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นายอาทิตย์ อินทรสิทธิ์

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# OPTION PRICING MODEL FOR A FRACTIONAL STOCHASTIC VOLATILITY WITH JUMPS

**Arthit Intarasit**

**A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics**

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# OPTION PRICING MODEL FOR A FRACTIONAL STOCHASTIC VOLATILITY WITH JUMPS

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Thesis Examining Committee

Prapasri Asawakun

(Assoc. Prof. Dr. Prapasri Asawakun)

Chairperson

P. Sattayatham

(Prof. Dr. Pairote Sattayatham)

Member (Thesis Advisor)

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ในวิทยานิพนธ์ฉบับนี้ได้เสนอแบบจำลองการประเมินราคาออปชันสำหรับความผันผวนสโตแคสติกเศษส่วนอย่างกระโดดอีกชนิดหนึ่ง ซึ่งมูลค่าของหุ้นเคลื่อนที่แบบบราวเนียนเรขาคณิตพร้อมด้วยกระบวนการปัวซงเชิงประกอบและความผันผวนสโตแคสติกที่ถูกรบกวนด้วยตัวก่อกวนเศษส่วน ลักษณะความจำระยะยาวในแบบจำลองความผันผวนสโตแคสติกดังกล่าวนี้ไม่มีในแบบจำลองความผันผวนสโตแคสติกแบบดั้งเดิม โดยผลหลักมูลของการประมาณตัวก่อกวนเศษส่วนในปริภูมิ $L^2$ ได้พิสูจน์ทฤษฎีบทเกี่ยวกับการลู่เข้าของผลเฉลยโดยประมาณ ได้มีการคำนวณสูตรออปชันแบบยุโรปโดยใช้เทคนิคบนพื้นฐานเมื่อรู้สูตรอย่างชัดแจ้งของฟังก์ชันแคแรกเทอริสติก ตัวอย่างการจำลองแสดงการลดลงของค่าคลาดเคลื่อนของวิถีตัวอย่างหนึ่งของแบบจำลองความผันผวนสโตแคสติกเศษส่วนอย่างกระโดดเปรียบเทียบกับแบบจำลองความผันผวนสโตแคสติกแบบดั้งเดิม

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา อาทิตย์ อินทรสิทธิ์

ปีการศึกษา 2553 ลายมือชื่ออาจารย์ที่ปรึกษา

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FRACTIONAL STOCAHASTIC VOLATILITY/ JUMP DIFFUSION MODEL/ FRACTIONAL BROWNIAN MOTION/ APPROXIMATE APPROACH

An alternative fractional stochastic volatility model with jumps is proposed in this thesis in which the stock prices follow a geometric Brownian motion combining compound Poisson processes and the stochastic volatility perturbed by a fractional noise. The proposed model exhibits a long term memory of a stochastic volatility model that is not expressed in the classical stochastic volatility model. Using a fundamental result on the $L^2$-approximation of a fractional noise, a convergence theorem is proved concerning an approximate solution. The formula of the European option is calculated by using the technique based on the characteristic function of an underlying asset which can be expressed in an explicit formula. A simulation example shows a reduction of error of a sample path in a fractional stochastic volatility model with jumps as compared to the classical stochastic volatility model.

School of Mathematics Student's Signature A. Intarasit

Academic Year 2010 Advisor's Signature P. Sattayatham

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Arthit Intarasit

# CONTENTS

# CONTENTS (Continued)

# CONTENTS (Continued)

# LIST OF FIGURES

**Figure** **Page**

# LIST OF TABLES

# CHAPTER I

# PRELIMINARIES ON FRACTIONAL STOCHASTIC VOLATILITY MODEL

## 1.1 Introduction to Option Pricing Problem

A *derivative security* (also known as a *contingent claim*) is a financial contract whose value at expiration time $T$ is precisely determined by the price of an underlying asset at time $T$. Options and futures are examples. However, options have become one of the most important and frequently traded derivative in mathematical finance.

An *option* is derivative security that gives its holder the right, *but not the obligation*, to buy or sell a certain amount of a financial asset, by a certain date, for a certain strike price. For example, a stock option is a derivative security whose value depends on the value of the underlying stock.

The writer of the option needs to specify:

- *the type of option*: the option to buy is called a *call* while the option to sell is a *put*;
- *the underlying asset*: typically, it can be a stock, a bond, a currency and so on;
- *the amount of an underlying asset* to be purchased or sold;
- *the expiration date*: if the option can be exercised at any time before maturity, it is called an *American* option but, if it can only be exercised at maturity, it is called a *European* option;
- *the exercise price*, the price at which the transaction is done if the option is exercised.

The price of the option is the *premium*. When the option is traded on an organized market, the premium is quoted by the market. Otherwise, the problem is to price the option. Also, even if the option is traded on an organized market, it can be interesting to detect some possible abnormalities in the market.

Note that there are two major categories of style: European-style and American-style. Whichever type one chooses depends on the way one wants to exercise them, and each has advantages one should know about to make wise investing decisions. A European call (or put) option allows the holder to exercise the option (i.e., to buy (or sell)) only on the option expiration date. An American call (or put) option allows exercise at any time during the life of the option.

Let us examine the case of a European call option on a stock, whose price at time $t$ is denoted by $S_t$. Let us call $T$ the expiration date and $K$ the exercise price. Obviously, if $K$ is greater that $S_T$, the holder of the option has no interest whatsoever in exercising the option. But, if $S_T > K$, the holder makes a profit of $S_T - K$ by exercising the option, i.e. buying the stock for $K$ and selling it back on the market at $S_T$. Therefore, the value of the call at maturity is given by

$$(S_T - K)_+ = \max(S_T - K, 0).$$

If the option is exercised, the writer must be able to deliver a stock at price $K$. It means that he or she must generate an amount $(S_T - K)_+$ at maturity. At the time of writing the option, which will be considered as the origin of time, $S_T$ is unknown and therefore two questions have to be asked:

1. How do we model the underling asset specific on a stock price?
2. How much should the buyer pay for the option? In other words, how should we price at time $t = 0$ an asset worth $(S_T - K)_+$ at time $T$? That is the problem of *pricing* the option.

### 1.1.1 The Behavior of Asset Prices

The model suggested by Black and Scholes (1973) to describe the behaviour of prices is a continuous-time model with one risky asset (a share with price $S_t$ at time $t$) and a riskless asset (with price $\widetilde{S}_t$ at time $t$). We suppose the behaviour of $\widetilde{S}_t$ to be encapsulated by the following (ordinary) differential equation:

$$d\widetilde{S}_t = r\widetilde{S}_t dt$$

where $r > 0$ is an instantaneous interest rate. We set $\widetilde{S}_0 = 1$, so that $\widetilde{S}_t = e^{rt}$ for $t \geq 0$.

We assume that the behavior of the stock price is determined by the following stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{1.1}$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion (Bm), $T$ is the maturity of the option, $\mu \in \Re$ is the instantaneous expected total return of the stock (possibly adjusted by a dividend yield), and $\sigma > 0$ is the instantaneous standard deviation of stock price returns, called the *volatility* in financial markets. This equation is known as *Black-Scholes model* or *diffusion model.*

By using Itô's lemma (for more details see Section 1.3), equation (1.1) implies that

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \tag{1.2}$$

called the *geometric Brownian motion* (gBm). In particular,

$$\log S_t \sim \mathcal{N}\big(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\big),$$

i.e., the process $\log S_t$ is distributed normally with mean $\mu$ and variance $\sigma^2$.

Note that the stock return $\mu$ could easily become time dependent without changing any of our arguments.

### 1.1.2 Pricing Option in the Black-Scholes Model

In 1973 Black and Scholes tackled the problem of pricing and hedging a European option (call or put) on a non-dividend paying stock. In this section, we briefly explain the main results. Firstly, we make the following assumptions.

***Assumption A:***

*(i) We have frictionless markets with continuous trading.*

*(ii) There are no transaction costs or taxes and no dividends during the life of the option.*

*(iii) No arbitrage opportunity.*

*(iv) The risk-free interest rate is deterministic and equal to* $r \geq 0$.

*(vi) Under the real-world or physical probability measure* $\mathbb{P}$ *the stock price process* $(S_t)$ *follows the diffusion model of equation (1.1).*

We note that an arbitrage opportunity is the opportunity to buy an asset at a low price then immediately selling it on a different market for a higher price. Less rigorously, an arbitrage opportunity is a "free lunch", that allows investors to make a gain for no risk.

Suppose that Assumption A holds. Standard derivative pricing theory offers two ways for computing the fair value $C(t, S_t)$ of a European call option at time $t \leq T$. Under the partial differential equation (PDE) approach the function $C(t, s)$ is computed by solving the PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} + rs\frac{\partial C}{\partial s} - rC = 0, \text{ for } t \in [0, T]. \tag{1.3}$$

This equation is the famous *Black-Scholes PDE* of European call option.

In order to obtain a unique solution for the Black-Scholes PDE we must consider final and boundary conditions. We will restrict our attention to a European call option, $C(t, s)$.

At maturity, $t = T$, a call option is worth:

$$C(T, s) = (s_T - K, 0)$$

where $K$ is the exercise price. So this will be the final condition.

The asset price boundary conditions are applied at $s = 0$ and as $s \to \infty$.

If $s = 0$ then $ds$ is also zero and therefore $s$ can never change. This implies on $s = 0$ we have:

$$C(t, 0) = 0.$$

Obviously, if the asset price increases without bound $s \to \infty$, then the option will be exercised indifferently how big the exercise price is. Thus as $s \to \infty$ the value of the option becomes that of the asset:

$$C(t, s) \approx s, \quad s \to \infty.$$

We have the following final-boundary value problem:

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} + rs\frac{\partial C}{\partial s} - rC = 0, \text{ for } t \in [0, T], \\ C(t, 0) = 0; \quad C(t, s) \approx s \text{ as } s \to \infty, \\ C(T, s) = \max(s_T - K, 0). \end{cases} \tag{1.4}$$

Alternatively, the value $C(t, S_t)$ can be computed as the expectation of the discounted pay-off under the risk-neutral measure $\mathbb{Q}$ (the so-called *risk-neutral pricing approach*). Under $\mathbb{Q}$, the process $(S_t)$ satisfies the stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

for a standard $\mathbb{Q}$-Brownian motion $\widetilde{W}$; in particular, the drift $\mu$ in equation (1.2) has been replaced by risk-free interest rate $r$.

The risk-neutral pricing rule now states that

$$C(t, S_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\max(S_T - K)|\mathcal{F}_t\right] \tag{1.5}$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes expectation with respect to $\mathbb{Q}$.

In order to take expectation of equation (1.5) under the risk neutral measure $\mathbb{Q}$, we need to change the physical probability $\mathbb{P}$ to the risk neutral measure $\mathbb{Q}$ in the stochastic differential equation, using Girsanov's Thorem. Girsanov's Theorem tells us how a stochastic differential equation (SDE) changes as the physical probability $\mathbb{P}$ changes. Essentially, Girsanov's Theorem tells us that change in $\mathbb{P}$ corresponds to a change in drift $\mu$ and the rest of the SDE remains unchanged.

The solution of the PDE (1.4), or the risk-neutral value of stock price obtained from (1.5), is simply given by the Black-Scholes price $C^{\mathrm{BS}}$ of a European call option. This yields

$$C^{\mathrm{BS}}(t, S_t; r, \sigma, T, K) := S_t\Phi(d_{t,1}) - Ke^{-r(T-t)}\Phi(d_{t,2}) \tag{1.6}$$

where

$$d_{t,1} = \frac{\ln S_t - \ln K + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_{t,2} = d_{t,1} - \sigma\sqrt{T-t}.$$

and $\Phi$ is the *cumulative distribution function* for the standard normal distribution. This equation know as *Black-Scholes formula* for a European call option.

Similarly, the price for a European put option is:

$$P(t, S_t) = -S_t\Phi(-d_{t,1}) + Ke^{-r(T-t)}\Phi(-d_{t,2}).$$

### 1.1.3 Implied Volatility

It is possible to deduce the *implied volatility* of call (or put) options by solving the reverse Black-Scholes equation, that is, find the volatility that would equal the Black-Scholes price to the market price of the option. This is a good way to see how derivatives markets *perceive* the underlying volatility.

More precisely, using Black-Scholes option pricing, call options $C$ are a function of $C(t, S; r, \sigma, T, K)$ where $t$ is the time at which $C$ is being priced, $T$ is the expiration date, $r$ is the risk free rate of return, and $K$ is the strike price. Note that all the independent variables are observable except $\sigma$. Since the quoted option price $C^{obs}$ is observable, using the Black-Scholes formula we can therefore calculate or imply the volatility that is consistent with the quoted options prices and observed variables. We can therefore define implied volatility $\mathcal{I}$ by:

$$C_{BS}(t, S; r, \mathcal{I}, T, K) = C^{obs},$$

where $C_{BS}$ is the option price calculated by the Black-Scholes equation (equation 1.6).

Implied volatility surfaces are graphs plotting $\mathcal{I}$ for each call options strike $K$ and expiration $T$. Theoretically options whose underlying asset is governed by gBm should have a flat implied volatility surface, since volatility is a constant; however in practice the implied volatility surface is not flat and $\mathcal{I}$ varies with $K$ and $T$.

Implied volatility plotted against strike prices from empirical data tends to vary in a “u-shaped” relationship, known as the *volatility smile*, with the lowest value normally at $S = K$ (called “at the money” options). The opposite graph shape to a volatility smile is known as a volatility frown due to its shape. The smile curve has become a prominent feature since the 1987 October crash (see for instance Bates (2000)).

### 1.1.4 An Extension of Black-Scholes Model

Since Black-Scholes formula models stock prices by using the geometric Brownian motion, thus there are various shortcomings of this model, such as

(i) the asymmetric leptokurtic features (also called *leptokurticity*), that is, the return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution;

(ii) the volatility smile, that is, the implied volatility is not a constant as assumed in the Black-Scholes model, and

(iii) the large random fluctuations such as crashes and rallies.

Therefore, many financial engineering studies have been undertaken to modify and improve the Black-Scholes formula to explain some or all of the above three empirical phenomena. The supporting details will be discussed in later of the thesis.

We note that that "tail" of the distribution are where the extreme values occur. Empirical distributions for stock prices and returns have found that the extreme values are more likely than would be predicted by the normal distribution. This means that, between periods where the market exhibits relatively modest changes in prices and returns, there will be periods where there are changes that, are much higher (i.e., crashes and booms) than predicted by the normal distribution. This is not only of concern to financial theorists, but also to practitioners. However, heavy or fat tails can help explain larger price fluctuations for stocks over short time periods than can be explained by changes in fundamental economic variable.

### 1.1.5 Jump-Diffusions

In addition to the volatility smile observable from the implied volatilities of the options, there is evidence that the assumption of a pure normal distribution (also called pure *diffusion*) for the stock return is not accurate. Indeed "fat tails" have been observed away from the mean of the stock return.

Some authors try to explain the volatility smile and the leptokurticity by changing the underlying stock distribution from a diffusion process to a jump-diffusion process. For *jump-diffusion* models, the "normal" evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. Here the jumps represent rare events – crashes and large drawdowns. Such an evolution can be represented by modeling the (log-)price as a Lévy process with a nonzero Gaussian component and a jump part, which is a compound Poisson process with finitely many jumps in every time interval. Merton (1976) was first to actually introduce jumps in the stock distribution. Recently, Kou (2002) proposed double exponential jump-diffusion models by using the same idea to explain both the existence of fat tails and the volatility smile. In subsection 1.2.5, we will see a model combining compound Poisson jumps and stochastic volatility: the Bate model (1996).

The Merton jump-diffusion model with Gaussian jumps (known as *an exponential Lévy model*) introduced by Merton (1976) is given by

$$S_t = S_0 \exp\left(\mu t + \sigma W_t + \sum_{n=1}^{N_t} Y_n\right)$$

where $(N_t)_{t\in[0,T]}$ is a Poisson process with intensity $\lambda$, and independent jumps $Y_n \sim \mathcal{N}(m, \delta^2)$. The Poisson process and the jumps are assumed to be independent of the Brownian Motion. The use of the Poisson process is economically motivated by two assumptions: the numbers of crashes in non overlapping time intervals should be independent and the occurrence of one crash should be roughly proportional to the length of the time interval.

In analogy to the Black-Scholes model, the parameter $\mu$ stands in the Merton model for the expected stock return and $\sigma$ is the volatility of regular shocks to the stock return. The jump component can be interpreted as a model for crashes. The parameter $\lambda$ is the expected number of crashes per year and $m$ and $\delta^2$ determine the distribution of a single jump.

The Merton model allows modelling of jumps and study of leptokurtic distributions. However in 2007, Sattayatham at el., extended the Merton model by introducing a *fractional Black-Scholes model with jumps*, in which the stock price has long-memory property.

The rest of this chapter is organized as follows. The literature review of stochastic volatility is presented in Section 1.2. Section 1.3 gives us a guide to stochastic calculus. We review basic definitions of Brownian motion, Lévy process and compound Poisson process. All mathematical tools such as Itô integration and its extensions will also be reviewed. The study of Poisson processes which naturally lead to the notion of Poisson random measure are introduced in this section. The need to study fractional Brownian motion and some developments for the stochastic integral with respect to fractional Brownian motion are detailed in Section 1.4. An approximate approach to fractional Brownian motion and fractional stochastic integration is also introduced in this section.

## 1.2 A Stochastic Volatility Model

A concise literature review of a stochastic volatility model is presented in this section. The most popular extension of stochastic volatility model is noted in the sequel.

### 1.2.1 The Volatility Problem

Although the Black-Scholes formula is often quite successful in explaining stock option prices (Black and Scholes (1973) and Merton (1976)). It does have known biases (Rubinstein (1895)). Its performance also is substantially worse on foreign currency options (Melino and Turnbull (1990; 1991) and Knoch (1992)). This is not surprising, since the Black-Scholes model makes the strong assumption that (continuously compounded) stock returns are normally distributed with known mean and variance. Since the Black-Scholes formula does not depend on the mean spot return, it cannot be generalized by allowing the mean to vary. But the variance assumption is somewhat dubious. Motivated by this theoretical consideration, Scott (1987), Hull and White (1987), and Wiggins (1987) have generalized the model to allow stochastic volatility. Melino and Turnbull (1990; 1991) report that this approach is successful in explaining the prices of currency options.

In fact, there are several kinds of stochastic volatility models, but a very popular one used with option models is due to Heston (1993) and originally came from interest rate models of Cox, Ingersoll and Ross (1985), called the *CIR model*. More details will be discussed in subsection 1.2.3. For recently, however, the full stochastic volatility, jump-diffusion (SVJD) model for the option pricing problem is given in Yan-Hanson (2006; 2007) and in Hanson (2008) for the SVJD optimal portfolio and consumption problem.

A common approach to measure the volatility uses the standard deviation of the returns of the last $T$ trading days. This more heuristic approach is based on the assumption that volatility changes only slightly over a certain period of time and can be treated as (almost) constant. Using a statistical model for describing the underlyings offers a wide range of more sophisticated techniques for measuring the volatility including GARCH and stochastic volatility models which allow the volatility to vary over time in a specific way. Thus, these models facilitate approaches to measure the volatility more appropriately. Besides these techniques the implied volatility obtained from option prices is often considered as a sensible measure because it reflects the opinion of the market on future volatility.

To summalize: we have

- *Historic volatility* (also known as *realised volatility*) is a measure of volatility using past empirical stock price data.
- *Implied volatility* is the volatility associated with empirical option prices.

The details of implied volatility have already discussed in Section 1.1.3. The remaining terms are addressed as follows.

### 1.2.2 Historic Volatility

In theory volatility should not depend on the method of measurement. However, in practice this is not the case. Volatility $\sigma$ can be *empirically measured* by two methods: historic volatility and implied volatility.

Historic volatility is calculated from empirical (and therefore discrete) stock price data $S_{t_0}, \ldots, S_{t_n}$ where $\Delta t =: t_i - t_{i-1}$ denotes the chosen sampling interval. To estimate historic volatility $\widehat{\sigma}$ we calculate the standard deviation of an asset's continuously compounded return per unit time:

$$\widehat{\sigma} = \frac{\sqrt{V_R}}{\sqrt{\Delta t}},$$

where the sample variance $V_R = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \overline{R})$, $R_i = \ln(S_{t_i}/S_{t_{i-1}})$, and the sample mean $\overline{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$.

Note that there are many way to calculate historic volatility. Another formula of historic volatility will introduce in Chapter IV.

### 1.2.3 Stochastic Volatility Models

Stochastic volatility models are one approach to resolve the shortcomings of the Black-Scholes model. In particular, the Black-Scholes model (1.1) assumes that the volatility is constant over a given time interval and unaffected by the changes in the stock price.

Several different stochastic processes have been suggested for the volatility. A popular one is the *Ornstein-Uhlenbeck* (OU) process:

$$d\sigma_t = -\alpha\sigma_t dt + \beta d\overline{W}_t$$

where $\alpha$, $\beta$ are two parameters and $\overline{W}_t$ is another standard Brownian motion, remembering the stock equation follows equation (1.1).

Note that there is a (usually negative) correlation $\rho$ between $dW_t$ and $d\overline{W}_t$, which can in turn be time or level dependent.

Heston (1993) and Stein (1991) were among those who suggested the use of this process. Using Itô's lemma, we can see that the stock-return variance $v_t = \sigma_t^2$ satisfies a *square-root* or Cox-Ingersoll-Ross (CIR) process

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}\overline{W}_t \tag{1.7}$$

with $\omega = \beta^2$, $\theta = 2\alpha$, and $\xi = 2\beta$.

Note that the OU process has a closed-form solution

$$\sigma_t = \sigma_0 e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} d\overline{W}_s$$

which means that $\sigma_t$ follows in law $\mathcal{N}\big(\sigma_0 e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1-e^{-2\alpha t})\big)$, with $\mathcal{N}$ is the normal distribution.

Heston and Nandi (1997) show that this process corresponds to a special case of the general auto regressive conditional heteroskedasticity (GARCH) model, which we will discuss next subsection. Another popular process for estimating stochastic volatility is the GARCH(1,1) process, where we would have

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t. \tag{1.8}$$

In general, the stochastic volatility models generalize to

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t,$$

$$dv_t = \alpha(S_t, t)dt + \beta(S_t, t)d\overline{W}_t,$$

where $\alpha(S_t, t)$ and $\beta(S_t, t)$ are some functions of $v_t$ while $d\overline{W}_t$ is another standard Brownian motion that is correlated with $dW_t$ with constant correlation fator $\rho$.

We note that a GARCH model of (1.8) assumes that the randomness of the variance process varies with the variance, as opposed to the square root of the variance as in the Heston model (1.7).

### 1.2.4 GARCH and Diffusion Limits

The most elementary GARCH process, called GARCH(1,1), was developed originally in the field of econometrics by Engle (2004) and Bollerslev (1986) in a discrete framework.

Firstly, consider the discrete equivalent of gBm (1.2) is

$$\ln S_{t+\Delta t} = \ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}W_t \tag{1.9}$$

where $(W_t)$ is a sequence of independent normal random variable with zero mean and variance of 1. The stock discrete equation (1.9) could be rewritten by taking $\Delta t = 1$ and $v_n = \sigma_n^2$ as

$$\ln S_{n+1} = \ln S_n + \left(\mu - \frac{1}{2}v_{n+1}\right) + \sqrt{v_{n+1}}W_{n+1}$$

calling the mean adjusted return

$$u_n = \ln\left(\frac{S_n}{S_{n-1}}\right) - \left(\mu - \frac{1}{2}v_n\right) = \sqrt{v_n}W_n$$

the variance process in GARCH(1,1) is supposed to be

$$v_{n+1} = \omega_0 + \beta v_n + \alpha u_n^2 = \omega_0 + \beta v_n + \alpha v_n W_n^2 \tag{1.10}$$

where $\alpha$ and $\beta$ are weight parameters and $\omega_0$ is a parameter related to the long-term variance.

Nelson (1990) shows that as the time interval length decreases and becomes infinitesimal, Equation (1.10) becomes precisely the previously cited equation (1.8). To be more accurate, there is a *weak convergence* of the discrete GARCH process to the continuous diffusion limit. For an explanation on weak convergence, see, for example, Varadham (2000). For a GARCH(1,1) continuous diffusion, the correlation between $dW_t$ and $d\overline{W}_t$ is zero.

Note that the discrete GARCH version of the square-root process (1.7) is

$$v_{n+1} = \omega_0 + \beta v_n + \alpha(W_n - c\sqrt{v_n})^2 \tag{1.11}$$

as Heston and Nandi show in (1997).

Also, note that having a diffusion process

$$dv_t = b(v_t)dt + a(v_t)d\overline{W}_t$$

we can apply an Euler approximation to discretize and obtain a Monte Carlo process, such as

$$v_{n+1} - v_n = b(v_n)\Delta t + a(v_n)\sqrt{\Delta t}\overline{W}_n.$$

It is important to note that if we use a GARCH process and go to the continuous diffusion limit, and then apply an Euler approximation, we will *not necessarily* find the original GARCH process again. Indeed, there are many different ways to discretize the continuous diffusion limit and the GARCH process corresponds to one special way. In particular, if we use (1.11) and allow $\Delta t \to 0$ to get to the continuous diffusion limit, we shall obtain equation (1.7).

In summary, the standard GARCH model has the following form for the variance differential:

$$dS_t = \mu S_t dt + v_t S_t dW_t,$$

$$dv_t = \theta(\omega - v_t)dt + \xi v_t d\overline{W}_t$$

where $\omega$ is the mean long-term volatility, $\theta$ is the rate at which the volatility revert toward its long-term mean, $\xi$ is the volatility of the volatility process (Nelson, (1990)).

The volatility exhibits long memory is well established in the recent empirical literature. For example see Baillie et al. (1996), Robinson (2001), and Andersen et al. (2003).

Baillie et al. (1996) suggests the *Fractionally Integrated GARCH* (FIGARCH) model in discrete time to capture the long memory present in volatility. For recent works, Plienpanich et al. (2009) introduced a FIGARCH with long memory properties in continuous time, that process of the form:

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t^H$$

where $\overline{W}_t^H$ is a fractional Brownian motion (fBm).

### 1.2.5 A Stochastic Volatility Model with Jumps

In 1996, Bates [?] introduced the jump-diffusion stochastic volatility model by adding proportional log-normal jumps to Heston stochastic volatility model. In the original formulation the model has the following form:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t}dW_t + dZ_t,$$

$$dv_t = \theta(\omega - v_t)dt + \xi\sqrt{v_t}d\overline{W}_t,$$

where $(W_t)$ and $(\overline{W}_t)$ are Brownian motions with correlation $\rho$, driving price and volatility, and $Z_t$ is a compound Poisson process with intensity $\lambda$ and log-normal distribution of jump size such that if $k$ is its jump size then

$$\ln(1+k) \sim \mathcal{N}\Big(\ln(1+\overline{k}) - \frac{1}{2}\delta^2, \delta^2\Big).$$

The combination of Bates's (gBm with compound Poisson process) and FIGARCH's model still an open problem. Hence in this thesis, we extend the main result of Sattayathem at el. (2007) by replacing a Poisson jump by a compound Poisson jump and assuming that the variance of the stock return follows a fractional stochastic volatility model (FIGARCH model). Using a fundamental result on the $L_2(\Omega)$ approximation of fractional Brownian motion by semimartingale, we shell also prove that the solution of our approximate models converges to the solution of the gBm model with compound Poisson processes and fractional stochastic volatility.

### 1.2.6 Another Application of Stochastic Volatility Model

There are two main advantages to focusing on stochastic volatility models. First, much asset pricing theory is built on continuous-time models. Within this class, stochastic volatility models tend to fit more naturally with a wide array of applications, including the pricing of currencies, options, and other derivatives, as well as the modeling of the term structure of interest rates. Second, the increasing use of high-frequency intraday data for construction of so-called realized volatility measures is also starting to push the GARCH models out of the limelight as the realized volatility approach is naturally linked to the continuous-time stochastic volatility framework of financial economics (Andersen and Benzoni (2008)).

As we discuss above, an important application of the stochastic volatility model is the pricing of options. However, the relationship of interest rate and stock returns has been widely examined by researchers. Changes in interest rates influence the value of a companys stocks and shares and thus the stock returns. With an increase in interest rate, risk and required rate of return of a particular investment goes up and profits of a firm tend to decrease (due to increased cost of capital) which in turn causes the stock value to fall down.

Interest rates are determined by monetary policy of a country according to its economic situation. High interest rates induce the investors to keep their money deposited in saving bank accounts to get high interest rather to put it into risky stock market. As the risk free returns come down, investors switch their money from bank accounts to stock market investments. Consequently, demand of stocks increases and the stock markets go up as a result of interest rate cut. Mishkin (1977) also proved that lower interest rates increase stock prices which in turn reduce the probability of financial distress.

The literature on interest rate volatlity models is vast and rapidly growing, and excellent surveys are available, e.g., Zafar, Urooj, and Durrani (2008) examined the relationship of interest rate volatility and stock return and volatility. Trolle and Schwartz (2008) states a general stochastic volatility model for the pricing of interest rate derivatives. And an empirical application of stochastic volatility models provided by Mahieu and Schotman (1998).

## 1.3 A Primer on Itô Stochastic Calculus

This section provides a brief exposition of all definitions and tools used. For readers familiar with stochastic calculus, we recommend Section 1.4, which gives an introduction to an approximate approach to fractional Brownian motion.

A *stochastic process* is a sequence of random variables $X = (X_t)_{t \geq 0}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that, by abuse of the standard notation, whenever we write $t \geq 0$ that means $t \in [0, T]$. A stochastic process $X$ induces a *probability transition function* of the form

$$\mathbb{P}[X_{t+1} = s_{t+1} | X_t = s_t, \ldots, X_0 = s_0].$$

That is the probability that the state at future time $t + 1$ is $s_{t+1}$, given that the states at past times $t, \ldots, 0$ where $s_t, \ldots, s_0$, respectively.

A *Markov process* is a stochastic process such that for all $t$, for all $s_0, \ldots, s_t, s_{t+1}$,

$$\mathbb{P}[X_{t+1} = s_{t+1} | X_t = s_t, \ldots, X_0 = s_0] = \mathbb{P}[X_{t+1} = s_{t+1} | X_t = s_t].$$

This equation is the *Markov property*, sometimes called the *memoryless property*; it implies that probability transitions to future states, such as $s_{t+1}$ depend only on the present state $s_t$, but are independent of the remote past, $s_{t-1}, \ldots, s_0$.

A *martingale* is a stochastic process $X = (X_t)_{t \geq 0}$ such that $X_t \in L_1(\Omega)$ for each $t$ and such that the conditional expectation satisfies the relation

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s, \quad (s < t)$$

where $\mathcal{F}_s$ is a $\sigma$-algebra representing all observable events before time $s$. This equation says that if $X_t$ follows a martingale, the best forecast of $X_t$ that could be constructed based on current information $\mathcal{F}_s$ would just equation $X_s$.

A stochastic process is called *Gaussian process* if all its joint probability distributions are Gaussian. If $X_t$ is a Gaussian process, $X_t \sim \mathcal{N}(\mu_t, \sigma_t^2)$ for all $t$. A Gaussian process is fully characterized by its mean and covariance function.

### 1.3.1 Standard Brownian motion

*A standard Brownian motion process* or *a Wiener process* $(W_t)_{t \in [0,\infty)}$ is a stochastic process on $\Re$ (denote the set of real numbers) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

(i) It starts at zero i.e. $W_0 = 0$.

(ii) It has stationary, independent increments, i.e. $W_{t+u} - W_t$, $\forall u > 0$ are stationary and independent.

(iii) For very $t > 0$, $W_t$ has a normal $\mathcal{N}(0, t)$ distribution.

(iv) It has continuous sample paths: "no jumps" i.e. There is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$, i.e. $X_t(\omega)$ is continuous in $t$ for every $\omega \in \Omega_0$.

Stationary increments of the condition (ii) mean that the distributions of increments $W_{t+u} - W_t$ do not depend on the time $t$, but they depend on the time-distance $u$ of two observations (i.e. interval of time).

For example, if one models a log stock price $\log(S_t)$ as a Brownian motion (with drift) process (equation (1.2)), the distribution of increment in year 2004 for the next one year $\log\big(S_{2004+1}\big) - \log\big(S_{2004}\big)$ is the same as in year 2050, $\log\big(S_{2050+1}\big) - \log\big(S_{2050}\big)$:

$$\log\big(S_{2004+1}\big) - \log\big(S_{2004}\big) \stackrel{d}{=} \log\big(S_{2050+1}\big) - \log\big(S_{2050}\big).$$

Recall that the conditional probability of the event $A$ given $B$ is assuming $\mathbb{P}(B) > 0$:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

If $A$ and $B$ are independent events:

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

Independent increments property mean that when modeling a log stock price $\log S_t$ as a Brownian motion (with drift) process, the probability distribution of a log stock price in year 2005 is not affected by whatever happens in year 2004 in the stock price (i.e. such as stock price crash):

$$\mathbb{P}\Big(\log S_{2005+1} - \log S_{2005} \big| \log S_{2004+1} - \log S_{2004}\Big) \;=\; \mathbb{P}\Big(\log S_{2005+1} - \log S_{2005}\Big).$$

The *Brownian motion* (or *Wiener process*) has three properties which make it of fundamental importance to the theory of stochastic processes: it is Gaussian, a Markov process, and a martingale. Let $W = (W_t(\omega))_{t\geq 0}$ denote a Brownian motion, in which $t$ is the time and each $\omega$ is a particle; then $W_t(\omega)$ represents the position of that particle at time $t$. One can show that except on a set of probability zero, every sample path (i.e., $W_t(\omega)$ as a function of $t$ for fixed $\omega$) is continuous but is of unbounded variation on every compact time set.

### 1.3.2 Itô Integral

Since the sample paths are of unbounded variation on every compact set, they cannot be differentials in the Stieltjes integral sense. Although Stieltjes integration with respect to the paths of the Brownian motion is not possible, the differential $dW$ does have an intuitive interpretation. Engineers think of $dW$ as *white nose*, and using generalized functions, one can define the quantity $dW$ rigorously (see Arnold (1974)). Wiener (1933) gave meaning to $dW$ in his definition of what is called the *Wiener integral*, but in such integrals the integrands are functions of time only (certain functions). It was Itô (1944) who first defined an integral for random integrands with respect to the Brownian motion. Itô used his integral to represent a large class of diffusions as solution of stochastic differential equations (SDE). In 1953 Doob extended Itô's work on integration by using martingales instead of Brownian motion. The integral was so constructed that integration with respect to a martingale yields a martingale.

The best known extension of the Itô integral is the *semimartingale integral*. If all the paths of an adapted process are right continuous and of finite variation on compact time sets, we call the process a *VF process*. If $V$ is a VF process and $H$ is a bounded predictable process ($H$ is $\mathcal{F}_t$-measurable) then, for each fixed $\omega$, we denote by $\int_0^t H_s(\omega)dV_s(\omega)$ the Lebesgue-Stieltjes integral.

A stochastic process is a *local martingale* if certain integrability condition in the definition of a martingale are relaxed. A stochastic process $X$ is a *semimartingale* if $X$ can be written in the form

$$X = L + V$$

where $L$ is a local martingale and $V$ is a VF process. If $H$ is bounded, predictable process, one can then define $\int_0^t H_s dX_s$ by

$$\int_0^t H_s dX_s = \int_0^t H_s dL_s + \int_0^t H_s dV_s.$$

We will refer to this stochastic integral as the *semimartingale integral.* In fact the semimartingale form the largest class of processes for which the Itô integral can be defined.

### 1.3.3 Long Memory and Short Memory

A stochastic process, in general, is characterized by two quantities, namely, the probability density and the correlation function. The probability density describes the random nature of the fluctuations while the correlation function describes how a fluctuation at a given time influences subsequent fluctuations. If the correlation between two observations that are far apart decreases fairly slowly and is summed up to infinity then this is interpreted as a *long memory.* In fact, if $X = (X_t)_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\rho(k) = \mathbb{E}[X_1(X_{k+1} - X_k)]$ and if

$$\sum_{k=0}^{\infty} \rho(k) = \infty$$

then the process $X$ is said to have *long memory* or *long-range dependence* or *strong aftereffect.* This means that the process today may influence the process at some time in the future. In other words, the process at long time before may influence the process today.

On the other hand, if the correlation between two observations that are far apart decreases fast enough so that they are summed up to a finite number then it is interpreted as a *short memory* or *short-range dependence.* For example, since the Brownian motion $W = (W_t)_{t\geq 0}$ has independent increments so that for all

$k \geq 1$, we have $\mathbb{E}[W_1(W_{(k+1)} - W_k)] = 0$ and hence,

$$\sum_{k=0}^{\infty} \mathbb{E}[W_1(W_{(k+1)} - W_k)] < \infty.$$

Therefore, Brownian motion, as well as the processes of martingale property and Markov process, has short memory.

### 1.3.4 Lévy Processes

In this section, we shall review the notion of the Lévy process and some of its properties. For more details see Kou (2008).

A stochastic process is called *cadlag* if it has almost surely right continuous paths and the limits from the left exist. A cadlag stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\Re^d$ such that $X_0 = 0$ is called *Lévy process* if it has independent and stationary increments and has a stochastically continuous sample path, i.e. for any $\varepsilon > 0$,

$$\lim_{h \downarrow 0} \mathbb{P}(|X_{t+h} - X_t| > \varepsilon) \to 0.$$

Note that (i) this condition does not imply that the path of Lévy process are continuous. It only requires that for a given time $t$, the probability of seeing a jump at $t$ is zero, i.e. jumps occur at random times and (ii) Lévy processes have a version with cadlag paths, i.e. paths which are right continuous and have limits from the left.

Therefore, Lévy processes provide a natural generalization of the sum of independent and identically distributed (i.i.d.) random variables. The simplest possible Lévy processes are the standard Brownian motion $W_t$, Poisson processes $N_t$, and compound Poisson processes $\sum_{n=1}^{N_t} Y_n$,where $Y_n$ are i.i.d. random variables.

Of course, one can combine the above processes to form other Lévy processes. For example, an important class of Lévy processes is the jump-diffusion process given by

$$\mu t + \sigma W_t + \sum_{n=1}^{N_t} Y_n$$

where $\mu$ and $\sigma$ are constants. Interestingly the famous Lévy-Itô decomposition says that the converse is also true. More precisely, any Lévy process can be written as a drift term $\mu t$, a Brownian motion with variance and covariance matrix $A$, and a possibly infinite sum of independent compound Poisson processes which are related to a intensity measure $\nu(dx)$. This implies that a Lévy process can be approximated by jump-diffusion processes. This has important numerical applications in finance, as jump-diffusion models are widely used in finance.

The triplet $(\mu, A, \nu)$ is also linked to the Lévy-Khinchin representation which states that the characteristic function of a Lévy process $X_t$ can be written in terms of $(\mu, A, \nu)$ as

$$\frac{1}{t} \log \mathbb{E}\left[e^{iz'X_t}\right] = -\frac{1}{2} z'Az + i\mu z + \int_{\Re^d} \left(e^{iz'x} - 1 - iz'xI_{|x|\leq 1}\right) v(dx).$$

The representation suggests that it is easier to study Lévy processes via Laplace transforms, and then numerically invert Laplace transforms.

There are two types of Lévy processes: jump-diffusion and infinite activity Lévy processes. In jump-diffusion processes, jumps are considered rare events, and in any given finite interval there are only finite many jumps. Examples of jump-diffusion models in finance include Merton's model (1976) in which the jump size $Y$ has a normal distribution, and the double exponential jump-diffusion model in Kou (2002). For infinite activity Lévy processes, in any finite time interval there are infinitely many jumps.

### 1.3.5 Compound Poisson Processes

In this section, we shall review the notion of the compound Poisson process and some of its properties which will be useful in the sequel; see Cont and Tankov (2004) for the details.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with $E \subset \Re^d$ and $\mu$ as a (positive) Radon measure on $(E, \mathcal{E})$. A *Poisson random measure* on $E$ with intensity measure on $\mu$ is an integer-valued random measure $M : \Omega \times \mathcal{E} \to \mathbb{N}$ such that:

- For (almost all) $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on $E$ and, for any bounded measurable set $A \subset E$, $M(\cdot, A) := M(A) < \infty$ is an integer-valued random variable.

- For each measurable set $A \subset E$, $M(A)$ is a Poisson random variable with parameter $\mu(A)$ such that

$$\forall k \in \mathbb{N},\ \mathbb{P}(M(A) = k) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}.$$

- For disjoint measurable sets $A_1, \ldots, A_n \in \mathcal{E}$, the random variables $M(A_1), \ldots,$ $M(A_n)$ are independent.

One can prove that for any Radon measure $\mu$ on $E \subset \Re^d$, there exists a Poisson random measure $M$ on $E$ with intensity $\mu$. Consequently, any Poisson random measure on $E$ can be represented as a counting measure associated with a random sequence of points in $E$, i.e. there exists $(T_n(\omega))_{n\geq 1}$, such that

$$\forall A \in \mathcal{E},\ M(\omega, A) = \sum_{n\geq 1} 1_A(T_n(\omega)) = \#\{n \geq 1, T_n(\omega) \in A\}. \tag{1.12}$$

Define a random variable $T_n = \sum_{i=1}^{n} \tau_i$ where $(\tau_i)_{i\geq 1}$ is a sequence of independent exponential random variables with parameter $\lambda$, that is $\mathbb{P}(\tau_i > t) = e^{\lambda t}$. The process $(N_t)_{t\geq 0}$ defined by

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n}$$

is called a *Poisson process* with intensity $\lambda$.

Moreover, by equation (1.12), the Poisson process may be expressed in terms of the Poisson random measure $M$ in the following way:

$$N_t(\omega) = M(\omega, [0,t]) = \int_{[0,t]} M(\omega, ds)$$

where $ds$ is the Lebesgue area element on $[0,t]$.

A *compound Poisson process* on $\Re^d$ with intensity $\lambda > 0$ and jump size distribution $f$ is a stochastic process $X_t$ defined as

$$X_t = \sum_{n=1}^{N_t} Y_n$$

where jump size $Y_n$ are independent and identically distributed (i.i.d.) with distribution $f$ and $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda$, independent from $(Y_n)_{n\geq 1}$. The Poisson process itself can be seen as a compound Poisson process on $\Re$ such that $Y_n := 1$. This explains the origin of the term "compound Poisson" in the definition.

For every compound Poisson process $(X_t)_{t\geq 0}$ on $\Re^d$ with intensity $\lambda$ and jump size distribution $f$, its jump measure

$$J_X(B) = \#\{(t, X_t - X_{t-}) \in B\}$$

is a Poisson random measure on $\Re^d \times [0,\infty)$ with intensity measure

$$\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt$$

where $B$ is a measurable subset of $\Re^d \times [0,\infty)$ and $\nu$ is *Lévy measure of the compound Poisson process.* This fact implies that every compound Poisson process can be represented in the following form:

$$X_t = \sum_{s \in [0,t]} \Delta X_s = \int_{\Re^d \times [0,t]} x J_X(dx \times ds)$$

where $J_X$ is a Poisson random measure with intensity measure $\nu(dx)dt$.

Let $E$ be a measurable subset on $\Re^d$. For a measurable function $f : [0,T] \times E \to \Re^d$, one can construct an integral with respect to the Poisson random measure $M$, given by the random variable

$$\int_{E \times [0,T]} f(y,t) M(\cdot, dy\ dt) = \sum_{n \geq 1} f(Y_n(\cdot), T_n(\cdot)).$$

### 1.3.6 Itô formula and its Extensions

In this section we review Itô formula and its extensions for the further uses.

**Lemma 1.1.** *(Itô's formula or Itô's lemma)*
*Assume that the process $X$ has a stochastic differential given by*

$$dX_t = \mu_t dt + \sigma_t dW_t$$

*where $\mu$ and $\sigma$ are adapted processes, and let $f$ be a $C^{1,2}$-function. Define the process $Y$ by $Y_t = f(t, X_t)$. Then $Y$ has a stochastic differential given by*

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW_t.$$

Note that the term $\mu \partial f / \partial x$, for example, is shorthand notation for

$$\mu_t \frac{\partial f}{\partial x}(t, X_t)$$

and correspondingly for the other terms.

In fact Itô's lemma provides a derivative chain rule for stochastic functions. Clarifying the relationship between a stochastic process and a function of that stochastic process. Itô's lemmas have many extension. The following Itô's lemma is the key step in establishing the main theorem of our thesis (for the proof see Cont and Tankov (2006)).

**Lemma 1.2.** *(Itô formula for jump-diffusion processes)*

*Let $X$ be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process:*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{n=1}^{N_t} \Delta X_n,$$

*where $b_t$ and $\sigma_t$ are continuous nonanticipating processes with*

$$\mathbb{E}\Big[\int_0^T \sigma_t^2 dt\Big] < \infty.$$

*Then, for any $C^{1,2}$ function, $f : [0,T] \times \Re \to \Re$, the process $Y_t = f(t, X_t)$ can be represented as:*

$$\begin{aligned} f(t, X_t) - f(0, X_0) = & \int_0^t \Big[\frac{\partial f}{\partial s}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s) b_s\Big] ds \\ & + \frac{1}{2}\int_0^t \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\sigma_s dW_s \\ & + \sum_{\{n \geq 1, T_n \leq t\}} [f(X_{T_n-} + \Delta X_n) - f(X_{T_n-})]. \end{aligned}$$

*In differential notation:*

$$\begin{aligned} dY_t = & \frac{\partial f}{\partial t}(t, X_t) dt + b_t \frac{\partial f}{\partial x}(t, X_t) dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t) dt \\ & + \frac{\partial f}{\partial x}(t, X_t)\sigma_t dW_t + [f(X_{t-} + \Delta X_t) - f(X_{t-})]. \end{aligned}$$

Note that a nonanticipating process is also called an *adapted process*: $(X_t)_{t\in[0,T]}$ is said to be $\mathcal{F}_t$-adapted, that is the random variable $X_t$ is $\mathcal{F}_t$-measurable.

## 1.4 Fractional Brownian Motion

In this section, we shall review definitions of a fractional Brownian motion (fBm) and its approximation by a semimartingale. Stochastic integrals with respect to fractional Brownian motion is also discussion here.

### 1.4.1 The Need to Study Fractional Brownian Motions

Almost all statistical analysis of economic and financial systems begins by assuming that the dynamics are primarily random. Models considered earlier in Mathematical finance assume that the price of an asset should follow a martingale property in which each price change is unaffected by its predecessor.

Stochastic differential equations driven by Brownian motion are traditionally used to model the dynamic of stock prices. It is well known that Brownian motion is a typical semimartingale with short-range dependence: where $H = 1/2$, the autocorrelation $\rho(n)$ of equation (1.14) is zero for all $n$, hence $\sum_{n=1}^{\infty} \rho(n) < \infty$. However, in recent years it has become increasingly obvious that long-range dependence phenomena are widespread in financial data. The dependence structure of the financial data have been studied using the so-called *Hurst index* (*Hurst parameter*) $H$. In the uncorrelated case one should have $H = 1/2$. If $H < 1/2$ the time series is *antipersistent*. This means that whenever the price has been up, it is more likely that it will be down in the close future. Conversely, if $H > 1/2$ one has *persistence* with positive correlations. This means that all price fluctuations are correlated with all future price fluctuations. Persistence implies that if the price has been up or down then the chances are that it will continue to be up or down in the future, respectively.

Many studies indicated Hurst indices $H > 1/2$. For example, for the monthly S&P500 index (from January, 1963 through December 1989) the estimated Hurst index is $H = 0.78$ (see Shiryaev (1999)). In 2002, Alvarez-Ramirez et al., studied the daily records of international crude oil prices and found that the rescaled range Hurst analysis provides evidence that the crude oil market is a persistent process with long memory effect. In fact, they found that the Hurst indices are all above 1/2 with different time scales.

### 1.4.2 Fractional Brownian Motion (fBm) and Its Properties

A fBm with *Hurst parameter* $H \in (0,1)$ is a centered Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with zero mean, and the covariance function is given by

$$R(t,s) := \mathbb{E}\big(B_t^H B_s^H\big) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big). \tag{1.13}$$

This process was introduced by Kolmogorov (1968) and studied further by Mandelbrot and Van Ness (1968).

Furthermore, fBm has a stationary increment, i.e. the increment of $B^H$ has a normal distribution with zero mean and tis variance is given by

$$\mathbb{E}\big[(B_t^H - B_s^H)^2\big] = |t-s|^{2H},$$

in an interval $[s,t]$.

If $H = 1/2$, then $R(t,s) = \min(t,s)$ and $B_t^H$ is the usual standard Brownian motion.

The following theorems show that the fractional Brownian motion $X = (X_t)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is neither a semimartingale (Theorem 1.4) nor a Markov process (Theorem 1.5).

Further we need the following definition. Recall here that a stochastic process $X = (X)_{t\geq 0}$ is *H-self-similar* with parameter $H > 0$ if

$$(X_{at})_{t\geq 0} \stackrel{d}{=} (a^H X_t)_{t\geq 0}$$

for all $a > 0$, where $\stackrel{d}{=}$ means equality in distributions.

Suppose that $Y = (Y_t)_{t\geq 0}$ is self-similar process with parameter $H$. Then

$$Y_t \stackrel{d}{=} t^H Y_1 \quad \text{for} \quad t > 0$$

and hence

$$Var(Y_t) = Var(t^H Y_1) = t^{2H} Var(Y_1).$$

In the following, we consider the values of $H \in (0,1)$, and in particular $Y_0 = 0$ with probability 1. Assume further that $Y_t$ has zero mean, is normalized so that $Var(Y_1) = 1$, and stationary increments, i.e., the random $n$-vectors $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})$ and $(Y_{t_1+h}, Y_{t_2+h}, \ldots, Y_{t_n+h})$, $h > 0$ are identically distributed.

**Theorem 1.3.** *A fractional Brownian motion $(W_t^H)_{t\geq 0}$ is $H$-self-similar with stationary increments. When $H \in (0,1)$, it has a stochastic integral representation:*

$$\frac{1}{C_H}\int_{\Re}\left[\left((t-s)^+\right)^{h-\frac{1}{2}} - \left((-s)^+\right)^{h-\frac{1}{2}}\right]dW_s, \quad t \geq 0$$

*where $H \in (0,1)$, $f^+ = \max\{f, 0\}$ and*

$$C_H = \left(\int_0^\infty \left[(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right]^2 ds + \frac{1}{2H}\right)^{\frac{1}{2}}.$$

*If $H = 1$, $W_t^1 = tW_1^1$ almost surely, Fractional Brownian motion is unique in the sense that the class of all fractional Brownian motions coincides with that of all Gaussian self-similar processes with stationary increments.*

*Proof.* See Embrechts et al. (2005). $\square$

Another basic property of fractional Brownian motion, $W_t^H$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is *long-range dependence*. In fact, for $n \geq 1$,

$$\begin{aligned}
\rho(n) &= \mathbb{E}[W_1^H(W_{n+1}^H - W_n^H)] \\
&= \mathbb{E}W_1^H W_{n+1}^H - \mathbb{E}W_1^H W_n^H \\
&= \frac{1}{2}\left(1^{2H} + (n+1)^{2H} - n^{2H}\right) - \frac{1}{2}\left(1^{2H} + n^{2H} - (n-1)^{2H}\right) \\
&= \frac{1}{2}\left((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right) \\
&= \frac{1}{2}n^{2H}g(n^{-1}),
\end{aligned}$$

where $g(x) = (1+x)^{2H} - 2 + (1-x)^{2H}$. If $0 < H < 1$ and $H \neq 1/2$, then the Taylor expansion of $g(x)$ about the origin gives

$$g(x) = 2H(2H-1)x^2 + o(x^4).$$

Therefore,

$$\rho(n) = \frac{1}{2}n^{2H}g(n^{-1}) = \frac{1}{2}n^{2H}\left[2H(2H-1)n^{-2} + o(n^{-4})\right]$$

and as $n$ tends to infinity,

$$p(n) = H(2H-1)n^{2H-2}. \tag{1.14}$$

Moreover, for $1/2 < H < 1$ the correlation decay to zero so slowly that

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

Hence, for $1/2 < H < 1$, fractional Brownian motion $W_t^H$ has *long-range dependence*. For $H = 1/2$ it can easily be seen, by equation (1.13), that the observations are uncorrelated. In fact, the fractional Brownian motion with Hurst index $H$ is semimartingale if and only if $H = 1/2$ (Roger (1997)). Finally, for $0 < H < 1/2$, we have $2 - 2H > 1$ and hence

$$\sum_{n=0}^{\infty} \rho(n) = H(2H-1)\sum_{n=0}^{\infty} \frac{1}{n^{2-2H}} < \infty.$$

Therefore in this case, the process exhibits short-range dependence.

**Theorem 1.4.** *(Rogers (1997)) The fBm is a semimartingale only if $H = 1/2$.*

*Proof.* Let $X = (X_t)_{t\geq 0}$ be a fractional Brownian motion with self-similar parameter $H \in (0,1)$. We know that when $H = 1/2$ fractional Brownian motion is in fact a standard Brownian motion and hence a semimartingale.

Now fix the parameter $H$ and consider for $p > 0$ fixed

$$Y_{n,p} := \sum_{j=1}^{2^n} |X_{j2^{-n}} - X_{(j-1)2^{-n}}|^p (2^n)^{pH-1}. \tag{1.15}$$

From self-similarity property we obtain that (1.15) has (for each $n$) the same law as

$$\begin{aligned}\sum_{j=1}^{2^n} |2^{-nH}X_j - 2^{-2H}X_{j-1}|^p (2^n)^{pH-1} &= \sum_{j=1}^{2^n} |X_j - X_{j-1}|^p 2^{-npH}(2^n)^{pH-1} \\ &= 2^{-n}\sum_{j=1}^{2^n} |X_j - X_{j-1}|^p.\end{aligned}$$

Noticing that the sequence $(X_k - X_{k-1})_{k\in\mathbb{Z}}$ is stationary and ergodic, the ergodic theorem tells us that

$$\widetilde{Y}_{n,p} := 2^{-n}\sum_{j=1}^{2^n} |X_j - X_{j-1}|^p \to \mathbb{E}|X_1 - X_0|^p =: \gamma_p \quad (n \to \infty)$$

almost surely and in $L_1$. Hence

$$Y_{n,p} \xrightarrow{d} \gamma_p \quad (n \to \infty),$$

and therefore $Y_{n,p} \xrightarrow{P} \gamma_p$. Hence,

$$V_{n,p} := \sum_{j=1}^{2^n} |X_{j2^{-n}} - X_{(j-1)2^{-n}}|^p \xrightarrow{\mathbb{P}} \begin{cases} 0 & \text{if } pH > 1, \\ \infty & \text{if } pH < 1. \end{cases}$$

If $H > 1/2$, we can choose $p \in (H^{-1}, 2)$ such that $V_{n,p} \to 0$ in probability, and therefore almost surely down a fast subsequence. This implies that the quadratic variation of $X$ is zero, and so (if $X$ were to be a semimartingale) $X$ must be a finite-variation process. But since for $p \in (1, H^{-1})$, $V_p := \lim_{n\to} V_{n,p}$ is almost surely in finite, and (by scaling) the $p$-variation on any interval is infinite almost surely, $X$ can not be finite variation. If $H < 1/2$, we can choose $p > 2$ such that $pH < 1$, and the $p$-variation of $X$ on $[0, 1]$ (and hence on any fixed interval) must be infinite. This contradicts the almost-sure finiteness of the quadratic variation of $X$, assuming $X$ is a semimartingale. In either way, if $H \neq 1/2$, $X$ is not a semimartingale. □

The following lemma need to prove for the next theorem.

**Lemma 1.5.** *Let $R(s, t)$ be covariance function of a centered Gaussian process, $Y_t$ is Markovian then for all $t, s, t_0$ such that $t > s > t_0$ we have*

$$R(t, t_0) = \frac{R(t, s)R(s, t_0)}{R(s, s)}$$

*(see for example, Wong and Hajek (1985)).*

**Theorem 1.6.** *Every fractional Brownian motion with Hurst index $H \neq 1/2$ is not a Markov process.*

*Proof.* (Huy (2003)) Let $(W_t^H)_{t\geq 0}$ be a fractional Brownian motion with $H \neq 1/2$. Suppose that it is a Markov process. Put

$$f_s(t) = \frac{R(t,s)}{s^\alpha} = \frac{1}{2}\left[\left(\frac{t}{s}\right)^\alpha + 1 - \left(\frac{t}{s} - 1\right)^\alpha\right], \quad t > s$$

where $\alpha = 2H$. Consider the derivative of $f_s(t)$ with respect to $t$:

$$f_s'(t) = \frac{1}{2}\frac{\alpha}{s}\left[\left(\frac{t}{s}\right)^{\alpha-1} - \left(\frac{t}{s} - 1\right)^{\alpha-1}\right], \quad t > s. \tag{1.16}$$

We see for $s < t$ that

$$f_s'(t) < 0 \quad \text{if} \quad \alpha < 1$$

$$f_s'(t) > 0 \quad \text{if} \quad \alpha > 1$$

$$f_s'(t) = 0 \quad \text{if} \quad \alpha = 1$$

So, if $\alpha \neq 1$, $f_s(t)$ is either decreasing or increasing. On the other hand, for $\alpha < 1$ we have

$$\begin{aligned}\lim_{t\to\infty} f_s(t) &= \frac{1}{2}\lim_{t\to\infty}\frac{\frac{1}{t^\alpha} + \frac{1}{s^\alpha} - \left(\frac{1}{s} - \frac{1}{t}\right)^\alpha}{\frac{1}{t^\alpha}} \\ &= \frac{1}{2}\lim_{t\to\infty}\left(1 + \left(\frac{t}{s} - 1\right)^{\alpha-1}\right) = \frac{1}{2}; \quad t > s.\end{aligned}$$

Hence for $\alpha < 1$, $f_s(t)$ is decreasing from 1 to 1/2 when $t$ varies from 0 to infinity. Now for $0 < r < s < t$ it follows from Lemma 1.5 that

$$\frac{R(t,r)}{r^\alpha} = \frac{R(t,s)}{s^\alpha}\frac{R(s,r)}{r^\alpha}$$

or

$$f_r(t) = f_s(t) \cdot f_r(s).$$

Taking the limit of both sides of the above equation when $t \to \infty$, we get

$$\frac{1}{2} = \frac{1}{2}f_r(s), \quad r < s$$

or

$$f_r(s) = 1, \quad r < s.$$

This is contrary to the property of the function $f_r(s)$ given by (1.16);

$$f_r(s) = \frac{1}{2}\left[\left(\frac{s}{r}\right)^{\alpha} + 1 - \left(\frac{s}{r} - 1\right)^{\alpha}\right]$$

Then $W_t^H$ should not be a Markov process. $\square$

### 1.4.3 Stochastic Integrals with Respect to Fractional Brownian Motion

In this subsection, we follow the approach outline in Saelim (2004). In order to apply fractional Brownian motion to study the market situations we need a stochastic calculus for fractional Brownian motion. Since for $H \neq \frac{1}{2}$, the fractional Brownian motion $B_t^H$ is neither a semimartingale (Theorem 1.4) nor a Markov process (Theorem 1.6), then the well developed stochastic calculus is not applicable. In particular, for $H > 1/2$, it is a long memory process. In other words, the behavior of a real process after a given time $t$ does not only depend on the situation at $t$ but also of the whole history of the process up to time $t$. This significant property makes fractional Brownian motion a natural candidate as a model of noise in mathematical finance (see, e.g., Rogers (1997)) and in communication networks (Leland et al. (1994)).

Many authors tried to understand what a stochastic integral of the form

$$\int_0^T f(t, \omega) dW_t^H$$

should mean. The most common constructions of such a stochastic integral are the following.

### 1.4.4 The Pathwise or Forward Integral

The integral is denoted by

$$\int_0^T \phi(t,\omega) d^- W_t^H.$$

If the integrand $\phi(t,\omega)$ is *caglad* (left-continuous with right sided limits) then this integral can be defined by Riemann sums, as follows:

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of $[0,T]$. Put $\Delta t_k = t_{k+1} - t_k$ and define

$$\int_0^T \phi(t,\omega) d^- W_t^H := \lim_{\Delta t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k) \Big( W_{t_{k+1}}^H - W_{t_k}^H \Big), \tag{1.17}$$

if the limit exists in probability.

Note that with this definition the integration takes place with respect to $t$ for each fixed "path" $\omega \in \Omega$. Therefore, this integral is often called *pathwise integral*. Using a classical integration theory due to Young (i.e., the Riemann-Stieltjes integral $\int f dg$ exists if $f(t)$ is a function of bounded $p$-variation and $g(t)$ function of bounded $q$-variation for $p, q > 0$ and $1/p + 1/q > 1$) one can prove that the pathwise integral (1.17) exists if the $p$-variation of $t \mapsto \phi(t,\omega)$ is finite for all $p > \frac{1}{1-H}$. Since $t \mapsto W_t^H$ has finite $q$-variation if and only if $q \geq 1/H$, we see that if $H < 1/2$ then this theory does not even include integrals like

$$\int_0^T W_s^H d^- W_s^H.$$

For this reason one often assumes that $H > 1/2$ when dealing with forward integrals with respect to $W_t^H$. In general

$$\mathbb{E} \int_0^T W_s^H d^- W_s^H \neq 0,$$

even if the forward integral belongs to $L_1(\Omega, \mathcal{F}, \mathbb{P})$.

For $H > 1/2$ the forward integral obeys *Stratonovich type* of integration rules. For example, if $f \in C^1(\Re)$ and

$$X_t := \int_0^t \phi(s,\omega) d^- W_s^H$$

exists for all $t \geq 0$ then

$$f(X_t) = f(0) + \int_0^t f'(X_s) d^- X_s, \tag{1.18}$$

where

$$d^- X_s = \phi(s,\omega) d^- W_s^H.$$

For this reason the forward integral is also sometimes called integral of Stratonovich type with respect to fractional Brownian motion. In fact, this is the Newton-Leibnitz's rule of integration.

As special case of (1.18) we note that

$$\int_0^T W_s^H d^- W_s^H = \frac{1}{2}\Big(W_T^H\Big)^2 \quad \text{for} \quad H > \frac{1}{2}.$$

Moreover, a slight extension of (1.18) gives that the unique solution $X_t$ of the fractional forward stochastic differential equation

$$d^- X_t = \alpha(t,\omega) X_t dt + \beta(t,\omega) X_t d^- W_t^H, \quad X_0 = x > 0 \tag{1.19}$$

is

$$X_t = x \exp\left(\int_0^t \alpha(s,\omega) ds + \int_0^t \beta(s,\omega) d^- W_s^H\right)$$

for $H > 1/2$, provided that the integrals on the right hand side exist.

### 1.4.5 The Skorohod (Wick-Itô integral)

Let

$$h_n(x) := (-1)^n \exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right),\ n = 0, 1, 2, \ldots$$

and

$$\xi_n(x) := \pi^{-1/4}\big((n-1)!\big)^{-1/2} h_{n-1}\big(\sqrt{2}x\big)\exp\left(-\frac{x^2}{2}\right),\ n = 1, 2, \ldots.$$

Further let $J$ be the set of all multi-indies $\alpha = (\alpha_1, ..., \alpha_m)$ of finite length, with $\alpha_i \in N \cap \{0\}$ for all $i$. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in J$ define

$$\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(<\omega, \xi_1>)h_{\alpha_2}(<\omega, \xi_2>)\ldots h_{\alpha_m}(<\omega, \xi_m>).$$

Moreover, the space $(S)^*$ of Hida distribution is the set of all formal expansions

$$G(w) = \sum_{\alpha\in J} b_\alpha \mathcal{H}_\alpha(\omega)$$

such that

$$\sum_{\alpha\in J} b^2 \alpha!(2\mathbb{N})^{-q\alpha} < \text{ for some } q \in \mathbb{N}.$$

**Definition 1.1.** *Let*

$$F(\omega) = \sum_{\alpha\in J} a_\alpha \mathcal{H}_\alpha(\omega) \in (S)^*$$

*and*

$$G(\omega) = \sum_{\beta\in J} b_\beta \mathcal{H}_\beta(\omega) \in (S)^*.$$

*The the Wick Product of $F$ and $G$, $F\Diamond G$, is defined by*

$$\begin{aligned}(F\Diamond G)(\omega) &= \sum_{\alpha,\beta\in J} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega)\\ &= \sum_{\gamma\in J}\Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta\Big)\mathcal{H}(\omega).\end{aligned} \tag{1.20}$$

The Skorohod (Wick-Itô integral) integral is denoted by

$$\int_0^T \phi(t,\omega)\delta W_t^H.$$

It is defined in terms of Riemann sums, as follows:

$$\int_0^T \phi(t,\omega)\delta W_t^H = \lim_{t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k) \Diamond \left( W_{t_{k+1}}^H - W_{t_k}^H \right), \tag{1.21}$$

where $\Diamond$ denotes the Wick product. The difference between this integral and the forward integral is the use of the Wick product instead of the ordinary product in the Riemann sums (1.21) and (1.17), respectively.

The Skorohod integral behaves in many ways like the Itô integral of classical Brownian motion. For example, we have

$$\mathbb{E} \int_0^T \phi(t,\omega)\delta W_t^H = 0$$

if the integral belongs to $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, if $f \in C^2(\Re)$ then we have the following Itô type formula

$$f(W_t^H) = f(0) + \int_0^t f'(W_s^H)\delta W_t^H + H \int_0^t \int_0^t f''(W_s^H) s^{2H} ds, \tag{1.22}$$

valid for all $H \in (0,1)$, provided that the left hand side and the last term on the right hand side both belong to $L_2(\Omega, \mathcal{F}, \mathbb{P})$.

Note that as special case of (1.22) we get

$$\int_0^T W_s^H \delta W_s^H = \frac{1}{2}(W_T^H)^2 - \frac{1}{2}\, T^{2H} \quad \text{for } H \in (0,1). \tag{1.23}$$

The Wick-Skorohod-Itô analogue of (1.19) is the equation

$$\delta X_t = \alpha(t,\omega) X_t dt + \beta(t,\omega) X_t \delta W_t^H, \quad X_0 = x > 0. \tag{1.24}$$

Assume that $\alpha(t,\omega) = a$ and $\beta(t,\omega) = b$ are constant. Then by a slight extension of the Itô formula (1.22) one obtains that the unique solution of (1.24) is

$$X_t = x \exp\left( \beta W_t^H + \alpha t \frac{1}{2} \beta^2 t^{2H} \right), \quad H \in (0,1). \tag{1.25}$$

Note that if $H = 1/2$ then the formulas (1.23) and (1.25) reduce to the formulas obtained by the Itô formula for the classical Brownian motion.

After the pathwise theory for fractional Brownian motion was developed (see, e.g., Lin (1995), and Decreusefond et al. (1998; 1999) it was proved that the market mathematical model driven by fractional Brownian motion could have arbitrage Cheridito (2003), Rogers (1997), Sottinen (2001) Sottinen and Valkeila (2001; 2003). However after the development of the Skorohod integral based on the Wick product (e.g., Duncan et al. (2000) and Hu and Oksendal (2003)) it was proved (Hu and Oksendal (2003)) that the corresponding Itô type fractional Black-Scholes market has no arbitrage. Unfortunately, this integral does not allow an economics interpretation. Worse still, these two types of definition (the pathwise and Skorohod integrals) are difficult to implement numerically.

In 2006, Thao tried to solve this problem. He proposed another definition of *fractional stochastic integral* motivated by a formulae of integration by parts and an approximate approach to fractional Brownian motion.

The next subsection, prepares mathematical tools for defining stochastic integral with respect to fractional Brownian motion via integration by parts. Moreover, in this thesis we choose to use the approximate approach, namely, using the $L_2$-convergence of a semimartingale to a fractional process. The Details of the discussion can be found in Thao (2006).

### 1.4.6 An Approximate Approach to Fractional Brownian Motion

We consider the fractional Brownian motion of Liouville form with parameter $H \in (0,1)$

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s,\ H \neq \frac{1}{2}. \tag{1.26}$$

For every $\varepsilon > 0$ we define

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s,\ H \neq \frac{1}{2},\ 0 < H < 1. \tag{1.27}$$

**Theorem 1.7.** *The process $(B_t^\varepsilon)_{t\geq 0}$ is a semimartingale.*

*Proof.* (Thao (2006)) Consider the stochastic process $\varphi_t^\varepsilon$ defined as

$$\varphi_t^\varepsilon = \int_0^t (t-u+\varepsilon)^{\alpha-1} dW_u$$

where $\alpha = H - 1/2$ (then $-1/2 < \alpha < 1/2$, since $0 < H < 1$).

An application of the stochastic theorem of Fubini gives us:

$$\begin{aligned}
\int_0^t \varphi_s^\varepsilon ds &= \int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u ds \\
&= \int_0^t \Big( \int_u^t (s-u+\varepsilon)^{\alpha-1} \Big) dW_u \\
&= \frac{1}{\alpha} \int_0^t \Big( (t-u+\varepsilon)^\alpha - \varepsilon^\alpha \Big) dW_u \\
&= \frac{1}{\alpha} \Big[ \int_0^t (t-u+\varepsilon)^\alpha dW_u - \int_0^t \varepsilon^\alpha dW_u \Big] \\
&= \frac{1}{\alpha} (B_t^\varepsilon - \varepsilon^\alpha W_t).
\end{aligned}$$

Hence

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t.$$

Since $\alpha \int_0^t \varphi_s^\varepsilon ds$ is of bounded variation and $W_t$ is a martingale so $B_t^\varepsilon$ is a semimartingale. □

**Theorem 1.8.** *$B_t^\varepsilon$ converges to $B_t$ in $L_2(\Omega)$ when $\varepsilon$ tends to $0$. This convergence is uniform with respect to $t \in [0,T]$.*

*Proof.* The Mean Value Theorem applied to the function $f(u) = u^\alpha$ yields:

$$\begin{aligned} \big|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big| &\leq |\alpha|\varepsilon \sup_{0\leq\theta\leq 1} \big|(t-s+\theta\varepsilon)^{\alpha-1}\big| \\ &= |\alpha|\varepsilon(t-s)^{\alpha-1},\ \alpha = H - \frac{1}{2}, \\ &\quad (0 < s < t). \end{aligned} \tag{1.28}$$

By virtue of Itô integration isometry we see that

$$\begin{aligned} \mathbb{E}|B_t^\varepsilon - B_t|^2 &= \mathbb{E}\bigg|\int_0^t \Big[(t-s+\varepsilon)^\alpha - (t-s)^\alpha\Big] dW_s\bigg|^2 \\ &= \int_0^t \big|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big|^2 ds. \end{aligned} \tag{1.29}$$

(i) (Thao (2006)) If $1/2 < H < 1$, that is, $0 < \alpha < 1/2$ we have from (1.28)

$$\begin{aligned} \int_0^t \big|(t-s+\varepsilon)^\alpha - (t-s)^\alpha\big|^2 ds &\leq \alpha^2\varepsilon^2 \int_0^t |t-s|^{2\alpha-2} ds \\ &= \alpha^2\varepsilon^2\bigg(\int_0^{t-\varepsilon} |t-s|^{2\alpha-2} ds + \int_{t-\varepsilon}^t |t-s|^{2\alpha-2} ds\bigg) \\ &\leq \alpha^2\varepsilon^2\frac{\varepsilon^{2\varepsilon-1}}{1-2\alpha} + \alpha^2\varepsilon^2\frac{\varepsilon^{2\alpha-1}}{1-2\alpha} \\ &= C_1(\alpha)\varepsilon^{2\alpha-1} \to 0 \end{aligned} \tag{1.30}$$

as $\varepsilon \to 0$, where $C_1(\alpha) = \frac{2\alpha^2}{1-2\alpha} > 0$.

(ii) (Thao et al. (2003)) If $0 < H < 1/2$, that is, $-1/2 < \alpha < 0$, we put $\alpha = -\beta$, so $0 < \beta < 1/2$ and we have

$$\begin{aligned} \big|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\big| &\leq \beta\varepsilon \sup_{0\leq\theta\leq 1} \big|(t-s+\theta\varepsilon)^{-\beta-1}\big| \\ &= \beta\varepsilon(t-s)^{-\beta-1}. \end{aligned} \tag{1.31}$$

From equation (1.29), we note that

$$\begin{aligned}
\mathbb{E}|B_t^\varepsilon - B_t|^2 &= \mathbb{E}\left|\int_0^t \left[(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right] dW_s\right|^2 \\
&= \int_0^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \\
&= \int_0^{t-\varepsilon} \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \\
&\quad + \int_{t-\varepsilon}^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds,
\end{aligned} \tag{1.32}$$

The evaluation of (1.31) applied to the first term of (1.32) gives us

$$\int_0^{t-\varepsilon} \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \leq \beta^2\varepsilon^2 \int_0^{t-\varepsilon} (t-s)^{-2\beta-2} ds. \tag{1.33}$$

For the second term of the right hand side of (1.32) we have

$$\int_{t-\varepsilon}^t \left|(t-s+\varepsilon)^{-\beta} - (t-s)^{-\beta}\right|^2 ds \leq \int_{t-\varepsilon}^t (t-s)^{-2\beta} ds. \tag{1.34}$$

It follows from (1.32), (1.33) and (1.34) that

$$\mathbb{E}|B_t^\varepsilon - B_t|^2 \leq \beta^2\varepsilon^2 \int_0^{t-\varepsilon} (t-s)^{-2\beta-2} ds + \int_{t-\varepsilon}^t (t-s)^{-2\beta} ds.$$

After some calculation we get

$$\mathbb{E}|B_t^\varepsilon - B_t|^2 \leq C_2(\beta\varepsilon^{1-2}) \to 0, \text{ as } \varepsilon \to 0, \tag{1.35}$$

where $C_2(\beta)$ is a positive constant depending only on $\beta$.

From (1.30) and (1.35) we see that in both cases ($H > 1/2$ and $H < 1/2$) there is an estimation for $||B_t^\varepsilon - B_t||^2 = \mathbb{E}|B_t^\varepsilon - B_t|^2$ as follows:

$$||B_t^\varepsilon - B_t||^2 \leq C_3(\alpha)\varepsilon^{1+2\alpha}, \tag{1.36}$$

where $0 < \alpha < 1/2$ for $1/2 < H < 1$ and $-1/2 < \alpha < 0$, for $0 < H < 1/2$, and $C_3(\alpha) = \max\{C_1(\alpha), C_2(\beta)\}$ depending only on $\alpha(= -\beta)$.

The relation (1.36) is valid for every $t \geq 0$, so

$$\sup_{0\leq t\leq T} ||B_t^\varepsilon - B_t|| \leq C(\alpha)\varepsilon^{\frac{1}{2}+\alpha} \to 0, \text{as}, \varepsilon \to 0,$$

where $C(\alpha) = \sqrt{C_3(\alpha)}$ which proves that $B_t^\varepsilon \to B_t$ in $L_2(\Omega)$ uniformly with respect to $t \in [0, T]$. $\square$

Since in the case $1/2 < H < 1$ the fractional Brownian motion exhibits statistical long range dependency in the sense that $\rho_n := \mathbb{E}[B_1^H(B_{n+1}^H - B_n^H)] > 0$ for all $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} \rho_n = \infty$ (see Oksendal (2003)). Hence, in financial modeling, one usually assumes that $H \in (1/2, 1)$. Put $\alpha = 1/2 - H$. It is known that a fractional Brownian motion $B_t^H$ can be decomposed as follows:

$$B_t^H = \frac{1}{\Gamma(1-\alpha)}\Big\{Z_t + \int_0^t (t-s)^{-\alpha} dW_s\Big\},$$

where $\Gamma$ is the gamma function,

$$Z_t = \int_{-\infty}^0 [(t-s)^{-\alpha} - s^{-\alpha}] dW_s,$$

and $W = (W_t)_{t \geq 0}$ is a standard Brownian motion. We suppose from now on that $0 < \alpha < 1/2$. Then $Z_t$ has absolutely continuous trajectories and it is the term

$$B_t = \int_0^t (t-s)^{-\alpha} dW_s, \tag{1.37}$$

that exhibits long range dependence. We will use $B_t$ instead of $B_t^H$ in fractional stochastic calculus.

Note that $B_t$ can be approximated by

$$B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{-\alpha} dW_s, \tag{1.38}$$

in the sense that $B_t^{\varepsilon}$ converges to $B_t$ in $L_p(\Omega)$ as $\varepsilon \to 0$ for any $p \geq 2$, uniformly with respect to $t \in [0, T]$ (Theorem 1.8 in case $p = 2$, other case see Dung (2007)).

Since $(B_t^{\varepsilon})_{t \in [0,T]}$ is a continuous semimartingale then the Itô calculus can be applied to the following stochastic differential equation (SDE)

$$dS_t^{\varepsilon} = S_t^{\varepsilon}(\mu dt + \sigma dB_t^{\varepsilon}), \;\; 0 \leq t \leq T.$$

Let $S_t^\varepsilon$ be the solution of the above SDE. Because of the convergence of $B_t^\varepsilon$ to the process $B_t$ in $L_2(\Omega)$ as $\varepsilon \to 0$, we shall define a solution of a fractional stochastic differential equation of the form

$$dS_t = S_t(\mu dt + \sigma dB_t),\ 0 \leq t \leq T,$$

to be a process $S_t^*$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the process $S_t^\varepsilon$ converges to $S_t^*$ in $L_2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to $t \in [0, T]$. This definition will be applied to the other similar fractional stochastic differential equations which will appear later. However, the following subsection will be explain more detail of an approximation approach to stochastic integration with respect to fractional Brownian motion.

### 1.4.7 An Approximation Approach to Fractional Stochastic Integration

The following definition and theorem can be found in Thao (2003). Let a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\geq 0}, \mathbb{P})$ be given where $\mathcal{F}_t^W$ is the $\sigma$-algebra generated by standard Brownian motion $(W_t)_{t\geq 0}$. Suppose that $f(t)$ is a deterministic function of bounded variation on $[0, T]$ and the fractional process $B_t$ is given as in (1.38):

$$B_t \ = \ \int_0^t (t-s)^\alpha \ W_s,\ \alpha \ = \ H - \frac{1}{2},\ 0 < H < 1.$$

Then the integral $\int_0^t B_s \ df(s)$ is well defined in the sense of Riemann-Stieltjes for almost all $\omega$.

**Definition 1.2.** *The fractional stochastic integral of $f(t)$ is a stochastic process $I_t$ defined as*

$$I_t := \int_0^t f(s) dB_s = f(t)B_t - \int_0^t B_s df(s).$$

Now suppose $(f(t,\omega))_{t\geq 0}$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ whose sample paths are of bounded variation on $[0,T]$ for almost every $\omega \in \Omega$.

**Definition 1.3.** *The fractional stochastic integral of $f(t,\omega)$ is a stochastic process $I(t)$ defined as*

$$I_t = \int_0^t f(s,\omega)dB(s) = f(t,\omega)B_t - \int_0^t B_s df(s,\omega) - [f,B]_t, \tag{1.39}$$

*where the notation $[.,.]$ stands for the quadratic variation of two processes given by a limit in probability:*

$$[f,B]_t = \mathbb{P} - \lim_{\max\ |t_{k+1}-t_k|\to 0} \sum_{k=0}^{n-1} \left[f(t_{k+1}) - f(t_k)\right]\left[B_{t_{k+1}} - B_{t_k}\right],$$

*for all partitions $\{0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots < t_n = T\}$ of $[0,T]$.*

**Remark 1.4.** $(i)$ *The pathwise integral on the right hand side of (1.39) exists in the sense of Riemann-Stieltjes for almost all $\omega$.*
$(ii)$ *If the function $f(t,\omega)$ has absolutely continuous sample paths (for instance, if it is Lipschitzian with respect to $t$) then it is of bounded variation and so its integral $I(t) = \int_0^t f(s,\omega)dB_s$ exists.*

**Theorem 1.9.**
*Suppose that the process $f(t,\omega)$ has continuous sample paths of bounded variation on $[0,T]$ such that $\mathbb{E}\int_0^T f^2(s,\omega)ds < \infty$. Then the stochastic integral*

$$I_t^{\varepsilon} = \int_0^t f(s,\omega)\ B_s^{\varepsilon},$$

*where $B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{\alpha} dW_s$, $\alpha = H - \frac{1}{2}$, $0 < H < 1$, converges in $L_2(\Omega)$ as $\varepsilon \to 0$ to $I_t = \int_0^t f(s,\omega)dB_s$ defined as in (1.39). This convergence is uniform with respect to $t \in [0,T]$.*

*Proof.* (Saelim (2004)). For a revised of this proof, see Dung (2010). Since

$$\begin{aligned}\mathbb{E}(B_t^\varepsilon)^2 &= \mathbb{E}\Big(\int_0^t (t-s+\varepsilon)^\alpha dW_s\Big)^2 = \mathbb{E}\int_0^t (t-s+\varepsilon)^{2\alpha} ds \\ &= -\frac{(t-s+\varepsilon)^{2\alpha+1}}{2\alpha+1}\bigg|_{s=0}^{s=t} = -\Big(\frac{\varepsilon^{2\alpha+1}}{2\alpha+1} - \frac{(t+\varepsilon)^{2\alpha+1}}{2\alpha+1}\Big) < \infty\end{aligned}$$

it follows from Theorem 1.8 that $B_t^\varepsilon$ is a square integrable martingale. Therefore the stochastic integral $I_t^\varepsilon = \int_0^t f(s,\omega)dB_t^\varepsilon$ exists. An application of the formula of integration by parts to $I_t^\varepsilon$ gives us

$$I_t^\varepsilon = \int_0^t f(s,\omega)dB_t^\varepsilon = f(t,\omega)B_t^\varepsilon - \int_0^t B_s^\varepsilon df(s,\omega) - [f, B^\varepsilon]_t.$$

Denote by $||\cdot||$ the norm in the space $L_2(\Omega)$ and taking account of properties of quadratic variations we have

$$\begin{aligned}||I_t - I_t^\varepsilon|| &= \bigg|\bigg|\int_0^t f(s,\omega)dB_s - \int_0^t f(s,\omega)dB_s^\varepsilon\bigg|\bigg| \\ &= \bigg|\bigg|\int_0^t f(s,\omega)d(B_s - B_s^\varepsilon)\bigg|\bigg| \\ &= \bigg|\bigg|f(s,\omega)(B_s - B_s^\varepsilon) - \int_0^t (B_s - B_s^\varepsilon)df(s,\omega) - [f, B - B^\varepsilon]_t\bigg|\bigg| \\ &\le ||f(s,\cdot)||\ ||B_t - B_t^\varepsilon|| + \bigg|\bigg|\int_0^t (B_s - B_s^\varepsilon)df(s,\omega)\bigg|\bigg| + ||[f, B - B^\varepsilon]_t||.\end{aligned}$$

An analogous argument as in the proof of Theorem 1.8 yields

$$\sup_{0\le t\le T} ||B_t - B_t^\varepsilon|| \le C\varepsilon^{\frac{1}{2}+\alpha}$$

where $\alpha = H - \frac{1}{2}$, $H \in (0,1)$ and $C > 0$ is some constant. Then

$$||f(t,\cdot)||\ ||B_t - B_t^\varepsilon|| \le MC\varepsilon^{\frac{1}{2}+\alpha} \tag{1.40}$$

where $M = \max_{0\le t\le T} ||f(t,\cdot)||$ (the maximum exists since $\mathbb{E}|f(t,\cdot)|^2$ is continuous with respect to $t \in [0,T]$). Moreover, we have

$$||[f, B - B^\varepsilon]_t|| \le ||f(t,\omega)||\ ||B_t - B_t^\varepsilon|| \le MC\varepsilon^{\frac{1}{2}+\alpha}. \tag{1.41}$$

On the other hand we see that

$$\begin{aligned}\left|\left|\int_0^t (B_s - B_s^\varepsilon) df(s,\omega)\right|\right| &\leq \left|\left|\int_0^t ||B_s - B_s^\varepsilon|| df(s,\omega)\right|\right| \\ &\leq C\varepsilon^{\frac{1}{2}+\alpha}\left|\left|\int_0^t df(s,\omega)\right|\right| \\ &\leq C\varepsilon^{\frac{1}{2}+\alpha}\big(||f(t,\cdot)|| + ||f(0,\cdot)||\big) \\ &\leq 2CM\varepsilon^{\frac{1}{2}+\alpha}.\end{aligned} \tag{1.42}$$

It follows from (1.40), (1.41) and (1.42) that

$$||I_t - I_t^\varepsilon|| \leq 4CM\varepsilon^{\frac{1}{2}+\alpha}.$$

Hence

$$\sup_{0\leq t\leq T} ||I_t - I_t^\varepsilon|| \leq 4CM\varepsilon^{\frac{1}{2}+\alpha} \to 0, \text{ as } \varepsilon \to 0.$$

Therefore, $I_t \to I_t^\varepsilon$ in $L_2(\Omega)$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0,T]$. $\square$

**Remark 1.5.** *Theorem 1.9 is proved for the $L_2$-convergence of $I_t^\varepsilon \to I_t$ in the case that $f$ is of bounded variation. This motivates us to define the fractional stochastic integral for any stochastic process $f(t,\omega)$ as follows.*

**Definition 1.6.** *Let $f(t,\omega)$ be a stochastic process with continuous paths. Then the fractional stochastic integral of $f(t,\omega)$ is defined by*

$$\int_0^t f(s,\omega)dB_s \ := \ L^2 - \lim_{\varepsilon\to 0}\int_0^t f(s,\omega)dB_s^\varepsilon,$$

*whenever the limit exists in $L_2(\Omega,\mathcal{F},\mathbb{P})$, where $B_t = \int_0^t (t-s)^{H-\frac{1}{2}}dW$ and $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}}dW_s$ for $0 < H < 1$.*

# CHAPTER II

# A FRACTIONAL STOCHASTIC VOLATILITY WITH JUMPS

## 2.1 Introduction

Assume that under the real-world or physical probability measure $\mathbb{P}$, a *geometric Brownian motion* (know as *Black-Scholes model* or a *diffusion model*) is the model of the form

$$dS_t = S_t(\mu dt + \sigma dW_t),\ t \in [0, T] \text{ and } T < \infty,$$

where $\mu \in \Re$, $\sigma > 0$, $S = (S_t)_{t\in[0,T]}$ is a process representing the price of the underling assets, and $(W_t)_{t\in[0,T]}$ is standard Brownian motion.

Over the last decade, many academic researchers have tried to extend and improve the classical geometric Brownian motion model in various directions. Some researchers represent rare events by jumps and introduce a model of jump diffusion (see Merton (1976) and Kou (2002)). Other authors try to provide a more realistic stochastic process for the underlying process (e.g. stock price) by introducing a stochastic process for the volatility, i.e. with the variance of the stock return as random. See, for example, Hull and White (1987), Stein and Stein (1991) and Heston (1993). Since there is an empirical study showing that the behaviour of stock price exhibits a long-range dependence, Thao (2006) replaced Browian motions (Bm) by fractional Brownian motions (fBm) in the diffusion model. Moreover, Sattayatham et. al., (2007) extended Thao's results by adding a Poisson jump into the model. In this thesis, we shall extend our investigations by replacing a Poisson jump by a compound Poisson jump and assuming that the variance of the stock return follows a fractional stochastic process.

In Section 2.2 an alternative stock price model is proposed in which the stock prices follow geometric Brownian motion (gBm) with compound Poisson jumps and fractional stochastic volatility. The convergence theorem of the approximate solution to the limit process is established in Section 2.3.

## 2.2 Description of the Model

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which are defined two standard Brownian motions $(W_t)_{t\geq 0}$ and $(\overline{W}_t)_{t\geq 0}$ with correlation $\rho$ and a compound Poisson process $(Z_t)_{t\geq 0}$ with intensity $\lambda$ and Gaussian distribution of jump sizes.

We assume that the $\sigma$-algebras generated respectively by $(Z_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$, and $(\overline{W}_t)_{t\geq 0}$ are independent.

Suppose that a single stock price $S_t$ and its volatility $v_t = \sigma_t^2$ satisfy the following stochastic differential equations:

$$dS_t = S_t(\mu dt + \sqrt{v_t} dW_t) + S_{t-} dZ_t \tag{2.1}$$

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t \tag{2.2}$$

with initial condition $S_{t(t=0)} = S_0 \in L_2(\Omega)$ and $v_{t(t=0)} = v_0 \in L_2(\Omega)$, where $\mu$ is the (deterministic) instantaneous drift of stock price returns, $\omega$ is the mean long-term volatility, $\theta$ is the rate at which the volatility reverts toward its long-term mean, and $\xi$ is the volatility of the volatility process.

The notation $S_{t-}$ means that whenever there is a jump, the value of the process before the jump is used on the left-hand side of the formula.

The last term of equation (2.1) is just a symbol. More precisely, it can be defined by a stochastic integral with respect to the Poisson random measure $N(\omega, \cdot)$

as the sum of jumps of a Poisson process $N_t$,

$$\int_0^t S_{s-} dZ_s = \int_0^t Y_s S_{s-} dN_s = \sum_{n=1}^{N_t} \Delta S_{T_n},$$

where $Y_t$ is the random jump amplitude.

We now assume that the $T_n$'s correspond to the jump times of a Poisson process $N_t$ and the $Y_n$ is a sequence of identically distributed random variables with values in $(-1, \infty)$. Let $S_t$ be a predictable process. At time $T_n$ the jump of the dynamics of $S_t$ is given by

$$\Delta S_{T_n} := S(T_n) - S(T_n-) = Y_n S(T_n-)$$

which, by the assumption $Y_n > -1$, leads always to positive values of the prices.

To solve equation (2.1), let us rewrite it into an integral form as follows:

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sqrt{v_s} S_s dW_s + \sum_{n=1}^{N_t} \Delta S_{T_n}. \tag{2.3}$$

Assume $\mathbb{E}[\int_0^T v_t S_t^2 dt] < \infty$. Then, by an application of Itô's lemma for the jump-diffusion process (see Lemma 1.2, Section 1.3) to equation (2.3) with $f(S_t, t) = \log(S_t)$, one gets

$$\log S_t = \log S_0 + \mu t - \frac{1}{2} \int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1 + Y_s) dN_s,$$

or, equivalently,

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2} \int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1 + Y_s) dN_s\right).$$

Since in may problems related to network traffic analysis, mathematical finance, and many other fields, the processes under study seem empirically to exhibit long-range dependent properties, their dynamics should be driven by a fractional Brownian process. Hence, instead of (2.2), we consider the fractional version of (2.2):

$$dS_t = S_t(\mu dt + \sqrt{v_t} dW_t) + S_{t-} Y_t dN_t \tag{2.4}$$

$$dv_t = (\omega - \theta v_t)dt + \xi v_t dB_t \tag{2.5}$$

where $B_t$ is as given in equation (1.37).

The corresponding approximately fractional model can be defined, for each $\varepsilon > 0$, by

$$dS_t^\varepsilon = S_t^\varepsilon(\mu dt + \sqrt{v_t^\varepsilon} dW_t) + S_{t-}^\varepsilon Y_t dN_t \tag{2.6}$$

$$dv_t^\varepsilon = (\omega - \theta v_t^\varepsilon)dt + \xi v_t^\varepsilon dB_t^\varepsilon \tag{2.7}$$

where $B_t^\varepsilon$ is as given in equation (1.38).

In the paper of Plienpanich at el. (2009) the solution of the approximate model (2.7) with initial condition $v_{t(t=0)} = v_0 \in L_2(\Omega)$ is given by

$$v_t^\varepsilon = \left(v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_t^\varepsilon) ds\right) \exp(\xi B_t^\varepsilon - \chi_\varepsilon t) \tag{2.8}$$

where $\chi_\varepsilon = \theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}$, $\varepsilon > 0, \alpha \in (0, 1/2)$, and $\theta, \xi$ are real constants.

Assume $\mathbb{E}[\int_0^T v_t^\varepsilon (S_t^\varepsilon)^2 dt] < \infty$. Using Itô's lemma for the jump-diffusion process again, the solution of the approximate model (2.6) is given by

$$\log S_t^\varepsilon = \log S_0 + \mu t - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1 + Y_s) dN_s,$$

or, equivalently,

$$S_t^\varepsilon = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1 + Y_s) dN_s\right).$$

## 2.3 Convergence of a Solution of an Approximate Model

Before stating the main theorem concerning the convergence $S_t^\varepsilon$ to a random variable $S_t^* \in L_2(\Omega)$ as $\varepsilon \to 0$, we first prove the convergence of the process $(v_t^\varepsilon : \varepsilon > 0)$ in $L_r(\Omega)$ as $\varepsilon \to 0$. Denoting the norm in $L_r(\Omega)$ by $\|\cdot\|_r$ where $r \in [1, \infty)$.

**Lemma 2.1.** *Let $p \in [1, \infty)$ and suppose that $v_0(\cdot)$ is a random variable such that $\mathbb{E}|v_0|^p < \infty$. Then for every $1 \le r \le p$, $\|v_t^\varepsilon\|_r \in L_r(\Omega)$ for all $t \in [0, T]$.*

*Proof.* Pick $q$ satisfy $1/r = 1/q + 1/q$. Then $q > 1$. Using the fact that

$$\|fg\|_r \le \|f\|_p \|g\|_q$$

where $p, q, r \in [0, \infty)$ and $1/r = 1/p + 1/q$ (see Jones (1993)), we have

$$\begin{aligned}
\|v_t^\varepsilon\|_r &= \left\| \left( v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \right) \exp(\xi B_t^\varepsilon - \chi_\varepsilon t) \right\|_r \\
&\le \|v_0 \exp(\xi B_t^\varepsilon - \chi_\varepsilon t)\|_r + |\omega| \left\| \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \exp(\xi B_t^\varepsilon - \chi_\varepsilon t) \right\|_r \\
&\le \|v_0\|_p \|\exp(\xi B_t^\varepsilon - \chi_\varepsilon t)\|_q + |\omega| \left\| \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \right\|_p \|\exp(\xi B_t^\varepsilon - \chi_\varepsilon t)\|_q \\
&\le \|v_0\|_p \Big( \exp \|\xi B_t^\varepsilon - \chi_\varepsilon t\|_q \Big) + |\omega| \int_0^t \|\exp(\chi_\varepsilon s - \xi B_s^\varepsilon)\|_p ds \; \|\exp(\xi B_t^\varepsilon - \chi_\varepsilon t)\|_q \\
&\le (\mathbb{E}|v_0|^p)^{1/p} \exp(-\chi_\varepsilon t) e^{q\xi^2\gamma_\varepsilon^2(t)/2} + |\omega| t \exp(\chi_\varepsilon t) e^{p\xi^2\gamma_\varepsilon^2(t)/2} \exp(-\chi_\varepsilon t) e^{q\xi^2\gamma_\varepsilon^2(t)/2} \\
&< \infty, \text{ for all } t \in [0, T].
\end{aligned}$$

Here, we have used the following computation of $\|\exp(\xi B_t^\varepsilon - \chi_\varepsilon t)\|_q$. Note that $B_t^\varepsilon$ is a Gaussian process with zero mean and finite variance. Let $\gamma_\varepsilon^2(t)$ be the variance of $B_t^\varepsilon$; we get $\gamma_\varepsilon^2(t) = \mathbb{E}|B_t^\varepsilon|^2 = \frac{(t+\varepsilon)^{2\alpha+1} - \varepsilon^{2\alpha+1}}{2\alpha+1}$ (see Dung (2007)). Hence

$$\begin{aligned}
\|\exp(\xi B_t^\varepsilon - \chi_\varepsilon t)\|_q &= \exp(-\chi_\varepsilon t) \Big[ \mathbb{E} \Big| e^{\xi B_t^\varepsilon} \Big|^q \Big]^{1/q} = \exp(-\chi_\varepsilon t) \Big[ \mathbb{E}(e^{q\xi B_t^\varepsilon}) \Big]^{1/q} \\
&= \exp(-\chi_\varepsilon t) \left[ \frac{1}{\sqrt{2\pi}\gamma_\varepsilon(t)} \int_{-\infty}^{\infty} e^{q\xi z} e^{\frac{-z^2}{2\gamma_\varepsilon^2(t)}} dz \right]^{1/q} \\
&= \exp(-\chi_\varepsilon t) \left[ \frac{1}{\sqrt{2\pi}\gamma_\varepsilon(t)} \int_{-\infty}^{\infty} e^{\frac{1}{2}q^2\xi^2\gamma_\varepsilon^2(t)} e^{\frac{-(z^2 - 2q\xi\gamma_\varepsilon^2(t)z + q^2\xi^2\gamma_\varepsilon^4(t))}{2\gamma_\varepsilon^2(t)}} dz \right]^{1/q} \\
&= \exp(-\chi_\varepsilon t) \left[ e^{q^2\xi^2\gamma_\varepsilon^2(t)/2} \frac{1}{\sqrt{2\pi}\gamma_\varepsilon(t)} \int_{-\infty}^{\infty} e^{\frac{-(z - q\xi\gamma_\varepsilon^2(t))^2}{2\gamma_\varepsilon^2(t)}} dz \right]^{1/q} \\
&= \exp(-\chi_\varepsilon t) e^{q\xi^2\gamma_\varepsilon^2(t)/2} < \infty, \text{ for all } t \in [0, T].
\end{aligned}$$

The other expressions can be computed similarly. This proves the lemma. □

**Lemma 2.2.** *Let $p \geq 2$. Suppose that $v_0(\cdot)$ is a random variable such that $\mathbb{E}|v_0|^p < \infty$. Then the process $(v_t^\varepsilon : \varepsilon > 0)$ converges to $v_t$ in $L_r(\Omega)$ as $\varepsilon \to 0$ for each $1 \leq r \leq p$. This convergence is uniform with respect to $t \in [0, T]$.*

*Proof.* Pick $q \geq 2$ satisfy $1/p + 1/q = 1/r$. Define a process $(v_t)_{t\in[0,T]}$ as follows:

$$v_t = \left(v_0 + \omega \int_0^t \exp(\theta s - \xi B_s) ds\right) \exp(\xi B_t - \theta t),$$

where all the parameters are as in equation (2.8).

Using the fact that $B_t$ is a Gaussian process with zero mean and finite variance $\gamma_t^2 := \mathbb{E}|B_t|^2 = \frac{t^{2\alpha+1}}{2\alpha+1}$, as in the proof of Lemma 2.1 one shows that $v_t \in L_r(\Omega)$. Next we compute

$$\begin{aligned}
v_t^\varepsilon - v_t &= \left(v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds\right) \exp(\xi B_t^\varepsilon - \chi_\varepsilon t) \\
&\quad - \left(v_0 + \omega \int_0^t \exp(\theta s - \xi B_s) ds\right) \exp(\xi B_t - \theta t) \\
&= v_0[\exp(\xi B_t^\varepsilon - \chi_\varepsilon t) - \exp(\xi B_t - \theta t)] \\
&\quad + \omega\left[\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \exp(\xi B_t^\varepsilon - \chi_\varepsilon t)\right. \\
&\qquad \left. - \int_0^t \exp(\theta s - \xi B_s) ds \exp(\xi B_t - \theta t)\right].
\end{aligned}$$

The second expression of the last equation is equal to

$$\begin{aligned}
&\omega\left(\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \exp(\xi B_t^\varepsilon - \chi_\varepsilon t) - \int_0^t \exp(\theta s - \xi B_s) ds \exp(\xi B_t - \theta t)\right) \\
&= \omega\left(\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \exp(\xi B_t^\varepsilon - \chi_\varepsilon t) - \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \exp(\xi B_t - \theta t)\right) \\
&+ \omega\left(\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds \exp(\xi B_t - \theta t) - \int_0^t \exp(\theta s - \xi B_s) ds \exp(\xi B_t - \theta t)\right) \\
&= \omega\left[\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds\right]\left[\big(\exp(\xi B_t - \theta t)\big)\left[\exp\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\right]\right] \\
&+ \omega\left[\int_0^t \big(\exp(\theta s - \xi B_s)\big)\left[\exp\big(\xi(B_s - B_s^\varepsilon) + s(\chi_\varepsilon - \theta)\big) - 1\right] ds\right] \exp(\xi B_t - \theta t).
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&v_t^\varepsilon - v_t = v_0\big(\exp(\xi B_t - \theta t)\big)\big[\exp\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\big] \\
&+\omega\Big[\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon)ds\Big]\Big[\big(\exp(\xi B_t - \theta t)\big)\big[\exp\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\big]\Big] \\
&+\omega\Big[\int_0^t \big(\exp(\theta s - \xi B_s)\big)\big[\exp\big(\xi(B_s - B_s^\varepsilon) + s(\chi_\varepsilon - \theta)\big) - 1\big]ds\Big]\exp(\xi B_t - \theta t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|v_t^\varepsilon - v_t\|_r \\
&\le \Big\| v_0\big(\exp(\xi B_t - \theta t)\big)\big[\exp\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\big]\Big\|_r \\
&+ \Big\|\omega\Big[\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon)ds\Big]\Big[\big(\exp(\xi B_t - \theta t)\big)\big[\exp\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\big]\Big]\Big\|_r \\
&+ \Big\|\omega\Big[\int_0^t \big(\exp(\theta s - \xi B_s)\big)\big[\exp\big(\xi(B_s - B_s^\varepsilon) + s(\chi_\varepsilon - \theta)\big) - 1\big]ds\Big]\exp(\xi B_t - \theta t)\Big\|_r \\
&\le \|v_0\|_p \Big\|\exp(\xi B_t - \theta t)\Big\|_{2q}\Big\|\exp\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\Big\|_{2q} \\
&+ |\omega|\Big\|\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon)ds\Big\|_p \Big\|\exp(\xi B_t - \theta t)\Big\|_{2q}\big\|\exp(\xi\big(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\big\|_{2q} \\
&+ |\omega|\Big\|\int_0^t \big(\exp(\theta s - \xi B_s)\big)\big[\exp\big(\xi(B_s - B_s^\varepsilon) + s(\chi_\varepsilon - \theta)\big) - 1\big]ds\Big\|_p \big\|\exp(\xi B_t - \theta t)\big\|_q.
\end{aligned} \tag{2.9}
$$

We aim to prove that $\|v_t^\varepsilon - v_t\|_r \to 0$ in $L_r(\Omega)$ as $\varepsilon \to 0$. To do this we note that $\|v_0\|_p = (\mathbb{E}|v_0^p|)^{1/2} < \infty$ and, recalling that $B_t$ is the Gaussian process with zero mean and finite variance $\gamma_t^2$, then we have

$$
\begin{aligned}
\|\exp(\xi B_t - \theta t)\|_{2q} &= \exp(-\theta t)\Big[\mathbb{E}\Big|e^{\xi B_t}\Big|^{2q}\Big]^{1/2q} \\
&= \exp(-\theta t)\Big[\mathbb{E}\,\Big|\,e^{2q\xi B_t}\,\Big|^{1/2q}\Big] \\
&= \exp(-\theta t)\Big[\frac{1}{\sqrt{2\pi}\gamma_t}\int_{-\infty}^{\infty} e^{2q\xi z}e^{\frac{-z^2}{2\gamma_t^2}}\,dz\Big]^{1/2q} \\
&= \exp(-\theta t)\Big[\frac{1}{\sqrt{2\pi}\gamma_t}\int_{-\infty}^{\infty} e^{2q^2\xi^2\gamma_t^2}e^{\frac{-(z^2-4q\xi\gamma_t^2 z+4q^2\xi^2\gamma^4 t)}{2\gamma_t^2}}\,dz\Big]^{1/2q} \\
&= \exp(-\theta t)\Big[e^{2q^2\xi^2\gamma_t^2}\frac{1}{\sqrt{2\pi}\gamma_t}\int_{-\infty}^{\infty} e^{\frac{-(z-2q\xi\gamma_t^2)^2}{2\gamma_t^2}}\,dz\Big]^{1/2q} \\
&= \exp(-\theta t)e^{q\xi^2\gamma_t^2} \le M < \infty, \text{ for some } M \text{ and for all } t \in [0,T].
\end{aligned}
$$

Next, it follows from the relation $e^{\|A\|_{2q}} - 1 = \|A\|_{2q} + o(\|A\|_{2q})$ that

$$\begin{aligned}\|\exp\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\|_{2q} &\leq \|\big(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta)\big) - 1\|_{2q} + R_1 \\ &\leq |\xi|\|(B_t^\varepsilon - B_t)\|_{4q} + |t|\|\chi_\varepsilon - \theta\|_{4q} + R_1 \qquad (2.10)\\ &\leq |\xi|\|(B_t^\varepsilon - B_t)\|_{4q} + t\big(\frac{1}{2}\xi^2\varepsilon^{2\alpha}\big) + R_1\end{aligned}$$

where $R_1 = o\Big(\|(\xi(B_t^\varepsilon - B_t) - t(\chi_\varepsilon - \theta))\|_{2q}\Big)$.

Since $\|(B_t^\varepsilon - B_t)\|_{4q} \to 0$ as $\varepsilon \to 0$ uniformly in $t \in [0, T]$ then the right hand side of equation (2.10) approaches zero as $\varepsilon \to 0$, after which we seen that the first expression of the equation (2.9) approaches zero as $\varepsilon \to 0$.

For the second expression of equation (2.9), we note that

$$\begin{aligned}\Big\|\int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds\Big\|_p &\leq \int_0^t \|\exp(\chi_\varepsilon s - \xi B_s^\varepsilon)\|_p ds \\ &= t\exp\big(\chi_\varepsilon t + p\xi^2\gamma_\varepsilon^2(t)/2\big) \leq M < \infty,\end{aligned}$$

for some $M$ and for all $t \in [0, T]$. Since (2.10) approaches zero as $\varepsilon \to 0$, the second expression of equation (2.9) approaches zero as $\varepsilon \to 0$ uniformly in $t \in [0, T]$.

Finally, for the third expression of equation (2.9), we have

$$\begin{aligned}&\Big\|\int_0^t \big(\exp(\theta s - \xi B_s)\big)\big[\exp\big(\xi(B_s - B_s^\varepsilon) + s(\chi_\varepsilon - \theta)\big) - 1\big] ds\Big\|_p \\ &\leq \int_0^t \Big\|\big(\exp(\theta s - \xi B_s)\big)\big[\exp\big(\xi(B_s - B_s^\varepsilon) + s(\chi_\varepsilon - \theta)\big) - 1\big]\Big\|_p ds \\ &\leq \int_0^t \big\|\exp(\theta s - \xi B_s)\big\|_{2p}\big\|\exp\big(\xi(B_s - B_s^\varepsilon) + s(\chi_\varepsilon - \theta)\big) - 1\big\|_{2p} ds \\ &\leq \big[\big\|\xi(B_t - B_t^\varepsilon) + t(\chi_\varepsilon - \theta)\big\|_{2p} + R_2\big] \times \int_0^t \big\|\exp(\theta s - \xi B_s)\big\|_{2p} ds \\ &\leq \big[|\xi|\|B_t - B_t^\varepsilon\|_{4p} + |t|\|\chi_\varepsilon - \theta\|_{4p} + R_2\big] \times \int_0^t \big\|\exp(\theta s - \xi B_s)\big\|_{2p} ds \\ &\leq te^{(\theta t + p\xi^2\gamma_t^2)}\Big[|\xi|\|(B_t - B_t^\varepsilon)\|_{4p} + t\big(\frac{1}{2}\xi^2\varepsilon^{2\alpha}\big) + R_2\Big] \to 0\end{aligned}$$

as $\varepsilon \to 0$ uniformly in $t \in [0, T]$ where $R_2 = o\big(\big\|\xi(B_t - B_t^\varepsilon) - s(\chi_\varepsilon - \theta)\big\|_{2p}\big)$. Thus the third expression of equation (2.9) approaches zero as $\varepsilon \to 0$, uniformly in $t \in [0, T]$.

Consequently, all expressions of the right hand side of equation (2.9) approach zero as $\varepsilon \to 0$. Therefore, $v_t^\varepsilon \to v_t$ in $L_r(\Omega)$ as $\varepsilon \to 0$ and this convergence is uniform with respect to $t \in [0,T]$. □

Now we ready to state and prove our main results. The solution of the approximated model (2.6) is given by

$$S_t^\varepsilon = S_0 \exp\Big(\mu t - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1+Y_s)dN_s\Big). \tag{2.11}$$

Define a stochastic process $S_t^*$ as follows:

$$S_t^* = S_0 \exp\Big(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s)dN_s\Big). \tag{2.12}$$

The following theorem shows that the process $S_t^*$ is the limit process $S_t^\varepsilon$ in $L_2(\Omega)$ as $\varepsilon \to 0$. Hence, by definition, $S_t^*$ will be the solution of equation (2.4)

**Theorem 2.3.** *Suppose that $S_0(\cdot)$ is a non-negative random variable such that $\mathbb{E}|S_0|^4$ is finite, and $v_0(\cdot) \neq 0$ a.s. The stochastic process $S_t^\varepsilon$ of equation (2.11) converges to the limit process $S_t^*$ in $L_2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to $t \in [0,T]$ whenever $0 < \alpha < 1/2$.*

*Proof.* It follows from equations (2.11) and (2.12) that

$$S_t^\varepsilon - S_t^* = S_0\Big(\frac{S_t^*}{S_0}\Big)\Big[\exp\Big(-\frac{1}{2}\int_0^t (v_s^\varepsilon - v_s)ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s})dW_s\Big) - 1\Big]$$

Then

$$\begin{aligned} \|S_t^\varepsilon - S_t^*\|_2 &\le \|S_0\|_4 \Big\| \Big(\frac{S_t^*}{S_0}\Big)\Big[\exp\Big(-\frac{1}{2}\int_0^t (v_s^\varepsilon - v_s)ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s})dW_s\Big) - 1\Big]\Big\|_4 \\ &\le \|S_0\|_4 \Big\|\frac{S_t^*}{S_0}\Big\|_8 \Big\|\exp\Big(-\frac{1}{2}\int_0^t (v_s^\varepsilon - v_s)ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s})dW_s\Big) - 1\Big\|_8. \end{aligned} \tag{2.13}$$

The following three parts show an approximation of norm $\|S_t^\varepsilon - S_t^*\|_2$. in equation (2.13).

(i) $\|S_0\|_4 = \mathbb{E}|S_0|^4 < \infty$ by the assumptions of the theorem.

(ii) We note that

$$\left\|\frac{S_t^*}{S_0}\right\|_8 = \left\|\exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s) dN_s\right)\right\|_8$$
$$= \left\|\exp\left(\mu t + \int_0^t \log(1+Y_s) dN_s\right) \times \exp\left(-\frac{1}{2}\int_0^t v_s ds\right) \times \exp\left(\int_0^t \sqrt{v_s} dW_s\right)\right\|_8$$
$$\leq \left\|\exp\left(\mu t + \int_0^t \log(1+Y_s) dN_s\right)\right\|_{16} \times \left\|\exp\left(-\frac{1}{2}\int_0^t v_s ds\right)\right\|_{32}$$
$$\times \left\|\exp\left(\int_0^t \sqrt{v_s} dW_s\right)\right\|_{32} \tag{2.14}$$

In order to get an approximation of equation (2.14), we firstly note that

$$\begin{aligned}\left\|\exp\left(\mu t + \int_0^t \log(1+Y_s) dN_s\right)\right\|_{16} &= \left\|\exp(\mu t) \times \exp\left(\sum_{n=1}^{N_t} \log(1+Y_n)\right)\right\|_{16} \\ &= \left\|\sum_{n=1}^{N_t}(1+Y_n)\right\|_{16} \exp(\mu t) \\ &\leq K \exp(|\mu| T)\end{aligned}$$

where $K$ is a constant. The last inequality follows from the fact that there are a finite number of jumps in the finite interval $[0, T]$. Moreover,

$$\begin{aligned}\left\|\exp\left(-\frac{1}{2}\int_0^t v_s ds\right)\right\|_{32} &\leq \exp\left\|-\frac{1}{2}\int_0^t v_s ds\right\|_{32} \\ &\leq \exp\left(\frac{1}{2}M_1 T\right) < \infty\end{aligned}$$

where $M_1 := \sup_{0\leq t\leq T} \|v_t\|_{32}$. The maximum exists since $v_t \in L_{32}(\Omega)$ (by Lemma 2.1) and

$$\|v_t\|_{32}^{32} = \mathbb{E}\left|\left(\exp(\xi B_t - \theta t)\right)\left(v_0 + \omega\int_0^t \exp(\theta s - \xi B_s) ds\right)\right|^{32}$$

is continuous with respect to $t \in [0, T]$.

For the remaining term, we note that

$$\begin{aligned}\left\|\exp\left(\int_0^t \sqrt{v_s} dW_s\right)\right\|_{32} &\leq \exp\left(\left\|\int_0^t \sqrt{v_s} dW_s\right\|_{32}\right) \\ &\leq \exp\left(M_2 \left\|W_t - W_0\right\|_{32}\right) \\ &= \exp(M_2 M_3) < \infty\end{aligned}$$

where $M_2 := \sup_{0\leq t\leq T} \|\sqrt{v_t}\|_{32}$ and $M_3 := \sup_{0\leq t\leq T} \|W_t\|_{32}$.

The maximum exists since

$$\|W_t\|_{32}^{32} = \mathbb{E}|W_t|^{32} = \frac{32!}{2^{16}\cdot 16!}t^{16} < \infty,$$

and this expression is continuous with respect to $t \in [0,T]$.
Consequently, we see that $\left\|\frac{S_t^*}{S_0}\right\|_8$ is finite.

(iii) The third factor on the right hand side of equation (2.13) is calculated by using the relation $\exp(A) - 1 = A + o(A)$. So we have

$$\begin{aligned}
&\left\|\exp\left(-\frac{1}{2}\int_0^t (v_s^\varepsilon - v_s)ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s})dW_s\right) - 1\right\|_8 \\
&\le \left\|-\frac{1}{2}\int_0^t (v_s^\varepsilon - v_s)ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s})dW_s\right\|_8 + R_3 \\
&\le \frac{1}{2}\int_0^t \|v_s^\varepsilon - v_s\|_8 ds + \int_0^t \frac{\|v_s^\varepsilon - v_s\|_8}{\|\sqrt{v_s^\varepsilon} + \sqrt{v_s}\|_8}dW_s + R_3 \\
&\le \frac{1}{2}t\|v_s^\varepsilon - v_s\|_8 + \frac{\|v_s^\varepsilon - v_s\|_8}{\|\sqrt{v_s^\varepsilon} + \sqrt{v_s}\|_8}\|W_t - W_0\|_8 + R_3
\end{aligned}$$

where $R_3 = \left\|o\Big(-\frac{1}{2}\int_0^t (v_s^\varepsilon - v_s)ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s})dW_s\Big)\right\|_8.$
Hence,

$$\begin{aligned}
&\left\|\exp\left(-\frac{1}{2}\int_0^t (v_s^\varepsilon - v_s)ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s})dW_s\right) - 1\right\|_8 \\
&\qquad\qquad \le \frac{1}{2}t\|v_s^\varepsilon - v_s\|_8 + \frac{\|v_s^\varepsilon - v_s\|_8}{\|\sqrt{v_s^\varepsilon} + \sqrt{v_s}\|_8}\widehat{M} + R_3
\end{aligned} \tag{2.15}$$

where $\widehat{M} := \max_{0\le t\le T}\|W_t\|_8$. Note that $0 < c \le \|\sqrt{v_s^\varepsilon} + \sqrt{v_s}\|_8$ for all $s \in [0,t]$ since we assume that $v_0 \neq 0$. Hence, by Lemma 2.2, the right hand side of equation (2.15) approaches zero as $\varepsilon \to 0$.

Therefore $S_t^\varepsilon \to S_t^*$ in $L_2(\Omega)$ as $\varepsilon \to 0$. This convergence does not depend on $t$ and is hence uniform with respect to $t \in [0,T]$. $\square$

# CHAPTER III

# OPTION PRICING MODEL FOR A FRACTIONAL STOCHASTIC VOLATILITY WITH JUMPS

## 3.1 Introduction

The aim of this chapter is to compute a European call option of the approximate model given in Chapter II. In this thesis, however it is quite straightforward to get options by inverting the characteristic function of a given approximate model if it is known in an explicit form. The chapter is structured as follows. A risk-neutral for geometric Brownian motion (gBm) model with a compound Poisson process and stochastic volatility model is described in Section 3.2. A risk-neutral for a gBm model with compound Poisson process and fractional stochastic volatility model is also introduced in this section. The relationship between the stochastic differential equation and partial integro-differential equation (PIDE) for the jump diffusion process with stochastic volatility is presented in Section 3.3. This relationship will play the role of the main theorem. Section 3.4 discusses the problem and method to evaluate the European call option, based on the explicit knowledge of the characteristic function. Finally, a closed-form solution for a European call option in terms of characteristic function is formulated in Section 3.5.

## 3.2 Risk-Neutral for a Fractional Stochastic Volatility Model with Jumps

In this section, a risk-neutral for a fractional stochastic volatility model is introduced. Its solution will also be discussed in this section.

It is assumed that a risk-neutral probability measure $\mathcal{M}$ exists; the asset process $S_t$, under this risk-neutral measure, follows a jump-diffusion process, with zero-mean at risk-free rate $r$, and stochastic variance $v_t$,

$$dS_t = S_t\Big((r - \lambda\mathbb{E}_{\mathcal{M}}(Y_t))dt + \sqrt{v_t}dW_t\Big) + S_{t-}Y_t dN_t. \tag{3.1}$$

It is only necessary to know that the risk-neutral measure exists (see, Cont and Tankov (2004)). Hence, all processes to be discussed after this chapter will be processes under the risk-neutral probability measure $\mathcal{M}$.

Let us rewrite equation (3.1) into integral form as follows:

$$S_t = S_0 + \int_0^t (r - \lambda\mathbb{E}_{\mathcal{M}}(Y_s))S_s ds + \int_0^t \sqrt{v_s}S_s dW_s + \int_0^t S_{s-}Y_s dN_s. \tag{3.2}$$

Note that the last term on the right hand side of equation (3.2) is defined by

$$\int_0^t S_{s-}Y_s dN_s = \sum_{n=1}^{N_t} \Delta S_n,$$

where

$$\Delta S_n := S_{T_n} - S_{T_n -} = S_{n-}Y_n.$$

The assumption $Y_n > 0$ always leads to positive values of the stock prices. The process $(Y_n)_{n\in\mathbb{N}}$ is assumed to be independently and identically distributed (i.i.d.) with density $\phi_Y(y)$ and $(T_n)_{n\in\mathbb{N}}$ is a sequence of jump times.

Using an initial condition $S_{t(t=0)} = S_0 \in L_2(\Omega)$, its solution is given by

$$S_t = S_0 \exp\bigg( \int_0^t \Big( (r - \lambda \mathbb{E}_{\mathcal{M}}(Y_s)) - \frac{1}{2} v_s \Big) ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1 + Y_s) dN_s \bigg),$$

where $v_t$ satisfies the following fractional SDE

$$dv_t = (\omega - \theta v_t)dt + \xi v_t dB_t, \tag{3.3}$$

with an initial condition $v_{t(t=0)} = v_0 \in L_p(\Omega)$, where $p > 2$.

For each $\varepsilon < 0$, consider an approximation model of equation (3.3);

$$dv_t^\varepsilon = (\omega - \theta v_t^\varepsilon)dt + \xi v_t^\varepsilon dB_t^\varepsilon. \tag{3.4}$$

Using lemma 2.2, it follows that solution $v_t^\varepsilon$ of equation (3.4) converges in $L_2(\Omega)$ to the process

$$v_t = \left( v_0 + \omega \int_0^t \exp(\theta s - \xi B_s) ds \right) \exp(\xi B_t - \theta t),$$

that is the solution of equation (3.3).

Now we consider an approximate model of equation (3.1);

$$dS_t^\varepsilon = S_t^\varepsilon \big( (r - \lambda \mathbb{E}_{\mathcal{M}}(Y_t)) dt + \sqrt{v_t^\varepsilon} dW_t \big) + S_{t-}^\varepsilon Y_t dN_t, \tag{3.5}$$

and by using the same initial condition as in equation (3.2), we have

$$S_t^\varepsilon = S_0 \exp\bigg( \int_0^t \Big( (r - \lambda \mathbb{E}_{\mathcal{M}}(Y_s)) - \frac{1}{2} v_s^\varepsilon \Big) ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1 + Y_s) dN_s \bigg). \tag{3.6}$$

By theorem 2.3 the process $S_t^\varepsilon$ converges to $S_t$ in $L_2(\Omega)$ as $\varepsilon \to 0$ and uniformly on $t \in [0, T]$, under approximate assumptions on $S_0$ and $v_0$.

## 3.3 Partial Integro-Differential Equations for Jump Diffusion Model with Stochastic Volatility

This section provides a link between the partial differential equation of a jump diffusion process and its expectation.

Consider the process $\overrightarrow{X}_t = (X_t^1, X_t^2)$ where $X_t^1$ and $X_t^2$ are processes in $\Re$ and satisfiy the following equations:

$$dX_t^1 = f_1(t)dt + g_1(t)dW_t + X_{t-}^1 Y_t dN_t, \tag{3.7}$$

$$dX_t^2 = f_2(t)dt + g_2(t)d\overline{W}_t, \tag{3.8}$$

where $f_1,\ g_1,\ f_2$, and $g_2$ are all continuous functions from $[0, T]$ into $\Re$.

Since every compound Poisson process can be represented as an integral form of Poisson random measure (Cont and Tankov (2004)) then the last term on the right hand side of equation (3.7) can be written as follows

$$\int_0^t X_{s-}^1 Y_s dN_s = \sum_{n=1}^{N_t} X_{n-}^1 Y_n = \sum_{n=1}^{N_t} [X_{T_n}^1 - X_{T_n-}^1] = \int_0^t \int_{\Re} X_{s-}^1 z J_z(ds\ dz)$$

where $Y_n$ are i.i.d. random variables with density $\phi_Y(y)$ and $J_Z$ is a Poisson random measure of the process $Z_t = \sum_{n=1}^{N_t} Y_n$ with intensity measure $\lambda\phi_Y(dz)dt$.

Let $U(\overrightarrow{x})$ be a bounded real function on $\Re^2$ and twice continuously differentiable in $\overrightarrow{x} = (x_1, x_2) \in \Re^2$ and

$$u(t, \overrightarrow{x}) = \mathbb{E}[U(\overrightarrow{X}_T)|\overrightarrow{X}_t = \overrightarrow{x}]. \tag{3.9}$$

By the two dimensional Dynkin's formula (Hanson (2007), Theorem 7.7), $u$ is a solution of the partial integro-differetial equation (PIDE)

$$0 = \frac{\partial u(t, \overrightarrow{x})}{\partial t} + \mathcal{A}u(t, \overrightarrow{x}) + \lambda \int_{\Re} [u(t, \overrightarrow{x} + y) - u(t, \overrightarrow{x})]\phi_Y(y)dy,$$

subject to the final condition $u(T, \overrightarrow{x}) = U(\overrightarrow{x})$.

The notation $\mathcal{A}$ is defined by

$$\begin{aligned}\mathcal{A}(t,\overrightarrow{x}) =& f_1(t)\frac{\partial u(t,\overrightarrow{x})}{\partial x_1} + f_2(t)\frac{\partial u(t,\overrightarrow{x})}{\partial x_2} \\ &+ \frac{1}{2}g_1^2(t)\frac{\partial^2 u(t,\overrightarrow{x})}{\partial x_1^2} + \rho g_1(t)g_2(t)\frac{\partial^2 u(t,\overrightarrow{x})}{\partial x_1 \partial x_2} + \frac{1}{2}g_2^2(t)\frac{\partial^2 u(t,\overrightarrow{x})}{\partial x_2^2},\end{aligned}$$

and the correlation $\rho$ is defined by $\rho = \mathrm{Corr}[dW_t, d\overline{W}_t]$.

## 3.4 Pricing European Call Option

Let $C$ denote the price at time $t$ of a *European style call option* on the current price of the underlying asset $S_t$ with strike price $K$ and expiration time $T$.

The *terminal payoff of a European call option* on the underline stock $S_t$ with strike price $K$ is

$$\max(S_t - K, 0).$$

This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \le K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate $r$ is constant over the lifetime of the option, the price of the European call at time $t$ is equal to the discounted conditional expected payoff

$$\begin{aligned}&C(t,S_t,v_t;T,K) \\ &\quad = e^{-r(T-t)}\ \mathbb{E}_{\mathcal{M}}[\max(S_T-K,0)|S_t,v_t] \\ &\quad = e^{-r(T-t)}\left(\int_K^\infty S_T\ \mathbb{P}_{\mathcal{M}}(S_T|S_t,v_t)dS_T - K\int_K^\infty \mathbb{P}_{\mathcal{M}}(S_T|S_t,v_t)dS_T\right) \\ &\quad = S_t\left(\frac{1}{e^{r(T-t)}S_t}\int_K^\infty S_T\ \mathbb{P}_{\mathcal{M}}(S_T|S_t,v_t)dS_T\right) - Ke^{-r(T-t)}\int_K^\infty \mathbb{P}_{\mathcal{M}}(S_T|S_t,v_t)dS_T \\ &\quad = S_t\left(\frac{1}{\mathbb{E}_{\mathcal{M}}[S_T|S_t,v_t]}\ \int_K^\infty S_T\ \mathbb{P}_{\mathcal{M}}(S_T|S_t,v_t)dS_T\right) \\ &\qquad - Ke^{-r(T-t)}\int_K^\infty \mathbb{P}_{\mathcal{M}}(S_T|S_t,v_t)dS_T \\ &\quad = S_t\ \mathbb{P}_1(t,S_t,v_t;T,K) - Ke^{-r(T-t)}\ \mathbb{P}_2(t,S_t,v_t;T,K)\end{aligned} \tag{3.10}$$

where $\mathbb{E}_{\mathcal{M}}$ is the expectation with respect to the risk-neural probability measure, $\mathbb{P}_{\mathcal{M}}(S_T|S_t, v_t)$ is the corresponding conditional density given $(S_t, v_t)$, and

$$\mathbb{P}_1(t, S_t, v_t; T, K) = \Big(\int_K^\infty S_T \mathbb{P}_{\mathcal{M}}(S_T|S_t, v_t) dS_T\Big)/\mathbb{E}[S_T|S_t, v_t].$$

Note that as $\mathbb{P}_1$ is the risk-neutral probability then $S_T > K$ (since the integrand is nonnegative and the integral over $[0, \infty)$ is one), and finally, that

$$\mathbb{P}_2(t, S_t, v_t; T, K) = \int_K^\infty S_T \mathbb{P}_{\mathcal{M}}(S_T|S_t, v_t) dS_T = \mathrm{Prob}(S_T > K|S_t, v_t)$$

is the risk-neutral in-the-money probability. Moreover, $\mathbb{E}_{\mathcal{M}}[S_T|S_t, v_t] = e^{r(T-t)S_t}$ for $t \geq 0$.

Note that we do not have a closed form solution for these probabilities. However, these probabilities are related to characteristic functions which have closed form solutions as will be seen in Lemma 3.1.

We would like to compute the price of a European call option with strike price $K$ and maturity $T$ of the model (3.1) for which its fractional stochastic volatility satisfies equation (3.3).

To do this, consider the logarithm of $S_t^\varepsilon$, namely, $L_t^\varepsilon$, i.e. $L_t^\varepsilon = \log(S_t^\varepsilon)$ where $S_t^\varepsilon$ satisfies equation (3.6) and its inverse $S_t^\varepsilon = \exp(L_t^\varepsilon)$. Denote by $\kappa = \log(K)$ the logarithm of the strike price.

We now refer to equation (3.4), since this approximate model is driven by a semimartingale $B_t^\varepsilon$ and hence there is no opportunity of arbitrage. This is the advantage of our approximate approach and we will use this model for pricing the European call option instead of equation (3.3).

Note that we can write

$$dB_t^\varepsilon = \alpha\varphi_t^\varepsilon dt + \varepsilon^\alpha d\overline{W}_t \tag{3.11}$$

where $\varphi_t^\varepsilon = \int_0^t (t-u+\varepsilon)^{1-\alpha} dW_u$, $\alpha = 1/2 - H$ and $0 < \alpha < 1/2$ (Thao (2006), Lemma 2.1).

Substituting equation (3.11) into equation (3.3), we obtain

$$dv_t^\varepsilon = \big(\omega + (\alpha\xi\varphi_t^\varepsilon - \theta)v_t^\varepsilon\big)dt + \xi\varepsilon^\alpha v_t^\varepsilon d\overline{W}_t. \tag{3.12}$$

Consider the SDE (3.1) and (3.12). Define a function $U$ on $\Re^2$ as follows:

$$U(x_1, x_2) = e^{-r(T-t)} \max(e^{x_1} - \kappa, 0).$$

By virtue of equation (3.9),

$$\begin{aligned} u(t, S_t) &= \mathbb{E}_{\mathcal{M}}[U(\overrightarrow{X}_T)|\overrightarrow{X}_t = \overrightarrow{x}] \\ &= e^{-r(T-t)}\ \mathbb{E}_{\mathcal{M}}[\max\big(\exp(L_t^\varepsilon) - \kappa, 0\big)|L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon] \\ &:= C(t, \ell^\varepsilon, v^\varepsilon; T, \kappa) \end{aligned}$$

satisfies the following PIDE:

$$\begin{aligned} 0 =& \frac{\partial C}{\partial t} + f_1\frac{\partial C}{\partial \ell^\varepsilon} + f_2\frac{\partial C}{\partial v^\varepsilon} + \frac{1}{2}g_1^2\frac{\partial^2 C}{\partial(\ell^\varepsilon)^2} + \rho g_1 g_2\frac{\partial^2 C}{\partial\ell^\varepsilon\partial v^\varepsilon} + \frac{1}{2}g_2^2\frac{\partial^2 C}{\partial(v^\varepsilon)^2} \\ & - rC + \lambda\int_{\Re}\big[C(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - C(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\big]\phi_Y(y)dy. \end{aligned} \tag{3.13}$$

In the current state variables, the last line of the equation (3.10) becomes

$$C(t, \ell^\varepsilon, v^\varepsilon; T, \kappa) = e^{\ell^\varepsilon}\mathbb{P}_1(t, \ell^\varepsilon, v^\varepsilon; T, \kappa) - e^{\kappa - r(T-t)}\mathbb{P}_2(t, \ell^\varepsilon, v^\varepsilon; T, \kappa). \tag{3.14}$$

The following lemma shows the relationship between $\mathbb{P}_1$ and $\mathbb{P}_2$ in the option value of equation (3.14).

**Lemma 3.1.** *The functions $\mathbb{P}_1$ and $\mathbb{P}_2$ in the option value of the equation (3.14) satisfy the PIDEs:*

$$\begin{aligned}0 = &\frac{\partial \mathbb{P}_1}{\partial t} + A[\mathbb{P}_1](t, \ell^\varepsilon, v^\varepsilon; T, \kappa) + v^\varepsilon \frac{\partial \mathbb{P}_1}{\partial \ell^\varepsilon} + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial \mathbb{P}_1}{\partial v^\varepsilon} \\ &+ \big(r - \lambda \mathbb{E}_{\mathcal{M}}(Y_t)\big)\mathbb{P}_1 + \lambda \int_{\Re} \big[(e^y - 1)\mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa)\big]\phi_Y(y)dy,\end{aligned}$$

*subject to the boundary condition at expiration time $t = T$;*

$$\mathbb{P}_1(T, \ell^\varepsilon, v^\varepsilon; T, \kappa) = 1_{\ell^\varepsilon > \kappa}. \tag{3.15}$$

*Moreover, $\mathbb{P}_2$ satisfies the equaion*

$$0 = \frac{\partial \mathbb{P}_2}{\partial t} + A[\mathbb{P}_2](t, \ell^\varepsilon, v^\varepsilon; T, \kappa) + r\mathbb{P}_2,$$

*subject to the boundary condition at expiration time $t = T$;*

$$\mathbb{P}_2(T, \ell^\varepsilon, v^\varepsilon; T, \kappa) = 1_{\ell^\varepsilon > \kappa}, \tag{3.16}$$

*where*

$$\begin{aligned}A[f](t, \ell^\varepsilon, v^\varepsilon; T, \kappa) := &\left(r - \lambda \mathbb{E}_{\mathcal{M}}(Y_t) - \frac{1}{2}v^\varepsilon\right)\frac{\partial f}{\partial \ell^\varepsilon} + \big(\omega + (\alpha\xi\varphi_t^\varepsilon - \theta)v^\varepsilon\big)\frac{\partial f}{\partial v^\varepsilon} \\ &+ \frac{1}{2}v^\varepsilon \frac{\partial^2 f}{\partial (\ell^\varepsilon)^2} + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2}\frac{\partial^2 f}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2}\xi\varepsilon^{2\alpha}(v^\varepsilon)^2 \frac{\partial^2 f}{\partial (v^\varepsilon)^2} \\ &- rf + \lambda \int_{\Re} \big[f(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - f(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\big]\phi_Y(y)dy.\end{aligned} \tag{3.17}$$

Note that $1_{\ell^\varepsilon > \kappa} = 1$ if $\ell^\varepsilon > \kappa$ and otherwise $1_{\ell^\varepsilon > \kappa} = 0$.

*Proof.* We plan to substitute equation (3.14) into equation (3.13). Firstly we compute

$$\begin{aligned}\frac{\partial C}{\partial t} &= e^{\ell^\varepsilon}\frac{\partial \mathbb{P}_1}{\partial t} - e^{\kappa - r(T-t)}\frac{\partial \mathbb{P}_2}{\partial t} - re^{\kappa - r(T-t)}\mathbb{P}_2 \\ \frac{\partial C}{\partial \ell^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial \mathbb{P}_1}{\partial \ell^\varepsilon} + e^{\ell^\varepsilon}\mathbb{P}_1 - e^{\kappa - r(T-t)}\frac{\partial \mathbb{P}_2}{\partial \ell^\varepsilon} \\ \frac{\partial C}{\partial v^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial \mathbb{P}_1}{\partial v^\varepsilon} - e^{\kappa - r(T-t)}\frac{\partial \mathbb{P}_2}{\partial v^\varepsilon}\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 C}{\partial(\ell^\varepsilon)^2} &= e^{\ell^\varepsilon}\frac{\partial^2 \mathbb{P}_1}{\partial(\ell^\varepsilon)^2} + 2e^{\ell^\varepsilon}\frac{\partial^2 \mathbb{P}_1}{\partial^2 \ell^\varepsilon} + \mathbb{P}_1 e^{\ell^\varepsilon} - e^{\kappa - r(T-t)}\frac{\partial^2 \mathbb{P}_2}{\partial(\ell^\varepsilon)^2}\\
\frac{\partial^2 C}{\partial \ell^\varepsilon \partial v^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial^2 \mathbb{P}_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + e^{\ell^\varepsilon}\frac{\partial \mathbb{P}_1}{\partial v^\varepsilon} - e^{\kappa - r(T-t)}\frac{\partial^2 \mathbb{P}_2}{\partial \ell^\varepsilon \partial v^\varepsilon}\\
\frac{\partial^2 C}{\partial(v^\varepsilon)^2} &= e^{\ell^\varepsilon}\frac{\partial^2 \mathbb{P}_2}{\partial(v^\varepsilon)^2} - e^{\kappa - r(T-t)}\frac{\partial^2 \mathbb{P}_2}{\partial(v^\varepsilon)^2}
\end{aligned}$$

and

$$\begin{aligned}
C(t,&\ell^\varepsilon + y, v^\varepsilon; T, \kappa) - C(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\\
&= \Big[e^{(\ell^\varepsilon + y)}\ \mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - e^{\kappa - r(T-t)}\ \mathbb{P}_2(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa)\Big]\\
&\quad - \Big[e^{\ell^\varepsilon}\ \mathbb{P}_1(t, \ell^\varepsilon, v^\varepsilon; T, \kappa) - e^{\kappa - r(T-t)}\ \mathbb{P}_2(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\Big]\\
&= \Big\{e^{\ell^\varepsilon}\Big[e^y\ \mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - \mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa)\Big]\\
&\qquad\quad + \Big[e^{\ell^\varepsilon}\ \mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - e^{\ell^\varepsilon}\mathbb{P}_1(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\Big]\Big\}\\
&\quad - e^{\kappa - r(T-t)}\Big[\mathbb{P}_2(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - \mathbb{P}_2(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\Big]\\
&= e^{\ell^\varepsilon}\Big(e^y - 1\Big)\ \mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa)\\
&\quad + e^{\ell^\varepsilon}\Big[\mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - \mathbb{P}_1(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\Big]\\
&\quad - e^{\kappa - r(T-t)}\Big[\mathbb{P}_2(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - \mathbb{P}_2(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\Big].
\end{aligned}$$

We substitute all terms above into equation (3.13) and separate it by assumed independent terms of $\mathbb{P}_1$ and $\mathbb{P}_2$. This gives two PIDEs for the risk-neutralized probability for $\mathbb{P}_j(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)$, $j = 1, 2$:

$$\begin{aligned}
0 &= \frac{\partial \mathbb{P}_1}{\partial t} + \left(r - \lambda \mathbb{E}_{\mathcal{M}}(Y_t) - \frac{1}{2}v^\varepsilon\right)\left(\frac{\partial \mathbb{P}_1}{\partial \ell^\varepsilon} + \mathbb{P}_1\right) + \big(\omega + (\alpha\xi\varphi_t^\varepsilon - \theta)v^\varepsilon\big)\frac{\partial \mathbb{P}_1}{\partial v^\varepsilon}\\
&+ \frac{1}{2}v^\varepsilon\left(\frac{\partial^2 \mathbb{P}_1}{\partial(\ell^\varepsilon)^2} + 2\frac{\partial \mathbb{P}_1}{\partial \ell^\varepsilon} + \mathbb{P}_1\right) + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2}\left(\frac{\partial^2 \mathbb{P}_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{\partial \mathbb{P}_1}{\partial v^\varepsilon}\right) + \frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^\varepsilon)^2\frac{\partial^2 \mathbb{P}_1}{\partial(v^\varepsilon)^2}\\
&- r\mathbb{P}_1 + \int_{\Re}\Big[(e^y - 1)\ \mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa)\\
&\qquad\qquad + (\mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - \mathbb{P}_1(t, \ell^\varepsilon, v^\varepsilon; T, \kappa)\Big]\phi_Y(y)dy
\end{aligned} \tag{3.18}$$

subject to the boundary condition at the expiration time $t = T$ according to equation (3.15).

By using the notation (3.17), PIDE of (3.18) becomes

$$
\begin{aligned}
0 = \frac{\partial \mathbb{P}_1}{\partial t} &+ \Bigg( A[\mathbb{P}_1](t, \ell^\varepsilon, v^\varepsilon; T, \kappa) + v^\varepsilon \frac{\partial \mathbb{P}_1}{\partial \ell^\varepsilon} + \rho \xi {\varepsilon^\alpha} (v^\varepsilon)^{3/2} \frac{\partial \mathbb{P}_1}{\partial v^\varepsilon} + \big(r - \lambda \mathbb{E}_{\mathcal{M}}(Y_t)\big)\ \mathbb{P}_1 \\
&\qquad + \lambda \int_{\Re} \big[ (e^y - 1)\ \mathbb{P}_1(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) \big] \phi_Y(y) dy \Bigg) \\
:= &\frac{\partial \mathbb{P}_1}{\partial t} + A_1[\mathbb{P}_1](t, \ell_t^\varepsilon, v^\varepsilon; T, \kappa).
\end{aligned}
$$

For $\mathbb{P}_2(t, \ell_t^\varepsilon, v^\varepsilon; T, \kappa)$:

$$
\begin{aligned}
0 = &\frac{\partial \mathbb{P}_2}{\partial t} + r\ \mathbb{P}_2 + \left( r - \lambda \mathbb{E}_{\mathcal{M}}(Y_t) - \frac{1}{2} v^\varepsilon \right) \left( \frac{\partial \mathbb{P}_2}{\partial \ell^\varepsilon} \right) + \big(\omega + (\alpha \xi \varphi_t^\varepsilon - \theta) v^\varepsilon\big) \frac{\partial \mathbb{P}_2}{\partial v^\varepsilon} \\
&+ \frac{1}{2} v^\varepsilon \frac{\partial^2 \mathbb{P}_2}{\partial (\ell^\varepsilon)^2} + \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial^2 \mathbb{P}_2}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} \xi^2 \varepsilon^{2\alpha} (v^\varepsilon)^2 \frac{\partial^2 \mathbb{P}_2}{\partial (v^\varepsilon)^2} \\
&- r \mathbb{P}_2 + \int_{\Re} \big[ \mathbb{P}_2(t, \ell^\varepsilon + y, v^\varepsilon; T, \kappa) - \mathbb{P}_2(t, \ell^\varepsilon, v^\varepsilon; T, \kappa) \big] \phi_Y(y) dy
\end{aligned}
\tag{3.19}
$$

subject to the boundary condition at expiration time $t = T$ according to equation (3.16). Again, by using the notation (3.17), PIDE of (3.19) becomes

$$
\begin{aligned}
0 &= \frac{\partial \mathbb{P}_2}{\partial t} + \Big( A[\mathbb{P}_2](t, \ell_t^\varepsilon, v^\varepsilon; T, \kappa) + r \mathbb{P}_2 \Big), \\
&:= \frac{\partial \mathbb{P}_2}{\partial t} + A_2[\mathbb{P}_2](t, \ell_t^\varepsilon, v^\varepsilon; T, \kappa)
\end{aligned}
$$

The proof is now completed. □

## 3.5 The Closed-Form Solution for European Call Options

For $j = 1, 2$, the characteristic functions for $\mathbb{P}_j(t, \ell^\varepsilon, v^\varepsilon; \kappa, T)$, with respect to the variable $\kappa$ are defined by

$$
f_j(t, \ell^\varepsilon, v^\varepsilon; T, x) := - \int_{-\infty}^{\infty} e^{ix\kappa} d\mathbb{P}_j(t, \ell^\varepsilon, v^\varepsilon; T, \kappa),
$$

with a minus sign to account for the negativity of the measure $d\mathbb{P}_j$. Note that $f_j$ also satisfies similar PIDEs

$$
\frac{\partial f_j}{\partial t} + A_j[f_j](t, \ell^\varepsilon, v^\varepsilon; \kappa, T) = 0, \tag{3.20}
$$

with the respective boundary conditions

$$f_j(T,\ell^\varepsilon,v^\varepsilon;x,T)=-\int_{-\infty}^{\infty}e^{ix\kappa}d\mathbb{P}_j(T,\ell^\varepsilon,v^\varepsilon;T,\kappa)=-\int_{-\infty}^{\infty}e^{ix\kappa}(-\delta(\ell^\varepsilon-\kappa)d\kappa)=e^{ix\ell^\varepsilon},$$

since

$$d\mathbb{P}_j(t,\ell^\varepsilon,v^\varepsilon;T,\kappa)=d1_{\ell^\varepsilon>\kappa}=dH(\ell^\varepsilon-\kappa)=-\delta(\ell^\varepsilon-\kappa)d\kappa.$$

The following lemma shows how to calculate the functions $\mathbb{P}_1$ and $\mathbb{P}_2$ as they appear in Lemma 3.1.

**Lemma 3.2.** *The functions $\mathbb{P}_1$ and $\mathbb{P}_2$ can be calculated by the inverse Fourier transforms of the characteristic function, i.e.*

$$\mathbb{P}_j(t,\ell^\varepsilon,t;v^\varepsilon;T,\kappa)=\frac{1}{2}+\frac{1}{\pi}\int_{0+}^{+\infty}Re\Big[\frac{e^{-ix\kappa}f_j(t,\ell^\varepsilon,v^\varepsilon;T,x)}{ix}\Big]dx,$$

*for $j=1,2$, with $Re[\cdot]$ denoting the real component of a complex number.*

*By letting $\tau=T-t$. (i) The characteristic function $f_1$ is given by*

$$f_1(t,\ell^\varepsilon,v^\varepsilon;t+\tau,x)=\exp\big(g_1(\tau)+v^\varepsilon h_1(\tau)+ix\ell^\varepsilon\big),$$

*where*

$$\begin{aligned}g_1(\tau)=&\big[(r-\lambda\mathbb{E}_{\mathcal{M}}(Y_t))ix-\lambda\mathbb{E}_{\mathcal{M}}(Y_t)\big]\tau+\lambda\tau\int_{\Re}\Big(e^{(ix+1)y}-1\Big)\phi_Y(y)dy\\&-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1-\frac{(\Delta_1+\eta_1)(1-e^{-\Delta_1\tau})}{2\Delta_1}\right)+(\Delta_1+\eta_1)\tau\right],\end{aligned}$$

$$h_1(\tau)=\frac{(\eta_1^2-\Delta_1^2)(e^{\Delta_1\tau}-1)}{\xi^2\varepsilon^{2\alpha}v^\varepsilon(\eta_1+\Delta_1-(\eta_1-\Delta_1)e^{\Delta_1\tau})},$$

$$\eta_1=\rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}(1+ix)+(\alpha\xi\varphi_t^\varepsilon-\theta),$$

*and*

$$\Delta_1=\sqrt{\eta_1^2-\xi^2\varepsilon^{2\alpha}v^\varepsilon ix(ix+1)}.$$

*(ii) The characteristic function $f_2$ is given by*

$$f_2(t,\ell^\varepsilon,v^\varepsilon,x,t+\tau)=\exp\big(g_2(\tau)+v^\varepsilon h_2(\tau)+ix\ell^\varepsilon+r\tau\big),$$

*where*

$$g_2(\tau) =\big[(r-\lambda\mathbb{E}_{\mathcal{M}}(Y_t))ix - r\big]\tau + \lambda\tau\int_{\Re}\Big(e^{ixy}-1\Big)\phi_Y(y)dy$$
$$-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\left[\log\left(1-\frac{(\Delta_2+\eta_2)(1-e^{-\Delta_2\tau})}{2\Delta_2}\right)+(\Delta_2+\eta_2)\tau\right],$$

$$h_2(\tau) = \frac{(\eta_2^2-\Delta_2^2)(e^{\Delta_2\tau}-1)}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}(\eta_2+\Delta_2-(\eta_2-\Delta_2)e^{\Delta_2\tau})},$$
$$\eta_2 = \rho\xi\varepsilon^{\alpha}\sqrt{v^{\varepsilon}} - (\alpha\xi\varphi_t^{\varepsilon}-\theta),$$

*and*

$$\Delta_2 = \sqrt{\eta_2^2 - \xi^2\varepsilon^{2\alpha}v^{\varepsilon}ix(ix-1)}.$$

*Proof.* Proof of (i). To solve for the characteristic explicitly, letting $\tau = T-t$ be the time-to-go, we conjecture that the function $f_1$ is given by

$$f_1(t,\ell^{\varepsilon},v^{\varepsilon};t+\tau,x) = \exp\big(g_1(\tau)+v^{\varepsilon}h_1(\tau)+ix\ell^{\varepsilon}\big), \tag{3.21}$$

and the boundary condition;

$$g_1(0) = 0 = h_1(0).$$

This conjecture exploits the linearity of the coefficient in PIDE (3.20). Note that the characteristic function of $f_1$ always exists.

In order to substitute equation (3.21) into equation (3.20), firstly, we compute

$$\frac{\partial f_1}{\partial t} = \big(-g_1'(\tau)-v^{\varepsilon}h_1'(\tau)\big)f_1, \quad \frac{\partial f_1}{\partial \ell^{\varepsilon}} = ixf_1, \quad \frac{\partial f_1}{\partial v^{\varepsilon}} = h_1(\tau)f_1,$$
$$\frac{\partial^2 f_1}{\partial(\ell^{\varepsilon})^2} = -x^2f_1, \quad \frac{\partial^2 f_1}{\partial\ell^{\varepsilon}\partial v^{\varepsilon}} = ixh_1(\tau)f_1, \quad \frac{\partial^2 f_1}{\partial(v^{\varepsilon})^2} = h_1^2(\tau)f_1,$$
$$f_1(t,\ell^{\varepsilon}+y,v^{\varepsilon};t+\tau,x) - f_1(t,\ell^{\varepsilon},v^{\varepsilon};t+\tau,x) = (e^{ixy}-1)f_1(t,\ell^{\varepsilon},v^{\varepsilon};t+\tau,x),$$

and

$$(e^y-1)f_1(t,\ell^{\varepsilon},v^{\varepsilon};t+\tau,x) =(e^y-1)e^{g_1(\tau)+v^{\varepsilon}h_1(\tau)+ix(\ell^{\varepsilon}+x)}$$
$$=(e^y-1)e^{ixy}f_1(t,\ell^{\varepsilon},v^{\varepsilon};t+\tau,x).$$

Substituting all the above terms into equation (3.20), after canceling the common factor of $f_1$, we get a simplified form as follows:

$$\begin{aligned}0 = &- g_1'(\tau) - v^\varepsilon h_1'(\tau) + \Big(r - \lambda \mathbb{E}_{\mathcal{M}}(Y_t) - \frac{1}{2}v^\varepsilon\Big)ix \\ &+ \Big(\big(\omega + (\alpha\xi\varphi_t^\varepsilon - \theta)v^\varepsilon\big) + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2}\Big)h_1(\tau) \\ &- \frac{1}{2}v^\varepsilon x^2 + \rho\xi\varepsilon^\alpha (v^\varepsilon)^{3/2} ixh_1(\tau) + \frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^\varepsilon)^2 h_1^2(\tau) \\ &- \lambda\mathbb{E}_{\mathcal{M}}(Y_t) + \lambda\int_{\Re}\Big(e^{(ix+1)y} - 1\Big)\phi_Y(y)dy.\end{aligned}$$

By separating the order $v^\varepsilon$ and ordering the remaining terms, we can reduce it to two ordinary differential equations (ODEs),

$$h_1'(\tau) \;=\; \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^\varepsilon h_1^2(\tau) + \big(\rho\xi\sqrt{v^\varepsilon}(1+ix) + (\alpha\xi\varphi_t^\varepsilon - \theta)\big)h_1(\tau) - \frac{1}{2}ix - \frac{1}{2}x^2, \tag{3.22}$$

$$g_1'(\tau) \;=\; \omega h_1(\tau) + \big(r - \lambda\mathbb{E}_{\mathcal{M}}(Y_t)\big)ix - \lambda\mathbb{E}_{\mathcal{M}}(Y_t) + \lambda\int_{\Re}\Big(e^{(ix+1)y} - 1\Big)\phi_Y(y)dy. \tag{3.23}$$

Let $\eta_1 = \big(\rho\xi\sqrt{v^\varepsilon}(1+ix) + (\alpha\xi\varphi_t^\varepsilon - \theta\big)$ and substitute it into equation (3.22). We get

$$\begin{aligned}h_1'(\tau) &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(h_1^2(\tau) + \frac{2\eta}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}h_1(\tau) + \frac{1}{\xi^2\varepsilon^{2\alpha v^\varepsilon}}ix(ix+1)\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}\left(h_1(\tau) + \frac{2\eta_1 + \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}v^\varepsilon ix(ix+1)}}{2\xi v^\varepsilon}\right) \\ &\quad\times\left(h_1(\tau) + \frac{2\eta_1 - \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}ix(ix+1)}}{2\xi^2\varepsilon^{2\alpha}}\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(h_1(\tau) + \frac{\eta_1 + \Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right)\left(h_1(\tau) + \frac{\eta_1 - \Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\right)\end{aligned}$$

where $\Delta_1 = \sqrt{\eta_1^2 - \xi^2\varepsilon^{2\alpha}v^\varepsilon ix(ix+1)}$.

By the method of variable separation, we have

$$\frac{2dh_1(\tau)}{\Big(h_1 + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\Big)\Big(h_1 + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\Big)} = \xi^2\varepsilon^{2\alpha}v^\varepsilon d\tau.$$

Using partial fractions, we get

$$\frac{1}{\Delta_1}\left(\frac{1}{h_1 + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}} - \frac{1}{h_1 + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}}\right)dh_1(\tau) = d\tau.$$

Integrating both sides, we obtain

$$\log\left(\frac{h_1(\tau)+\frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}}{h_1(\tau)+\frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}}\right)=\Delta_1\tau+C.$$

Using the boundary condition $h_1(\tau=0)=0$ we get $C=\log\left(\frac{\eta_1-\Delta_1}{\eta_1+\Delta_1}\right)$.

Solving for $h_1$, we obtain

$$h_1=\frac{(\eta_1^2-\Delta_1^2)(e^{\Delta_1\tau}-1)}{\xi^2\varepsilon^{2\alpha}v^\varepsilon\Big(\eta_1+\Delta_1-(\eta_1-\Delta_1)e^{\Delta_1\tau}\Big)}.$$

In order to solve $g_1(\tau)$ explicitly, substituting $h_1$ in to equation (3.23) and integrate with respect to $\tau$ on the both sides. Then we get

$$\begin{aligned}g_1(\tau)=&[(r-\lambda\mathbb{E}_{\mathcal{M}}(Y_t))ix-\lambda\mathbb{E}(Y_t)]\tau+\lambda\tau\int_{\Re}\Big(e^{(ix+1)y}-1\Big)\phi_Y(y)dy\\&-\frac{2\omega}{v^\varepsilon\xi^2\varepsilon^{2\alpha}}\left[\log\left(1-\frac{(\Delta_1+\eta_1)(1-e^{-\Delta_1\tau})}{2\Delta_1}\right)+(\Delta_1+\eta_1)\tau\right].\end{aligned}$$

Proof of (ii). The details of the proof are similar to case (i). Hence, we have

$$f_2(t,\ell^\varepsilon,v^\varepsilon;t+\tau,x)=\exp\big(g_2(\tau)+v^\varepsilon h_2(\tau)+iy\ell^\varepsilon+r\tau\big),$$

where $g_2(\tau)$, $h_2(\tau)$, $\eta_2$ and $\Delta_2$ are as given in the Lemma.

We can thus evaluate the characteristic functions in closed form. However, we are interested in the risk-neutral probabilities $\mathbb{P}_j$. These can be inverted from the characteristic functions by performing the following integration

$$\begin{aligned}\widehat{\mathbb{P}}_j(S_t^\varepsilon,v_t^\varepsilon,t;T,K)&=\mathbb{P}_j(\ell^\varepsilon,v^\varepsilon,t;\kappa,T)\\&=\frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty}\mathrm{Re}\left[\frac{e^{-ix\kappa}f_j(t,\ell^\varepsilon,v^\varepsilon;T,x)}{ix}\right]dx\end{aligned}\tag{3.24}$$

for $j=1,2$, where $\ell^\varepsilon=\log(S_t^\varepsilon)$, $v^\varepsilon=\log(v_t^\varepsilon)$, and $\kappa=\log(K)$.

To verify equation (3.24), firstly we note that

$$\begin{aligned}&\mathbb{E}_{\mathcal{M}}\Big[e^{ix(\log(S_t^\varepsilon)-\log(K))}\mid\log(S_t^\varepsilon)=L_t^\varepsilon,v_t^\varepsilon=v^\varepsilon\Big]\\&=\mathbb{E}_{\mathcal{M}}\Big[e^{ix(\ell^\varepsilon-\kappa)}\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\Big]\\&=\int_{-\infty}^{+\infty}e^{ix(\ell^\varepsilon-\kappa)}d\mathbb{P}_j(t,\ell^\varepsilon,v^\varepsilon;T,\kappa)\end{aligned}$$

$$
\begin{aligned}
&= e^{-ix\kappa} \int_{-\infty}^{+\infty} e^{ix\ell^\varepsilon} d\mathbb{P}_j(t, \ell^\varepsilon, v^\varepsilon; T, \kappa) \\
&= e^{-ix\kappa} \int_{-\infty}^{+\infty} e^{ix\kappa} \big( -\delta(\ell^\varepsilon - \kappa) d\kappa \big) \\
&= e^{-ix\kappa} f_j(t, \ell_t^\varepsilon, v_t^\varepsilon; T, x).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2} + \frac{1}{\pi} \int_{0+}^{+\infty} \mathrm{Re}\left[ \frac{e^{-ix\kappa} f_j(t, \ell^\varepsilon, v^\varepsilon; T, x)}{ix} \right] dx \\
&= \frac{1}{2} + \frac{1}{\pi} \int_{0+}^{+\infty} \mathrm{Re}\left[ \frac{\mathbb{E}_{\mathcal{M}}[e^{ix(\log(S_t^\varepsilon) - log(K))} \mid \log(S_t) = L_t^\varepsilon, v_t^\varepsilon = v^\varepsilon]}{ix} \right] dx \\
&= \mathbb{E}_{\mathcal{M}}\left[ \frac{1}{2} + \frac{1}{\pi} \int_{0+}^{+\infty} \mathrm{Re}\left[ \frac{e^{ix(\ell^\varepsilon - \kappa)}}{ix} \right] dx \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon \right] \\
&= \mathbb{E}_{\mathcal{M}}\left[ \frac{1}{2} + \frac{1}{\pi} \int_{0+}^{+\infty} \frac{\sin(x(\ell^\varepsilon - \kappa))}{x} dx \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon \right] \\
&= \mathbb{E}_{\mathcal{M}}\left[ \frac{1}{2} + sgn(\ell^\varepsilon - \kappa) \frac{1}{\pi} \int_{0+}^{+\infty} \frac{\sin(x)}{x} dx \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon \right] \\
&= \mathbb{E}_{\mathcal{M}}\left[ \frac{1}{2} + \frac{1}{2} sgn(\ell^\varepsilon - \kappa) \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon \right] \\
&= \mathbb{E}_{\mathcal{M}}\left[ 1_{\ell^\varepsilon \geq \kappa} \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon \right]
\end{aligned}
$$

where we have used the Dirichlet formula $\int_{-\infty}^{+\infty} \frac{\sin(x)}{x} dx = 1$, and the $sgn$ function is defined as $sgn(x) = 1$ if $x > 0$, $0$ if $x = 0$ and $-1$ if $x < 0$. □

In summary, we have just proved the following main theorem.

**Theorem 3.3.** *For each $\varepsilon > 0$, the value of a European call option of SDE (3.4) is*

$$
\widehat{C}(t, S_t^\varepsilon, v_t^\varepsilon; T, K) = S_t^\varepsilon \ \widehat{\mathbb{P}}_1(t, S_t^\varepsilon, v_t^\varepsilon; T, K) - K e^{-r(T-t)} \ \widehat{\mathbb{P}}_2(t, S_t^\varepsilon, v_t^\varepsilon; T, K),
$$

*where $\mathbb{P}_1$, $\mathbb{P}_2$ are as given in Lemma 3.2, and*

$$
\begin{aligned}
\widehat{C}(t, S_t^\varepsilon, v_t^\varepsilon; T, K) &:= C(t, \log(S_t^\varepsilon), v^\varepsilon; T, log(K)), \\
\widehat{\mathbb{P}}_1(t, S_t^\varepsilon, v_t^\varepsilon; T, K) &:= \mathbb{P}_1(t, \log(S_t^\varepsilon), v^\varepsilon; T, log(K)), \\
\widehat{\mathbb{P}}_2(t, S_t^\varepsilon, v_t^\varepsilon; T, K) &:= \mathbb{P}_2(t, \log(S_t^\varepsilon), v^\varepsilon; T, log(K)).
\end{aligned}
$$

**Remark 3.4.** *In numerical computation, we first choose a real number* $\varepsilon > 0$ *and then compute the value of* $\widehat{C}(t, S_t^\varepsilon, v_t^\varepsilon; T, K)$ *according to the formula as given in Theorem 3.3. The solution that we get is the value of a call option of the approximation model (3.4) and this value can be used as an approximating value of a call option of the fraction model (3.1) as* $\varepsilon$ *approaches zero.*

# CHAPTER IV

# SIMULATION EXAMPLE

Let us consider the Petroleum Authority of Thailand (PTT) stock market. Figure 4.1 shows the daily prices of the data set consisting of close-prices (Baht) of the PTT between October 10, 2009 and March 19, 2010. The empirical data set for these stock prices were obtained from `http://www.set.or.th/`. Figure 4.2 shows that log returns of the stock prices in the period.

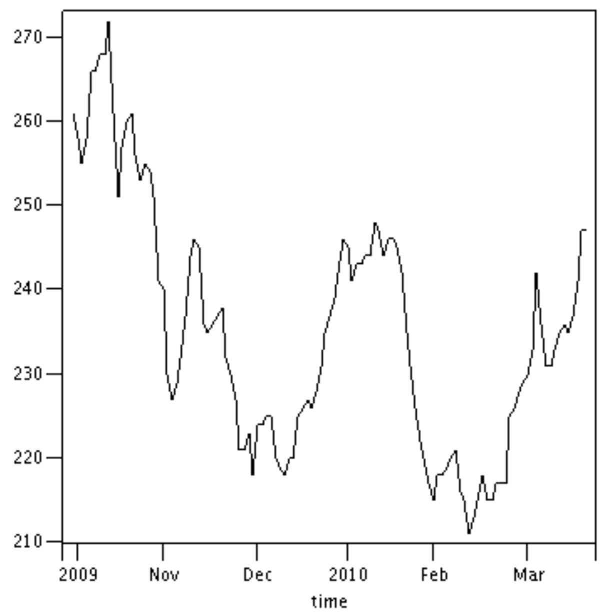


**Figure 4.1** *Stock prices trading daily of PTT between October 10, 2009 and March 19, 2010.*

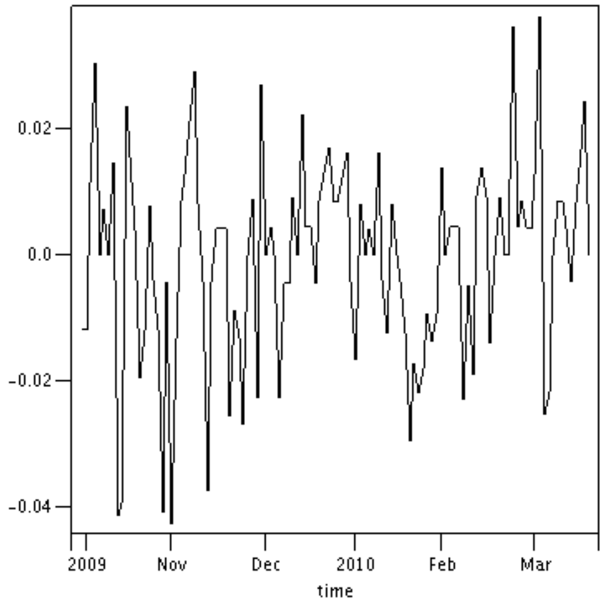


**Figure 4.2** *Log returns on the stock prices of PTT between October 10, 2009 and March 19, 2010.*

The statistic of stock prices and log returns are given in Table 4.1.

**Table 4.1** *Statistic of PPT data set.*

| | Stock prices | Log returns |
|---|---|---|
| Data Amount | 115 | 114 |
| Mean | 235.17391 | -0.000483617 |
| Standard Deviation | 14.57052 | 0.00025190 |
| Skewness | 0.48403 | -0.424870 |
| Kurtosis | 2.43137 | 3.40700 |

Figure 4.3 shows the *historical volatility*, that is the annualized standard deviation of the returns, namely

$$\sigma_{\text{hist}} = \sqrt{\frac{252}{N-1}\sum_{n=0}^{N-1}(R_n - \overline{R})^2},$$

where $R_n$ be the stock price return between two days computed by $R_n = \ln(S_{n+1}/S_n)$ with the sequence of known historic daily stock close prices $S_1, \ldots, S_n$ and $\overline{R}$ the mean return. The factor 252 supposes that there are approximately 252 business days in a year, because we work with annualized quantities, and we use daily stock closing prices.

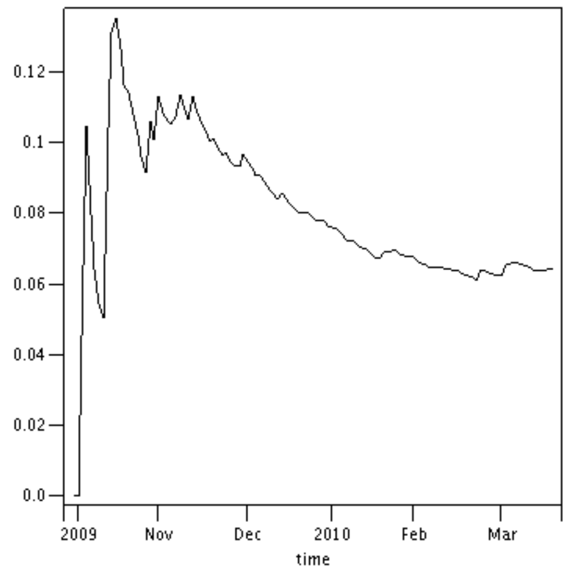


**Figure 4.3** *The historical volatility of PTT between October 10, 2009 and March 19, 2010, simulated by*

$$\sigma_{hist} = \sqrt{\frac{252}{N-1}\sum_{n=0}^{N-1}(R_n - \overline{R})^2}.$$

*As can be seen, the volatility is clearly non-constant.*

It is easily seen that the historic volatility of PTT is *not* constant over time (Figure 4.3).

Figure 4.4 shows the empirical data of PTT closed-price as compared to the price simulated by the classical geometric Brownian motion with compound Poisson process and a stochastic volatility. The simulated model is

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s) dN_s\right)$$

with stochastic volatility model

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t.$$

The model parameters $\mu = -0.00048361605$, $v_0 = 0.0002518958471$. The mean of jumps $= 0.001250$, the standard deviation of jumps $= 0.0375$, and the intensity $\lambda = 17$. The model parameters for stochastic volatility are $\omega = 0.001500$, $\xi = 0.775000$, and $\theta = 0.000125$. For comparative purpose, we compute the *Average Relative Percentage Error* (ARPE). By definition,

$$\text{ARPE} = \frac{1}{N}\sum_{k=1}^{N} \frac{|X_k - Y_k|}{X_k} \times 100,$$

where $N$ is the number of prices, $X = (X_k)_{k\geq 1}$ is the market price and $Y = (Y_k)_{k\geq 1}$ is the model price. After working 250 trails we compute ARPE for Figure 4.4 which will be denotedy by ARPE(4).

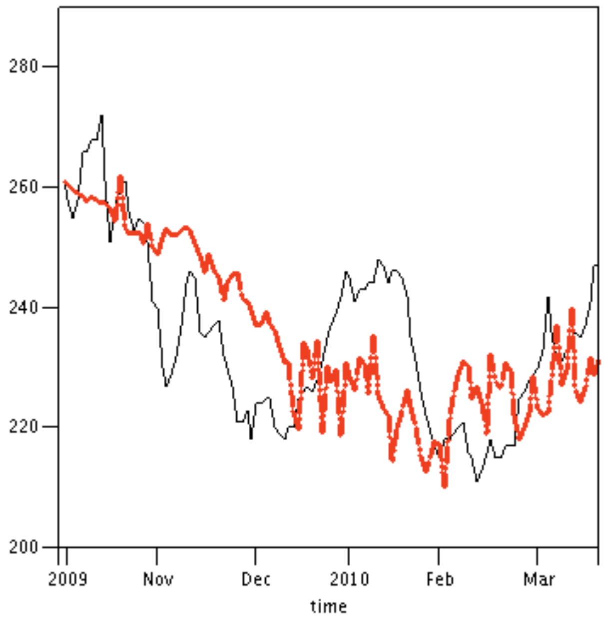


**Figure 4.4** *Price behavior of PTT, between October 10, 2009 and March 19, 2010, as compared with a scenario simulated from geometric Brownian motion adding jumps and a stochastic volatility model. (solid line:=empirical data, dash line:=simulated by*

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s) dN_s\right)$$

*with stochastic volatility model*

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t,$$

$$N = 250,\ ARPE(4) = 4.62363494).$$

*where* $ARPE(4)$ *is the* $ARPE$ *for Figure 4.4).*

Figure 4.5 shows the volatility of PTT simulated by the stochastic volatility model,

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t$$

in Figure 4.4.

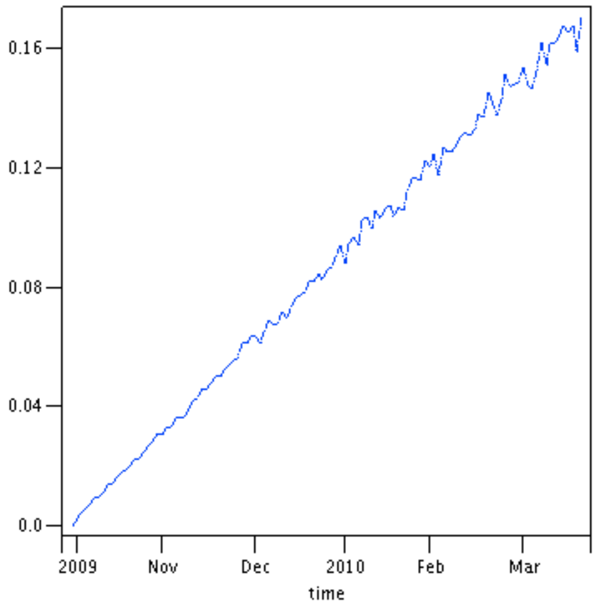


**Figure 4.5** *The PTT volatility simulated by stochastic volatility model*

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t$$

*in Figure 4.4.*

Figure 4.6 shows the empirical data of PTT closed-price as compared to the price simulated by geometric Brownian motion with compound Poisson process and an approximation of fractional stochastic volatility model. The simulated model is

$$S_t^\varepsilon = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1+Y_s)dN_s\right)$$

with fractional stochastic volatility model

$$v_t^\varepsilon = \left(v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon)ds\right)\exp(\xi B_t^\varepsilon - \chi_\varepsilon t).$$

The value of $\mu$, $v_0$ and the parameter for volatility and jumps are the same as Figure 4.3. For the remaining data, we choose $\varepsilon = 0.00001$ and $\alpha = 0.00125$.

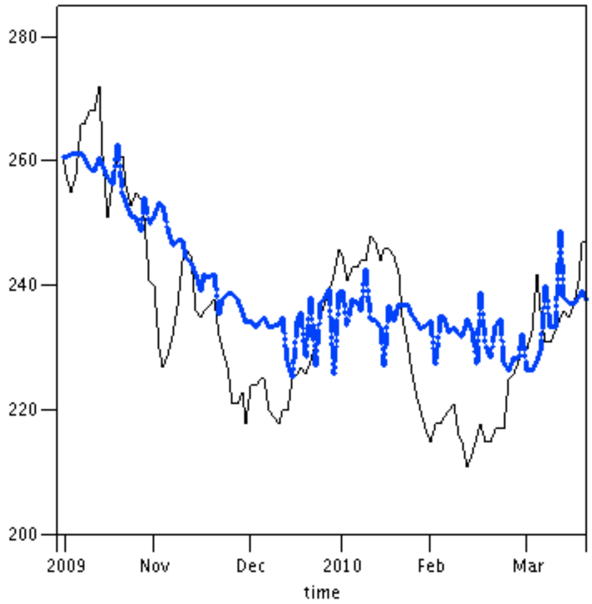


**Figure 4.6** *Price behavior of PPT, between October 10, 2009 and March 19, 2010, as compared with a scenario simulated from geometric Brownian motion adding jumps and an approximate fractional stochastic volatility model. (solid line:=empirical data, dash line:=simulated by*

$$S_t^{\varepsilon} = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s^{\varepsilon} ds + \int_0^t \sqrt{v_s^{\varepsilon}} dW_s + \int_0^t \log(1+Y_s) dN_s\right)$$

*with fractional stochastic volatility model*

$$v_t^{\varepsilon} = \left(v_0 + \omega \int_0^t \exp(\chi_{\varepsilon} s - \xi B_s^{\varepsilon}) ds\right) \exp(\xi B_t^{\varepsilon} - \chi_{\varepsilon} t),$$

$$N = 250, \ \ ARPE(6) = 3.719528807$$

*where $ARPE(6)$ is the ARPE for Figure 4.6).*

Figure 4.7 shows the volatility of PTT simulated by an approximation fractional stochastic volatility model,

$$v_t^\varepsilon = \left(v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds\right) \exp(\xi B_t^\varepsilon - \chi_\varepsilon t)$$

in Figure 4.6

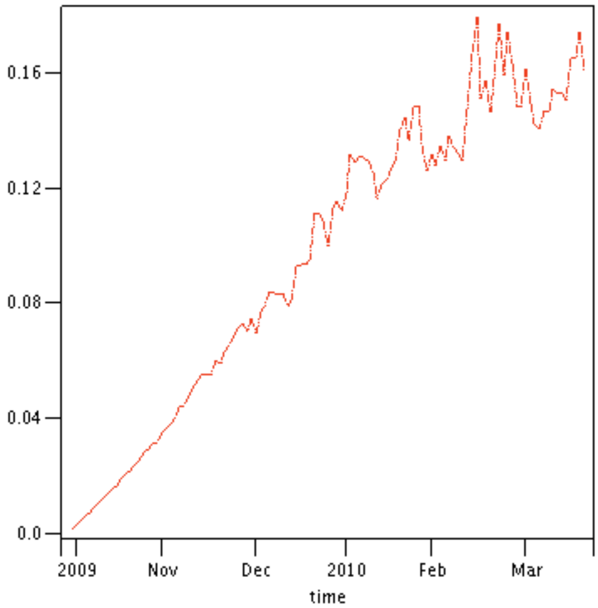


**Figure 4.7** *The PTT volatility simulated by an approximation fractional stochastic volatility model*

$$v_t^\varepsilon = \left(v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon) ds\right) \exp(\xi B_t^\varepsilon - \chi_\varepsilon t)$$

*in Figure 4.6.*

The Figure 4.8 shows the tendency of $ARPE(4)$ and $ARPE(6)$, working out for 25, 45, 55, 100, 150, 350, 550 and 750.

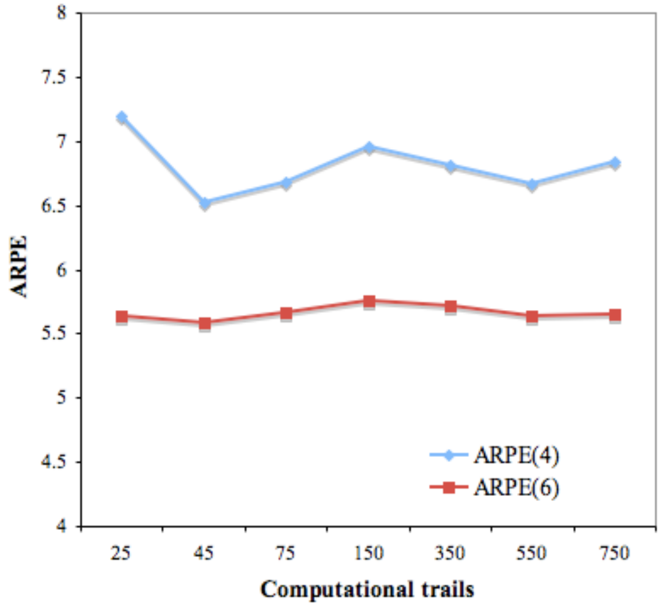


**Figure 4.8** *Tendency of $ARPE(4)$ and $ARPE(6)$ with $N$=25, 45, 55, 95, 100, 150, 350, 550 and 750.*

**Remark 4.1.** *However, in any case, the results depend on what data one uses. For some sets of data the theoretical price of classical gBm with jumps and the stochastic volatility model is better and for some data the gBm with jumps and the fractional stochastic volatility is better.*

The Figure 4.9 shows the forecasts of PTT one week after March 20, 2009 simulated by

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s) dN_s\right)$$

with stochastic volatility model

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t.$$

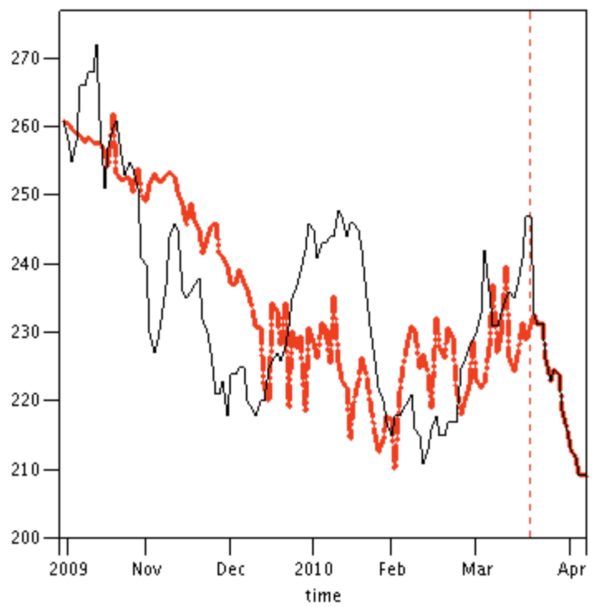


**Figure 4.9** *Stock Price forecasts of PPT, between March 22, 2010 and April 16, 2010, simulated by*

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s) dN_s\right)$$

*with stochastic volatility model*

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\overline{W}_t.$$

The figure 4.10 shows the forecasts of PTT one week after March 20, 2009 simulated by

$$S_t^\varepsilon = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1+Y_s)dN_s\right)$$

with fractional stochastic volatility model

$$v_t^\varepsilon = \left(v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon)ds\right) \exp(\xi B_t^\varepsilon - \chi_\varepsilon t).$$

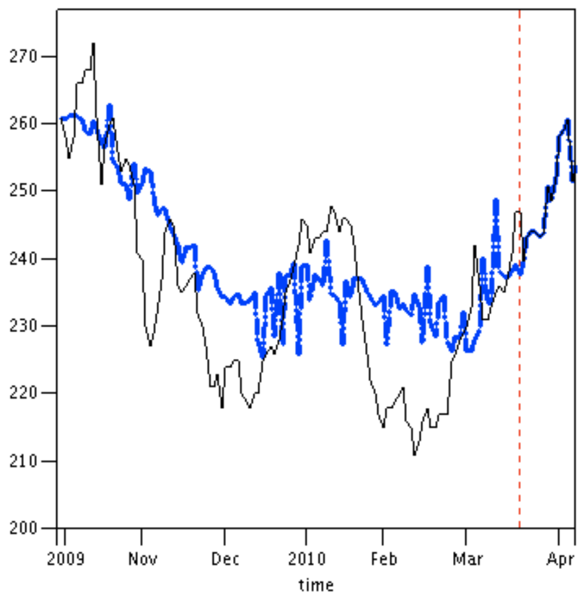


**Figure 4.10** *Stock price forecasts of PPT, between March 22, 2010 and April 16, 2010, simulated by*

$$S_t^\varepsilon = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1+Y_s)dN_s\right)$$

*with fractional stochastic volatility model*

$$v_t^\varepsilon = \left(v_0 + \omega \int_0^t \exp(\chi_\varepsilon s - \xi B_s^\varepsilon)ds\right) \exp(\xi B_t^\varepsilon - \chi_\varepsilon t).$$

# CHAPTER V

# CONCLUSION AND RESEARCH POSSIBILITY

## 5.1 Conclusion

The aim of this thesis is to introduce an alternative model of stochastic volatility of jump diffusion in which the stock prices follow a geometric Brownian motion, with the addition of the compound Poisson process and stochastic volatility perturbed by a fractional noise. This model exhibits a long term dependence of stochastic volatility that is not expressed in the classical stochastic volatility model. The following procedure are investigated:

(i) We investigated the solution of this model by studying its corresponding approximate model. By using Itô's lemma for the jump-diffusion process, the approximate model was solved.

(ii) By using a fundamental result on the $L^2$-approximation of a fractional noise, we proved a convergence theorem concerning an approximation solution. Based on the approximate approach, we found that the solution of the approximate model converges to the solution of the original model.

(iii) The mathematical formula of the European options is formulated by inverting the characteristic function of the approximate model. In order to solve the characteristic function explicitly, we proved the lemma that established a relationship between stochastic volatility and partial differential equations in the general case. Then we got the explicit formula of characteristic function. And the formula of the European option can be expressed in terms of the probability function.

(iv) A simulation example shows the sample paths simulated by geometric Brownian motion adding compound Poisson process with the fractional stochastic volatility and a geometric Brownian motion adding compound Poisson process with the stochastic volatility against the empirical data.

## 5.2 Research Possibility

In this section, we provide possible extension of the geometric Brownian motion adding compound Poisson process with the fractional stochastic volatility. It is hoped that the following aspects can be explored:

(i) Schoutens (2003) showed that Lévy models give a much better fit to the data and lead to a significant improvement with respect to the Black-Scholes model. Thus the stock price model can be extended to Lévy model. For example, the Gamma process and the VG process.

(ii) In this thesis, after the stock price has jumped, the volatility will stay unchanged because the jump process is uncorrelated with the volatility process. Thus, it is possible to add jumps effect into the fractional stochastic volatility model.

(iii) In practice, interest rates are determined by monetary policy of a country according to its economic situation. In this thesis, the interest rates are calculated by assuming that volatility remains constant over the period of analysis. Thus, we may extend this analysis to the case where we have a stochastic volatility for interest rate.

(iv) In order to study a numerical solution of European option, we can apply the Discrete Fourier transform (DFT) or the fast Fourier transform (FFT) for more higher accuracy.

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# REFERENCES


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# CURRICULUM VITAE

**Name** Mr. Arthit Intarasit.

**Office Address**

Faculty of Science and Technology

Prince of Songkla University, Pattani Campus

181 Charoenpradit Rd., Rusamelae, Meaung, Pattani 94000, Thailand.

Tel. 073-312179, Fax. 073-312179.

**Education Background**

2002 Bachelor of Science (Applied Mathematics) Second Class Honors

Prince of Songkla University (Pattani Campus), Thailand.

2004 Master of Science (Applied Mathematics), Suranaree University of Technology, Nakhon Ratchasima, Thailand.

**Published**

Sattayatham, P., Intarasit, A. and Chaiyasena, A. P. 2007. A Fractional Black-Scholes Model with Jumps. **Vietnam Journal of Mathematics.** 35(3): 1-15.

Intarasit, A. and Sattayatham, P. 2010. A geometric Brownian motion model with compound Poisson process and Fractional Stochastic Volatility. **Advances and Applications in Statistics.** 16(1): 25-47.

**Workshops and Presentation**

(i) CARISMA-IIM Calcutta Workshop, Optimisation Methods and its Financial Applications, High Frequency Finance, 10-13 March, 2010 at Financial Research and Trading Lab, IIM Calcutta, India.

(ii) Financial Mathematics Seminar, 25-26 February, 2010 at Department of Mathematics, Chiangmai University Thailand.