$$B_1(Lu_t+ALu_x)=B_1Lf,$$
$$B_2(Lu_t+ALu_x)=B_2Lf.$$

Here it is used that $\lambda=B_2AB_2'$ and conditions (15).

Therefore, for constructing a generalized simple wave one needs to satisfy the differential constraints on some curve $x_0(t)$ which is not the characteristic $x_0'\neq\lambda$, then the solution can be found by integrating the system of ordinary differential equations (17).

By the same way one can construct a solution of a problem with the initial data on a characteristic curve of the overdetermined system $(S\Phi)$ and with a singularity of the rarefaction wave type at the point $(0,0)$ [10]. In fact there exists an unique solution of the system (S) in some domain $V \in R^2$ that satisfies the following conditions.

1. On the characteristic curve $\Pi : x = x_0(t)$ the value $u(x_0(t), t) = u_\lambda(t)$ satisfy (17).

2. The point $(0,0) \in \Pi \subset V$ is singular: the solution is multiply defined at this point. The value $u = u_0(a)$ of the solution at this point depends on the parameter a, $(u_0(0) = u_\lambda(0))$ and it defines the curve in the space R^m satisfying the equations

$$(B_1 + \Psi_y(u_0(a), 0, 0)B_2)L(u_0(a), 0, 0)u_0'(a) = 0,$$

$$\frac{\partial \lambda}{\partial u}(u_0(a), 0, 0)u_0'(a) < 0, \ (0 \le a \le a_0). \tag{18}$$

Here the parameter a plays role of the variable ξ at the point of singularity $(0,0)$.

The solution of this problem generalizes the well-known rarefaction wave in gas dynamics.

3 Systems in Riemann variables

In this section we study system (3) written in the Riemann variables (4)

$$\begin{cases} r_t + \lambda(r,s)r_x = f(r,s), \\ s_t + \mu(r,s)s_x = g(r,s), \end{cases} \tag{19}$$

Equations (19) can be considered up to the equivalence transformation

$$\tilde{r} = h(r), \quad \tilde{s} = q(s).$$

Following to the analysis presented in section 2 a generalized simple wave of system (19) is described by the differential constraint

$$r_x = p(r,s). \tag{20}$$

appended to (19)[2]. In the present case the consistency conditions (10), (11) are specialized to

$$p_s(\lambda - \mu) = f_s - p\lambda_s, \tag{21}$$

$$p_r f = p f_r - p^2 \lambda_r - g p_s. \tag{22}$$

[2]We restrict our consideration by autonomous differential constraint.

Once the function $p(r,s)$ has been determined from (21) and (22), then generalized simple wave solutions of (19) and (20) can be obtained by integrating the system of ordinary differential equations (17):

$$\frac{dx}{dt} = \mu, \ \frac{dr}{dt} = f - p(\lambda - \mu), \ \frac{ds}{dt} = g. \tag{23}$$

This allows finding a solution of system (19), (20) by integrating system of ordinary differential equations (23) along the characteristic curves corresponding to the eigenvalue $\mu(r,s)$.

3.1 The case $f = 0$

First let us consider the case $f = 0$. Equations (21), (22) accept the form

$$p_s = -p(\lambda - \mu)^{-1}\lambda_s, \tag{24}$$
$$p(\lambda_r p - g(\lambda - \mu)^{-1}\lambda_s) = 0. \tag{25}$$

Relations (24), (25) are identically satisfied if $p = 0$, which corresponds to the well-known Riemann wave.

Assume that $p \neq 0$. From (25) we get

$$\lambda_r p = g(\lambda - \mu)^{-1}\lambda_s. \tag{26}$$

There are two cases depending on $\lambda_r \neq 0$ or $\lambda_r = 0$.

In the first case $\lambda_r \neq 0$, from (26) we obtain

$$p = g(\lambda - \mu)^{-1}\lambda_s\lambda_r^{-1} \tag{27}$$

and, in turn, from (24) the following relation involving the functions $\lambda(r,s)$, $\mu(r,s)$, $g(r,s)$ is derived

$$\left(g\lambda_s\lambda_r^{-1}\right)_s = -g\lambda_s\lambda_r^{-1}\frac{\mu_s}{\lambda - \mu} \tag{28}$$

Once (28) is satisfied, taking into account (27), exact solutions of (19), (20) are obtained by solving the system of ordinary differential equations

$$\frac{dx}{dt} = \mu(r,s), \ \frac{ds}{dt} = g(r,s), \ \frac{d\lambda}{dt} = 0 \tag{29}$$

Let us consider the second case $\lambda_r = 0$. Equation (26) becomes

$$g\lambda_s = 0. \tag{30}$$

The case $\lambda_s = 0$ corresponds to a travelling wave solution for the function r (i.e. $r = h(x - \lambda t)$. If $\lambda_s \neq 0$, then without loss of generality we can assume $\lambda = s$. Hence, $g = 0$, while the function $p = p(r, s)$ is obtained by solving the equation

$$p_s + p(\lambda - \mu)^{-1} = 0. \tag{31}$$

3.2 The case $f \neq 0$

For $f \neq 0$ we can append to system (23) the equation for the function p:

$$\frac{dp}{dt} = p\left(f_r - gf_sf^{-1}\right) + p^2\left(g\lambda_s - \lambda_r - (\lambda - \mu)f_rf^{-1}\right) + p^3\lambda_r(\lambda - \mu)f^{-1}. \tag{32}$$

After appending this equation, system (23), (32) is a closed system. Therefore, if a solution at the initial time t_0 satisfies the differential constraint (20), then it can be found by integrating the system of ordinary differential equations (23), (32).

By taking mixed derivatives of the function p, the consistency of the pair of equations (21), (22) yields

$$H_2p^2 + H_1p + H_0 = 0. \tag{33}$$

where

$$H_2 = (\lambda - \mu)(f_s\lambda_r - f\lambda_{rs}) + \lambda_r\lambda_s.$$

$$H_1 = (\lambda - \mu)\left(f_{rs}f - f_rf_s\right) - f_s(2\lambda_rf + \lambda_sg) +$$
$$+ f\left(g_s\lambda_s + \lambda_{rs}f + \lambda_{ss}g + \lambda_s(-\lambda_rf - \lambda_sg + \mu_rf + \mu_sg)(\lambda - \mu)^{-1}\right),$$

$$H_0 = -f\left(f_{rs}f - f_rf_s\right) - f_{ss}fg + f_s^2g - f_sg_sf +$$
$$+ f_sf(\lambda_rf + \lambda_sg - \mu_rf - \mu_sg)(\lambda - \mu)^{-1}.$$

If $H_2^2 + H_1^2 \neq 0$, then from equation (33) one can find p and after substituting it into (21), (22) the conditions for existence of a generalized simple wave are found. These conditions are equations on the functions $\lambda(r, s), \mu(r, s), f(r, s)$ and $g(r, s)$. The expressions are very cumbersome, but for the known functions these conditions can be checked easily by using computer systems for symbolic calculations.

If $H_2 = H_1 = 0$, then equation (33) is satisfied identically and the conditions for existence of a generalized simple wave are the relations

$$H_2 = 0,\ H_1 = 0,\ H_0 = 0.$$

Hereafter we are concerned with systems of the last type.

First we assume that $\lambda_s = 0$ so that equation $H_2 = 0$ gives $f_s \lambda_r = 0$. Taking into account (21), (22), the case $f_s = 0$ corresponds to travelling wave solution for the function $r = r(x - Dt)$, (D is a constant). If $\lambda_r = 0$, then without loss of generality one can assume that $\lambda = 0$, and the equations $H_1 = 0, H_0 = 0$ give

$$f = f_1(r) f_2(s) \tag{34}$$

$$g(r,s) = \frac{f_2(s)}{\mu(r,s) f_2'}(g_0(r) + \hat{h}(r,s) f_1(r)) \tag{35}$$

where $\hat{h}_s = -\mu_r f_2'$. In (34), (35) by virtue of an equivalence transformation with $q' = f_1^{-1}(r)$ one can assume $f_1 = 1$.

Assume that $\lambda_s \neq 0$ and let us study the case $\lambda_r \neq 0$. By determining g_s and f_s, respectively, from the equations $H_1 = 0$ and $H_2 = 0$ we are leading to

$$a_1 a_2 f + a_3 g = 0, \tag{36}$$

where

$$a_1 = \lambda_{rs}(\lambda - \mu) - 2\lambda_r \lambda_s,\ a_2 = (\lambda_{rrs}\lambda_r - \lambda_{rs}\lambda_{rr})(\lambda - \mu)^2 - \lambda_r^2 (2\lambda_{rs}(\lambda - \mu) - \lambda_s(2\lambda_r - \mu_r)),$$
$$a_3 = -\lambda_r^2 (\lambda_{rss}\lambda_s - \lambda_{rs}\lambda_{ss})(\lambda - \mu)^2 + \lambda_r^2 \lambda_s^2 (2\lambda_{rs}(\lambda - \mu) + \lambda_r(-2\lambda_s + \mu_s)).$$

If $a_1 a_2 \neq 0$, then equation (36) gives $f = \alpha g$, where $\alpha = -a_3(a_1 a_2)^{-1}$. After substituting it into representation for the derivative f_s there is the equation

$$\left(\ln(\alpha \lambda_s^{-1})\right)_s = (\lambda - \mu)\alpha\left(\psi_r + \psi^2 \lambda_r\right) + (\mu_r \alpha + \mu_s - \lambda_s)(\lambda - \mu)^{-1}, \tag{37}$$

where $\psi \equiv \lambda_{rs} - \lambda_r \lambda_s (\lambda - \mu)^{-1}$. Hence, in this case the functions λ and μ satisfy equation (37), $f = \alpha g$ and the function $g(r,s)$ is found by integrating the equation $H_1 = 0$.

If $a_1 a_2 = 0$, then equation (36) is satisfied if $a_1 = 0,\ a_3 = 0$ or $a_2 = 0,\ a_3 = 0$.

Let us consider first the case $a_1 = 0,\ a_3 = 0$, so that we get

$$\begin{array}{c} \lambda_{rs} - 2\frac{\lambda_r\lambda_s}{\lambda-\mu},\ \mu_s = 0,\ f_s = f\frac{\lambda_s}{\lambda-\mu}, \\ g_s = -2f\frac{\mu_r}{\lambda-\mu} - g\frac{\lambda_{ss}(\lambda-\mu)-2\lambda_s^2}{\lambda_s(\lambda-\mu)}, \end{array} \tag{38}$$

which can be integrated

$$\begin{array}{c} \lambda_r = \lambda_1(r)(\lambda-\mu)^2,\ \mu = \mu(r),\ f = f_1(r)(\lambda-\mu), \\ g = \lambda_s^{-1}(\lambda-\mu)\left(g_1(r)(\lambda-\mu) + 2\mu' f_1(r)\right) \end{array}$$

with arbitrary functions $f_1(r),\ \mu(r),\ \lambda_1(r)$ and $g_1(r)$. By solving equations (21), (22), the following form of $p(r,s)$ is obtained:

$$p = (p_1 + \lambda f_1)(\lambda-\mu)^{-1}, \tag{39}$$

where the functions $p_1(r),\ f_1(r),\ \mu(r),\ \lambda_1(r)$ and $g_1(r)$ are related by the equation

$$\frac{d}{dr}\left(p_1 f_1^{-1} + 2\mu\right) + \lambda_1(p_1 f_1^{-1} + \mu)^2 - g_1 f_1^{-1}(p_1 f_1^{-1} + \mu) = 0.$$

We note that by using an equivalence transformation one can assume $f_1 = 1$.

Assume $a_2 = 0,\ a_3 = 0$, which can be reduced to

$$\psi_r = -\lambda_r\psi^2,\ \psi_s = -\lambda_s\psi^2, \tag{40}$$

where

$$\psi \equiv \lambda_{rs} - \lambda_r\lambda_s(\lambda-\mu)^{-1}. \tag{41}$$

A trivial solution of (40) is $\psi = 0$ which is tantamount to require λ and μ satisfy the equation:

$$\lambda_{rs} - \lambda_r\lambda_s(\lambda-\mu)^{-1} = 0. \tag{42}$$

In this case from $H_1 = 0,\ H_2 = 0$ we obtain

$$f = f(r),\ g = h\frac{\lambda-\mu}{\lambda_s}, \tag{43}$$

where the function $h(r,s)$ satisfies the equation

$$h_s = -f\frac{\lambda_s\mu_r}{(\lambda-\mu)^2}.$$

Also in this case by means of an equivalence transformation one can assume $f = 1$.

If $\psi \neq 0$, then integration of (40) gives

$$\psi = (\lambda + c)^{-1}$$

with an arbitrary constant c, while equations $H_1 = 0$, $H_2 = 0$ lead to

$$f = f_1(r)(\lambda + c), \; g_s = -\lambda_s^{-1}\lambda_{ss} g + \lambda_s g \frac{2\lambda - \mu + c}{(\lambda + c)(\lambda - \mu)} - \frac{g\mu_s + \mu_r f_1(\lambda + c)}{\lambda - \mu},$$

where by virtue of equivalence transformation we can assume $f_1 = 1$.

We note that by changing $r = \tilde{r} + x + ct$ the first equation of system (19) becomes

$$\tilde{r}_t + \lambda \tilde{r}_x = 0,$$

while the differential constraint (20) accepts the form

$$\tilde{r}_x = p - 1.$$

In the present case, taking into account (21), (22), one of possible forms for the function p involved into the differential constraint (20) is $p = 1$.

Finally we consider the case $\lambda_s \neq 0$ and $\lambda_r = 0$, so that equation $H_2 = 0$ is satisfied identically. Because $\lambda_s \neq 0$ without loss of generality one can assume that $\lambda = s$. From the equation $H_1 = 0$ the derivative g_s is found. Substituting it into $H_0 = 0$ one obtains the equation

$$f_{ss} f g + ((\mu - s) f_s + f)(f_{rs} f - f_r f_s) = 0. \tag{44}$$

If $f_{ss} \neq 0$, then from (44) one can find g and after substituting it into the representation for the derivative g_s one obtains the equation for the functions $f(r,s), \mu(r,s)$ (we omit this equation because of its cumbersome expression).

If $f_{ss} = 0$, then $f = f_0(r) + s f_1(r)$ and equation (44) becomes

$$(f_0' f_1 - f_1' f_0)(f_0 + f_1 \mu) = 0. \tag{45}$$

It could be of a certain interest to note that if in the present case we further require $\mu = r$ and $f(r,s) = g(r,s)$ we get

$$f(r,s) = g(r,s) = \alpha(r)(\beta_0 + \beta_1 s), \tag{46}$$

where $\alpha(r)$ is an arbitrary function and β_0, β_1 are arbitrary constants. From (21), (22) we obtain

$$p = \beta_1\alpha(r) + k_0\frac{\alpha(r)}{s-r}. \tag{47}$$

Then exact solutions to (19), (20) can be found by integrating the equations

$$r_t + rr_x = \alpha(r)(\beta_0 - k_0 + \beta_1 r), \tag{48}$$

$$r_x = p(r,s). \tag{49}$$

Here by solving (48) one can obtain $r = r(x,t)$, and the function $s = s(x,t)$ is obtained from (49).

4 Traffic Flow

Within the framework of the continuum theory traffic flow modeling is usually based upon a pair of equations of the form:

$$\frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0, \tag{50}$$

$$\frac{\partial v}{\partial t} + \frac{p'(\rho)}{\rho}\frac{\partial\rho}{\partial x} + v\frac{\partial v}{\partial x} = \frac{1}{\tau}[V(\rho) - v], \tag{51}$$

where ρ denotes the traffic density (car concentration), v is the travel speed, the "pressure–like" function $p(\rho)$ with $p'(\rho) > 0$ characterizes the anticipation of the driver reaction to stimuli,$V(\rho)$ is the "equilibrium" velocity while τ is a relaxation time accounting for non equilibrium situation, which is assumed to be constant.

System of equations (50), (51) for different form of $p = p(\rho)$ includes a number of continuous models widely adopted for simulating a traffic flow [11], [12] (see also references quoted therein). Moreover when $\tau \to 0$, the governing system (50), (51) is reduced to the pioneering model of Lighthill, Whitham and Richards [13], [14].

The governing system (50), (51) can be rewritten in the Riemann form (19) with $\lambda = v - c(\rho)$, $\mu = v + c(\rho)$, $c(\rho) = \sqrt{p'(\rho)}$, $f(r,s) = g(r,s) = \frac{1}{\tau}(V(\rho) - v)$, whereas the Riemann variables r and s are

$$r(\rho,v) = v - \int\frac{c(\rho)}{\rho}\,d\rho, \tag{52}$$

$$s(\rho,v) = v + \int\frac{c(\rho)}{\rho}\,d\rho. \tag{53}$$

According to the approach worked out in section 3 we assume

$$\lambda_r = 0, \tag{54}$$

which leads to the requirement that the material response function $c(\rho)$ adopts the form

$$c(\rho) = \frac{c_0}{\rho}, \tag{55}$$

where c_0 is an arbitrary constant. Taking into account (52), (53), we obtain $\lambda = s$, $\mu = r$. Equation (46) is satisfied if $\beta_1 = 0$ and

$$\alpha(r) = \frac{V_0 - r}{\tau \beta_0},$$

with an arbitrary constant V_0, which gives

$$V(\rho) = V_0 - \frac{c_0}{\rho}. \tag{56}$$

Finally exact solution of (50), (51) can be obtained by solving (48) and (49).

If $q_0 = \frac{k_0}{\tau\beta_0} \neq \frac{1}{\tau}$, then we determine

$$r = V_0 + (\phi(\xi) - V_0)e^{-t/T}, \tag{57}$$

$$s = V_0 + (\phi(\xi) - V_0)e^{-t/T} - q_0 \frac{(\phi(\xi) - V_0)(e^{-t/T} - 1)}{\phi'(\xi)}, \tag{58}$$

$$\xi = x - V_0 t + T(\phi(\xi) - V_0)(e^{-t/T} - 1), \tag{59}$$

where $\frac{1}{T} = \frac{1}{\tau} - q_0$.

If $q_0 = \frac{1}{\tau}$, then we obtain

$$r = \phi(\xi), \tag{60}$$

$$s = \phi(\xi) + \frac{V_0 - \phi(\xi)}{\tau \phi'(\xi)} \left(1 + \phi'(\xi) t\right), \tag{61}$$

$$\xi = x - \phi(\xi) t, \tag{62}$$

In both cases the initial value $\phi(\xi) = r(x, 0)$ can be obtained in terms of the initial data for the car density $\rho(x, 0)$ through the relation

$$\frac{2c_0 \phi'(x)}{q_0 \left(\phi(x) - V_0\right)} = \rho(x, 0) \tag{63}$$

5 Rate–type material

Within the framework of viscoelasticity, in order to describe the experimental behavior of special classes of materials under loading–unloading one–dimensional processes, the following quasilinear model [15] has been considered

$$w_\tau - \sigma_x = 0, \tag{64}$$
$$\sigma_\tau - \Phi(\tau,\sigma) w_x = \Psi(\tau,\sigma), \tag{65}$$

where w is the Lagrangian velocity, σ is the stress, τ and x are time and space coordinates, respectively. The functions $\Phi(\tau,\sigma)$ and $\Psi(\tau,\sigma)$ denote material response functions measuring, respectively, the noninstantaneous and the instantaneous response of the material to an increment of the stress. Moreover the strain ϵ must obey the equation $\epsilon_\tau = w_x$. We notice that the strict hyperbolicity of equations (64), (65) is assured if $\Phi > 0$.

In [16] it has been proven that the nonautonomous model (64), (65) can be reduced to the autonomous form

$$u_t - v_x = ku, \tag{66}$$
$$v_t - \phi^2(v) u_x = \psi(v) + kv, \tag{67}$$

by means of the change of variables

$$t = F(\tau), \tag{68}$$
$$u = e^{kF(\tau)}\left(w - \frac{\beta}{k}x\right), \tag{69}$$
$$v = T(\tau)e^{kF(\tau)}\sigma - \int \delta(\tau)e^{kF(\tau)}d\tau, \tag{70}$$

provided that the functions Φ, Ψ adopt the form

$$\Phi = \frac{\phi^2(v)}{T(\tau)}, \tag{71}$$
$$\Psi = \frac{e^{kF(\tau)}}{T^2(\tau)}\left(\int h(t,v)T^2 e^{kF(\tau)}d\tau + \psi(v)\right), \tag{72}$$
$$h(t,v) = \frac{\delta'(\tau) - \beta\Phi}{T} - \frac{T''}{T^2}e^{-kF(\tau)}\left(\int \delta(\tau)e^{kF(\tau)}d\tau + v\right). \tag{73}$$

In (68)–(70) and (71)–(73) $T(\tau)$ and $\delta(\tau)$ are not specified, $F'(\tau) = \frac{1}{T(\tau)}$, k and β are constants.

In order to apply the analysis developed in section 3 it is convenient to rewrite the autonomous model (66), (67) in the Riemann form (19), where

$$r = u + \int \frac{dv}{\phi(v)}, \quad s = u - \int \frac{dv}{\phi(v)} \tag{74}$$

Moreover $\lambda = -\phi(v)$, $\mu = \phi(v)$, and

$$f = ku + \frac{\psi(v) + kv}{\phi(v)}, \tag{75}$$

$$g = ku - \frac{\psi(v) + kv}{\phi(v)}. \tag{76}$$

Here we are dealing with the case $a_2 = 0$, $a_3 = 0$ presented in section 3.

By integrating (42) and requiring (43) to be satisfied we get

$$\phi(v) = (\phi_0 v + \phi_1)^{2/3}, \tag{77}$$

$$\psi(v) = -kv + \frac{3k}{\phi_0}(\phi_0 v + \phi_1), \tag{78}$$

where ϕ_0 and ϕ_1 are constants. From (21), (22) we get

$$p(r,s) = \frac{p_0(r)}{r-s}, \qquad p_0(r) = 36k\frac{r^2}{c_1 - \phi_0 r^2} \tag{79}$$

with an arbitrary constant c_1.

Finally exact solutions to (66), (67) can be obtained by solving the system of ordinary differential equations (23). One of possible solutions can be obtained for $c_1 = 0$, so that by integrating (23) we get:

$$r = s_0(\xi)(e^{kt} - e^{-kt}) + r_0(\xi)e^{-kt}, \tag{80}$$

$$s = s_0(\xi)e^{kt}, \tag{81}$$

$$\xi = x - \frac{\phi_0^2}{72k}(r_0 - s_0)^2(1 - e^{-2kt}), \tag{82}$$

where $s_0(x) = s(x, 0)$ while

$$r_0(\xi) = s_0(\xi) - \frac{36k}{\phi_0^2 s_0'(\xi)} \tag{83}$$

It could be of a certain interest to note that whence an initial value problem for v is given, then from

$$\phi_0 v(x,0) + \phi_1 = -\left(\frac{6k}{\phi_0 s_0'(x)}\right)^3 \tag{84}$$

$s_0(\xi)$ can be calculated.

Let us present a rarefaction wave at the point x_0. To obtain this solution one needs to integrate system (23) with the initial conditions ($t = 0$)

$$x = x_0,\ r = r_0,\ s = s_0(a),$$

where r_0 is an arbitrary constant, $s_0(a)$ satisfies the rarefaction wave condition (18)

$$\frac{d\,\mu}{d\,a} = -\frac{\phi_0}{3}(r_0 - s_0)s_0' < 0.$$

The general solution is

$$\begin{array}{c} (x - x_0)\frac{72k}{\phi_0^2} = (1 - e^{-2kt})(r_0 - s_0(a))^2, \\ r = s_0(a)(e^{kt} - e^{-kt}) + r_0 e^{-kt}, s_0 = s_0(a)e^{kt}. \end{array} \tag{85}$$

The representation of the solution in (x, t) variables is obtained by substituting

$$s_0(a) = r_0 + \frac{6\sqrt{2k}}{\phi_0}\sqrt{1 - e^{-2kt}}(x - x_0)^{1/2},$$

found from the first relation of (85) (except $t = 0$).This solution can be applied to the problem of pulling one of the end of a bar with finite velocity.

6 Conclusion

Here we classified equations written in the Riemann variables, which admit one differential constraint of first order. There is one class of solutions which has properties similar to simple waves for autonomous systems. These solutions can be obtained by integrating a system of ordinary differential equations. At the same time they have one arbitrary function in the general solution of the Cauchy problem. The existence of an arbitrary function allows solving more general initial value problems. These solutions can not be obtained by group analysis method. In the nonclassical approach the number of differential constraints is equal to the number of the dependent variables and for Cauchy type systems with two independent variables there is no arbitrary functions in solution.

Here we analyzed two models from continuum mechanics. For these models we constructed generalized simple waves.

Our analysis was restricted by autonomous systems. But it can be extended for nonautonomous systems and for systems with more than two dependent variables.

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