Suranaree University of Technology
433 251 Engineering Economy

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Engineering

It is the profession in which knowledge of the mathematical and natural sciences gained by study, experience and practice is applied with judgment to develop ways to utilize efficiently and effectively the materials and forces of nature for the benefit of mankind.

Engineering Economy

Mathematical approach to evaluate and compare engineering projects and proposals from an economic point of view in which the time value of money is usually considered. In an engineering study we must decide how the physical environment will be altered to produce services and goods in a way that is considered attractive from the economic environment point of view. An economic study will tell the final word about the feasibility of the intended project.

Steps of an Economic Study

- Research, explore, investigate (creative step: discover opportunities)
- Objectives, synthesis, requirements (definition step: identify alternatives)
- Appraisal, analysis, communication (conversion step: evaluate alternatives)
- Presentation, comparison, risk analysis (decision step: selection)

Course Overview

A typical Engineering Economy course would include the following topics of the textbook. However, since INEN 302 is a two-credit hour course, we will only cover levels 1 and 3 and an **overview** of some other relevant topics on levels 3 through 5, covered in Chapters 10, 11, 13, and 16.

Level 1: Basic computational skills, time value of money (Chapters: 1, 2, 3, 4)
Level 2: Basic techniques for evaluating alternatives (Chapters 5, 6, 7, 8, 9)
Level 3: Additional methods for evaluating alternatives (Chapters 10, 11, 12)
Level 4: Depreciation and taxation (Chapters 13, 15)
Level 5: Risk and sensitivity analysis (Chapters 16, 19)

Algebraic Background

<u>Arithmetic Series</u>
A series consisting of the following *n* terms:

$$a \qquad a+d \qquad a+2d \quad \ldots \quad a+(n-1)d$$

Geometric Series

A series consisting of the following *n* terms:

$$a \qquad ar \qquad ar^2 \qquad \ldots \qquad ar^{n-1}$$

Sum of the Terms of an Arithmetic Series

$$S_n = a + (a+d) + (a+2d) + \ldots + [a+(n-1)d] = na + \tfrac{1}{2}(n-1)nd$$

Sum of the Terms of a Geometric Series

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} = a(1-r^n)/(1-r)$$

When $|r| < 1$ the sum converges to

$$S = a/(1-r)$$

Example

(a) As an illustration, verify that for $a = 2$, $n = 5$, and $d = 4$, the sum of the terms of the arithmetic progression is $S_5 = 50$.
(b) Also, for $a = 2$, $n = 5$, and $r = ¼$, the sum of the terms of the geometric progression is $S_5 = 341/128$.
(c) For an infinite geometric progression, $S = 2/(1-¼) = 8/3$.

Exercises

1. Prove the formulas for the sum of terms of both arithmetic and geometric progressions.
2. Provide adequate definitions for engineering and engineering economy. Give examples of meaningful projects that illustrate the definitions.
3. Discuss the steps of a typical engineering economy study.

Chapter 1
Terminology and Cash-Flow Diagrams

- Interest
- Interest Rate
    - Simple Interest
    - Compound Interest
    - Time Value of Money
- Equivalence
- Cash Flow Diagrams

Interest

Interest is a fee charged for the use of an amount of money borrowed (called *principal*). Similarly, it is the gain from an amount of money invested. More specifically, for an investor, interest is defined as the difference between the total amount accumulated minus the original investment. Similarly, it is defined as the difference between the present amount owed minus the original loan, in the case of a borrower. In general, the interest depends on both the amount and the length of the period.

Interest Rate

The interest rate (as a percentage of the original amount) is defined as

$$i = \frac{Interest\ per\ Unit\ Time}{Original\ Amount} x 100\%$$

Simple Interest

Fixed percentage of the principal multiplied by the number of periods in the life of a loan. Let P be the present amount. Therefore, the interest I can be calculated as

$$\mathbf{I = niP}$$

Since the future amount, F, can be written as F = I+P, we conclude that

$$\mathbf{F = P(1+ni)}$$

Compound Interest

The total period is subdivided into several *interest periods* (one year, for example), interest is credited at the end of each interest period, and it is allowed to accumulate from one interest period to the next. Here,

$$\mathbf{F = P(1+i)^n}$$

Therefore, $\mathbf{P(1+i)^n = P + I}$. From this, we conclude that

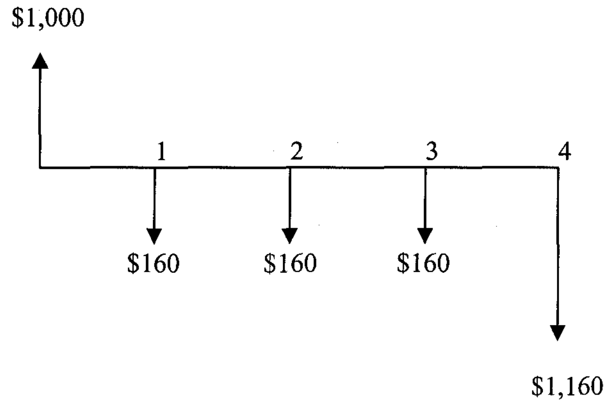
$$\mathbf{I = [(1+i)^n - 1]P}$$

Time Value of Money

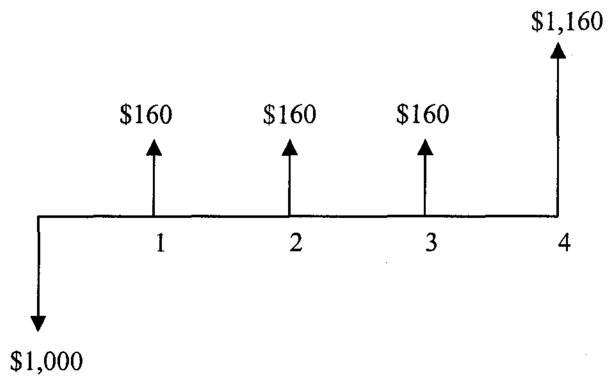
Since money has the ability to earn interest, its value increases with time. For example, \$100 today is equivalent to $F = \$100(1+0.07)^5 = \$140.26$ five years from now at an interest rate of 7% per year compounded annually.

Cashflow Diagrams

Graphical representation of a sequence of *receipts* and *disbursements* along a period of time. In this diagram, the horizontal axis represents time periods, upward arrows at a particular time represent receipts, and downward arrows represent disbursements. Receipts are revenues and income. Disbursements are expenses and costs. A cashflow diagram for a borrower is the mirror image of a cashflow diagram for the lender in the same transaction. As an illustration, let us consider the following transaction: a person borrows $1,000 and pays back the loan in four periods. The amount at the end of periods 1, 2, and 3 is equal to $160 (interest) and the amount at the end of period 4 is $1,160 (principal plus interest). The cashflow diagram for the **borrower** is shown below:



The cashflow diagram for the **lender** is shown below. Note that it is a *mirror image* of the above diagram:



The following symbols are used in drawing cashflow diagrams:

P = present amount
F = Future amount
A = Equal amount per period
n = number of periods
i = interest rate
t = label of any time period (t=1,2,...,n)

Examples

Examples 1.3, 1.4, 1.9, 1.15.

Chapter 2
Engineering Economy Factors

- Single-Payment Transactions
- Uniform-Series Transactions
- Gradient-Series Transactions

Single-Payment Transactions

In this transaction we know the interest rate *i* and the number of periods *n*. There are two scenarios. The first scenario assumes that we now the future amount *F* and want the present amount *P*. The second scenario assumes that we know *P* and want to find *F*.

Find P Given F

We can reason as follows. The future worth of P after one period is $F_1 = P(1+i)$; after two periods, it will be $F_2 = F_1(1+i) = P(1+i)^2$; continuing like this, we conclude that $F = F_n = P(1+i)^n$.

$$P = F[1/(1+i)^n]$$
$$= F\ (P/F,i,n) \qquad \text{(using tables)}$$

Find F Given P

From the previous result, we conclude that

$$F = P(1+i)^n$$
$$= P\ (F/P,i,n) \qquad \text{(using tables)}$$

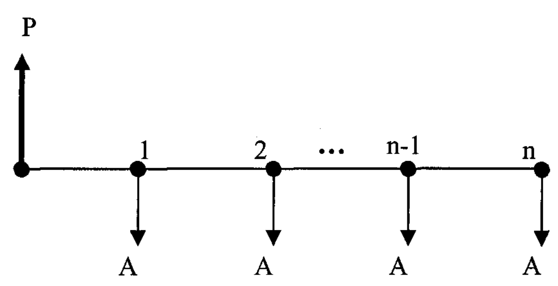
Exercise

Compute the worth of \$100 five years ago and five years from now, assuming an interest rate of 8% compounded annually. Do the calculations from both formulas and using tables.

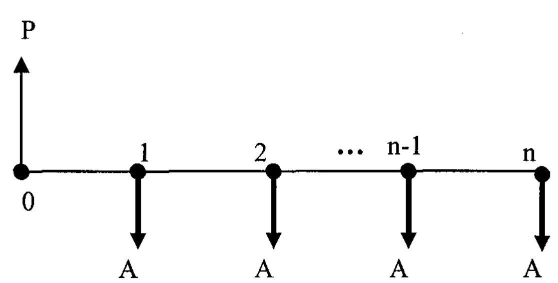
Uniform-Series Transactions

In this transaction we know the interest rate *i* and the number of periods *n*. There is a payment equal to *A* at the end of each period. There are four cases or scenarios to be considered:

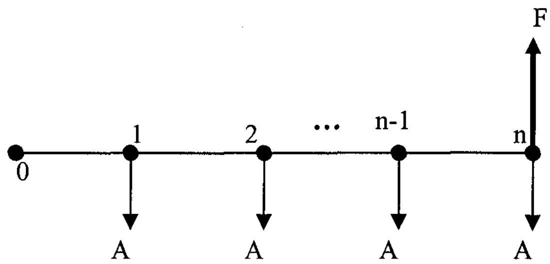
Scenario 1: Find *P* given *A*



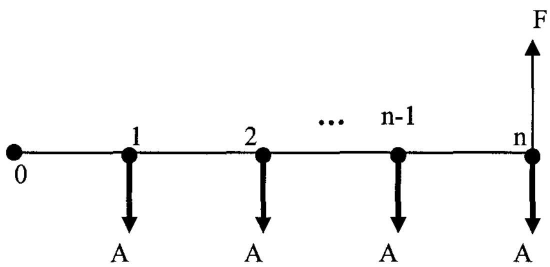
Scenario 2: Find *A* given *P*



Scenario 3: Find F given A *(compound-amount factor)*



Scenario 4: Find A given F *(sinking-fund factor)*



Find *P* given *A*

Recall that there is a payment equal to A at the end of each period. The present worth (at time zero) of the first payment (at the end of period 1) is equal to

$$P_1 = A/(1+i)$$

Similarly, the present worth of the first payment (at the end of period 2) is equal to

$$P_2 = A/(1+i)^2$$

In general, the present worth of the tth payment (at the end of period t) is equal to

$$P_t = A/(1+i)^t$$

Therefore, $P = P_1 + P_2 + \ldots + P_n$. Equivalently,

$$P = A/(1+i) + A/(1+i)^2 + \ldots + A/(1+i)^t + \ldots + A/(1+i)^n$$

$$= A/(1+i)\ [1 + 1/(1+i) + 1/(1+i)^2 + \ldots + 1/(1+i)^{t-1} + \ldots + 1/(1+i)^{n-1}]$$

As can be noted, the expression in brackets is the sum of n terms of a geometric progression with first term equal to 1 and common ratio equal to 1/(1+i). Thus,

$$P = \frac{A}{1+i}\left[\frac{1-(\frac{1}{1+i})^n}{1-\frac{1}{1+i}}\right]$$

Simplifying we obtain the result

$$P = A[\frac{(1+i)^n - 1}{(1+i)^n i}]$$

Using tables, we get the value of the factor (P/A,i,n) and compute

P = A (P/A,i,n)

Find *A* given *P*

The case is well known as *capital recovery.* From the formula to determine *P* given *A*, we conclude directly that

$$A = P[\frac{i(1+i)^n}{(1+i)^n - 1}]$$

Using tables, we get the value of the factor (A/P,i,n) and compute

A = P (A/P,i,n)

Find *F* given *A*

This is known as the *compound-amount* case. The easiest way to reason in this case is that we already know *P* given *A*. Therefore, all we need to do is to determine the future worth *F* of *P*. From the single-payment case,

$$\mathbf{F = P(1+i)^n}$$

Therefore,

$$F = A[\frac{(1+i)^n - 1}{(1+i)^n i}](1+i)^n$$

Simplifying the above relationship, we obtain:

$$F = A[\frac{(1+i)^n - 1}{i}]$$

Using tables,

F = A (F/A,i,n)

Find *A* given *F*

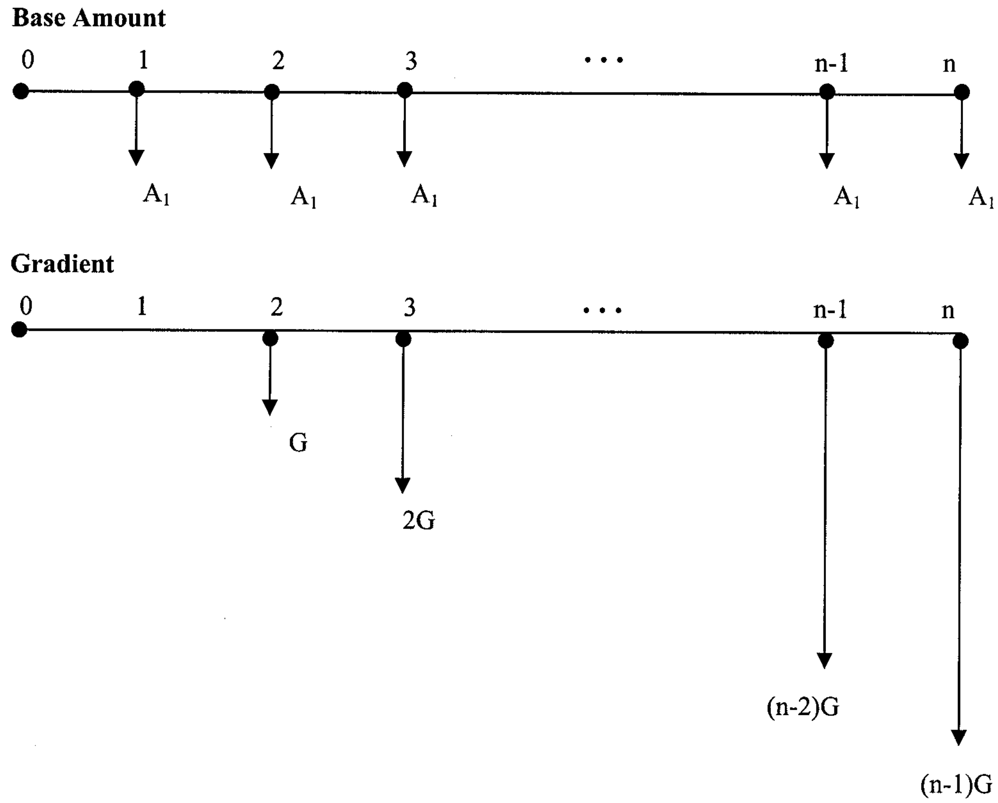
$$A = F[\frac{i}{(1+i)^n - 1}]$$

Using tables,

A = F (A/F,i,n)

Gradient-Series Transactions

Consider the following situation: A *base amount* is paid every year starting with the **first** year. Starting with the **second** year, each payment is greater than the previous one by a constant amount referred to as a **uniform gradient**. The value of the gradient is represented by G. The cashflows (assumed to be payments) corresponding to the **uniform-gradient** series are shown below:



We will study the following three cases:

<u>Case 1</u>: Find *P* given *G*
<u>Case 2</u>: Find F given *G*
<u>Case 3</u>: Find A given *G*

Find *P* given *G* and A_1

The present worth can be found as the sum of the present values of both the series of base payments and the series of gradient payments:

$$\mathbf{P = P_B + P_G}$$

First, we can find the present worth of the **base payment** for each period 1 through n. As we know, this can be calculated as

$$P_B = A_1\left[\frac{(1+i)^n - 1}{(1+i)^n i}\right]$$

$$= \mathbf{A_1\ (P/A,i,n)} \qquad \textbf{(using tables)}$$

Now we proceed to find the present worth of the **gradient series.**

$$P_G = G[1/(1+i)^2 + 2/(1+i)^3 + 3/(1+i)^4 + \ldots + (n-2)/(1+i)^{n-1} + (n-1)/(1+i)^n]$$

The above series can be written as a collection of sums of terms of *n-1* **geometric progressions**, as follows. Each sum S_j has *j* terms:

$S_{n-1} = G[1/(1+i)^2 + 1/(1+i)^3 + 1/(1+i)^4 + \ldots + 1/(1+i)^{n-1} + 1/(1+i)^n]$
$S_{n-2} = G[\quad 1/(1+i)^3 + 1/(1+i)^4 + \ldots + 1/(1+i)^{n-1} + 1/(1+i)^n]$
$S_{n-3} = G[\quad 1/(1+i)^4 + \ldots + 1/(1+i)^{n-1} + 1/(1+i)^n]$
.
.
.
$S_2 = G[\quad 1/(1+i)^{n-1} + 1/(1+i)^n]$
$S_1 = G[\quad 1/(1+i)^n]$

Note that

$$S_{n-1} = G\frac{1}{(1+i)^2}\left[\frac{1-\frac{1}{(1+i)^{n-1}}}{1-\frac{1}{(1+i)}}\right] = G\left[\frac{(1+i)^{n-1}-1}{i(1+i)^n}\right]$$

Similarly,

$$S_{n-2} = G\left[\frac{(1+i)^{n-2}-1}{i(1+i)^n}\right]$$

$$S_{n-3} = G\left[\frac{(1+i)^{n-3}-1}{i(1+i)^n}\right]$$

.
.
.

$$S_1 = G\left[\frac{(1+i)^1-1}{i(1+i)^n}\right]$$

Therefore, the present worth of the gradient series, P_G, can be found as the sum

$$\mathbf{P_G = S_1 + S_2 + \ldots + S_{n-2} + S_{n-1}}$$

$$P_G = \frac{G}{i(1+i)^n}[(1+i) + (1+i)^2 + \ldots + (1+i)^{n-1} - (n-1)]$$

Note that the sum of all the quantities within brackets, except the last one, is equal to the sum of terms of a geometric progression with first term (1+i), common ratio (1+i), and *n-1 terms.* Thus, after simplifying, we get:

$$P_G = \frac{G}{i(1+i)^n}\left[\frac{(1+i)^n-1}{i} - n\right]$$

$$= \mathbf{G\ (P/G,i,n)} \qquad \textbf{(using tables)}$$

Now we can calculate $\mathbf{P = P_B + P_G}$.

Find *F* given *G* and A_1

Since we can find F given P, by using $F = P(1+i)^n$ we conclude that

$$F_G = \frac{G}{i}\left[\frac{(1+i)^n-1}{i} - n\right]$$

$$= G\ (F/G,i,n) \qquad \textbf{(using tables)}$$

Also, since we can find F_B given A_1

$$F_B = A_1[\frac{(1+i)^n - 1}{i}]$$

$$= A\ \ (F/A,i,n) \qquad \textbf{(using tables)}$$

Now we can calculate $\mathbf{F = F_B + F_G}$.

Find *A* given *G* and *A₁*

Note that $\mathbf{A = A_G + A_1}$.

We can determine the equal-payment amount A_G the gradient series from (A/P,i,n) or (A/F,i,n) factor. The following result is obtained:

$$A_G = G[\frac{1}{i} - \frac{n}{(1+i)^n - 1}]$$

$$= G\ (A/G,i,n) \qquad \textbf{(using tables)}$$

Examples

Examples 2.1, 2.2, 2.9, 2.10 from the textbook.

Exercises

1. A certain sum of money will be deposited in a savings account that pays annual interest at the rate of 6% compounded each year. If all the money is allowed to accumulate, how much must be the initial deposit (at time 0) so that there will be $5000 after 10 years?

2. A student plans to deposit $600 each year in a savings account paying an annual interest of 6% compounded each year. How much will the student have at the end of 10 years?

3. Suppose that a fixed amount of money will be deposited in a savings account at the end of each year for 20 years. If the bank pay 6% per year, compounded annually, find the amount in the case that a total of $50,000 will be accumulated at the end of the 20-year period.

4. An engineer who is about to retire has accumulated $50,000 in a savings account that pays 6% per year, compounded annually. What is the *maximal* equal-amount the engineer will be able to withdraw each year during the next 10 years?

5. An engineer who is planning his retirement has decided that he will have to withdraw $10,000 from his savings account at the end of each year of a 12-year retirement period. How much must he have in the bank at the beginning of his retirement, if the bank pays 6% per year compounded annually?

6. An engineer is planning for a 15-year retirement. After careful planning, he has decided that he needs to withdraw $5,000 at the end of the *first* year, and to increase the withdrawals by $1,000 at the end of each successive year. Assuming an annual interest rate of 6% compounded every year, how much must he have in his savings account at the start of his retirement?

7. How much money must initially be deposited in a savings account paying 6% per year, compounded annually, to provide for 10 withdrawals that start at $6,000 and *decrease* by $500 each year?

Chapter 3
Nominal and Effective Interest Rates

- Nominal and Effective Interest Rates
- Interest Periods Equal to Payment Periods
- Interest Periods Smaller Than Payment Periods
- Continuous Compounding
- Interest Periods Larger Than Payment Periods

Nominal and Effective Interest Rates

Payment Period and Interest Period

Many times interest is compounded more often than once a year. The interval between two successive times of compounding is known as the *interest period.* For example, a bank claims to pay interest to its depositors at the rate of 6% per year, compounded *quarterly.* In this case the *payment period* is one year, but the interest period is one quarter.

Nominal and Effective Interest Rates

In the previous example, interest is compounded after each quarter, and it is allowed to accumulate until the next quarter, and so on until the payment period is over. The interest rate, expressed on an *annual basis*, is known as the *nominal interest rate.* The *effective interest rate* corresponding to the actual compounding or interest period (one quarter in the example) is $i = 6/4 = 1.5\%$.

Notation

r	nominal interest rate (expressed on an *annual* basis)
m	number of interest periods per year (number of times interest is compounded in one year)
i	effective interest rate (corresponds to interest period used in the formulas developed in Ch.2)

Interest Periods Equal to Payment Periods

In this case all the results obtained in Chapter 2 are correct after setting **i = r/m** and replacing ***n*** by ***mn.***

Example

An engineer plans to borrow $3,000 from his company credit union, to be paid in 24 equal monthly installments. The credit union charges interest at the rate of 1% per month on the unpaid balance. How much money must the engineer repay each month? Here both the payment and interest periods are equal to one month. Therefore, we can use the interest tables introduced in Chapter 2. Note that the following figures are given directly: (a) $i = r/m = .01$, and (b) $mn = 24$. Therefore, A = P (A/P, 1%, 24) = $3,000 (0.04707) = $141.21.

Interest Periods Smaller Than Payment Periods

Method 1: Determine the effective interest rate for the given **interest period** and treat each payment separately.

Method 2: Calculate an effective interest rate for the **payment period** (usually one year) and then proceed as when the interest period and payment period coincide. Recall, from Chapter 1, that $\mathbf{I = P[(1+i)^n - 1]}$. In words,

$$\textbf{Interest = Principal [(1 + interest rate per interest period)}^{\#\text{ of interest periods}} - 1]$$

If the payment period is equal to one year, then from the above relationship we can conclude that

$$I = P[(1+\frac{r}{m})^m - 1]$$

Since **I = Pi**, we can determine **i as i = I/P**. From this we obtain the following result:

$$i = (1+\frac{r}{m})^m - 1$$

This is the *effective* annual *interest rate.* If the payment period is not equal to one year, we can generalized the previous result as

$$i = (1+\frac{r}{\alpha})^\alpha - 1$$

where r is the nominal interest rate for the given payment period, $\forall$ is the number of interest periods per payment period, and i is the effective interest rate per payment period.

Example

An engineer deposits $1,000 in a savings account at the end of each year. If the bank pays interest at the rate of 6% per *year*, compounded *quarterly*, how much money will be accumulated in the account after 5 years?

Method 1

m=4, n = 5, mn = 20, r = 0.06, payment period = 1 year, interest period = 1 quarter. First, we calculate i=r/m = 0.06/4 = 0.015 (effective interest rate per quarter).

Using formulas
$F = F_1 + F_2 + F_3 + F_4 + F_5 = 1{,}000(1+0.015)^{16} + 1{,}000(1+0.015)^{12} + 1{,}000(1+0.015)^8 + 1{,}000(1+0.015)^4 + 1{,}000(1+0.015)^0$ = $5,652.46.

Using tables
F = 1,000(F/P,1.5%,16) + 1,000(F/P,1.5%,12) + 1,000(F/P,1.5%,8) + 1,000(F/P,1.5%,4) + 1,000 = 1,000 (1.2690 + 1.1956 + 1.1265 + 1.0614 + 1) = $5,652.50.

Method 2

$i = (1 + r/m)^m - 1 = (1 + 0.06/4)^4 - 1 = 0.06136$ (*effective* interest rate per year). Now, we can find the future worth as F = $1,000(F/A, 6.136%,5). Note that there are no tables for i = 6.136%. We can do one of two things. Either use the formula or interpolate values from tables. Using the formula, we get

$$F = A[\frac{(1+i)^n - 1}{i}] = \$1{,}000\,\frac{(1+0.06136)^5 - 1}{0.06136} = \$5{,}652.40$$

Exercise: Find the value of F using the tables.

Example

See Example 3.5, textbook.

Continuous Compounding

From book, pages 93-94:

$$i = e^r - 1$$

See **Example 3.3** for an application of this result.

Interest Periods Larger Than Payment Periods

Case 1

Interest is earned only by those payments that have been deposited or invested for the entire interest period. This is the usual mode of operation of banks and lending institutions. The following rules are applied:

1. All deposits made during an interest period are "moved" to the end of the interest period.
2. All withdrawals made during an interest period are "moved" to the beginning of the interest period.
3. Proceed as in the case where interest periods and payment periods coincide.

Case 2

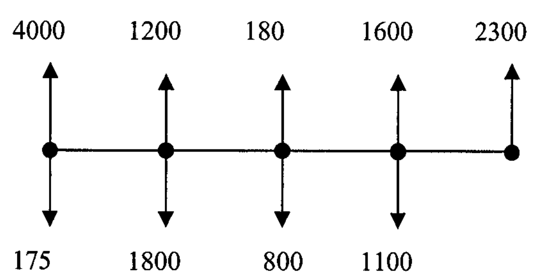
Any amount of money that is deposited between compounding times earns simple interest. If M is the number of payment periods, N the number of payment periods prior to the end of an interest period, and i is the interest rate per interest period, each deposit made during a compounding period should be multiplied by $(M/N)i$.

Example

A person has $4,000 in a savings account at the beginning of a calendar year. The bank pays interest at 6% per year, compounded quarterly. Assume the transactions shown below. The first interest period goes from January 1 to March 31; the second period from April 1 to June 30; the third period from July 1 to September 30; and the fourth period from October 1 to December 31.

IP	Date	Effective Date	Deposit	Withdrawal
1	01-10	01-01		175
1	02-20	03-31	1,200	
2	04-12	04-01		1,500
2	05-05	06-30	65	
2	05-13	06-30	115	
2	05-24	04-01		50
2	06-21	04-01		250
3	08-10	09-30	1,600	
3	09-12	07-01		800
4	11-27	10-01		350
4	12-17	12-31	2,300	
4	12-29	10-01		750

To calculate the future worth (balance in the account at the end of the year), we determine the effective interest rate per **quarter**, as 6%/4 = 1.5%. The cashflow for the example is shown below:



Using the information on the previous cashflow diagram, we obtain the result

F = (4000-175) (F/P,1.5%,4) + (1200-1800) (F/P,1.5%,3) + (180-800) (F/P,1.5%,2) + (1600-1100) (F/P,1.5%,1) + 2300

$$F = 3825(1.0614) - 600(1.0457) - 620(1.0302) + 500(1.0150) + 2300 = \$5601.21$$

Exercise

How much money must be deposited in a savings account each month to accumulate $10,000 at the end of 5 years, if the bank pays interest at the rate of 6% per year, compounded (a) monthly? (b) semiannually? (c) quarterly? (d) daily?

Examples

Examples 3.1, 3.2, 3.3, 3.4, 3.5.

Chapter 4
Use of Multiple Factors

- Uniform series starting after Period 1
- Uniform series and non-uniform cashflows
- Shifted gradients
- Case study: financing a home purchase

Uniform Series Starting After Period 1

Suppose that the uniform series consists of *m* payments equal to A and starts at (the end of) period *k*. We can determine either the **present worth** at period *k-1* or the **future worth** at period *k+m-1*. If we want the present worth **at time zero** we can proceed in two different (although equivalent) ways.

Method 1

Let the present worth at time *k-1* be represented by P_{k-1}. If we consider this as a future value from the point of view of time zero, we obtain the result

$$\mathbf{P_0 = P_{k-1}\ (P/F,i,k-1)}$$

where

$$\mathbf{P_{k-1} = A\ (P/A,i,m)}$$

Method 2

Let the future worth at time *k+m-1* be represented by F_{k+m-1}. If we consider this as a future value from the point of view of time zero, we obtain the result

$$\mathbf{P_0 = F_{k+m-1}\ (P/F,i,k+m-1)}$$

where

$$\mathbf{F_{k+m-1} = A\ (F/A,i,m)}$$

Example 4.2

See book, pages 113,114. Here *k=3, m=6, k+m-1 = 3+6-1 = 8.* A summary of the results is shown below:

Method 1

$\mathbf{P_{k-1} = P_2 = A\ (P/A,i,m)}$ = 800 (P/A,16%,6) = \$2,947.76
$\mathbf{P_0 = P_{k-1}\ (P/F,i,k-1)}$ = P_2 (P/F,16%,2) = 2,947.76 (P/F,16%,2) = \$2,190.78
A = 2,190.78 (A/P,16%,8) = \$504.36

Method 2

$\mathbf{F_{k+m-1} = F_8 = A\ (F/A,i,m)}$ = 800 (F/A,16%,6) = \$7,182
$\mathbf{P_0 = F_{k+m-1}\ (P/F,i,k+m-1)}$ = F_8 (P/F,16%,8) = \$2,190.51
A = 2,190.51 (A/P,16%,8) = \$504.30
A = 7,182 (A/F,16%,8) = \$504.32

Uniform Series And Non-Uniform Cashflows

The uniform series is considered as in the previous section. Each non-uniform payment is considered as an individual payment.

Example 4.3

See book, pages 114, 115.

Shifted Gradient Series

The **present worth** of the series can be calculated **two periods** before the gradient shows up the first time. After this, we can compute either the present worth at time zero, the future worth at time n, or the equivalent uniform annual worth for periods 1,2,...,n.

Example 4.9

See book, pages 119, 120. In the following calculations (not shown in the textbook) we will consider an interest rate of 5%. First, $P_3=P_B+P_G$. We developed the formulas for both P_B and P_G in Chapter 2. Consolidating the two formulas into one, we obtain the following result:

$$P_3 = (A_1 + \frac{G}{i})[\frac{(1+i)^n - 1}{i(1+i)^n}] - \frac{nG}{i(1+i)^n}$$

Now, $A_1 = 100$, $G = 50$ and $i = 0.05$. Using these values and the previous formula, we can get

$$P_3 = (100 + \frac{50}{0.05})[\frac{(1.05)^5 - 1}{0.05(1.05)^5}] - \frac{250}{0.05(1.05)^5} = \$844.79$$

This value can also be obtained from the tables as P_3 = 100 (P/A,5%,5) + 50 (P/G,5%,5) = 100 (4.3295) + 50 (8.237) = $844.80. Finally, we can calculate P_0 as

$$P_0 = 100/(1.05) + 100/(1.05)^2 + 100/(1.05)^3 + 844.79/(1.03)^3 = 1{,}002.09.$$

Exercise

See **Example 4.12** on page 124-126 of the textbook. Now, repeat this example using formulas for (a) and (b) instead of using the tables.

Case Study: Financing a Home Purchase

You get credit by promising to pay in the future for something you receive in the present. Credit is (in a way) a convenience. It lets you take out a loan to buy a house. With credit, you can enjoy your new house while you are paying for it. Or you can buy a house when you are lacking ready cash. But there are strings attached to credit too. It usually costs something. And of course, what is borrowed must be paid back.

If you are thinking of borrowing, your first step should be to figure out how much it will cost you and whether you can afford it. Then you should shop around for the best terms. Perhaps your next consideration is the matter of paying for a new house. Here are two ways many people use to estimate ability to make house payments:

1. The price of the house generally should not exceed two times your annual family income.
2. A homeowner usually should not pay more than 38% of income after Federal tax for monthly housing expense (payment on the mortgage loan plus average cost of heat, utilities, repair and maintenance).

A house is probably the single largest credit purchase for most consumers-and one of the most complicated. By law the lender must give you, in advance, certain information about the costs you will pay when you close the loan. This event is called settlement or closing. Typically, the closing costs include: origination fee, appraisal, survey, attorney's fee, processing, and others (recording, credit report, etc.)

When you are ready to apply for credit, you should know what creditors think is important in deciding whether you are credit-worthy. Creditors look for an ability to repay debt and a willingness to do so. Additionally, a little extra security to protect their loans. They speak of the three C's of credit: *capacity, character, and collateral.* Creditors ask for employment information and your expenses. They will look at your credit history and signs of stability. Creditors want to know what you have that could be used to back up or secure your loan.

Exercise

A 15-year old house is being sold for $75,900. The owner of the house still owes $34,400 to the mortgage company. He wants to buy a new house for $122,000. The real estate company selling the old house charges 6% of the selling price as a selling fee. The remaining quantity plus an additional $10,000 will be used as down payment for the new house. In order to pay for the difference between the price of the new house and the down payment a 15-year bank loan will be used. If the bank charges a nominal rate of 8% per year compounded monthly, how much must be paid each month? (a) Solve this problem using the effective interest rate per month. (b) Solve again using the effective interest rate per year.

Chapter 5
Present Worth and Capitalized Cost Evaluation

Purpose

To compare alternatives from the point of view of the total sum of present worth components of a series of cashflows included in a common evaluation period known as the *planning horizon*. Generally speaking, the cashflows to be considered are: first cost, annual costs, non-recurring costs, and salvage value. The series can be *finite* or *infinite*.

Finite Series of Cashflows

When the series is finite there are two cases to be considered:

- All alternatives have the same life.
  In this case, the planning horizon is defined as the life of the alternatives.
  Example 5.1.

- Alternatives have different lives.
  In this case, the planning horizon is defined as the *least common multiple* (LCM) of the lives of the alternatives. AS an illustration, if we want to compare two alternatives with lives equal to 6 and 9, the planning horizon should be the LCM of 6 and 9, which is equal to 18.
  Example 5.2.

Infinite Series of Cashflows

When the series is infinite, the present worth of all its cashflows is known as the *capitalized cost.* The basic formula to determine capitalized costs can be derived as follows. Let A be the cashflow at (the end of) each period of a planning horizon consisting of periods 1, 2, 3, ..., n, **where** $n = \infty$. The present worth of the cashflow taking place at period 1 is $A/(1+I)$. Similarly, the present worth of the cashflow taking place at period 2 is equal to $A/(1+i)^2$. The sum of present-worth terms for all periods (infinitely many) is equal to the sum of the terms of the following infinite geometric progression with first term equal to $A/(1+i)$ and common ratio equal to $1/(1+i)$:

$$\mathbf{P} = A/(1+i) + A/(1+i)^2 + \ldots + A/(1+i)^m + \ldots$$

$$= [A/(1+i)]/[1 - 1/(1+i)]$$

$$= A/i$$

Similarly, if the cashflows take place every f years,

$$\mathbf{P} = A/[(1+i)^f - 1]$$

Example

Find the capitalized cost of an infinite series of annual payments equal to $10,000 using an interest rate of 8% compounded annually. Here A = 1,000 and $i = 0.08$. Therefore, P = 10,000/0.08 = $125,000. Just as an illustration, suppose that we are interested in the present worth of the first 100 terms of this series. This is equal to 10,000 (P/A,8%,100) = 10,000 x 12.4943 = $124,943. Note that this value is approaching the limit of $125,000.

Example

Find the capitalized cost of an infinite series of payments equal to $50,000 *every 5 years*, using an interest rate of 8% compounded *annually*. In this case, $f = 5$, $i = 0.08$, and $A = 50,000$. Therefore, $P = 50,000/[(1+0.08)^5 - 1] = \$106,535.28$.

Examples

1. Example 5.3. This example shows how to compute the capitalized cost of a series of recurring and non-recurring payments.

2. Example 5.4. This example shows how to compare alternatives using the capitalized cost criterion.

Exercises

1. A new bridge with a 100-year life is expected to have an initial cost of $20 million. This bridge must be resurfaced every five years, at a cost of $1 million. The annual inspection and operating costs are estimated to be $50,000. Determine the capitalized cost using an interest rate of 10% per year compounded annually.

2. The ABC Company is currently earning an average before-tax return of 25% on its total investment. The board of directors of ABC is considering three proposals as given in the following table (in $1000s):

End of Year	0	1	2	3	4
Proposal A	-$40	18	18	18	18
Proposal B	-$60	25	25	25	25
Proposal C	-$50	27	27	27	27

Proposal C involves a great deal of risk. For this reason, ABC believes that the net present worth for the proposal should be computed using a 40% rate of return. Which of these proposals are acceptable?

3. Compute the present worth of the following cashflows at (a) i = 6% per year; (b) $i = 15\%$ per year compounded annually. Explain the results.

End of Year	0	1	2	3	4
Cashflow,$1000	-40	12	12	12	12

4. A company can purchase either of two alternative machines, A and B, with the following characteristics:

	P	**A**	**n**
Machine A	$15000	$6000	5
Machine B	9000	3000	3

Here P is the initial outlay, A is the annual net cashflow, and n is the service life. Which machine should be purchased, assuming an interest rate of 10%?

Chapter 6
Equivalent Uniform Annual Worth Calculation

- Salvage Sinking-Fund Method
- Salvage Present-Worth Method
- Capital-Recovery-Plus-Interest Method

Positive and Negative Cashflows

Positive cashflows: Revenues
Negative cashflows: Costs

Salvage Sinking-Fund Method

EUAW = – P(A/P,i%,n) + SV (A/F,i%,n) (Method 1)

Salvage Present-Worth Method

EUAW = [– P + SV(P/F,i%,n)] (A/P,i%,n) (Method 2)

Capital-Recovery-Plus-Interest Method

EUAW = – (P– SV) (A/P,i%,n) – SV(i) (Method 3)

Examples

Examples 6.1, 6.2, 6.3.

Comparing Alternatives by EUAW

The annual-worth method for comparing alternatives is the easiest of all relevant procedures since only one life cycle for each alternative is needed. The alternative to be selected should be the one resulting in the lowest equivalent uniform annual worth. In the case of an alternative that has an infinite (or very long) life and requires an initial investment of P, the equivalent uniform annual worth is equal to Pi.

Examples

Examples 6.4, 6.5, 6.6.

Exercises

1. The ABC Company is considering the replacement of a current milling machine. This machine has 15 more years of service life, a current salvage value of \$500, and a salvage value of \$100 at the end of its service life. The original purchase cost was \$5000 and the annual cost is \$1000. An improved machine has a service life of 10 years, a purchase cost of \$7000, an annual cost of \$375, and salvage value of \$500. Assuming that any estimates beyond 10 years are not accurate enough for decision-making, should the company replace the machine? ? Let i = 15%.

2. Suppose that in Problem 1 accurate forecasts of annual costs cannot be accepted beyond five years into the future. All other data remain the same. Should the company replace the machine

Proof of the Result for Third Method

PW = – P + SV (P/F,i%,n)
EUAC = [–P + SV (P/F,i,n)] (A/P,i%,n) = –P (A/P,i%,n) + SV (P/F,i%,n) (A/P,i%,n)
= –P (A/P,i%,n) + SV (A/F,i%,n)

Now, note that (A/P,i%,n) = (A/F,i%,n) + i. This can be proved as follows:

$(A/F,i\%,n) + i = i / [(1+i)^n - 1] + i = i\,(1+i)^n / [(1+i)^n - 1] = (A/P,i\%,n)$

Therefore,

$$
\begin{aligned}
EUAC &= -P\,(A/P,i\%,n) + SV\,(A/F,i\%,n) = -P\,(A/P,i\%,n) + SV\,[(A/P,i\%,n) - i] \\
&= -P\,(A/P,i\%,n) + SV\,(A/P,i\%,n) - SV(i) \\
&= (SV - P)\,(A/P,i\%,n) - SV(i)
\end{aligned}
$$

Summary of Procedures (Based on Example 6.4)

Method 1

$$\mathbf{EUAW_A = -26{,}000\ (A/P,15\%,6) + 2{,}000\ (A/F,15\%,6)} - 11{,}800$$

$$= -26{,}000\ (0.26424) + 2{,}000\ (0.11424) - 11{,}800 = -\$18{,}442$$

$$\mathbf{EUAW_B = -36{,}000\ (A/P,15\%,10) + 3{,}000\ (A/F,15\%,10)} - 9{,}900$$

$$= -36{,}000\ (0.19925) + 3{,}000\ (0.04925) - 9{,}900 = -\$16{,}925$$

Method 2

$$\mathbf{EUAW_A = [-26{,}000 + 2{,}000\ (P/F,15\%,6)]\ (A/P,15\%,6)} - 11{,}800$$

$$= [-26{,}000 + 2{,}000(0.4323)](0.26424) - 11{,}800 = -\$18{,}442$$

$$\mathbf{EUAW_B = [-36{,}000 + 3{,}000\ (P/F,15\%,10)]\ (A/P,15\%,10)} - 9{,}900$$

$$= [-36{,}000 + 3{,}000\ (0.2472)](0.19925) - 9{,}900 = -\$16{,}925$$

Method 3

$$\mathbf{EUAW_A = -\ (26{,}000 - 2{,}000)\ (A/P,15\%,6) - 2{,}000\ (0.15)} - 11{,}800$$

$$= -\ (26{,}000 - 2{,}000)\ (0.26424) - 2{,}000\ (0.15) - 11{,}800 = -\$18{,}442$$

$$\mathbf{EUAW_B = -\ (36{,}000 - 3{,}000)\ (A/P,15\%,10) - 3{,}000\ (0.15)} - 9{,}900$$

$$= -\ (36{,}000 - 3{,}000)\ (0.19925) - 3{,}000\ (0.15) - 9{,}900 = -\$16{,}925$$

Chapter 7
Rate of Return Computations for a Single Project

- Definition, Concept, Basic Equations
- Descartes' Rules of Signs
- Single Rate-of-Return Value
- Multiple Rate-of-Return Values

Definition, Concept, Basic Equations

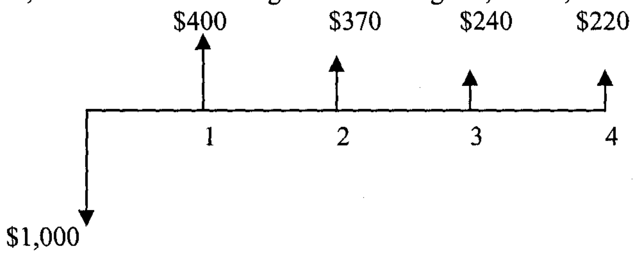
Rate of Return

Interest rate applied to a transaction (loan or investment) so that the final cashflow (payment or receipt) brings the balance to zero. In other words, assuming an investment situation, it is the percentage or rate of interest earned on the *unpaid* or *unrecovered* balance of the investment so that this balance becomes equal to zero at the end of the investment period. The unrecovered balance of an investment is the portion of the initial investment that remains to be recovered after interest payments have been added and receipts have been deducted, up to the point in time being considered.

Concept

An illustration of this definition is given in Example 7.1, Table 7.1. Note that for the interest rate to be called *rate of return* it must be charged on the balance and the last payment must bring the balance to zero. Table 7.2 shows an incorrect interpretation of the rate of return.

As another illustration, consider the following cash-flow diagram, which, as we will see, has an ROR = 10%:



	n = 0	n = 1	n = 2	n = 3	n = 4
• Amount owed	-1,000	-1,000	-700	-400	-200
• Interest owed (10%)	0	-100	-70	-40	-20
• Amount paid	0	400	370	240	220
• **Unpaid balance**	**-1,000**	**-700**	**-400**	**-200**	**0**

As a result of the above analysis, this investment is fully recovered at the end of the last payment, using an interest rate of 10%. That is, ROR = 10%. Now, suppose that you want to decide if it is a good idea to go ahead with the investment. The answer to the question can be reached if we can answer this other question first: *Do you have a better opportunity to invest the $1,000?* Keep in mind that the future worth of the net cash-flows generated by the investment is equal to

$$F_1 = 400(1.10)^3 + 370(1.10)^2 + 240(1.10) + 220 = 1,464$$

If there is another opportunity, for example a limited partnership, that offers an interest rate of 13%, then the future worth of the initial amount available for investment would be

$$F_2 = 1000(1.13)^4 = 1,630.47$$

Since $F_2 > F_1$, original investment is not *attractive.*

Let us calculate the unpaid balance at the end of the 4th year.

Interest Rate = 9%

n = 0	n = 1	n = 2	n = 3	n = 4

• Amount owed	-1,000	-1,000	-690	-382.10	-176.48
• Interest owed (9%)	0	-90	-62.10	-34.39	-15.88
• Amount paid	0	400	370	240	220
• **Unpaid balance**	**-1,000**	**-690**	**-382.10**	**-176.48**	**27.63**

The above *unpaid* balance is *positive.* This is a *surplus* balance. Recall that we are defining balance as income minus debt. Therefore, the last payment is larger that what was needed to have the balance equal zero, and then $i = 0.09$ is not the ROR. The unpaid balance is always equal to the future worth of the NPW calculation. Let us see how it works:

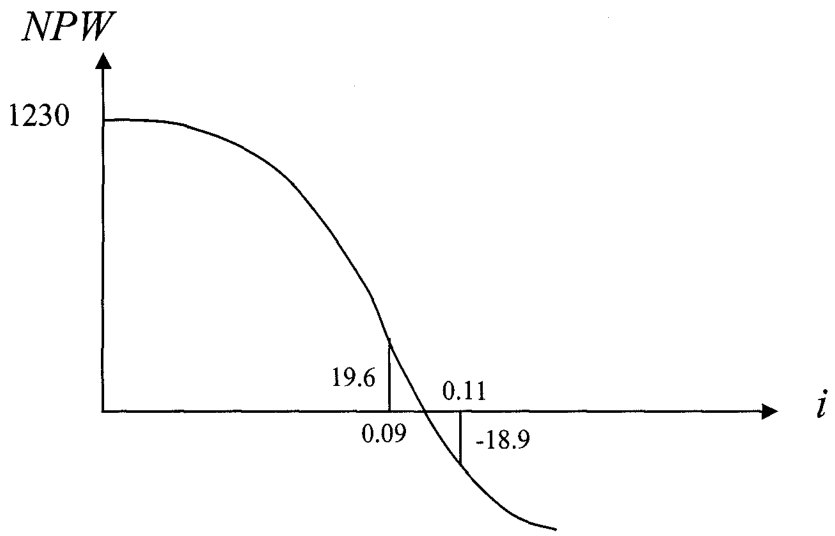
$$NPW(0.09) = -1{,}000 + 400/(1.09) + 370/(1.09)^2 + 240/(1.09)^3 + 220/(1.09)^4 = 19.57166$$
$$F(0.09) = 19.57166\,(1.09)^4 = 27.63$$

Interest Rate = 11%

	n = 0	n = 1	n = 2	n = 3	n = 4
• Amount owed	-1,000	-1,000	-710	-418.10	-224.09
• Interest owed (11%)	0	-110	-78.10	-45.99	-24.65
• Amount paid	0	400	370	240	220
• **Unpaid balance**	**-1,000**	**-710**	**-418.10**	**-224.09**	**-28.74**

The above *unpaid* balance is *negative.* Thus, the debt is larger than the income. This means that after the last payment an unpaid balance still exists. Again, $i = 0.11$ is not the ROR. This is a *slack* balance. As efore, we can compute it as follows:

$$NPW(0.11) = -1{,}000+400/(1.11)+370/(1.11)^2+240/(1.11)^3+220/(1.11)^4 = -18.93259$$
$$F(0.11) = -18.93259(1.11)^4 = -28.74$$



Note

A common misinterpretation of what the rate of return (ROR) of a project measures is to view it as the rate of interest earned on the initial outlay required by the project.

Basic Equations

Let CF_j be the cashflow corresponding to the *end* of year j, where $j = 0, 1, 2, \ldots, n$. Usually, CF_0 is an initial investment taking place at the beginning of the planning horizon (end of year 0). **In this case, $CF_0 < 0$.** In general, each cashflow can be either positive, negative, or zero. The net present worth of the transaction, here represented by the notation *NPW*, is given by

$$NPW(i) = CF_0 + \frac{CF_1}{1+i} + \frac{CF_2}{(1+i)^2} + \ldots + \frac{CF_n}{(1+i)^n}$$

In the special case where $CF_j = A$ for $j = 1,2,\ldots,n$, the net present worth is given by the following relationship:

$$NPW(i) = CF_0 + \frac{A}{1+i} + \frac{A}{(1+i)^2} + \ldots + \frac{A}{(1+i)^n}$$

Note that this equation can be rewritten as

$$NPW(i) = CF_0 + A(P/A, i\%, n)$$

The rate of return is the solution $i = i^*$ of the equation

$$NPW(i) = 0$$

Descartes' Rules of Signs

An upper bound on the number of positive rates of return is the number of sign changes in the sequence of coefficients $CF_0, CF_1, CF_2, \ldots, CF_n$.

Single Rate-of-Return Value

Example 7.2

$$NPW(i) = -5000 + \frac{100}{1+i} + \frac{100}{(1+i)^2} + \ldots + \frac{100}{(1+i)^9} + \frac{7100}{(1+i)^{10}} = 0$$

This equation can be expressed as

$$NPW(i) = -5000 + 100(P/A, i\%, 10) + 7000(P/F, i\%, 10) = 0$$

Equivalently, using the formulas for the factors:

$$NPW(i) = -5000 + 100\frac{1-(1+i)^{-10}}{i} + 7000(1+i)^{-10} = 0$$

Bisection Method

We can solve this non-linear equation using the *bisection method.* We start by finding two values at which the left-hand side of the above equation will have opposite signs. For example, choose $i = 0.05$ and i = 0.06. We can verify that for these values, NPW(0.05) = 69.56 and NPW(0.06) = -355.23. Instead of computing these values from the equation, we could use tables. Doing so, we would obtain NPW(0.05) = 69.46 and NPW(0.06) = -355.19.

The bisection method works in the following manner. At each iteration it shortens the interval containing the solution, by halving the current interval and determining the portion of it corresponding to two values of NPW(i) having opposite signs. This procedure is repeated until the rate of return is available within a specified level of precision (decimal places). The following is an illustration of this procedure:

Iteration	Interval [A;B]	Mid-Point M	NPW(A)	NPW(B)	NPW(M)
0	[0.05;0.06]	0.055	69.56	-355.23	-148.22
1	[0.05;0.055]	0.0525	69.56	-148.22	-40.71
2	[0.05;0.0525]	0.05125	69.56	-40.71	14.074
3	[0.05125;0.0525]	0.051875	14.074	-40.71	-13.41
4	[0.05125;0.051875]	0.0515625			

Note that based on these results we can establish that the ROR is $i^* = 0.0515625$.

Interpolation

This is less computationally intensive and less precise than the bisection method. From the tables we have the following two points (X,Y): (0.05, 69.46) and (0.06, -355.19). Note that here the X-value is the interest rate *i* and the Y-value is *NPW(i).*

The equation of the straight line through these two points is

$$\mathbf{Y = 69.46 - 42{,}465(X - 0.05)}$$

Setting Y=0 and solving for X we obtain X = 0.0516. Therefore, i^* = 0.0516.

Exercise

Solve Example 7.3, textbook. The procedure to be used in this example is the EUAW method. Thus, we want to find the value of *i* such that EUAW(i) = 0. Here the equation is –5000 (A/P,i,10) +100 + 700 (A/F,i,10) = 0. Verify that the solution is *i = 5.16%.*

Multiple Rate-of-Return Values

When multiple ROR's exist, there is no rational means for deciding which one is most appropriate for conducting an ROR analysis. Since the most often applied methods are not designed to consider multiple ROR's, the most common practice is to abandon the use of the ROR concept as a basis of comparison. It would be better in this case to evaluate the NPW for various assumed values of the interest rate.

Example 7.4

The equation to be solved is:

$$NPW(i) = 2000 - \frac{500}{1+i} - \frac{8100}{(1+i)^2} + \frac{6800}{(1+i)^3} = 0$$

Note that the sequence of coefficients (cashflows) has two sign changes. **Therefore, according to Descartes'rules, there are at most two positive rate-of-return values.** Let us define X = 1/(1+i). In this case, the above equation can be reformulated as indicated below, after dividing all coefficients by 100:

$$\mathbf{68X^3 - 81X^2 - 5X + 20 = 0}$$

The roots (solutions) of this cubic polynomial equation are X = -0.44679, X = 0.93057, and X = 0.70746. Since X = 1/(1+i), we can obtain *i* from

$$i = (1/X) - 1$$

In conclusion, *i* = -3.23819 (which is not acceptable), i^* = 0.07461 or **7.47%**, and i^* = 0.41351 or **41.35%**. Figure 7.4, book, page 212, shows the *NPW(i)* as a function of *i.*

Composite Rate of Return

Definition

The composite rate of return (*i'*) is a rate calculated under the assumption that positive cashflows (money available) can be reinvested at a specified interest rate equal to *c*.

Iterative Relationship

Let $\mathbf{F_t}$ be the net investment value in year *t*. This can obtained from the value of the net investment $\mathbf{F_{t-1}}$ corresponding to year *t-1,* using the F/P factor, and the net cashflow $\mathbf{C_t}$ in year *t:*

$$F_t = F_{t-1}(1+i) + C_t$$

where t = 1, 2, …, n. Here $i = c$ if $\mathbf{F_{t-1}} > 0$, or $i = i'$, otherwise.

Composite Rate of Return Equation

$$F_n(i') = 0$$

Illustration

Year	Cash Flow
0	50
1	-200
2	50
3	100

$F_0 = 50$, $C_1 = -200$, $C_3 = 100$. Assume c = 0.15 (reinvestment interest rate). The net amount available in year 1 is equal to $F_1 = 50\ (1 + 0.15) - 200 = -\ 142.50$. Now, we can calculate the net amount for year 2. Since the net amount for year 1 was negative, we use the (unknown) composite rate of return. Thus, $F_2 = -142.50\ (1 + i') + 50 < 0$. The net amount for year 3 is calculated as $F_3 = F_2\ (1 + i') + C_3 = -142.50(1 + i')^2 + 50(1 + i') + 100$. The composite rate of return is the solution of the following equation:

$$\mathbf{-142.50(1 + i')^2 + 50(1 + i') + 100 = 0}$$

Let X = 1+i'. Using this transformation, the above equation can be reformulated as

$$\mathbf{-142.50\ X^2 + 50\ X + 100 = 0}$$

which is a quadratic equation having the solutions X = 1.03132 and X = -0.680442. In conclusion, the composite rate of return is i = 0.03132 or 3.13%.

Exercise

See Example 7.5, book, pages 218-220.

Chapter 8
Rate-of-Return Evaluation for Selecting Alternatives

Comparison of Two Alternatives

Let us consider two alternatives *A* and *B*, and let us assume that Alternative A has a lower initial investment than alternative B. Moreover, let CF_{At} and CF_{Bt} the cashflows for activities A and B, respectively, at time t = 0, 1, 2, ..., n. Note that by definition, $CF_{B0} > CF_{A0}$. Now let us define the *net* or *incremental* cashflow at period *t* as:

$$NCF_t = CF_{Bt} - CF_{At}$$

From this point on, we view NCF_t as the cashflow at time *t* of an alternative. Its cashflow diagram will be referred to as the *Incremental Cashflow Diagram.* We want to determine the ROR for this incremental cashflow diagram to compare it against a *Minimal Attractive Rate of Return* (MARR) and decide which of the two original alternatives *A* and *B* should be selected for investment.

The minimal attractive rate of return to be used in judging the economic attractiveness of a proposed investment is normally a policy matter to be determined by the top management of an organization. The MARR is usually set after considering a list of factors including:

- Availability of funds for investment and their sources (equity or borrowing)
- Competing investment opportunities
- Risks
- Time required to recover investments (short, medium, long)
- The "going price of money" (interest rates, prime rate paid by large banks, government notes and bonds)

It should be recognized that there is no way for one person to tell another person or an organization what its minimal attractive rate of return *should be.*

PW Procedure

1. Prepare individual cashflow diagrams.
2. Generate the incremental cashflow diagram along a period equal to the least common multiple (LCM) of the lives of the alternatives.
3. Formulate the present-worth equation $NPW_{B-A}(i) = 0$ and solve it to derive the value of the rate of return, i^*_{B-A}.
4. If $i^*_{B-A} <$ MARR select alternative *A*. Otherwise, select alternative *B*.

EUAW Procedure

1. Prepare individual cashflow diagrams for one cycle of each alternative.
2. Generate the incremental cashflow diagram along a period equal to the least common multiple (LCM) of the lives of the alternatives.
3. Formulate the basic equation and solve it to derive the value of the rate of return, i^*_{B-A}.

    - Method 1

      From individual cycles of alternatives *A* and *B*, formulate and solve the basic equation $EUAW_B(i) - EUAW_A(i) = 0$.

    - Method 2

      From the incremental cashflow diagram for the LCM, formulate and solve the equation $EUAW_{B-A}(i)$.

4. If $i^*_{B-A} <$ MARR select alternative *A*. Otherwise, select alternative *B*.

Examples

We will consider Examples 8.3 and 8.6 to illustrate the application of these methods.

Example 8.3

The incremental cashflow diagram is shown in Figure 8-1, book, page 239. The **Present Worth Equation** is formulated below:

-5000 + 1900(P/A,i%,10) – 11000(P/F,i%,5) + 2000(P/F,i%,10) = 0

Using the formulas for the above factors:

$$-5000 + 1900\,\frac{1-(1+i)^{-10}}{i} - 11000\,(1+i)^{-5} + 2000\,(1+i)^{-10} = 0$$

If we set $X = 1 + i$ and divide by 100, we can rewrite the above equation as

$$f(X) = -50 + 19\,\frac{1-X^{-10}}{X-1} - 110X^{-5} + 20X^{-10} = 0$$

The following results are obtained using the bisection method, starting with $a = 1.12$ and $b = 1.15$. Note that $f(a) = 1.38$ and $f(b) = -4.39$.

$X_1 = (a+b)/2 = 1.135$	$f(X_1) = -1.6919$
$X_2 = (a+X_1)/2 = 1.1275$	$f(X_2) = -0.2077$
$X_3 = (a+X_2)/2 = 1.12375$	$f(X_3) = 0.5715$
$X_4 = (X_2+X_3)/2 = 1.125625$	$f(X_4) = 0.1787$
$X_5 = (X_2+X_4)/2 = 1.126562$	

Based on the above results, we can conclude that $i^*_{B-A} = X_5 - 1 =$ 0.12656 or 12.656%. Since (in this example) MARR = 15%, we choose the semiautomatic machine (alternative *A*).

Exercise

Solve this example using the interpolation method.

Remark

If we multiply the equation f(x) = 0 given above by X-1 and then by X^{10}, we obtain the following *polynomial* equation:

$$-50X^{11} + 69X^{10} - 110X^{6} + 110X^{5} + 20X - 39 = 0$$

Exercise

Plot the 11th degree polynomial given above.

Alternative Basic Equation

$$NPW(i) = CF_0 + \frac{CF_1}{1+i} + \frac{CF_2}{(1+i)^2} + \ldots + \frac{CF_n}{(1+i)^n}$$

Dividing all given cashflows by 100:

$$-50 + \frac{19}{1+i} + \frac{19}{(1+i)^2} + \frac{19}{(1+i)^3} + \frac{19}{(1+i)^4} - \frac{91}{(1+i)^5} + \frac{19}{(1+i)^6} + \frac{19}{(1+i)^7} + \frac{19}{(1+i)^8} + \frac{19}{(1+i)^9} + \frac{39}{(1+i)^{10}} = 0$$

If we set $X = 1/(1+i)$, the above equation can be re-written as:

$$39X^{10} + 19X^9 + 19X^8 + 19X^7 + 19X^6 - 91X^5 + 19X^4 + 19X^3 + 19X^2 + 19X - 50 = 0$$

Exercise

Plot the 10^{th} degree polynomial found above.

Alternatively, setting $X = 1+i$, multiplying by X^{10}, and finally multiplying by –1, we obtain the following polynomial equation:

$$50X^{10} - 19X^9 - 19X^8 - 19X^7 - 19X^6 + 91X^5 - 19X^4 - 19X^3 - 19X^2 - 19X - 39 = 0$$

Exercise

Plot the 10^{th} degree polynomial found above.

Example 8.6

We want to use the EUAW method using the data given for Example 8.3. See Table 8-4, book, page 239.

Method 1

$EUAW_f$ = -13000(A/P,i%,5) + 2000(A/F,i%,5) – 1600 = 0
$EUAW_s$ = -8000(A/P,i%,10) – 3500 = 0
Basic equation: $EUAW_f$ – EUAWs = 0

-13000(A/P,i%,5) + 2000(A/F,i%,5) + 8000(A/P,i%,10) +1900 = 0

Dividing all terms by 100 and using the formulas for the above factors, we obtain the following result:

$$-130\frac{i(1+i)^5}{(1+i)^5-1} + 20\frac{i}{(1+i)^5-1} + 80\frac{i(1+i)^{10}}{(1+i)^{10}-1} + 19 = 0$$

Verify that the solution is $i = 0.126486$.

Exercise

Solve the above equation using (a) the bisection method, and (b) interpolation.

Method 2

In this case LCM = 10. The basic equation obtained from the incremental cashflow diagram (see book, Figure 8-1, page 239) is given below:

$EUAW_{B\text{-}A}(i)$ = -5000(A/P,i%,10) + 1900 – 11000(P/F,i%,5)(A/P,i%,10) + 2000(A/F,i%,10) = 0

After dividing by 100 and replacing the factors by formulas, we obtain:

$$-50\frac{i(1+i)^{10}}{(1+i)^{10}-1} - 110\left[\frac{i}{(1+i)^5}\right]\left[\frac{i(1+i)^{10}}{(1+i)^{10}-1}\right] + 20\frac{i}{(1+i)^{10}-1} + 19 = 0$$

Exercises

1. Find the polynomial equation corresponding to the equation given above.
2. Solve the polynomial equation by the bisection method. Find the ROR.
3. Solve the polynomial equation by interpolation.

Chapter 9
Benefit/Cost Ratio Analysis

Benefit Cost Ratio

The most commonly used method for selecting public work projects funded by federal, state, and municipal government agencies is the *benefit cost ratio*, defined as

$$BCR = (B - D) / C$$

where B is the dollar-value of total benefits associated with a project, D is the dollar-value of all its disbenefits or disadvantages, and C is the total project cost. For a project to be desirable, BCR > 1.

Net Benefit Value

The net benefit value is defined as

$$NBV = B - D - C$$

Again, for a project to be desirable, NBV > 0.

Modified Benefit Cost Ratio

If maintenance and operation (M&O) costs are separated from the initial investment I_0, the modified benefit cost ratio is defined as

$$MBCR = (B - D - M\&O) / I_0$$

Remarks

1. Benefit and disbenefit quantification is usually not very precise. Furthermore, the distinction between disbenefits and costs is somewhat conjectural. If questions arise about the classification of disbenefits, it is better to use the NBV approach.

2. Either present worth (NPW), future worth, or annual worth (EUAW) approach may be used to evaluate B, D, and C, provided that the same method is used for the three terms.

Example

A city is close to a major seaport. Currently, a congested two-way highway connects the city to the port. A proposal for building a superhighway is being considered. The construction cost of the superhighway is $280 million, and the annual maintenance cost is $1.5 million. Furthermore, the new project offers the following benefits:

1. Additional business having a value of $50 million per year.
2. Future economic growth resulting in an increase of $5 million beginning in the second year and ending in the tenth year.
3. Reduction of accidents, resulting in savings of (approximately) $0.8 million per year.

The following disadvantages are associated with the new project:

1. Destruction of valuable farmland currently totaling $1.3 million per year.
2. Decrease in commercial activity along the present highway, valued at $0.7 million per year.

Assume a lifetime of 30 years and an interest rate of 7% compounded annually.

<u>Solution</u>

Over the entire 30-year period, the yearly net benefit, B–D, is calculated as follows, using the EUAW method:

$$B - D = \$50 + \$5\,(A/G,7\%,10)\,(P/A,7\%,10)\,(A/P,7\%,30) + \$0.8 - (\$1.3 + \$0.7)$$
$$= \$60.0 \text{ million}$$

The yearly costs are equal to

$$C = \$280\,(A/P,7\%,30) + \$1.5 = \$24.1 \text{ million}$$

The benefit cost ratio is

$$BCR = 60.0/24.1 = 2.49$$

Since BCR > 1, the proposal is desirable.

Examples

(a) Example 9.1.
(b) Example 9.2.
(c) Example 9.3. This example shows how to select alternatives.

Chapter 10
Special Topics

- Replacement Analysis (Book, Ch. 10)
- Bonds (Book, Ch. 11)
- Depreciation Models (Book, Ch. 13)
- Break-Even Point (Book, Ch. 16)

Replacement Analysis

Replacement Decision

This decision depends on the following factors: (a) physical condition, (b) obsolescence, (c) external economic conditions. Only current and future costs and investments are relevant.

Economic Life

Consider an asset with *service life* equal to *n* years. As operating equipment ages, the capital costs typically decline while operating costs increase. The annual *capital recovery cost* (see Chapter 6) CR(j) after *j* years (j = 1,2,...,n) is given by:

$$CR(j) = (P - SV)\,(A/P,i\%,n) + SV(i)$$

If the annual *operating cost* is represented by *A(j),* the total annual cost, *EUAC(j)*, is calculated as

$$EUAC(j) = CR(j) + A(j)$$

The *economic life* is given by the value *j** at which EUAC(j) is minimized.

Example

A machine has an initial cost of $10,000, and a salvage value, at any time, of $500 because it is a special-purpose custom-built machine. The service life is 10 years. Annual operating costs are $2000 for first and second years, with an increase of $600 per year thereafter. MARR = 10%. Find the economic life.

$$CR(j) = (10{,}000 - 500)\,(A/P,10\%,j) + (0.10)(500)$$
$$A(j) = 1400 + 600\,(A/G,10\%,j) + 600\,(P/F,10\%,1)\,(A/P,10\%,j)$$

The above formula for *A(j)* was derived by extending the gradient series backwards writing the first year cost as $1400 + $600. Calculations for different values of *j* are tabulated below. *Economic life* = 7 years.

Years of Service Life, *j*	*CR(j)*	*A(j)*	*EUAC(j) = CR(j)+A(j)*
1	$10,500.00	$2,000.00	$12,500.00
2	5,523.81	2,000.00	7,523.81
3	3,870.05	2,181.24	6,051.29
4	3,046.97	2,400.77	5,447.74
5	2,556.10	2,629.99	5,186.09
6	2,231.30	2,859.40	5,090.70
7	2,0001.40	3,085.07	5,086.47 = min
8	1,830.68	3,304.85	5,135.53
9	1,699.58	3,518.12	5,217.70
10	1,596.13	3,724.16	5,320.29

Exercise: Plot the CR curve, the A curve, and the EUAC curve.

Bonds

Definition

A bond is an economic instrument which has a *face value* guaranteed to be paid to its holder by the issuing company when the instrument reaches *maturity*. Additionally, periodic *dividends* are usually paid at a specified *interest rate.*

Interest Paid

V = face value
b = bond interest (usually per year)
c = number of periods per year
I = interest (dividend) paid per period

$$I = Vb/c$$

Company's Cost

S = selling price
n = number of years to reach maturity
Q = quantity received by the company
i = company's cost (rate)

The present worth of all payments must be equal to the quantity received by the company, when using an interest rate equal to i. In symbols,

$$Q = I\,(P/A,i\%,nc) + V\,(P/F,i\%,nc)$$

Illustration

The ABC Corporation sells 4% \$1000 bonds that will pay dividends twice a year and will mature in 5 years. Bonds are sold at \$830, but after broker's fees the company receives \$760. What is the company's cost of the capital raised through the sale of these bonds?

From the statement of the problem: V = \$1000, b = 0.04, and c = 2. Using this, we can compute I = \$1000(0.04)/2 = \$20. Furthermore, S = \$830, Q = \$760, n = 5, and nc = 10. Therefore, we want to find the interest rate i in the following equation:

$$760 = 20\,(P/A,i\%,10) + 1000\,(P/F,i\%,10)$$

We can determine the solution by interpolation (or by the bisection method, or by the use of a spread sheet). Note that for i = 0.05 and i = 0.06, we get the following results:

$$20\,(P/A,5\%,10) + 1000\,(P/F,5\%,10) = 20(7.7216) + 1000(0.61392) = 768.35$$
$$20\,(P/A,6\%,10) + 1000\,(P/F,6\%,10) = 20(7.3601) + 1000(0.55840) = 705.60$$

Let x = 0.06 - i_s, where i_s is the effective semiannual interest rate. From the above results, 62.75/0.01 = 54.40/x. Therefore, x = 0.008669, which implies that i_s^* = 0.05133. Thus, the semiannual rate is 5.13%. The corresponding effective interest rate *per year* is $i_a = (1.05133)^2 - 1 = 0.105296$, or 10.52%.

Investor's Point of View

Let us introduce the following additional notation:
P = maximal amount the investor can bid for the bond
i = expected effective rate of return *per year*
i_x = expected effective rate of return *per payment period*

Since $(1 + i_x)^c = 1 + i$, we can determine $i_x = (1 + i)^{1/c} - 1$. The equivalent value at time 0 of the investor's expected receipt is

$$P = I\,(P/A, i_x\%, nc) + V\,(P/F, i_x\%, nc)$$

If the market value $S > P$ the investor should look for another business opportunity.

Illustration

Let us continue with the previous case. Furthermore, let us assume that $i = 0.09$. Therefore, $i_x = (1.09)^{1/2} - 1 = 0.044$, or 4.40% per semester. From the above equation

$$P = 20\,(P/A, 4.40\%, 10) + 1000\,(P/F, 4.40\%, 10)$$
$$= 809.12$$

Since $S > P$ the investor should look for another business opportunity.

Exercises

Examples 11.1, 11.2, 11.3, 11.4, 11.5, textbook.

Depreciation Models

Definitions

- *Depreciation* is a way of accounting for the cost of an asset when income is determined for tax purposes. Depreciation consists in amortizing the cost treated as a prepayment for future service (Note: the word *amortize* means to provide for gradual extinguishment).
- *Useful life* is the period over which an asset is depreciated. This is different from service life and economic life.
- *Salvage value* is the estimated proceeds that will be realized from the sale of the asset or its disposition when it is retired.
- *Book value* is the difference between the original cost and the accumulated depreciation at the end of any year of the useful life.
- *Traditional methods* include: straight-line method (SLM), declining-balance method (DBM).
- *IRS* uses the Modified Accelerated Cost Recovery System (MACRS).

Straight-Line Method

B = first cost
SV = salvage value
n = number of years in useful life
D_t = annual depreciation charge in year t.
$(BV)_t$ = book value at the end of year t.

In this method we use the following relationships:

$$D_t = (B - SV)/n$$

If we rewrite this equation as

$$D_t = d\,(B - SV)$$

then $d = 1/n$ can be referred to as a *uniform depreciation rate per year*. Note that d is a rate applied to the total cumulative depreciated amount. The book value after t years is:

$$(BV)_t = B - t\,D_t$$

Example

See Example 13.1, textbook.

Declining-Balance Method

This is also known as the *Fixed-Percentage Method.* In this method we will use the following relationships:

$$D_t = d(BV)_{t-1}$$

where d is known as the *uniform depreciation rate per year.* Note that this is a rate applied to the book value at the end of year $t-1$. The allowed range is $0 < d \leq 2/n$. The book value at the end of year t is:

$$(BV)_t = B(1-d)^t$$

When $d=2/n$ this method is known as the *Double Declining-Balance Method* (DDBM).

Example

See Example 13.2, book.

Modified Accelerated Cost Recovery System

- Switching from DBM to SLM. See Example 13.5.
- Only 50% of first year DDB depreciation applies for tax purposes. The remaining 50% is taken in years 2 through $n+1$ of service life.
- $D_t = d_t B$. Table 13.2 has the d_t values.

Example

See Example 13.4, textbook.

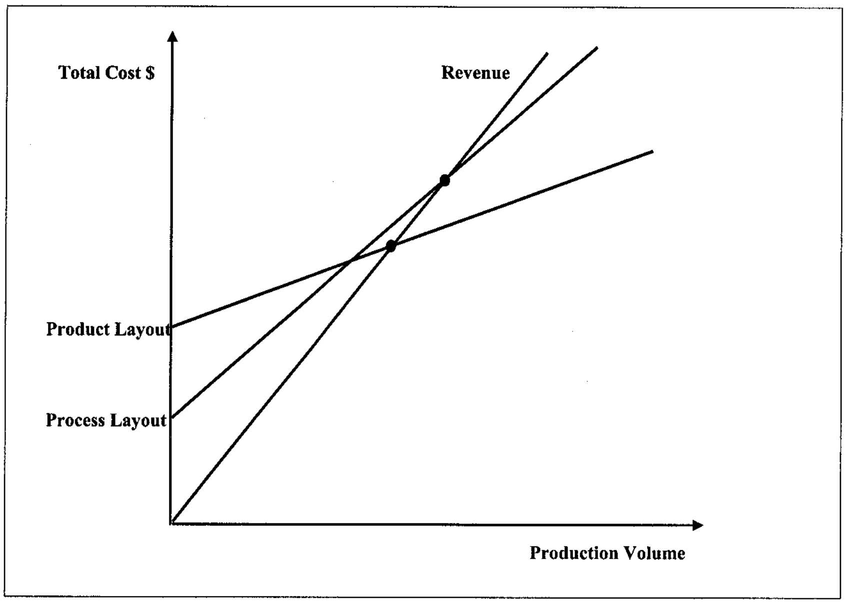
Break-Even Point

If F is the fixed cost, v the variable cost per unit, and r the selling price per unit, the production level Q corresponding to the break-even point can be determined from the relationship

$$F + vQ = rQ$$

Illustration

A graphical break-even point comparison between the process and product layouts is shown below. In this graph we consider production volumes, costs (fixed and variable), and revenues. We show the total-cost lines for both layouts, the revenue line, and the break-even points for the layouts. As can be seen from this graph, the product layout is preferred for high production volumes and the process layout is preferred for low production volumes.



As an illustration, let $1,800,000 and $1,200,000 be the fixed cost for the production and process layouts, respectively. Similarly, let $220 and $300 be the variable costs for the two layouts, respectively. Assume that the selling price per unit equal to $380. (a) Find the level of production at which the two layouts are identical from a cost point of view. (b) For each layout find the level of production at which the production of the item starts being profitable. Using the relationship $F + vQ = rQ$, we get the following results:

(a) $(F + vQ)_{product} = (F + vQ)_{process}$; $Q = 600{,}000/80 = 7{,}500$.
(b) $Q_{product} = 1{,}800{,}000/(380\text{-}220) = 11{,}250$; $Q_{process} = 1{,}200{,}000/((380\text{-}300) = 15{,}000$.

Example

See Example 16.1, book.

Example

See Example 16.2, book.

APPENDIX

IMPORTANT FORMULAS

Sum of the Terms of an Arithmetic Series

$S_n = a + (a+d) + (a+2d) + ... + [a+(n-1)d] = na + \frac{1}{2}(n-1)nd$

Sum of the Terms of a Geometric Series

$S_n = a + ar + ar^2 + ... + ar^{n-1} = a(1-r^n)/(1-r)$ If $|r| < 1$ S_n converges to $S = a/(1-r)$

Simple Interest

$I = niP$ $F = P(1+ni)$

Compound Interest

$F = P(1+i)^n$ $I = [(1+i)^n - 1]P$

Single-Payment Transactions

$P = F[1/(1+i)^n] = F\ (P/F,i,n)$ $F = P(1+i)^n = P\ (F/P,i,n)$

Uniform-Series Transactions

$$P = A\left[\frac{(1+i)^n - 1}{(1+i)^n i}\right] = A\ (P/A,i,n) \qquad A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P\ (A/P,i,n)$$

$$F = A\left[\frac{(1+i)^n - 1}{i}\right] = A\ (F/A,i,n) \qquad A = F\left[\frac{i}{(1+i)^n - 1}\right] = F\ (A/F,i,n)$$

Gradient-Series Transactions

Base amount (paid every period) = A_1

Uniform gradient (starts at the second period) = G.

$P = P_B + P_G$ $F = F_B + F_G$ $A = A_G + A_1$

$$P_B = A_1\left[\frac{(1+i)^n - 1}{(1+i)^n i}\right] = A_1\ (P/A,i,n)$$

$$P_G = \frac{G}{i(1+i)^n}\left[\frac{(1+i)^n - 1}{i} - n\right] = G\ (P/G,i,n)$$

$$F_G = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i} - n\right] = G\ (F/G,i,n)$$

$$F_B = A_1\left[\frac{(1+i)^n - 1}{i}\right] = A\ (F/A,i,n)$$

$$A_G = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] = G\ (A/G,i,n)$$

Nominal and Effective Interest Rates

r = nominal annual interest rate; m = number of interest periods per year; i = effective interest rate

Interest Periods Equal to Payment Periods

Results obtained in Chapter 2 are correct after setting $i = r/m$ and replacing n by mn.

Interest Periods Smaller Than Payment Periods

Method 1: Determine the effective interest rate for the given interest period and treat each payment separately. Method 2: Calculate an effective interest rate for the payment period and then proceed as when the interest period and payment period coincide. If the payment period is equal to one year, the *effective annual interest rate* is equal to

$$i = \left(1 + \frac{r}{m}\right)^m - 1$$

Continuous Compounding

$$i = e^r - 1$$

Uniform Series Starting at the End of Period *k*

Method 1 $P_0 = A\ (P/A,i,m)\ (P/F,i,k-1)$

Method 2 $P_0 = A\ (F/A,i,m)\ (P/F,i,k+m-1)$

Non-Uniform Cashflows

Each non-uniform payment is considered as an individual payment.

Shifted Gradient Series

The **present worth** of the series can be calculated **two periods** before the gradient shows up the first time. After this, we can compute either the present worth at time zero, the future worth at time n, or the equivalent uniform annual worth for periods $1,2,...,n$.

Present Worth and Capitalized Cost Evaluation

Finite Series of Cashflows

- All alternatives have the same life: the planning horizon is defined as the life of the alternatives.
- Alternatives have different lives: the planning horizon is defined as the *least common multiple* (LCM) of the lives of the alternatives.

Infinite Series of Cashflows

If the cashflows take place every year: $P = A/i$

If the cashflows take place every *f* years: $P = A/[(1+i)^f-1]$

Equivalent Uniform Annual Worth Calculation

$EUAW = -P(A/P,i\%,n) + SV(A/F,i\%,n)$

$EUAW = [-P + SV(P/F,i\%,n)]\,(A/P,i\%,n)$

$EUAW = -(P-SV)\,(A/P,i\%,n) - SV(i)$

Rate of Return Calculation

$NPW(i) = 0$ or $EUAW(i) = 0$

Bisection Method

At each iteration the bisection method shortens the interval containing the solution, by halving the current interval and determining the portion of it corresponding to two values of *NPW(i)* or *EUAW(i)* having opposite signs.

Interpolation

From the tables we get the coordinates for two points (X,Y). The X-value is the interest rate *i* and the Y-value is *NPW(i)*. From the equation of the straight line through these two points we can obtain X for Y=0.

Composite Rate of Return

Let F_t be the net investment value in year *t*, and the net cash flow C_t in year *t*: $F_t = F_{t-1}(1+i) + C_t$ where $t = 1, 2, ..., n$. Here $i = c$ if $F_{t-1} > 0$, or $i = i'$, otherwise. Find *i'* from the solution of the equation $F_n(i') = 0$

Comparison of Two Alternatives

Let us consider two alternatives *A* and *B*, and let us assume that Alternative A has a lower initial investment than alternative B. Moreover, let CF_{At} and CF_{Bt} the cashflows for activities A and B, respectively, at time t = 0, 1, 2, …, n. $NCF_t = CF_{Bt} - CF_{At}$

Benefit Cost Ratio

BCR = (B – D) / C

where B is the dollar-value of total benefits associated with a project, D is the dollar-value of all its disbenefits or disadvantages, and C is the total project cost. For a project to be desirable, BCR > 1.

Net Benefit Value

NBV = B – D – C

Again, for a project to be desirable, NBV > 0.

Modified Benefit Cost Ratio

If maintenance and operation (M&O) costs are separated from the initial investment I_0, the modified benefit cost ratio is defined as

MBCR = (B – D – M&O) / I_0

Economic Life

Consider an asset with *service life* equal to *n* years. As operating equipment ages, the capital costs typically decline while operating costs increase. The annual *capital recovery cost* (see Chapter 6) CR(j) after *j* years (j = 1,2,…,n) is given by:

CR(j) = (P – SV) (A/P,i%,n) + SV(i)

If the annual *operating cost* is represented by A(j), the total annual cost, EUAC (j), is calculated as

EUAC(j) = CR(j) + A(j)

The *economic life* is given by the value j* at which EUAC(j) is minimized.

Bond Analysis

V = face value

b = bond interest (usually per year)

c = number of periods per year

I = interest (dividend) paid per period

$I = Vb/c$

<u>Company's Cost</u>

S = selling price

n = number of years to reach maturity

Q = quantity received by the company

i = company's cost (rate)

te equal to *i*. In symbols,

$Q = I\,(P/A,i\%,nc) + V\,(P/F,i\%,nc)$

<u>Investor's Point of View</u>

P = maximal amount the investor can bid for the bond

i = expected effective rate of return *per year*

i_x = expected effective rate of return *per payment period*

$i_x = (1 + i)^{1/c}$

$$P = I\,(P/A, i_x\%, nc) + V\,(P/F, i_x\%, nc)$$

If the market value $S > P$ the investor should look for another business opportunity.

Depreciation Models

Straight-Line Method (SLM)

B = first cost

SV = salvage value

n = number of years in useful life

D_t = annual depreciation charge in year t.

$(BV)_t$ = book value at the end of year t.

$$D_t = (B - SV)/n$$

$$D_t = d\,(B - SV)$$

then d can be referred to as a *uniform depreciation rate per year.* Note that $d = 1/n$ is a rate applied to the total cumulative depreciated amount. The book value after t years is:

$$(BV)_t = B - t\,D_t$$

Declining-Balance Method (DBM)

This is also known as the *Fixed-Percentage Method.* In this method we will use the following relationships:

$$D_t = d(BV)_{t-1}$$

where d is known as the *uniform depreciation rate per year.* Note that this is a rate applied to the book value at the end of year $t-1$. The book value at the end of year t is:

$$(BV)_t = B(1-d)^t$$

When $d=2/n$ this method is known as the *Double Declining-Balance Method* (DDBM).

Break-Even Point

If F is the fixed cost, v the variable cost per unit, and r the selling price per unit, the production level Q corresponding to the break-even point can be determined from the relationship

$$F + vQ = rQ$$