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SOLUTIONS OF ONE-DIMENSIONAL CONVECTION-DIFFUSION EQUATIONS BY THE WAVELET GALERKIN METHOD

Somchai Suk-In

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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SOLUTIONS OF ONE-DIMENSIONAL CONVECTION-DIFFUSION EQUATIONS BY THE WAVELET GALERKIN METHOD

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์นี้ ศึกษาการประยุกต์วิธีเวฟเลตกาเลอร์คิน เพื่อหาผลเฉลยของสมการเชิงอนุพันธ์ย่อยไม่เชิงเส้น ที่มีเงื่อนไขขอบไม่เป็นคาบ และเงื่อนไขเริ่มต้นที่มีภาวะไม่ต่อเนื่อง การศึกษาเริ่มจากสมการเบอร์เกอร์ 1 มิติ การหาผลเฉลยใช้กระบวนการผลต่างกึ่งโดยปริยายในทิศทางของเวลา ตามด้วยการประมาณกาเลอร์คินบนปริภูมิ โดยใช้ฟังก์ชันมาตราคอยเฟทเป็นฟังก์ชันฐานหลัก จากนั้นใช้หลักการเดียวกันเพื่อหาผลเฉลยของสมการที่มีความหนืดไม่เชิงเส้น

$$u_t - \frac{1}{2}u_x = \left(u^2 u_x\right)_x$$

ระดับความถูกต้องของผลเฉลยเชิงตัวเลขประเมินโดยการเปรียบเทียบกับผลเฉลยเชิงวิเคราะห์ที่แท้จริง และผลเฉลยที่ได้จากกระบวนการผลต่างจำกัดด้วยตัวจำกัดค่าฟลักซ์

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GALERKIN METHOD / WAVELETS / BURGERS EQUATION / NONLINEAR VISCOSITY

The application of the Wavelet-Galerkin method to the solution of nonlinear partial differential equations with non-periodic boundary conditions and discontinuities in the initial conditions is investigated. First the one-dimensional Burgers equation is considered. A semi-implicit difference scheme in time direction is followed by Galerkin approximation in space, using Coiflet scaling functions as basis functions. This is followed by the study of an equation with nonlinear viscosity,

$$u_t - \frac{1}{2}u_x = (u^2 u_x)_x.$$

Accuracy of the numerical results is determined by comparison with exact solutions and with solutions by finite difference schemes with flux limiter.

School of Mathematics

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CHAPTER I

INTRODUCTION

1.1 Background

Nonlinear partial differential equations, and in particular the equations of fluid dynamics, are difficult to solve analytically. Instead, one relies on numerical methods for their solution, such as the finite difference or finite element methods.

Very often, the solutions of these equations are irregular, such as the shock waves which can form in compressible gas flow for example. The standard numerical methods deal poorly with this type of behaviour. Low-order schemes tend to smooth-out the steep gradients, while high-order schemes give solutions which exhibit overshoots or oscillations. In oder to obtain accurate solutions, special finite difference techniques such as flux limiters had to be designed (Harten (1983); LeVeque (1996); Roe (1985); Roe and Sidilkover (1992); Van Leer (1997) and Van Leer (1997)).

The development of the theory of wavelets was initially motivated by applications in signal processing (Daubechies (1988); Grossman and Morlet (1984); Mallat (1989); Mallat and Zhong (1992)). However, it was realized early on that their particular features should make them a useful tool for the solution of partial differential equations (Glowinski et al. (1990); Latto and Tenenbaum (1990)). This is because wavelet methods provide for basis functions which are well localized in space, and are of arbitrarily small scale, and they should thus represent the irregular solutions of nonlinear equations with good accuracy.

As an example, consider the Burgers equation

$$u_t + uu_x = \nu u_{xx} \qquad (\nu > 0)$$

in the spatial domain $x \in [0,1]$, with initial condition $u(0,x) = u_o(x)$ and boundary conditions $u(t,0) = u_l$, $u(t,1) = u_r$. This relatively simple nonlinear equation is frequently used to test the accuracy of numerical schemes, as its solutions may develop shocks at low values of viscosity ν, while many exact solutions are known. Discretizing the Burgers equation in time, one obtains an ordinary differential equation

$$\frac{u^{k+1} - u^k}{\Delta t} + u^k (u^{k+1})_x = \nu (u^{k+1})_{xx}.$$

It can be formulated in variational form as

$$a(w,v) = L(v) \tag{1.1}$$

where a is the bilinear form given by

$$a(w,v) = \int_0^1 \left(\nu \Delta t w' v' + \Delta t u^k w' v + wv\right) \, dx \tag{1.2}$$

and L the linear form given by

$$L(v) = \int_0^1 u^k \, v \, dx.$$

The usual Galerkin method consists of finding an approximate solution to (1.1) in some finite dimensional subspace V of $L^2[0,1]$, which after choosing a basis of V, simply means solving a system of linear equations.

The particular feature of the wavelet techniques which is of importance here is the construction of a compactly supported and differentiable function φ, called scaling function, so that for each "scale" $j \in \mathbb{Z}$, the collection $\{\varphi_{j,m}\}_{m \in \mathbb{Z}}$ forms an orthonormal family, where $\varphi_{j,m}(x) = 2^{/j/2} \varphi(2^j x - m)$. In addition, for sufficiently large j, the space $V_j = \text{span}\{\varphi_{j,m}\}_{m \in \mathbb{Z}}$ is a good approximation to the space $L^2(\mathbb{R})$.

Only finitely many of the basis functions of V_j do not vanish on $[0, 1]$, and in the Wavelet-Galerkin method, these will be used as basis functions for the Galerkin solution of (1.1). After expressing the functions w, u^k and v in (1.2) as linear combinations of these basis vectors $\varphi_{j,m}$, solving (1.1) then amounts to solving of a linear system of equations whose coefficients are determined by integrals such as

$$\int_0^1 \varphi_{j,l}(x)\varphi'_{j,m}(x)\varphi_{j,n}(x)$$

and which are called connection coefficients. Although the scaling functions φ can usually not be represented in explicit form, it is still possible to compute the connection coefficients as outlined in Lin and Zhou (2001).

The Wavelet-Galerkin method for the solution of the Burgers equation has already been studied by several authors beginning with Latto and Tenenbaum (1991), and usually with periodic boundary conditions. Periodic conditions allow one to use the connection coefficients of Latto et al. (1991) whose computation is simpler and faster than the coefficients given above. Lin and Zhou (2001) are the only authors known to us who have solved the Burgers equation for various non-periodic initial conditions. Their numerical experiments have shown that the Wavelet-Galerkin method can produce substantially better approximations to the exact solution than the traditional Galerkin method for this equation. However, their numerical experiments have involved only medium to high levels of viscosity, and it is not clear how this method performs at low viscosity.

Kuma and Mehra (2005) have chosen a forward second order Taylor expansion for discretization in time,

$$u_t^k = \frac{u^{k+1} - u^k}{\Delta t} - \frac{\Delta t}{2}(u_{tt})^k,$$

in order to obtain higher accuracy at each time step. Their solutions of some periodic boundary value problems using a Daubechies scaling function exhibits very good correspondence with the exact shockwave solution, again at a medium-high level of viscosity. These authors have applied the same method to the Kortweg-de Vries equation (Kuma and Mehra, 2005b) as well. Thus, the partial differential equations studied so far by employing the Wavelet-Galerkin method involve non-linearity in the first derivative term, yet are still linear in the second derivative.

1.2 Objectives

In this thesis, we apply the Wavelet-Galerkin method to the study of an equation with nonlinear diffusion term, that is, with nonlinearity in the second derivative term, namely the equation

$$u_t - \frac{u_x}{2} = (u^2 u_x)_x$$

by using the Coiflet scaling functions of Daubechies (1993) as basis functions. We impose a non-smooth initial condition and non-periodic boundary conditions, and, by means of numerical experiments, compare the wavelet solution with both, the exact solution and solutions by various flux limiter finite difference schemes. In addition, we investigate how scale and time step size affect the accuracy of the solution.

Along the way we revisit the Burgers equation, and verify by numerical experiments that this method works well at lower viscosities. We further investigate how the choice of scale and scaling function influences the accuracy of the solution of the Burgers equation, and compare the wavelet solutions of the Burgers equation with solutions obtained by finite difference flux limiter schemes.

This thesis is organized as follows. In Chapter II we review the basic background from wavelet theory and from the Galerkin method that are required throughout. Connection coefficients and the techniques for their computation are discussed in Chapter III. Chapter IV is used to obtain the solution of the Burgers equation with initial discontinuity by the wavelet method. This solution is compared with solutions obtained by traditional numerical methods, and by numerical experiments it is investigated how choice of viscosity, of wavelet scale and scaling function influence the accuracy of the solution. In Chapter V the same investigations are applied to a partial differential equation with nonlinear viscosity. The final chapter then summarizes the results of this study.

CHAPTER II

BASIC BACKGROUND

In this chapter, we review the mathematical background and tools required, which consist mainly of the theory of compactly supported wavelets, and the Galerkin method for the numerical solution of differential equations. Detailed proofs can be found in standard references, such as Hernandez and Weiss (1996) and Glowinski (1984) for example.

2.1 Discrete Wavelets

The basic idea of wavelet theory is to decompose the Hilbert space $L^2(\mathbb{R})$ into an infinite direct sum $\{W_j\}_{j\in\mathbb{Z}}$ of subspaces of special form: For each j, the dilation operator $D : f(x) \mapsto \sqrt{2}f(2x)$ is a Hilbert space isomorphism of W_j onto W_{j+1}, and each W_j has an orthonormal basis consisting of the integer translates of a single function.

2.1.1 Wavelets from Multiresolution Analysis

Definition 2.1. (Wavelets)

Fix $\psi \in L^2(\mathbb{R})$, and consider the family of functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ obtained from ψ by dilations and translations,

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$, then each $\psi_{j,k}$ is called a wavelet and ψ the *mother wavelet*. In this case, by Parseval's identity, every $f \in L^2(\mathbb{R})$

can be expressed as

$$f = \sum_{j,k\in\mathbb{Z}} d_{j,k}\psi_{j,k} \tag{2.1}$$

with convergence in the mean square norm, where

$$d_{j,k} =< f, \psi_{j,k} >= 2^{j/2} \int_{\mathbb{R}} f(x)\overline{\psi(2^j x - k)}\, dx.$$

The sequence $\{d_{j,k}\} \in \ell^2(\mathbb{Z}\times\mathbb{Z})$ is called the *discrete wavelet transform* of f, and the $d_{j,k}$ are called the *wavelet coefficients* of f. The identity (2.1) thus shows how f can be reconstructed form its wavelet transform, and is therefore called the *inverse wavelet transform.*

Almost all wavelets, and in particular, all wavelets of compact support (Hernandez, 1996) can be obtained by a process called multiresolution analysis as introduced by Mallat (1989).

Definition 2.2. (Multiresolution analysis)
A *multiresolution analysis* (MRA) on $L^2(\mathbb{R})$ is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ satisfying the following properties:
$(M1)$: $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$,
$(M2)$: $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,
$(M3)$: $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
$(M4)$: $f(x) \in V_0$ if and only if $f(2^j x) \in V_j$, for all j,
$(M5)$: there exists a function $\varphi(x) \in L^2(\mathbb{R})$, called the scaling function, such that the collection of integer translates $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0.

Let us briefly outline how wavelets can be obtained from a multiresolution analysis. By (M4), the dilation operator D is a linear isometry of V_j onto V_{j+1} for all j. Applying the dilation operator j-times, it follows that the collection

$$\{D^j\varphi_{0,k}\}_{k\in\mathbb{Z}} = \{\varphi_{j,k}\}_{k\in\mathbb{Z}} = \{2^{j/2}\varphi(2^j x - k)\}_{k\in\mathbb{Z}}$$

is an orthonormal basis of V_j, for all j. In particular, when $j = 1$ then

$$\{\varphi_{1,k}\}_{k\in\mathbb{Z}} = \{\sqrt{2}\varphi(2x-k)\}_{k\in\mathbb{Z}}$$

is an orthonormal basis of V_1. Now by (M1), $\varphi \in V_1$ as well, and thus it can be expressed in terms of this basis,

$$\varphi = \sum_{k\in\mathbb{Z}} g_k \varphi_{1,k} \tag{2.2}$$

where

$$g_k = < \varphi, \varphi_{1,k} > = \sqrt{2}\int_{-\infty}^{\infty} \varphi(x)\overline{\varphi(2x-k)}\, dx. \tag{2.3}$$

The sequence $\{g_k\}_{k\in\mathbb{Z}}$ is called the *scaling filter* and is an element of $\ell^2(\mathbb{Z})$, in fact by Parseval's identity,

$$\|\{g_k\}\|_{l^2(\mathbb{Z})} = \|\varphi\|_{L^2(\mathbb{R})} = 1. \tag{2.4}$$

It follows that the sequence $\{h_k\}_{k\in\mathbb{Z}}$ defined by

$$h_k = (-1)^k \overline{g_{1-k}}$$

is also an element of $\ell^2(\mathbb{Z})$ of norm one, so that the function ψ defined by

$$\psi = \sum_{k\in\mathbb{Z}} h_k \varphi_{(1,k)} \tag{2.5}$$

is an element of V_1. The sequence $\{h_k\}$ is called the *wavelet filter.*

Now express the integer translates of φ and ψ in terms of the basis vectors $\varphi_{1,k}$ of V_1. By (2.2),

$$\begin{aligned}\varphi(x-l) &= \sum_{k\in\mathbb{Z}} g_k \varphi_{1,k}(x-l) = \sum_{k\in\mathbb{Z}} g_k \sqrt{2}\varphi(2(x-l)-k) \\ &= \sum_{k\in\mathbb{Z}} g_k \sqrt{2}\varphi\left(2x-(k+2l)\right) = \sum_{k\in\mathbb{Z}} g_{k-2l}\varphi_{1,k}(x)\end{aligned}$$

and similarly by (2.5),

$$\psi(x-l) = \sum_{k\in\mathbb{Z}} h_{k-2l}\varphi_{1,k}(x).$$

It follows by Parseval's equality that

$$<\varphi(x-m),\psi(x-l)>=\sum_{k\in\mathbb{Z}} g_{k-2m}\overline{h_{k-2l}}=\sum_{k\in\mathbb{Z}} g_{k-2m}(-1)^{k-2l}g_{1-k+2l}$$
$$=\sum_{k\in\mathbb{Z}}(-1)^{k+m-l}g_{k+(l-m)}g_{1-k+(l-m)}$$
$$=\sum_{k=1}^{\infty}(-1)^{k+m-l}g_{k+(l-m)}g_{1-k+(l-m)}+\sum_{k=0}^{\infty}(-1)^{-k+m-l}g_{-k+(l-m)}g_{1+k+(l-m)}$$
$$=\sum_{k=1}^{\infty}(-1)^{k+m-l}g_{k+(l-m)}g_{1-k+(l-m)}+\sum_{k=1}^{\infty}(-1)^{1-k+m-l}g_{1-k+(l-m)}g_{k+(l-m)}$$
$$=\sum_{k=1}^{\infty}\left[(-1)^{k}-(-1)^{-k}\right](-1)^{m-l}g_{k+(l-m)}g_{1-k+(l-m)}=0.$$

Similar computations give

$$<\psi(x-m),\psi(x-l)>=\sum_{k\in\mathbb{Z}} h_{k-2m}\overline{h_{k-2l}}$$
$$=\sum_{k\in\mathbb{Z}}(-1)^{k-2m}\overline{g_{1-k+2m}}\,(-1)^{k-2l}g_{1-k+2l}=\sum_{k\in\mathbb{Z}} g_{1-k+2l}\overline{g_{1-k+2m}}$$
$$=\sum_{k\in\mathbb{Z}} g_{k+2l}\overline{g_{k+2m}}=<\varphi(x+l),\varphi(x+m)>=\delta_{l,m}$$

by (M5). This shows that the collection $\{\varphi_{0,k},\psi_{0,k}\}_{k\in\mathbb{Z}}$ is orthonormal in V_1. Thus, if we let W_0 denote the orthonormal complement of V_0 in V_1,

$$V_1=V_0\oplus W_0$$

then $\{\psi_{0,k}\}_{k\in\mathbb{Z}}$ will be an orthonormal set in W_0. One can show that this collection is total in W_0, that is, is a basis of W_0.

In general, for each j, let W_j denote the orthogonal complement of V_j in V_{j+1}, that is,

$$V_{j+1}=V_j\oplus W_j.$$

Applying induction, we have for each $n > j_o$,

$$
\begin{aligned}
V_n &= V_{n-1} \oplus W_{n-1} \\
&= V_{n-2} \oplus W_{n-2} \oplus W_{n-1} \\
&= V_{n-3} \oplus W_{n-3} \oplus W_{n-2} \oplus W_{n-1} \\
&\;\;\vdots \\
&= V_{j_o} \oplus W_{j_o} \oplus W_{j_o+1} \oplus \cdots \oplus W_{n-1}.
\end{aligned} \tag{2.6}
$$

One can show that (M1)–(M3) imply that for each $j_o \in \mathbb{Z}$,

$$L^2(\mathbb{R}) = V_{j_o} \oplus \bigoplus_{j=j_0}^{\infty} W_j$$

and also

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Now since D^j maps W_0 isomorphically onto W_j, the collection $\{\psi_{j,k}\}_{k\in\mathbb{Z}} = \{D^j\psi_{0,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j, for each j. This shows that $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is a basis of $L^2(\mathbb{R})$, and hence ψ is a mother wavelet.

Remark. One can show that if the scaling function and the wavelet are also elements of $L^1(\mathbb{R})$ (Walnut, 2002), for example if φ has compact support, then

$$\left|\int_{-\infty}^{\infty} \varphi(x)\, dx\right| = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \psi(x)\, dx = 0. \tag{2.7}$$

Since for all scalars $|\alpha| = 1$, the function $\alpha\varphi$ remains a scaling function for the given MRA, we may replace φ by a suitable $\alpha\varphi$ and obtain

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1. \tag{2.8}$$

Then in addition (Walnut, 2002),

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = 1 \qquad \text{a.e.} \tag{2.9}$$

Remark. Written out, scaling relation (2.2) becomes

$$\varphi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_k\varphi(2x-k).$$

Because the factor $\sqrt{2}$ is inconvenient, many authors prefer to modify the scaling filter. Setting

$$a_k = \sqrt{2}g_k = 2\int_{-\infty}^{\infty}\varphi(x)\overline{\varphi(2x-k)}\,dx \tag{2.10}$$

the scaling relation is simplified to

$$\varphi(x) = \sum_{k\in\mathbb{Z}} a_k\varphi(2x-k). \tag{2.11}$$

Similarly, one sets

$$b_k = \sqrt{2}h_k = (-1)^n\overline{a_{1-k}}$$

so that (2.5) becomes

$$\psi(x) = \sum_{k\in\mathbb{Z}} b_k\varphi(2x-k).$$

Furthermore, by (2.4) one has

$$\sum_{k\in\mathbb{Z}}|a_k|^2 = \|\{a_k\}\|^2_{\ell^2(\mathbb{Z})} = \|\sqrt{2}\{g_k\}\|^2_{\ell^2(\mathbb{Z})} = 2$$

and hence also

$$\sum_{k\in\mathbb{Z}}|b_k|^2 = \sum_{k\in\mathbb{Z}}|a_k|^2 = 2.$$

2.1.2 The Pyramidal Algorithm

The coefficients of a function f with regards to the wavelet basis can be computed efficiently by the pyramidal algorithm.

First consider the decomposition $V_j = V_{j-1}\oplus W_{j-1}$. Given $f\in L^2(\mathbb{R})$, let f_j denote the projection of f along V_j. Since $\{\varphi_{j,m}\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_j, then

$$f_j = \sum_{m\in\mathbb{Z}} c_{j,m}\varphi_{j,m} \tag{2.12}$$

where the coefficients $c_{j,m} = \langle f, \varphi_{j,m} \rangle$ in this basis are called the *scaling coefficients of f at scale j*. On the other hand, since $\{\varphi_{j-1,m}\}_{m\in\mathbb{Z}}$ and $\{\psi_{j-1,m}\}_{m\in\mathbb{Z}}$ are orthonormal bases of V_{j-1} and W_{j-1}, respectively, then

$$f_j = \sum_{m\in\mathbb{Z}} c_{j-1,m}\varphi_{j-1,m} + \sum_{m\in\mathbb{Z}} d_{j-1,m}\psi_{j-1,m}, \tag{2.13}$$

where now $c_{j-1,m} =< f, \varphi_{j-1,m} >$ and $d_{j-1,m} =< f, \psi_{j-1,m} >$ are the scaling coefficients, respective *wavelet coefficients*, of f at scale $j-1$.

One can compute the scaling coefficients $c_{j-1,m}$ and the wavelet coefficients $d_{j-1,m}$ at scale $j-1$ from the collection of scaling coefficients $c_{j,m}$ at scale j by

$$\begin{aligned}
c_{j-1,n} &=< f_j, \varphi_{j-1,n} >= \sum_{m\in\mathbb{Z}} c_{j,m} < \varphi_{j,m}, \varphi_{j-1,n} > \\
&= \sum_{m\in\mathbb{Z}} c_{j,m} < D^{j-1}\varphi_{1,m}, D^{j-1}\varphi_{0,n} > \\
&= \sum_{m\in\mathbb{Z}} c_{j,m}\overline{< \varphi_{0,n}, \varphi_{1,m}, >} = \sum_{m\in\mathbb{Z}} \overline{g_{m-2n}}\, c_{j,m}, \\
d_{j-1,n} &=< f_j, \psi_{j-1,n} >= \sum_{m\in\mathbb{Z}} c_{j,m} < \varphi_{j,m}, \psi_{j-1,n} > \\
&= \sum_{m\in\mathbb{Z}} c_{j,m} < D^{j-1}\varphi_{1,m}, D^{j-1}\psi_{0,n} > \\
&= \sum_{m\in\mathbb{Z}} c_{j,m}\overline{< \psi_{0,n}, \varphi_{1,m}, >} = \sum_{m\in\mathbb{Z}} \overline{h_{m-2n}}\, c_{j,m}
\end{aligned} \tag{2.14}$$

since

$$\begin{aligned}
<\varphi_{0,n}, \varphi_{1,m} > &= \int_{-\infty}^{\infty} \sqrt{2}\varphi(x-n)\overline{\varphi(2x-m)}\, dx \\
&= \int_{-\infty}^{\infty} \sqrt{2}\varphi(x)\overline{\varphi(x-(m-2n))}\, dx =< \varphi, \varphi_{1,m-2n} >= g_{m-2n},
\end{aligned}$$

and a similar identity holds for $< \psi_{0,n}, \varphi_{1,m} >$.

In reverse, the scaling coefficients $c_{j,m}$ at scale j can be obtained from the collection $c_{j-1,m}$ and $d_{j-1,m}$ of scaling and wavelet coefficients at scale $j-1$. In fact,scaling relation (2.2) modifies to other scaling levels by

$$\begin{aligned}
\varphi_{j-1,m}(x) &= \left[D^{j-1}\varphi_{0,m}\right](x) = 2^{(j-1)/2}\varphi(2^{j-1}x-m) \\
&= 2^{(j-1)/2}\sum_{n\in\mathbb{Z}} g_n\varphi_{1,n}\left(2^{j-1}x-m\right) \\
&= 2^{(j-1)/2}\sum_{n\in\mathbb{Z}} g_n\sqrt{2}\varphi\left(2(2^{j-1}x-m)-n\right) \\
&= 2^{j/2}\sum_{n\in\mathbb{Z}} g_n\varphi\left(2^{j}x-(n+2m)\right) = \sum_{n\in\mathbb{Z}} g_{n+2m}\varphi_{j,n}(x),
\end{aligned}$$

and by a similar computation,

$$\psi_{j-1,m}(x) = \left[D^{j-1}\psi_{0,m}\right](x) = \sum_{n\in\mathbb{Z}} h_{n+2m}\varphi_{j,n}(x).$$

Now by (2.13) and the above

$$\begin{aligned}
f_j &= \sum_{m\in\mathbb{Z}} c_{j-1,m}\sum_{n\in\mathbb{Z}} g_{n+2m}\varphi_{j,n} + \sum_{m\in\mathbb{Z}} d_{j-1,m}\sum_{n\in\mathbb{Z}} h_{n+2m}\varphi_{j,n} \\
&= \sum_{n\in\mathbb{Z}}\left[\sum_{m\in\mathbb{Z}} c_{j-1,m}g_{n+2m} + d_{j-1,m}h_{n+2m}\right]\varphi_{j,n}.
\end{aligned}$$

Comparing with (2.12) we see that

$$c_{j,n} = \sum_{m\in\mathbb{Z}}\left(c_{j-1,m}g_{n+2m} + d_{j-1,m}h_{n+2m}\right). \tag{2.15}$$

The pyramidal algorithm consists of repeated application of the above decompositions and recompositions. In practice, one does not work in $L^2(\mathbb{R})$ but in a space V_n which sufficiently approximates $L^2(\mathbb{R})$, and uses a decomposition of V_n of form (2.6) with sufficiently small j_o:

$$V_n = V_{j_o} \oplus W_{j_o} \oplus W_{j_o+1} \oplus \cdots \oplus W_{n-1}. \tag{2.16}$$

Given $f \in L^2(\mathbb{R})$, the projection of f along V_n will be

$$f_n = \sum_{m\in\mathbb{Z}} c_{n,m}\varphi_{n,m},$$

as $\{\varphi_{n,m}\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_n. Then using the algorithm (2.14), one successively computes the scaling and wavelet coefficients of f first in V_{n-1}, respectively W_{n-1}, then those of f in V_{n-2}, respectively W_{n-2}, and continues, until one has obtained the coefficients of f in all of the component spaces.

The reconstruction of the coefficients $\{c_{n,m}\}_{m\in\mathbb{Z}}$ of f in V_n from the wavelet coefficients $d_{j,k}$ $(j_o \leq j < n)$ and the scaling coefficients $c_{j_o,k}$ can be done by successively applying (2.15).

2.1.3 Compactly Supported Wavelets

For the purpose of computations, it is useful to have scaling functions φ of compact support: then by (2.3), the scaling filter, and hence the wavelet filter, will be of finite length, that is have only finitely many nonzero terms. By (2.5), the mother wavelet ψ will also have compact support. In addition, one wishes these functions to be smooth. Daubechies (1992) has shown that there exist no wavelets which are infinitely differentiable and have compact support. However, for each positive integer r there exist a scaling function φ and associated wavelet ψ which are both r-times differentiable and have compact support (Hernandez and Weiss, 1996).

There is a connection between the length of the scaling filter $\{a_k\}$ (or $\{g_k\}$) of φ, and the support of φ:

Theorem 2.3. (Daubechies, 1992) Let $N_1 < N_2$ be two integers, and suppose that $a_k = 0$ for all $k \notin [N_1, N_2]$. Then $\mathrm{supp}(\varphi) \subset [N_1, N_2]$.

Throughout, we will choose N_1 and N_2 so that $[N_1, N_2]$ is the smallest interval satisfying the assumption of the Theorem.

2.1.3.1 Vanishing Moments

Another useful property of wavelets is that of vanishing moments. Recall that given a function f defined on $\mathbb{R}$, its *p-th moment* ($p \in \mathbb{N}_0$) is defined by

$$M^p(f) = \int_{-\infty}^{\infty} x^p f(x)\,dx$$

provided that this integral converges. We say that f *has vanishing moments of order* $p \leq L$, if $M^p(f) = 0$ for all $0 \leq p \leq L$.

Let φ be a compactly supported scaling function with corresponding wavelet ψ. For each $p \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, set

$$M_k^p = \int_{-\infty}^{\infty} x^p\,\overline{\varphi(x-k)}\,dx.$$

It turns out that when ψ has vanishing moments of order $p \leq L$, then any polynomial of degree less or equal to L can be reconstructed from the translates of the scaling function:

Theorem 2.4. Suppose,

$$\int_{-\infty}^{\infty} x^p \psi(x)\,dx = 0$$

for $p = 0, \ldots, L$. Then for each p, $0 \leq p \leq L$,

$$x^p = \sum_{k \in \mathbb{Z}} M_k^p \varphi(x-k) \tag{2.17}$$

with convergence in $L^2(I)$ for every bounded interval I, and with pointwise convergence.

Proof. (Pro-forma proof) Let p be given, $0 \leq p \leq L$. Then for each $k \in \mathbb{Z}$,

$$\begin{aligned}\int_{-\infty}^{\infty} x^p\,\overline{\psi(x-k)}\,dx &= \int_{-\infty}^{\infty} (x+k)^p \overline{\psi(x)}\,dx \\ &= \sum_{i=0}^{p} \binom{p}{i} k^{p-i} \int_{-\infty}^{\infty} x^i\,\overline{\psi(x)}\,dx \\ &= k^p \overline{\int_{-\infty}^{\infty} \psi(x)\,dx} = 0.\end{aligned}$$

Moving to dilation level $j \geq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} x^p\, \overline{\psi_{j,k}(x)}\, dx &= \int_{-\infty}^{\infty} 2^{j/2} x^p\, \overline{\psi(2^j x - k)}\, dx \\ &= \int_{-\infty}^{\infty} 2^{-j/2} (2^{-j} x)^p\, \overline{\psi(x-k)}\, dx \\ &= 2^{-j(p+1/2)} \int_{-\infty}^{\infty} x^p\, \overline{\psi(x-k)}\, dx = 0.\end{aligned}$$

This shows that $x^p \perp W_j$ for all j. Now as $L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots$ it follows that $x^p \in V_0$. That is,

$$x^p = \sum_{k \in \mathbb{Z}} c_k \varphi(x-k)$$

where $c_k = < x^p, \varphi(x-k) > = M_k^p$. Now since φ has compact support, the sum (2.17) is locally finite, and hence converges pointwise as well. $\square$

Since $x^p \notin L^2(\mathbb{R})$, the above argument is only formally correct. We can, however, explain why the theorem is still is correct.

Let $I = [M_1, M_2]$ be a given bounded interval, pick an integer M greater than the length of the supports of φ and ψ, and set $J = [M_1 - M, M_2 + M]$. Then the restriction of x^p to J, denoted $x^p\, \mathbf{1}_J$, is an element of $L^2(\mathbb{R})$, and hence it can be expressed as

$$x^p\, \mathbf{1}_J = \sum_k c_{0,k} \varphi_{0,k}(x) + \sum_{j=0}^{\infty} \sum_k d_{j,k} \psi_{j,k}(x) \tag{2.18}$$

where

$$\begin{aligned} c_{0,k} &= < x^p\, \mathbf{1}_J, \varphi_{0,k} > = \int_J x^p \overline{\varphi_{0,k}(x)}\, dx \\ d_{j,k} &= < x^p\, \mathbf{1}_J, \psi_{j,k} > = \int_J x^p \overline{\psi_{j,k}(x)}\, dx. \end{aligned}$$

The important observation is that those $\varphi_{0,k}$ and $\psi_{j,k}$ whose supports intersect I are supported inside J, so that

$$\begin{aligned} c_{0,k} &= \int_J x^p \overline{\varphi_{0,k}(x)}\, dx = \int_{-\infty}^{\infty} x^p \overline{\varphi_{0,k}(x)}\, dx = M_k^p \\ d_{j,k} &= \int_J x^p \overline{\psi_{j,k}}\, dx = \overline{\int_{-\infty}^{\infty} x^p \psi_{j,k}\, dx} = 0 \end{aligned}$$

for these $\varphi_{0,k}$ and $\psi_{j,k}$. Thus, if we restrict x^p further to I, then (2.18) becomes

$$x^p\,\mathbf{1}_I = \sum_k c_k \varphi_{0,k}(x). \tag{2.19}$$

which converges in $L^2(I)$. Since this is a finite sum, it also converges pointwise.

Remark. Let φ be a compactly supported scaling function. Then by (2.8), $M_k^0 = 1$ for all k. Now by (2.7), ψ has vanishing zero moment. Applying the Theorem to the case $p = 0$, it follows that

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = \sum_{k\in\mathbb{Z}} M_k^0 \varphi(x-k) = 1. \tag{2.20}$$

Applying this identity to $x = 0$ and $x = 1/2$, and using scaling relation (2.11) we obtain

$$1 = \sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_k \varphi(2n-k) = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_k \varphi(2n-k)$$

and

$$1 = \sum_{n\in\mathbb{Z}} \varphi(n+1/2) = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_k \varphi(2n+1-k) = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_k \varphi(2n+1-k)$$

Adding both equations and using (2.20) gives

$$2 = \sum_{k\in\mathbb{Z}} a_k \sum_{n\in\mathbb{Z}} (\varphi(2n-k) + \varphi(2n+1-k))) = \sum_{k\in\mathbb{Z}} a_k \sum_{n\in\mathbb{Z}} \varphi(n-k) = \sum_{k\in\mathbb{Z}} a_k$$

and we have shown that

$$\sum_{k\in\mathbb{Z}} a_k = 2. \tag{2.21}$$

A large number of vanishing moments ensures that the wavelet coefficients of a smooth function f decrease rapidly. In fact, one has:

Theorem 2.5. (Walnut (2002)) Let $g \in L^2(\mathbb{R})$ be compactly supported, with $\|g\|_2 = 1$, and suppose that

$$\int_{-\infty}^{\infty} x^p g(x)\,dx = 0 \qquad (0 \le p \le L-1).$$

If $f(x)$ is C^L on $\mathbb{R}$, and $f^{(L)}(x)$ is bounded, then there exists a constant C (depending on f and L only) so that

$$|< f, g_{j,k} >| \leq C2^{-j(L+1/2)}$$

for all j and k, where $g_{j,k}(x) = 2^{-j/2}g(2^j x - k)$.

2.1.3.2 The Daubechies Wavelets

The Daubechies wavelets (Daubechies, 1988) are the most commonly used wavelets. This is a family of wavelets with compact support, labelled $D\{2N\}$, N a positive integer. The scaling function of a $D\{2N\}$-wavelet is supported on the interval $[0, N-1]$, and the scaling filter has length N. The corresponding wavelets have vanishing p-th moments for $0 \leq p \leq N-1$. Table 2.1 shows the scaling coefficients of the $D2$–$D10$ wavelets. There is no closed form known for these wavelets, but their values at dyadic rationals can be computed by applying the algorithm outlined in subsection 2.1.4. Figure 2.1 shows the graphs of various Daubechies scaling functions. They become smoother with increasing N, in fact, they are C^1 functions for $N \geq 3$.

Table 2.1 The nonzero scaling coefficients for the $D2$ - $D10$ wavelets.(Source: Daubechies (1992)).

	$D2$	$D4$	$D6$	$D8$	$D10$
a_0	1	0.6830127	0.4704672	0.3258034	0.2264190
a_1	1	1.1830127	1.1411169	1.0109457	0.8539435
a_2		0.3169873	0.6503650	0.8922014	1.0243269
a_3		−0.1830127	−0.1909344	−0.0395750	0.1957670
a_4			−0.1208322	−0.2645072	−0.3426567
a_5			0.0498175	0.0436163	−0.0456011
a_6				0.0465036	0.1097027
a_7				−0.0149870	−0.0088268
a_8					−0.0177919
a_9					0.0047174

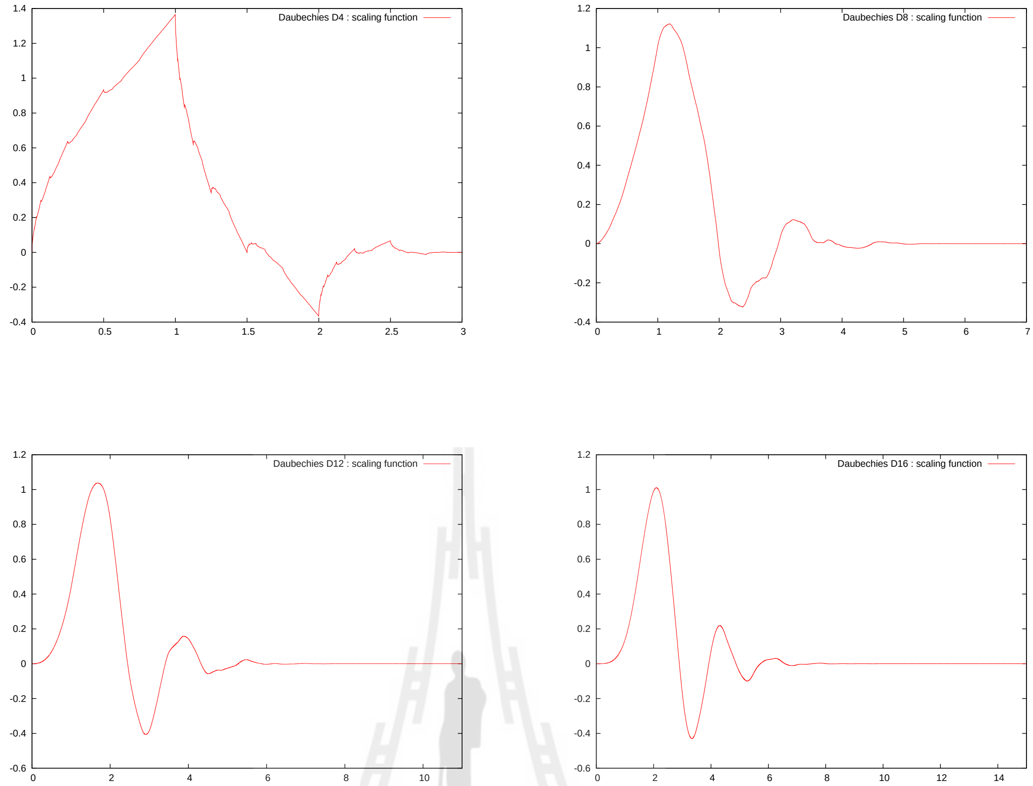

Figure 2.1 The Daubechies scaling functions $D4$, $D8$, $D12$ and $D16$.

2.1.3.3 The Coiflets

The Coiflets (Daubechies, 1993) are another frequently used family of compactly supported wavelets. The scaling function of the coiflet $C\{3N\}$, N an even positive integer, is supported on the interval $[-N, 2N-1]$, and the scaling filter has length $3N$. The special feature of the Coiflets is that the scaling functions have vanishing p-th moments, for $1 \leq p \leq N$. On the other hand, the wavelet has vanishing p-th moments for $0 \leq p \leq N-1$. Table 2.2 shows the scaling coefficients of the $C6$–$C24$ Coiflets, and Figure 2.2 shows the graphs of various Coiflet scaling functions, computed by applying the algorithm outlined in subsection 2.1.4.

Table 2.2 The nonzero scaling coefficients for the $C6$ - $C24$ Coiflets. (Source: Daubechies (1992)).

	$C6$	$C12$	$C18$	$C24$
a_{-8}				0.00126192
a_{-7}				−0.00230445
a_{-6}			−0.00536484	−0.01038905
a_{-5}			0.01100625	0.02272492
a_{-4}		0.02317519	0.03316712	0.03773448
a_{-3}		−0.05864028	−0.09301553	−0.11492848
a_{-2}	−0.10285946	−0.09527918	−0.08644153	−0.07930531
a_{-1}	0.47785946	0.54604209	0.57300667	0.58733481
a_0	1.20571891	1.14936479	1.12257051	1.10625291
a_1	0.54428109	0.58973439	0.60596714	0.61431462
a_2	−0.102859466	−0.10817121	−0.10154028	−0.09422548
a_3	−0.022140543	−0.08405296	−0.11639250	−0.13607623
a_4		0.03348882	0.04886819	0.05562727
a_5		0.00793577	0.02245848	0.03547166
a_6		−0.002578416	−0.01273920	−0.02151263
a_7		−0.001019011	−0.00364093	−0.00800202
a_8			0.00158041	0.00530533
a_9			0.00065933	0.00179119
a_{10}			−0.00010039	−0.00083300
a_{11}			−0.00004893	−0.00036766
a_{12}				0.00008816
a_{13}				0.00004417
a_{14}				−0.00000461
a_{15}				−0.00000252

2.1.4 Computation of the Values of the Scaling Function and the Mother Wavelet

Neither the scaling functions nor the wavelets of the Daubechies or Coiflet families can be written in closed form. However, it is possible to compute their values on the set of dyadic rationals.

More generally, let φ be a compactly supported scaling function with associated compactly supported wavelet ψ. We present the algorithm for computing the values of the scaling function φ, in case of ψ one simply replaces the scaling filter by the wavelet filter. As it turns out, the computations involve solving systems of linear equations.

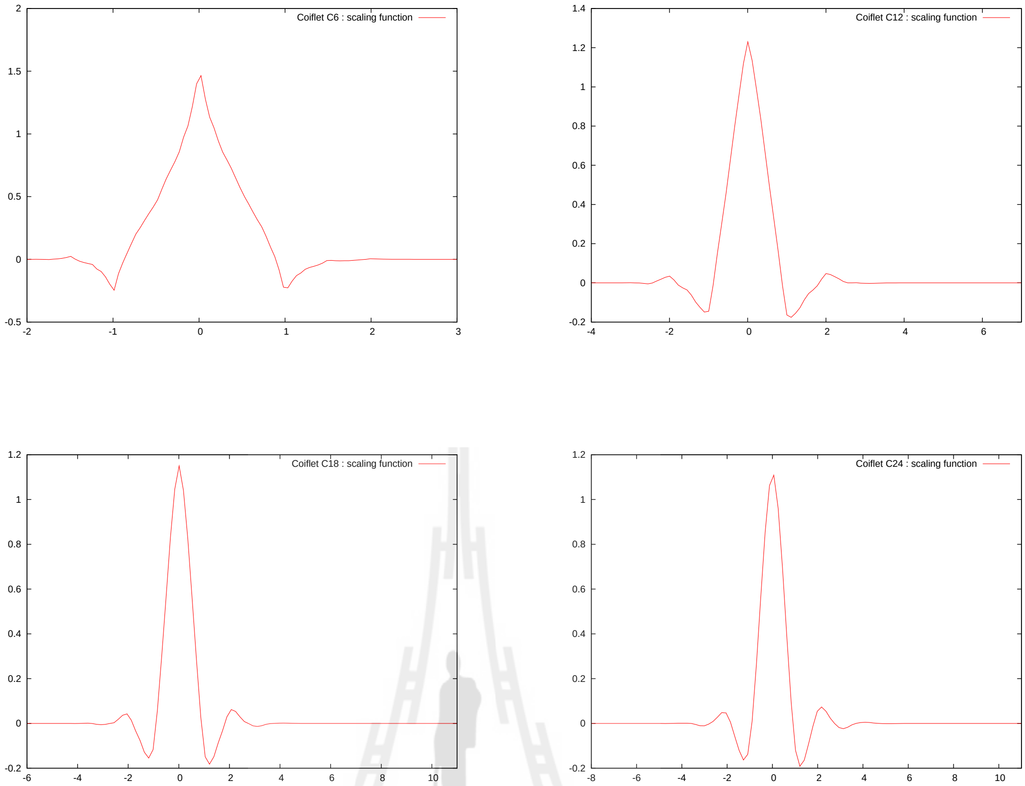

Figure 2.2 The Coiflet scaling functions $C6$, $C12$, $C18$ and $C24$.

Suppose, the scaling coefficients a_k are nonzero for $N_1 \leq k \leq N_2$ only. The starting point is the scaling relation (2.11) which reduces to

$$\varphi(x) = \sum_{k=N_1}^{N_2} a_k \varphi(2x - k). \tag{2.22}$$

The first step is to compute the values of φ at all integers inside the interval $[N1, N2]$. Substituting these integers into (2.22) and making use of the fact that

$\mathrm{supp}(\varphi) \subset [N_1, N_2]$ one obtains the following system of $N = N_2 - N_1 + 1$ equations,

$$
\begin{aligned}
\varphi(N_1) &= a_{N_1}\varphi(N_1) \\
\varphi(N_1+1) &= a_{N_1}\varphi(N_1+2) + a_{N_1+1}\varphi(N_1+1) + a_{N_1+2}\varphi(N_1) \\
\vdots \qquad & \quad \vdots \\
\varphi(N_1+k) &= a_{N_1}\varphi(N_1+2k) + a_{N_1+1}\varphi(N_1+2k-1) + \dots \\
&\qquad \dots + a_{N_1+j}\varphi(N_1+2k-j) + \dots + a_{N_1+2k}\varphi(N_1) \\
\vdots \qquad & \quad \vdots \\
\varphi(N_2-1) &= a_{N_2-2}\varphi(N_2) + a_{N_2-1}\varphi(N_2-1) + a_{N_2}\varphi(N_2-2) \\
\varphi(N_2) &= a_{N_2}\varphi(N_2).
\end{aligned}
$$

Expressed in matrix form,

$$X = A_o X$$

where $X = \{\varphi(n)\}_{n=N_1}^{N_2}$ is the vector of unknown values of φ, and $A_o = [\alpha_{kl}]$ is the $N \times N$ band-matrix whose entries are given by

$$
\alpha_{kl} = \begin{cases} a_{N_1+2k-l-1} & \text{if } 0 \le 2k-l-1 \le N_2 - N_1 \\ 0 & \text{else.} \end{cases}
$$

Thus, X is an eigenvector of this equation belonging to the eigenvalue one. As eigenvectors are not unique one augments this system by adding the coefficient equation (2.21),

$$\sum_{k=N_1}^{N_2} a_k = 2$$

and solves the nonhomogeneous system

$$AX = B$$

where A is an $(N+1) \times N$-matrix and B an $N+1$-vector,

$$
A = \begin{bmatrix} A_o - I_N \\ a_{N_1}\ a_{N_1+1} \dots a_{N_2} \end{bmatrix}, \qquad B = \begin{bmatrix} 0_N \\ 2 \end{bmatrix}.
$$

Solving this system for the unknown vector X, one obtains the values $\varphi(N_1), \ldots, \varphi(N_2)$ of φ on on the set of integers inside $[N_1, N_2]$.

Next one computes the values of $\varphi(x)$ at the dyadic points $\frac{m}{2^j}$ inside $[N_1, N_2]$ inductively. Suppose one has already obtained the values $\varphi(\frac{m}{2^{j-1}})$ for $2^{j-1}N_1 \leq m \leq 2^{j-1}N_2$. Then the scaling relation (2.22) immediately gives

$$\varphi\left(\frac{m}{2^j}\right) = \sum_{k=N_1}^{N_2} a_k \, \varphi\left(\frac{m}{2^{j-1}} - k\right) = \sum_{k=N_1}^{N_2} a_k \, \varphi\left(\frac{m - 2^{j-1}k}{2^{j-1}}\right).$$

for all $2^j N_1 \leq m \leq 2^j N_2$.

2.2 The Galerkin Method

Many boundary value problems involving nonlinear partial differential equations have only weak solutions. The Galerkin method is one way to obtain approximations to weak solutions, by expressing a given problem in variational form as a linear equation in Hilbert space, which then is approximated by a linear equation in a finite dimensional vector space. We first review the theory underlying this method.

2.2.1 The Lax-Milgram Theorem

Definition 2.6. A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ on a normed linear space V is said to be

1. *bounded* (or continuous), if there exists $C > 0$ such that $|a(v,w)| \leq C\|v\|\|w\| \quad \forall v, w \in V$, and

2. *coercive*, if there exists $\alpha > 0$ such that $a(v,v) \geq \alpha\|v\|^2 \quad \forall v \in V$.

Note that it is not required that the form $a(\cdot,\cdot)$ be symmetric. In fact, if it is symmetric, then V must be an inner product space:

Proposition 2.7. Let V be a real normed linear space and suppose that $a(\cdot,\cdot)$ is a symmetric, bounded and coercive bilinear form on V. Then $a(\cdot,\cdot)$ is an inner product, and the norm determined by this inner product is equivalent to the given norm on V.

Proof. Let $\|v\|_a = \sqrt{a(v,v)}$ denote the semi-norm determined by the symmetric bilinear form $a(\cdot,\cdot)$. The assumptions give that for all $v \in V$,

$$\alpha\|v\|^2 \leq \|v\|_a^2 = a(v,v) \leq C\|v\|^2$$

which proves that $a(\cdot,\cdot)$ is definite, i.e. an inner product, so that $\|\cdot\|_a$ is a norm. In addition, this inequality shows that both norms are equivalent. □

Theorem 2.8. (Lax-Milgram Theorem) Let H be a real Hilbert space, and $a(\cdot,\cdot)$ a (not necessarily symmetric) bilinear form having the property that

i) $\exists C > 0$ such that $|a(u,v)| \leq C\|u\|\,\|v\| \quad \forall\, u, v \in H$ (boundedness).

ii) $\exists \alpha > 0$ such that $a(v,v) \geq \alpha\|v\|^2 \quad \forall\, v \in H$ (coerciveness).

Then for each bounded linear functional F on H, there exists a unique $u \in H$ such that

$$a(u,v) = F(v) \qquad \forall\, v \in H. \tag{2.23}$$

Proof. Because of the importance of this theorem, we give a brief sketch of its proof. We note that when $a(\cdot,\cdot)$ is symmetric, then (2.23) follows directly from Proposition 2.7 and the Riesz Representation Theorem. In the general case, one invokes the Contraction Principle.

By the Riesz Representation Theorem, there exists a unique element $w \in H$ such that

$$F(v) =< w, v > \qquad (\forall\, v \in H),$$

and $\|w\| = \|F\|$. On the other hand, by bilinearity and boundedness of $a(\cdot,\cdot)$, every $u \in H$ defines a bounded linear functional f_u on H by

$$f_u(v) = a(u,v) \qquad (\forall\, v \in H)$$

of norm $\|f_u\| \leq C\|u\|$. Applying the Riesz Representation Theorem again, there exists a unique element $S_u \in H$ so that

$$a(u,v) = f_u(v) =< S_u, v > \qquad (\forall\, v \in H)$$

and $\|S_u\| = \|f_u\| \leq C\|u\|$. As both, $a(\cdot,\cdot)$ and $< \cdot,\cdot >$ are linear in the first component, the map $S : u \mapsto S_u$ is linear. We thus need to show that there exists a unique $u \in H$ so that $S_u = w$.

Define a (nonlinear) mapping $T : H \to H$ by

$$Tu = u - \rho(S_u - w)$$

where $\rho > 0$. We claim that T is a contraction provided that ρ is sufficiently small. In fact, given $u_1, u_2 \in H$, set $u = u_1 - u_2$. Then by coerciveness and linearity of the map S,

$$\begin{aligned}
\|Tu_1 - Tu_2\|^2 &= \|u - \rho S_u\|^2 \\
&=< u, u > -2\rho < S_u, u > +\rho^2 < S_u, S_u > \\
&\leq \|u\|^2 - 2\rho a(u,u) + \rho^2 C^2 \|u\|^2 \\
&\leq \|u\|^2 - 2\rho\alpha\|u\|^2 + \rho^2 C^2 \|u\|^2 \\
&= (1 - 2\alpha\rho + C^2\rho^2)\|u_1 - u_2\|^2.
\end{aligned}$$

It follows that T is a contraction provided that $\rho < 2\alpha/C^2$, which proves the claim.

The Contraction Principle (also called the Banach Fixed Point Theorem) immediately yields the existence of a unique fixed point of T, i.e. a unique element $u \in H$ such that $S_u = w$. $\square$

2.2.2 Application of the Lax-Milgram Theorem to Variational Problems

This theorem is often used to prove the existence and uniqueness of weak solutions to partial differential equations. As an example, consider a second order linear equation with Dirichlet boundary conditions

$$-u'' + b(x)u' + c(x)u = f(x), \qquad u(0) = u(1) = 0 \tag{2.24}$$

where $b, c \in L^\infty[0,1]$ and $f \in H = L^2[0,1]$.

If u is a twice continuously differentiable solution of this problem, then

$$< -u'' + bu' + cu, v >_{L^2[0,1]} = < f, v >_{L^2[0,1]}$$

for all $v \in L^2[0,1]$. That is

$$\int_0^1 (-u''v + bu'v + cuv)\, dx = \int_0^1 fv\, dx.$$

Applying integration by parts and using the fact that $u(0) = u(1) = 0$,

$$\int_0^1 (u'v' + bu'v + cuv)\, dx = \int_0^1 fv\, dx \tag{2.25}$$

for all $v \in L^2[0,1]$. This is the variational form of problem (2.24). Note that because of the boundary conditions, one looks for a solution in the subspace $H^1(0,1)$ of $L^2[0,1]$. Recall here the definition of the *Sobolev spaces* $H^1(0,1)$ and $H_o^1(0,1)$:

$$H^1(0,1) = \{\, f \in L^2[0,1] \;:\; f \text{ is absolutely continuous and } f' \in L^2[0,1] \,\}$$

with inner product

$$\langle f, g\rangle_{H^1(0,1)} = < f, g >_{L^2[0,1]} + < f', g' >_{L^2[0,1]}\ .$$

Then $H^1(0,1)$ is a Hilbert space in the norm determined by this inner product, and clearly $\|f\|_2 \le \|f\|_{H^1(0,1)}$ for all $f \in H^1(0,1)$. Obviously, $C_c^\infty(0,1) \subset H^1(0,1)$, and $H_o^1(0,1)$ denotes its closure in $H^1(0,1)$.

Thus, (2.25) gives a reformulation of the boundary value problem (2.24) as a problem in $H = H_o^1(0,1)$,

$$a(u,v) = F(v) \qquad \forall\, v \in H \tag{2.26}$$

where $a(\cdot,\cdot)$ is the bilinear form defined on H by

$$a(u,v) = \int_0^1 (u'v' + bu'v + cuv)\, dx \qquad (u, v \in H)\,, \tag{2.27}$$

and $F(v) =< f, v >_{L^2[0,1]}$. By the Lax-Milgram theorem, this problem will have a unique solution in H provided that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive, and that F is continuous on H.

Continuity of F on H is obvious since $\|v\|_{L^2[0,1]} \leq \|v\|_H$ for all $v \in H$. Similarly, $\|v'\|_{L^2[0,1]} \leq \|v\|_H$, and applying Hölder's inequality, it follows that $a(\cdot,\cdot)$ is continuous in the norm of H. Coerciveness of $a(\cdot,\cdot)$ is the nontrivial part, and at times can be established immediately, and sometimes by applying the following theorem:

Proposition 2.9. (Poincaré-Friedrichs) Let $v \in H^1(0,1)$. Then

$$\|v\|^2_{L^2[0,1]} \leq 2\left(v(0)^2 + \|v'\|^2_{L^2[0,1]}\right).$$

Proof. Note that for almost all $0 \leq x \leq 1$,

$$v(x) = v(0) + \int_0^x v'(t)\, dt.$$

Using the identity $(a+b)^2 \leq 2(a^2+b^2)$ and Hölder's inequality one obtains

$$\begin{aligned} v(x)^2 &\leq 2\left(v(0)^2 + \left[\int_0^x |v'(t)|\ dt\right]^2\right) \\ &\leq 2\left(v(0)^2 + \left[\int_0^1 |v'(t)|^2\ dt\right]\right) \leq 2\left(v(0)^2 + \|v'\|^2_{L^2[0,1]}\right). \end{aligned}$$

Integrating from 0 to 1, the assertion follows. □

2.2.3 The Galerkin Method

The idea of this method is to find an approximate solution of problem (2.26) in some finite dimensional subspace of H.

Thus, let $a(\cdot,\cdot)$ be a continuous and coercive bilinear form on a Hilbert space H, and F a continuous linear functional on H. Consider the problem of finding $u \in H$ so that

$$a(u,v) = F(v) \qquad \forall v \in H. \tag{2.28}$$

Choose a finite dimensional subspace V of H. When restricted to V this problem becomes one of finding $\tilde{u} \in V$ so that

$$a(\tilde{u},\tilde{v}) = F(\tilde{v}) \qquad \forall \tilde{v} \in V. \tag{2.29}$$

Since $a(\cdot,\cdot)$ is bounded and coercive, so is its restriction to V. Hence by the Lax-Milgram Theorem, equation (2.29) has a unique solution in V.

2.2.3.1 Error Estimate

One can give an estimate for the error of approximating the solution u of (2.28) by the solution $\tilde{u}$ of (2.29): For each $\tilde{v} \in V$

$$a(u-\tilde{u},\tilde{v}) = a(u,\tilde{v}) - a(\tilde{u},\tilde{v}) = F(\tilde{v}) - F(\tilde{v}) = 0, \tag{2.30}$$

Then by bilinearity, coerciveness and boundedness of $a(\cdot,\cdot)$ and by (2.30), for each $\tilde{v} \in V$,

$$\alpha\|u-\tilde{u}\|^2 \le a(u-\tilde{u},u-\tilde{u}) \overset{(2.30)}{=} a(u-\tilde{u},u-\tilde{v}) \le C\|u-\tilde{u}\|\,\|u-\tilde{v}\|.$$

Dividing by $\|u-\tilde{u}\|$ (which is nonzero unless $\tilde{u}$ is already the correct solution) we obtain $\alpha\|u-\tilde{u}\| \le C\|u-\tilde{u}\|$ for all $\tilde{v} \in V$ and hence

$$\|u-\tilde{u}\| \le \inf_{\tilde{v}\in V} \frac{C}{\alpha}\|u-\tilde{v}\| = \frac{C}{\alpha}\|(I-P)u\| \le \frac{C}{\alpha}\|u\| \tag{2.31}$$

where P denotes the orthogonal projection of H onto V.

2.2.3.2 Computing the Approximate Solution

The approximate solution $\tilde{u}$ can be found by simply solving a linear system: Choose a basis $\{e_1, \ldots, e_N\}$ of V. By linearity, (2.29) is equivalent to system of N equations

$$a(\tilde{u}, e_k) = F(e_k) \qquad k = 1, \ldots, N.$$

Expressing $\tilde{u}$ as a linear combination of the basis vectors, $\tilde{u} = \sum_{m=1}^{N} \alpha_m e_m$, we then obtain

$$\sum_{m=1}^{N} \alpha_m a(e_m, e_k) = F(e_k) \qquad k, m = 1, \ldots, N$$

or in matrix form,

$$AX = B$$

where $A = \Big[a(e_m, e_k)\Big]_{km}$, $X = [\alpha_1, \ldots, \alpha_N]^T$ and $B = [F(e_1), \ldots, F(e_N)]^T$.

2.2.4 The Wavelet-Galerkin Method

Let a MRA $\{V_j\}_{j\in\mathbb{Z}}$ with compactly supported scaling function φ and wavelet ψ be given. Since the sequence of spaces V_j is increasing and their union is dense in $L^2(\mathbb{R})$, it would be natural to use one of the spaces V_j as the approximation space in the Galerkin method, for sufficiently large j. Recall that the collection $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ of scaling functions at level j is an orthonormal basis of V_j. Thus, the spaces V_j are not finite dimensional.

Next consider a boundary value problem over $[0, 1]$ as in (2.24) for example, and its variational formulation (2.26) in the Hilbert space $H = H^1(0, 1)$. If φ is sufficiently differentiable, then all functions $\varphi_{j,k}$ will be elements of H. (To be precise, the restrictions $\varphi_{j,k_{|[0,1]}}$ will be in H.) Let us set $V_j[0, 1] = \{f_{|[0,1]} : f \in V_j\}$. Clearly, $\{V_j[0, 1]\}_{j\in\mathbb{Z}}$ is an increasing sequence of closed subspaces of $L^2[0, 1]$ whose union is dense in $L^2[0, 1]$. In addition, the collection $\{\varphi_{j,k_{|[0,1]}}\}_{k\in\mathbb{Z}}$ spans all of $V_j[0,1]$, for each j. Since φ is compactly supported, only finitely many of the functions $\varphi_{j,k|[0,1]}$ will be nonzero, forming a collection $\{\varphi_{j,k|[0,1]}\}_{k=L_1}^{L_2}$ for some integers L_1, L_2 which depend on j. It is not difficult to verify that this remaining collection is linearly independent, since the $\varphi_{j,k}$ are linearly independent in $L^2(\mathbb{R})$ and are translates of another.

One can now apply the Galerkin method to the finite dimensional subspace $V_j[0,1]$ of $H^1(0,1)$ by employing the basis $\{\varphi_{j,k|[0,1]}\}_{k=L_1}^{L_2}$, and obtain an approximate solution of problem (2.26) in $H^1(0,1)$.

Remark. Many authors apply the decomposition

$$V_j = V_{j_o} \oplus W_{j_o} \oplus W_{j_o+1} \oplus \cdots \oplus W_{j-1}, \qquad (j_o < j) \tag{2.32}$$

into wavelet spaces, and use the basis

$$\{\varphi_{j_o,k}\}_{k\in\mathbb{Z}},\ \{\psi_{j_o,k}\}_{k\in\mathbb{Z}}, \{\psi_{j_o+1,k}\}_{k\in\mathbb{Z}}, \ldots \{\psi_{j-1,k}\}_{k\in\mathbb{Z}}$$

of V_j consisting of scaling functions at lower level j_o, together with the wavelets at all intermediate scales. If one restricts each of the above spaces to $[0,1]$ one obtains a decomposition

$$V_j[0,1] = V_{j_o}[0,1] + W_{j_o}[0,1] + W_{j_o+1}[0,1] + \cdots + W_{j-1}[0,1]$$

of spaces which are no longer mutually orthogonal. This can be overcome in two ways:

1. Periodization. If j_o is sufficiently large so that $\varphi_{j_o,k}$ and $\psi_{j,k}$ have supports of length less than $1/2$, then the 1-periodic functions

$$\tilde{\varphi}_{j_o,k}(x) = \sum_{m\in\mathbb{Z}} \varphi_{j_o,k}(x-m) \quad \text{and} \quad \tilde{\psi}_{j,k}(x) = \sum_{m\in\mathbb{Z}} \psi_{j,k}(x-m)$$

will be mutually orthonormal in $L^2[0,1]$. Note that by compactness of supports, these are locally finite sums, hence they are well defined, and only finitely many periodized functions will be distinct, so that the above is really a finite collection. Correspondingly, (2.32) is modified to a sum of finite dimensional subspaces of $H^1(0,1)$

$$\tilde{V}_j = \tilde{V}_{j_o} \oplus \tilde{W}_{j_o} \oplus \tilde{W}_{j_o+1} \oplus \cdots \oplus \tilde{W}_{j-1},$$

each consisting of 1-periodic functions. One can now apply the Galerkin Method to problems with periodic boundary conditions. This approach was taken in Kumar and Mehra (2005), Kumar and Mehra (2005b), and Nielsen (1998).

2. Orthogonalization. If j_o is sufficiently large, then almost all of the basis functions of V_j whose restrictions to $[0,1]$ is nonzero will have support inside $[0,1]$, hence they are equal to their restrictions: $\varphi_{j_o,k_{|[0,1]}} = \varphi_{j_o,k}$ and $\psi_{j,k_{|[0,1]}} = \psi_{j,k}$ and thus form an orthonormal family in $L^2[0,1]$. Those whose supports cross the endpoints 0 or 1 can be rendered orthonormal on $[0,1]$ by the Gram-Schmidt process. This process leads to a modification of the spaces $W_j[0,1]$, and one obtains a modified decomposition

$$V_j[0,1] = V_{j_o}[0,1] \oplus \tilde{W}_{j_o}[0,1] \oplus \tilde{W}_{j_o+1}[0,1] \oplus \cdots \oplus \tilde{W}_{j-1}[0,1]$$

into orthogonal subspaces. This approach was taken by Monasse and Perrier (1998).

In chapters IV and V, by choosing to work with the basis of V_j consisting of the scaling functions $\{\varphi_{j,k_{|[0,1]}}\}_{k=L_1}^{L_2}$ we will avoid the complexities introduced by this orthogonalization.

CHAPTER III

CONNECTION COEFFICIENTS

Consider again the variational boundary value problem (2.27),

$$a(u,v) = F(v) \qquad \forall v \in H^1(0,1),$$

where

$$a(u,v) = \int_0^1 (u'v' + bu'v + cuv)\,dx \qquad (u,v \in H)\,,$$

and for simplicity, b and c are constant. Suppose, we want to find a solution of this problem by the Galerkin method in some finite-dimensional subspace $V_j[0,1]$. Expressing u and v in terms of the basis $\{\varphi_{j,k}\}_{k=L_1}^{L_2}$ of $V_j[0,1]$,

$$u = \sum_k c_k \varphi_{j,k} \qquad \text{and} \qquad v = \sum_l b_l \varphi_{j,l}$$

the left-hand side becomes

$$\sum_{k,l} a_k b_l \left[\int_0^1 \varphi'_{j,k}\varphi'_{j,l}\,dx + b\int_0^1 \varphi'_{j,k}\varphi_{j,l}\,dx + c\int_0^1 \varphi_{j,k}\varphi_{j,l} dx \right].$$

Since the functions φ are not known in closed form, one can not evaluate the derivatives and integrals directly. In this chapter, we will study how they can be computed, based on the ideas of Lin and Zhou (2001).

3.1 Definition of the Connection Coefficients

In generality, let $\varphi_{j,l}^{(n)}$ denote the n-th derivative of the function $\varphi_{j,l}$ and consider integrals of the form

$$\int_0^1 \varphi_{j,l_1}^{(d_1)}(x)\varphi_{j,l_2}^{(d_2)}(x)\,dx \qquad (l_1, l_2 \in \mathbb{Z}) \tag{3.1}$$

and

$$\int_0^1 \varphi_{j,l_1}^{(d_1)}(x)\varphi_{j,l_2}^{(d_2)}(x)\varphi_{j,l_3}^{(d_3)}(x)\,dx \qquad (l_1, l_2, l_3 \in \mathbb{Z}) \tag{3.2}$$

where j, d_1, d_2, d_3 are fixed non-negative integers, assuming that φ is sufficiently many times differentiable. Now by the chain rule,

$$\varphi_{j,l}^{(n)}(x) = \frac{d^n\,[\varphi_{j,l}(x)]}{dx^n} = \frac{d^n\left[2^{j/2}\varphi(2^j x - l)\right]}{dx^n} = 2^{j/2}2^{jn}\varphi^{(n)}(2^j x - l). \tag{3.3}$$

Applying (3.3) and replacing $2^j x$ by x, the above integrals become

$$2^{jd}\int_0^{2^j} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\,dx \tag{3.4}$$

where $d = d_1 + d_2$ and

$$2^{j(d+1/2)}\int_0^{2^j} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\varphi^{(d_3)}(x-l_3)\,dx \tag{3.5}$$

where $d = d_1+d_2+d_3$, respectively. In order to simplify the formulas, one drops the leading constants and defines the *two-term* and *three-term connection coefficients of* φ *on* $[0,1]$ *at level* j by

$$\Gamma_{l_1,l_2}^{j,d_1,d_2} = \int_0^{2^j} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\,dx \tag{3.6}$$

and

$$\Gamma_{l_1,l_2,l_3}^{j,d_1,d_2,d_3} = \int_0^{2^j} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\varphi^{(d_3)}(x-l_3)\,dx, \tag{3.7}$$

respectively. Thus,

$$\int_0^1 \varphi_{j,l_1}^{(d_1)}(x)\varphi_{j,l_2}^{(d_2)}(x)\,dx = 2^{jd}\Gamma_{l_1,l_2}^{j,d_1,d_2} \tag{3.8}$$

where $d = d_1 + d_2$, while

$$\int_0^1 \varphi_{j,l_1}^{(d_1)}(x)\varphi_{j,l_2}^{(d_2)}(x)\varphi_{j,l_3}^{(d_3)}(x)\,dx = 2^{j(d+1/2)}\Gamma_{l_1,l_2,l_3}^{j,d_1,d_2,d_3} \tag{3.9}$$

where $d = d_1 + d_2 + d_3$.

Remark. Latto, Resnikoff and Tenenbaum (1991) gave a definition of connection coefficients over unbounded intervals, such as

$$\Lambda_{l_1,l_2}^{d_1,d_2} = \int_{-\infty}^{\infty} \varphi^{(d_1)}(x - l_1)\varphi^{(d_2)}(x - l_2)\, dx$$

for example. These are much easier to compute, as by shift invariance of the integral,

$$\Lambda_{l_1,l_2}^{d_1,d_2} = \Lambda_{0,l_2-l_1}^{d_1,d_2}$$

and by integration by parts,

$$\Lambda_{0,l}^{d_1,d_2} = -\Lambda_{0,l}^{d_1+1,d_2-1}.$$

Since we will be dealing with boundary value problems over the unit interval, we have chosen the type of connection coefficients introduced by Lin and Zhou (2001).

3.2 Properties of the Connection Coefficients

Before explaining how these connection coefficients can be computed, we list some of their properties.

1. *Symmetry.* Clearly,

$$\Gamma_{l_1,l_2}^{j,d_1,d_2} = \Gamma_{l_2,l_1}^{j,d_2,d_1}$$

by definition. For three-term connection coefficients, several kinds of symmetry exist, for example,

$$\Gamma_{l_1,l_2,l_3}^{j,d_1,d_2,d_3} = \Gamma_{l_2,l_1,l_3}^{j,d_2,d_1,d_3}.$$

2. *Level-up* The connection coefficients at level $j + 1$ can be computed from

those at level j by a simple calculation. In fact by (3.6),

$$\begin{aligned}
\Gamma^{(j+1),d_1,d_2}_{l_1,l_2} &= \int_0^{2^j} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\,dx \\
&\qquad\qquad + \int_{2^j}^{2^{j+1}} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\,dx \\
&= \int_0^{2^j} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\,dx \\
&\qquad\qquad + \int_0^{2^j} \varphi^{(d_1)}(x-(l_1-2^j))\varphi^{(d_2)}(x-(l_2-2^j))\,dx \\
&= \Gamma^{j,d_1,d_2}_{l_1,l_2} + \Gamma^{j,d_1,d_2}_{l_1-2^j,l_2-2^j}.
\end{aligned}$$

Similar computations give that

$$\begin{aligned}
\Gamma^{(j+1),d_1,d_2,d_3}_{l_1,l_2,l_3} &= \int_0^{2^j} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\varphi^{(d_3)}(x-l_3)\,dx \\
&\qquad\qquad + \int_{2^j}^{2^{j+1}} \varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\varphi^{(d_3)}(x-l_3)\,dx \\
= \int_0^{2^j} &\varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2)\varphi^{(d_3)}(x-l_3)\,dx \\
&+ \int_0^{2^j} \varphi^{(d_1)}(x-(l_1-2^j))\varphi^{(d_2)}(x-(l_2-2^j))\varphi^{(d_3)}(x-(l_3-2^j))\,dx \\
= \Gamma^{j,d_1,d_2,d_3}_{l_1,l_2,l_3} &+ \Gamma^{j,d_1,d_2,d_3}_{l_1-2^j,l_2-2^j,l_3-2^j}.
\end{aligned}$$

As will be evident from the discussion in the next section, the calculation of the connection coefficients at level $j = 0$ is substantially simpler than at higher levels of the dilation parameter j. This property allows us to reduce the calculation of the connection coefficients to the simplest level $j = 0$.

3. *Finite collection.* Since φ has compact support, then for each j, only finitely many connection coefficients are nonzero.

4. *Partial sums.* One way to verify whether the computed connection coefficients are correct is by verifying partial sums. For example, by (2.8) and

(2.9), for all $l_1, l_2 \in \mathbb{Z}$,

$$\begin{aligned}\sum_{l_3} \Gamma^{j,1,0,0}_{l_1,l_2,l_3} &= \sum_{l_3} \int_0^{2^j} \varphi'(x-l_1)\varphi(x-l_2)\varphi(x-l_3)\,dx \\ &= \int_0^{2^j} \varphi'(x-l_1)\varphi(x-l_2)\left[\sum_{l_3}\varphi(x-l_3)\right]dx \\ &= \int_0^{2^j} \varphi'(x-l_1)\varphi(x-l_2)\,dx = \Gamma^{j,1,0}_{l_1,l_2}.\end{aligned}$$

3.3 The Moment Equations

Before we can explain how to compute the connection coefficients, we need to discuss the moment equations.

Let the wavelet ψ satisfy the assumption of Theorem 2.4. That is

$$x^p = \sum_{k\in\mathbb{Z}} M^p_k \varphi(x-k)$$

for all $p = 0, \ldots, L$, and this sum is locally finite. Differentiating n times ($n \le p$), provided that φ is sufficiently differentiable, we obtain

$$\frac{p!}{(p-n)!}x^{p-n} = \sum_{k\in\mathbb{Z}} M^p_k \varphi^{(n)}(x-k). \tag{3.10}$$

Now consider a two-term connection coefficient $\Gamma^{j,d_1,d_2}_{l_1,l_2}$ with $d_i \le p_i \le L$. Then by (3.10),

$$\begin{aligned}&\left[\frac{p_1!}{(p_1-d_1)!}x^{p_1-d_1}\right]\left[\frac{p_2!}{(p_2-d_2)!}x^{p_2-d_2}\right] \\ &\qquad = \left[\sum_{l_1\in\mathbb{Z}} M^{p_1}_{l_1}\varphi^{(d_1)}(x-l_1)\right]\left[\sum_{l_2\in\mathbb{Z}} M^{p_2}_{l_2}\varphi^{(d_2)}(x-l_2)\right],\end{aligned}$$

that is,

$$\frac{p_1!p_2!}{(p_1-d_1)!(p_2-d_2)!}x^{p_1+p_2-d}\sum_{l_1,l_2} M^{p_1}_{l_1}M^{p_2}_{l_2}\varphi^{(d_1)}(x-l_1)\varphi^{(d_2)}(x-l_2).$$

Integrate from 0 to 2^j,

$$\frac{p_1!p_2!2^{j(p_1+p_2-d+1)}}{(p_1-d_1)!(p_2-d_2)!(p_1+p_2-d+1)} = \sum_{l_1,l_2} M_{l_1}^{p_1} M_{l_2}^{p_2} \Gamma_{l_1,l_2}^{j,d_1,d_2}. \tag{3.11}$$

These equations are called the *moment equations*. For the three-term connection coefficients, we obtain similarly a family of moment equations,

$$\begin{aligned}\frac{p_1!p_2!p_3!2^{j(p_1+p_2+p_3-d+1)}}{(p_1-d_1)!(p_2-d_2)!(p_3-d_3)!(p_1+p_2+p_3-d+1)}&\\ = \sum_{l_1,l_2,l_3} M_{l_1}^{p_1} M_{l_2}^{p_2} M_{l_3}^{p_3} \Gamma_{l_1,l_2,l_3}^{j,d_1,d_2,d_3}.&\end{aligned} \tag{3.12}$$

In practical computations, it is advisable to use low orders p_i to avoid a rapid increase of the size of the moments M_k^p as $|k|$ increases. We also observe that in the case of Coiflets $C\{3N\}$, the moments are easily obtained: Since in this case the scaling function φ also has vanishing moments for $p = 1, \ldots N$ then

$$\begin{aligned} M_k^p &= \int_{-\infty}^{\infty} x^p \varphi(x-k)\, dx = \int_{-\infty}^{\infty} (x+k)^p \varphi(x)\, dx \\ &= \sum_{i=0}^{p} \binom{p}{i} k^{p-i} \int_{-\infty}^{\infty} x^i \varphi(x)\, dx = k^p \int_{-\infty}^{\infty} \varphi(x)\, dx = k^p \end{aligned}$$

for all $p = 1, \ldots, N$. Furthermore, (3.11) and (3.12) hold for $0 \le d_i \le p_i \le N-1$, as ψ has vanishing moments of order $\le N-1$.

3.4 Computation of the Connection Coefficients

We are now ready to discuss computation of the connection coefficients. The starting point is again the scaling relation (2.22),

$$\varphi(x) = \sum_{k=N_1}^{N_2} a_k \varphi(2x-k).$$

Differentiating n times,

$$\varphi^{(n)}(x) = \sum_{k=N_1}^{N_2} 2^n a_k \varphi^{(n)}(2x-k).$$

Substitute this representation of the derivatives into (3.6),

$$\begin{aligned}
\Gamma_{l_1,l_2}^{j,d_1,d_2} &= \int_0^{2^j} \left[\sum_{k=N_1}^{N_2} 2^{d_1} a_k \varphi^{(d_1)}(2x-2l_1-k)\right]\left[\sum_{m=N_1}^{N_2} 2^{d_2} a_m \varphi^{(d_2)}(2x-2l_2-m)\right] dx \\
&= 2^d \sum_{k,m=N_1}^{N_2} a_k a_m \int_0^{2^j} \varphi^{(d_1)}\big(2x-2l_1-k\big)\,\varphi^{(d_2)}\big(2x-2l_2-m\big)\,dx \quad (2x \to x) \\
&= 2^{d-1} \sum_{k,m=N_1}^{N_2} a_k a_m \int_0^{2^{j+1}} \varphi^{(d_1)}\big(x-(k+2l_1)\big)\,\varphi^{(d_2)}\big(x-(m+2l_2)\big)\,dx \\
&= 2^{d-1} \sum_{k,m=N_1}^{N_2} a_k a_m \left[\int_0^{2^j} \varphi^{(d_1)}\big(x-(k+2l_1)\big)\,\varphi^{(d_2)}\big(x-(m+2l_2)\big)\,dx\right. \\
&\qquad\qquad \left. + \int_{2^j}^{2^{j+1}} \varphi^{(d_1)}\big(x-(k+2l_1)\big)\,\varphi^{(d_2)}\big(x-(m+2l_2)\big)\right] dx \\
&= 2^{d-1} \sum_{k,m=N_1}^{N_2} a_k a_m \left[\Gamma_{k+2l_1,m+2l_2}^{j,d_1,d_2}\right. \\
&\qquad\qquad \left. + \int_0^{2^j} \varphi^{(d_1)}\big(x-(k+2l_1-2^j)\big)\,\varphi^{(d_2)}\big(x-(m+2l_2-2^j)\big)\right] dx \\
&= 2^{d-1} \sum_{k,m=N_1}^{N_2} a_k a_m \left[\Gamma_{k+2l_1,m+2l_2}^{j,d_1,d_2} + \Gamma_{k+2l_1-2^j,m+2l_2-2^j}^{j,d_1,d_2}\right],
\end{aligned} \tag{3.13}$$

where as usual, $d = d_1 + d_2$. In a similar way, one obtains

$$\Gamma_{l_1,l_2,l_3}^{j,d_1,d_2.d_3} = 2^{d-1} \sum_{k,m,n=N_1}^{N_2} a_k a_m a_n \left[\Gamma_{k+2l_1,m+2l_2,n+2l_3}^{j,d_1,d_2,d_3} + \Gamma_{k+2l_1-2^j,m+2l_2-2^j,n+2l_3-2^j}^{j,d_1,d_2,d_3}\right] \tag{3.14}$$

where $d = d_1 + d_2 + d_3$.

3.4.1 Computation of the Two-term Connection Coefficients Without Symmetry

Let us first consider the computation of the two-term coefficients. Most computations which follow are valid for all choices of d_1 and d_2, so we will frequently simply write Γ_{l_1,l_2}^{j} or even Γ_{l_1,l_2} instead of $\Gamma_{l_1,l_2}^{j,d_1,d_2}$. Since φ is compactly supported, all but finitely many of the connection coefficients will be zero. Let us determine which coefficients vanish with certainty.

In fact, for the integral (3.6) to be nonzero, two necessary conditions must be satisfied: The supports of $\varphi^{(d_1)}(x-l_1)$ and $\varphi^{(d_2)}(x-l_2)$ must

1. overlap on a set of positive measure, and
2. both intersect the interval $(0, 2^j)$.

Recall first that $\mathrm{supp}(\varphi) \subset [N_1, N_2]$ where N_1 and N_2 are as in Theorem 2.3. In particular, $\varphi(x)$ vanishes at the endpoints of this interval, and hence

$$\mathrm{supp}(\varphi^{(d_i)}(x-l_i)) \subset [N_1+l_i, N_2+l_i] \tag{3.15}$$

for all n and l_i, and these functions vanish at the two endpoints N_1+l_i and N_2+l_i.

Hence, condition 1. is satisfied only when

$$(N_1+l_1, N_2+l_1) \cap (N_1+l_2, N_2+l_2) \neq \emptyset,$$

that is,

$$(0, N_2-N_1) \cap (l_2-l_1, N_2-N_1+l_2-l_1) \neq \emptyset$$

which is equivalent to

$$|l_2-l_1| \leq N_2-N_1-1 = N-2. \tag{3.16}$$

As for condition 2, note that by (3.15),

$$\mathrm{supp}(\varphi^{(d_i)}(x-l)) \cap (0, 2^j) \neq \emptyset$$

can only hold when $-N_2+1 \leq l \leq 2^j-N_1-1$. For further reference, we observe that

$$\mathrm{supp}(\varphi_{j,l}) = \mathrm{supp}(\varphi(2^j x - l)) \subset \left[\frac{N_1+l}{2^j}, \frac{N_2+l}{2^j}\right],$$

and hence

$$\operatorname{supp}(\varphi_{j,l}) \cap (0,1) \neq \emptyset \Leftrightarrow \left[\frac{N_1+l}{2^j}, \frac{N_2+l}{2^j}\right] \cap (0,1) \neq \emptyset$$
$$\Leftrightarrow [N_1+l, N_2+l] \cap (0,2^j) \neq \emptyset$$

which by the above requires that $-N_2+1 \leq l \leq 2^j - N_1 - 1$.

Condition 2. is thus satisfied only when $N_2 + l_i > 0$ and $N_1 + l_i < 2^j$, that is, when

$$-N_2+1 \leq l_i \leq 2^j - N_1 - 1 \qquad (i=1,2). \tag{3.17}$$

Combining (3.16) and (3.17) we see that a connection coefficient $\Gamma^j_{l_1,l_2}$ can be nonzero only if the pair (l_1,l_2) lies in the set

$$\Delta^j = \big\{(l_1,l_2) \in \mathbb{Z}\times\mathbb{Z} : -N_2+1 \leq l_1 \leq 2^j - N_1 - 1 \quad \text{and}$$
$$\min(-N_2+1, N_1-N+2) \leq l_2 \leq \max(2^j-N_1-1, N_1+N-2)\big\}.$$

A simple computation shows that the set Δ^j has cardinality

$$K = K(j) = [2^j - 2N_2 + 1][2N-3] + 3N^2 - 9N$$

provided that $2^{j-1} \geq N_2$.

Observe that equations (3.13) can be written as

$$\Gamma^{j,d_1,d_2}_{l_1,l_2} = 2^{d-1} \sum_{(k,m)\in\Delta^j} \left[a_{k-2l_1}a_{m-2l_2} + a_{k-2l_1+2^j}a_{m-2l_2+2^j}\right]\Gamma^{j,d_1,d_2}_{k,m} \tag{3.18}$$

for $(l_1,l_2) \in \Delta^j$, where by Theorem 2.3, $a_i = 0$ whenever $i \notin [N_1,N_2]$. In order to replace the double index (l_1,l_2) by a single index, fix a bijection $\gamma : \Delta^j \to \{1,2,\ldots,K\}$. If we set $p = \gamma(l_1,l_2)$ and $r = \gamma(k,m)$ then the system (3.18) of K equations in K unknowns can be written as

$$\Gamma^{j,d_1,d_2}_p = 2^{d-1}\sum_{r=1}^{K} a_{p,r}\Gamma^{j,d_1,d_2}_r \qquad (p=1,\ldots,K) \tag{3.19}$$

where we have set

$$a_{p,r} = a_{k-2l_1}a_{m-2l_2} + a_{k-2l_1+2^j}a_{m-2l_2+2^j}.$$

In matrix form,

$$X = 2^{d-1}AX$$

where $A = [a_{p,r}]$ is a $K \times K$ matrix, and $X = (\Gamma_1^{j,d_1,d_2}, \ldots, \Gamma_K^{j,d_1,d_2})^T$ the vector of connection coefficients. This vector is thus a solution of the linear system

$$\left[A - 2^{1-d}I\right]X = 0, \tag{3.20}$$

I denotes the identity matrix. As this system is homogeneous, there is no unique solution. One must add one or several nonhomogeneous equations until one has an augmented matrix of rank K. If φ has vanishing first moments, this can be done by applying the moment equations.

Table B.2 in Appendix B shows the two-term connection coefficients $\Gamma^{j,1,0}$ of the coiflet $C12$ for $j = 0$, computed without symmetry.

3.4.2 Computation of the Two-term Connection Coefficients Using Symmetry

When $d_1 = d_2$, then symmetry of the connection coefficients, $\Gamma_{l_1,l_2}^{j,d_1,d_1} = \Gamma_{l_2,l_1}^{j,d_1,d_1}$, allows one to reduce the number of equations by nearly one-half, and also reduce the number of nonhomogeneous equations that need to be added. Here one replaces Δ^j by

$$\Delta_s^j = \{\,(l_1, l_2) \in \Delta^j \,:\, l_1 \leq l_2\,\}.$$

Then $K_s := \text{card}(\Delta_s^j) = 2^j(N-1) + 3N_2 - 3N - 2 - 2NN_2 + 3N^2/2$, and system of equations (3.18) simplifies to the system

$$\Gamma_{l_1,l_2}^{j,d_1,d_2} = 2^{d-1}\Bigg[\sum_{(k,k)\in\Delta_s^j}\left[a_{k-2l_1}a_{k-2l_2} + a_{k-2l_1+2^j}a_{k-2l_2+2^j}\right]\Gamma_{k,k}^{j,d_1,d_2} + 2\sum_{\substack{(k,m)\in\Delta_s^j\\ k<m}}\left[a_{k-2l_1}a_{m-2l_2} + a_{k-2l_1+2^j}a_{m-2l_2+2^j}\right]\Gamma_{k,m}^{j,d_1,d_2}\Bigg]$$

for $(l_1, l_2) \in \Delta_s^j$. Thus, after fixing a bijection $\gamma : \Delta_s^j \to \{1, 2, \ldots, K_s\}$ and setting $p = \gamma(l_1, l_2)$ and $r = \gamma(k, m)$, this system can be written as

$$\Gamma_p^{j,d_1,d_2} = 2^{d-1}\sum_{r=1}^{K} a_{p,r}\Gamma_r^{j,d_1,d_2} \qquad (p = 1, \ldots, K_s)$$

where now

$$a_{p,r} = \begin{cases} a_{k-2l_1}a_{k-2l_2} + a_{k-2l_1+2^j}a_{k-2l_2+2^j} & \text{if } k = m \\ 2\left[a_{k-2l_1}a_{m-2l_2} + a_{k-2l_1+2^j}a_{m-2l_2+2^j}\right] & \text{else.} \end{cases}$$

System (3.20) is now of reduced form; in particular, the matrix $A - 2^{1-d}I$ is of lower rank deficiency so that fewer moment equations need to be added.

Table B.1 in Appendix B shows the connection coefficients $\Gamma^{j,0,0}$ and $\Gamma^{j,1,1}$ of the coiflet $C12$ for $j = 0$, computed using symmetry.

3.4.3 Computation of the Three-term Connection Coefficients

The three-term connection coefficients can be computed similarly. In order for the integral (3.7) to be nonzero, the two necessary conditions required to be satisfied become now the following: The supports of $\varphi^{(d_i)}(x - l_i)$ $(i = 1, 2, 3)$ must

1. overlap on a set of positive measure, and

2. all three intersect the interval $(0, 2^j)$.

By (3.15), condition 1. is satisfied only when

$$|l_j - l_i| \leq N - 2 \qquad \text{for all } 1 \leq i < j \leq 3. \tag{3.21}$$

Condition 2 is satisfied only when

$$-N_2 + 1 \leq l_i \leq 2^j - N_1 - 1 \qquad (i = 1, 2, 3). \tag{3.22}$$

As before, we let

$$\Delta^j = \left\{ (l_1, l_2, l_3) \in \mathbb{Z}^3 : (l_1, l_2, l_3) \text{ satisfy conditions (3.21) and (3.22)} \right\},$$

and we let $K = K(j) = \text{card}(\Delta^j)$. Then equations (3.14) can be written as

$$\begin{aligned} \Gamma^{j,d_1,d_2,d_3}_{l_1,l_2,l_3} &= 2^{d-1} \\ &\times \sum_{(k,m,n)\in\Delta^j} \left[a_{k-2l_1} a_{m-2l_2} a_{n-2l_e} + a_{k-2l_1+2^j} a_{m-2l_2+2^j} a_{n-2l_3+2^j}\right] \Gamma^{j,d_1,d_2,d_3}_{k,m,n} \end{aligned} \tag{3.23}$$

for $(l_1, l_2, l_3) \in \Delta^j$. After fixing a bijection $\gamma : \Delta^j \to \{1, 2, \ldots, K\}$, then (3.23) can be written as

$$\Gamma^{j,d_1,d_2,d3}_{p} = 2^{d-1} \sum_{r=1}^{K} a_{p,r} \Gamma^{j,d_1,d_2,d3}_{r} \qquad (p = 1, \ldots, K). \tag{3.24}$$

where now $p = \gamma(l_1, l_2, l_3)$, $r = \gamma(k, m, n)$, and where we have set

$$a_{p,r} = a_{k-2l_1} a_{m-2l_2} a_{n-2l_3} + a_{k-2l_1+2^j} a_{m-2l_2+2^j} a_{n-2l_3+2^j}.$$

In matrix form,

$$X = 2^{d-1} AX$$

where $A = [a_{p,r}]$ is a $K \times K$ matrix, and $X = (\Gamma^{j,d_1,d_2,d3}_1, \ldots, \Gamma^{j,d_1,d_2,d3}_K)^T$. Equivalently,

$$\left[A - 2^{1-d} I\right] X = 0. \tag{3.25}$$

Again, this system is not homogeneous and one must, for example, add one or several moment equations. In addition, under presence of symmetry, for example if $d_2 = d_3$, the complexity of this system can be reduced as explained in the case of two-term coefficients.

Table B.3 in Appendix B shows the three-term connection coefficients $\Gamma^{j,1,0,0}$ and $\Gamma^{j,0,0,1}$ of the coiflet $C12$ for $j = 0$, computed using symmetry.

The computation of all connection coefficients was performed with mixed C/C++ code and using the LAPACK numerical library.

CHAPTER IV

BURGERS EQUATION

In this chapter we discuss the application of the Wavelet-Galerkin Method to the Burgers equation. Our presentation essentially follows Lin and Zhou (2001). The purpose is to verify validity of our code by comparing our results with those of Lin and Zhou (2001), and in addition to evaluate by numerical experiments how the choice of coiflets and scaling level influences accuracy of the solution.

4.1 Formulation of the Problem

Consider the homogeneous Burgers equation

$$u_t + uu_x = \nu u_{xx}. \tag{4.1}$$

This equation was first discussed by Burgers (1948) in the modelling of one-dimensional fluid flow. Here, $u(x,t)$ denotes the velocity of a fluid and the constant ν is its viscosity. It is well known that solutions of this equation can be linearized to the one-dimensional heat equation by means of the Hopf-Cole transformation (Cole,1951; Hopf,1950), and hence some exact solutions are known. For this reason, the Burgers equation is often used to verify the accuracy of numerical schemes, a discussion of which can be found in Fletcher (1991).

We begin by imposing the following Dirichlet boundary conditions:

$$u(x,0) = u^0(x) = \begin{cases} 1, & 0 \le x \le 1/2, \\ 0, & 1/2 < x \le 1, \end{cases} \tag{4.2}$$

$$u(0,t) = 1, \qquad u(1,t) = 0 \qquad (t > 0).$$

Next apply the semi-implicit scheme with regards to time t,

$$\begin{aligned} &\frac{u^{k+1}-u^k}{\Delta t} + u^k \frac{\partial u^{k+1}}{\partial x} = \nu \frac{\partial^2 u^{k+1}}{\partial x^2} \qquad (0 < x < 1) \\ &u^k(0) = 1 \qquad u^k(1) = 0 \end{aligned} \tag{4.3}$$

for $k = 0, 1, 2, \ldots$ and where $u^k = u(x, k\Delta t)$. Setting $w = u^{k+1}$ and $g = u^k$, then at each time step k we have to solve a boundary value problem

$$-\nu \Delta t w'' + \Delta t g w' + w = g \tag{4.4}$$

$$w(0) = 1, \quad w(1) = 0. \tag{4.5}$$

The variational form of equation (4.4) is

$$a(u, v) = F(v) \qquad \big(u \in H^1(0,1),\ v \in H_0^1(0,1)\big)$$

where the bilinear form $a(\cdot, \cdot)$ is given by

$$a(u, v) = \int_0^1 (-\nu \Delta t u'' v + \Delta t g u' v + uv)\, dx$$

or after integrating the first term by parts

$$a(u, v) = \int_0^1 (\nu \Delta t u' v' + \Delta t g u' v + uv)\, dx, \tag{4.6}$$

and

$$F(v) = < g, v > = \int_0^1 g v\, dx. \tag{4.7}$$

Clearly, $a(\cdot, \cdot)$ and $L(\cdot)$ are continuous on $H^1(0,1)$. To verify that $a(\cdot, \cdot)$ is coercive, we note that

$$\begin{aligned} a(v, v) &= \int_0^1 \left(\nu \Delta t \,(v')^2 + \Delta t g v' v + v^2\right) dx \\ &= \int_0^1 \left(\left[\nu \Delta t - (\Delta t g^2\right] (v')^2 + \left[\Delta t g v' + \frac{v}{2}\right]^2 + \frac{3}{4} v^2\right) dx \\ &\geq \int_0^1 \left(\Delta t \left[\nu - (\Delta t g)^2\right] (v')^2 + \frac{3}{4} v^2\right) dx. \end{aligned}$$

Now since the solution $g = u^k$ is bounded, then $\nu - (\Delta t g)^2 \geq \beta > 0$ provided that Δt is sufficiently small (this can be verified at each time step in the computation) and hence we obtain

$$a(v,v) \geq \int_0^t \left(\Delta t \beta \left(v'\right)^2 + \frac{3}{4} v^2 \right) dx \geq \int_0^1 \Delta t \beta \left(\left(v'\right)^2 + v^2 \right) dx = \Delta t \beta \|v\|^2_{H^1(0,1)}$$

provided that Δt is sufficiently small, which shows that this bilinear form is coercive. Hence by the Lax-Milgram Theorem, the equation (4.7) has a unique solution in $H^1_0(0,1)$. Observe however, that because of the given boundary conditions, the solution u can only be an element of the larger space $H^1(0,1)$.

In order to obtain a solution satisfying the given boundary conditions, now split the variational problem into 3 parts:

1. Find the solution u_0 to the problem

$$a(u,v) = F(v) \qquad \forall\, v \in H^1(0,1) \tag{4.8}$$

2. Find the solution u_1 to the problem

$$a(u,v) = v(0) \qquad \forall\, v \in H^1(0,1) \tag{4.9}$$

3. Find the solution u_2 to the problem

$$a(u,v) = v(1) \qquad \forall\, v \in H^1(0,1). \tag{4.10}$$

Note that $v(0)$ and $v(1)$ are well defined, as all elements of $H^1(0,1)$ are continuous. Next set

$$u = u_0 + \lambda_1 u_1 + \lambda_2 u_2 \tag{4.11}$$

and find λ_1, λ_2 so that

$$u(0) = 1, \quad u(1) = 0. \tag{4.12}$$

This can be easily done, since by (4.11), λ_1 and λ_2 are solutions to the system

$$\begin{aligned} u_1(0)\lambda_1 + u_2(0)\lambda_2 &= 1 - u_0(0) \\ u_1(1)\lambda_1 + u_2(1)\lambda_2 &= -u_0(1). \end{aligned} \tag{4.13}$$

Now the function u of (4.11) satisfies the boundary value problem

$$a(u,v) = F(v) + \lambda_1 v(0) + \lambda v(1), \qquad u(0) = 1, \quad u(1) = 0 \qquad \big(v \in H^1(0,1)\big)$$

so that in particular, for all $v \in H_o^1(0,1)$,

$$a(u,v) = F(v), \qquad u(0) = 1, \quad u(1) = 0.$$

4.2 Solution by the Wavelet Galerkin Method

We now discuss the solution of problem (4.8). We begin by choosing a real valued compactly supported scaling function φ which has N nonzero scaling filter coefficients and whose support is contained in an interval of length $N-1$, as outlined in Chapter II. In our experiments this is a coiflet scaling function, as it has vanishing first moments, but this is no restriction for the algorithm below. Next choose an approximation space V_j, so that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j. We also let $V_j[0,1]$ denote the finite dimensional space obtained by restricting the functions in V_j to $[0,1]$. Then those basis functions of V_j whose supports intersect the unit interval, say the collection $\{\varphi_{j,k}\}_{k=L_1}^{L_2}$ will be a basis of $V_j[0,1]$ and its dimension will be $L = L_2 - L_1 + 1$.

Replacing g by its projection onto $V_j[0,1]$, we thus have to solve the problems

$$a(u,v) = F(v), \qquad \forall\, v \in V_j[0,1] \tag{4.14}$$

$$a(u,v) = v(0), \qquad \forall\, v \in V_j[0,1] \tag{4.15}$$

$$a(u,v) = v(1), \qquad \forall\, v \in V_j[0,1]. \tag{4.16}$$

Now we express u and g in the basis of $V_j[0,1]$,

$$u = \sum_{k=L_1}^{L_2} c_k \varphi_{j,k} \qquad \text{and} \qquad g = \sum_{m=L_1}^{L_2} b_m \varphi_{j,m}$$

Then (4.14) becomes

$$\int_0^1 \left(\nu \Delta t \left[\sum_{k=L_1}^{L_2} c_k \varphi_{j,k} \right]' \varphi'_{j,l} + \Delta t \left[\sum_{m=L_1}^{L_2} b_m \varphi_{j,m} \right] \left[\sum_{k=L_1}^{L_2} c_k \varphi_{j,k} \right]' \varphi_{j,l} \right.$$
$$\left. + \left[\sum_{k=L_1}^{L_2} c_k \varphi_{j,k} \right] \varphi_{j,l} \right) dx = \int_0^1 \left[\sum_{m=L_1}^{L_2} b_m \varphi_{j,m} \right] \varphi_{j,l} \, dx$$

for all $l = L_1, \ldots, L_2$. That is,

$$\sum_{k=L_1}^{L_2} c_k \left(\nu \Delta t \int_0^1 \varphi'_{j,k} \varphi'_{j,l} \, dx + \Delta t \sum_{m=L_1}^{L_2} b_m \int_0^1 \varphi'_{j,k} \varphi_{j,l} \varphi_{j,m} \, dx + \int_0^1 \varphi_{j,k} \varphi_{j,l} \, dx \right)$$
$$= \sum_{m=L_1}^{L_2} b_m \int_0^1 \varphi_{j,m} \varphi_{j,l} \, dx.$$

By (3.4) and (3.5) this becomes

$$\sum_{k=L_1}^{L_2} c_k \left(2^{2j} \nu \Delta t \Gamma_{k,l}^{j,1,1} + 2^{3j/2} \Delta t \sum_{m=L_1}^{L_2} b_m \Gamma_{k,l,m}^{j,1,0,0} + \Gamma_{k,l}^{j,0,0} \right) = \sum_{m=L_1}^{L_2} b_m \Gamma_{m,l}^{j,0,0}$$

for all $l = L_1, \ldots, L_2$. This can be expressed as a matrix equation

$$AU_o = B_o \tag{4.17}$$

where A is the matrix of size $L \times L$ whose entries are

$$a_{lk} = 2^{2j} \nu \Delta t \Gamma_{k,l}^{j,1,1} + 2^{3j/2} \Delta t \sum_{m=L_1}^{L_2} b_m \Gamma_{k,l,m}^{j,1,0,0} + \Gamma_{k,l}^{j,0,0},$$

U_o is the vector of unknowns, $U_o = \{c_l\}_{l=L_1}^{L_2}$, and B_o the column vector whose l-th entry b_l is

$$b_l = \sum_{m=L_1}^{L_2} b_m \Gamma_{m,l}^{j,0,0}.$$

Here all vector and matrix indices range from L_1 to L_2. Observe that A is a band matrix whose upper and lower bands have width $N-2$ each, as $\Gamma_{kl} = 0$ whenever

$|k-l| > N-2$. System (4.17) can be solved by LU decomposition for example. In our experiments we have used the LAPACK routine `DBGSV` for band matrices. The problems (4.15) and (4.16) can be solved in a similar way, by solving the systems

$$AU_1 = B_1 \qquad \text{and} \qquad AU_2 = B_2,$$

where $B_1 = \{\varphi_{j,l}(0)\}_{l=L_1}^{L_2}$ and $B_2 = \{\varphi_{j,l}(1)\}_{l=L_1}^{L_2}$. Computing the values of λ_1 and λ_2 from (4.13), then the vector

$$U = U_0 + \lambda_1 U_1 + \lambda_2 U_2$$

will contain the coefficients of the solution u in $V_j[0,1]$ of problem (4.6) – (4.7).

4.3 Solution by Finite Difference Schemes

In order to compare the wavelet solution with traditional methods, numerical experiments using a finite difference scheme with and without flux limiter were performed. Partitioning the interval $[0,1]$ into subintervals of length Δx each with grid points $x_i = i\Delta x$, one naturally obtains the scheme

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} + \frac{u_i^k\left(u_{i+1}^k - u_{i-1}^k\right)}{2\Delta x} = \frac{\nu(u_{i+1}^k - 2u_i^k + u_{i-1}^k)}{\Delta x^2},$$

where $u_i^k = u(x_i, t_k)$ with $t_k = k\Delta t$. However, because we will also use schemes with flux limiters, we modify the above.

Consider an equation without diffusion term,

$$\frac{\partial Q}{\partial t} + \frac{\partial (vQ)}{\partial x} = 0 \tag{4.18}$$

where $v = v(x,t)$ is a velocity field and $Q = Q(x,t)$ is a conserved quantity. For example, in the inviscid Burgers equation ($\nu = 0$) we have $v = u/2$ and $Q = u$. The flux of quantity Q is defined by $F = vQ$. To maintain the conservative property of (4.18) the numerical flux $\widehat{F}(x,t)$ is implicitly defined by

$$F(x,t) = \frac{1}{\triangle x}\int_{x-\triangle x/2}^{x+\triangle x/2} \widehat{F}(x,t)dx,$$

hence the derivative $\partial F/\partial x$ is calculated exactly by the formula

$$\frac{\partial F}{\partial x}(x,t) = \frac{\widehat{F}(x+\triangle x/2,t) - \widehat{F}(x-\triangle x/2,t)}{\triangle x}.$$

Discretization of equation (4.18) is now obtained in conservative form and expressed explicitly in time as

$$\frac{Q_i^{k+1} - Q_i^k}{\Delta t} + \frac{\widehat{F}_{i+1/2}^k - \widehat{F}_{i-1/2}^k}{\Delta x} = 0 \tag{4.19}$$

where $Q_i^k = Q(x_i, t_k)$, and $\widehat{F}_{i\pm1/2}^k$ are the numerical Q fluxes through the right and left boundaries of the i-th grid cell, respectively. Different numerical approximations of the $\widehat{F}_{i\pm1/2}$ give finite difference schemes with different properties. The most straightforward approximation to $\widehat{F}_{i+1/2}^k$ is certainly a linear approximation,

$$\widehat{F}_{i+1/2}^k = v_{i+1/2}^k Q_{i+1/2}^k = v_{i+1/2}^k \frac{Q_i^k + Q_{i+1}^k}{2} \tag{4.20}$$

with a similar approximation of $\widehat{F}_{i-1/2}^k$. This gives an approximation of the partial derivative $\partial(vQ)/\partial x$ by central differences, and (4.19) becomes

$$\frac{Q_i^{k+1} - Q_i^k}{\Delta t} + \frac{v_{i+1/2}^k(Q_{i+1}^k + Q_i^k) - v_{i-1/2}^k(Q_i^k + Q_{i-1}^k)}{2\Delta x} = 0 \tag{4.21}$$

Applying this concept of numerical flux to the Burgers equation (4.1) ($v = u/2$, $Q = u$) we obtain

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} + \frac{\hat{F}_{i+1/2} - \hat{F}_{i-1/2}}{\Delta x} = \frac{\nu(u_{i+1}^k - 2u_i^k + u_{i-1}^k)}{\Delta x^2}.$$

The scheme (4.21) becomes

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} + \frac{u_{i+1/2}^k(u_{i+1}^k + u_i^k) - u_{i-1/2}^k(u_i^k + u_{i-1}^k)}{4\Delta x} = \frac{\nu(u_{i+1}^k - 2u_i^k + u_{i-1}^k)}{\Delta x^2}.$$

Using linear approximation for $u_{i+1/2}^k$ and $u_{i-1/2}^k$ then we have a second order approximation scheme

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} + \frac{(u_{i+1}^k + u_i^k)^2 - (u_i^k + u_{i-1}^k)^2}{8\Delta x} = \frac{\nu(u_{i+1}^k - 2u_i^k + u_{i-1}^k)}{\Delta x^2}.$$

4.4 Flux Limiters

The choice of second order numerical flux (4.20), however, leads to the appearance of spurious oscillations in the numerical solution. One strategy to avoid nonphysical oscillations and excessive numerical diffusion is a hybrid method which uses the second order numerical flux in smooth regions and limits the solution in vicinity of discontinuities by using the monotonic upwind method in these regions. This procedure is carried out by introducing a flux-limiter based on the local gradient of the solution. We write the interface value $Q^k_{i+1/2}$ as the sum of the diffusive first order upwind term and an "anti-diffusive" one. The higher order antidiffusive part is multiplied by the flux limiter, which depends locally on the nature of the solution by means of the non-linear function $\theta_{i+1/2}$. This function is expressed by the slope ratios at the neighborhood of the interfaces in the upwind direction,

$$\theta_{i+1/2} = \begin{cases} \dfrac{Q^k_i - Q^k_{i-1}}{Q^k_{i+1} - Q^k_i} = \theta^+_{i+1/2} & \text{if } v^k_{i+1/2} \geq 0, \\ \dfrac{Q^k_{i+2} - Q^k_{i+1}}{Q^k_{i+1} - Q^k_i} = \theta^-_{i+1/2} & \text{if } v^k_{i+1/2} < 0. \end{cases} \tag{4.22}$$

Introduction of this new parameter θ and the limiter function Ψ, leads to the flux limiter version of the hybrid scheme as

$$Q^k_{i+1/2} = \begin{cases} Q^k_i + \frac{1}{2}(Q^k_{i+1} - Q^k_i)\Psi\left(\theta^+_{i+1/2}\right) & \text{if } v^k_{i+1/2} \geq 0, \\ Q^k_{i+1} - \frac{1}{2}(Q^k_{i+1} - Q^k_i)\Psi\left(\theta^-_{i+1/2}\right) & \text{if } v^k_{i+1/2} < 0. \end{cases} \tag{4.23}$$

The interface value $Q^k_{i-1/2}$ is obtained by using the same formula as for $Q^k_{i+1/2}$, by replacing the index i with $i-1$. From Eq. (4.23), one can see that if $\Psi = 0$ we find the upwind scheme, and if $\Psi = 1$ the scheme is reduced to the central one.

The following limiter functions are used in this study (LeVeque, 1996):

$$
\begin{aligned}
\text{Minmod}: \ & \Psi(\theta) = \max(0, \min(1, \theta)), \\
\text{Superbee}: \ & \Psi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta)), \\
\text{Van Leer}: \ & \Psi(\theta) = (\theta + |\theta|)/(1 + |\theta|).
\end{aligned}
\tag{4.24}
$$

The Minmod and Superbee limiters were introduced by Roe (1985) and Roe and Sidilkover (1992). The Van Leer limiter was introduced in Van Leer (1974).

4.5 Results of Numerical Experiments

In this section we present the results of the numerical experiments with the boundary value problem (4.2), for Reynold numbers $Re = 200$ and $Re = 2000$, respectively. (Here we have set $Re = 1/\nu$.)

4.5.1 The Case $Re = 200$

In case $Re = 200$ viscosity is sufficiently large so that the initial jump at $x = 0.5$ smoothes out quickly.

Figures 4.1–4.8 show plots of the wavelet solutions by C12 Coiflets for scales $j = 6$ to $j = 9$ and time scales $\Delta t = 10^{-2}$ to 10^{-5}, as a set of four pairs. The first plot in each pair shows the solution on the jump interval, while the second plot gives a zoomed image into the center of the jump.

The graphs show that at time steps 10^{-4} or smaller, the wavelet solution approximates the exact solution very well; the wavelet and the exact solutions are visually indistinguishable when $j = 8$ or $j = 9$ and $\Delta t = 10^{-5}$. This fact is corroborated in Table 4.1 which shows the errors of approximation, in both the supremum and the mean-square norms.

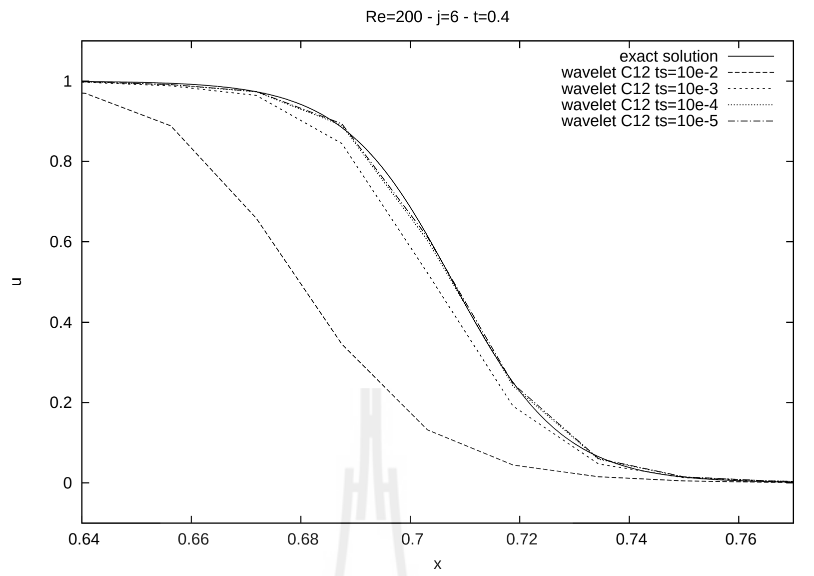

Figure 4.1 Wavelet solution of the Burgers equation for scale $j = 6$ at various time resolutions, for $Re = 200$ ($t = 0.4$).

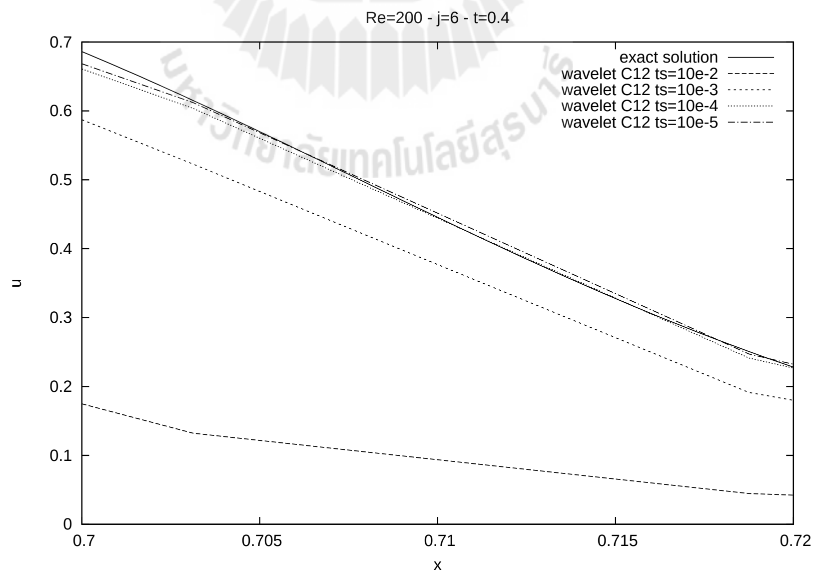

Figure 4.2 Wavelet solution of the Burgers equation for scale $j = 6$ at various time resolutions, for $Re = 200$, enlarged ($t = 0.4$).

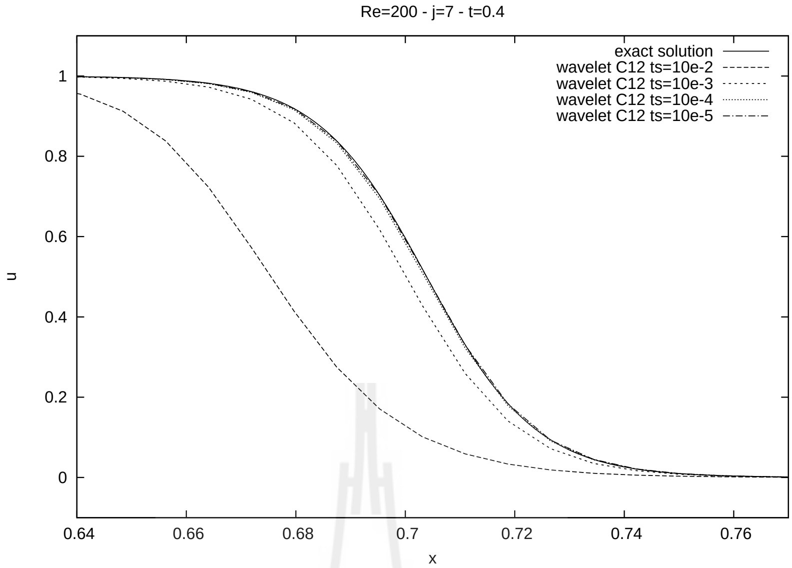

Figure 4.3 Wavelet solution of the Burgers equation for scale $j = 7$ at various time resolutions, for $Re = 200$ $(t = 0.4)$.

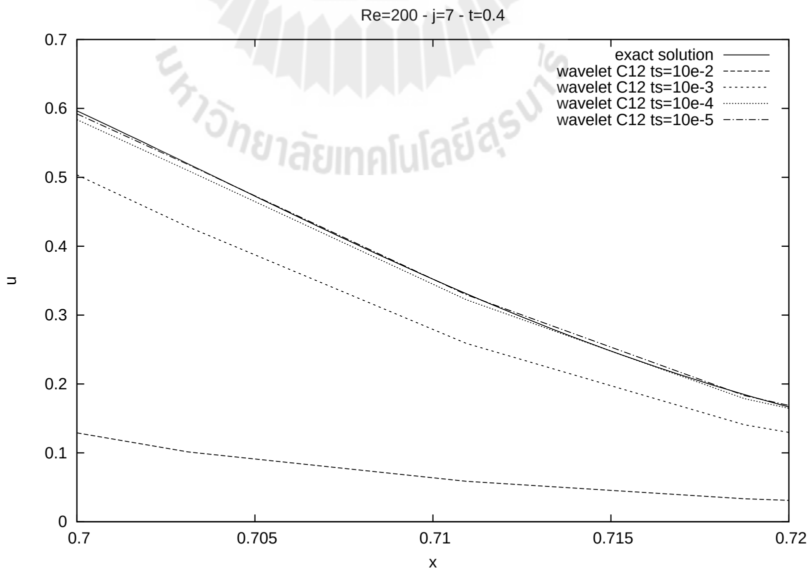

Figure 4.4 Wavelet solutions of the Burgers equation for scale $j = 7$ at various time resolutions, for $Re = 200$, enlarged $(t = 0.4)$.

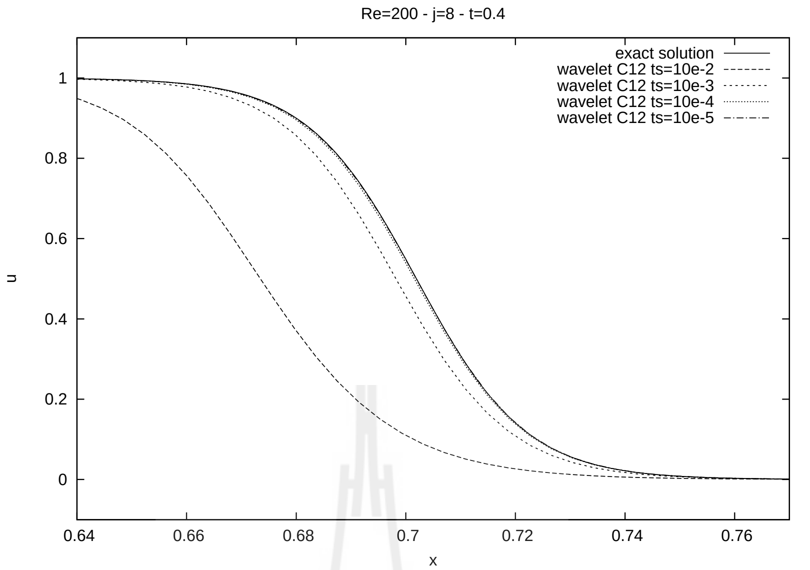

Figure 4.5 Wavelet solutions of the Burgers equation for scale $j = 8$ at various time resolutions, for $Re = 200$ ($t = 0.4$).

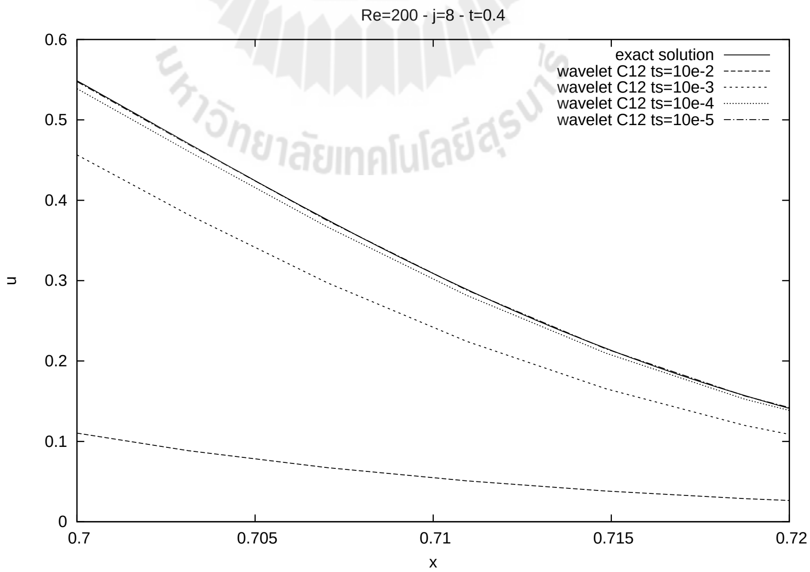

Figure 4.6 Wavelet solutions of the Burgers equation for scale $j = 8$ at various time resolutions, for $Re = 200$, enlarged ($t = 0.4$).

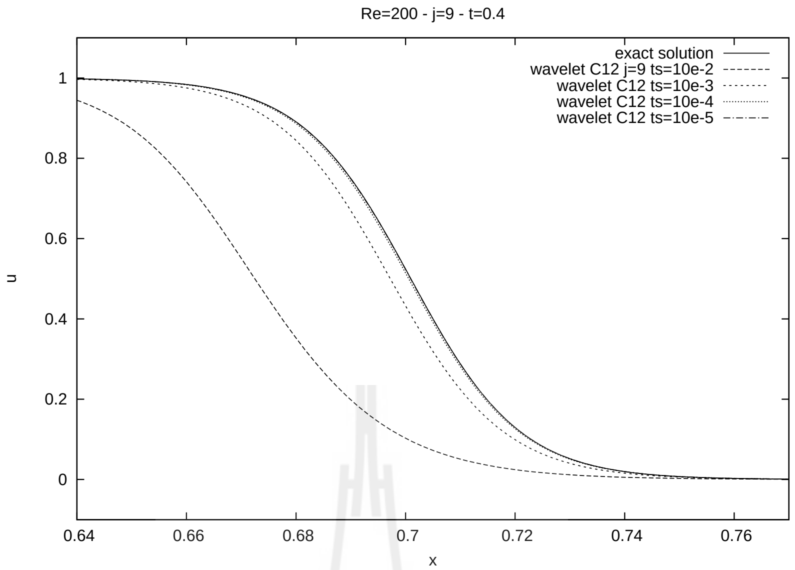

Figure 4.7 Wavelet solutions of the Burgers equation for scale $j = 9$ at various time resolutions, for $Re = 200$ ($t = 0.4$).

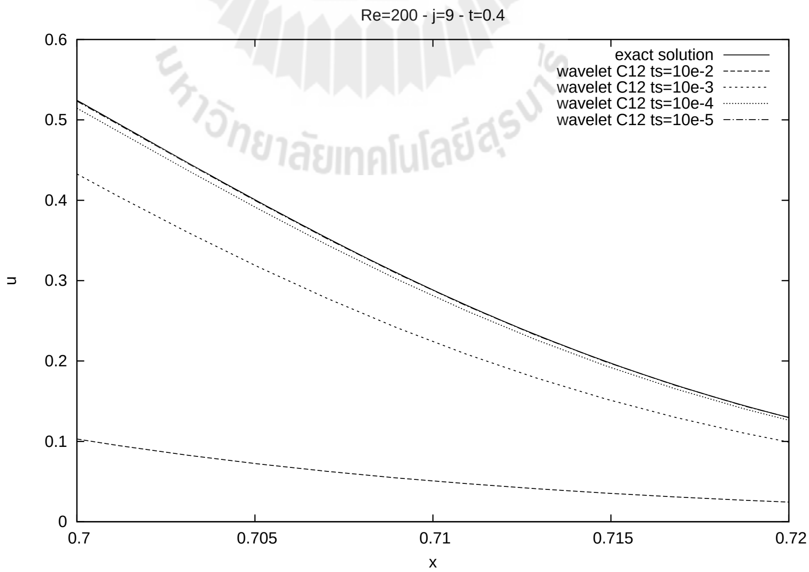

Figure 4.8 Wavelet solutions of the Burgers equation for scale $j = 9$ at various time resolutions, for $Re = 200$, enlarged ($t = 0.4$).

Table 4.1 Errors computing the solution of the Burgers equation by the wavelet method at various time steps and scales for $Re = 200$ using $C12$ Coiflets. ($t = 0.4$).

Scale	Norm	$\Delta t = 10^{-3}$	$\Delta t = 10^{-4}$	$\Delta t = 10^{-5}$
$j = 6$	$\|u - u_e\|_\infty$	0.092354	0.011483	0.009464
	$\|u - u_e\|_2$	0.014836	0.002194	0.001544
	$\frac{\|u - u_e\|_2}{\|u_e\|_2}$	0.017661	0.002612	0.001838
$j = 7$	$\|u - u_e\|_\infty$	0.091507	0.009812	0.001445
	$\|u - u_e\|_2$	0.015025	0.001569	0.000214
	$\frac{\|u - u_e\|_2}{\|u_e\|_2}$	0.017986	0.001878	0.000256
$j = 8$	$\|u - u_e\|_\infty$	0.092935	0.009705	0.000989
	$\|u - u_e\|_2$	0.015096	0.001576	0.000158
	$\frac{\|u - u_e\|_2}{\|u_e\|_2}$	0.018122	0.001892	0.000190
$j = 9$	$\|u - u_e\|_\infty$	0.092974	0.009794	0.000983
	$\|u - u_e\|_2$	0.015107	0.001580	0.000159
	$\frac{\|u - u_e\|_2}{\|u_e\|_2}$	0.018161	0.001899	0.000191

It can be noticed that when the time steps are relatively large, a change of scale j has little effect on accuracy. On the other hand, at the smallest time step $\Delta t = 10^{-5}$, increasing scale improves accuracy, at least until $j = 8$, so that the error falls below 0.001.

The pair of Figures 4.9 and 4.10 represents a visual comparison of the wavelet solution with solutions by various finite difference methods, at low scale $j = 6$ and for time step $\Delta t = 10^{-4}$. The first of the two plots again show the graphs over the overall interval on which the solution decreases to zero, while the second plot is a zoomed view into the center of this interval.

The scheme by central differences does not exhibit monotonicity, while the first order monotone scheme gives a solution with smoothed-out gradient. On the other hand, all three flux limiter schemes give good approximations. The wavelet solution is closest to the exact solution.

Table 4.2 shows some of the data used for these two plots, together with the errors of approximation in the supremum and mean-square norms. The error of the wavelet solution is noticeably below that of any of the finite difference solutions.

Finally, the pair of Figures 4.11–4.12 as well as Table 4.3 are a comparison of the wavelet solutions for various choices of Coiflet scaling functions $C\{3N\}$, namely for $N = 4, 6, 8$ and 10, at $j = 6$ and $\Delta t = 10^{-4}$. The choice of N does not affect the solution noticeably. Since increasing N leads to a larger number of connection coefficients and hence to a larger computation time, the Coiflets $C12$ are a reasonably good choice.

Table 4.2 Comparison of the exact solution of the Burgers equation with solutions by the wavelet method and by various finite difference schemes for $Re = 200$. ($t = 0.4$, $\Delta t = 10^{-4}$, $\Delta x = 2^{-6}$).

x	Exact solution u_e	CoifletC12 $j = 6$ u_{w6}	Central difference u_{cd}	First order monotone u_{1m}	Maxmod limiter u_{mm}	Super Bee limiter u_{sb}	van-Leer limiter u_{vl}
0.000000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.54688	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.56250	1.00000	1.00000	1.00000	0.99999	1.00000	1.00000	1.00000
0.57813	1.00000	1.00000	0.99997	0.99996	1.00000	1.00000	1.00000
0.59375	0.99999	0.99999	1.00002	0.99984	0.99998	1.00000	1.00000
0.60938	0.99995	0.99995	1.00028	0.99945	0.99991	0.99998	0.99997
0.62500	0.99975	0.99977	0.99957	0.99814	0.99959	0.99991	0.99984
0.64063	0.99881	0.99811	0.99729	0.99379	0.99816	0.99943	0.99911
0.65625	0.99432	0.99087	1.00671	0.97986	0.99190	0.99661	0.99513
0.67188	0.97336	0.97328	1.02369	0.93744	0.96532	0.98013	0.97417
0.68750	0.88424	0.88968	0.90299	0.82395	0.86532	0.89383	0.87760
0.70313	0.61518	0.60370	0.55394	0.59593	0.60323	0.59640	0.59809
0.71875	0.25073	0.24134	0.22445	0.31241	0.26474	0.24360	0.25431
0.73438	0.06543	0.05894	0.06888	0.11587	0.08378	0.06873	0.07696
0.75000	0.01441	0.01487	0.01877	0.03420	0.02308	0.01820	0.02074
0.76563	0.00304	0.00489	0.00492	0.00917	0.00607	0.00474	0.00542
0.78125	0.00063	0.00109	0.00127	0.00238	0.00157	0.00122	0.00140
0.79688	0.00013	0.00007	0.00033	0.00061	0.00040	0.00032	0.00036
0.81250	0.00003	0.00000	0.00008	0.00016	0.00010	0.00008	0.00009
0.82813	0.00001	0.00000	0.00002	0.00004	0.00003	0.00002	0.00002
0.84375	0.00000	0.00000	0.00001	0.00001	0.00001	0.00001	0.00001
0.85938	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
max error		0.01148	0.06125	0.06168	0.01893	0.01878	0.01710
at $x =$		0.703125	0.703125	0.718750	0.687500	0.703125	0.703125
$\|u - u_e\|_2$		0.00219	0.01084	0.01387	0.00431	0.00300	0.00288
$\frac{\|u - u_e\|_2}{\|u_e\|_2}$		0.00261	0.01290	0.01651	0.00513	0.00357	0.00343

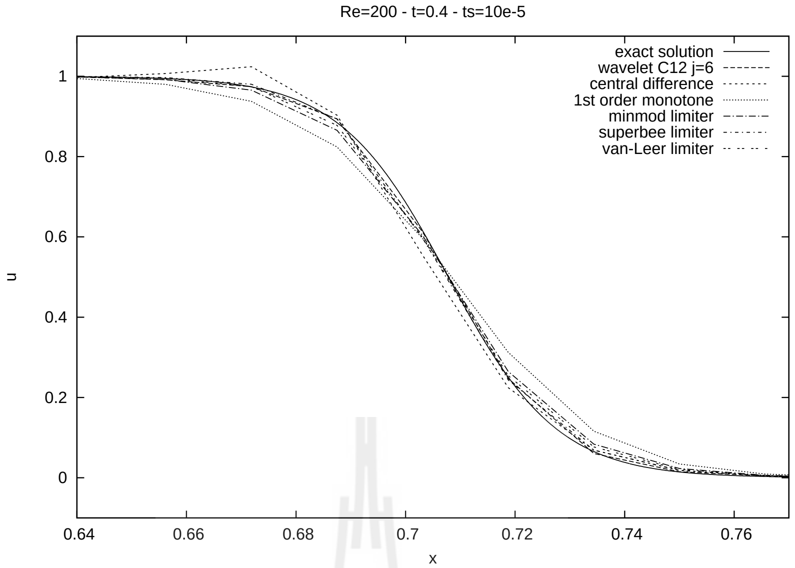

Figure 4.9 Comparison of the solutions of Burgers equation by the wavelet method with solutions by various finite difference schemes for $Re = 200$ and $0.64 < x < 0.77$. ($t = 0.4$, $\Delta t = 10^{-4}$, $\Delta x = 2^{-6}$).

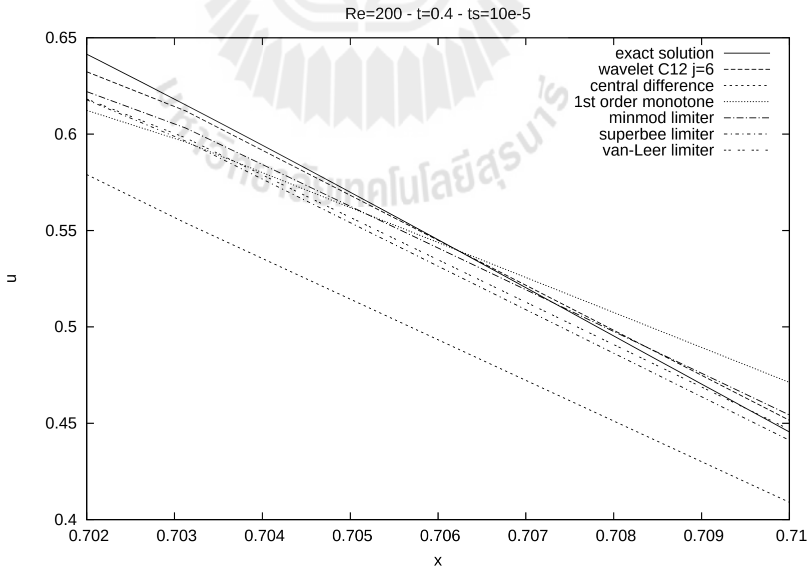

Figure 4.10 Comparison of the solutions of Burgers equation by the wavelet method with solutions by various finite difference schemes for $Re = 200$ and $0.702 < x < 0.71$. ($t = 0.4$, $\Delta t = 10^{-4}$, $\Delta x = 2^{-6}$).

4.5.2 The Case $Re = 2000$

As the Reynold numbers increases, the initial jump smoothes out very slowly with time, and thus fine spatial grids are required for a good solution. Figure 4.13 shows that a grid spacing of $\Delta x = 2^{-8}$ still produces oscillations in vicinity of the jump for all schemes. As all limiter schemes perform similarly, only the solution by van-Leer limiter is shown. At the finer grid $\Delta x = 2^{-9}$ of Figure 4.14, only the basic difference scheme retains oscillations. The enlarged graphs around the jump interval in Figure 4.15 show that the limiter schemes produce the best results. Table 4.4 lists the approximation errors of the various schemes.

4.5.3 Summary

The results of the numerical simulation can be summarized as follows:

1. The Wavelet-Galerkin method can produce accurate solutions in the presence of jumps.

2. In order to increase accuracy, the scaling level j should be increased, and concurrently, the time step size Δt should be decreased.

3. The choice of Coiflet type does not affect accuracy of the solution.

4. The Wavelet-Galerkin method performs better than the various finite difference schemes without flux limiter.

5. The Wavelet-Galerkin method produces solutions of similar accuracy as the finite flux limiter schemes.

6. At same required accuracy, time step sizes in the Wavelet-Galerkin method may be up to two orders of magnitude larger than in the flux-limiter schemes.

Table 4.3 Wavelet solutions of the Burgers equation by various Coiflets for $Re = 200$ and $j = 6$. ($t = 0.4$, $\Delta t = 10^{-4}$).

x	Exact solution u_e	Coiflet $C12$ u_{w12}	Coiflet $C18$ u_{w18}	Coiflet $C24$ u_{w24}	Coiflet $C30$ u_{w30}
0.000000	1.00000	1.00000	1.00000	1.00000	1.00000
0.562500	1.00000	1.00000	1.00000	0.99999	0.99999
0.578125	1.00000	1.00000	1.00000	1.00002	1.00001
0.593750	0.99999	0.99999	0.99998	0.99996	0.99999
0.609375	0.99995	0.99995	1.00000	0.99998	0.99990
0.625000	0.99975	0.99977	0.99967	0.99970	0.99991
0.640625	0.99881	0.99811	0.99921	0.99867	0.99819
0.656250	0.99432	0.99087	0.99379	0.99518	0.99565
0.671875	0.97336	0.97328	0.96907	0.96904	0.96944
0.687500	0.88424	0.88968	0.88535	0.88342	0.88248
0.703125	0.61518	0.60370	0.60470	0.60488	0.60494
0.718750	0.25073	0.24134	0.24232	0.24307	0.24347
0.734375	0.06543	0.05894	0.06292	0.06383	0.06407
0.750000	0.01441	0.01487	0.01558	0.01489	0.01441
0.765625	0.00304	0.00489	0.00303	0.00256	0.00258
0.781250	0.00063	0.00109	0.00037	0.00060	0.00076
0.796875	0.00013	0.00007	0.00012	0.00018	0.00013
0.812500	0.00003	0.00000	0.00003	0.00001	0.00001
0.828125	0.00001	0.00000	0.00000	0.00001	0.00001
1.00000	0.00000	0.00000	0.00000	0.00000	0.00000

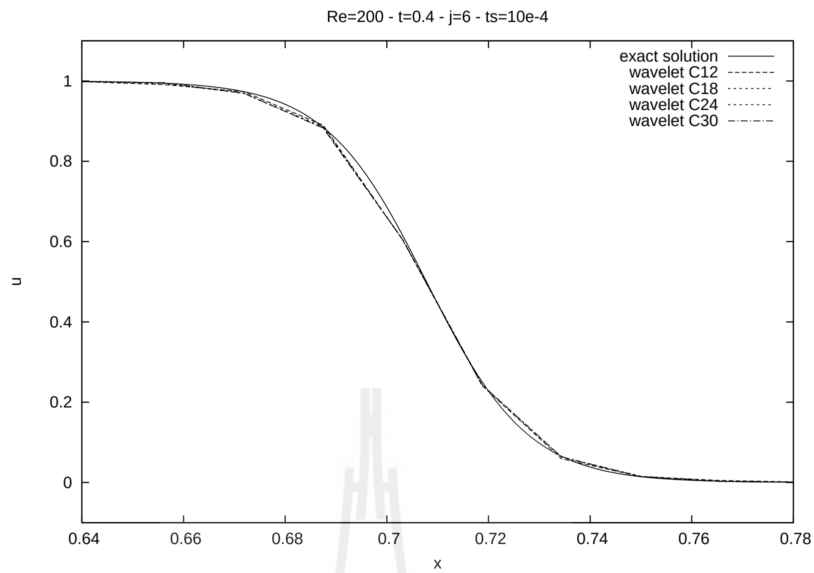

Figure 4.11 Comparison of the wavelet solutions for various choices of Coiflets, for $Re = 200$. ($t = 0.4$, $\Delta t = 10^{-4}$).

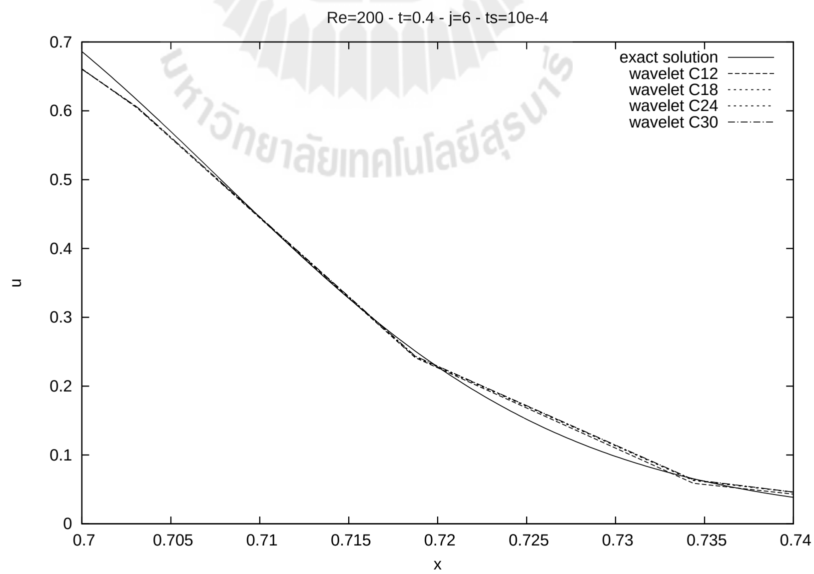

Figure 4.12 Comparison of the wavelet solutions for various choices of Coiflets, for $Re = 200$, enlarged. ($t = 0.4$, $\Delta t = 10^{-4}$).

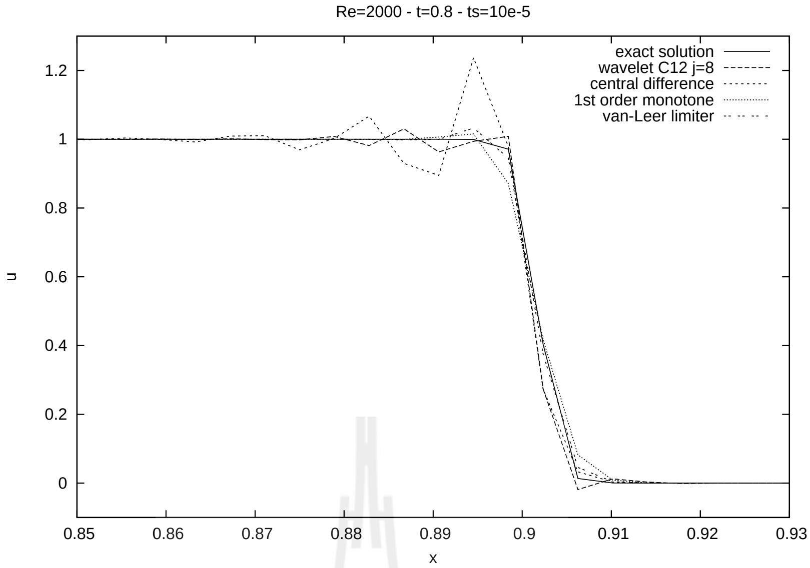

Figure 4.13 Comparison of the solutions of Burgers equation by the wavelet method with solutions by various finite difference schemes for $Re = 2000$ and $0.85 < x < 0.93$. ($t = 0.8$, $\Delta t = 10^{-5}$, $\Delta x = 2^{-8}$).

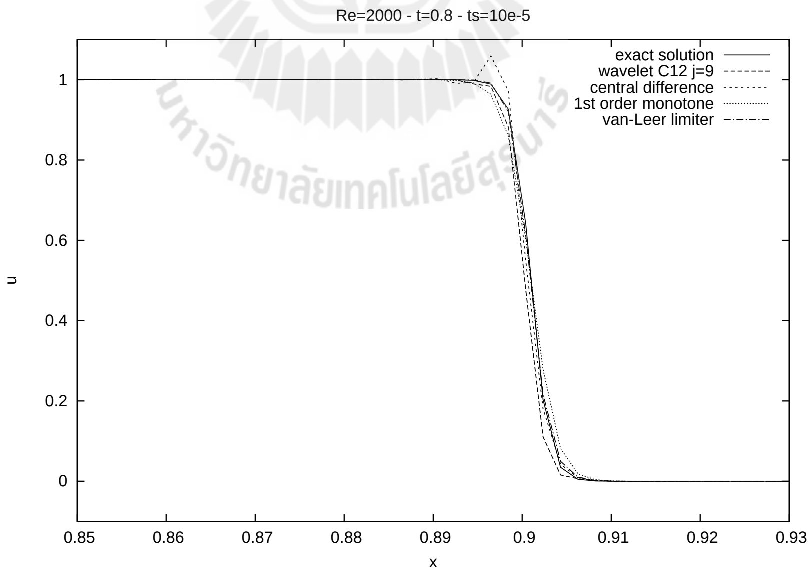

Figure 4.14 Comparison of the solutions of Burgers equation by the wavelet method with solutions by various finite difference schemes for $Re = 2000$ and $0.85 < x < 0.93$. ($t = 0.8$, $\Delta t = 10^{-5}$, $\Delta x = 2^{-9}$).

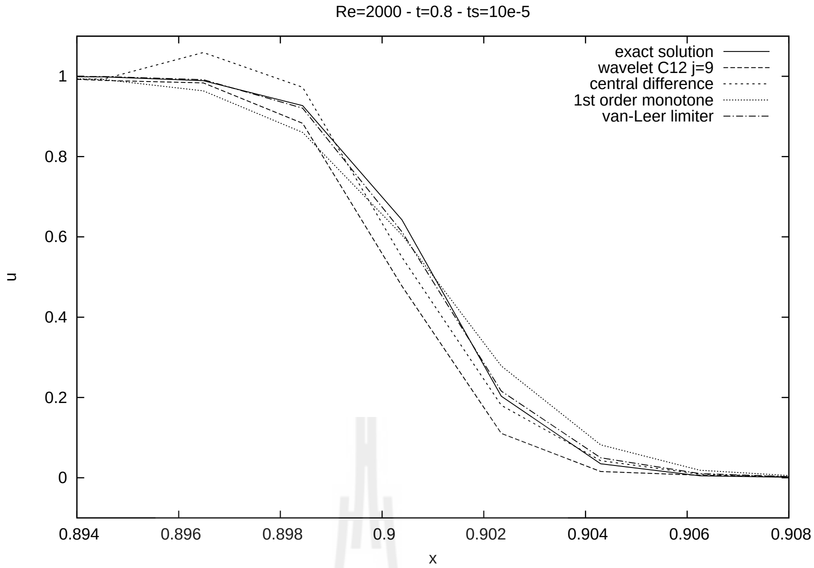

Figure 4.15 Comparison of the solutions of Burgers equation by the wavelet method with solutions by various finite difference schemes for $Re = 2000$ and $0.894 < x < 0.908$. ($t = 0.8$, $\Delta t = 10^{-5}$, $\Delta x = 2^{-9}$).

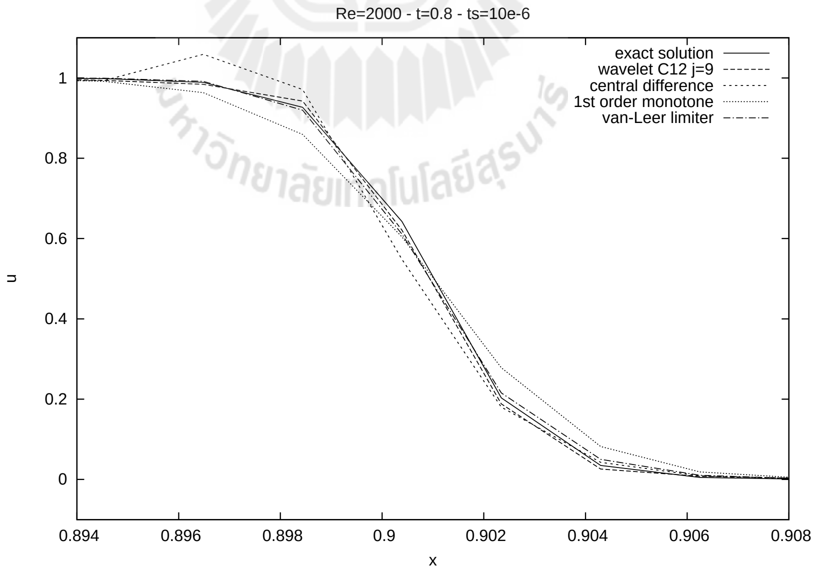

Figure 4.16 Comparison of the solutions of Burgers equation by the wavelet method with solutions by various finite difference schemes for $Re = 2000$ and $0.894 < x < 0.908$. ($t = 0.8$, $\Delta t = 10^{-6}$).

Table 4.4 Comparison of the exact solution of the Burgers equation with solutions by the wavelet method with solutions by various finite difference schemes for $Re = 2000$. ($t = 0.8$, $\Delta t = 10^{-6}$, $\Delta x = 2^{-9}$).

x	Exact solution u_e	CoifletC12 $j = 9$ u_{w9}	Central difference u_{cd}	First order monotone u_{1m}	Maxmod limiter u_{mm}	Super Bee limiter u_{sb}	van-Leer limiter u_{vl}
0.000000000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.878906250	1.00000	1.00000	0.99998	1.00000	1.00000	1.00000	1.00000
0.880859375	1.00000	1.00000	0.99999	1.00000	1.00000	1.00000	1.00000
0.882812500	1.00000	1.00000	1.00011	1.00000	1.00000	1.00000	1.00000
0.884765625	1.00000	1.00000	0.99993	1.00000	1.00000	1.00000	1.00000
0.886718750	1.00000	1.00002	0.99942	0.99998	1.00000	1.00000	1.00000
0.888671875	1.00000	0.99999	1.00105	0.99992	1.00000	1.00000	1.00000
0.890625000	0.99997	0.99990	1.00248	0.99961	0.99997	1.00000	1.00000
0.892578125	0.99977	1.00029	0.99110	0.99818	0.99975	0.99996	0.99994
0.894531250	0.99841	0.99379	0.99367	0.99167	0.99804	0.99952	0.99928
0.896484375	0.98893	0.98415	1.05900	0.96333	0.98520	0.99386	0.99152
0.898437500	0.92684	0.94241	0.97105	0.85923	0.90463	0.93323	0.91957
0.900390625	0.64243	0.62085	0.54810	0.60413	0.61509	0.61344	0.61111
0.902343750	0.20307	0.18829	0.18014	0.27825	0.22655	0.20447	0.21544
0.904296875	0.03488	0.02612	0.04265	0.08209	0.05578	0.04411	0.05001
0.906250000	0.00510	0.00783	0.00901	0.01870	0.01192	0.00908	0.01046
0.908203125	0.00073	0.00298	0.00185	0.00391	0.00246	0.00186	0.00214
0.910156250	0.00010	0.00010	0.00038	0.00080	0.00050	0.00038	0.00044
0.912109375	0.00001	−0.00016	0.00008	0.00016	0.00010	0.00008	0.00009
0.914062500	0.00000	0.00000	0.00002	0.00003	0.00002	0.00002	0.00002
0.916015625	0.00000	0.00001	0.00000	0.00001	0.00000	0.00000	0.00000
1.000000000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
max error		0.02158	0.09433	0.07517	0.02735	0.02899	0.03132
at $x =$		0.900390625	0.900390625	0.902343750	0.900390625	0.900390625	0.900390625
$u - u_e$ 2		0.00144	0.00567	0.00538	0.00212	0.00141	0.00169
$\frac{\|u - u_e\|_2}{\|u_e\|_2}$		0.00152	0.00598	0.00567	0.00223	0.00148	0.00178

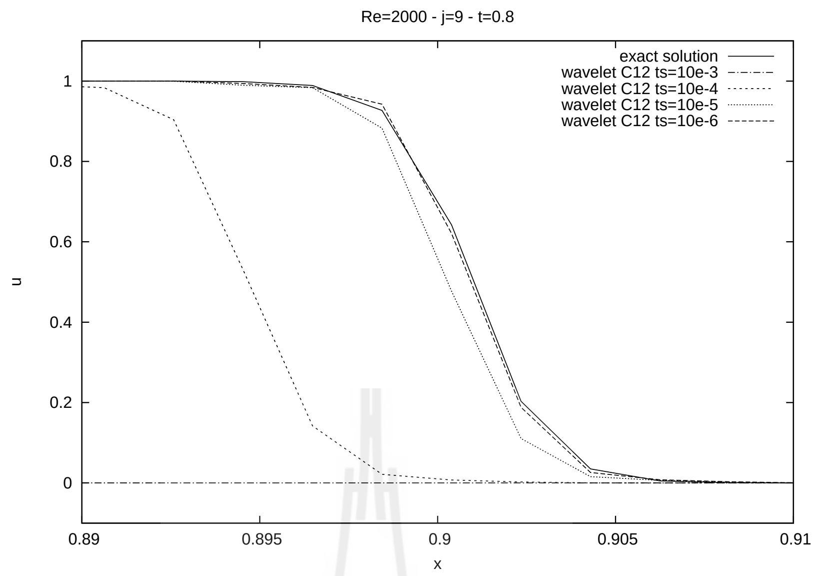

Figure 4.17 Solutions of Burgers equation by the wavelet method at various time step sizes, $Re = 2000$ and $0.894 < x < 0.908$. ($j = 9$, $t = 0.8$).

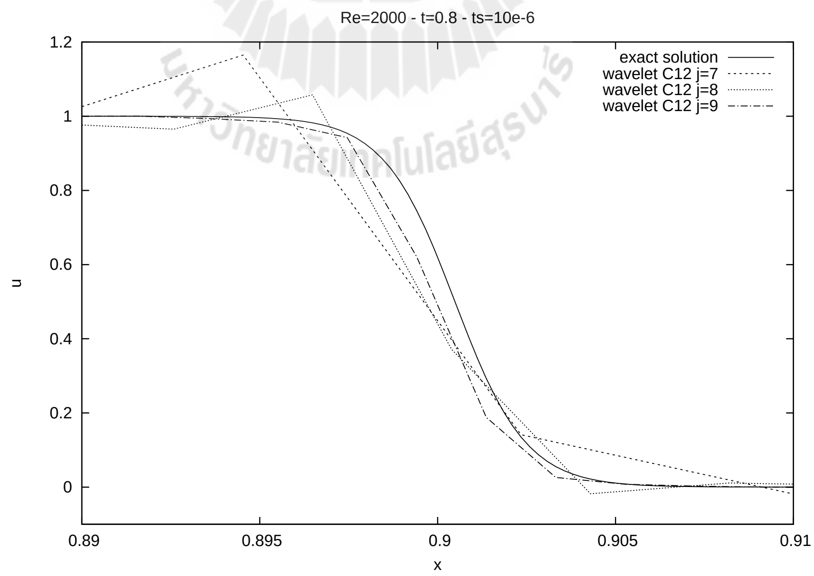

Figure 4.18 Solutions of Burgers equation by the wavelet method at various scale levels for $Re = 2000$ and $0.894 < x < 0.908$. ($t = 0.8$, $\Delta t = 10^{-6}$).

CHAPTER V

AN EQUATION WITH NONLINEAR DIFFUSION TERM

In this chapter we investigate suitability of the Wavelet-Galerkin method for a partial differential equation with nonlinear viscosity. We compare the wavelet solution with the exact solution and solutions by finite difference schemes.

5.1 Formulation of the Problem

Consider an equation

$$u_t + [f(u)]_x = [c(u)u_x]_x \tag{5.1}$$

where $f(x)$ and $c(x)$ are continuously differentiable functions. We impose the initial condition

$$u(x,0) = u^0(x) \qquad (0 < x < 1)$$

while the boundary conditions at $x = 0$ and $x = 1$ remain to be specified. Note that (5.1) can be rewritten as

$$u_t + f'(u)u_x = [c(u)u_x]_x$$

In order to obtain a variational problem, we apply the semi-implicit scheme

$$\frac{u^{k+1} - u^k}{\Delta t} + f'(u^k)\left(u^{k+1}\right)_x = \left[c(u^k)\left(u^{k+1}\right)_x\right]_x \tag{5.2}$$

where $u^k(x) = u(x, k\Delta t)$, $k = 0, 1, \ldots$. For ease of notation, at each time step Δt we set $w = u^{k+1}$ and $g = u^k$ and obtain the nonlinear ordinary boundary value problem

$$-\Delta t\,[c(g)w']' + \Delta t f'(g)w' + w = g \qquad (0 < x < 1).$$

The variational form of this problem is

$$a(u,v) = F(v) \qquad \forall v \in H^1(0,1) \tag{5.3}$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a(u,v) = \int_0^1 \big(-\Delta t\,[c(g)u']'\,v + \Delta t f'(g)u'v + uv\big)dx \tag{5.4}$$

and

$$F(v) =< g, v >_{L^2[0,1]} = \int_0^1 gv\,dx.$$

Applying integration by parts, the bilinear form (5.4) can be expressed as

$$a(u,v) = \int_0^1 (\,\Delta t c(g)u'v' + \Delta t f'(g)u'v + uv\,)\,dx \tag{5.5}$$

provided that either $v \in H_o^1(0,1)$, or $u' \in H_o^1(0,1)$.

As for existence and uniqueness of solutions, we have:

Theorem 5.1. Suppose that $m := \min_{0\le x\le 1} c(g)(x) > 0$ and let $M = \max_{0\le x\le 1} |f'(g)(x)|$. Then for every $\Delta t < 4m^2/M$,

$$a(u,v) = F(v) \qquad \forall v \in H$$

has a unique solution $u \in H$, where $H = H^1(0,1)$ or $H = H_o^1(0,1)$.

Proof. We verify the assumptions of the Lax-Milgram theorem. Clearly, the bilinear form $a(u,v)$ is continuous. To verify coerciveness, note that for each $\alpha > 0$,

$$\begin{aligned}
a(v,v) &= \int_0^1 \Big(\Delta t c(g)\,(v')^2 + \Delta t f'(g)v'v + v^2\Big)\,dx \\
&= \int_0^1 \left(\Delta t\,\big[c(g) - \alpha^2 f'(g)^2\big]\,(v')^2 + \Delta t\left[f'(g)\alpha v' + \frac{v}{2\alpha}\right]^2 + \left[1 - \frac{\Delta t}{4\alpha^2}\right]v^2\right)dx \\
&\ge \int_0^1 \left(\Delta t\,\big[c(g) - \alpha^2 M^2\big]\,(v')^2 + \left[1 - \frac{\Delta t}{4\alpha^2}\right]v^2\right)dx.
\end{aligned}$$

As m is positive, the first bracket will be greater than some positive constant when $\alpha < m/\sqrt{M}$. The second bracket will be greater than some positive constant when $\Delta t < 4\alpha^2$, thus yielding coerciveness of $a(\cdot,\cdot)$. The assertion thus follows from the Lax-Milgram Theorem. $\square$

Remark: In practice, the solutions u^k are usually known a-priori to be bounded by some common constant M. On the other hand, the value of $c(g)$ may need to be evaluated at each iteration step, unless the function $c(u)$ is bounded below by some positive constant.

5.2 The Specific Equation

We now make a particular choice of $f(u)$ and $c(u)$, by letting $f(u) = -u/2$ and $c(u) = u^2$. We thus obtain the equation

$$u_t - \frac{1}{2}u_x = \left(u^2 u_x\right)_x . \tag{5.6}$$

This equation was discussed by Pinkuchov and Shu (2000). It is easy to check that each function in the family

$$u_\alpha(x,t) = [\,\max(0, x+t+\alpha)\,]^{1/2} \qquad (\alpha \text{ const}) \tag{5.7}$$

is a solution of this equation on the domain $-\infty < x < \infty$.

In order to investigate suitability of the Wavelet-Galerkin method, we thus impose boundary conditions corresponding to these solutions,

$$u(x,0) = u^0(x) = [\,\max(0, x+\alpha)\,]^{1/2} \qquad (0 < x < 1) \tag{5.8}$$

$$u(0,t) = f_0(t) \qquad (0 \le t \le T) \tag{5.9}$$

$$u(1,t) = f_1(t) \qquad (0 \le t \le T) \tag{5.10}$$

for some $T > 0$, where $f_0(t) = \sqrt{\max(0, t+\alpha)}$, $f_1(t) = \sqrt{1+t+\alpha}$ and $\alpha > -1$. By including the boundary conditions, scheme (5.2) becomes

$$\frac{u^{k+1} - u^k}{\Delta t} - \frac{1}{2}\left(u^{k+1}\right)_x = \left[\left(u^k\right)^2 \left(u^{k+1}\right)_x\right]_x$$
$$u^{k+1}(0) = f_0((k+1)\Delta t)$$
$$u^{k+1}(1) = f_1((k+1)\Delta t).$$

We thus obtain the nonlinear ordinary boundary value problem going from k to $k+1$,

$$-\Delta t \left(g^2 w'\right)' - \frac{\Delta t}{2} w' + w = g \qquad (0 < x < 1) \tag{5.11}$$
$$w(0) = f_0((k+1)\Delta t) \tag{5.12}$$
$$w(1) = f_1((k+1)\Delta t), \tag{5.13}$$

and the bilinear form (5.5) is

$$a(u, v) = \int_0^1 \left(\Delta t g^2 u'v' - \frac{\Delta t}{2} u'v + uv\right) dx. \tag{5.14}$$

5.3 Solution by the Wavelet Galerkin Method

We now discuss the numerical solution of problem (5.11)–(5.13). Similar to the Burgers equation, we let u_0, u_1 and u_2 denote the solutions of the problems,

$$a(u, v) = F(v) \qquad \forall v \in H^1(0, 1) \tag{5.15}$$
$$a(u, v) = v(0) \qquad \forall v \in H^1(0, 1) \tag{5.16}$$
$$a(u, v) = v(1) \qquad \forall v \in H^1(0, 1) \tag{5.17}$$

respectively. We then find λ_1 and λ_2, so that $u = u_0 + \lambda_1 u_1 + \lambda_2 u_2$ satisfies

$$u(0) = f_0((k+1)\Delta t), \qquad u(1) = f_1((k+1)\Delta t)$$

by solving the system

$$\begin{aligned} u_1(0)\lambda_1 + u_2(0)\lambda_2 &= f_0((k+1)\Delta t) - u_0(0) \\ u_1(1)\lambda_1 + u_2(1)\lambda_2 &= f_1((k+1)\Delta t) - u_0(1). \end{aligned} \tag{5.18}$$

Problems (5.15)–(5.17) are now solved using the Wavelet-Galerkin method. As before, we choose an approximation space V_j, and let $\{\varphi_{j,k}\}_{k=L_1}^{L_2}$ be the basis of $V_j[0,1]$ obtained by restricting the functions $\varphi_{j,k}$ to $[0,1]$. We replace g and g^2 by their projections onto $V_j[0,1]$, which can be expressed as

$$g = \sum_{m=L_1}^{L_2} b_m \varphi_{j,m}, \qquad g^2 = \sum_{m=L_1}^{L_2} d_m \varphi_{j,m}$$

and proceed at working in the finite dimensional space $V_j[0,1]$ by looking for solutions of the form $u = \sum_{k=L_1}^{L_2} c_k \varphi_{j,k}$. For example, (5.15) becomes

$$\int_0^1 \left(\Delta t \left[\sum_{m=L_1}^{L_2} d_m \varphi_{j,m} \right] \left[\sum_{k=L_1}^{L_2} c_k \varphi_{j,k} \right]' \varphi'_{j,l} - \frac{\Delta t}{2} \left[\sum_{k=L_1}^{L_2} c_k \varphi_{j,k} \right]' \varphi_{j,l} + \left[\sum_{k=L_1}^{L_2} \varphi_{j,k} \right] \varphi_{j,l} \right) dx = \int_0^1 \left[\sum_{m=L_1}^{L_2} b_m \varphi_{j,m} \right] \varphi_{j,l}\, dx$$

for all $l = L_1, \ldots, L_2$. That is,

$$\sum_{k=L_1}^{L_2} c_k \left(\Delta t \sum_{m=L_1}^{L_2} d_m \int_0^1 \varphi_{j,m} \varphi'_{j,k} \varphi'_{j,l}\, dx - \frac{\Delta t}{2} \int_0^1 \varphi'_{j,k} \varphi_{j,l}\, dx + \int_0^1 \varphi_{j,k} \varphi_{j,l}\, dx \right) = \sum_{m=L_1}^{L_2} b_m \int_0^1 \varphi_{j,m} \varphi_{j,l}\, dx.$$

By (3.4) and (3.5) this becomes

$$\sum_{k=L_1}^{L_2} c_k \left(2^{5j/2} \Delta t \left[\sum_{m=L_1}^{L_2} d_m \Gamma_{m,k,l}^{j,0,1,1} \right] - 2^{j-1} \Delta t \Gamma_{k,l}^{j,1,0} + \Gamma_{k,l}^{j,0,0} \right) = \sum_{m=L_1}^{L_2} b_m \Gamma_{m,l}^{j,0,0}$$

for all $l = L_1, \ldots, L_2$. This can be expressed as a matrix equation

$$AU_o = B_o \tag{5.19}$$

where A is the matrix of size $L \times L$ $(L = L_2 - L_1 + 1)$ whose entries are

$$a_{lk} = 2^{5j/2}\Delta t \left[\sum_{m=L_1}^{L_2} d_m \Gamma^{j,0,1,1}_{m,k,l} \right] - 2^{j-1}\Delta t \Gamma^{j,1,0}_{k,l} + \Gamma^{j,0,0}_{k,l},$$

U_o is the vector of unknowns, $U_o = \{c_l\}_{l=L_1}^{L_2}$, and B_o the column vector with entries $b_l = \sum_{m=L_1}^{L_2} b_m \Gamma^{j,0,0}_{m,l}$. Similarly, problems (5.16) and (5.17) become the systems

$$AU_1 = B_1 \qquad \text{and} \qquad AU_2 = B_2,$$

where $B_1 = \{\varphi_{j,l}(0)\}_{l=L_1}^{L_2}$ and $B_2 = \{\varphi_{j,l}(1)\}_{l=L_1}^{L_2}$. Computing the values of λ_1 and λ_2 from (5.18), then the vector $U = \{u_k\}_{k=L_1}^{L_2}$ given by

$$U = U_0 + \lambda_1 U_1 + \lambda_2 U_2$$

will contain the coefficients of the solution u of problem (4.6) – (4.7) in $V_j[0,1]$. Since the values of the scaling function at the dyadic rationals are known, the solution u can by computed by

$$u(x) = \sum_{k=L_1}^{L_2} u_k \varphi_{j,k}(x)$$

at the dyadic rationals in $[0,1]$.

5.4 Solution by Finite Difference with Flux Limiter

For the purpose of comparison, we again performed computations to solve system (5.6), (5.8)–(5.10) with and without flux limiter, now by the scheme

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} + \frac{\hat{F}_{i+1/2} - \hat{F}_{i-1/2}}{\Delta x} = \frac{\left(u^k_{i+1/2}\right)^2 (u^k_{i+1} - u^k_i) - \left(u^k_{i-1/2}\right)^2 (u^k_i - u^k_{i-1})}{\Delta x^2}. \tag{5.20}$$

Fluxes on the boundary of the computational domain were computed according to the exact solution.

5.5 Results of Numerical Experiments

In this section we present the results of the numerical experiments with equation (5.6) at time $t = 0.25$. Choosing $\alpha = -0.5$, the boundary conditions imposed were thus

$$
\begin{aligned}
u(x,0) = u^0(x) &= \sqrt{\max(0, x-0.5)} && (0 < x < 1) \\
u(0,t) &= \sqrt{\max(0, t-0.5)} && (t \geq 0) \\
u(1,t) &= \sqrt{t+0.5} && (t \geq 0).
\end{aligned}
$$

Figures 5.1–5.3 show the solutions by finite difference method ($\Delta t = 10^{-4}$, $\Delta x = 2^{-6}$) and by the wavelet method for scaling level $j = 6$ and various choices of Δt at $t = 0.25$. The graphs on the left depict the solutions over the whole interval $[0, 1]$, and the graphs on the right are magnifications around the point of singularity $x = 0.25$. Table 5.1 lists the errors of approximation in each case. u_e denotes the exact solution, u_w the solution by the wavelet method, and u_d the solution by finite difference scheme. The last column lists the point at which the maximum uniform error occurs. For $\Delta t = 10^{-3}$ the wavelet solution gives a good approximation of the exact solution. Observe, however, that when $\Delta t = 10^{-4}$ the wavelet solution shows a large error at the left-hand endpoint, while the approximation at the point of singularity is still good.

Figure 5.4 shows how the choice of time step Δt influences accuracy of approximation at the point of singularity. A value of $\Delta t = 10^{-2}$ is certainly not sufficient: all plots show a smoothed-out and delayed ascent at this level. A value of $\Delta t = 10^{-3}$ appears sufficient at all scales, except at scale $j = 10$, where a delayed ascent can still be observed at this level. A value of $\Delta t = 10^{-4}$ gives best results at all scales. We caution that each plot is drawn at a different scale.

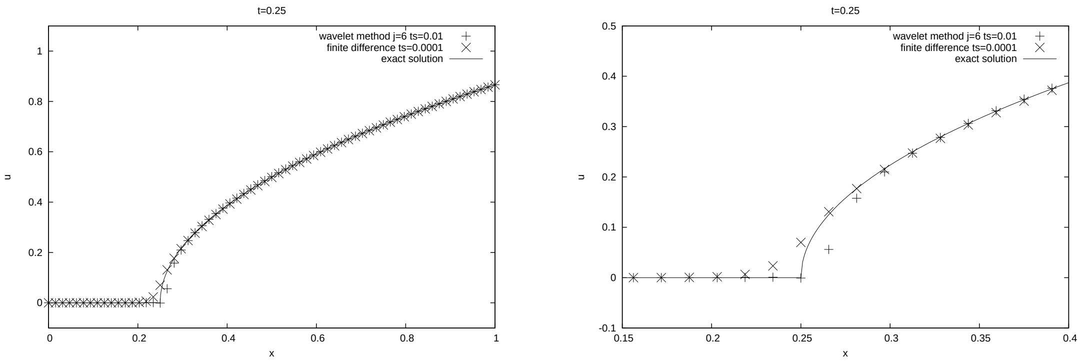

Figure 5.1 Sketch of finite difference solution ($\Delta t = 10^{-4}$, $\Delta x = 2^{-6}$) and wavelet solution ($j = 6$, $\Delta t = 10^{-2}$).

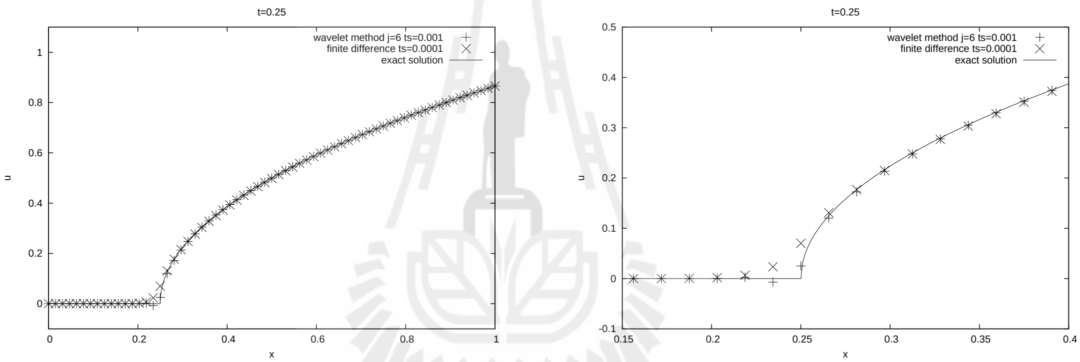

Figure 5.2 Sketch of finite difference solution ($\Delta t = 10^{-4}$, $\Delta x = 2^{-6}$) and wavelet solution ($j = 6$, $\Delta t = 10^{-3}$).

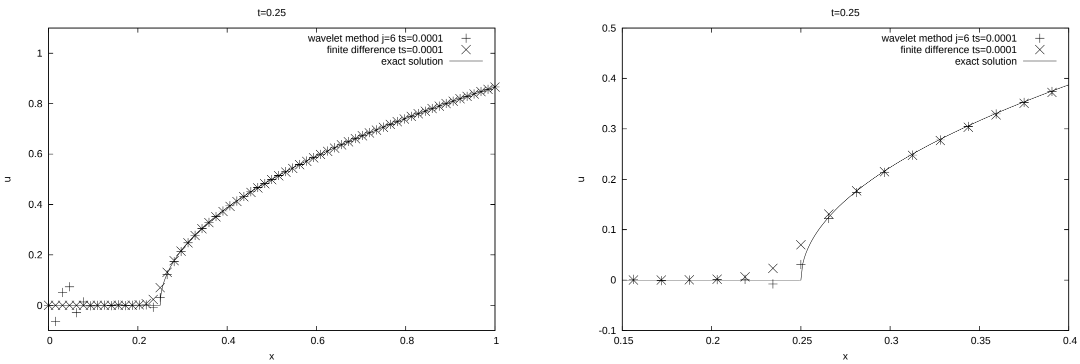

Figure 5.3 Sketch of finite difference solution ($\Delta t = 10^{-4}$, $\Delta x = 2^{-6}$) and wavelet solution ($j = 6$, $\Delta t = 10^{-4}$).

Table 5.1 Approximation errors: solution by finite difference versus wavelet solution ($j = 6$, various time steps Δt).

Solution u	$\|u - u_e\|_2$	$\frac{\|u-u_e\|_2}{\|u_e\|_2}$	$\|u - u_e\|_\infty$	max. error at $x =$
$u_d,\ \Delta t = 10^{-4}$	$6.645e - 03$	$1.754e - 02$	$7.003e - 02$	0.250000
$u_w,\ \Delta t = 10^{-2}$	$9.027e - 03$	$1.702e - 02$	$6.903e - 02$	0.265625
$u_w,\ \Delta t = 10^{-3}$	$3.417e - 03$	$6.443e - 03$	$2.498e - 02$	0.250000
$u_w,\ \Delta t = 10^{-4}$	$1.488e - 02$	$2.806e - 02$	$7.345e - 02$	0.046875

Figure 5.5 compares the approximate solutions in the vicinity of the point of singularity for various scaling levels, at time steps $\Delta t = 10^{-3}$, respectively $\Delta t = 10^{-4}$. Tables 5.2 and 5.3 show the respective errors of approximation over $[0, 1]$. For $j \leq 8$ there are substantial oscillations close to the left-hand endpoint. These oscillations vanish as j increases.

Finally, Figure 5.6 shows the best approximation obtained, choosing $j = 10$ and $\Delta t = 10^{-5}$. The graphs of both, the exact and approximate solution, merge to one single graph, and as table 5.4 shows, the mean square error of approximation is of order 10^{-3}. However, a mild oscillation can be observed at the point of singularity in all graphs.

Table 5.5 lists the results of computations by the wavelet method and by several finite difference methods, including flux limiter schemes, at coarse spatial resolution. Table 5.7 does the same at higher spatial resolutions, and Table 5.6 shows the error of approximation of each method. The accuracy of wavelet method is comparable with that of the central difference method, while the flux limiter schemes do not show improved accuracy.

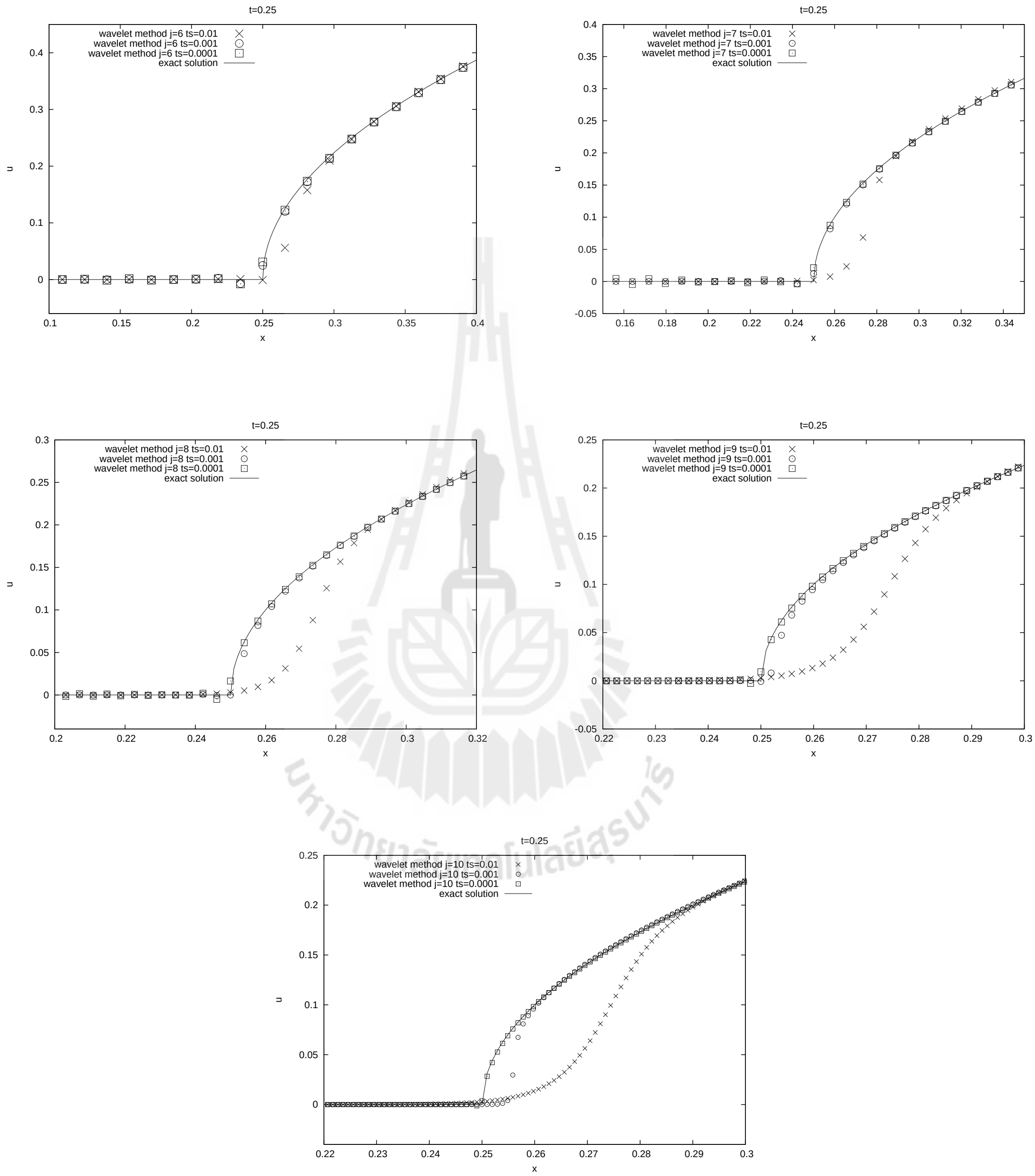

Figure 5.4 Influence of time step size on approximation error at various scales j.

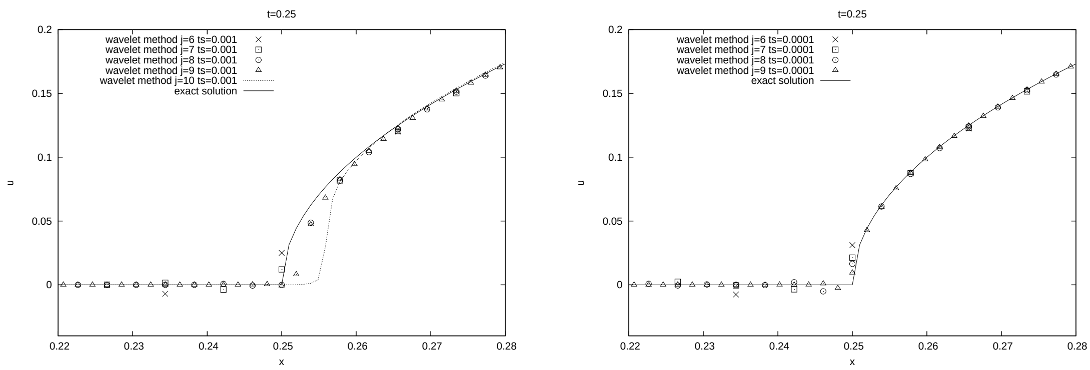

Figure 5.5 Influence of scale on approximation error ($\Delta t = 10^{-3}$ and $\Delta t = 10^{-4}$).

Table 5.2 Approximation errors at various scaling levels j ($\Delta t = 10^{-3}$).

j	$\|u_w - u_e\|_2$	$\frac{\|u_w-u_e\|_2}{\|u_e\|_2}$	$\|u_w - u_e\|_\infty$	max. error at $x =$
6	$3.417e-03$	$6.443e-03$	$2.498e-02$	0.2500000000
7	$1.382e-03$	$2.607e-03$	$1.209e-02$	0.2500000000
8	$1.046e-03$	$1.973e-03$	$1.388e-02$	0.2539062500
9	$1.827e-03$	$3.444e-03$	$0.251e-02$	0.2519531250
10	$4.025e-03$	$7.590e-03$	$6.590e-02$	0.2548828125

Table 5.3 Approximation errors at various scaling levels j ($\Delta t = 10^{-4}$).

j	$\|u_w - u_e\|_2$	$\frac{\|u_w-u_e\|_2}{\|u_e\|_2}$	$\|u_w - u_e\|_\infty$	max. error at $x =$
6	$1.488e-02$	$2.806e-02$	$7.345e-02$	0.0468750
7	$5.238e-03$	$9.877e-03$	$3.501e-02$	0.0234375
8	$2.268e-02$	$4.278e-02$	$1.046e-01$	0.0312500
9	$4.514e-04$	$8.512e-04$	$9.390e-03$	0.2500000
10	$1.878e-04$	$3.540e-04$	$3.687e-03$	0.2500000

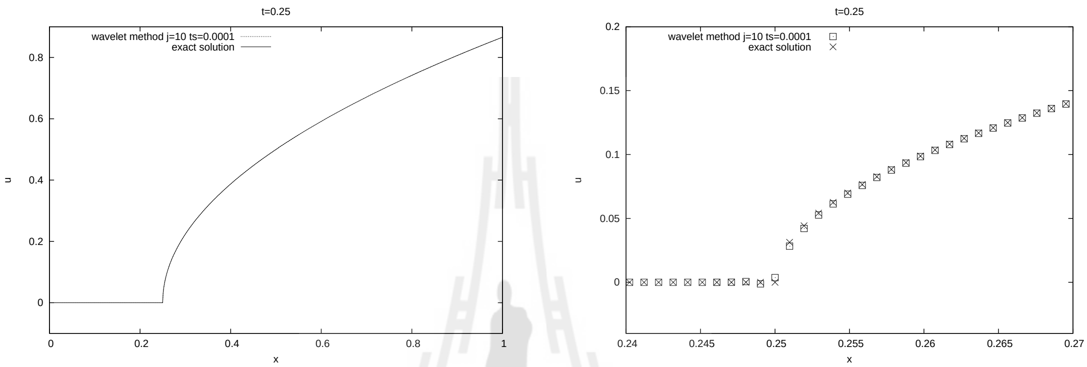

Figure 5.6 The best approximation is achieved at $j = 10$.

Table 5.4 Approximation errors for $j = 10$ at various time steps Δt.

Δt	$\|u - u_e\|_2$	$\frac{\|u-u_e\|_2}{\|u_e\|_2}$	$\|u - u_e\|_\infty$	max. error at $x =$
10^{-2}	$1.248e-02$	$2.353e-02$	$9.300e-02$	0.2646484375
10^{-3}	$4.025e-03$	$7.590e-03$	$6.590e-02$	0.2548828125
10^{-4}	$1.878e-04$	$3.540e-04$	$3.687e-03$	0.2500000000

Table 5.5 Comparison of the exact solution with solutions by the wavelet method with solutions by various finite difference schemes. ($\Delta t = 10^{-3}$ for the wavelet solutions and $\Delta t = 10^{-4}$ for the finite-difference solutions; $\Delta x = 2^{-6}$).

x	Exact solution u_e	CoifletC12 $j=6$ u_{w6}	Coiflet$C$12 $j=8$ u_{w8}	Central difference u_{cd}	First order monotone u_{1m}	Maxmod limiter u_{mm}	Super Bee limiter u_{sb}	van-Leer limiter u_{vl}
0.000000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.140625	0.00000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000	0.00000
0.156250	0.00000	0.00001	0.00000	0.00000	0.00003	0.00000	0.00000	0.00000
0.171875	0.00000	0.00003	0.00000	0.00000	0.00013	0.00000	0.00000	0.00000
0.187500	0.00000	−0.00019	0.00000	0.00000	0.00049	0.00001	0.00000	0.00000
0.203125	0.00000	0.00016	0.00000	0.00004	0.00180	0.00005	0.00000	0.00000
0.218750	0.00000	0.00247	0.00000	0.00035	0.00662	0.00048	0.00000	0.00000
0.234375	0.00000	−0.00710	−0.00004	0.00308	0.02367	0.00424	0.00001	0.00090
0.250000	0.00000	0.02498	−0.00006	0.02606	0.07043	0.03491	0.01868	0.02906
0.265625	0.12500	0.12009	0.12202	0.11066	0.13104	0.11797	0.11263	0.11685
0.281250	0.17678	0.17240	0.17566	0.17315	0.17710	0.17247	0.17437	0.17314
0.296875	0.21651	0.21349	0.21598	0.21431	0.21488	0.21325	0.21529	0.21403
0.312500	0.25000	0.24780	0.24975	0.24837	0.24755	0.24732	0.24916	0.24806
0.328125	0.27951	0.27782	0.27941	0.27822	0.27670	0.27723	0.27887	0.27790
0.343750	0.30619	0.30483	0.30618	0.30513	0.30324	0.30421	0.30569	0.30482
0.359375	0.33072	0.32960	0.33077	0.32984	0.32776	0.32898	0.33031	0.32953
0.375000	0.35355	0.35261	0.35364	0.35280	0.35065	0.35202	0.35322	0.35251
0.390625	0.37500	0.37420	0.37511	0.37435	0.37218	0.37363	0.37472	0.37408
0.406250	0.39528	0.39460	0.39541	0.39472	0.39257	0.39405	0.39504	0.39446
0.421875	0.41458	0.41399	0.41471	0.41408	0.41198	0.41347	0.41437	0.41384
0.437500	0.43301	0.43250	0.43315	0.43257	0.43054	0.43200	0.43282	0.43235
0.453125	0.45069	0.45025	0.45083	0.45030	0.44834	0.44978	0.45053	0.45009
0.468750	0.46771	0.46732	0.46785	0.46735	0.46548	0.46687	0.46756	0.46716
0.484375	0.48412	0.48378	0.48426	0.48380	0.48201	0.48336	0.48399	0.48362
0.500000	0.50000	0.49970	0.50013	0.49971	0.49800	0.49930	0.49987	0.49954
0.562500	0.55902	0.55883	0.55913	0.55881	0.55743	0.55851	0.55892	0.55868
0.625000	0.61237	0.61226	0.61247	0.61222	0.61113	0.61200	0.61229	0.61212
0.687500	0.66144	0.66137	0.66151	0.66132	0.66047	0.66116	0.66137	0.66125
0.750000	0.70711	0.70707	0.70716	0.70701	0.70637	0.70690	0.70704	0.70696
0.812500	0.75000	0.74998	0.75004	0.74992	0.74946	0.74984	0.74994	0.74988
0.875000	0.79057	0.79056	0.79059	0.79050	0.79020	0.79045	0.79051	0.79048
0.937500	0.82916	0.82915	0.82917	0.82909	0.82893	0.82907	0.82910	0.82908
1.000000	0.86603	0.86603	0.86603	0.86601	0.86601	0.86601	0.86601	0.86601
max error		0.02498	0.00298	0.02606	0.07043	0.03491	0.01868	0.02906
at $x=$		0.250000	0.265625	0.250000	0.250000	0.250000	0.250000	0.250000

Table 5.6 L^{∞} and L^{2} errors for solutions by the wavelet method and by various finite difference schemes using the data from Table 5.7 ($\Delta t = 10^{-4}$ for the wavelet solutions and $\Delta t = 10^{-6}$ for the finite-difference solutions; $\Delta x = 2^{-9}$).

x	Exact solution u_e	CoifletC12 $j = 9$ u_{w9}	Central difference u_{cd}	First order monotone u_{1m}	Maxmod limiter u_{mm}	Super Bee limiter u_{sb}	van-Leer limiter u_{vl}
0.000000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.218750000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.234375000	0.000000	0.000000	0.000000	0.000003	0.000000	0.000000	0.000000
0.238281250	0.000000	0.000002	0.000000	0.000033	0.000000	0.000000	0.000000
0.240234375	0.000000	0.000010	0.000000	0.000116	0.000000	0.000000	0.000000
0.242187500	0.000000	-0.000068	0.000002	0.000409	0.000003	0.000000	0.000000
0.244140625	0.000000	0.000061	0.000015	0.001443	0.000023	0.000000	0.000000
0.246093750	0.000000	0.000895	0.000131	0.005045	0.000200	0.000000	0.000001
0.248046875	0.000000	-0.002631	0.001156	0.016290	0.001764	0.000004	0.000388
0.250000000	0.000000	0.009390	0.009685	0.036972	0.014057	0.006679	0.011294
0.251953125	0.044194	0.042681	0.039873	0.055031	0.043360	0.039920	0.042386
0.253906250	0.062500	0.061017	0.061521	0.069243	0.061919	0.061677	0.061763
0.255859375	0.076547	0.075472	0.075987	0.081323	0.076077	0.076128	0.076066
0.257812500	0.088388	0.087569	0.087974	0.091994	0.087974	0.088096	0.088008
0.259765625	0.098821	0.098165	0.098489	0.101649	0.098442	0.098595	0.098495
0.261718750	0.108253	0.107701	0.107975	0.110527	0.107901	0.108070	0.107964
0.263671875	0.116927	0.116450	0.116687	0.118785	0.116598	0.116774	0.116666
0.265625000	0.125000	0.124580	0.124788	0.126537	0.124690	0.124870	0.124760
0.267578125	0.132583	0.132209	0.132393	0.133864	0.132289	0.132469	0.132361
0.269531250	0.139754	0.139418	0.139582	0.140827	0.139476	0.139653	0.139547
0.273437500	0.153093	0.152814	0.152946	0.153851	0.152839	0.153010	0.152909
0.277343750	0.165359	0.165123	0.165230	0.165890	0.165125	0.165289	0.165192
0.281250000	0.176777	0.176573	0.176661	0.177139	0.176559	0.176715	0.176624
0.285156250	0.187500	0.187322	0.187395	0.187732	0.187296	0.187445	0.187358
0.289062500	0.197642	0.197486	0.197546	0.197773	0.197451	0.197593	0.197510
0.292968750	0.207289	0.207150	0.207200	0.207339	0.207108	0.207244	0.207165
0.296875000	0.216506	0.216381	0.216423	0.216492	0.216335	0.216464	0.216389
0.300781250	0.225347	0.225235	0.225269	0.225279	0.225184	0.225308	0.225236
0.304687500	0.233854	0.233752	0.233780	0.233742	0.233698	0.233817	0.233748
0.308593750	0.242061	0.241969	0.241992	0.241914	0.241913	0.242027	0.241961
0.312500000	0.250000	0.249916	0.249934	0.249822	0.249858	0.249968	0.249904
0.375000000	0.353553	0.353529	0.353516	0.353240	0.353470	0.353535	0.353498
0.437500000	0.433013	0.433006	0.432987	0.432730	0.432957	0.432999	0.432975
0.500000000	0.500000	0.500000	0.499981	0.499767	0.499960	0.499988	0.499972
0.562500000	0.559017	0.559019	0.559001	0.558830	0.558987	0.559007	0.558995
0.625000000	0.612372	0.612375	0.612359	0.612226	0.612349	0.612363	0.612355
0.687500000	0.661438	0.661441	0.661426	0.661325	0.661419	0.661429	0.661424
0.750000000	0.707107	0.707109	0.707097	0.707022	0.707092	0.707098	0.707095
0.812500000	0.750000	0.750002	0.749991	0.749939	0.749988	0.749992	0.749990
0.875000000	0.790569	0.790571	0.790561	0.790529	0.790559	0.790562	0.790560
0.937500000	0.829156	0.829157	0.829149	0.829133	0.829148	0.829149	0.829148
1.000000000	0.866025	0.866025	0.866019	0.866019	0.866019	0.866019	0.866019

Table 5.7 Comparison of the exact solution with solutions by the wavelet method and by various finite difference schemes ($\Delta t = 10^{-4}$ for the wavelet solutions and $\Delta t = 10^{-6}$ for the finite-difference solutions; $\Delta x = 2^{-9}$).

	CoifletC12 $j = 9$ u_{w9}	Central difference u_{cd}	First order monotone u_{1m}	Maxmod limiter u_{mm}	Super Bee limiter u_{sb}	van-Leer limiter u_{vl}
$\|u - u_e\|_\infty$	0.009390	0.009685	0.036972	0.014057	0.006679	0.011294
max error at $x =$	0.250000	0.250000	0.250000	0.250000	0.250000	0.250000
$\|u - u_e\|_\infty$ (excluding $x = 0.25$)	0.002631	0.004322	0.016290	0.001764	0.004275	0.001809
max error at $x =$	0.248047	0.251953	0.248047	0.248047	0.251953	0.251953
$\|u - u_e\|_2$	0.000451	0.000476	0.001923	0.000632	0.000354	0.000510
$\frac{\|u - u_e\|_2}{\|u_e\|_2}$	0.000850	0.000896	0.003622	0.001190	0.000666	0.000960

The results of the numerical simulation can be summerized as follows:

1. The Wavelet-Galerkin method can produce accurate solutions for equations with nonlinear viscocity and jumps in the dervative of the solution.

2. The Wavelet-Galerkin method produces solutions of similar accurcay as the finite difference schemes, but requires fewer time steps.

3. The Wavelet-Galerkin solutions exhibit mild oscillations at the point of singularity which the flux limiter solutions do not.

4. Increasing the scaling level j does not improve accuracy, unless time step size Δt is decreased concurrently. Conversely, a decrease in time step size needs to be accompanied by an increase in scale.

CHAPTER VI

CONCLUSION

In this thesis, the application of the Wavelet-Galerkin method to nonlinear partial differential equations was studied by means of numerical experiments with two examples, using Coiflet scaling functions as basis functions. The emphasis was on obtaining solutions which either are discontinuous, or have discontinuous derivatives.

The first equation studied is the Burgers equation, with non-periodic boundary conditions and a discontinuity in the initial condition. The Wavelet-Galerkin method produced approximate solutions of good accuracy not only in case of high viscosity as previously shown in the literature, but also at low viscosity. As viscosity decreases, the number of basis functions has to increase while time step size has to decrease in order to preserve accuracy, which is in line with expectations. The type of Coiflets chosen has no noticeable effect on the accuracy of the solution, however. The solutions by the Wavelet-Galerkin method were then compared with solutions by finite difference schemes. It was found that the Wavelet-Galerkin solutions are of better accuracy than solutions by basic finite difference schemes, and comparable in accuracy with solutions by flux limiter schemes, at equal spatial grid sizes.

The second equation studied involves a nonlinearity in the diffusion term. Initial and boundary conditions were chosen so that the solution has a point of non-differentiability. The Wavelet-Galerkin method gives accurate solutions provided the scaling level is sufficiently large. Again, it was found that the Wavelet-Galerkin solutions are of better accuracy than solutions by basic finite difference schemes, and comparable in accuracy with solutions by flux limiter schemes, at equal spatial grid sizes. However, the Wavelet-Galerkin method allows for significantly larger time steps than the finite difference schemes.

Altogether, these experiments demonstrate that wavelets can be applied successfully to simulate solutions with discontinuities. Future work can go in two directions. To deal with singularities in the solution, wavelets instead of scaling functions could be used as basis functions. At regions of smoothness in the solution only few basis functions at low scaling level would be required, while at regions of large gradients, wavelets at a finer scale would be employed. This should give good approximate solutions, yet require fewer basis functions.

Secondly, the Wavelet-Galerkin Method lends itself to solving problems in two or more space dimensions, where other more elaborate numerical techniques such as finite difference schemes are much more difficult to implement.

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APPENDICES

APPENDIX A

EXACT SOLUTION OF THE BURGERS EQUATION

In this first appendix, we show how the exact solution of Burgers equation can be obtained by means of the Cole-Hopf transformation, and compute the solution for the Cauchy problem considered in chapter IV.

A.1 The General Solution of the Burgers Equation

Consider an initial value problem

$$u_t + uu_x - \nu u_{xx} = 0 \qquad (-\infty < x < \infty,\ t > 0) \tag{A.1}$$

$$u(x,0) = f(x) \qquad (-\infty < x < \infty). \tag{A.2}$$

where $\nu > 0$. Cole and Hopf showed independently (Cole, 1951; Hopf, 1950) that this problem can be transformed to an initial value problem involving the one-dimensional heat equation. In fact us set

$$u(x,t) = -2\nu\frac{w_x}{w}. \tag{A.3}$$

Taking partial derivatives and assuming sufficient differentiability,

$$\begin{aligned}
u_t &= -2\nu\frac{w_{tx}}{w} + 2\nu\frac{w_t w_x}{w^2} \\
u_x &= -2\nu\frac{w_{xx}}{w} + 2\nu\frac{(w_x)^2}{w^2} \\
u_{xx} &= -2\nu\frac{w_{xxx}}{w} + 6\nu\frac{w_{xx}w_x}{w^2} - 4\nu\frac{(w_x)^3}{w^3}.
\end{aligned}$$

This substitution changes equation (A.1) to

$$\left[-2\nu\frac{w_{tx}}{w}+2\nu\frac{w_t w_x}{w^2}\right]+\left[-2\nu\frac{w_x}{w}\right]\left[-2\nu\frac{w_{xx}}{w}+2\nu\frac{(w_x)^2}{w^2}\right]$$
$$-\nu\left[-2\nu\frac{w_{xxx}}{w}+6\nu\frac{w_{xx}w_x}{w^2}-4\nu\frac{(w_x)^3}{w^3}\right]=0,$$

which can be expressed, after multiplying by $-w/2\nu$, as

$$[w_t-\nu w_{xx}]_x-\frac{w_x}{w}[w_t-\nu w_{xx}]=0. \tag{A.4}$$

Thus, if $w(x,t)$ is a solution of the heat equation

$$w_t-\nu w_{xx}=0$$

then obviously, w will solve (A.4) and hence, u will solve the Burgers equation (A.1).

Next we transform the initial condition (A.2). Applying (A.3) we obtain the ordinary differential equation

$$f(x)=-2\nu\frac{w_x(x,0)}{w(x,0)}$$

whose general solution is

$$w(x,0)=g(x)=Ce^{-\frac{1}{2\nu}\int_0^x f(s)\,ds}. \tag{A.5}$$

For simplicity, we choose $C=1$. Now it is well known that the homogeneous heat equation

$$\begin{aligned} &w_t-\nu w_{xx}=0 \qquad (-\infty<x<\infty,\ t>0)\\ &w(x,0)=g(x) \qquad (-\infty<x<\infty)\end{aligned}$$

has general solution

$$w(x,t)=\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4\nu t}}g(y)\,dy.$$

Hence the initial value problem (A.1)–(A.2) has solution

$$u(x,t) = -2\nu \frac{w(x,t)_x}{w(x,t)} = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{1}{2\nu}\int_0^y f(s)\,ds}\,dy}{\displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{1}{2\nu}\int_0^y f(s)\,ds}\,dy}. \tag{A.6}$$

A.2 A Particular Solution Given an Initial Discontinuity

Next we solve problem (A.1)–(A.2) where

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0. \end{cases}$$

Then

$$\int_0^y f(s)\,ds = \begin{cases} y & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

so that solution (A.6) becomes

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{0} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{y}{2\nu}}\,dy + \int_0^{\infty} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\nu t}}\,dy}{\displaystyle\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{y}{2\nu}}\,dy + \int_0^{\infty} e^{-\frac{(x-y)^2}{4\nu t}}\,dy}. \tag{A.7}$$

We compute each of these integrals separately. The second integral in the denominator can easily computed by substitution,

$$\begin{aligned} \int_0^{\infty} e^{-\frac{(x-y)^2}{4\nu t}}\,dy &= \sqrt{4\nu t}\int_{-x/\sqrt{4\nu t}}^{\infty} e^{-u^2}\,du \\ &= \sqrt{\nu t\pi}\,\frac{2}{\sqrt{\pi}}\int_{-x/\sqrt{4\nu t}}^{\infty} e^{-u^2}\,du = \sqrt{\nu t \pi}\,\operatorname{erfc}\left(-\frac{x}{\sqrt{4\nu t}}\right) \end{aligned} \tag{A.8}$$

where erfc denotes the complementary error function,

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}\,du.$$

The first integral in the denominator can now be computed by completing the square and using (A.8),

$$
\begin{aligned}
\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{y}{2\nu}}\, dy &= \int_{0}^{\infty} e^{-\frac{(x+y)^2}{4\nu t}} e^{\frac{y}{2\nu}}\, dy = \int_{0}^{\infty} e^{-\frac{x^2+2xy+y^2-2ty}{4\nu t}}\, dy \\
&= \int_{0}^{\infty} e^{-\frac{([t-x]-y)^2+2tx-t^2}{4\nu t}}\, dy = \int_{0}^{\infty} e^{-\frac{([t-x]-y)^2}{4\nu t}} e^{\frac{t-2x}{4\nu}}\, dy \\
&= \sqrt{\nu t\pi}\, e^{\frac{t-2x}{4\nu}} \operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right).
\end{aligned}
$$

The integrals in the numerator are computed by substitution,

$$
\int_{0}^{\infty} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\nu t}}\, dy = -\frac{1}{2t}\int_{x^2}^{\infty} e^{-\frac{u}{4\nu t}}\, du = 2\nu e^{-\frac{u}{4\nu t}}\Big]_{x^2}^{\infty} = -2\nu e^{-\frac{x^2}{4\nu t}}
$$

and

$$
\begin{aligned}
\int_{-\infty}^{0} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{y}{2\nu}}\, dy &= \int_{0}^{\infty} \frac{x+y}{t} e^{-\frac{(x+y)^2}{4\nu t}} e^{\frac{y}{2\nu}}\, dy \\
&= \int_{0}^{\infty} \frac{x+y}{t} e^{-\frac{([t-x]-y)^2}{4\nu t}} e^{\frac{t-2x}{4\nu}}\, dy \\
&= e^{\frac{t-2x}{4\nu}} \int_{0}^{\infty} \left[1-\frac{t-x-y}{t}\right] e^{-\frac{([t-x]-y)^2}{4\nu t}}\, dy \\
&= e^{\frac{t-2x}{4\nu}} \int_{0}^{\infty} e^{-\frac{([t-x]-y)^2}{4\nu t}}\, dy + e^{\frac{t-2x}{4\nu}} \int_{0}^{\infty} \frac{[t-x]-y}{t} e^{-\frac{([t-x]-y)^2}{4\nu t}}\, dy \\
&= e^{\frac{t-2x}{4\nu}} \sqrt{\nu t\pi}\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right) + 2\nu e^{\frac{t-2x}{4\nu}} e^{-\frac{(t-x)^2}{4\nu t}}.
\end{aligned}
$$

Combining all four integrals, the solution (A.7) becomes

$$
\begin{aligned}
u(x,t) &= \frac{e^{\frac{t-2x}{4\nu}}\sqrt{\nu t\pi}\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right) + 2\nu e^{\frac{t-2x}{4\nu}} e^{-\frac{(t-x)^2}{4\nu t}} - 2\nu e^{-\frac{x^2}{4\nu t}}}{\sqrt{\nu t\pi}\, e^{\frac{t-2x}{4\nu}}\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right) + \sqrt{\nu t\pi}\operatorname{erfc}\left(-\frac{x}{\sqrt{4\nu t}}\right)} \\
&= \frac{e^{\frac{t-2x}{4\nu}}\sqrt{\nu t\pi}\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right) + 2\nu\left[e^{-\frac{2tx-t^2+t^2-2tx+x^2}{4\nu t}} - e^{-\frac{x^2}{4\nu t}}\right]}{\sqrt{\nu t\pi}\, e^{\frac{t-2x}{4\nu}}\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right) + \sqrt{\nu t\pi}\operatorname{erfc}\left(-\frac{x}{\sqrt{4\nu t}}\right)} \\
&= \frac{e^{\frac{t-2x}{4\nu}}\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right)}{e^{\frac{t-2x}{4\nu}}\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right) + \operatorname{erfc}\left(-\frac{x}{\sqrt{4\nu t}}\right)} \\
&= \frac{\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right)}{\operatorname{erfc}\left(\frac{x-t}{\sqrt{4\nu t}}\right) + e^{\frac{2x-t}{4\nu}}\operatorname{erfc}\left(-\frac{x}{\sqrt{4\nu t}}\right)}.
\end{aligned}
\tag{A.9}
$$

APPENDIX B

TABLES OF CONNECTION COEFFICIENTS

This appendix lists some of the connection coefficients which we have computed. Only coefficients for the Coiflet $C12$ at level $j = 0$ are shown. The coefficients at level $j > 0$ can easily be derived from these by using the "level-up" property. Coefficients which are not listed are all zero.

Table B.1 shows the two-term connection coefficients $\Gamma^{j,0,0}$ and $\Gamma^{j,1,1}$ computed using symmetry. For the coefficients $\Gamma^{j,0,0}$, one moment equation was required, while for the coefficients $\Gamma^{j,1,1}$, two moment equations were required.

Table B.2 shows the two-term connection coefficients $\Gamma^{j,1,0}$. Two moment equations were required for their computation.

Table B.3 shows the three-term connection coefficients $\Gamma^{j,1,0,0}$ and $\Gamma^{j,0,1,1,}$ computed using symmetry. Three moment equations were required for the computation of $\Gamma^{j,1,0,0}$, and four moment equations for the computation of $\Gamma^{j,0,1,1}$.

Table B.1 The connection coefficients $\Gamma^{0,0,0}$ and $\Gamma^{0,1,1}$ for the Coiflets $C12$.

l_1	l_2	$\Gamma^{0,0,0}_{l_1,l_2}$	$\Gamma^{0,1,1}_{l_1,l_2}$
−6	−6	0.00000000000000	0.00000000000000
−6	−5	0.00000000000000	−0.00000000000002
−6	−4	−0.00000000000001	0.00000000000015
−6	−3	0.00000000000080	0.00000000000208
−6	−2	−0.00000000001362	−0.00000000023256
−6	−1	0.00000000005252	0.00000000030935
−6	0	−0.00000000035551	−0.00000000309039
−6	1	0.00000000002162	0.00000000357846
−6	2	−0.00000000000698	−0.00000000054555
−6	3	−0.00000000000008	−0.00000000002154
−6	4	0.00000000000000	0.00000000000002
−5	−5	0.00000000000118	0.00000000002741
−5	−4	0.00000000003053	0.00000000034329
−5	−3	−0.00000000150870	−0.00000000845307
−5	−2	0.00000002342231	0.00000027903730
−5	−1	−0.00000009849357	−0.00000068867968
−5	0	0.00000065099074	0.00000365418315
−5	1	0.00000001214765	−0.00000376428555
−5	2	0.00000000595667	0.00000050623617
−5	3	−0.00000000019226	0.00000002193387
−5	4	0.00000000000008	−0.00000000034288
−4	−4	0.00000000123911	0.00000006526979
−4	−3	−0.00000004341001	−0.00000086709471
−4	−2	0.00000066667700	0.00001136301153
−4	−1	−0.00000261442351	−0.00003626851589
−4	0	0.00001615321007	0.00006757457049
−4	1	−0.00000272521677	−0.00003981195446
−4	2	0.00000043706395	−0.00000747640347
−4	3	−0.00000000308766	0.00000544731941
−4	4	0.00000000019924	−0.00000002654616
−3	−3	0.00000246915018	0.00001697983048
−3	−2	−0.00003384435323	−0.00024955063059
−3	−1	0.00016313350162	0.00089973867461
−3	0	−0.00109451531379	−0.00397284543005

Table B.1 (Continued)

l_1	l_2	$\Gamma^{0,0,0}_{l_1,l_2}$	$\Gamma^{0,1,1}_{l_1,l_2}$
-3	1	-0.00025664173611	0.00362316204067
-3	2	0.00001830831802	-0.00032539583809
-3	3	-0.00000027970416	0.00000703518643
-3	4	-0.00000000289063	0.00000175171222
-2	-2	0.00049775750769	0.00505403577627
-2	-1	-0.00222197189913	-0.01392226364246
-2	0	0.01465981373947	0.07154617537182
-2	1	0.00083141674719	-0.07041326460790
-2	2	0.00007156383649	0.00761942482804
-2	3	-0.00001912685212	0.00036981246409
-2	4	-0.00000016915193	-0.00001601137522
-1	-1	0.01113375815006	0.06879767816885
-1	0	-0.07462935683210	-0.32960483070033
-1	1	-0.01304635782232	0.30221802249227
-1	2	0.00058905708129	-0.03609600599428
-1	3	0.00017769505100	0.00750247926106
-1	4	0.00000289270760	0.00024213862437
0	0	0.50739284923247	2.00795133850641
0	1	0.13600075787929	-1.95442390071702
0	2	-0.00978766328703	0.26107173247811
0	3	-0.00041754161142	-0.05146755916392
0	4	-0.00000867185426	-0.00117133600578
1	1	0.47350244515946	1.93960516578098
1	2	-0.05805785282816	-0.27051580994627
1	3	0.00802262977287	0.04896228617501
1	4	0.00009417409713	0.00098791144420
2	2	0.00726960244236	0.04728949690463
2	3	-0.00106027416485	-0.00892890220379
2	4	-0.00001222107290	-0.00010756951663
3	3	0.00020106653537	0.00348968856187
3	4	0.00000258557766	0.00005969048758
4	4	0.00000005058212	0.00000345151828

Table B.2 The connection coefficients $\Gamma^{0,1,0}$ for the coiflets $C12$.

l_1	l_2	$\Gamma^{0,1,0}_{l_1,l_2}$
−6	−6	0.00000000000000
−6	−5	0.00000000000000
−6	−4	0.00000000000015
−6	−3	−0.00000000000684
−6	−2	0.00000000010788
−6	−1	−0.00000000044784
−6	0	0.00000000290264
−6	1	−0.00000000001790
−6	2	0.00000000003399
−6	3	−0.00000000000027
−6	4	0.00000000000000
−5	−6	0.00000000000000
−5	−5	−0.00000000000320
−5	−4	−0.00000000012292
−5	−3	0.00000000586104
−5	−2	−0.00000007677644
−5	−1	0.00000038090674
−5	0	−0.00000246386908
−5	1	−0.00000040358077
−5	2	0.00000002086285
−5	3	0.00000000381359
−5	4	−0.00000000002072
−4	−6	−0.00000000000009
−4	−5	0.00000000005920
−4	−4	−0.00000000031377
−4	−3	0.00000003880063
−4	−2	0.00000021937713
−4	−1	0.00000320368593
−4	0	−0.00001894379758
−4	1	−0.00000803076688
−4	2	0.00000014665070
−4	3	0.00000071976634
−4	4	−0.00000000146898
−3	−6	0.00000000000034
−3	−5	0.00000000053371
−3	−4	0.00000002489481
−3	−3	−0.00000319314932
−3	−2	0.00002788963419
−3	−1	−0.00022085963722
−3	0	0.00157102103325
−3	1	0.00130730486702
−3	2	−0.00013390115445
−3	3	0.00000398172341
−3	4	0.00000014775379
−2	−6	0.00000000001321
−2	−5	−0.00000004236091
−2	−4	−0.00000141131504
−2	−3	0.00009104373006
−2	−2	−0.00110534219209
−2	−1	0.00613682765294
−2	0	−0.04134950865729
−2	1	−0.01461364126370
−2	2	0.00119228023391
−2	3	0.00003794649809
−2	4	−0.00000117725503
−1	−6	0.00000000002885
−1	−5	0.00000003119862
−1	−4	0.00000101732481
−1	−3	−0.00018967316043
−1	−2	0.00141503221938
−1	−1	−0.01216168362796
−1	0	0.09278776288774
−1	1	0.14298743264767
−1	2	−0.01653911445972
−1	3	0.00167535291958
−1	4	0.00002199350382
0	−6	0.00000000026151
0	−5	−0.00000064884473
0	−4	−0.00001244571378
0	−3	0.00153417261609
0	−2	−0.01616905811550
0	−1	0.09997468254022
0	0	−0.74356559108211
0	1	−0.81727091191582
0	2	0.09233081087159
0	3	−0.00993851752323
0	4	−0.00010848477729
1	−6	−0.00000000036020
1	−5	0.00000077241830
1	−4	0.00001484550931
1	−3	−0.00164787073891
1	−2	0.01842668583761
1	−1	−0.10900778891562
1	0	0.79771277229102
1	1	0.74602903298581
1	2	−0.08267859123512
1	3	0.00839791233244
1	4	0.00008134843933
2	−6	0.00000000005416
2	−5	−0.00000010721929
2	−4	−0.00000138170599
2	−3	0.00021682717156
2	−2	−0.00243472426162
2	−1	0.01520101723351
2	0	−0.11055235270305
2	1	−0.09315723503447
2	2	0.01021930087433
2	3	−0.00078586064876
2	4	0.00000143698505
3	−6	0.00000000000220
3	−5	−0.00000000580050
3	−4	−0.00000065192531
3	−3	−0.00000122226359
3	−2	−0.00015990667935
3	−1	0.00006088155866
3	0	0.00343106351053
3	1	0.03388458936996
3	2	−0.00427919720364
3	3	0.00058719496811
3	4	0.00000445529982
4	−6	0.00000000000000
4	−5	0.00000000001879
4	−4	0.00000000336773
4	−3	−0.00000012886028
4	−2	−0.00000071915120
4	−1	0.00001333905070
4	0	−0.00001376251646
4	1	0.00084186270945
4	2	−0.00011175547450
4	3	0.00002126615071
4	4	0.00000028154020

Table B.3 The connection coefficients $\Gamma^{0,1,0,0}$ and $\Gamma^{0,0,1,1}$ for the coiflets $C12$.

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−6	−6	−6	−0.000000000000000010	0.000000000000000226
−6	−6	−5	−0.000000000000000005	−0.000000000000000001
−6	−6	−4	−0.000000000000000259	0.000000000000000248
−6	−6	−3	−0.000000000000000077	−0.000000000000000023
−6	−6	−2	0.000000000000000054	0.000000000000000644
−6	−6	−1	−0.000000000000000012	0.000000000000000918
−6	−6	0	0.000000000000000252	−0.000000000000001506
−6	−6	1	−0.000000000000000331	0.000000000000001264
−6	−6	2	−0.000000000000000044	0.000000000000001720
−6	−6	3	0.000000000000000110	−0.000000000000000373
−6	−6	4	−0.000000000000000141	−0.000000000000000186
−6	−5	−5	−0.000000000000000068	0.000000000000000051
−6	−5	−4	−0.000000000000000044	0.000000000000000552
−6	−5	−3	0.000000000000000265	0.000000000000000104
−6	−5	−2	0.000000000000000130	−0.000000000000000532
−6	−5	−1	−0.000000000000001150	−0.000000000000000530
−6	−5	0	0.000000000000006833	−0.000000000000001508
−6	−5	1	−0.000000000000000188	0.000000000000000372
−6	−5	2	0.000000000000000064	0.000000000000000414
−6	−5	3	0.000000000000000136	0.000000000000000064
−6	−5	4	0.000000000000000292	−0.000000000000000379
−6	−4	−4	−0.000000000000000056	0.000000000000000466
−6	−4	−3	−0.000000000000000492	−0.000000000000000172
−6	−4	−2	0.000000000000006371	0.000000000000000205
−6	−4	−1	−0.000000000000026288	0.000000000000021331
−6	−4	0	0.000000000000169039	0.000000000000055225
−6	−4	1	−0.000000000000000753	−0.000000000000090300
−6	−4	2	0.000000000000002099	0.000000000000013333
−6	−4	3	0.000000000000000020	−0.000000000000000654
−6	−4	4	0.000000000000000203	−0.000000000000000805
−6	−3	−3	0.000000000000018320	−0.000000000000002105
−6	−3	−2	−0.000000000000287199	0.000000000000007243
−6	−3	−1	0.000000000001192390	−0.000000000000137631
−6	−3	0	−0.000000000007727504	−0.000000000000136690
−6	−3	1	0.000000000000051631	0.000000000000310948
−6	−3	2	−0.000000000000091038	−0.000000000000041990
−6	−3	3	0.000000000000001013	−0.000000000000002672
−6	−3	4	0.000000000000000069	0.000000000000000665
−6	−2	−2	0.000000000004570852	−0.000000000001213713
−6	−2	−1	−0.000000000018864373	0.000000000001639871
−6	−2	0	0.000000000122381367	−0.000000000015971684
−6	−2	1	−0.000000000001454262	0.000000000018522187
−6	−2	2	0.000000000001537373	−0.000000000002883660
−6	−2	3	−0.000000000000009120	−0.000000000000101458

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−6	−2	4	−0.000000000000000235	0.000000000000000558
−6	−1	−6	−0.000000000000000012	0.000000000000000918
−6	−1	−5	−0.000000000000001150	−0.000000000000000530
−6	−1	−4	−0.000000000000026288	0.000000000000021331
−6	−1	−3	0.000000000001192390	−0.000000000000137631
−6	−1	−2	−0.000000000018864373	0.000000000001639871
−6	−1	−1	0.000000000078110414	−0.000000000016236204
−6	−1	0	−0.000000000506524640	−0.000000000004077858
−6	−1	1	0.000000000004347298	0.000000000021801802
−6	−1	2	−0.000000000006111843	−0.000000000002803700
−6	−1	3	0.000000000000042239	−0.000000000000206881
−6	−1	4	−0.000000000000000200	−0.000000000000000327
−6	0	0	0.000000003284436888	−0.000000000259686050
−6	0	1	−0.000000000029713901	0.000000000332445615
−6	0	2	0.000000000039887056	−0.000000000050720338
−6	0	3	−0.000000000000272670	−0.000000000001905412
−6	0	4	0.000000000000000435	0.000000000000000714
−6	1	−6	−0.000000000000000331	0.000000000000001264
−6	1	−5	−0.000000000000000188	0.000000000000000372
−6	1	−4	−0.000000000000000753	−0.000000000000090300
−6	1	−3	0.000000000000051631	0.000000000000310948
−6	1	−2	−0.000000000001454262	0.000000000018522187
−6	1	−1	0.000000000004347298	0.000000000021801802
−6	1	0	−0.000000000029713901	0.000000000332445615
−6	1	1	0.000000000010855908	−0.000000000442583475
−6	1	2	−0.000000000001957673	0.000000000066958980
−6	1	3	−0.000000000000028110	0.000000000002632903
−6	1	4	0.000000000000000137	−0.000000000000000723
−6	2	2	0.000000000000727180	−0.000000000010120128
−6	2	3	0.000000000000000796	−0.000000000000401034
−6	2	4	0.000000000000000040	0.000000000000000017
−6	3	3	0.000000000000000190	−0.000000000000016434
−6	3	4	0.000000000000000060	−0.000000000000000222
−6	4	4	0.000000000000000271	−0.000000000000000263
−5	−6	−6	−0.000000000000000022	−0.000000000000000078
−5	−6	−5	−0.000000000000000228	0.000000000000000069
−5	−6	−4	0.000000000000000007	0.000000000000000387
−5	−6	−3	−0.000000000000000050	−0.000000000000000718
−5	−6	−2	−0.000000000000000014	−0.000000000000000961
−5	−6	−1	−0.000000000000000138	0.000000000000000684
−5	−6	0	0.000000000000000705	−0.000000000000006984
−5	−6	1	−0.000000000000000154	0.000000000000007970
−5	−6	2	−0.000000000000000084	−0.000000000000001481
−5	−6	3	−0.000000000000000153	0.000000000000000159

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−5	−6	4	0.000000000000000057	0.000000000000000084
−5	−5	−5	0.000000000000000076	0.000000000000000432
−5	−5	−4	−0.000000000000000423	0.000000000000000147
−5	−5	−3	0.000000000000008357	−0.000000000000005331
−5	−5	−2	−0.000000000000115317	0.000000000000329429
−5	−5	−1	0.000000000000527219	−0.000000000000714038
−5	−5	0	−0.000000000003344236	0.000000000004416502
−5	−5	1	−0.000000000000281222	−0.000000000004764549
−5	−5	2	0.000000000000002162	0.000000000000722543
−5	−5	3	0.000000000000001907	0.000000000000014986
−5	−5	4	−0.000000000000000044	0.000000000000000181
−5	−4	−4	−0.000000000000008157	0.000000000000051003
−5	−4	−3	0.000000000000325905	−0.000000000000450496
−5	−4	−2	−0.000000000004481967	0.000000000002196204
−5	−4	−1	0.000000000020395350	−0.000000000031145327
−5	−4	0	−0.000000000128923803	−0.000000000012990025
−5	−4	1	−0.000000000010483296	0.000000000053357227
−5	−4	2	0.000000000000235113	−0.000000000010994762
−5	−4	3	0.000000000000024668	−0.000000000000022466
−5	−4	4	0.000000000000000443	−0.000000000000000328
−5	−3	−3	−0.000000000014381452	0.000000000006498836
−5	−3	−2	0.000000000192730290	−0.000000000102087204
−5	−3	−1	−0.000000000920497954	0.000000000378968029
−5	−3	0	0.000000005888667056	−0.000000001107407540
−5	−3	1	0.000000000752766264	0.000000000909058911
−5	−3	2	−0.000000000032796034	−0.000000000070787876
−5	−3	3	−0.000000000005794922	−0.000000000013838371
−5	−3	4	0.000000000000008549	0.000000000000051547
−5	−2	−2	−0.000000002621128972	0.000000004138480139
−5	−2	−1	0.000000012307204794	−0.000000008018817391
−5	−2	0	−0.000000078338889610	0.000000055433468115
−5	−2	1	−0.000000008675813946	−0.000000060097029876
−5	−2	2	0.000000000303843816	0.000000008231323131
−5	−2	3	0.000000000060227833	0.000000000412781587
−5	−2	4	−0.000000000000017001	−0.000000000000643273
−5	−1	−1	−0.000000059282444294	0.000000031420404660
−5	−1	0	0.000000380329476503	−0.000000079989754448
−5	−1	1	0.000000051144360078	0.000000064336316055
−5	−1	2	−0.000000002245101503	−0.000000007533918323
−5	−1	3	−0.000000000447682567	−0.000000000567029313
−5	−1	4	0.000000000000498566	0.000000000005689526
−5	0	0	−0.000002443460530426	0.000000781124875221
−5	0	1	−0.000000347373774612	−0.000000882821588589
−5	0	2	0.000000016108672465	0.000000120574582554

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−5	0	3	0.000000003114673639	0.000000006824893983
−5	0	4	−0.000000000005111014	−0.000000000030487484
−5	1	1	−0.000000107960440929	0.000001028130757086
−5	1	2	0.000000007365872329	−0.000000142836061304
−5	1	3	0.000000001195069095	−0.000000007698966636
−5	1	4	−0.000000000018039522	0.000000000028913734
−5	2	2	−0.000000000560393885	0.000000020637673454
−5	2	3	−0.000000000079714482	0.000000001012425392
−5	2	4	0.000000000002229592	−0.000000000004963821
−5	3	3	−0.000000000022921066	0.000000000028342828
−5	3	4	−0.000000000000289466	0.000000000001398424
−5	4	4	−0.000000000000001957	0.000000000000041780
−4	−6	−6	0.000000000000000172	0.000000000000000148
−4	−6	−5	0.000000000000000077	−0.000000000000000174
−4	−6	−4	0.000000000000000116	0.000000000000000551
−4	−6	−3	0.000000000000000231	−0.000000000000000219
−4	−6	−2	−0.000000000000003916	−0.000000000000013057
−4	−6	−1	0.000000000000014652	0.000000000000017441
−4	−6	0	−0.000000000000102852	−0.000000000000180232
−4	−6	1	0.000000000000009218	0.000000000000208937
−4	−6	2	−0.000000000000002644	−0.000000000000031925
−4	−6	3	−0.000000000000000041	−0.000000000000001076
−4	−6	4	0.000000000000000209	−0.000000000000000171
−4	−5	−5	0.000000000000000362	0.000000000000001520
−4	−5	−4	0.000000000000003393	0.000000000000001156
−4	−5	−3	−0.000000000000185558	−0.000000000000195546
−4	−5	−2	0.000000000003737773	0.000000000012467530
−4	−5	−1	−0.000000000012133079	−0.000000000026707450
−4	−5	0	0.000000000085041156	0.000000000168666484
−4	−5	1	−0.000000000020975980	−0.000000000182600251
−4	−5	2	0.000000000003577473	0.000000000027759688
−4	−5	3	0.000000000000133682	0.000000000000605814
−4	−5	4	0.000000000000000040	0.000000000000000449
−4	−4	−4	−0.000000000000005460	0.000000000000656802
−4	−4	−3	−0.000000000000512069	−0.000000000009220688
−4	−4	−2	0.000000000033691457	0.000000000062270074
−4	−4	−1	−0.000000000049687400	−0.000000000580021302
−4	−4	0	0.000000000415738958	0.000000001351250658
−4	−4	1	−0.000000000828182714	−0.000000000904321979
−4	−4	2	0.000000000117591966	0.000000000128680939
−4	−4	3	−0.000000000002445521	−0.000000000049857364
−4	−4	4	0.000000000000034187	0.000000000000562336
−4	−3	−3	−0.000000000001000363	0.000000000174076329
−4	−3	−2	−0.000000001788163841	−0.000000003360242181

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−4	−3	−1	−0.000000000653019608	0.000000007397569638
−4	−3	0	0.000000000348737886	−0.000000035024901256
−4	−3	1	0.000000046774775013	0.000000034503611548
−4	−3	2	−0.000000005240558902	−0.000000002593813733
−4	−3	3	−0.000000000640631099	−0.000000001087816150
−4	−3	4	0.000000000001183787	0.000000000000931952
−4	−2	−2	0.000000051198275317	0.000000138652853520
−4	−2	−1	−0.000000113857949045	−0.000000251854066545
−4	−2	0	0.000000834211306741	0.000001818524674690
−4	−2	1	−0.000000641645527417	−0.000001989530372770
−4	−2	2	0.000000086185009355	0.000000268592864211
−4	−2	3	0.000000005041760450	0.000000018886409609
−4	−2	4	−0.000000000005008044	0.000000000013156242
−4	−1	−1	−0.000000100702855718	0.000000527299224122
−4	−1	0	0.000000466876582508	−0.000001650836088388
−4	−1	1	0.000003360314886409	0.000001571448971860
−4	−1	2	−0.000000361012786212	−0.000000097141914336
−4	−1	3	−0.000000047355149480	−0.000000106317886819
−4	−1	4	0.000000000138030103	0.000000000610903717
−4	0	0	−0.000001532256734078	0.000013443505446895
−4	0	1	−0.000021322318883261	−0.000015683109592897
−4	0	2	0.000002246396053631	0.000001186567309403
−4	0	3	0.000000363413770066	0.000000923766631291
−4	0	4	−0.000000000969087456	−0.000000004913220668
−4	1	1	0.000012167498315006	0.000018626494240747
−4	1	2	−0.000002084479558528	−0.000001581685373739
−4	1	3	0.000000444659181631	−0.000000981894920270
−4	1	4	−0.000000000720917276	0.000000004860149493
−4	2	2	0.000000313112910464	0.000000044552244944
−4	2	3	−0.000000048504308903	0.000000182418499672
−4	2	4	0.000000000072771971	−0.000000000866224387
−4	3	3	0.000000003140901865	−0.000000036014334919
−4	3	4	0.000000000013125522	0.000000000292670723
−4	4	4	0.000000000000883041	0.000000000001071161
−3	−6	−6	0.000000000000000208	−0.000000000000000372
−3	−6	−5	−0.000000000000000134	0.000000000000000077
−3	−6	−4	−0.000000000000000007	−0.000000000000000373
−3	−6	−3	−0.000000000000000949	−0.000000000000005664
−3	−6	−2	0.000000000000016561	0.000000000000613259
−3	−6	−1	−0.000000000000058604	−0.000000000000800089
−3	−6	0	0.000000000000424141	0.000000000008188432
−3	−6	1	−0.000000000000048845	−0.000000000009500412
−3	−6	2	0.000000000000011584	0.000000000001447447
−3	−6	3	0.000000000000000131	0.000000000000057628

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−3	−6	4	−0.000000000000000032	−0.000000000000000011
−3	−5	−5	−0.000000000000000233	−0.000000000000068006
−3	−5	−4	0.000000000000020955	−0.000000000000474024
−3	−5	−3	−0.000000000000890019	0.000000000014979461
−3	−5	−2	0.000000000004088152	−0.000000000623220799
−3	−5	−1	−0.000000000058461628	0.000000001461771721
−3	−5	0	0.000000000346303104	−0.000000008293772397
−3	−5	1	0.000000000274134437	0.000000008723964282
−3	−5	2	−0.000000000029408508	−0.000000001244462570
−3	−5	3	−0.000000000002071933	−0.000000000038896337
−3	−5	4	0.000000000000000291	0.000000000000179515
−3	−4	−4	0.000000000002626402	−0.000000000101696747
−3	−4	−3	−0.000000000079905465	0.000000001349294175
−3	−4	−2	0.000000000930204630	−0.000000015696026371
−3	−4	−1	−0.000000004501134761	0.000000066837816115
−3	−4	0	0.000000026239307528	−0.000000133802027507
−3	−4	1	0.000000002632843643	0.000000080840314646
−3	−4	2	−0.000000000346738201	0.000000004692486579
−3	−4	3	0.000000000017637989	−0.000000004138576025
−3	−4	4	−0.000000000000054223	0.000000000018887639
−3	−3	−3	0.000000005380438629	−0.000000024635033040
−3	−3	−2	−0.000000050916801222	0.000000382616427875
−3	−3	−1	0.000000365688633018	−0.000001193173921248
−3	−3	0	−0.000002433330133454	0.000005015968980896
−3	−3	1	−0.000001197455712256	−0.000004625156774865
−3	−3	2	0.000000109807024088	0.000000392930254861
−3	−3	3	0.000000007776281044	0.000000050570400908
−3	−3	4	−0.000000000018255166	−0.000000000484596122
−3	−2	−2	0.000000446860053204	−0.000009420552811246
−3	−2	−1	−0.000003403585510458	0.000022342108878282
−3	−2	0	0.000021852218800074	−0.000126347368336117
−3	−2	1	0.000010078331241998	0.000129403274502665
−3	−2	2	−0.000000930921352979	−0.000015249069790413
−3	−2	3	−0.000001030611311376	−0.000001100312103727
−3	−2	4	−0.000000000225235476	0.000000005621859378
−3	−1	−1	0.000025801100434669	−0.000077986098433583
−3	−1	0	−0.000172871828154965	0.000316185781638067
−3	−1	1	−0.000078269222447457	−0.000292963352451845
−3	−1	2	0.000006588448372525	0.000033904235161251
−3	−1	3	0.000000931068398475	−0.000000306173221299
−3	−1	4	0.000000003252706330	−0.000000051626437344
−3	0	0	0.001171661693295060	−0.002078714454351630
−3	0	1	0.000612181544902940	0.002160720613671040
−3	0	2	−0.000053993832774121	−0.000281348869361149

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−3	0	3	−0.000005398575157940	0.000004409596388837
−3	0	4	−0.000000003443565716	0.000000220819019935
−3	1	1	0.000850133676893315	−0.002294016138178510
−3	1	2	−0.000095406171552007	0.000305503505123219
−3	1	3	0.000009616016229169	−0.000003922242002757
−3	1	4	0.000000165240539010	−0.000000190058682215
−3	2	2	0.000011167812546226	−0.000045512724184986
−3	2	3	−0.000001414624116721	0.000002279615793253
−3	2	4	−0.000000021296457835	0.000000026927527469
−3	3	3	0.000000338937900220	−0.000001396423515863
−3	3	4	0.000000004169624600	−0.000000010454324468
−3	4	4	0.000000000074491260	−0.000000000763434859
−2	−6	−6	−0.000000000000000047	0.000000000000000234
−2	−6	−5	−0.000000000000000096	−0.000000000000000865
−2	−6	−4	0.000000000000000509	0.000000000000007005
−2	−6	−3	−0.000000000000035270	0.000000000000081713
−2	−6	−2	0.000000000000565408	−0.000000000009531041
−2	−6	−1	−0.000000000002330589	0.000000000011667174
−2	−6	0	0.000000000015041388	−0.000000000127464769
−2	−6	1	−0.000000000000222966	0.000000000148790106
−2	−6	2	0.000000000000197654	−0.000000000022616487
−2	−6	3	−0.000000000000001524	−0.000000000000933607
−2	−6	4	0.000000000000000002	0.000000000000000440
−2	−5	−5	−0.000000000000070971	0.000000000000917569
−2	−5	−4	−0.000000000002255554	0.000000000004513089
−2	−5	−3	0.000000000104286742	−0.000000000169923113
−2	−5	−2	−0.000000001494434951	0.000000008095356330
−2	−5	−1	0.000000006800078541	−0.000000018565434545
−2	−5	0	−0.000000043923139324	0.000000107947062959
−2	−5	1	−0.000000003867991771	−0.000000114543997485
−2	−5	2	−0.000000000011184306	0.000000016788021633
−2	−5	3	0.000000000033803336	0.000000000444275001
−2	−5	4	−0.000000000000006974	−0.000000000000792140
−2	−4	−4	−0.000000000097248917	0.000000001261719954
−2	−4	−3	0.000000003853948048	−0.000000014677956805
−2	−4	−2	−0.000000053502171921	0.000000140035592720
−2	−4	−1	0.000000239073065217	−0.000000840625331920
−2	−4	0	−0.000001508201305598	0.000001022087230143
−2	−4	1	−0.000000092562478871	−0.000000215175034887
−2	−4	2	0.000000000154656701	−0.000000113850471305
−2	−4	3	−0.000000000034455270	0.000000021024347168
−2	−4	4	0.000000000003201673	−0.000000000084615002
−2	−3	−3	−0.000000198901745580	0.000000247566791648
−2	−3	−2	0.000002578183969730	−0.000003892021801881

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−2	−3	−1	−0.000013086581513172	0.000012321397657017
−2	−3	0	0.000085716603760247	−0.000046476150716886
−2	−3	1	0.000017219208942577	0.000041861166894599
−2	−3	2	−0.000001051246234711	−0.000003390493766319
−2	−3	3	−0.000000137726408753	−0.000000656417349235
−2	−3	4	0.000000000231087410	−0.000000000199910733
−2	−2	−2	−0.000034802896199959	0.000112812434191689
−2	−2	−1	0.000168244733918278	−0.000246356727904345
−2	−2	0	−0.001092069112379360	0.001497773136867340
−2	−2	1	−0.000157457342020968	−0.001570410391286170
−2	−2	2	0.000006734775652709	0.000196505961052647
−2	−2	3	0.000001483324930889	0.000013429035780985
−2	−2	4	0.000000001136083752	−0.000000009548310752
−2	−1	−1	−0.000873175135073395	0.000835018895291719
−2	−1	0	0.005748126258568430	−0.002715137706596240
−2	−1	1	0.001188346173879300	0.002411128465208930
−2	−1	2	−0.000068979425509067	−0.000268226929491796
−2	−1	3	−0.000012882375363243	−0.000027724847915869
−2	−1	4	−0.000000011866776351	−0.000000163367290201
−2	0	0	−0.038015717020198400	0.020586232727045300
−2	0	1	−0.008700806064834610	−0.022420673429984900
−2	0	2	0.000543321556274217	0.002895203367839710
−2	0	3	0.000083527519079142	0.000200976578530336
−2	0	4	−0.000000056288165872	0.000000971569410275
−2	1	1	−0.007713813547486430	0.025046472434239300
−2	1	2	0.000793131925012661	−0.003299576672386310
−2	1	3	−0.000038925726112166	−0.000207713412466311
−2	1	4	−0.000001239460396473	−0.000000758589461073
−2	2	2	−0.000088494900516849	0.000459673144630726
−2	2	3	0.000007459183463718	0.000019886793293410
−2	2	4	0.000000158222102240	0.000000021913846863
−2	3	3	−0.000002548938215039	0.000001839778446754
−2	3	4	−0.000000028762633625	−0.000000058976013650
−2	4	4	−0.000000000469522393	−0.000000002716863470
−1	−6	−6	0.000000000000000117	0.000000000000000076
−1	−6	−5	0.000000000000000011	0.000000000000002876
−1	−6	−4	0.000000000000000820	−0.000000000000028389
−1	−6	−3	−0.000000000000074471	−0.000000000000348603
−1	−6	−2	0.000000000001467072	0.000000000039958766
−1	−6	−1	−0.000000000004926482	−0.000000000051031486
−1	−6	0	0.000000000036782728	0.000000000533570686
−1	−6	1	−0.000000000005628746	−0.000000000620375366
−1	−6	2	0.000000000001202414	0.000000000094469608
−1	−6	3	0.000000000000030035	0.000000000003782952

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−1	−6	4	0.000000000000000155	−0.000000000000002179
−1	−5	−6	0.000000000000000011	0.000000000000002876
−1	−5	−5	−0.000000000000011674	−0.000000000004321338
−1	−5	−4	0.000000000002116471	−0.000000000034414095
−1	−5	−3	−0.000000000062837991	0.000000001029475421
−1	−5	−2	0.000000000303029849	−0.000000040609646481
−1	−5	−1	−0.000000003940022149	0.000000095978242900
−1	−5	0	0.000000020612904049	−0.000000540033483917
−1	−5	1	0.000000016263547220	0.000000566109940923
−1	−5	2	−0.000000001876713264	−0.000000079781761868
−1	−5	3	−0.000000000103424712	−0.000000002664589116
−1	−5	4	0.000000000000035724	0.000000000010553903
−1	−4	−4	0.000000000202954934	−0.000000007350900251
−1	−4	−3	−0.000000005209239866	0.000000098559266699
−1	−4	−2	0.000000067623408911	−0.000001166735464208
−1	−4	−1	−0.000000280965998210	0.000004803843596430
−1	−4	0	0.000001524770636563	−0.000009941278567818
−1	−4	1	−0.000000322532489751	0.000006161375109706
−1	−4	2	0.000000032173875237	0.000000377975244941
−1	−4	3	0.000000001223715121	−0.000000328250328003
−1	−4	4	0.000000000035834491	0.000000001896485608
−1	−3	−3	0.000000309162875239	−0.000001778139882153
−1	−3	−2	−0.000002898911132449	0.000027360831310730
−1	−3	−1	0.000020664365139288	−0.000084273617275168
−1	−3	0	−0.000135685431282551	0.000349775004685515
−1	−3	1	−0.000080227990895895	−0.000322929676954189
−1	−3	2	0.000008099430537655	0.000027656860902447
−1	−3	3	0.000000073086937740	0.000004087269266910
−1	−3	4	−0.000000001600455111	0.000000001879543689
−1	−2	−2	0.000024999596837399	−0.000644198510326639
−1	−2	−1	−0.000191052681904289	0.001548968025358280
−1	−2	0	0.001159185804601130	−0.008576565444457280
−1	−2	1	0.000476065552575104	0.008737531555016220
−1	−2	2	−0.000044830240579942	−0.001012602235160520
−1	−2	3	−0.000006463080618977	−0.000079431982874564
−1	−2	4	−0.000000041748303690	0.000000145066334906
−1	−1	−1	0.001476052718464520	−0.005121151133342140
−1	−1	0	−0.009685799275106760	0.020555338341117100
−1	−1	1	−0.004198389269206190	−0.019324922564269600
−1	−1	2	0.000357208281944873	0.002359063303953720
−1	−1	3	0.000059242429548417	0.000059540142105916
−1	−1	4	0.000000674714108686	0.000002537731668624
−1	0	0	0.065086575301724000	−0.136384912933810000
−1	0	1	0.040465985679451700	0.143788077867533000

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
−1	0	2	−0.003896518044209350	−0.019777036123467900
−1	0	3	−0.000205862458695466	0.000073468978975384
−1	0	4	−0.000001664109056910	−0.000017664907247763
−1	1	1	0.118735970152774000	−0.154110998050521000
−1	1	2	−0.014486222851725100	0.021382320503053000
−1	1	3	0.002048890453103610	−0.000170617968564981
−1	1	4	0.000025667196210718	0.000014811468207069
−1	2	2	0.001801055051123210	−0.003176586960752560
−1	2	3	−0.000274599398844840	0.000197731826592286
−1	2	4	−0.000003336986336438	−0.000000845463023037
−1	3	3	0.000053387504582875	−0.000085436974071312
−1	3	4	0.000000683263244880	0.000000989619709142
−1	4	4	0.000000012738541424	0.000000022697769757
0	−6	−6	0.000000000000000053	0.000000000000000281
0	−6	−5	0.000000000000000660	−0.000000000000017559
0	−6	−4	0.000000000000013824	0.000000000000182326
0	−6	−3	−0.000000000000693116	0.000000000002274431
0	−6	−2	0.000000000011564256	−0.000000000259608487
0	−6	−1	−0.000000000045738518	0.000000000332060525
0	−6	0	0.000000000304234492	−0.000000003459452913
0	−6	1	−0.000000000013102209	0.000000004021479445
0	−6	2	0.000000000005205095	−0.000000000612212099
0	−6	3	0.000000000000030507	−0.000000000024718979
0	−6	4	−0.000000000000000031	0.000000000000013686
0	−5	−5	−0.000000000001188978	0.000000000027700723
0	−5	−4	−0.000000000034500285	0.000000000250135483
0	−5	−3	0.000000001632668561	−0.000000007016054559
0	−5	−2	−0.000000024323337762	0.000000265242908348
0	−5	−1	0.000000106245173799	−0.000000636181628813
0	−5	0	−0.000000693368716722	0.000003514657379473
0	−5	1	−0.000000036047306900	−0.000003666948597435
0	−5	2	−0.000000003288983525	0.000000512334752918
0	−5	3	0.000000000341614136	0.000000017728716594
0	−5	4	−0.000000000000156817	−0.000000000095294271
0	−4	−4	−0.000000001612477955	0.000000052528600673
0	−4	−3	0.000000051688551292	−0.000000691868301891
0	−4	−2	−0.000000783558038446	0.000008264770495056
0	−4	−1	0.000003035525508618	−0.000033626357854754
0	−4	0	−0.000018256181107325	0.000065427986272793
0	−4	1	0.000004086105961853	−0.000038413074834669
0	−4	2	−0.000000580109659525	−0.000003565554796078
0	−4	3	0.000000002723537990	0.000002565949178017
0	−4	4	−0.000000000261565539	−0.000000014629069309
0	−3	−3	−0.000002991373212933	0.000012495871980968

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
0	−3	−2	0.000038845169549706	−0.000189917858250347
0	−3	−1	−0.000198178565296466	0.000606548222768433
0	−3	0	0.001327207597733390	−0.002504003898777890
0	−3	1	0.000402058492367463	0.002295831351710170
0	−3	2	−0.000033223521793579	−0.000197202087216332
0	−3	3	0.000000396488474721	−0.000023202319010714
0	−3	4	0.000000005007739151	0.000000149598849908
0	−2	−2	−0.000544595338540862	0.004338136206725930
0	−2	−1	0.002549633754527040	−0.010703348374863000
0	−2	0	−0.016661888089210300	0.058199292056752000
0	−2	1	−0.001575878431801170	−0.058893927825286000
0	−2	2	−0.000003072496624468	0.006739385415692850
0	−2	3	0.000028478850555920	0.000504199966204242
0	−2	4	0.000000226335859257	−0.000002349340050316
0	−1	−1	−0.013642653022761800	0.038438016660720100
0	−1	0	0.091379462623008200	−0.156485086285383000
0	−1	1	0.021491536209525800	0.145327483089982000
0	−1	2	−0.001353328886266740	−0.017615142377454900
0	−1	3	−0.000251111189846025	0.000459320278789596
0	−1	4	−0.000003820107622901	0.000006470987565661
0	0	0	−0.622204862458768000	1.025392693870120000
0	0	1	−0.215687591397413000	−1.064688806886120000
0	0	2	0.017691928953702200	0.146250829159046000
0	0	3	0.000598056082359742	−0.006237736910352090
0	0	4	0.000011044852059717	0.000003879702464999
0	1	1	−0.695325127031950000	1.126664279250840000
0	1	2	0.085106188241167300	−0.157297244850630000
0	1	3	−0.011556970215203400	0.006639706173073500
0	1	4	−0.000129177828348943	−0.000052244291833089
0	2	2	−0.010628464332917000	0.024359935110589500
0	2	3	0.001534537077199190	−0.002235366049964460
0	2	4	0.000016829230597190	−0.000021404883348893
0	3	3	−0.000288383540792243	0.000891425848770047
0	3	4	−0.000003524141167631	−0.000000930640639683
0	4	4	−0.000000067864686229	0.000000179196328236
1	−6	−6	−0.000000000000000114	0.000000000000000085
1	−6	−5	−0.000000000000000907	−0.000000000000000883
1	−6	−4	−0.000000000000018315	−0.000000000000012484
1	−6	−3	0.000000000000953008	0.000000000000062014
1	−6	−2	−0.000000000016125999	−0.000000000001386122
1	−6	−1	0.000000000062878189	0.000000000016230311
1	−6	0	−0.000000000422364157	−0.000000000009330509
1	−6	1	0.000000000022365125	−0.000000000006158982
1	−6	2	−0.000000000007816875	−0.000000000000051871

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
1	−6	3	−0.000000000000069827	0.000000000000643261
1	−6	4	−0.000000000000000255	0.000000000000006039
1	−5	−5	0.000000000001504278	0.000000000003337715
1	−5	−4	0.000000000040817801	0.000000000136206403
1	−5	−3	−0.000000001967117391	−0.000000002529183753
1	−5	−2	0.000000030067794506	0.000000050348301662
1	−5	−1	−0.000000128160568993	−0.000000141656731277
1	−5	0	0.000000841615311036	0.000000619430310915
1	−5	1	0.000000024688761418	−0.000000593332060332
1	−5	2	0.000000006428329405	0.000000060745389305
1	−5	3	−0.000000000296694415	0.000000007143370984
1	−5	4	0.000000000000162540	−0.000000000288941222
1	−4	−4	0.000000001757443418	0.000000020614980187
1	−4	−3	−0.000000058684084397	−0.000000284708801322
1	−4	−2	0.000000900633580235	0.000004568684813580
1	−4	−1	−0.000003494236067257	−0.000007052401378205
1	−4	0	0.000021283185862096	0.000011916401102779
1	−4	1	−0.000004440781284105	−0.000008079714742511
1	−4	2	0.000000659357234789	−0.000004645371749570
1	−4	3	−0.000000005983216765	0.000003571764235310
1	−4	4	0.000000000219037714	−0.000000015404655013
1	−3	−3	0.000003334669415234	0.000006651219403652
1	−3	−2	−0.000044870229022753	−0.000091879265258768
1	−3	−1	0.000220484777434063	0.000404387763410176
1	−3	0	−0.001476401228798630	−0.001970067661954120
1	−3	1	−0.000378699650660606	0.001789417647540150
1	−3	2	0.000028722273529119	−0.000170192004108606
1	−3	3	−0.000000377318282874	0.000030175517428942
1	−3	4	−0.000000003382271106	0.000001794021450849
1	−2	−2	0.000648716443752163	0.001368769260541370
1	−2	−1	−0.002938550145521220	−0.004993599451437050
1	−2	0	0.019299876525227600	0.022489569347420900
1	−2	1	0.001435356836152440	−0.020538374775093300
1	−2	2	0.000049564826908565	0.001855652948225720
1	−2	3	−0.000024166741453089	−0.000079283744898396
1	−2	4	−0.000000172363681952	−0.000015473351113573
1	−1	−1	0.015026327214997800	0.038517480170287800
1	−1	0	−0.100861882091266000	−0.213157445186717000
1	−1	1	−0.021971217823112900	0.194160649199176000
1	−1	2	0.001322037439149490	−0.023052661994965500
1	−1	3	0.000195701994787672	0.007866899956941040
1	−1	4	0.000002932051627075	0.000261483584092412
1	0	0	0.686561423454117000	1.225044721602250000
1	0	1	0.211068926480356000	−1.127188528919980000

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
1	0	2	−0.016470108327300500	0.147129331787108000
1	0	3	−0.000422474596941267	−0.051061673747300300
1	0	4	−0.000008712302872626	−0.001298443030838730
1	1	1	0.621822783048911000	1.049244629039260000
1	1	2	−0.075741332758791900	−0.146480790047057000
1	1	3	0.009700304084902640	0.047923832519760400
1	1	4	0.000097328838187830	0.001097838380224580
1	2	2	0.009436962750623020	0.028598602234395400
1	2	3	−0.001292354908452150	−0.007757573870719450
1	2	4	−0.000012748308527165	−0.000117784430191791
1	3	3	0.000238613930858808	0.003007091453430960
1	3	4	0.000002672166999172	0.000066953007354530
1	4	4	0.000000051520671179	0.000003647512626322
2	−6	−6	0.000000000000000150	0.000000000000000166
2	−6	−5	0.000000000000000099	−0.000000000000000113
2	−6	−4	0.000000000000002443	0.000000000000003427
2	−6	−3	−0.000000000000143354	0.000000000000014722
2	−6	−2	0.000000000002415831	−0.000000000002632485
2	−6	−1	−0.000000000009452169	0.000000000001307350
2	−6	0	0.000000000063367620	−0.000000000036154113
2	−6	1	−0.000000000003185978	0.000000000044460421
2	−6	2	0.000000000001147336	−0.000000000006634687
2	−6	3	0.000000000000009567	−0.000000000000364433
2	−6	4	0.000000000000000232	−0.000000000000000423
2	−5	−5	−0.000000000000223682	−0.000000000000143332
2	−5	−4	−0.000000000005995404	−0.000000000011716514
2	−5	−3	0.000000000281412912	0.000000000196876407
2	−5	−2	−0.000000004377579012	−0.000000003056888740
2	−5	−1	0.000000018269746503	0.000000009225829089
2	−5	0	−0.000000119591968382	−0.000000034127644995
2	−5	1	−0.000000000577700822	0.000000030450274315
2	−5	2	−0.000000001234121626	−0.000000002061358458
2	−5	3	0.000000000017183421	−0.000000000651302134
2	−5	4	−0.000000000000040761	0.000000000036074137
2	−4	−4	−0.000000000257855489	−0.000000001572676464
2	−4	−3	0.000000007759072633	0.000000021104268848
2	−4	−2	−0.000000126037086966	−0.000000385611331534
2	−4	−1	0.000000451925020617	0.000000187284214085
2	−4	0	−0.000002667509317628	0.000000338566675319
2	−4	1	0.000001094325822973	−0.000000334327835557
2	−4	2	−0.000000144003147164	0.000000571301385005
2	−4	3	0.000000002109102081	−0.000000398332804613
2	−4	4	−0.000000000011604002	0.000000001599817595
2	−3	−3	−0.000000438635890906	−0.000000575555179711

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
2	−3	−2	0.000006070452399564	0.000007725030451928
2	−3	−1	−0.000028974883631680	−0.000038743033037349
2	−3	0	0.000194399542202048	0.000198807145344274
2	−3	1	0.000049394154125883	−0.000180496494453374
2	−3	2	−0.000003827116258243	0.000017890079082343
2	−3	3	0.000000195668917342	−0.000004394839167807
2	−3	4	−0.000000000050641217	−0.000000233634197267
2	−2	−2	−0.000090404385775245	−0.000101164027768215
2	−2	−1	0.000394242536773058	0.000434093565112245
2	−2	0	−0.002586488460754370	−0.001882450484626370
2	−2	1	−0.000148301023789967	0.001661862665574580
2	−2	2	−0.000010725643268963	−0.000140843886383219
2	−2	3	0.000001015580687350	0.000019175037434147
2	−2	4	−0.000000002905642477	0.000001990771052912
2	−1	−1	−0.001950102603427280	−0.004227417313234310
2	−1	0	0.013247951815681100	0.024375066315650600
2	−1	1	0.003826565859334050	−0.022155627775967000
2	−1	2	−0.000295974722760960	0.002660958944479350
2	−1	3	0.000006788655946279	−0.001014040275574970
2	−1	4	0.000000050390288502	−0.000034486938630054
2	0	0	−0.091170725217452700	−0.139298499001726000
2	0	1	−0.032969315201099700	0.126463611440291000
2	0	2	0.002830894154548120	−0.016473114935154800
2	0	3	−0.000096197631030875	0.006443900964207050
2	0	4	−0.000000084667262118	0.000172374151181341
2	1	1	−0.071791372418013600	−0.115824824649865000
2	1	2	0.008663612249959720	0.016189433121281000
2	1	3	−0.000790476998519630	−0.006007408332548970
2	1	4	0.000001564598599773	−0.000146246137298849
2	2	2	−0.001072600644566900	−0.003213731393093160
2	2	3	0.000108195664937536	0.000942808538478769
2	2	4	−0.000000127832144936	0.000016030297995841
2	3	3	−0.000015420127729737	−0.000370691176993228
2	3	4	0.000000036411736533	−0.000008950931310045
2	4	4	0.000000001051765794	−0.000000479214685452
3	−6	−6	−0.000000000000000135	−0.000000000000000058
3	−6	−5	−0.000000000000000020	−0.000000000000000063
3	−6	−4	−0.000000000000000115	−0.000000000000000086
3	−6	−3	−0.000000000000005652	−0.000000000000000117
3	−6	−2	0.000000000000101491	0.000000000000036813
3	−6	−1	−0.000000000000386211	−0.000000000000103980
3	−6	0	0.000000000002616633	0.000000000000440641
3	−6	1	−0.000000000000185698	−0.000000000000449888
3	−6	2	0.000000000000055739	0.000000000000077485

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
3	−6	3	0.000000000000000465	−0.000000000000000687
3	−6	4	0.000000000000000055	0.000000000000000098
3	−5	−5	−0.000000000000008787	−0.000000000000019156
3	−5	−4	−0.000000000000206589	−0.000000000000957229
3	−5	−3	0.000000000012663444	0.000000000021005668
3	−5	−2	−0.000000000183212412	−0.000000000372771123
3	−5	−1	0.000000000856285373	0.000000001090880490
3	−5	0	−0.000000005777979835	−0.000000005600530001
3	−5	1	−0.000000000727688681	0.000000005470450912
3	−5	2	0.000000000010406883	−0.000000000576389259
3	−5	3	0.000000000009237232	−0.000000000027030857
3	−5	4	0.000000000000006287	−0.000000000004638950
3	−4	−4	0.000000000004634622	−0.000000000110787279
3	−4	−3	0.000000000672925665	0.000000003159504678
3	−4	−2	−0.000000006116120113	−0.000000042525904509
3	−4	−1	0.000000053293385318	0.000000193759440213
3	−4	0	−0.000000403512383495	−0.000001058965767894
3	−4	1	−0.000000329187643319	0.000000991421096792
3	−4	2	0.000000032996262477	−0.000000106241389563
3	−4	3	−0.000000000091014132	0.000000019452828129
3	−4	4	0.000000000014845276	0.000000000051936646
3	−3	−3	−0.000000020323501680	−0.000000036972434452
3	−3	−2	0.000000327900356987	0.000000676048806595
3	−3	−1	−0.000001272465801470	0.000000642341232521
3	−3	0	0.000007194659002908	−0.000005649923786521
3	−3	1	−0.000008455672441132	0.000003900429030867
3	−3	2	0.000001157077181243	−0.000000542796580782
3	−3	3	−0.000000153965733234	0.000000967620748062
3	−3	4	−0.000000000158239367	0.000000040072472232
3	−2	−2	−0.000004411382348243	−0.000011066002917409
3	−2	−1	0.000021022636028492	0.000016222982218145
3	−2	0	−0.000141363781419171	−0.000058671367402833
3	−2	1	−0.000038292249218413	0.000064109857983711
3	−2	2	0.000003049724243215	−0.000003787123594462
3	−2	3	−0.000000223480469387	−0.000007124028659815
3	−2	4	−0.000000009747296867	−0.000000317467793033
3	−1	−1	−0.000062404598364344	0.000427004923651088
3	−1	0	0.000344437691086184	−0.002461042674065830
3	−1	1	−0.000276854267031471	0.002066528268096090
3	−1	2	0.000034719965893078	−0.000213364330676967
3	−1	3	0.000001013907697166	0.000157542840877920
3	−1	4	0.000000164539868335	0.000006270798375128
3	0	0	−0.001422209211621990	0.014523345382304300

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
3	0	1	0.005244801061164100	−0.012405556192984200
3	0	2	−0.000649014687505744	0.001326214083640570
3	0	3	0.000048138200474946	−0.000885310424893841
3	0	4	−0.000000511132901176	−0.000032264316509892
3	1	1	0.032468184989704000	0.010770687937131700
3	1	2	−0.004113350887497220	−0.001312302496833520
3	1	3	0.000603514779042885	0.000784269530559433
3	1	4	0.000005371531754377	0.000027365775442937
3	2	2	0.000523587519218677	0.000305363902049934
3	2	3	−0.000078667734315860	−0.000098616910758433
3	2	4	−0.000000711187580487	−0.000002857509612122
3	3	3	0.000013424710836408	0.000046573204263387
3	3	4	0.000000148632348840	0.000001678742058197
3	4	4	0.000000002807014848	0.000000083858268458
4	−6	−6	−0.000000000000000008	0.000000000000000227
4	−6	−5	−0.000000000000000134	−0.000000000000000114
4	−6	−4	0.000000000000000065	0.000000000000000218
4	−6	−3	0.000000000000000196	0.000000000000000159
4	−6	−2	0.000000000000000250	−0.000000000000000063
4	−6	−1	0.000000000000000176	0.000000000000000310
4	−6	0	−0.000000000000000546	−0.000000000000001030
4	−6	1	−0.000000000000000510	0.000000000000000695
4	−6	2	0.000000000000000037	0.000000000000000140
4	−6	3	−0.000000000000000345	0.000000000000000047
4	−6	4	−0.000000000000000195	0.000000000000000043
4	−5	−5	−0.000000000000000076	0.000000000000000041
4	−5	−4	−0.000000000000000371	−0.000000000000000206
4	−5	−3	−0.000000000000009052	−0.000000000000044405
4	−5	−2	0.000000000000028740	0.000000000000459630
4	−5	−1	−0.000000000000624969	−0.000000000005190090
4	−5	0	0.000000000005582205	0.000000000030749546
4	−5	1	0.000000000015503069	−0.000000000028158171
4	−5	2	−0.000000000001905014	0.000000000003496200
4	−5	3	0.000000000000217252	−0.000000000001290655
4	−5	4	−0.000000000000000232	−0.000000000000022033
4	−4	−4	−0.000000000000063544	−0.000000000000155079
4	−4	−3	−0.000000000001081618	−0.000000000002314663
4	−4	−2	−0.000000000002991837	0.000000000024892424
4	−4	−1	−0.000000000084460745	−0.000000000245246970
4	−4	0	0.000000000921323564	0.000000002237269594
4	−4	1	0.000000002837933615	−0.000000002447477751
4	−4	2	−0.000000000340312780	0.000000000528124025
4	−4	3	0.000000000037113785	−0.000000000099586517
4	−4	4	0.000000000000270953	0.000000000004494692

Table B.3 (Continued)

l_1	l_2	l_3	$\Gamma^{0,1,0,0}_{l_1,l_2,l_3}$	$\Gamma^{0,0,1,1}_{l_1,l_2,l_3}$
4	−3	−3	0.000000000036986364	0.000000000294264027
4	−3	−2	−0.000000000053600333	−0.000000002549970000
4	−3	−1	−0.000000000762635888	0.000000040997406945
4	−3	0	−0.000000004642160396	−0.000000209781511212
4	−3	1	−0.000000138613320777	0.000000167360792378
4	−3	2	0.000000018569460631	−0.000000005662020518
4	−3	3	−0.000000003363761079	0.000000008885773830
4	−3	4	−0.000000000030156882	0.000000000457624028
4	−2	−2	0.000000002520502348	0.000000024178700649
4	−2	−1	−0.000000035678699374	−0.000000325899202597
4	−2	0	0.000000138900329397	0.000001701555998387
4	−2	1	−0.000000921350343439	−0.000001409362186892
4	−2	2	0.000000123484633695	0.000000085996895871
4	−2	3	−0.000000026494300340	−0.000000070805878714
4	−2	4	−0.000000000476758302	−0.000000003139709050
4	−1	−1	0.000000314232911302	0.000006153363380819
4	−1	0	−0.000002271893328463	−0.000030978468929552
4	−1	1	0.000014870875467031	0.000024111362712364
4	−1	2	−0.000001907836824072	−0.000000432167736363
4	−1	3	0.000000363311622449	0.000001354224368949
4	−1	4	0.000000006887270948	0.000000076838434177
4	0	0	−0.000002174108415279	0.000152246996545518
4	0	1	−0.000012512380146030	−0.000116179653829710
4	0	2	0.000001227682650994	0.000000346923673145
4	0	3	−0.000000155068260106	−0.000006524790370101
4	0	4	−0.000000011934036711	−0.000000405049584434
4	1	1	0.000941181580556842	0.000089281824128280
4	1	2	−0.000124542630945891	−0.000001428564611848
4	1	3	0.000023601752338335	0.000005129500040787
4	1	4	0.000000320622410481	0.000000330008575344
4	2	2	0.000016474191246362	0.000001688413236393
4	2	3	−0.000003106675849826	−0.000000235577399882
4	2	4	−0.000000041916654129	−0.000000019893658262
4	3	3	0.000000584404578602	0.000000318837660106
4	3	4	0.000000008247011284	0.000000019826682577
4	4	4	0.000000000140842452	0.000000000947163263

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- Invited Lecturer in Discrete Structures, Automata and Formal Languages, School of Computer Engineering, Suranaree University of Technology.
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