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Construction of Frames

คณะผู้วิจัย

หัวหน้าโครงการ

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ผลงานวิจัยเป็นความรับผิดชอบของหัวหน้าโครงการวิจัยแต่เพียงผู้เดียว

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บทคัดย่อ

ให้ A เป็นเมทริกซ์ที่หาตัวผกผันได้ขนาด $n \times n$ สำหรับ $k \in \boldsymbol{Z}$, $\vec{x} \in \boldsymbol{R}^n$ และ $w \in L^2(\boldsymbol{R}^n)$ กำหนดให้ $w_{k,\vec{x}}(\cdot) = |\det(A)|^{-k/2} w(A^{-k} \cdot)$ เราแสดงได้ว่ามีฟังก์ชัน w ซึ่งชุดของฟังก์ชัน $\{w_{k,\vec{x}} : k \in \boldsymbol{Z}, \vec{x} \in \boldsymbol{R}^n\}$ ก่อกำเนิดกรอบสำหรับฟังก์ชันใน $L^2(\boldsymbol{R}^n)$ ก็ต่อเมื่อ $|\det(A)| \neq 1$ นอกจากนี้ถ้า A^t เป็นเมทริกซ์ที่ถูกนิยามสำหรับทุกจำนวนจริง t แล้วชุดของฟังก์ชัน $\{w_{t,\vec{x}} : t \in \boldsymbol{R}, \vec{x} \in \boldsymbol{R}^n\}$ ก่อกำเนิดกรอบสำหรับฟังก์ชันใน $L^2(\boldsymbol{R}^n)$ ก็ต่อเมื่อ $|\det(A)| \neq 1$

Abstract

Let A be a fixed invertible $n \times n$ matrix. For $k \in \boldsymbol{Z}$, $\vec{x} \in \boldsymbol{R}^n$ and $w \in L^2(\boldsymbol{R}^n)$, set $w_{k,\vec{x}}(\cdot) = |\det(A)|^{-k/2} w(A^{-k} \cdot)$. We show that there exists a function w such that the family $\{w_{k,\vec{x}} : k \in \boldsymbol{Z}, \vec{x} \in \boldsymbol{R}^n\}$ forms a frame for $L^2(\boldsymbol{R}^n)$ if and only if $|\det(A)| \neq 1$. Furthermore, if A^t is defined for all real numbers t, then the family $\{w_{t,\vec{x}} : t \in \boldsymbol{R}, \vec{x} \in \boldsymbol{R}^n\}$ forms a frame for $L^2(\boldsymbol{R}^n)$ if and only if $|\det(A)| \neq 1$.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background and Rationale

Continuous wavelet transforms are now widely used to analyze functions, signals and images in Euclidean space. Often termed "windowed Fourier transform", the continuous wavelet transform permits analysis of signals in both spatial and time domains.

In the general mathematical setting one starts with a closed group H of invertible $n \times n$ matrices acting as linear transformations on Euclidean space $\mathbf{R}^n$, and considers the semi-direct product G of the two groups. There is a natural representation π of G on the space of square integrable functions $L^2(\mathbf{R}^n)$ given by $\pi(a, z) = D_a T_z$, where D_a denotes the dilation operator associated with a matrix a of H, and T_z the translation operator determined by a vector z in $\mathbf{R}^n$. For a given square integrable function w, the wavelet transform W associated with w maps an element f in $L^2(\mathbf{R}^n)$ to a function Wf on G by means of the inner product, $Wf(a,z) = \langle f, \pi(a, z)w \rangle$.

It is now natural to ask under what conditions the original function can be reconstructed from its wavelet transform. A frequently used sufficient condition is that W be an isometry with regards to the L^2 norms, in which case we call w a *tight frame generator*. In this case, f can be expressed as a weak integral in $L^2(\mathbf{R}^n)$,

$$f = \int_G Wf(a,z)\pi(a,z)w \; d(a,z).$$

If the representation π is square-integrable, then the existence of tight frame generators is guaranteed by a theorem of Dufflo and Moore [4]. This is the approach taken in the initial paper on the continuous wavelet transform in $L^2(\mathbf{R})$ by Grossman, Morlet and Paul [6], where π was split into the sum of two square-integrable representations. The expository paper by Heil and Walnut [7] showed that one actually need not use Dufflo-Moore's theorem, but can directly obtain conditions for a function w to be a tight frame generator. This idea was generalized to the multidimensional setting (n>1) by Bernier and Taylor [1] who showed that if H possesses open, free orbits under its natural action on $\boldsymbol{R}^n$, then π decomposes into a finite sum of square-integrable representations, so that tight frame generators exist. This orbit condition requires that the topological dimension of H be n, which greatly restricts the groups that can be considered. Fuehr [4] could generalize the orbit condition from a topological to a measure-theoretic one by requiring that orbits be of non-zero measure and the corresponding stabilizer subgroups be compact.

The simplest and most interesting generalization of the one-dimensional wavelet transform to the n-dimensional setting is by choosing H to be a one-parameter group, $H = \{A^t : t \in \boldsymbol{R}\}$ where A is a fixed, invertible matrix. Recall here that A^t is defined in general only if A is the exponential of some matrix B, and is given by $A^t = e^{tB}$. Fuehr's results do not apply here as all orbits have measure zero if n>1. That tight frame generators should exist is suggested by an analogous result for the discrete wavelet series by Dai, Larson and Speegle [3], who showed that if A is equivalent to an expanding matrix with integer entries, then discrete tight wavelet frames exist, and Calogero [2] has further given a characterization of generators of discrete tight wavelet frames.

Mallat and Zhong [8] have discussed a semi-discrete wavelet transform in $L^2(\boldsymbol{R})$, where the translation parameter is still continuous, but the dilation parameter lies in a discrete, cyclic subgroup of $\boldsymbol{R}^+$. The obvious generalization of this construction to $L^2(\boldsymbol{R}^n)$ involves a discrete one-parameter group of dilations, $H = \{A^k : k \in \boldsymbol{Z}\}$ with A an arbitrary invertible matrix, and has not been studied yet.

It is thus natural to ask under what conditions on the matrix A there exist tight frame generators associated with either the continuous or the discrete one-parameter subgroup H generated by A.

1.2 Research Objectives

The objective of this project was as follows: Given a fixed, invertible matrix A and either the continuous or the discrete one-parameter group H generated by A,

1. give a characterization of tight frame generators w,
2. find necessary and sufficient conditions on A so that tight frame generators exist,
3. investigate relationships between the tight frame generators for continuous and discrete H.

1.3 Scope and Limitations

In practical applications one often wishes the tight frame generators w to have special properties, to be smooth or have compact support, for example. This project did not investigate the existence of such nicely behaved functions, but focused on the question of existence of tight frame generators.

1.4 Benefits from Research

This project adds to the variety of methods for continuous wavelet analysis in Euclidean space. Its results help clarify the theory of the wavelet transform, and may be applied by engineers and scientists requiring data analysis and compression tools.

CHAPTER 2

METHODOLOGY

2.1 Constructions

The starting point was to show that the characterization of tight frame generators given in [5] applies in this situation as well, that is:

1. A function $w \in L^2(\boldsymbol{R}^n)$ is a tight frame generator for the continuous group $\{A^t : t \in \boldsymbol{R}\}$ if and only if $\int_{\boldsymbol{R}} |\tilde{w}(\vec{\gamma}A^t)|^2 \, dt = 1$ for almost all $\vec{\gamma} \in \boldsymbol{R}^n$ where $\tilde{w}$ denotes the Fourier transform of w.
2. A function $w \in L^2(\boldsymbol{R}^n)$ is a tight frame generator for the discrete group $\{A^k : k \in \boldsymbol{Z}\}$ if and only if $\sum_k |\tilde{w}(\vec{\gamma}A^k)|^2 = 1$ for almost all $\vec{\gamma} \in \boldsymbol{R}^n$.

In the case of a discrete group, there is a natural candidate for a tight frame generator: If S is a Borel cross-section of finite measure for the natural action of $\{A^k : k \in \boldsymbol{Z}\}$ on $\boldsymbol{R}^n$, then its characteristic function χ_S is both square-integrable and satisfies 2. above, that is, the inverse Fourier transform of χ_S is a tight frame generator. Thus, we tried to determine under what conditions on the matrix A there exist cross-sections of finite measure, and construct such cross-sections explicitly.

Finally, we studied relationships between tight frame generators for the continuous and the discrete cases in order to apply the results from the discrete case to the continuous case.

CHAPTER 3

RESULTS

3.1 Main Results

Besides the characterizations 1. and 2. of tight frame generators above, we have been able to establish the following results:

1. If w is a tight frame generator for the discrete group generated by A, and if A is an exponential, then w is also a tight frame generator for the continuous group generated by A.
2. If w is a tight frame generator for the continuous group generated by A, then w can be modified to a tight frame generator for the discrete group generated by A.
3. If A is an exponential, then there exists a cross-section for the action of the continuous group generated by A if and only if A is not equivalent to an orthogonal matrix.
4. If A is an invertible matrix, then exists a cross-section for the action of the discrete group generated by A if and only if A is not equivalent to an orthogonal matrix.
5. There exists a cross-section of finite measure for the action of the discrete group generated by A if and only if $|\det(A)| \neq 1$.
6. There exists a bounded cross-section for the action of the discrete group generated by A if and only if the eigenvalues of A have all either modulus less than one or modulus greater than one.

Using all the above, we thus could show:

7. There exists a tight frame generator in $L^2(R^n)$ for the continuous (respectively the discrete) one-parameter group generated by A if and only if $|\det(A)| \neq 1$.

Further details can be found in the preprint in Appendix I. The major part of the preprint was incorporated into and published as a research article as shown in Appendix II.

3.2 Discussion

The main results show that most invertible (respectively exponential) matrices A, namely those with $|\det(A)| \neq 1$, can be used for semi-discrete and continuous wavelet analysis. This is different from the discrete case as discussed in [3], where the eigenvalues of the matrix A were required to all have modulus greater than one.

Furthermore, these results illustrate that square integrability of the representation π or its subrepresentations is not required for the existence of tight wavelet frames.

CHAPTER 4

AN APPLICATION

4.1 Data Denoising

To illustrate how multidimensional wavelet frames can be practically applied, a data denoising example is now presented.

Figure 1 shows a 257x257 pixel array of vertical lines overlaid with noise. The origin of the coordinate system is located at the center, at pixel (129,129). Each pixel is assigned an integer value in the range of 0 to 63, representing the values *f(x,y)* of the signal *f*. The dark red color represents large pixel values, while the blue color represent values close to zero. The matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$$

was chosen as dilation matrix. An admissible function w was constructed following the procedure outlined in the preprint: First, a circle was chosen as cross-section for the action of A on the plane. This curve was enlarged to an annulus, whose characteristic function was then smoothed at the edges. Averaging the values of this smoothed function over orbits, the Fourier transform $\tilde{w}$ of the wavelet w was obtained. Figures 2 and 3 show the function $\tilde{w}$ and its inverse Fourier transform w, respectively. The wavelet coefficients $W(t,z) = \langle f, \pi(A^t, z)w \rangle$ were computed for $t=1$ and $z=(z_1, z_2)$ in the range of $-128 \le z_1, z_2 \le 127$. Figure 4 shows the dilated frame generator $D_1 w$. The wavelet coefficients obtained for various values of (z_1, z_2) are shown in Figure 5. While the noise is still visible, its amplitude is decreased. Setting coefficients below some threshold value to zero, one obtains the wavelet coefficients of Figure 6. All computations were done using Matlab.

For denoising of general data with a wide range of data values *f(x,y)* one would need to compute the wavelet coefficients on a fine grid of dilation parameters, and reassemble the denoised set of coefficients using the inverse wavelet transform. For reasons of computational complexity, we have not done so.

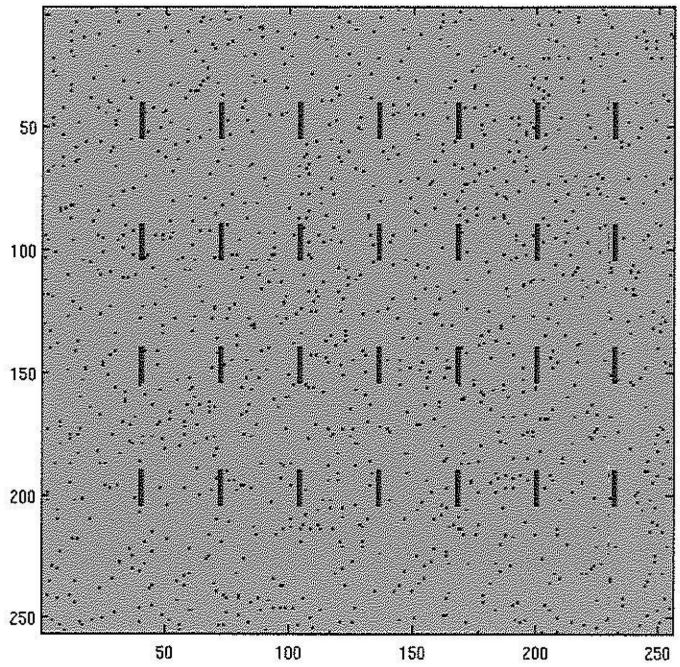

Figure 1: Noisy Image

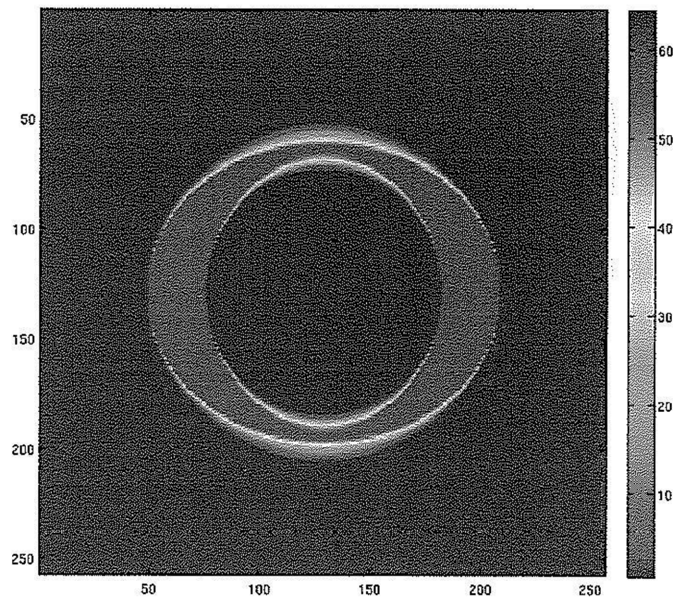

Figure 2: Wavelet in Fourier Space (scaled by a factor of 25)

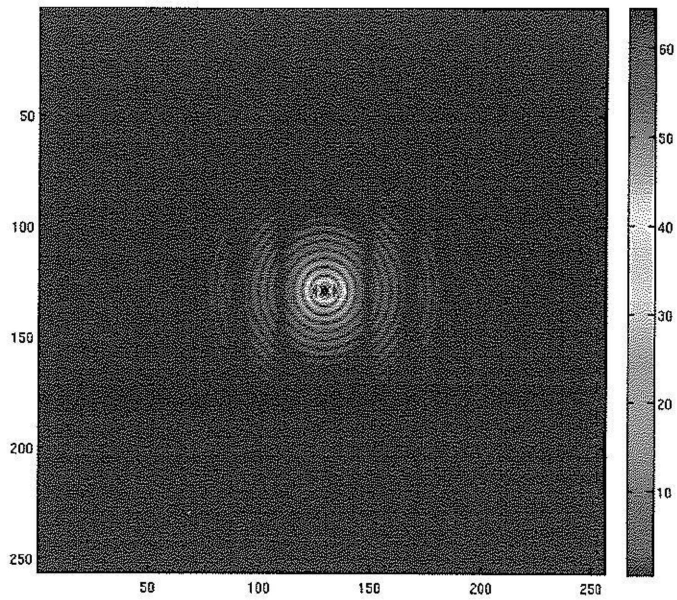

Figure 3: The Wavelet w

(scaled by a factor of 3.6)

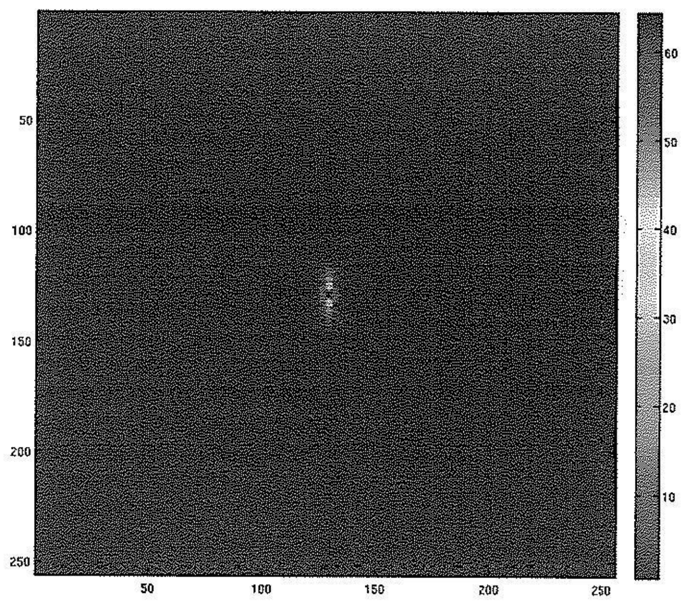

Figure 4 : The Dilated Wavelet

(scaled by a factor of 1.5)

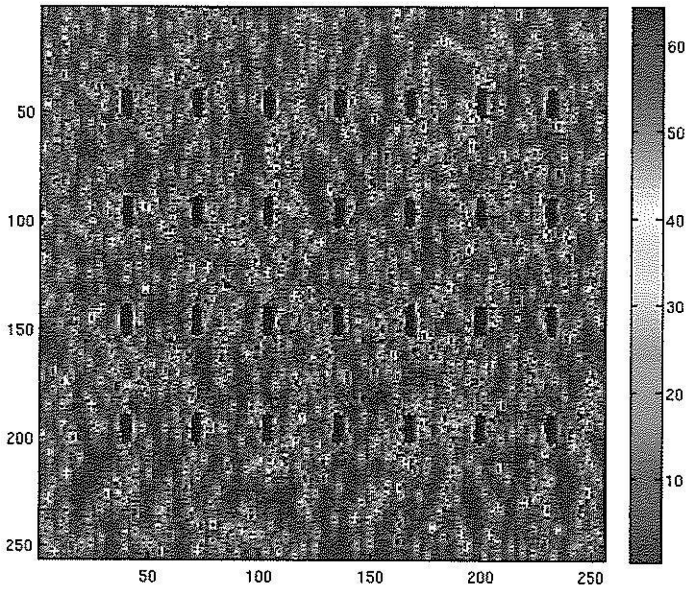

Figure 5 : The Wavelet Coefficients

(The point $(z_1, z_2) = (0,0)$ is at the center)

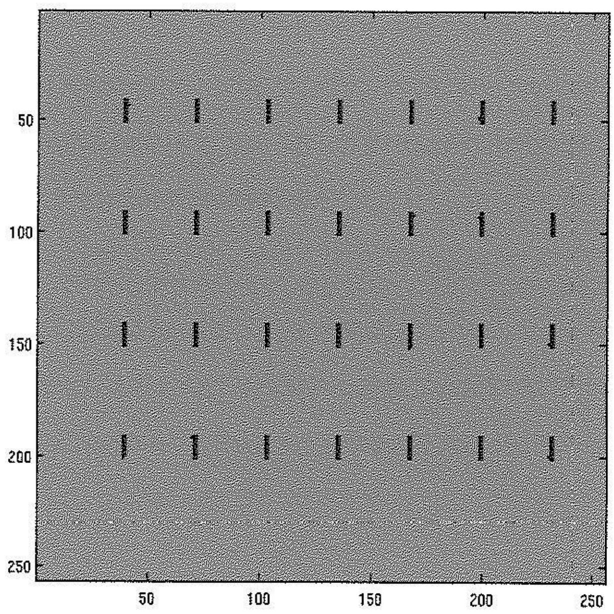

Figure 6 : The Wavelet Coefficients greater than the threshold value

CHAPTER 5

CONCLUSION

5.1 Summary

The main result of this research project is that there exist tight frame generators for the semi-discrete and continuous wavelet analysis associated with an invertible matrix A if and only if $|\det(A)| \neq 1$. Along the way we showed that the two problems, the existence of semi-discrete and of continuous tight frame generators are essentially equivalent. Furthermore, we constructed explicit examples of tight frame generators which are the inverse Fourier transforms of characteristic functions of some measurable sets.

5.2 Recommendations

A couple of further questions which merit attention arise now naturally. One may investigate under what conditions on A there exist tight frame generators with "nice" properties, for example, which are smooth, have compact support or vanish rapidly. Also, one may try to generalize the above results to n-parameter abelian matrix group. This is a much more difficult endeavor, because the generators need not have a common Jordan basis.

After the main investigation of this project was completed, we became aware of a preprint [9] which discusses the same questions in a more general setting, namely for arbitrary dilation group H. However, our proofs give a concrete construction of tight frame generators which the proofs in [9] do not.

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APPENDIX I

UNPUBLISHED PREPRINT OF THE RESEARCH RESULTS

CONTINUOUS AND SEMI-DISCRETE TIGHT FRAMES ON $\mathbb{R}^n$

ABSTRACT. Let $A \in GL_n(\mathbb{R})$. We show that there exists a function $w \in L^2(\mathbb{R}^n)$ whose dilates by integer powers of A and translates by elements of $\mathbb{R}^n$ form a semi-discrete tight frame on $L^2(\mathbb{R}^n)$ if and only if $|\det(A)| \neq 1$. If in addition, A lies in the exponential group of $GL_n(\mathbb{R})$ then there exists $w \in L^2(\mathbb{R}^n)$ whose dilates by arbitrary powers of A and translates by elements of $\mathbb{R}^n$ form a continuous tight frame on $L^2(\mathbb{R}^n)$ if and only if $\det(A) \neq 1$.

1. INTRODUCTION

In multidimensional discrete wavelet analysis the usual approach is to fix a dilation matrix A. One takes A to be a strictly expanding matrix, that is, a matrix whose eigenvalues all have modulus greater than one and assumes that A preserves some lattice Γ in $\mathbb{R}^n$. It was shown in [2] that under these assumptions there always exists a function $w \in L^2(\mathbb{R}^n)$ whose discrete dilates and translates $w_{k,\vec{\gamma}}(\vec{y}) = |\det(A)|^{-k/2} w(A^{-k}\vec{y} - \vec{\gamma})$, $k \in \mathbb{Z}, \vec{\gamma} \in \Gamma$ form an orthonormal basis of $L^2(\mathbb{R}^n)$. A complete characterization of such functions, called wavelets, was given in [1].

The focus of this paper is on the semi-discrete and continuous situation. One still needs to fix a matrix A to play the role of the basic dilation, but the goal here is to find the weakest conditions on A that permit the existence of a continuous or a semi-discrete tight frame, respectively.

To be more precise, let $A \in GL_n(\mathbb{R})$ and set $\delta = |\det(A)|$. For $w \in L^2(\mathbb{R}^n)$, define $T_{\vec{x}}w(\vec{y}) = w(\vec{y} - \vec{x})$, for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, and $D_k w(\vec{y}) = \delta^{-k/2} w(A^{-k}\vec{y})$, for $k \in \mathbb{Z}, \vec{y} \in \mathbb{R}^n$. Since $\int_{\mathbb{R}^n} f(A\vec{x})\, d\vec{x} = \delta^{-1} \int_{\mathbb{R}^n} f(\vec{x})\, d\vec{x}$ for $f \in L^1(\mathbb{R}^n)$, the dilations D_k and translations $T_{\vec{x}}$ constitute unitary operators on $L^2(\mathbb{R}^n)$. A function $w \in L^2(\mathbb{R}^n)$ is called a *semi-discrete tight frame generator*, if

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |< f, D_k T_{\vec{x}} w >|^2\, d\vec{x} \;=\; \|f\|_2^2 \tag{1}$$

for all $f \in L^2(\mathbb{R}^n)$. The collection $\{D_k T_{\vec{x}} w : k \in \mathbb{Z},\ \vec{x} \in \mathbb{R}^n\}$ then forms a semi-discrete tight frame in $L^2(\mathbb{R}^n)$. Such frames were used in the one-dimensional setting by Mallat and Zhong [4].

In order to consider continuous tight frames one must be be able to define A^t for arbitrary $t \in \mathbb{R}$. This requires A to be an exponential, $A = e^B$ for some $B \in M_n(\mathbb{R})$. Now for $w \in L^2(\mathbb{R}^n)$, let $D_t w(\vec{y}) = \delta^{-t/2} w(A^{-t}\vec{y})$, for $t \in \mathbb{R}, \vec{y} \in \mathbb{R}^n$. A function $w \in L^2(\mathbb{R}^n)$ is called a *continuous tight frame generator*, if

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |< f, D_t T_{\vec{x}} w >|^2\, d\vec{x}\, dt \;=\; \|f\|_2^2 \tag{2}$$

for all $f \in L^2(\mathbb{R}^n)$.

These notions generalize the continuous wavelet transform as discussed, for example, in [3] to the multidimensional setting. For if w is a semi-discrete tight frame generator, then the wavelet transform $f \to < f, D_k T_{\vec{x}} w >$ is a partial isometry from $L^2(\mathbb{R}^n)$ onto a subspace of $L^2(\mathbb{Z} \times \mathbb{R}^n)$, and we have the reconstruction formula

$$f = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} < f, D_k T_{\vec{x}} w > \; D_k T_{\vec{x}} w \; d\vec{x}$$

which holds weakly in $L^2(\mathbb{R}^n)$. Similarly, for a continuous tight frame generator w the reconstruction formula

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}^n} < f, D_t T_{\vec{x}} w > \; D_t T_{\vec{x}} w \; d\vec{x} \; dt$$

holds weakly in $L^2(\mathbb{R}^n)$.

Our aim is to determine the class of matrices for which tight frame generators exist. It turns out that this class is rather large, in fact, we will see that both kinds of frame generators exist if and only if $|\det(A)| \neq 1$.

2. Tight Frame Generators in Fourier Space

It will be more convenient to work with Fourier transforms. As usual, $\widehat{\mathbb{R}^n}$ will denote the dual group of $\mathbb{R}^n$ which can be identified with $\mathbb{R}^n$ itself through the pairing $(\vec{\gamma}, \vec{x}) \to e^{-2i\pi\vec{\gamma}\cdot\vec{x}}$, where elements $\vec{x}$ of $\mathbb{R}^n$ are written as column vectors and elements $\vec{\gamma}$ of $\widehat{\mathbb{R}^n}$ as row vectors. The Fourier transform $f \to \hat{f} = \int_{\mathbb{R}^n} f(\vec{x}) e^{-2i\pi\vec{\gamma}\cdot\vec{x}} \, d\vec{x}$ maps $L^1(\mathbb{R}^n)$ into $C_o(\widehat{\mathbb{R}^n})$, the set of continuous functions vanishing at infinity, and its restriction to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ extends to a Hilbert space isomorphism between $L^2(\mathbb{R}^n)$ and $L^2(\widehat{\mathbb{R}^n})$, also denoted by $f \to \hat{f}$, which takes the translation operator $T_{\vec{x}}$ to the phase shift operator $E_{-\vec{x}}$ and the dilation operator D_t to the dilation operator D_{-t}. Here, $E_{-\vec{x}} \hat{w}(\vec{\gamma}) = e^{2i\pi\vec{\gamma}\cdot(-\vec{x})} \hat{w}(\vec{\gamma})$ and $D_{-t}\hat{w}(\vec{\gamma}) = \delta^{t/2} \hat{w}(\vec{\gamma} A^t)$ for $\hat{w} \in L^2(\widehat{\mathbb{R}^n})$, $\vec{x} \in \mathbb{R}^n$, $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ and $t \in \mathbb{Z}$ or $t \in \mathbb{R}$, respectively.

There is a simple characterization of tight frame generators through their Fourier transforms:

Proposition 1. (1) *Let $A \in GL_n(\mathbb{R})$. Then $w \in L^2(\mathbb{R}^n)$ is a semi-discrete tight frame generator if and only if*

$$\sum_{k \in \mathbb{Z}} |\hat{w}(\vec{\gamma} A^k)|^2 = 1 \tag{3}$$

for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

(2) *Let $A = e^B \in GL_n(\mathbb{R})$. Then $w \in L^2(\mathbb{R}^n)$ is a continuous tight frame generator if and only if*

$$\int_{\mathbb{R}} |\hat{w}(\vec{\gamma} A^t)|^2 \, dt = 1 \tag{4}$$

for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$.

Proof. Note that for all $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} | < \hat{f}, D_{-k}E_{-x}\hat{w} > |^2 \, d\vec{x} &= \int_{\mathbb{R}^n} \left| \int_{\widehat{\mathbb{R}^n}} \delta^{k/2} \hat{f}(\vec{\gamma}) \, \overline{\hat{w}(\vec{\gamma}A^k)} \, e^{2i\pi\vec{\gamma}A^k\cdot\vec{x}} \, d\vec{\gamma} \right|^2 d\vec{x} \\ &= \int_{\mathbb{R}^n} \left| \int_{\widehat{\mathbb{R}^n}} \delta^{-k/2} \hat{f}(\vec{\gamma}A^{-k}) \, \overline{\hat{w}(\vec{\gamma})} \, e^{2i\pi\vec{\gamma}\cdot\vec{x}} \, d\vec{\gamma} \right|^2 d\vec{x} \\ &= \delta^{-k} \int_{\mathbb{R}^n} \left| \int_{\widehat{\mathbb{R}^n}} \varphi_k(\vec{\gamma}) \, e^{2i\pi\vec{\gamma}\cdot\vec{x}} \, d\vec{\gamma} \right|^2 d\vec{x}\end{aligned}$$

where we have set $\varphi_k(\vec{\gamma}) = \hat{f}(\vec{\gamma}A^{-k}) \, \overline{\hat{w}(\vec{\gamma})} \in L^1(\widehat{\mathbb{R}^n})$. Now the inner integral is precisely the inverse Fourier transform $\check{\varphi}_k$ of φ_k, so that by Plancherel's formula

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} | < \hat{f}, D_{-k}E_{-x}\hat{w} > |^2 \, d\vec{x} &= \sum_{k\in\mathbb{Z}} \delta^{-k} \int_{\mathbb{R}^n} |\check{\varphi}_k(\vec{x})|^2 \, d\vec{x} \\ &= \sum_{k\in\mathbb{Z}} \delta^{-k} \int_{\widehat{\mathbb{R}^n}} |\varphi_k(\vec{\gamma})|^2 \, d\vec{\gamma} \\ &= \sum_{k\in\mathbb{Z}} \delta^{-k} \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma}A^{-k}) \, \overline{\hat{w}(\vec{\gamma})}|^2 \, d\vec{\gamma} \\ &= \sum_{k\in\mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \, |\hat{w}(\vec{\gamma}A^k)|^2 \, d\vec{\gamma}\end{aligned}$$

where the integrals may possibly be infinite. It follows that w is a semi-discrete tight frame generator if and only if

$$\|\hat{f}\|_2^2 = \int_{\widehat{\mathbb{R}^n}} |\hat{f}(\vec{\gamma})|^2 \left(\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^k)|^2 \right) d\vec{\gamma}$$

for all $f \in L^2(\mathbb{R}^n)$. As this identity holds if and only if $\sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^k)|^2 = 1$ almost everywhere, the first assertion follows.

The second assertion is proved in a similar way. □

Proposition 2. *Let $A \in GL_n(\mathbb{R})$ be an exponential. If $w \in L^2(\mathbb{R}^n)$ is a semi-discrete tight frame generator for A, then w is also a continuous tight frame generator for A.*

Proof. Let $\mathcal{S}$ denote the set of all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ for which (3) does not hold, and $\mathcal{T}$ the set of those $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ whose stabilizer with respect to the discrete action of A is infinite. Thus, $\mathcal{S}$ has measure zero and $\mathcal{T} \subset \mathcal{S}$. Note that if $\vec{\gamma} \in \mathcal{S}\backslash\mathcal{T}$, then we can modify the values of $\hat{w}$ on the orbit of $\vec{\gamma}$ so that (3) holds for all elements of this orbit, and thus we may assume that $\mathcal{S} = \mathcal{T}$.

It is easy to see that $\vec{\gamma} \in \mathcal{T}$ if and only if $\vec{\gamma}A^t \in \mathcal{T}$ for all $t \in \mathbb{R}$. Hence for all $\vec{\gamma} \notin \mathcal{T}$,

$$\int_{\mathbb{R}} |\hat{w}(\vec{\gamma}A^t)|^2 \, dt \; = \; \int_0^1 \sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}A^tA^k)|^2 \, dt \; = \; \int_0^1 1 \, dt \; = \; 1.$$

That is, (4) holds. □

Every continuous tight frame generator can be modified to a discrete tight frame generator:

Proposition 3. *Let $A \in GL_n(\mathbb{R})$ be an exponential. Fix a real number a so that $\int_a^{a+1} \delta^{-t}\,dt = 1$. If w is a continuous tight frame generator for A then the inverse Fourier transform of the function $\tilde{w}$ given by*

$$\tilde{w}(\vec{\gamma}) = \left[\int_a^{a+1} \left|\hat{w}(\vec{\gamma}A^t)\right|^2 dt\right]^{1/2}$$

is a semi-discrete tight frame generator for A of the same norm as w.

Proof. First we show that $\tilde{w}$ and $\hat{w}$ have identical norms. In fact,

$$\begin{aligned}\|\tilde{w}\|_2^2 &= \int_{\widehat{\mathbb{R}^n}} |\tilde{w}(\vec{\gamma})|^2\,d\vec{\gamma} = \int_{\widehat{\mathbb{R}^n}}\int_a^{a+1} \left|\hat{w}(\vec{\gamma}A^t)\right|^2 dt\,d\vec{\gamma} \\ &= \int_a^{a+1}\int_{\widehat{\mathbb{R}^n}} \left|\hat{w}(\vec{\gamma}A^t)\right|^2 d\vec{\gamma}\,dt \\ &= \int_a^{a+1} \delta^{-t}\int_{\widehat{\mathbb{R}^n}} |\hat{w}(\vec{\gamma})|^2\,d\vec{\gamma}\,dt = \|\hat{w}\|_2^2.\end{aligned}$$

Finally, for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$,

$$\sum_{k\in\mathbb{Z}} \left|\tilde{w}(\vec{\gamma}A^k)\right|^2 = \sum_{k\in\mathbb{Z}} \int_a^{a+1} \left|\hat{w}(\vec{\gamma}A^kA^t)\right|^2 dt = \int_{\mathbb{R}} \left|\hat{w}(\vec{\gamma}A^t)\right|^2 dt = 1$$

which proves our assertion. □

We note that by modifying the above constant a we can obtain tight frame generators of arbitrary norm.

Remark 1. The property of existence of tight frame generators is invariant under similarity transformations. For example, let $w \in L^2(\mathbb{R}^n)$ be a semi-discrete tight frame generator for the matrix A. Given $C \in GL_n(\mathbb{R})$, set $\tilde{w}(\vec{\gamma}) = \hat{w}(\vec{\gamma}C) \in L^2(\widehat{\mathbb{R}^n})$ and set $\tilde{A} = CAC^{-1}$. Then

$$\sum_{k\in\mathbb{Z}} |\tilde{w}(\vec{\gamma}\tilde{A}^k)|^2 = \sum_{k\in\mathbb{Z}} |\hat{w}(\vec{\gamma}CA^k)|^2 = 1 \quad \text{a.e.}$$

and a similar computation holds for continuous tight frame generators.

3. Cross-sections

In order to unify notation, let $G = \mathbb{Z}$ or $G = \mathbb{R}$, respectively. Then for fixed $A \in GL_n(\mathbb{R})$ (which must be an exponential in the latter case), the map $\vec{\gamma} \to \vec{\gamma}A^t$ with $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ and $t \in G$ defines a continuous action of the group G on $\widehat{\mathbb{R}^n}$. Recall that a Borel set $S \subset \widehat{\mathbb{R}^n}$ is called a *cross-section* for this action provided that

(1) $\bigcup_{t\in G} SA^t = \widehat{\mathbb{R}^n}\backslash N$ for some set N of measure zero,
(2) $SA^{t_1} \cap SA^{t_2} = \emptyset$ whenever $t_1 \neq t_2 \in G$.

In this section we will discuss the existence of such cross-sections. It is easy to see that the property of existence of cross-sections is invariant under similarity transformations so that we can restrict our attention to matrices of particularly nice form. In fact, in theorems 1 and 2 we will see that cross-sections exist if and only if A is not similar to an orthogonal matrix.

The next lemma will be used in the case where A is similar to an orthogonal matrix:

Lemma 1. *Let $\Pi^s = \{\Theta = (e^{i\theta_1}, \ldots, e^{i\theta_s}) : 0 \leq \theta_i < 2\pi\}$ be the s-torus, let $\Theta \to \Theta \cdot t$ denote an action of the reals on Π^s by rotation and let $f \in L^1(\Pi^s)$.*

(1) *If*

$$\sum_{k\in\mathbb{Z}} |f(\Theta \cdot k)| < \infty \quad a.e.$$

then $f = 0$ almost everywhere.

(2) *If*

$$\int_{\mathbb{R}} |f(\Theta \cdot t)|\, dt < \infty \quad a.e.$$

then $f = 0$ almost everywhere.

Proof. Note that if $\Theta = (e^{i\theta_1}, \ldots, e^{i\theta_s}) \in \Pi^s$ then $\Theta \cdot t = (e^{i(\theta_1+\beta_1 t)}, \ldots, e^{i(\theta_s+\beta_s t)})$ for some fixed $\beta_1, \ldots \beta_s$. This action is measure preserving.

In the first case as Π^s has finite measure, Birkhoff's Ergodic Theorem shows that the sequence of functions

$$f_n(\Theta) = \frac{1}{n}\sum_{k=0}^{n-1} |f(\Theta \cdot k)|$$

converges almost everywhere to a function $f^* \in L^1(\Pi^s)$ with $\|f^*\|_1 = \|f\|_1$. Now by assumption,

$$\lim_{n\to\infty} f_n(\Theta) \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k\in Z} |f(\Theta \cdot k)| = 0 \quad \text{a.e.}$$

which shows that $\|f^*\|_1 = 0$. Thus, $f = 0$ almost everywhere.

The second assertion is proved in a similar way using the continuous form of Birkhoff's theorem. □

To begin with the continuous action, let $A = e^B \in GL_n(\mathbb{R})$ be given. Applying a similarity transformation, we may assume that B is in real Jordan normal form, so that B is a block diagonal matrix and a block corresponding to a real eigenvalue α is of the form

$$B_0 = \begin{pmatrix} \alpha & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \alpha \end{pmatrix}$$

while a block corresponding to a complex pair of eigenvalues $\alpha \pm i\beta$ with $\beta > 0$ is of the form

$$B_0 = \begin{pmatrix} D & I_2 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ (0) & & & D \end{pmatrix}$$

where

$$D = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this basis, A^t is again a block diagonal matrix, and its blocks are of the form

$$e^{tB_0} = \begin{pmatrix} \lambda^t E(t) & t\lambda^t E(t) & \frac{t^2}{2!}\lambda^t E(t) & \cdots\cdots & \cdots\cdots & \frac{t^{m-1}}{(m-1)!}\lambda^t E(t) \\ & \lambda^t E(t) & t\lambda^t E(t) & & & \vdots \\ & & \ddots & & & \vdots \\ & & & \ddots & t\lambda^t E(t) & \frac{t^2}{2!}\lambda^t E(t) \\ & & & & \lambda^t E(t) & t\lambda^t E(t) \\ (0) & & & & & \lambda^t E(t) \end{pmatrix} \tag{5}$$

with $\lambda = e^\alpha$ and $E(t) = 1$ or $E(t) = E_\beta(t) = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$ depending on whether this block corresponds to a real eigenvalue or a pair of complex eigenvalues of B. The eigenvalues of A corresponding to such a block are thus e^α or $e^\alpha e^{\pm i\beta}$, respectively.

In what follows, $\vec{v}_1, \ldots, \vec{v}_n$ will denote a Jordan basis of B chosen so that the block of A under discussion is its first block, and $(x_1, \ldots, x_n)$ will denote the components of a vector $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ in this basis.

Theorem 1. *Let $A = e^B$, where $B \in M_n(\mathbb{R})$ is in Jordan normal form. There exists a cross-section for the continuous action $\vec{\gamma} \to \vec{\gamma}A^t$ if and only if A is not orthogonal.*

Proof. Assume that A is not orthogonal. Then at least one of the following four situations, formulated in terms of the eigenvalues of B, will always apply.

Case 1: B has a real eigenvalue $\alpha \neq 0$. Then a corresponding block of A^t, which we may assume to be its first block, is of form (5) with $\lambda \neq 1$ and $E(t) = 1$. One easily checks that

$$S = \{\ (x_1, \ldots, x_n) \in \widehat{\mathbb{R}^n}\ :\ x_1 \in \{\pm 1\}\ \}.$$

is a cross-section with $\bigcup_{t \in \mathbb{R}} SA^t = \{(x_1, \ldots, x_n) \in \widehat{\mathbb{R}^n} : x_1 \neq 0\}$.

Case 2: B has a complex pair of eigenvalues $\alpha \pm i\beta$ with $\alpha \neq 0$ and $\beta > 0$. Then at least one block of A^t is of form (5) with $\lambda = e^{\alpha} \neq 1$ and $E(t) = E_\beta(t)$ a rotation matrix. Then

$$S = \{\ (x_1, \dots, x_n) \in \widehat{\mathbb{R}^n}\ :\ 1 \leq x_1 < \lambda^{2\pi/\beta},\ x_2 = 0\ \}$$

is a cross-section with $\bigcup_{t\in\mathbb{R}} SA^t = \{(x_1, \dots, x_n) \in \widehat{\mathbb{R}^n} : x_1^2 + x_2^2 \neq 0\}$.

Case 3: B has an eigenvalue $\alpha = 0$ and at least one of the blocks of B belonging to this eigenvalue has nontrivial nilpotent part. Then the corresponding block of A^t is of form (5) with $\lambda = 1$ and $E(t) = 1$, and is of at least size 2×2. We set

$$S = \{\ (x_1, \dots, x_n) \in \widehat{\mathbb{R}^n}\ :\ x_1 \neq 0,\ x_2 = 0\ \}$$

so that S is a cross-section with $\bigcup_{t\in\mathbb{R}} SA^t = \{(x_1, \dots, x_n) \in \widehat{\mathbb{R}^n} : x_1 \neq 0\}$.

Case 4: B has a purely imaginary pair of eigenvalues $\pm i\beta$, $\beta > 0$, and at least one of the blocks of B belonging to this pair has nontrivial nilpotent part. Then the corresponding block of A^t is of form (5) with $\lambda = 1$ and $E(t) = E_\beta(t)$ a rotation matrix, and is of at least size 4×4. Set

$$S = \{\ (x_1, \dots, x_n) \in \widehat{\mathbb{R}^n}\ :\ x_1 > 0,\ x_2 = 0,\ 0 \leq x_3 < \frac{2\pi}{\beta} x_1\ \}.$$

Since this is the least intuitive case, let us verify in detail that S is a cross-section. For convenience, we group the first four coordinates of a vector $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ into two pairs, and write

$$\vec{\gamma} = \big(\ (x_1, x_2),\ (x_3, x_4),\ x_5, x_6\ \dots, x_n\big)$$

so that

$$\vec{\gamma}A^t \ =\ \big(\ (x_1, x_2)E_\beta(t),\ t(x_1, x_2)E_\beta(t) + (x_3, x_4)E_\beta(t), \dots\big).$$

Now $E_\beta(t)$ acts by rotation through the angle βt, so whenever $x_1^2 + x_2^2 \neq 0$ then there exists $t_1 \in \mathbb{R}$ such that

$$(x_1, x_2)E_\beta(t_1) \ =\ (p, 0)$$

for some $p > 0$. Then

$$\vec{\gamma}A^{t_1} \ =\ (p, 0, t_1 p + y_3, y_4, \dots)$$

where $(y_3, y_4) = (x_3, x_4)E(t_1)$. Now if we set $t_2 = t_1 + k\,\frac{2\pi}{\beta}$ for some integer k, then

$$\vec{\gamma}A^{t_2} \ =\ (p, 0, k\frac{2\pi p}{\beta} + t_1 p + y_3, y_4, \dots).$$

There exists a k such that

$$0 \leq k\frac{2\pi p}{\beta} + t_1 p + y_3 < \frac{2\pi p}{\beta}$$

and for this choice of k, $\vec{\gamma}A^{t_2} \in S$. We conclude that

$$\bigcup_{t\in\mathbb{R}} SA^t \ =\ \{(x_1, \dots, x_n) \in \widehat{\mathbb{R}^n} : x_1^2 + x_2^2 \neq 0\}.$$

Suppose now that

$$\vec{\gamma}_1 A^{t_1} = \vec{\gamma}_2 A^{t_2}$$

for some $\vec{\gamma}_1, \vec{\gamma}_2 \in S$, $t_1, t_2 \in \mathbb{R}$. Equivalently,

$$\vec{\gamma}_1 = \vec{\gamma}_2 A^t$$

for some t. If $\vec{\gamma}_1 = (x_1, 0, x_3, x_4, \dots)$ and $\vec{\gamma}_2 = (\tilde{x}_1, 0, \tilde{x}_3, \tilde{x}_4, \dots)$ then

$$\big((x_1,0),\ (x_3,x_4), \dots \big) = \big((\tilde{x}_1,0)E_\beta(t),\ t(\tilde{x}_1,0)E_\beta(t) + (\tilde{x}_3,\tilde{x}_4)E_\beta(t), \dots \big)$$

so that

$$(x_1,0) = (\tilde{x}_1,0)E_\beta(t)$$
$$(x_3,x_4) = t(\tilde{x}_1,0)E_\beta(t) + (\tilde{x}_3,\tilde{x}_4)E_\beta(t).$$

The first equality gives $x_1 = \tilde{x}_1$ and $t = \frac{2\pi}{\beta}k$ for some integer k. Then the second equality reads

$$(x_3,x_4) = (\frac{2\pi}{\beta}\, kx_1 + \tilde{x}_3, \tilde{x}_4)$$

which gives $x_4 = \tilde{x}_4$ and because $0 \le x_3, \tilde{x}_3 < \frac{2\pi}{\beta}x_1$, also that $k = 0$ and $x_3 = \tilde{x}_3$. Thus, S is indeed a cross-section.

Now suppose to the contrary that A is orthogonal, but there exists a cross-section S. We may write A^t as

$$\begin{pmatrix} I_q & & & (0) \\ & E_{\beta_1}(t) & & \\ & & \ddots & \\ (0) & & & E_{\beta_s}(t) \end{pmatrix}$$

where I_q denotes the $q \times q$ identity matrix, and the rotations E_{β_i} are nontrivial. We do a change of coordinates by introducing polar coordinates into the part of $\widehat{\mathbb{R}^n}$ on which A acts by rotations. If

$$\vec{\gamma} = (x_1, \dots, x_q, x_{q+1}, x_{q+2}, \dots, x_{q+2s-1}, x_{q+2s})$$

we set

$$\begin{cases} x_{q+2i-1} = r_i \cos\theta_i \\ x_{q+2i} = r_i \sin\theta_i \end{cases} \qquad 0 \le \theta_i < 2\pi,\ 0 < r_i < \infty,\ i = 1 \dots s. \tag{6}$$

In this way we can identify the Borel space $\widehat{\mathbb{R}^n}$ with $\widehat{\mathbb{R}^q} \times (0,\infty)^s \times \Pi^s$ up to a set of measure zero, and A acts non-trivially only on the s-torus part Π^s and in the form of rotations determined by the multi-angle $(\beta_1, \dots, \beta_s)$. Then for $\hat{f} \in L^1(\widehat{\mathbb{R}^n})$,

$$\int_{\widehat{\mathbb{R}^n}} \hat{f}(\vec{\gamma})\, d\vec{\gamma} = \int_{\widehat{\mathbb{R}^q}} \int_0^\infty \cdots \int_0^\infty \Big[\int_0^{2\pi} \cdots \int_0^{2\pi}$$
$$\hat{f}(x_1, \dots, x_q, r_1, \dots, r_s, \theta_1, \dots, \theta_s)\, d\theta_1 \dots d\theta_s \Big]\, r_1 r_2 \dots r_s\, dr_1 \dots dr_s\, dx_1 \dots dx_q.$$

Set $T = \{SA^t : 0 \leq t < 1\}$ so that T is a cross-section for the discrete action of A on $\widehat{\mathbb{R}^n}$. For fixed $\vec{x} = (x_1, \ldots, x_q) \in \widehat{\mathbb{R}^q}$ and $\vec{r} = (r_1, \ldots, r_s) \in (0, \infty)^s$, we identify the set $\{\vec{x}\} \times \{\vec{r}\} \times \Pi^s$ with the s-torus Π^s and for simplicity, denote the restriction of the action of A to this torus by $\Theta \to \Theta A^k$. Set

$$T_{\vec{x},\vec{r}} = \{\Theta \in \Pi^s : (\vec{x}, \vec{r}, \Theta) \in T\}.$$

Then $T_{\vec{x},\vec{r}}A^k \cap T_{\vec{x},\vec{r}}A^j = \emptyset$ if $k \neq j$, and $T_{\vec{x},\vec{r}}$ is a cross-section for the discrete action of A on Π^s for almost all $(\vec{x}, \vec{r})$. In particular, $T_{\vec{x},\vec{r}}$ has finite, positive measure for almost all $(\vec{x}, \vec{r})$. Now if $\chi_{T_{\vec{x},\vec{r}}}$ denotes the characteristic function of the set $T_{\vec{x},\vec{r}}$, then by lemma 1, we have $\chi_{T_{\vec{x},\vec{r}}} = 0$ a.e. for almost all $(\vec{x}, \vec{r})$, which is a contradiction. This shows that there cannot exist a cross-section S in case A is orthogonal. □

Remark 2. The cross-sections constructed in the proof above allow for a change of variables to integrate along the orbits.

For example in case 3, each $\vec{\gamma} = (x_1, x_2, \ldots, x_n) \in \widehat{\mathbb{R}^n}$ with $x_1 \neq 0$, can be written uniquely as

$$\vec{\gamma} = \vec{\gamma}(t, s, a_3, \ldots, a_n) = (s, 0, a_3, \ldots, a_n)A^t$$

where t, s, a_i are real and $s \neq 0$. The Jacobian of this transformation is

$$\begin{vmatrix} (s,0,a_3,\ldots,a_n)B \\ \vec{v}_1 \\ \vec{v}_3 \\ \vdots \\ \vec{v}_n \end{vmatrix} \det(A)^t = \begin{vmatrix} 0 & s & * & \ldots & * \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{vmatrix} \det(A)^t = -s\delta^t \neq 0$$

so that for $g \in L^1(\widehat{\mathbb{R}^n})$,

$$\int_{\widehat{\mathbb{R}^n}} g(\vec{\gamma})\, d\vec{\gamma} = \int_{\widehat{\mathbb{R}^{n-2}}} \int_{\widehat{\mathbb{R}}\setminus\{0\}} \int_{\mathbb{R}} g(\, (s, 0, a_3, \ldots, a_n)A^t\,)\, |s|\delta^t\, dt\, ds\, da_3 \ldots da_n.$$

In case 4, each $\vec{\gamma} = (x_1, x_2, \ldots, x_n) \in \widehat{\mathbb{R}^n}$ with $x_1^2 + x_2^2 \neq 0$ can be written uniquely as

$$\vec{\gamma} = \vec{\gamma}(t, p, q, s, a_5, \ldots, a_n) = (p, 0, q, s, a_5, \ldots, a_n)A^t$$

where t, p, q, s, a_i are real, $p > 0$ and $0 \leq q < 2\pi p/\beta$. The Jacobian of this transformation is

$$\begin{vmatrix} (p,0,q,s,a_5,\ldots,a_n)B \\ \vec{v}_1 \\ \vec{v}_3 \\ \vdots \\ \vec{v}_n \end{vmatrix} \det(A)^t = \begin{vmatrix} 0 & \beta p & * & * & * & \ldots & * \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & \ldots & 1 \end{vmatrix} \det(A)^t$$

$$= -\beta p\delta^t \neq 0$$

since $\beta \neq 0$. Thus for $g \in L^1(\widehat{\mathbb{R}^n})$,

$$\int_{\widehat{\mathbb{R}^n}} g(\vec{\gamma})\, d\vec{\gamma} = \int_{\widehat{\mathbb{R}^{n-4}}} \int_0^\infty \int_{\widehat{\mathbb{R}}} \int_0^{2\pi p/\beta} \int_{\mathbb{R}} g(\,(p,0,q,s,a_5,\ldots,a_n)A^t\,)\, \beta p \delta^t \; dt\, dq\, ds\, dp\, da_5 \ldots da_n .$$

Let us now turn to the the discrete action. Any invertible matrix A gives rise to a discrete action on $\widehat{\mathbb{R}^n}$, and can be brought into Jordan normal form through a similarity transformation.

Theorem 2. *Let $A \in GL_n(\mathbb{R})$ be in Jordan normal form, and consider the discrete action $\vec{\gamma} \to \vec{\gamma}A^k$.*

1. *There exists a cross-section if and only if A is not orthogonal.*
2. *There exists a cross-section of finite measure if and only if $|\det(A)| \neq 1$.*
3. *There exists a bounded cross-section if and only if the eigenvalues of A have all modulus > 1 or all modulus < 1.*

Proof. To prove the first assertion, note that A^k is a block diagonal matrix whose blocks are of the form

$$\begin{pmatrix} \lambda^k E_\beta(k) & \binom{k}{1}\lambda^{k-1}E_\beta(k-1) & \ldots\;\ldots & \binom{k}{m-1}\lambda^{k-m+1}E_\beta(k-m+1) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \binom{k}{1}\lambda^{k-1}E_\beta(k-1) \\ (0) & & & \lambda^k E_\beta(k) \end{pmatrix}$$

where $E_\beta = 1$ if this block corresponds to a real eigenvalue λ and E_β is a rotation if it belongs to a complex pair $\lambda e^{\pm i\beta}$ of eigenvalues of A. By a slight change of basis we can simplify these blocks to

$$\begin{pmatrix} \lambda^k E_\beta(k) & \binom{k}{1}\lambda^{k}E_\beta(k) & \ldots\;\ldots & \binom{k}{m-1}\lambda^{k}E_\beta(k) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \binom{k}{1}\lambda^{k}E_\beta(k) \\ (0) & & & \lambda^k E_\beta(k) \end{pmatrix}. \tag{7}$$

In the following, $\vec{v}_1, \ldots, \vec{v}_n$ will denote such a basis chosen so that the block of A^k under discussion is its first block, and $(x_1, \ldots, x_n)$ will denote the components of a vector $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ in this basis. Now if A is not orthogonal then at least one of the following cases must be true.

Case 1: A has a real eigenvalue λ with $|\lambda| \neq 1$. Replacing A by A^{-1} if necessary we may assume that $|\lambda| > 1$. A corresponding block of A^k is an $m \times m$ upper diagonal matrix of form (7) with $E_\beta = 1$, and one easily checks that

$$S = \{\ (x_1, \ldots, x_n) \in \widehat{\mathbb{R}^n}\ :\ 1 \leq |x_1| < |\lambda|\ \} \tag{8}$$

is a cross-section.

Case 2: A has a complex pair of eigenvalues $\lambda e^{\pm i\beta}$ with $\lambda \neq 1$, $0 < \beta < 2\pi$. We may again assume that $\lambda > 1$. A corresponding block of A^k is a $2m \times 2m$ upper diagonal matrix of form (7) with E_β a proper rotation so that

$$S = \{ \ (x_1, \ldots, x_n) \in \widehat{\mathbb{R}^n} \ : \ (x_1, x_2) = (s, 0)\lambda^t E_\beta(t), \ 1 \leq s < \lambda^{2\pi/\beta}, \ 0 \leq t < 1 \ \} \tag{9}$$

is a cross-section.

Case 3: A has a real eigenvalue $\lambda = \pm 1$ and at least one of the blocks of A belonging to this eigenvalue has nontrivial nilpotent part. Then the corresponding block of A^k is of the form (7) with $E_\beta = 1$ and is of at least size 2×2. It follows that the set

$$S = \{ \ (x_1, \ldots, x_n) \in \widehat{\mathbb{R}^n} \ : \ x_1 \neq 0, \ 0 \leq \frac{x_2}{x_1} < 1 \ \}$$

is a cross-section.

Case 4: A has a complex pair of eigenvalues $e^{\pm i\beta}$, $0 < \beta < 2\pi$, of modulus one and at least one of the blocks of A belonging to this pair has nontrivial nilpotent part. Then the corresponding block of A^k is of form (7) with $\lambda = 1$ and E_β a proper rotation, and is at least of size 4×4. It is easy to verify that

$$S = \{ \ (x_1, \ldots, x_n) \in \widehat{\mathbb{R}^n} \ : \ (x_1, x_2, x_3, x_4) = (p, 0, q, s) \begin{pmatrix} E_\beta(t) & tE_\beta(t) \\ 0 & E_\beta(t) \end{pmatrix},$$
$$p > 0, \ 0 \leq q < \frac{2\pi}{\beta} p, \ -\infty < s < \infty, \ 0 \leq t < 1 \ \}$$

is the desired cross-section.

Now suppose to the contrary that A is orthogonal, but there exists a cross-section S. Then $T = S \cup SA$ will be a cross-section for the discrete action of $\tilde{A} = A^2$, and we may write $\tilde{A}^k$ as

$$\begin{pmatrix} I_q & & & (0) \\ & E_{\beta_1}(k) & & \\ & & \ddots & \\ (0) & & & E_{\beta_s}(k) \end{pmatrix}.$$

Arguing as in the last part of the proof of theorem 1, we arrive at a contradiction. This proves the first assertion.

The remaining assertions are obvious if $n = 1$, or if $n = 2$ and A has complex eigenvalues. We thus can exclude this situation in what follows, so that the cross-section S constructed above has infinite measure.

Let us now prove the second assertion. In order to show that $|\det(A)| \neq 1$ is a sufficient condition, we only need to distinguish between the first two of the above cases. In the first case, we take the cross-section (8) constructed above, partition $span(\vec{v}_2, \ldots, \vec{v}_n)$ into a collection $\{T_k\}_{k=1}^{\infty}$ of measurable sets of positive, finite measure each, and set

$$S_k = \{ \ \vec{\gamma} \in S \ : \ (x_2, \ldots, x_n) \in T_k \ \}, \qquad k = 1, 2, \ldots$$

In the second case, we take the cross-section (9) constructed above, partition $span(\vec{v}_3, \dots, \vec{v}_n)$ into a collection $\{T_k\}_{k=1}^{\infty}$ of measurable subsets of finite, positive measure each, and set

$$S_k = \{ \vec{\gamma} \in S \; : \; (x_3, \dots, x_n) \in T_k \}, \qquad k = 1, 2 \dots$$

In both cases, $\{S_k\}_{k=1}^{\infty}$ is a partition of S into measurable subsets of positive, finite measure. Pick a collection of positive numbers $\{d_k\}_{k=1}^{\infty}$ so that $\sum_{k=1}^{\infty} d_k = 1$, and pick $n_k \in \mathbb{Z}$ such that $\delta^{n_k} \leq \frac{d_k}{\mu(S_k)}$ where $\delta = |\det(A)|$. Then

$$\tilde{S} := \bigcup_{k=1}^{\infty} S_k A^{n_k}$$

is a cross-section for the discrete action such that

$$\mu(\tilde{S}) = \sum_{k=1}^{\infty} \delta^{n_k} \mu(S_k) \leq \sum_{k=1}^{\infty} d_k = 1.$$

Thus, we have shown sufficiency.

To prove the necessity implication, suppose there exists a cross-section P of finite measure for the discrete action while $|\det(A)| = 1$. Let S denote the cross-section of infinite measure constructed in part 1 above. Then,

$$\begin{aligned}
\mu(P) = \int_{\widehat{\mathbb{R}^n}} \chi_P(\vec{\gamma})\, d\vec{\gamma} \; &= \; \sum_{i\in\mathbb{Z}} \int_S \chi_P(\vec{\gamma}A^i)\, d\vec{\gamma} \\
&= \sum_{i\in\mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} \chi_P(\vec{\gamma}A^i)\, \chi_S(\vec{\gamma})\, d\vec{\gamma} \\
&= \sum_{i\in\mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} \chi_P(\vec{\gamma})\, \chi_S(\vec{\gamma}A^{-i})\, d\vec{\gamma} \\
&= \sum_{i\in\mathbb{Z}} \int_P \chi_S(\vec{\gamma}A^{-i})\, d\vec{\gamma} \\
&= \int_{\widehat{\mathbb{R}^n}} \chi_S(\vec{\gamma})\, d\vec{\gamma} \; = \; \mu(S) \; = \; \infty
\end{aligned}$$

which is impossible. Thus, there cannot exist a cross-section of finite measure.

Finally we will prove the last assertion. For sufficiency, it is enough to assume that all eigenvalues of A have modulus $|\lambda| < 1$ so that

$$\lim_{k\to\infty} \|A^k\| = 0.$$

Choosing the above sets T_k to be bounded we may assume that the sets S_k are bounded, so that there exist integers n_k such that $S_k A^{n_k}$ is contained in the unit ball. Then $\tilde{S} = \bigcup_{k=1}^{\infty} S_k A^{n_k}$ is the desired cross-section.

For necessity, suppose to the contrary that there exists a bounded cross-section $\tilde{S}$, but A has an eigenvalue $|\lambda_1| < 1$ and an eigenvalue $|\lambda_2| \geq 1$. (The case where

$|\lambda_1| \leq 1$ and $|\lambda_2| > 1$ is treated similarly). Using the block decomposition of A it is easy to see that for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$, either

$$\lim_{|k| \to \infty} \|\vec{\gamma} A^k\| = \infty$$

or, in the special case where no eigenvalue of A lies outside of the unit circle,

$$\lim_{k \to -\infty} \|\vec{\gamma} A^k\| = \infty$$

while $\{\vec{\gamma} A^k : k > 0\}$ is bounded below away from zero. Thus, for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$ there exists a constant $M = M(\vec{\gamma})$ so that

$$\|\vec{\gamma} A^k\| > M \qquad \forall k \in \mathbb{Z}.$$

Fix any such $\vec{\gamma}$. Then for sufficiently large scalars c, the orbit $\{(c\vec{\gamma})A^k : k \in \mathbb{Z}\}$ does not pass through $\tilde{S}$, which is a contradiction to the choice of $\tilde{S}$. □

We note that in the proof of the second assertion, the sets T_k can be chosen so that the cross-section $\tilde{S}$ has unit measure. For example, if $\delta < 1$ we simply choose the sets T_k so that $\mu(S_k) = (\sum_{i=1}^{\infty} \delta^i)^{-1}$ for all k, and set $n_k = k$.

4. Existence of Tight Frames

We are now ready to prove our main theorems.

Theorem 3. *Let $A \in GL_n(\mathbb{R})$. Then there exists a semi-discrete tight frame generator if and only if $|\det(A)| \neq 1$.*

Proof. By remark 1, we may assume that A is in Jordan normal form.

Suppose that $|\det(A)| \neq 1$ so that there exists a bounded cross-section S. Then the characteristic function χ_S is square integrable, and

$$\sum_{k \in \mathbb{Z}} |\chi_S(\vec{\gamma} A^k)|^2 = 1$$

for almost all $\vec{\gamma} \in \widehat{\mathbb{R}^n}$. By proposition 1, the inverse Fourier transform w of χ_S will be a semi-discrete tight frame generator.

Assume now that w is a semi-discrete tight frame generator, but $|\det(A)| = 1$. If A is not orthogonal, then as shown in the proof of theorem 2, there exists a cross-section S of infinite measure, and we have

$$\begin{aligned}\|\hat{w}\|_2^2 &= \sum_{k \in \mathbb{Z}} \int_{SA^k} |\hat{w}(\vec{\gamma})|^2 \, d\vec{\gamma} = \sum_{k \in \mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} \chi_{SA^k}(\vec{\gamma}) \, |\hat{w}(\vec{\gamma})|^2 \, d\vec{\gamma} \\ &= \sum_{k \in \mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} \chi_{SA^k}(\vec{\gamma} A^k) \, |\hat{w}(\vec{\gamma} A^k)|^2 \, d\vec{\gamma} \\ &= \int_S \sum_{k \in \mathbb{Z}} |\hat{w}(\vec{\gamma} A^k)|^2 \, d\vec{\gamma} = \int_S 1 \, d\vec{\gamma} = \infty\end{aligned}$$

which is impossible.

If on the other hand A is orthogonal, then A^2 is of the form

$$\begin{pmatrix} I_q & & & (0) \\ & E_{\beta_1} & & \\ & & \ddots & \\ (0) & & & E_{\beta_s} \end{pmatrix}.$$

If the rotation part is trivial, there obviously can not exist tight frame generators. Otherwise, we introduce polar coordinates into the part of $\widehat{\mathbb{R}^n}$ on which A acts by rotations, as shown in (6). For fixed $\vec{x} =\in \widehat{\mathbb{R}^q}$ and $\vec{r} \in (0,\infty)^s$, let us set

$$\hat{w}_{\vec{x},\vec{r}}(\Theta) = \hat{w}(\vec{x},\vec{r},\Theta).$$

Then $\hat{w}_{\vec{x},\vec{r}} \in L^2(\Pi^s)$ for almost all $(\vec{x},\vec{r})$. Now A^2 acts non-trivially only on the torus Π^s, and as w is a semi-discrete tight frame generator,

$$\sum_{k\in\mathbb{Z}} |\hat{w}_{\vec{x},\vec{r}}(\Theta A^{2k})|^2 \leq \sum_{k\in\mathbb{Z}} |\hat{w}(\,(\vec{x},\vec{r},\Theta)A^k\,)|^2 = 1 \quad \text{a.e.}$$

for almost all $(\vec{x},\vec{r})$. Then by lemma 1, $\hat{w}_{\vec{x},\vec{r}} = 0$ for almost all $(\vec{x},\vec{r})$ which is impossible. This proves the theorem.

□

Theorem 4. *Let $A = e^B$ for some $B \in M_n(\mathbb{R})$. Then there exists a continuous tight frame generator if and only if* $\det(A) \neq 1$.

Proof. Suppose that $\det(A) \neq 1$. Then by the above theorem there exists a semi-discrete tight frame generator $w \in L^2(\mathbb{R}^n)$ which by proposition 2 is also a continuous tight frame generator.

Now assume to the contrary that there exists a continuous tight frame generator w, but $\det(A) = 1$. As before, we may assume that B is in Jordan normal form.

If A is not orthogonal then A is as discussed in cases 3 and 4 of the proof of theorem 1. In case 3, when a cross-section is chosen as in remark 2, we have

$$\begin{aligned} \|\hat{w}\|_2^2 &= \int_{\widehat{\mathbb{R}^{n-2}}} \int_{\widehat{\mathbb{R}}\setminus\{0\}} \int_{\mathbb{R}} \left|\hat{w}(\,(s,0,a_3,\dots,a_n)A^t\,)\right|^2 |s|\,dt\,ds\,da_3\dots da_n \\ &= \int_{\widehat{\mathbb{R}^{n-2}}} \int_{\widehat{\mathbb{R}}\setminus\{0\}} |s|\,ds\,da_3\dots da_n \;=\; \infty \end{aligned}$$

which is impossible. In case 4 we have

$$\begin{aligned} \|\hat{w}\|_2^2 &= \int_{\widehat{\mathbb{R}^{n-4}}} \int_0^\infty \int_{\widehat{\mathbb{R}}} \int_0^{2\pi p/\beta} \int_{\mathbb{R}} \left|\hat{w}(\,(p,0,q,s,a_5,\dots,a_n)A^t\,)\right|^2 \beta p \\ &\qquad\qquad dt\,dq\,ds\,dp\,da_5\dots da_n \\ &= \int_{\widehat{\mathbb{R}^{n-4}}} \int_0^\infty \int_{\widehat{\mathbb{R}}} 2\pi p^2\,ds\,dp\,da_5\dots da_n \;=\; \infty \end{aligned}$$

which again is impossible.

If on the other hand A is orthogonal, then we again introduce polar coordinates into the part of $\widehat{\mathbb{R}^n}$ on which A acts by nontrivial rotations, as shown in (6). For fixed $\vec{x} \in \widehat{\mathbb{R}^q}$ and $\vec{r} \in (0, \infty)^s$, let us set again

$$\hat{w}_{\vec{x},\vec{r}}(\Theta) = \hat{w}(\vec{x}, \vec{r}, \Theta).$$

Then for almost all $(\vec{x}, \vec{r})$,

$$\hat{w}_{\vec{x},\vec{r}} \in L^2(\Pi^s) \quad \text{and} \quad \int_{\mathbb{R}} |\hat{w}_{\vec{x},\vec{r}}(\Theta A^t)|^2 \, dt = 1 \quad \text{a.e.}$$

By Lemma 1 it follows that $\hat{w}_{\vec{x},\vec{r}} = 0$ for almost all $(\vec{x}, \vec{r})$ which is impossible. This proves the theorem. □

Note that by choosing a cross-section S of unit measure in the proof of theorem 3, one can obtain tight frame generators of norm one.

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APPENDIX II

PUBLISHED ARTICLE WHICH INCLUDES THE RESEARCH RESULTS

The research results were incorporated into section I.2 of the following research article:

Title:	Explicit Cross-sections of Singly Generated Group Actions
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1

Explicit cross sections of singly generated group actions

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Summary. We consider two classes of actions on $\mathbb{R}^n$ - one continuous and one discrete. For matrices of the form $A = e^B$ with $B \in M_n(\mathbb{R})$, we consider the action given by $\gamma \longrightarrow \gamma A^t$. We characterize the matrices A for which there is a cross-section for this action. The discrete action we consider is given by $\gamma \longrightarrow \gamma A^k$, where $A \in GL_n(\mathbb{R})$. We characterize the matrices A for which there exists a cross-section for this action as well. We also characterize those A for which there exist special types of cross-sections; namely, bounded cross-sections and finite measure cross-sections. Explicit examples of cross-sections are provided for each of the cases in which cross-sections exist. Finally, these explicit cross-sections are used to characterize those matrices for which there exist MSF wavelets with infinitely many wavelet functions. Along the way, we generalize a well-known aspect of the theory of shift-invariant spaces to shift-invariant spaces with infinitely many generators.

1.1 Introduction

In discrete wavelet analysis on the line, the classical approach is to dilate and translate a single function, or *wavelet*, so that the resulting system is an orthonormal basis for $L^2(\mathbb{R})$. More precisely, a wavelet is a function $\psi \in L^2(\mathbb{R})$ such that

$$\{2^{j/2}\psi(2^j x + k) : k, j \in \mathbb{Z}\}$$

forms an orthonormal basis of $L^2(\mathbb{R})$.

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In multidimensional discrete wavelet analysis, the approach is similar. Fix a matrix $A \in GL_n(\mathbb{R})$ and a full rank lattice Γ. A collection of functions $\{\psi^i : i = 1, \dots, N\}$ is called an (A, Γ) *orthonormal wavelet of order* N if dilations by A and translations by Γ,

$$\{|\det A|^{j/2}\psi^i(A^j x + k) : i = 1, \dots, N, j \in \mathbb{Z}, k \in \Gamma\},$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$. In this generality, there is no characterization (in terms of A and Γ) of when wavelets exist. It was shown in [10] that, if A is expansive (that is, a matrix whose eigenvalues all have modulus greater than 1) then there does exist an orthonormal wavelet. A complete characterization of such wavelets in terms of the Fourier transform was given in [13]. The non-expansive case remains problematic.

It is also possible to study the continuous version of wavelet analysis. Consider the full affine group of motions given by $GL_n(\mathbb{R}) \times \mathbb{R}^n$ with multiplication given by $(a, b)(c, d) = (ac, c^{-1}b + d)$. We are interested in subgroups of the full affine group of motions of the form

$$G = \{(a, b) : a \in D, b \in \mathbb{R}^n\},$$

where D is a subgroup of $GL_n(\mathbb{R})$. In this case, G is the semi-direct product $D \times_s \mathbb{R}^n$. Now, if we define the unitary operator T_g for $g \in G$ by

$$T_g\psi = |\det a|^{-1/2}\psi(g^{-1}(x)),$$

then the continuous wavelet transform is given by

$$\langle f, \psi_g \rangle := \int_{\mathbb{R}^n} f(x)\overline{(T_g\psi)(x)}\, dx,$$

which is, of course, a function on G. The function ψ is a D-continuous wavelet if it is possible to reconstruct all functions f in $L^2(\mathbb{R}^n)$ via the following reconstruction formula:

$$f(x) = \int_G \langle f, \psi_g \rangle \psi_g(x)\, d\lambda(g),$$

where λ is Haar measure on G.

There is a simple characterization of continuous wavelets, given in [22].

Theorem 1. *[22] Let G be a subgroup of the full affine group of the form $D \times_s \mathbb{R}^n$. A function $\psi \in L^2(\mathbb{R}^n)$ is a D-continuous wavelet if and only if the Calderón condition*

$$\int_D |\hat{\psi}(\xi a)|^2\, d\mu(a) = 1 \quad a.e.\ \xi \text{ in } \widehat{\mathbb{R}^n} \tag{1.1}$$

holds, where μ is left Haar measure for D.

In this paper, we will always assume one of the two following cases, which for the purposes of this paper will be the *singly generated subgroups* of $GL_n(\mathbb{R})$.

1. $D = \{A^k : k \in \mathbb{Z}\}$ for some $A \in GL_n(\mathbb{R})$, or
2. $D = \{A^t : t \in \mathbb{R}\}$ for some $A = e^B$, where $B \in M_n(\mathbb{R})$.

We will say that D is generated by the matrix A. Applying Theorem 1 to these cases gives the following characterizations.

Proposition 1. *1. Let $A \in GL_n(\mathbb{R})$ and denote the dilation group $D = \{A^k : k \in \mathbb{Z}\}$. Then, $\psi \in L^2(\mathbb{R}^n)$ is a D-continuous wavelet if and only if*

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi A^k)|^2 = 1$$

for almost all $\xi \in \widehat{\mathbb{R}^n}$.

2. Let $D = \{A^t : t \in \mathbb{R}\}$ for some $A = e^B$, where $B \in M_n(\mathbb{R})$. Then $\psi \in L^2(\mathbb{R}^n)$ is a D-continuous wavelet if and only if

$$\int_{\mathbb{R}} |\hat{\psi}(\xi A^t)|^2 \, dt = 1$$

for almost all $\xi \in \widehat{\mathbb{R}^n}$.

In the case that D is generated by a single matrix as above, a complete characterization of matrices for which there exists a continuous wavelet is given in [17].

Theorem 2. *Consider the dilation group D as in case 1 or 2 above. There exists a continuous wavelet if and only if $|\det(A)| \neq 1$.*

The wavelets constructed in [17] are of the form $\hat{\psi} = \chi_K$, for some set K. One drawback to the proof in [17] is that, while the proof is constructive, the sets K that are constructed are written as the countable union of set differences of sets consisting of those points whose orbits land in a prescribed closed ball a positive, finite number of times. Hence, it is not clear whether the set constructed in the end can be chosen to be "nice" or easily described.

The purpose of this article is two-fold. First, we will give explicitly defined, easily verified sets K such that χ_K is the Fourier transform of a continuous wavelet. Here, we will exploit the fact that we are in the singly generated group case to a very large extent. We will also obtain a characterization of matrices such that the set K can be chosen to be bounded as well as a characterization of matrices such that the only sets K that satisfy (1.1) have infinite measure.

Second, we will show how to use these explicit forms to characterize those matrices such that there exists a discrete wavelet of order infinity. Note that this seems to be a true application of the form of the sets K in section 1.2, as it is not clear to the authors how to use the proof in [17] (or the related proof in [16]) to achieve the same result.

1.2 Cross-sections

Throughout this section, we will use vector notation to denote elements of $\widehat{\mathbb{R}^n}$, and m will denote the Lebesgue measure on $\mathbb{R}^n$. Multiplication of a vector with a matrix will be given by γA, and we will reserve the notation A^t as "A raised to the t-power". In the few places we need the transpose of a matrix, we will give it a separate name.

Definition 1. *A Borel set $S \subset \widehat{\mathbb{R}^n}$ is called a* cross-section *for the continuous action $\gamma \to \gamma A^t$ $(t \in \mathbb{R})$ if*

1. $\bigcup_{t\in\mathbb{R}} SA^t = \widehat{\mathbb{R}^n}\backslash N$ *for some set N of measure zero and*
2. $SA^{t_1} \cap SA^{t_2} = \emptyset$ *whenever $t_1 \neq t_2 \in \mathbb{R}$.*

Similarly, a Borel set $S \subset \widehat{\mathbb{R}^n}$ is called a cross-section *for the discrete action $\gamma \to \gamma A^k$ $(k \in \mathbb{Z})$ if*

1. $\bigcup_{k\in\mathbb{Z}} SA^k = \widehat{\mathbb{R}^n}\backslash N$ *for some set N of measure zero and*
2. $SA^j \cap SA^k = \emptyset$ *whenever $j \neq k \in \mathbb{Z}$.*

Note that we have defined cross-sections using left products, which will eliminate the need for taking transposes in section 1.3.

Note also that if S is a cross-section for the continuous action, then $\{\gamma A^t : \gamma \in S, 0 \leq t < 1\}$ is a cross-section for the discrete action. Cross-sections are sometimes referred to as multiplicative tiling sets.

Remark 1. Let S be a cross-section for the action $\gamma \to \gamma A^k$. Then, SJ^{-1} is a cross-section for the action $\gamma \to \gamma JA^kJ^{-1}$, and similarly, for the continuous action $\gamma \to \gamma A^t$, where $A = e^B$, SJ^{-1} is a cross-section for the continuous action $\gamma \to \gamma \tilde{A}^t$, where $\tilde{A} = e^{JBJ^{-1}}$.

To begin with cross-sections for the continuous action, let $A = e^B \in GL_n(\mathbb{R})$ be given, where by the preceding remark we may assume that B is in real Jordan normal form. Then, B is a block diagonal matrix, and a block corresponding to a real eigenvalue α_i is of the form

$$B_i = \begin{pmatrix} \alpha_i & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ (0) & & & \alpha_i \end{pmatrix}$$

while a block corresponding to a complex pair of eigenvalues $\alpha_i \pm i\beta_i$ with $\beta_i \neq 0$ is of the form

$$B_i = \begin{pmatrix} D_i & I_2 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ (0) & & & D_i \end{pmatrix} \quad \text{with} \quad \begin{aligned} D_i &= \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix} \\ I_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

In this basis, A^t is again a block diagonal matrix, and its blocks are of the form

$$A_i = e^{tB_i} = \begin{pmatrix} \lambda_i^t E_i(t) & t\lambda_i^t E_i(t) & \frac{t^2}{2!}\lambda_i^t E_i(t) & \cdots\cdots & \frac{t^{m-1}}{(m-1)!}\lambda_i^t E_i(t) \\ & \lambda_i^t E_i(t) & t\lambda_i^t E_i(t) & & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & t\lambda_i^t E_i(t) & \frac{t^2}{2!}\lambda_i^t E_i(t) \\ & & & \lambda_i^t E_i(t) & t\lambda_i^t E_i(t) \\ (0) & & & & \lambda_i^t E_i(t) \end{pmatrix}$$

with $\lambda_i = e^{\alpha_i}$ and $E_i(t) = 1$ or $E_i(t) = E_{\beta_i}(t) = \begin{pmatrix} \cos\beta_i t & \sin\beta_i t \\ -\sin\beta_i t & \cos\beta_i t \end{pmatrix}$ depending on whether this block corresponds to a real eigenvalue or a pair of complex eigenvalues of B. The eigenvalues of A are thus e^{α_i} and $e^{\alpha_i}e^{\pm i\beta_i}$, respectively.

For ease of notation, when referring to a specific block A_i of A we will drop the index i. Furthermore, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ will denote a Jordan basis of $\widehat{\mathbb{R}^n}$ chosen so that this block under discussion is the first block, and $(x_1, \ldots, x_n)$ will denote the components of a vector γ in this basis.

Theorem 3. *Let $A = e^B$, where $B \in M_n(\mathbb{R})$ is in Jordan-normal form. There exists a cross-section for the continuous action $\gamma \to \gamma A^t$ if and only if A is not orthogonal.*

Proof. Assume that A is not orthogonal. Then at least one of the following four situations, formulated in terms of the eigenvalues of B, will always apply.

Case 1: B has a real eigenvalue $\alpha \neq 0$. A corresponding block of A^t, which we may assume to be the first block, is of the form

$$\begin{pmatrix} \lambda^t & t\lambda^t & \ldots & \frac{t^{m-1}}{(m-1)!}\lambda^t \\ & \ddots & \ddots & \\ & & \ddots & t\lambda^t \\ (0) & & & \lambda^t \end{pmatrix}$$

with $\lambda = e^\alpha \neq 1$. Set

$$S = \{\pm\mathbf{v}_1\} \times span(\mathbf{v}_2, \ldots, \mathbf{v}_n).$$

Then S is a cross-section and $\bigcup_{t\in\mathbb{R}} SA^t = \{(x_1, \ldots, x_n) \in \widehat{\mathbb{R}^n} : x_1 \neq 0\}$.

Case 2: B has a complex pair of eigenvalues $\alpha \pm i\beta$ with $\alpha \neq 0$, $\beta > 0$. At least one block of A^t is then of the form

$$\begin{pmatrix} \lambda^t E_\beta(t) & t\lambda^t E_\beta(t) & \dots & \frac{t^{m-1}}{(m-1)!}\lambda^t E_\beta(t) \\ & \ddots & \ddots & \\ & & \ddots & t\lambda^t E_\beta(t) \\ (0) & & & \lambda^t E_\beta(t) \end{pmatrix}, \tag{1.2}$$

and replacing B with $-B$ if necessary, we may assume that $\lambda = e^\alpha > 1$. One easily checks that

$$S = \{s\mathbf{v}_1 : 1 \le s < \lambda^{2\pi/\beta}\} \times span(\mathbf{v}_3, \dots, \mathbf{v}_n)$$

is a cross-section and $\bigcup_{t\in\mathbb{R}} SA^t = \{(x_1, \dots, x_n) \in \widehat{\mathbb{R}^n} : x_1^2 + x_2^2 \neq 0\}$.

Case 3: B has an eigenvalue $\alpha = 0$ and at least one of the blocks of B belonging to this eigenvalue has nontrivial nilpotent part. Then the corresponding block of A^t is of the form

$$\begin{pmatrix} 1 & t & & (*) \\ & \ddots & \ddots & \\ & & \ddots & t \\ (0) & & & 1 \end{pmatrix} \tag{1.3}$$

and is of at least size 2×2. We set

$$S = \{\ s\mathbf{v}_1 : s \in \mathbb{R}\backslash\{0\}\ \} \times span(\mathbf{v}_3, \dots, \mathbf{v}_n)$$

so that S is a cross-section and $\bigcup_{t\in\mathbb{R}} SA^t = \{(x_1, \dots, x_n) \in \widehat{\mathbb{R}^n} : x_1 \neq 0\}$.

Case 4: B has a purely imaginary pair of eigenvalues $\pm i\beta$, $\beta > 0$, and at least one of the blocks of B belonging to this pair has nontrivial nilpotent part. Then the corresponding block of A^t is of the form

$$\begin{pmatrix} E_\beta(t) & tE_\beta(t) & & (*) \\ & \ddots & \ddots & \\ & & \ddots & tE_\beta(t) \\ (0) & & & E_\beta(t) \end{pmatrix} \tag{1.4}$$

and is of at least size 4×4. Set

$$S = \{p\mathbf{v}_1 + q\mathbf{v}_3 + s\mathbf{v}_4 : p > 0,\ 0 \le q < \frac{2\pi}{\beta}p,\ s \in \mathbb{R}\} \times span(\mathbf{v}_5, \dots, \mathbf{v}_n).$$

Since this is the least intuitive case, let us verify in detail that S is a cross-section. For convenience, we group the first four coordinates of a vector $\gamma \in \widehat{\mathbb{R}^n}$ into two pairs, and write

$$\gamma = \big(\ (x_1, x_2),\ (x_3, x_4),\ x_5, x_6 \dots, x_n\big),$$

so that

$$\gamma A^t = \big((x_1, x_2)E_\beta(t),\ t(x_1, x_2)E_\beta(t) + (x_3, x_4)E_\beta(t), \dots \big).$$

Now $E_\beta(t)$ acts by rotation through the angle βt, so whenever $x_1^2 + x_2^2 \neq 0$ then there exists $t_1 \in \mathbb{R}$ such that

$$(x_1, x_2)E_\beta(t_1) = (p, 0)$$

for some $p > 0$. Then

$$\gamma A^{t_1} = (p, 0, t_1 p + y_3, y_4, \dots)$$

where $(y_3, y_4) = (x_3, x_4)E(t_1)$. So if we set $t_2 = t_1 + k\,\frac{2\pi}{\beta}$ for some integer k, then

$$\gamma A^{t_2} = (p, 0, k\frac{2\pi p}{\beta} + t_1 p + y_3, y_4, \dots).$$

Now there exists a k such that

$$0 \leq k\frac{2\pi p}{\beta} + t_1 p + y_3 < \frac{2\pi p}{\beta}$$

and for this choice of k, $\gamma A^{t_2} \in S$. We conclude that

$$\bigcup_{t \in \mathbb{R}} SA^t = \{(x_1, \dots, x_n) \in \widehat{\mathbb{R}^n} : x_1^2 + x_2^2 \neq 0\}.$$

Suppose now that

$$\gamma_1 A^{t_1} = \gamma_2 A^{t_2}$$

for some $\gamma_1, \gamma_2 \in S,\ t_1, t_2 \in \mathbb{R}$. Equivalently,

$$\gamma_1 = \gamma_2 A^t$$

for some t. If $\gamma_1 = (p_1, 0, q_1, s_1, \dots)$ and $\gamma_2 = (p_2, 0, q_2, s_2, \dots)$ then

$$\big((p_1, 0),\ (q_1, s_1), \dots \big) = \big((p_2, 0)E_\beta(t),\ t(p_2, 0)E_\beta(t) + (q_2, s_2)E_\beta(t), \dots \big)$$

so that

$$\begin{aligned} (p_1, 0) &= (p_2, 0)E_\beta(t) \\ (q_1, s_1) &= t(p_2, 0)E_\beta(t) + (q_2, s_2)E_\beta(t). \end{aligned}$$

The first equality gives $p_1 = p_2$ and $t = \frac{2\pi}{\beta}\,k$ for some integer k. Then the second equality reads

$$(q_1, s_1) = (\frac{2\pi}{\beta}\,kp_1 + q_2, s_2)$$

which gives $s_1 = s_2$ and because $0 \le q_2, q_1 < \frac{2\pi p}{\beta}$, also that $k = 0$ and $q_1 = q_2$. Thus, S is indeed a cross-section.

Now suppose to the contrary that A is orthogonal, but there exists a cross-section S. Then

$$T = \{\gamma A^t : \gamma \in S,\ 0 \le t < 1\}$$

is a cross-section for the discrete action of A on $\widehat{\mathbb{R}^n}$. Note that A maps the closed unit ball $B_1(0)$ onto itself, so if $T_o = T \cap B_1(0)$ then

$$B_1(0) = \bigcup_{k\in\mathbb{Z}} T_o A^k,$$

except for a set of measure zero, and this union is disjoint. Then

$$m\big(B_1(0)\big) = \sum_{k\in\mathbb{Z}} m(T_o A^k) = \sum_{k\in\mathbb{Z}} m(T_o) \in \{0, \infty\}$$

which is impossible.

Remark 2. The cross-sections constructed in the proof above allow for a change of variables to integrate along the orbits.

For example in case 3), given $\gamma = (x_1, x_2, \ldots, x_n) \in \widehat{\mathbb{R}^n}$ with $x_1 \ne 0$, we set

$$\gamma = F(t, s, a_3, \ldots, a_n) = (s, 0, a_3, \ldots, a_n)A^t$$

where $s \ne 0$. The Jacobian of this transformation is

$$\begin{vmatrix} (s,0,a_3,\ldots,a_n)B \\ \mathbf{v}_1 \\ \mathbf{v}_3 \\ \vdots \\ \mathbf{v}_n \end{vmatrix} \det(A)^t = \begin{vmatrix} 0 & s & * & \ldots & * \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \end{vmatrix} \det(A)^t = -s\delta^t \ne 0,$$

so that for $\hat{f} \in L^2(\widehat{\mathbb{R}^n})$,

$$\int_{\widehat{\mathbb{R}^n}} \hat{f}(\gamma)\,d\gamma = \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} \hat{f}(\,(s,0,a_3,\ldots,a_n)A^t\,)\,|s|\delta^t\,dt\,ds\,da_3\cdots da_n.$$

In case 4), given $\gamma = (x_1, x_2, \ldots, x_n) \in \widehat{\mathbb{R}^n}$ with $x_1^2 + x_2^2 \ne 0$, we set

$$\gamma = F(t, p, q, s, a_5, \ldots, a_n) = (p, 0, q, s, a_5, \ldots, a_n)A^t$$

where $p > 0$, $0 \le q < 2\pi p/\beta$. The Jacobian of this transformation is

$$\begin{vmatrix} (p,0,q,s,a_5,\ldots,a_n)B \\ \mathbf{v}_1 \\ \mathbf{v}_3 \\ \vdots \\ \mathbf{v}_n \end{vmatrix} \det(A)^t = \begin{vmatrix} 0 & \beta p & * & * & * & \ldots & * \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & 0 & & 0 & \ldots & 1 \end{vmatrix} \det(A)^t$$

$$= -\beta p \delta^t \neq 0$$

since $\beta \neq 0$. Thus,

$$\int_{\widehat{\mathbb{R}^n}} \hat{f}(\gamma)\, d\gamma = \int_{\mathbb{R}^{n-4}} \int_0^\infty \int_{\mathbb{R}} \int_0^{2\pi p/\beta} \int_{\mathbb{R}} \hat{f}(\,(p,0,q,s,a_5,\ldots,a_n)A^t\,)\, |\beta| p \delta^t \; dt\, dq\, ds\, dp\, da_5 \ldots da_n.$$

Any invertible matrix gives rise to a discrete action on $\widehat{\mathbb{R}^n}$, and nearly always there will exist a cross-section for this action:

Theorem 4. *Let $A \in GL_n(\mathbb{R})$ be in Jordan normal form, and consider the discrete action $\gamma \to \gamma A^k$.*

1. *There exists a cross-section if and only if A is not orthogonal.*
2. *There exists a cross-section of finite measure if and only if $|\det(A)| \neq 1$.*
3. *There exists a bounded cross-section if and only if the (real or complex) eigenvalues of A have all modulus > 1 or all modulus < 1.*

Proof. To prove the first assertion, choose a Jordan basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ so that the Jordan block of A under discussion is the first block. Each Jordan block will be an upper diagonal matrix of the form

$$\begin{pmatrix} \lambda^k E_\beta(k) & \binom{k}{1}\lambda^{k-1}E_\beta(k-1) & \ldots & \ldots & \binom{k}{m-1}\lambda^{k-m+1}E_\beta(k-m+1) \\ & \ddots & & \ddots & \vdots \\ & & & \ddots & \binom{k}{1}\lambda^{k-1}E_\beta(k-1) \\ (0) & & & & \lambda^k E_\beta(k) \end{pmatrix}$$

where $E_\beta = 1$ if this block corresponds to a real eigenvalue λ, and E_β is a rotation if it belongs to a complex pair $\lambda e^{\pm i\beta}$ of eigenvalues. By a change of basis, we can always simplify this block to

$$\begin{pmatrix} \lambda^k E_\beta(k) & \binom{k}{1}\lambda^k E_\beta(k) & \ldots & \ldots & \binom{k}{m-1}\lambda^k E_\beta(k) \\ & \ddots & & \ddots & \vdots \\ & & & \ddots & \binom{k}{1}\lambda^k E_\beta(k) \\ (0) & & & & \lambda^k E_\beta(k) \end{pmatrix}. \tag{1.5}$$

Now if A is not orthogonal then at least one of the following cases will be true.

Case 1: A has a real eigenvalue λ with $|\lambda| \neq 1$. Replacing A by A^{-1} if necessary we may assume that $|\lambda| > 1$. A corresponding block of A^k is an $m \times m$ upper diagonal matrix of the form (1.5) with $E_\beta = 1$, and one easily checks that

$$S = \{ s\mathrm{v}_1 : 1 \leq |s| < |\lambda| \} \times span(\mathrm{v}_2, \dots, \mathrm{v}_n)$$

is a cross-section.

Case 2: A has a complex pair of eigenvalues $\lambda e^{\pm i\beta}$ with $\lambda \neq 1$, $0 < \beta < \pi$. We may again assume that $\lambda > 1$. A corresponding block of A^k is a $2m \times 2m$ upper diagonal matrix of form (1.5) with E_β a proper rotation. Then

$$S = \{ s\mathrm{v}_1 \lambda^t E_\beta(t) : 1 \leq s < \lambda^{2\pi/\beta},\ 0 \leq t < 1 \} \times span(\mathrm{v}_3, \dots, \mathrm{v}_n)$$

is a cross-section, which can be checked by using case 2 in Theorem 3 and keeping in mind the note immediately following definition 1.

Case 3: A has a real eigenvalue $\lambda = \pm 1$ and at least one of the blocks of A belonging to this eigenvalue has nontrivial nilpotent part. Then the corresponding block of A^k is of the form (1.5) with $E_\beta = 1$ and is of at least size 2×2. On easily verifies that the set

$$S = \{ s(\mathrm{v}_1 + t\mathrm{v}_2) : s \in \mathbb{R}\backslash\{0\},\ 0 \leq t < 1 \} \times span(\mathrm{v}_3, \dots, \mathrm{v}_n)$$

is a cross-section.

Case 4: A has a complex pair of eigenvalues $e^{\pm i\beta}$, $0 < \beta < \pi$, of modulus one and at least one of the blocks of A belonging to this pair has nontrivial nilpotent part. Then the corresponding block of A^k is of the form (1.5), with $\lambda = 1$ and E_β a proper rotation, and

$$S = \Big\{ (p\mathrm{v}_1 + q\mathrm{v}_3 + s\mathrm{v}_4) \begin{pmatrix} E_\beta(t) & tE_\beta(t) \\ 0 & E_\beta(t) \end{pmatrix} : p > 0,\ 0 \leq q < \frac{2\pi}{\beta} p,\ s \in \mathbb{R},\ 0 \leq t < 1 \Big\} \times span(\mathrm{v}_5, \dots, \mathrm{v}_n)$$

is the desired cross-section, which can be checked by using case 4 in Theorem 3 and keeping in mind the note immediately following definition 1.

The argument at the end of the proof of theorem 3 shows that if A is orthogonal, then there can not exist a cross-section. This proves the first assertion.

The remaining assertions are obvious if $n = 1$, or if $n = 2$ and A has complex eigenvalues. We thus can exclude this situation in what follows, so that the cross-section S constructed above has infinite measure.

Next let use prove the second assertion. In order to show that $|\det(A)| \neq 1$ is a sufficient condition, we only need to distinguish between the first two of the above cases.

We begin by considering the first case, and we may assume that $|\lambda| > 1$. Take the cross-section constructed above,

$$S = \{ s\mathbf{v}_1 + \mathbf{v} : 1 \leq |s| < |\lambda|, \ \mathbf{v} \in span(\mathbf{v}_2, \dots, \mathbf{v}_n) \},$$

partition $span(\mathbf{v}_2, \dots, \mathbf{v}_n)$ into a collection $\{T_k\}_{k=1}^{\infty}$ of measurable sets of positive, finite measure each, and set

$$S_k = \{ s\mathbf{v}_1 + \mathbf{v} : 1 \leq |s| < |\lambda|, \ \mathbf{v} \in T_k \}, \qquad k = 1, 2, \dots$$

Then $\{S_k\}_{k=1}^{\infty}$ is a partition of S into measurable subsets of positive, finite measure. Pick a collection of positive numbers $\{d_k\}_{k=1}^{\infty}$ so that $\sum_{k=1}^{\infty} d_k = 1$, and pick $n_k \in \mathbb{Z}$ such that $\delta^{n_k} \leq \frac{d_k}{m(S_k)}$ where $\delta = |\det(A)|$. It follows that

$$\tilde{S} := \bigcup_{k=1}^{\infty} S_k A^{n_k}$$

is a cross-section for the discrete action such that

$$m(\tilde{S}) = \sum_{k=1}^{\infty} \delta^{n_k} m(S_k) \leq \sum_{k=1}^{\infty} d_k = 1.$$

In the second case, we may assume that $\lambda > 1$. Start with the above constructed cross-section,

$$S = \{ s\mathbf{v}_1 \lambda^t E_\beta(t) + \mathbf{v} : 1 \leq s < \lambda^{2\pi/\beta}, \ 0 \leq t < 1, \ \mathbf{v} \in span(\mathbf{v}_3, \dots, \mathbf{v}_n) \},$$

partition $span(\mathbf{v}_3, \dots, \mathbf{v}_n)$ into a collection $\{T_k\}_{k=1}^{\infty}$ of measurable subsets of finite, positive measure each, and set

$$S_k = \{ s\mathbf{v}_1 \lambda^t E_\beta(t) + \mathbf{v} : 1 \leq s < \lambda^{2\pi/\beta}, \ 0 \leq t < 1, \ \mathbf{v} \in T_k \}, \qquad k = 1, 2 \dots$$

so that $\{S_k\}_{k=1}^{\infty}$ is a partition of S into measurable subsets of positive, finite measure. Continuing as in the first case we have shown sufficiency.

To prove the necessity implication, suppose there exists a cross-section P of finite measure for the discrete action and $|\det(A)| = 1$. Let S denote the cross-section for the discrete action constructed in part 1 above. Then,

$$\begin{aligned} m(P) = \int_{\widehat{\mathbb{R}^n}} \chi_P(\gamma)\, d\gamma \; &= \; \sum_{i\in\mathbb{Z}} \int_S \chi_P(\gamma A^i)\, d\gamma \\ &= \sum_{i\in\mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} \chi_P(\gamma A^i)\, \chi_S(\gamma)\, d\gamma \\ &= \sum_{i\in\mathbb{Z}} \int_{\widehat{\mathbb{R}^n}} \chi_P(\gamma)\, \chi_S(\gamma A^{-i})\, d\gamma \\ &= \sum_{i\in\mathbb{Z}} \int_P \chi_S(\gamma A^{-i})\, d\gamma \\ &= \int_{\widehat{\mathbb{R}^n}} \chi_S(\gamma)\, d\gamma \; = \; m(S) \; = \; \infty \end{aligned}$$

which is impossible. Thus, there can not exist a cross-section of finite measure.

Finally we will prove the last assertion. For sufficiency, it is enough to assume that all eigenvalues of A have modulus $|\lambda| < 1$ so that

$$\lim_{k\to\infty} \|A^k\| = 0.$$

Choosing each of the above sets T_k to be bounded we may assume that the sets S_k are bounded, so that there exist integers n_k such that $S_k A^{n_k}$ is contained in the unit ball. Then $\tilde{S} = \bigcup_{k=1}^{\infty} S_k A^{n_k}$ is the desired bounded cross-section.

For necessity, suppose to the contrary that there exists a bounded cross-section $\tilde{S}$, but A has an eigenvalue $|\lambda_1| < 1$ and an eigenvalue $|\lambda_2| \geq 1$. (The case where $|\lambda_1| \leq 1$ and $|\lambda_2| > 1$ is treated similarly). Using the block decomposition of A it is easy to see that for almost all $\gamma \in \widehat{\mathbb{R}^n}$, either

$$\lim_{|k|\to\infty} \|\gamma A^k\| = \infty$$

or, in the special case where no eigenvalue of A lies outside of the unit circle,

$$\lim_{k\to-\infty} \|\gamma A^k\| = \infty$$

while $\{\gamma A^k : k \geq 0\}$ is bounded below away from zero. Thus, for almost all $\gamma \in \widehat{\mathbb{R}^n}$ there exists a constant $M = M(\gamma)$ so that

$$\|\gamma A^k\| > M \qquad\qquad \forall k \in \mathbb{Z}.$$

Fix any such γ. Then for sufficiently large scalars c, the orbit of $c\gamma$ does not pass through $\tilde{S}$, contradicting the choice of $\tilde{S}$.

We note that in the proof of the second assertion, the sets T_k can be chosen so that the cross-section $\tilde{S}$ has unit measure.

Remark 3. In [17], it was obtained as a corollary of their general work that

1. For $A \in GL_n(\mathbb{R})$ and $D = \{A^k : k \in \mathbb{Z}\}$, there is a continuous wavelet if and only if $|\det(A)| \neq 1$.
2. For $A = e^B$ and $D = \{A^t : t \in \mathbb{R}\}$, there is a continuous wavelet if and only if $|\det(A)| \neq 1$.

It is possible to recover these results using the ideas in this section. We mention only how to do so in the case that continuous wavelets exist. Let $A \in GL_n(\mathbb{R})$, and let S be a cross-section of Lebesgue measure 1 for the discrete action $\gamma \to \gamma A^k$. Then, the function ψ whose Fourier transform equals χ_S is a continuous wavelet for the group $\{A^k : k \in \mathbb{Z}\}$. If in addition, $A = e^B$, then ψ is also a continuous wavelet for the group $\{A^t : t \in \mathbb{R}\}$ since

$$\int_{\mathbb{R}} |\chi_S(\gamma A^t)|^2 \, dt \;=\; \int_0^1 \sum_{k\in\mathbb{Z}} |\chi_S(\gamma A^t A^k)|^2 \, dt \;=\; \int_0^1 1 \, dt \;=\; 1.$$

We note here that the method of proof in [17], while ostensibly constructive, does not easily yield cross-sections of a desirable form such as the ones constructed above.

1.3 Shift-invariant Spaces and Discrete Wavelets

Let $A \in GL_n(\mathbb{R})$ and $\Gamma \subset \mathbb{R}^n$ be a full-rank lattice. An (A, Γ) orthonormal [resp. Parseval, Bessel] wavelet of order N is a collection of functions $\{\psi^i\}_{i=1}^N$ (where here we allow the possibility of $N = \infty$) such that

$$\{|\det A|^{j/2}\psi^i(A^j \cdot + k) : j \in \mathbb{Z}, k \in \Gamma, i = 1, \dots, N\}$$

is an orthonormal basis [resp. Parseval frame, Bessel system] for $L^2(\mathbb{R}^n)$. There has been much work done on determining for which pairs (A, Γ) orthonormal wavelets of finite order exist, often with extra desired properties such as fast decay in time or frequency.

This is not necessary for the proofs that we present. Of particular importance in determining when orthonormal wavelets exist are the MSF (minimally supported frequency) wavelets, which are intimately related to wavelet sets. An (A, Γ) multi-wavelet set K of order L is a set that can be partitioned into subsets $\{K_i\}_{i=1}^L$ such that $\{\frac{1}{|\det(B)|^{1/2}}\chi_{K_i}\}_{i=1}^L$ is the Fourier transform of an (A, Γ) orthonormal wavelet, where $\Gamma = B\mathbb{Z}^n$. (Here and in what follows, B denotes the transpose of A.) When the order of a multi-wavelet set is 1, we call it a wavelet set. These have been studied in detail in [1, 2, 3, 10, 14, 15, 19, 21]. The following fundamental question in this area remains open, even in the case $L = 1$.

Question 1. For which pairs (A, Γ) and orders L do there exist (A, Γ) wavelet sets of order L?

It is known that if A is expansive and Γ is any full-rank lattice, then there exists an (A,Γ) wavelet set of order 1 [10]. One can also modify the construction to obtain (A,Γ) wavelet sets of any finite order along the lines in Theorem 10 below. Diagonal matrices A for which there exist $(A,\mathbb{Z}^n)$ multi-wavelet sets of finite order were characterized in [20]. Theorem 4, part 2 above implies that, in order for an (A,Γ) multi-wavelet set of finite order to exist, it is necessary that A not have determinant one. There is currently no good conjecture as to what the condition on (A,Γ) should be for wavelet sets to exist. It is known that $|\det(A)| \neq 1$ is not sufficient and that all eigenvalues greater than or equal to 1 in modulus is not necessary.

We begin with the following.

Theorem 5. *Let $A \in GL_n(\mathbb{R})$ and $\Gamma \subset \mathbb{R}^n$ be a full-rank lattice with dual Γ^*. The set K is a multi-wavelet set of order L if and only if*

$$\sum_{\gamma\in\Gamma^*} \chi_K(\xi+\gamma) = L \quad a.e.\ \xi \in \widehat{\mathbb{R}^n}, \tag{1.6}$$

$$\sum_{j\in\mathbb{Z}} \chi_K(\xi A^j) = 1 \quad a.e.\ \xi \in \widehat{\mathbb{R}^n}. \tag{1.7}$$

Proof. The forward direction is very similar to the arguments presented in [10], so we sketch the proof only. Let K be a multi-wavelet set of order L. Partition K into $\{K_i\}_{i=1}^{\infty}$ such that χ_{K_i} is an (A,Γ) multi-wavelet of order L. Then, since $\chi_{K_i}(\xi A^j)$ is orthogonal to $\chi_{K_k}(\xi A^l)$ for each $(i,j) \neq (k,l)$, it follows that $K_iA^j \cap K_kA^l$ is a null-set when $(i,j) \neq (k,l)$. Therefore, $\sum_{j\in\mathbb{Z}} \chi_K(\xi A^j) \le 1$ a.e. $\xi \in \widehat{\mathbb{R}^n}$. Moreover, since every L^2 function can be written as the combination of functions supported on $\cup_{j=1}^{\infty} KA^j$, it follows that $\sum_{j\in\mathbb{Z}} \chi_K(\xi A^j) = 1$ a.e. $\xi \in \widehat{\mathbb{R}^n}$, proving (1.7). To see (1.6), since K_i is disjoint from K_jA^k for all $(j,k) \neq (i,0)$, it follows that $\frac{1}{|\det(B)|^{1/2}} e^{2\pi i\langle\xi,\gamma\rangle}$ must be an orthonormal basis for $L^2(K_i)$. This implies (1.6).

For the reverse direction, it is clear that what is needed is to partition K into $\{K_i\}_{i=1}^{L}$ so that each K_i satisfies $\sum_{\gamma\in\Gamma^*} \chi_K(\xi+\gamma) = 1$ a.e. $\xi \in \widehat{\mathbb{R}^n}$. This will follow from repeated application of the following fact. Given a measurable set K such that $\sum_{\gamma\in\Gamma^*} \chi_K(\xi+\gamma) \ge 1$ a.e. $\xi \in \widehat{\mathbb{R}^n}$, there exists a set $U = U(K) \subset K$ such that

$$\sum_{\gamma\in\Gamma^*} \chi_U(\xi+\gamma) = 1, \quad a.e. \quad \xi \in \widehat{\mathbb{R}^n}. \tag{1.8}$$

Now, let $\{V_i\}_{i=1}^{\infty}$ be a partition of $\widehat{\mathbb{R}^n}$ consisting of fundamental regions of Γ^*; that is, the sets V_i satisfy $\sum_{\gamma\in\Gamma^*} \chi_{V_i}(\xi+\gamma) = 1$ a.e. $\xi \in \widehat{\mathbb{R}^n}$. For a set $M \subset \widehat{\mathbb{R}^n}$ we define $M^t = \cup_{\gamma\in\Gamma^*}(M+\gamma)$. Let

$$L_0 = K.$$

Let

$$K_1 = (V_1 \cap L_0) \cup \big(U(L_0) \setminus (V_1 \cap L_0)^t\big),$$

where $U(L_0)$ is the subset of L_0 satisfying (1.8). Let $L_1 = L_0 \setminus K_1$, and notice that L_1 satisfies (1.6) with the right hand side reduced by 1. In general, let

$$K_i = (V_i \cap L_{i-1}) \cup \big(U(L_{i-1}) \setminus (V_i \cap L_{i-1})^t\big),$$

and

$$L_i = L_{i-1} \setminus K_i.$$

In the case that L is finite, this procedure will continue for L steps, resulting in a partition of K with the desired properties. In this case, the initial partition $\{V_i\}$ was not necessary. In the case $L = \infty$, since the V_i's partition $\widehat{\mathbb{R}^n}$, the union of the K_i's will contain K. Since the K_i's were constructed to be disjoint and to satisfy (1.8), the proof is complete.

There is also a soft proof of the reverse direction of Theorem 5, that yields slightly less information about wavelets, but provides some interesting facts about shift-invariant spaces. Before turning to the applications of Theorem 5, we provide this second proof.

When L is finite, we call an (A, Γ) orthonormal wavelet $\{\psi^i\}_{i=1}^L$ an (A, Γ) *combined MSF wavelet* if $\cup_{i=1}^L \mathrm{supp}(\hat{\psi}^i)$ has minimal Lebesgue measure. This terminology was introduced in [6], where it was shown that the minimal Lebesgue measure is L. It was also shown that if $\{\psi^i\}_{i=1}^L$ is a combined MSF wavelet, then there is a multi-wavelet set K of order L such that $K = \cup_{i=1}^L \mathrm{supp}(\hat{\psi}^i)$.

When $L = \infty$, it is not clear what the significance is for the union of the supports of $\hat{\psi}^i$ to have minimal Lebesgue measure. For this reason, we adopt the following definition. An (A, Γ) orthonormal wavelet $\{\psi^i\}_{i=1}^L$ is an (A, Γ) combined MSF wavelet if $K = \cup_{i=1}^L \mathrm{supp}(\hat{\psi}^i)$ is a multi-wavelet set of order L. This definition agrees with the previous definition in the case L is finite.

Let us begin by recalling some of the basic notions of shift-invariant spaces. A closed subspace $V \subset L^2(\mathbb{R}^n)$ is called *shift-invariant* if whenever $f \in V$ and $k \in \mathbb{Z}^n$, $f(x+k) \in V$. The shift-invariant space generated by the collection of functions $\Phi \subset L^2(\mathbb{R}^n)$ is denoted by $\mathcal{S}(\Phi)$ and given by

$$\overline{\mathrm{span}}\{\phi(x+k) : k \in \mathbb{Z}^n, \phi \in \Phi\}.$$

Given a shift-invariant space, if there exists a finite set $\Phi \subset L^2(\mathbb{R}^n)$ such that $V = \mathcal{S}(\Phi)$, then we say V is finitely generated. In the case Φ can be chosen to be a single function, we say V is a principal shift-invariant (PSI) space. For further basics about shift-invariant spaces, we recommend [7, 11, 12]. We will follow closely the development in [7].

Proposition 2. *The map* $\mathcal{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$ *defined by*

$$\mathcal{T} f(x) = (\hat{f}(x+k))_{k\in\mathbb{Z}^n}$$

is an isometric isomorphism between $L^2(\mathbb{R}^n)$ *and* $L^2(\mathbb{T}^n, \ell^2(\mathbb{Z}^n))$, *where* $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ *is identified with its fundamental domain, e.g.* $[0,1)^n$.

In what follows, as in Proposition 2 we will always assume that $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is identified with $[0,1)^n$.

A *range function* is a mapping

$$J : \mathbb{T}^n \to \{E \subset \ell^2(\mathbb{Z}^n) : E \text{ is a closed linear subspace}\}.$$

The function J is measurable if the associated orthogonal projections $P(x) : \ell^2(\mathbb{Z}^n) \to J(x)$ are weakly operator measurable. With these preliminaries, we can state an important theorem in the theory of shift-invariant spaces, due to Helson [7].

Theorem 6. *A closed subspace* $V \subset L^2(\mathbb{R}^n)$ *is shift-invariant if and only if*

$$V = \{f \in L^2(\mathbb{R}^n) : \mathcal{T} f(x) \in J(x) \ \textit{for a.e. } x \in \mathbb{T}^n\},$$

where J *is a measurable range function. The correspondence between* V *and* J *is one-to-one under the convention that the range functions are identified if they are equal a.e. Furthermore, if* $V = \mathcal{S}(\Phi)$ *for some countable* $\Phi \subset L^2(\mathbb{R}^n)$, *then*

$$J(x) = \overline{\operatorname{span}}\{\mathcal{T}\phi(x) : \phi \in \Phi\}.$$

Definition 2. *The* dimension function *of a shift-invariant space* V *is the mapping* $\dim_V : \mathbb{T}^n \to \mathbb{N} \cup \{0, \infty\}$ *given by*

$$\dim_V(x) = \dim J(x), \tag{1.9}$$

where J *is the range function associated with* V. *The spectrum of* V *is defined by* $\sigma(V) = \{x \in \mathbb{T}^n : J(x) \neq \{0\}\}$.

We are now ready to state the main result from [7] that we will need in this paper.

Theorem 7. *Suppose* V *is a shift-invariant subspace of* $L^2(\mathbb{R}^n)$. *Then* V *can be decomposed as an orthogonal sum*

$$V = \bigoplus_{i\in\mathbb{N}} \mathcal{S}(\phi_i), \tag{1.10}$$

where $\{\phi_i(x+k) : k \in \mathbb{Z}^n\}$ *is a Parseval frame for* $\mathcal{S}(\phi_i)$ *and* $\sigma(\mathcal{S}(\phi_{i+1})) \subset \sigma(\mathcal{S}(\phi_i))$ *for all* $i \in \mathbb{N}$. *Moreover,* $\dim_{\mathcal{S}(\phi_i)}(x) = \|\mathcal{T}\phi_i(x)\| \in \{0,1\}$ *for* $i \in \mathbb{N}$, *and*

$$\dim_V(x) = \sum_{i\in\mathbb{N}} \|\mathcal{T}\phi_i(x)\| \qquad \textit{for a.e. } x \in \mathbb{T}^n. \tag{1.11}$$

Finally, there is a folk-lore fact about dimension functions that we recall here. See Theorem 3.1 in [8] for discussion and references.

Proposition 3. *Suppose V is a shift-invariant space such that there exists a set Φ such that*

$$\{\phi(\cdot + k) : k \in \mathbb{Z}^n, \phi \in \Phi\}$$

is a Parseval frame for V. Then

$$\dim_V(\xi) = \sum_{\phi\in\Phi} \sum_{k\in\mathbb{Z}^n} |\hat{\phi}(\xi + k)|^2. \tag{1.12}$$

The following theorem is a relatively easy application of Theorem 7, which was certainly known in the case $N < \infty$, and probably known to experts in the theory of shift-invariant spaces in this full generality. It seems to be missing from the literature, so we include a proof.

Theorem 8. *Let V be a shift-invariant subspace of $L^2(\mathbb{R}^n)$. There exists a collection $\Phi = \{\phi_i\}_{i=1}^N \subset L^2(\mathbb{R}^n)$ such that*

$$\{\phi_i(x + k) : i \in \{1, \dots N\}, k \in \mathbb{Z}^n\}$$

is an orthonormal basis for V if and only if $\dim_V(x) = N$ a.e. $x \in \mathbb{T}^n$.

Proof. For the forward direction, it suffices to show that if $\{\phi_i(x + k) : k \in \mathbb{Z}^n, i = 1, \dots , N\}$ is an orthonormal basis for the (necessarily shift-invariant) space V, then $\dim_V(x) = N$ for a.e. $x \in \mathbb{T}^n$. It is easy to see that if $\{f(x+k) : k \in \mathbb{Z}^n\}$ is an orthonormal sequence, then $\sum_{k\in\mathbb{Z}^n} |\hat{f}(\xi + k)|^2 = 1$ a.e. Thus, by Proposition 3, $\dim_V(x) = N$ a.e.

For the reverse direction, assume V is a shift-invariant space satisfying $\dim_V(x) = N$ a.e. $x \in \mathbb{T}^n$. Let $\{\phi_i\}_{i=1}^\infty$ be the collection of functions such that (1.10) is satisfied. Using the facts that $\sigma(\mathcal{S}(\phi_{i+1})) \subset \sigma(\mathcal{S}(\phi_i))$ for all i, $\sigma(V) = \mathbb{T}^n$ and (1.11), it follows that

$$\sigma(\mathcal{S}(\phi_i)) = \begin{cases} \mathbb{T}^n & i \le N, \\ 0 & i > N. \end{cases} \tag{1.13}$$

By equation 1.10, we have for $1 \le i \le N$,

$$\dim_{\mathcal{S}(\phi_i)}(\xi) = 1 = \|\mathcal{T}\phi_i(\xi)\|^2 = \sum_{k\in\mathbb{Z}^n} |\hat{\phi}_i(\xi + k)|^2$$

Thus, $\{\phi_i(x+k) : k \in \mathbb{Z}^n\}$ is an orthonormal basis for $\mathcal{S}(\phi_i)$. Since the spaces $\mathcal{S}(\phi_i)$ are orthogonal, $\{\phi_i(x + k) : i \in \{1, \dots N\}, k \in \mathbb{Z}^n\}$ is an orthonormal basis for V, as desired.

We include a proof of the following proposition for completeness.

Proposition 4. *Let* $V = \{f \in L^2(\mathbb{R}^n) : supp(\hat{f}) \subset W\}$. *Then,* V *is shift-invariant and* $\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \chi_W(\xi+k) = \#\{k \in \mathbb{Z}^n : \xi + k \in W\}$ *a.e.*

Proof. Clearly, V so defined is shift-invariant. Let $\{e_k : k \in \mathbb{Z}^n\}$ be the standard basis for $\ell^2(\mathbb{Z}^n)$, and let ψ_k be defined by $\hat{\psi}_k = \chi_{(\mathbb{T}^n+k)\cap W}$, again for $k \in \mathbb{Z}^n$. It is easy to see that $V = S(\Psi)$, where $\Psi = \{\psi_k : k \in \mathbb{Z}^n\}$. Therefore, by Theorem 6, $J(\xi) = \overline{\text{span}}\{T\psi_k(\xi) : k \in \mathbb{Z}^n\} = \overline{\text{span}}\{e_k : \xi + k \in W\}$. The result then follows from the definition of dimension function in (1.9).

Corollary 1. *Let* $A \in GL_n(\mathbb{R})$, *and* K *be a measurable subset of* $\widehat{\mathbb{R}^n}$. *If*

$$\sum_{j\in\mathbb{Z}} \chi_K(\xi A^j) = 1 \quad a.e.\ \xi \text{ in } \widehat{\mathbb{R}^n}, \tag{1.14}$$

and

$$\sum_{k\in\mathbb{Z}^n} \chi_K(\xi + k) = N \quad a.e.\ \xi \text{ in } \widehat{\mathbb{R}^n},$$

then there is an $(A, \mathbb{Z}^n)$ *orthonormal wavelet of order* N *with* $\cup_{i=1}^N \text{supp}(\hat{\psi}^i) = K$.

Proof. By Proposition 4 and Theorem 8, there exists $\Psi = \{\psi^i\}_{i=1}^N$ such that $\{M_k\hat{\psi}^i : k \in \mathbb{Z}^n,\ i = 1,\dots,N\}$ is an orthonormal basis for $L^2(K)$, where M_k denotes modulation by k. Thus, by (1.14), Ψ is an $(A, \mathbb{Z}^n)$ wavelet.

The main theorem in this section is given in Theorem 10. Before stating this theorem, we give three results that will be useful in its proof.

Lemma 1. *Let* $C \subset \mathbb{R}^n$ *be a cone with non-empty interior,* $\Gamma \subset \mathbb{R}^n$ *be a full-rank lattice, and* $T \in \mathbb{N}$. *Then, the cardinality of* $C \cap \Gamma \cap (\mathbb{R}^n \setminus B_T(0))$ *is infinity.*

Proof. Let l be a line through the origin contained in the interior of C. The set $U = \{x \in \mathbb{R}^n : \text{dist}(x, l) < \epsilon\}$ is a centrally symmetric convex set, and

$$((C \cap B_T(0)) \setminus U) \text{ is bounded.} \tag{1.15}$$

By Minkowski's theorem (see, for example Theorem 1, Chapter 2, Section 7 in [18] and discussion thereafter), the cardinality of $U \cap \Gamma$ is infinity. Hence, by (1.15), the result follows.

The following proposition was proven in the setting of wave packets in $L^2(\mathbb{R})$ in [9]. We sketch the proof here in our setting of wavelets.

Proposition 5. *Suppose* $A \in GL_n(\mathbb{R})$ *has the following property: for all* $Z \subset \widehat{\mathbb{R}^n}$ *with positive measure and all* $q \in \mathbb{N}$, *there exist* $x_1 \dots, x_q \in \mathbb{Z}$ *such that*

$$m\Big(\bigcap_{i=1}^{q} ZA^{x_i}\Big) > 0.$$

Then, for every non-zero $\psi \in L^2(\mathbb{R}^n)$, ψ *is not an* $(A, \mathbb{Z}^n)$ *Bessel wavelet.*

Proof. Let $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$. Then there exists a set $Z \subset \widehat{\mathbb{R}^n}$ of positive measure such that $|\hat{\psi}(\xi)| \geq C > 0$ for all $\xi \in Z$. By reducing to a subset, we may assume that there exists a constant $K > 0$ such that, for every function $f \in L^2(\widehat{\mathbb{R}^n})$ with support in Z, we have

$$\sum_{k \in \mathbb{Z}^n} |\langle f, M_k \hat{\psi} \rangle|^2 \geq K \|f\|^2.$$

Since the operator $Df = |\det(A)|^{1/2} f(\cdot A)$ is unitary, for every $j \in \mathbb{Z}$ and for each function $f \in L^2(\widehat{\mathbb{R}^n})$ supported in $A^{-j}(Z)$, we obtain

$$\sum_{k \in \mathbb{Z}^n} |\langle f, D^j M_k \hat{\psi} \rangle|^2 \geq K \|f\|^2. \tag{1.16}$$

By hypothesis, there exist $x_1 \dots, x_q \in \mathbb{Z}$ such that for $U := \left(\bigcap_{i=1}^q ZA^{x_i}\right)$, we have $m(U) > 0$. This implies

$$\begin{aligned}\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle \chi_U, D^j M_k \hat{\psi} \rangle|^2 &\geq \sum_{i=1}^q \sum_{k \in \mathbb{Z}^n} |\langle \chi_U, D^{x_i} M_k \hat{\psi} \rangle|^2 \\ &\geq \sum_{i=1}^q K \|\chi_U\|^2 \\ &= qK \|\chi_U\|^2.\end{aligned}$$

Thus, since q is arbitrary, ψ is not an $(A, \mathbb{Z}^n)$ Bessel wavelet.

Theorem 9. *(Bonferroni's Inequality) If $\{A_i\}_{i=1}^N$ are measurable subsets of the measurable set B and k is a positive integer such that*

$$\sum_{i=1}^N |A_i| > k|B|,$$

then there exist $1 \leq i_1 < i_2 < \cdots < i_{k+1} \leq N$ such that

$$\left|\bigcap_{j=1}^k A_{i_j}\right| > 0.$$

Theorem 10. *Let $A \in GL_n(\mathbb{R})$ with real Jordan form J. The following statements are equivalent.*

1. *For every full-rank lattice $\Gamma \subset \mathbb{R}^n$, there exists a (J, Γ) orthonormal wavelet of order ∞.*
2. *There exists an $(A, \mathbb{Z}^n)$ orthonormal wavelet of order ∞.*
3. *For every full-rank lattice $\Gamma \subset \mathbb{R}^n$, there exists an (A, Γ) orthonormal wavelet of order ∞.*

4. There exists a (non-zero) $(A, \mathbb{Z}^n)$ Bessel wavelet of order 1.
5. J is not orthogonal.
6. The matrix A is not similar (over $M_n(\mathbb{C})$) to a unitary matrix.

Proof. Let us begin by summarizing the known results and obvious implications. The implication (2) $\implies$ (6) was proven Theorem 4.2 [16]. The implications (3) $\implies$ (2) $\implies$ (4) are obvious, and (5) $\iff$ (6) is standard.
(5) $\implies$ (1). Let Γ be a full-rank lattice with convex fundamental region Y for Γ^*. By Theorem 5, it suffices to show that there is a measurable cross-section S for the discrete action $\xi \to \xi J^k$ satisfying (1.7). As in Theorems 3 and 4, we break the analysis into cases.
Case 1: There is an eigenvalue of J not equal to 1 in modulus. WLOG, we assume there is an eigenvalue of modulus greater than 1. In this case, J can be written as a block diagonal matrix

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \tag{1.17}$$

where J_1 is expansive, and we allow the possibility that $\text{rank}(J_1) = \text{rank}(J)$. Let S be an open cross-section for the discrete action $\xi \to \xi J_1^k$. Partition S into disjoint open subsets $\{S_i : i \in \mathbb{N}\}$. For each i, choose k_i such that there exists $\gamma_i \in \Gamma^*$ such that $S_i A^{k_i} \times \mathbb{R}^{\text{rank}(J_2)} \supset (Y + \gamma_i)$. Then,

$$\cup_{i=1}^{\infty} (S_i A^{k_i} \times \mathbb{R}^{\text{rank}(J_2)})$$

is a cross-section satisfying (1.7).
Case 2: All eigenvalues of J have modulus 1. This means that we are in case 3 or case 4 of Theorem 4. We show that in either of these cases, the cross-section exhibited in Theorem 4 satisfies (1.7). First, note that S in these cases is a cone of infinite measure with a dense, open subset S°. Let $B \subset S^\circ$ be an open ball bounded away from the origin satisfying $\overline{B} \subset S^\circ$. Let $\delta = \text{diam}(Y)$. There exists a T such that

$$S_T := \{tb : t \geq T, b \in B\}$$

satisfies $\text{dist}(S_T, \mathbb{R}^n \backslash S) > \delta$. By Lemma 1, $\Gamma^* \cap S_T$ has infinite cardinality, and by choice of δ, $Y + \gamma \subset S$ for each $\gamma \in \Gamma^* \cap S_T$. Therefore, S is a cross-section satisfying (1.7).
(1) $\implies$ (2) $\implies$ (3). This follows from the following two facts. First, Γ is a full-rank lattice if and only if there is an invertible matrix B such that $\Gamma = B\mathbb{Z}^n$. Second, if $B \in GL_n(\mathbb{R})$, Ψ is an (A, Γ) orthonormal wavelet if and only if $\Psi_B := \{\frac{1}{|\det(B)|^{1/2}} \psi(B^{-1} \cdot) : \psi \in \Psi\}$ is a $(BAB^{-1}, B\Gamma)$ orthonormal wavelet. Indeed, (1) $\implies$ (2) is then immediate.

To see (2) $\implies$ (3), recall that (2) $\implies$ (6). Thus, if (2) is satisfied, then $J = B^{-1}AB$ is not orthogonal. Let Γ be a full-rank lattice. There exists a $(J, B^{-1}\Gamma)$ orthonormal wavelet of order ∞, so there exists an (A, Γ) orthonormal wavelet of order ∞.

(4) $\Longrightarrow$ *(6)*. Suppose that the real Jordan form of A is orthogonal. Then, for any bounded set $Z \subset \widehat{\mathbb{R}^n}$, there exists M such that for every $k \in \mathbb{Z}, z \in Z$, we have $\|zA^k\| \leq M$. Furthermore, if Z has positive measure, then

$$\sum_{k\in\mathbb{Z}} m(ZA^k \cap B_M(0)) = \infty.$$

Therefore, by Bonferroni's inequality, for every $q \in \mathbb{N}$, there exist $k_1, \ldots k_q$ such that

$$m(\cap_{j=1}^q ZA^{k_j}) > 0.$$

By Proposition 5, this says that for every non-zero ψ, ψ is not an $(A, \mathbb{Z}^n)$ Bessel wavelet.

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