การจำแนกประเภทเชิงกลุ่มของสมการของไหลหนึ่งมิติ ซึ่งมี
พลังงานภายในขึ้นอยู่กับความหนาแน่นและเกรเดียนต์ของความหนาแน่น

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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GROUP CLASSIFICATION OF ONE-DIMENSIONAL EQUATIONS OF FLUIDS WITH INTERNAL ENERGY DEPENDING ON THE DENSITY AND THE GRADIENT OF THE DENSITY

Mrs. Prakrong Voraka

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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ประครอง วรกา : การจำแนกประเภทเชิงกลุ่มของสมการของไหลหนึ่งมิติ ซึ่งมีพลังงานภายในขึ้นอยู่กับความหนาแน่นและเกรเดียนต์ของความหนาแน่น (GROUP CLASSIFICATION OF ONE-DIMENSIONAL EQUATIONS OF FLUIDS WITH INTERNAL ENERGY DEPENDING ON THE DENSITY AND THE GRADIENT OF THE DENSITY) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร. เซอร์เก เมเลชโก, 54 หน้า

การวิเคราะห์เชิงกลุ่ม สามารถสร้างกระบวนการสำหรับแบบจำลองทางคณิตศาสตร์ที่ใช้อธิบายสมบัติของของไหล โดยการจำแนกสมการเชิงอนุพันธ์เทียบกับสมาชิกใด ๆ วิทยานิพนธ์นี้นำเสนอการจำแนกประเภทเชิงกลุ่มของสมการของไหลหนึ่งมิติซึ่งมีพลังงานภายใน ε เป็นฟังก์ชันของความหนาแน่น ρ และเกรเดียนต์ของความหนาแน่น ρ_x :

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + \Pi)_x = 0,$$
$$\Pi = P + \rho\lambda\alpha, \quad P = \rho^2\varepsilon_\rho - \rho(\rho\lambda\rho_x)_x, \quad \lambda = 2\varepsilon_\alpha, \quad \alpha = \rho_x^2$$

ได้นำเสนอกลุ่มสมมูลของลีและกลุ่มยอมรับของลีในงานวิจัยนี้ การจำแนกประเภทเชิงกลุ่มแบ่งแบบจำลองออกเป็น 21 แบบที่แตกต่างกันตามกลุ่มยอมรับของลี วิทยานิพนธ์นี้ได้ให้ผลเฉลยยืนยงของแบบจำลองเฉพาะหนึ่งแบบ

สาขาวิชาคณิตศาสตร์	ลายมือชื่อนักศึกษา ____________________
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PRAKRONG VORAKA : GROUP CLASSIFICATION OF ONE-DIMENSIONAL EQUATIONS OF FLUIDS WITH INTERNAL ENERGY DEPENDING ON THE DENSITY AND THE GRADIENT OF THE DENSITY. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D., 54 PP.

FLUIDS WITH INTERNAL ENERGY / GROUP CLASSIFICATION / INVARIANT SOLUTION

Group analysis provides a regular procedure for mathematical modeling describing the behavior of fluids by classifying differential equations with respect to arbitrary elements. This thesis presents the group classification of one-dimensional equations of fluids where the internal energy ε is a function of the density ρ and the gradient of the density ρ_x:

$$\rho_t + (\rho u)_x = 0, \;\; (\rho u)_t + (\rho u^2 + \Pi)_x = 0,$$
$$\Pi = P + \rho\lambda\alpha, \;\; P = \rho^2\varepsilon_\rho - \rho\left(\rho\lambda\rho_x\right)_x, \;\; \lambda = 2\varepsilon_\alpha, \;\; \alpha = \rho_x^2.$$

The equivalence Lie group and the admitted Lie group are provided. The group classification separates all models into 21 different classes according to the admitted Lie group. Invariant solutions of one particular model are obtained.

School of Mathematics | Student's Signature ____________

Academic Year 2008 | Advisor's Signature ____________

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Prakrong Voraka

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CHAPTER I

INTRODUCTION

Many important physical processes in nature are governed by partial differential equations (PDEs). For this reason, the knowledge of the mathematical character and properties of the governing equations are required. Properties of PDEs can be effectively studied by using their exact solutions. Therefore, there is interest in finding exact solutions of PDEs. In general, it is not easy to obtain exact solutions of PDEs. One of the methods for obtaining exact solutions is the group analysis method (Ovsiannikov, 1978). It is well-known that the group analysis method is a powerful and direct approach to constructing exact solutions of PDEs. Furthermore, based on the group analysis method, many other types of exact solutions of PDEs can be obtained, such as traveling wave solutions, soliton solutions, fundamental solutions (Ovsiannikov, 1978), (Olver, 1986), (Handbook, 1994-1996), (Meleshko, 2005).

The general problem in the study of the group properties of differential equations was set up and investigated in the second half of the 19th century by the Norwegian mathematician Sophus Lie. He found a new method for integrating differential equations. This method is universal and effective for solving nonlinear differential equations analytically. It involves the study of symmetries of differential equations, with the emphasis on using the symmetries to find solutions. Symmetry means that any solution of a given system of differential equations is transformed by a Lie group of transformations into a solution of the same system. L. V. Ovisiannikov further developed the group analysis method and applied this method to problems of continuous mechanics. Many applications of Lie group analysis to partial differential equations are collected in Handbook (1994-1996).

Group analysis, besides constructing exact solutions provides a regular procedure for mathematical modelling by classifying differential equations with respect to arbitrary elements. This is the main motivation to use this method for the analysis of differential equations considered in this thesis.

In this thesis the group analysis method is applied to one class of the dispersive models (Gavrilyuk and Shugrin, 1996), (Anderson and McFadden and Wheeler, 1998), (Gavrilyuk and Teshukov, 2001)*. The equations describing the behavior of a dispersive continuum are obtained on the basis of the Euler-Lagrange principle with the Lagrangian

$$L = L(\rho, \frac{\partial \rho}{\partial t}, \nabla \rho, u),$$

where t is time, ∇ is the gradient operator with respect to the space variables, ρ is the fluid density, u is the velocity field. The density ρ and the velocity u satisfy the mass conservation equation and the equation of conservation of linear momentum

$$\rho_t + div(\rho u) = 0, \quad \rho \dot{u} + \nabla p = 0, \tag{1.1}$$

where $\dot{()} = \partial()/\partial t + u\nabla()$. In the literature there are two classes of dispersive models. One class of models is constructed assuming that the Lagrangian is of the form

$$L = \frac{1}{2}\rho|u|^2 - W(\rho, \dot{\rho}),$$

where $W(\rho, \dot{\rho})$ is a given potential. In this case the pressure p is given by the formula

$$p = \rho \left(W_\rho - (W_{\dot{\rho}})_t - div(W_{\dot{\rho}} u)\right) - W. \tag{1.2}$$

These models include the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles and the dispersive shallow water model. Group classification of equations (1.1), (1.2) was studied in Hematulin, Meleshko and Gavrilyuk (2007), Siriwat and Meleshko (2008). Invariant and partially invariant solutions were considered in Hematulin, Meleshko and Gavrilyuk (2007), Siriwat and Meleshko (2008), Bagderina and Chupakhin (2005), Hematulin and Siriwat (2008).

This thesis deals with another class of models describing the behavior of dispersive continuum. These models are constructed, assuming that the internal energy ε depends

*See also references therein.

on the density ρ and the gradient of the density $|\nabla\rho|$, and include the models studied in Cahn and Hilliard (1959), Pratz (1981), Truskinovsky (1993), Ngan and Truskinovsky (2002), and Gouin (2005). A Review of these models can be found in Gavrilyuk and Shugrin (1996), Anderson, McFadden and Wheeler (1998) and references therein.

The one-dimensional equations of fluids (1.1) with internal energy $\varepsilon = \varepsilon(\rho,\alpha)$ satisfy the following system (Gavrilyuk and Shugrin, 1996)

$$\rho_t + (\rho u)_x = 0, \;\; (\rho u)_t + (\rho u^2 + \Pi)_x = 0, \tag{1.3}$$

and

$$\Pi = P + \rho\lambda\alpha, \;\; P = \rho^2\varepsilon_\rho - \rho\left(\rho\lambda\rho_x\right)_x, \;\; \lambda = 2\varepsilon_\alpha, \;\; \alpha = \rho_x^2,$$

where P is the pressure.

The thesis is devoted to group classification of one-dimensional equations (1.3), where $\varepsilon \neq \sqrt{\alpha}\varphi_1(\rho) + \varphi_2(\rho)$, φ_1 and φ_2 are arbitrary functions. Notice that for $\varepsilon = \sqrt{\alpha}\varphi_1(\rho) + \varphi_2(\rho)$, equations (1.3) are similar to the gas dynamics equations. This case was completely studied in Chirkunov (1990). Another part of the thesis is devoted to the analysis of invariant solutions.

An application of the group analysis method requires to carry out a lot of complicated symbolic manipulations. Because this is a very laborious part, we used a computer for these tasks. All calculations were done with the REDUCE program (Hearn, 1999).

The outline of the thesis is as follows. The first chapter is this introduction. The second chapter introduces equations describing the behavior of fluid with internal inertia. The third chapter deals with the notations of group analysis and provides references to known facts on application of group analysis to partial differential equations. Complete group classification of one-dimensional equations of fluids with internal energy ϵ depending on the density ρ and the gradient of the density α, where $\varepsilon \neq \sqrt{\alpha}\varphi_1(\rho)+\varphi_2(\rho)$ is presented in the fourth chapter. In the fifth chapter the invariant solutions of the one-dimensional case of the model with $\varepsilon = \varphi(\rho)\alpha^p$ are presented. The conclusion of the thesis is presented in the last chapter.

CHAPTER II

FLUIDS WITH INTERNAL ENERGY

In this chapter, we introduce the equations describing the behavior of fluids with internal energy (Gavrilyuk and Shugrin, 1996), (Anderson, McFadden and Wheeler, 1998), (Gavrilyuk and Teshukov, 2001).

These equations are obtained as the Euler-Lagrange equation with the Lagrangian of the form

$$L = L(\rho, \frac{\partial \rho}{\partial t}, \nabla \rho, u),$$

where t is time, ∇ is the gradient operator with respect to the space variables, ρ is the fluid density, u is the velocity field. The density ρ and the velocity u satisfy the mass conservation equation

$$\dot{\rho} + \rho \, div \, u = 0, \tag{2.1}$$

and the equation of conservation of linear momentum

$$\rho \dot{u} + \nabla p = 0, \tag{2.2}$$

where p is the pressure, and "dot" denotes the material time derivative: $\dot{()} = \partial()/\partial t + u\nabla()$.

Among fluids with internal energy two classes of models have been intensively studied. One class of models is constructed assuming that the Lagrangian is of the form

$$L = \frac{1}{2}\rho|u|^2 - W(\rho, \dot{\rho}),$$

where $W(\rho, \dot{\rho})$ is a given potential. In this case the pressure p is given by the formula

$$p = \rho\left(W_\rho - (W_{\dot{\rho}})_t - div(W_{\dot{\rho}} u)\right) - W. \tag{2.3}$$

These models include the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski, 1960), and the dispersive shallow water model (Green-Naghdi, 1975).

This thesis is devoted to the study of another class of models. These models are constructed, assuming that the internal energy depends on the density ρ and the gradient of the density. The mass conservation equation (2.1) and the equation of conservation of momentum (2.3) are

$$\begin{gathered}\frac{\partial \rho}{\partial t}+\sum_{k=1}^{3}\frac{\partial}{\partial x^k}(\rho u^k)=0,\\ \frac{\partial}{\partial t}(\rho u^j)+\sum_{k=1}^{3}\frac{\partial}{\partial x^k}(\rho u^j u^k+\Pi^{jk})=0;\quad (j=1,2,3),\end{gathered}\tag{2.4}$$

where $\alpha=|\nabla\rho|^2$,

$$\Pi^{jk}\equiv P\delta^{jk}+\rho\lambda\frac{\partial \rho}{\partial x^j}\frac{\partial \rho}{\partial x^k},\quad P=\rho^2\frac{\partial \varepsilon}{\partial \rho}-\rho\sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left(\rho\lambda\frac{\partial \rho}{\partial x^i}\right),\quad \lambda=2\frac{\partial \varepsilon}{\partial \alpha},$$

t is time, $\nabla\rho$ is the gradient of ρ with respect to the space variable x, P is the pressure and $\varepsilon(\rho,\alpha)$ is the internal energy. Equations (2.4) include the models studied in Cahn and Hilliard (1959), Pratz (1981), Truskinovsky (1993), Ngan and Truskinovsky (2002), Gouin (2005). Review of these models can be found in Gavrilyuk and Shugrin (1996), Anderson, McFadden and Wheeler (1998) and references therein. For example, Chan and Hilliard (1959) studied

$$\varepsilon=F(\rho,T)+\frac{C(\rho,T)}{2\rho}(\nabla\rho)^2,$$

where T is the temperature. Partz (1981) proposed

$$\varepsilon=F(\rho,S)+\frac{C}{2\rho}(\nabla\rho)^2,$$

where S is the entropy and C is a constant.

Notice that if $\varepsilon=\sqrt{\alpha}\varphi_1(\rho)+\varphi_2(\rho)$, these equations are similar to the isentropic gas dynamics equations. A complete study of group properties of the one-dimensional isentropic gas dynamics equations was done in Chirkunov (1990) (see also Meleshko, 1998).

This thesis is devoted to group classification of the one-dimensional motions of fluids. The corresponding equations (2.4) become

$$\begin{gathered}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0,\\ \frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+\Pi)=0,\end{gathered}\tag{2.5}$$

where

$$\Pi=P+\rho\lambda\rho_x^2,\ \ P=\rho^2\frac{\partial\varepsilon}{\partial\rho}-\rho\frac{\partial}{\partial x}\left(\rho\lambda\rho_x\right),\ \ \lambda=2\frac{\partial\varepsilon}{\partial\alpha},\ \ \alpha=\rho_x^2,$$

t is time, ρ_x is the gradient of ρ with respect to the space variable x, P is the pressure and $\varepsilon(\rho,\alpha)$ is the internal energy.

CHAPTER III

THE GROUP ANALYSIS METHOD

In this chapter, the group analysis method is discussed. An introduction to this method can be found in various textbooks, e.g. L. V. Ovsyannikov (1978), P. Olver (1986), N. H. Ibragimov (1999).

3.1 Lie Groups

Let V be an open subset in $\mathbb{R}^N$, Δ a symmetric interval in $\mathbb{R}^1$. Consider a set of invertible transformations defined by equations of the form

$$\bar{z}^i = \varphi^i(z; a), \ a \in \Delta, \ z \in V, \tag{3.1}$$

where $i = 1, 2, \dots, N$, a is a parameter.

If $z = (x, u)$, then one uses the notation $\varphi = (f, g)$. Here $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is the vector of the independent variables, and $u = (u^1, u^2, \dots, u^m) \in \mathbb{R}^m$ is the vector of the dependent variables. The transformation of the independent variables x, and the dependent variables u has the form

$$\begin{aligned} \bar{x}_i &= f^i(x, u; a), \ (i = 1, 2, \dots, n), \\ \bar{u}^j &= g^j(x, u; a), \ (j = 1, 2, \dots, m), \end{aligned} \tag{3.2}$$

where $(x, u) \in V \subset \mathbb{R}^n \times \mathbb{R}^m$, and the set V is open in $\mathbb{R}^n \times \mathbb{R}^m$.

3.1.1 One-Parameter Lie-Group of Transformations

Definition 3.1. A set of transformations (3.1) is called a *local one-parameter Lie group* if it has the following properties:

1. $\varphi(z; 0) = z$ for all $z \in V$.

2. $\varphi(\varphi(z; a), b) = \varphi(z; a + b)$ for all $a, b, a + b \in \Delta, z \in V$.

3. If $a \in \Delta$ and $\varphi(z; a) = z$ for all $z \in V$, then $a = 0$.

The transformations (3.2) are called a *one-parameter Lie group of point transformations*.

3.1.2 Infinitesimal Transformations

For Lie groups of point transformations, let us expand the functions $f^i(x, u; a)$ and $g^j(x, u; a)$ into Taylor series with respect to the parameter a in a neighborhood of $a = 0$

$$\begin{aligned} \bar{x}_i &= x_i + a \left. \frac{\partial f^i}{\partial a} \right|_{a=0} + O(a^2), \\ \bar{u}^j &= u^j + a \left. \frac{\partial g^j}{\partial a} \right|_{a=0} + O(a^2). \end{aligned}$$

Then, invoking the first property of Lie group one obtains

$$\begin{aligned} &\bar{x}_i \approx x_i + a\xi^i(x, u), \ (i = 1, 2, \ldots, n), \\ &\bar{u}^j \approx u^j + a\zeta^j(x, u), \ (j = 1, 2, \ldots, m), \end{aligned} \tag{3.3}$$

where

$$\xi^i(x, u) = \left. \frac{\partial f^i(x, u; a)}{\partial a} \right|_{a=0}, \quad \zeta^j(x, u) = \left. \frac{\partial g^j(x, u; a)}{\partial a} \right|_{a=0}. \tag{3.4}$$

The transformations $x_i + a\xi^i(x, u)$ and $u^j + a\zeta^j(x, u)$ are called *infinitesimal transformations* of the Lie group of transformations (3.2) and

$$X = \sum_{i=1}^{n} \xi^i(x, u) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} \zeta^j(x, u) \frac{\partial}{\partial u^j} \tag{3.5}$$

is called an *infinitesimal generator* of the Lie group (3.2).

Example 3.1. The infinitesimal transformation of the rotation group

$$\bar{x} = x \cos a + u \sin a, \quad \bar{u} = u \cos a - x \sin a,$$

has the form $\bar{x} \approx x + ua$, $\bar{u} \approx u - xa$, and its infinitesimal generator is

$$X = u \frac{\partial}{\partial x} - x \frac{\partial}{\partial u}.$$

3.1.3 The Lie equations

The following theorem, due to Lie, asserts that local one-parameter Lie groups are determined by their infinitesimal transformations.

Theorem 3.1. *Let functions $f^i(x,u;a)$, $i=1,\dots,n$ and $g^j(x,u;a)$, $j=1,\dots,m$ satisfy the group properties and have the expansion*

$$\bar{x}_i = f^i(x,u;a) \approx x_i + a\xi^i(x,u),$$

$$\bar{u}^j = g^j(x,u;a) \approx u^j + a\zeta^j(x,u).$$

Then the functions f^i and g^j solve the system of first-order ordinary differential equations (known as the Lie equations)

$$\begin{aligned} \frac{d\bar{x}_i}{da} &= \xi^i(\bar{x},\bar{u}), \\ \frac{d\bar{u}^j}{da} &= \zeta^j(\bar{x},\bar{u}), \end{aligned} \tag{3.6}$$

with the initial conditions

$$\bar{x}_i\mid_{a=0} = x_i, \ \ \bar{u}^j\mid_{a=0} = u^j. \tag{3.7}$$

3.1.4 Prolongations

When applying the group of transformations (3.2) to differential equations one needs to know the transformations of the derivatives. For the sake of simplicity, we explain the basic idea by the case where $n=1$, and $m=1$.

Let $u_0(x)$ be a given function. The transformed function $u_a(x)$ can be obtained in the following way. As a first step, we have to solve the equation

$$\bar{x} = f(x,u_0(x);a)$$

with respect to x. Since the Jacobian is nonzero,

$$\left.\frac{\partial\bar{x}}{\partial x}\right|_{a=0} = \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{du_0}{dx}\right]_{a=0} = 1,$$

by the inverse function theorem one can express x as a function of $\bar{x}$ and a in some neighborhood of $a=0$,

$$x = \theta(\bar{x}, a). \tag{3.8}$$

Note that there is the identity

$$\bar{x} = f(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{3.9}$$

The transformed function $u_a(x)$ is

$$u_a(\bar{x}) = g(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{3.10}$$

Differentiating the last expression with respect to $\bar{x}$, one gets

$$\frac{du_a(\bar{x})}{d\bar{x}} = \frac{\partial g}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial g}{\partial u}\frac{du_0}{dx}\frac{\partial \theta}{\partial \bar{x}} = \left[\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}u_0'(x)\right]\frac{\partial \theta}{\partial \bar{x}}. \tag{3.11}$$

The derivative $\partial\theta/\partial\bar{x}$ is obtained by differentiating (3.9) with respect to $\bar{x}$

$$1 = \frac{\partial f}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial f}{\partial u}\frac{du_0}{dx}\frac{\partial \theta}{\partial \bar{x}} = \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right]\frac{\partial \theta}{\partial \bar{x}}. \tag{3.12}$$

Because $\partial f/\partial x + u_0'(x)\partial f/\partial u \neq 0$ in some neighborhood of $a = 0$, one has

$$\frac{\partial \theta}{\partial \bar{x}} = 1\Big/\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right]$$

and

$$\begin{aligned}\bar{u}_{\bar{x}} &= \left[\frac{\partial g(x, u_0; a)}{\partial x} + \frac{\partial g(x, u_0; a)}{\partial u}u_0'(x)\right]\Big/\left[\frac{\partial f(x, u_0; a)}{\partial x} + \frac{\partial f(x, u_0; a)}{\partial u}u_0'(x)\right] \\ &= h(x, u_0(x), u_0'(x); a).\end{aligned}$$

Transformation (3.2) together with

$$\bar{u}_{\bar{x}} = h(x, u, u_x; a) \tag{3.13}$$

is called the *prolongation* of (3.2). As before, the function h can be written by Taylor expansion with respect to the parameter a in some neighborhood of the point $a = 0$:

$$\frac{\partial \bar{u}}{\partial \bar{x}} \approx u_x + a\zeta^{u_x}(x, u, u_x), \tag{3.14}$$

where

$$\zeta^{u_x}(x,u,u_x) = \left.\frac{\partial h(x,u,u_x,a)}{\partial a}\right|_{a=0}, \quad h|_{a=0} = u_x.$$

Using the definition of the function h, equation (3.11) can be rewritten

$$h\left[\frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u}\right] = \frac{\partial g}{\partial x} + u_x \frac{\partial g}{\partial u}.$$

Differentiating this equation with respect to the group parameter a and substituting $a = 0$, one finds

$$\left[\frac{\partial h}{\partial a}\left(\frac{\partial f}{\partial x} + u_x \frac{\partial f}{\partial u}\right) + h\left(\frac{\partial^2 f}{\partial x \partial a} + u_x \frac{\partial^2 f}{\partial u \partial a}\right)\right]_{a=0} = \left[\frac{\partial^2 g}{\partial x \partial a} + u_x \frac{\partial^2 g}{\partial u \partial a}\right]_{a=0}$$

or

$$\begin{aligned}
\zeta^{u_x}(x,u,u_x) = \left.\frac{\partial h}{\partial a}\right|_{a=0} &= \left[\frac{\partial^2 g}{\partial x \partial a} + u_x \frac{\partial^2 g}{\partial u \partial a}\right]_{a=0} - h|_{a=0}\left[\frac{\partial^2 f}{\partial x \partial a} + u_x \frac{\partial^2 f}{\partial u \partial a}\right]_{a=0} \\
&= \left(\frac{\partial \zeta^u}{\partial x} + u_x \frac{\partial \zeta^u}{\partial u}\right) - u_x\left(\frac{\partial \xi^x}{\partial x} + u_x \frac{\partial \xi^x}{\partial u}\right) \\
&= D_x(\zeta^u) - u_x D_x(\xi^x),
\end{aligned}$$

where

$$\xi^x = \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad \zeta^u = \left.\frac{\partial g}{\partial a}\right|_{a=0}, \quad \zeta^{u_x} = \left.\frac{\partial h}{\partial a}\right|_{a=0},$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \dots \ .$$

The first prolongation of the generator (3.5) is given by

$$\underset{1}{X} = X + \zeta^{u_x}(x,u,u_x)\frac{\partial}{\partial u_x}.$$

Similar, one obtains the infinitesimal transformation of the second derivative

$$\bar{u}_{\bar{x}\bar{x}} \approx u_{xx} + a\zeta^{u_{xx}}(x,u,u_x,u_{xx}),$$

where

$$\zeta^{u_{xx}} = D_x(\zeta^{u_x}) - u_{xx} D_x(\xi^x).$$

The second prolongation of generator (3.5) is

$$\underset{2}{X} = \underset{1}{X} + \zeta^{u_{xx}}(x, u, u_x, u_{xx})\frac{\partial}{\partial u_{xx}}.$$

In case $n, m \geq 2$ one proceeds similarly.

Let $x = (x_1, \dots, x_n)$ be the independent variables and $u = (u^1, \dots, u^m)$ the dependent variables. We will use the notion $u_{(1)} = \{u_i^j\}, u_{(2)} = \{u_{is}^j\}, \dots$ for partial derivatives of first, second, etc. order:

$$u_i^j = D_i(u^j), \quad u_{is}^j = D_s(u_i^j),$$

where

$$D_i = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + u_{is}^j \frac{\partial}{\partial u_s^j} + u_{isk}^j \frac{\partial}{\partial u_{sk}^j} + \dots. \tag{3.15}$$

Let a generator of a Lie group be

$$X = \xi^i(x, u)\frac{\partial}{\partial x_i} + \eta^j(x, u)\frac{\partial}{\partial u^j}.$$

The generator of the prolonged Lie group is

$$\underset{1}{X} = X + \zeta_i^j(x, u, u_{(1)})\frac{\partial}{\partial u_i^j},$$

where

$$\zeta_i^j = D_i(\eta^j) - u_s^j D_i(\xi^s), \quad (i, s = 1, \dots, n;\ j = 1, \dots, m).$$

The generator of the second prolongation is

$$\underset{2}{X} = \underset{1}{X} + \zeta_{i_1 i_2}^j(x, u, u_{(1)}, u_{(2)})\frac{\partial}{\partial u_{i_1 i_2}^j},$$

where

$$\zeta_{i_1 i_2}^j = D_{i_2}(\zeta_{i_1}^j) - u_{s i_1}^j D_{i_2}(\xi^s), \quad (i_1, i_2, s = 1, \dots, n;\ j = 1, \dots, m).$$

The generator of l-th prolongation is

$$\underset{l}{X} = \underset{l-1}{X} + \zeta_{i_1 \cdots i_l}^j(x, u, u_{(1)}, \dots, u_{(l)})\frac{\partial}{\partial u_{i_1 \cdots i_l}^j},$$

where

$$\zeta_{i_1 \cdots i_l}^j = D_{i_l}(\zeta_{i_1 \cdots i_{l-1}}^j) - u_{s i_1 \cdots i_{l-1}}^j D_{i_l}(\xi^s),\ (i_1, \dots, i_l, s = 1, \dots, n;\ j = 1, \dots, m). \tag{3.16}$$

3.1.5 Admitted Lie group of Transformations

Consider a system of l-th order differential equations

$$F_\alpha(x, u, u_{(1)}, u_{(2)}, \dots, u_{(l)}) = 0, \quad (\alpha = 1, 2, \dots, t), \tag{3.17}$$

where the order l refers to the highest derivative appearing in (3.17).

Definition 3.2. A Lie group of transformations (3.2) which transforms a solution $u_0(x)$ of (3.17) into a solution $u_a(x)$ of the same system of equations is called an *admitted Lie group* of transformations.

Hence one has

$$F_\alpha(\bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \dots, \bar{u}_{(l)}) = 0, \quad (\alpha = 1, 2, \dots, t).$$

Differentiating these equations with respect to the parameter a, and substituting $a = 0$, one finds

$$\left[\frac{\partial F_\alpha}{\partial x_i}\frac{\partial \bar{x}_i}{\partial a} + \frac{\partial F_\alpha}{\partial u^j}\frac{\partial \bar{u}^j}{\partial a} + \frac{\partial F_\alpha}{\partial u^j_{i_1}}\frac{\partial \bar{u}^j_{i_1}}{\partial a} + \cdots + \frac{\partial F_\alpha}{\partial u^j_{i_1 i_2 \cdots i_l}}\frac{\partial \bar{u}^j_{i_1 i_2 \cdots i_l}}{\partial a} \right]_{a=0} = 0$$

or

$$\xi^i \frac{\partial F_\alpha}{\partial x_i} + \zeta^j \frac{\partial F_\alpha}{\partial u^j} + \zeta^j_{i_1} \frac{\partial F_\alpha}{\partial u^j_{i_1}} + \zeta^j_{i_1 i_2} \frac{\partial F_\alpha}{\partial u^j_{i_1 i_2}} + \cdots + \zeta^j_{i_1 i_2 \cdots i_l} \frac{\partial F_\alpha}{\partial u^j_{i_1 i_2 \cdots i_l}} = 0,$$

where

$$\xi^i = \left.\frac{\partial \bar{x}_i}{\partial a}\right|_{a=0}, \zeta^j = \left.\frac{\partial \bar{u}^j}{\partial a}\right|_{a=0}, \zeta^j_{i_1} = \left.\frac{\partial \bar{u}^j_{i_1}}{\partial a}\right|_{a=0}, \zeta^j_{i_1 \cdots i_l} = \left.\frac{\partial \bar{u}^j_{i_1 \cdots i_l}}{\partial a}\right|_{a=0}.$$

Note that the coefficients $\zeta^j_{i_1 \cdots i_l}$ are given by (3.16). The last equation can be expressed as an action of the prolonged infinitesimal generator

$$\underset{l}{X} F_\alpha(x, u, u_{(1)}, u_{(2)}, \dots, u_{(l)})\Big|_{(3.17)} = 0, \quad (\alpha = 1, 2, \dots, t), \tag{3.18}$$

where

$$\underset{l}{X} = \xi^i \frac{\partial}{\partial x_i} + \zeta^j \frac{\partial}{\partial u^j} + \zeta^j_{i_1} \frac{\partial}{\partial u^j_{i_1}} + \zeta^j_{i_1 i_2} \frac{\partial}{\partial u^j_{i_1 i_2}} + \cdots + \zeta^j_{i_1 i_2 \cdots i_l} \frac{\partial}{\partial u^j_{i_1 i_2 \cdots i_l}}.$$

The symbol $|_{(3.17)}$ means that the equations $\underset{l}{X} F_\alpha(x, u, u_{(1)}, u_{(2)}, \dots, u_{(l)}) = 0$ are considered on any solution $u_0(x)$ of equations (3.17).

Theorem 3.2. *A Lie group of transformations (3.2) is admitted by the system* (3.17) *(or the system (3.17) admits the Lie group of transformations (3.2)) if and only if it satisfies (3.18).*

3.1.6 Equivalence Lie group of transformations

Let $p = (u_{(1)}, u_{(2)}, \ldots, u_{(l)})$. A non-degenerate change of the dependent variables u, the independent variables x and arbitrary elements ϕ which transfers any system of differential equations of a given class

$$F_\alpha(x, u, p, \phi) = 0, \quad (\alpha = 1, 2, \ldots, t) \tag{3.19}$$

into a system of equations of the same class, but with different arbitrary elements ϕ, is called an *equivalence transformation.* Let equivalence transformations compose a Lie group

$$\bar{x}_i = f^i(x, u, \phi; a), \;\; \bar{u}^j = g^j(x, u, \phi; a), \;\; \bar{\phi}^k = h^k(x, u, \phi; a), \tag{3.20}$$

$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m; k = 1, 2, \ldots, r).$$

Generators of this Lie group have the form

$$X^e = \xi^i \frac{\partial}{\partial_{x_i}} + \zeta^j \frac{\partial}{\partial_{u_j}} + \zeta^{\phi^k} \frac{\partial}{\partial_{\phi_k}},$$

where

$$\xi^i = \xi^i(x, u, \phi) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0},$$

$$\zeta^j = \zeta^j(x, u, \phi) = \left.\frac{\partial g^i}{\partial a}\right|_{a=0},$$

$$\zeta^{\phi^k} = \zeta^{\phi^k}(x, u, \phi) = \left.\frac{\partial h^k}{\partial a}\right|_{a=0}.$$

Transformation of the arbitrary elements is obtained in the following way. Let $\phi_0(x, u)$ be given. By virtue of the inverse function theorem one can solve the equations

$$\bar{x} = f(x, u, \phi_0(x, u); a), \bar{u} = g(x, u, \phi_0(x, u); a)$$

with respect to x and u:

$$x = \bar{f}(\bar{x}, \bar{u};a), \ \ u = \bar{g}(\bar{x}, \bar{u};a).$$

The transformed arbitrary elements are

$$\phi_a(\bar{x}, \bar{u}) = h(\bar{f}(\bar{x}, \bar{u}; a), \bar{g}(\bar{x}, \bar{u}; a), \phi_0(\bar{f}(\bar{x}, \bar{u}; a), \bar{g}(\bar{x}, \bar{u}; a)); a).$$

Transformation of a function $u_0(x)$ is given in a different way. If $u_0(x)$ is a solution of system (3.19) with $\phi_0(x, u)$, by the equation

$$\bar{x} = f(x, u_0(x), \phi_0(x, u_0(x)); a),$$

one finds $x = \bar{f}(\bar{x}; a)$, and the transformed function is

$$u_a(\bar{x}) = g(\bar{f}(\bar{x}, a), u_0(\bar{f}(\bar{x}, a)), \phi_0(\bar{f}(\bar{x}, a), u_0(\bar{f}(\bar{x}, a))); a). \tag{3.21}$$

Differentiating (3.21) with respect to $\bar{x}$, we get the transformation of derivatives $\bar{p}_a = q(x, u, p, \phi, \ldots; a)$. By the assumption that $u_a(\bar{x})$ is a solution of the same system of equations with transformed arbitrary elements $\phi_a(\bar{x}, \bar{u})$, the equations

$$F_\alpha(\bar{x}, u_a(\bar{x}), \bar{p}_a(\bar{x}), \phi_a(\bar{x}, u_a(\bar{x}))) = 0, \quad (\alpha = 1, 2, \ldots, t)$$

are satisfied for an arbitrary $\bar{x}$. Because of a one-to-one correspondence between x and $\bar{x}$ one has

$$F_\alpha(f(z(x), a), g(z(x), a), q(z_p(x), a), h(z(x))) = 0, \quad (\alpha = 1, 2, \ldots, t),$$

where $z(x) = (x, u_0(x), \phi(x, u_0(x))), \ z_p(x) = (x, u_0(x), \phi(x, u_0(x)), p_0(x), \ldots)$. Differentiating these equations with respect to the group parameter a, one obtains the determining equations

$$\tilde{X}^e F_\alpha(x, u, p, \phi)\Big|_{(3.19)} = 0, \quad (\alpha = 1, 2, \ldots, t).$$

The prolonged operator for the equivalence Lie group

$$\tilde{X}^e = X^e + \varsigma^{u_x}\partial_{u_x} + \varsigma^{\phi_x}\partial_{\phi_x} + \varsigma^{\phi_u}\partial_{\phi_u} + \cdots \tag{3.22}$$

where

$$\varsigma^{u_\lambda} = D^e_\lambda \varsigma^u - u_x D^e_\lambda \xi^x, \quad D^e_\lambda = \partial_\lambda + u_\lambda \partial_u + (\phi_u u_\lambda + \phi_\lambda)\partial_\phi,$$

$$\varsigma^{\phi_\lambda} = \tilde{D}^e_\lambda \varsigma^\phi - \phi_x \tilde{D}^e_\lambda \xi^x - \phi_u \tilde{D}^e_\lambda \varsigma^u, \quad \tilde{D}^e_\lambda = \partial_\lambda + \phi_\lambda \partial_\phi.$$

The sign $|_{(3.19)}$ means that the equations $\tilde{X}^e F_\alpha(x, u, p, \varphi) = 0$ are considered on any solution $u_0(x)$ of equations (3.19).

3.1.7 Multi-Parameter Lie-Group of Transformations

Let O be a ball in the space $\mathbb{R}^r$ with center at the origin. Assume that ψ is a mapping, $\psi : O \times O \longrightarrow \mathbb{R}^r$. The pair (O, ψ) is called a local multi-parameter Lie group with the multiplication law ψ if it has the following properties:

1. $\psi(a, 0) = \psi(0, a) = a$ for all $a \in O$.
2. $\psi(\psi(a, b), c) = \psi(a, \psi(b, c))$ for all $a, b, c \in O$ for which $\psi(a, b), \psi(b, c) \in O$.
3. $\psi \in C^\infty(O, O)$.

Let V be an open set in Z. Consider transformations

$$\bar{z}^i = \varphi^i(z; a), \tag{3.23}$$

where $i = 1, 2, ..., N$, $z \in V \subset Z = R^N$, and the vector-parameter $a \in O$.

Definition 3.3. The set of transformations (3.23) is called a local r-parameter Lie group G^r if it has the following properties:

1. $\varphi(z, 0) = z$ for all $z \in V$.
2. $\varphi(\varphi(z, a), b) = \varphi(z, \psi(a, b))$ for all $a, b, \psi(a, b) \in O,\ z \in V$.
3. If for $a \in O$ one has $\varphi(z, a) = z$ for all $z \in V$, then $a = 0$.

Note that if one fixes all parameters except one, for example a_k, then the multi-parameter Lie group of transformations (3.23) composes a one-parameter Lie group. Conversely, in group analysis it is proven that any r-parameter group is a union of one-parameter subgroups belonging to it. Infinitesimal generators of these groups compose a Lie algebra.

Let G^r be a Lie group admitted by the system of partial differential equations

$$F^k(x, u, p) = 0,\ k = 1, ..., s.$$

Assume that $\{X_1, X_2, ..., X_r\}$ is a basis of the Lie algebra L^r, which corresponds to the Lie group G^r.

Definition 3.4. A function $\Phi(x,u)$ is called an invariant of a Lie group G^r if

$$\Phi(\bar{x},\bar{u}) = \Phi(x,u).$$

Theorem 3.3. *A function $\Phi(x,u)$ is an invariant of the group G^r with the generators $X_i,\ (i=1,...,r)$ if and only if,*

$$X_i\Phi(x,u) = 0,\ \ (i=1,...,r). \tag{3.24}$$

In order to find an invariant, one needs to solve the overdetermined system of linear equations (3.24). Any invariant Φ can be expressed through this set

$$\Phi = \phi\left(\ J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u)\ \right).$$

where n,m is the numbers of independent and dependent variables, respectively and r_* is the total rank of the matrix composed by the coefficients of the generators $X_i,\ (i=1,2,...,r)$. A set of functionally independent invariants

$$J = \left(\ J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u)\ \right)$$

is called an universal invariant.

Definition 3.5. A set M is said to be invariant with respect to the group G^r, if the transformation (3.23) carries every point z of M to a point of M.

Definition 3.6. Let V be an open subset of R^N, and $\Psi : V \longrightarrow R^t,\ t \leq N$ a mapping belonging to the class $C^1(V)$. The system of equations $\Psi(z) = 0$ is called regular, if for any point $z \in V$:

$$rank\ \left(\frac{\partial(\ \psi^1,...,\psi^t)}{\partial\ (\ z_1,...,z_N\ )}\right) = t$$

where $\Psi = (\psi^1,...,\psi^t)$.

If a system $\Psi(z) = 0$ is regular, then for each $z_0 \in V$ with $\Psi(z_0) = 0$ there exists a neighborhood U of z_0 in V such that

$$M = \{\ z \in U\ :\ \Psi(z) = 0\ \}$$

is a manifold. Such a manifold is called a regularly assigned manifold.

Theorem 3.4. *A regularly assigned manifold M is an invariant manifold with respect to a Lie group G^r with the generator X_i, $(i = 1, ..., r)$, if*

$$X_i \psi^k(z)\big|_M = 0, \ (i = 1, ..., r), \ k = 1, ..., t.$$

3.2 Lie algebras

Definition 3.7. A *Lie algebra* is a vector space L of operators $X = \zeta^\alpha(z)\, \partial/\partial z_\alpha$ with the following property. If the operators

$$X_1 = \zeta_1^\alpha(z)\frac{\partial}{\partial z_\alpha}, \quad X_2 = \zeta_2^\alpha(z)\frac{\partial}{\partial z_\alpha}$$

are elements of L, then their commutator

$$[X_1, X_2] \equiv X_1 X_2 - X_2 X_1 = \sum_{\alpha=1}^{N} (X_1(\zeta_2^\alpha) - X_2(\zeta_1^\alpha))\, \frac{\partial}{\partial z_\alpha} \tag{3.25}$$

is also an element of L. The Lie algebra is denoted by the same letter L, and the dimension dim$L$ of the Lie algebra is the dimension of the vector space L. We will use the symbol L^r to denote an $r-$dimensional Lie algebra.

It follows from (3.25) that the commutator satisfies the following properties:

1.(bilinearity): for any $X_1, X_2, X_3 \ \in L$ and $a, b \ \in \mathbb{R}$,

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3],$$

$$[X_1, aX_2 + bX_3] = a[X_1, X_2] + b[X_1, X_3].$$

2.(skew-symmetry): for any $X_1, X_2 \ \in L$,

$$[X_1, X_2] = -[X_2, X_1].$$

3.(the Jacobi identity): for any $X_1, X_2, X_3 \ \in L$,

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0.$$

Definition 3.8. A vector space $S \subset L$ is called a *subalgebra* of a Lie algebra L if $[X_1, X_2] \in S$ for any $X_1, X_2 \in S$.

Definition 3.9. A subalgebra $I \subset L$ is called an *ideal* of the Lie algebra L if for any $X \in L,\ Y \in I$ then $[X, Y] \in I$.

Consider a Lie algebra L^r of a finite dimension r with basis $X_1, X_2, \ldots, X_r$: i.e., any vector $X \in L^r$ can be decomposed as

$$X = \sum_{k=1}^{r} x_k X_k$$

where x_k are the coordinates of the vector X in the basis $\{X_1, \ldots, X_r\}$. Then

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k; \quad i,\ (j = 1, 2, \ldots, r) \tag{3.26}$$

with real constants c_{ij}^k. The numbers c_{ij}^k are called the *structural constants* of the Lie algebra L^r for the basis $\{X_1, \ldots, X_r\}$.

Given any set of operators $X_1, X_2, \ldots, X_r$, their linear span will be denoted by $\langle X_1, X_2, \ldots, X_r \rangle$. For example, a Lie algebra L^r with the basis $X_1, X_2, \ldots, X_r$ is written $L^r = \langle X_1, X_2, \ldots, X_r \rangle$.

In applications, it is convenient to use the relations (3.26) written in the form of a *table of commutators* of the basis $X_1, X_2, \ldots, X_r$. Consider an example.

Example 3.2. Let L^6 be the vector space with the following basis

$$\begin{gathered} X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = y\frac{\partial}{\partial u}, \\ X_5 = x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}, \quad X_6 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \end{gathered} \tag{3.27}$$

Thus, L^6 is the linear span of the operators (3.27), $L^6 = \langle X_1, X_2, \ldots, X_6 \rangle$. One can readily calculate, by definition (3.25), the commutators $[X_i, X_j]$ of the basic operators (3.27) and verify that they satisfy the test (3.26) for a Lie algebra. The result becomes directly visual if the commutators are disposed as in the following table, where the intersection of the X_i row with the X_j column represents the commutator $[X_i, X_j]$.

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	X_1	0
X_2	0	0	0	X_3	0	X_2
X_3	0	0	0	0	$3X_3$	$-2X_3$
X_4	0	$-X_3$	0	0	$3X_4$	$-3X_4$
X_5	$-X_1$	0	$-3X_3$	$-3X_4$	0	0
X_6	0	$-X_2$	$2X_3$	$3X_4$	0	0

3.2.1 Classification of Subalgebras

One of the aims of group analysis is to find exact solutions of differential equations. All solutions can be divided into equivalence classes of solutions as follows.

Definition 3.10. Two solutions u_1 and u_2 of a differential equation are said to be equivalent with respect to the admitted Lie group G if one of the solutions can be transformed into the other by a transformation belonging to the group G.

The problem of classification of exact solutions is equivalent to the classification of subalgebras of the admitted Lie algebra L. Because there is a one-to-one correspondence between Lie groups and Lie algebras let us explain here the classification of subalgebras. In order to give a method for classification of subalgebras, we need to give some definitions.

Definition 3.11. Let L and K be Lie algebras and let $\dim L = \dim K$. A linear one-to-one map f of L onto K is called an *isomorphism* if it preserves commutators

$$f\left([X_1, X_2]_L\right) = [f(X_1), f(X_2)]_K\,,\ \forall\ X_1,\ X_2\ \in L,$$

where the indices L and K denote the commutators in the corresponding algebras. An isomorphism of L onto itself is called an *automorphism* of the Lie algebra L. This mapping will be denoted by the symbol $A : L \to L$.

Theorem 3.5. *Two finite-dimensional Lie algebras are isomorphic if and only if one can choose bases for the algebras such that the algebras have, in theses bases, equal structure constants, i.e. the same table of commutators.*

Let L^r be an r-dimensional Lie algebra with basis $\{X_1, X_2, \ldots, X_r\}$. Then one has

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k, \quad (i, j = 1, 2, \ldots, r),$$

where c_{ij}^k are the structural constants. One constructs a one-parameter family of automorphism, A_i, $(i = 1, \ldots, r)$ on L^r,

$$A_i : \sum_{j=1}^{r} x_j X_j \rightarrow \sum_{j=1}^{r} \bar{x}_j X_j$$

where $\bar{x}_i = \bar{x}_i(a)$, as follows. Consider the system

$$\frac{d\bar{x}_j}{da} = \sum_{\beta=1}^{r} c_{\beta i}^j \bar{x}_\beta, \quad (j = 1, 2, \ldots, r).$$

Initial values for this system are $\overline{x}_j = x_j$ at $a = 0$. The set of solutions of these equations induces an automorphism A_i, (i = 1, 2, ..., r) of the Lie algebra L_r.

The set of all subalgebras is divided into equivalence classes with respect to these automorphisms. A list of representatives, where each element of this list is one representative from every class is called an *optimal system* of subalgebras.

Because of the difficulties in constructing the optimal system of subalgebras for Lie algebras of large dimension, there is a two-step algorithm (Ovsiannikov, 1993), which reduces this problem to the problem for constructing an optimal system of algebras of lower dimensions. In brief, let us consider an algebra L^r with basis $\{X_1, X_2, \ldots, X_r\}$. According to the algorithm, assume that the algebra L^r is decomposed as $I_1 \oplus N_1$, where I_1 is an ideal of L^r and N_1 is a subalgebra of the algebra L^r. In the same way, the subalgebra N_1 can also be decomposed as $N_1 = I_2 \oplus N_2$. Repeating the same process $(\alpha - 1)$ times one ends up with an algebra N_α. Since the algebra N_α has lower dimension, an optimal system of the subalgebras for it can be more simply constructed. By gluing the ideals I_l and subalgebras N_l starting from $l = \alpha$ to $l = 1$, together one constructs the optimal system of subalgebras for the algebra L^r. Note that for every subalgebra N_l one needs to check the subalgebra conditions and use the automorphisms to simplify the coefficients of these systems. Therefore, the problem for constructing an optimal system of subalgebras of the algebra L^r by this method is reduced to the problem of classification of algebras of lower dimensions.

After constructing the optimal system, one can start seeking invariant solutions of subalgebras from the optimal system.

3.3 Invariant Solutions

Representations of invariant solutions are obtained with the help of the admitted Lie group. These solutions can be constructed in the following way. Assume that G is a group admitted by the system (3.17). Let L be the Lie algebra which corresponds to G, and $L^r \subset L$ a r subalgebra. The algorithm of finding invariant solutions with respect to the subalgebra L^r consists of the following.

Let L^r be a Lie algebra with the basis

$$X_k = \xi_k^i(x,u)\frac{\partial}{\partial x_i} + \zeta_k^j(x,u)\frac{\partial}{\partial u^j}, \quad (k = 1, 2, \ldots, r).$$

Find the general solution of the system of linear homogeneous equations

$$\xi_k^i\frac{\partial J}{\partial x_i} + \zeta_k^j\frac{\partial J}{\partial u^j} = 0, \quad (k = 1, 2, \ldots, r),$$

where $J = J(x,u)$. The set of functionally independent solutions of these equations is called an *universal invariant* of the Lie algebra L^r. The universal invariant J consists of $s = m + n - r_*$ functionally independent invariants

$$J = \left(\ J^1(x,u), J^2(x,u), \ldots, J^{m+n-r_*}(x,u)\ \right),$$

where r_* is the total rank of the matrix composed by the coefficients of the generators X_k, $(k = 1, 2, \ldots, r)$.

Choosing m invariants such that the rank of the Jacobi matrix $\dfrac{\partial(J^1, \ldots, J^m)}{\partial(u^1, \ldots, u^m)}$ is equal to m, one composes the relations

$$J^i = \Phi^i \left(J^{m+1}, J^{m+2}, \ldots, J^{m+n-r_*}\right), \quad (i = 1, 2, \ldots, m). \tag{3.28}$$

Equations (3.28) form the representation of the invariant solution with respect to the algebra L^r. The representation of an invariant solution is obtained by solving the invariants $J^1, J^2, \ldots, J^m$ with respect to the dependent variables $u^1, \ldots, u^m$:

$$u^i = \phi^i(x, u^{m+1}, u^{m+2}, \ldots, u^{m+n-r_*}), \quad (i = 1, \ldots, m).$$

The last set of relations is called the *representation* of the solution invariant with respect to L^r.

CHAPTER IV

DETAILED GROUP CLASSIFICATION

The purpose of this chapter is to give a complete classification of one-dimensional equations of fluids with internal energy depending on the density ρ and the gradient of the density,

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + \Pi)_x = 0,$$

and

$$\Pi = P + \rho\lambda\alpha, \quad P = \rho^2\varepsilon_\rho - \rho\left(\rho\lambda\rho_x\right)_x, \quad \lambda = 2\varepsilon_\alpha, \quad \alpha = \rho_x^2,$$

where P is the pressure, t is time, ρ_x is the gradient of ρ with respect to the space variable x and $\varepsilon(\rho, \alpha)$ is the internal energy.

Equivalence Lie group, Admitted Lie group and complete group classification of one-dimensional equations of fluids with internal energy depending on the density and the gradient of the density are presented in this chapter.

4.1 Equivalence Lie Group

In this section, we will determine the equivalence Lie group for the equations (1.3). Since the arbitrary element ε depends on ρ and α only, the system of equations (1.3) has to be supplemented with auxiliary equations

$$\varepsilon_x = 0, \ \varepsilon_t = 0, \ \varepsilon_u = 0.$$

The infinitesimal generator of the equivalence Lie group has the representation (Meleshko, 2005)

$$X^e = \xi^x\partial_x + \xi^t\partial_t + \zeta^\rho\partial_\rho + \zeta^u\partial_u + \zeta^\alpha\partial_\alpha + \zeta^\varepsilon\partial_\varepsilon,$$

where all coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^\alpha$ and ζ^ε are functions of $(x, t, \rho, u, \alpha, \varepsilon)$. Let us denote

$$u^1 = \rho,\ u^2 = u, u^3 = \alpha$$

and

$$z^1 = x,\ z^2 = t,\ z^3 = \rho,\ z^4 = u,\ z^5 = \alpha.$$

The coefficients of the prolonged generator

$$\tilde{X}^e = X^e + \sum_i (\zeta^{u^i_x}\partial_{u^i_x} + \zeta^{u^i_t}\partial_{u^i_t}) + \sum_j \zeta^{\varepsilon_{zj}}\partial_{\varepsilon_{zj}} + \cdots.$$

are obtained by using the prolongation formulae:

$$\zeta^{u^i_x} = D^e_x\zeta^{u^i} - u^i_x D^e_x\xi^x - u^i_t D^e_x\xi^t$$
$$\zeta^{u^i_t} = D^e_t\zeta^{u^i} - u^i_x D^e_t\xi^x - u^i_t D^e_t\xi^t$$
$$\zeta^{\varepsilon_{zj}} = \tilde{D}^e_{zj}\zeta^{\varepsilon} - \sum_\gamma \varepsilon_{z^\gamma}\tilde{D}^e_{zj}\zeta^{z^\gamma}.$$

The operators D^e_x and D^e_t are operators of the total derivatives with respect to x and t, respectively in the space of the independent variables x and t:

$$D^e_x = \partial_x + \rho_x\partial_\rho + u_x\partial_u + \alpha_x\partial_\alpha + (\varepsilon_x + \varepsilon_\rho\rho_x + \varepsilon_\alpha\alpha_x)\partial_\varepsilon,$$
$$D^e_t = \partial_t + \rho_t\partial_\rho + u_t\partial_u + \alpha_t\partial_\alpha + (\varepsilon_t + \varepsilon_\rho\rho_t + \varepsilon_\alpha\alpha_t)\partial_\varepsilon.$$

The operators $\tilde{D}^e_{zj}$ are operators of the total derivatives with respect to $z^1, \ldots, z^5$ in the space of the independent variables x, t, ρ, u and α:

$$\tilde{D}^e_{zj} = \partial_{z^j} + (\delta_{3j}\varepsilon_\rho + \delta_{5j}\varepsilon_\alpha)\partial_\varepsilon,$$

where δ_{ij} is the Kronecker delta symbol.

For constructing the determining equations and solving them, the symbolic computer Reduce program was applied (Hearn, 1987). Calculations give the following basis of generators of the equivalence Lie group:

$$X^e_1 = x\partial_x + \rho\partial_\rho + u\partial_u + 2\varepsilon\partial_\varepsilon, \quad X^e_2 = \rho\partial_\rho + 2\alpha\partial_\alpha,$$
$$X^e_3 = t\partial_t + x\partial_x + \rho\partial_\rho, \quad X^e_4 = \partial_t, \quad X^e_5 = t\partial_x + \partial_u,$$
$$X^e_6 = \partial_x, \quad X^e_7 = \rho^{-1}\partial_\varepsilon, \quad X^e_8 = \partial_\varepsilon, \quad X^e_9 = f(\rho)\sqrt{\alpha}\partial_\varepsilon,$$

where the function $f(\rho)$ is an arbitrary function.

Since the equivalence transformations corresponding to the operators X^e_7, X^e_8

and X_9^e are applied for simplifying the function ε in the classification process, let us present these transformations. The Lie equations for finding equivalence transformation corresponding to X_7^e, X_8^e and X_9^e are

$$X_7^e : \begin{cases} \dfrac{d\tilde{x}}{da} = 0 \\ \dfrac{d\tilde{t}}{da} = 0 \\ \dfrac{d\tilde{\rho}}{da} = 0 \\ \dfrac{d\tilde{u}}{da} = 0 \\ \dfrac{d\tilde{\alpha}}{da} = 0 \\ \dfrac{d\tilde{\varepsilon}}{da} = \tilde{\rho}^{-1} \end{cases} \qquad X_8^e : \begin{cases} \dfrac{d\tilde{x}}{da} = 0 \\ \dfrac{d\tilde{t}}{da} = 0 \\ \dfrac{d\tilde{\rho}}{da} = 0 \\ \dfrac{d\tilde{u}}{da} = 0 \\ \dfrac{d\tilde{\alpha}}{da} = 0 \\ \dfrac{d\tilde{\varepsilon}}{da} = 1 \end{cases} \qquad X_9^e : \begin{cases} \dfrac{d\tilde{x}}{da} = 0 \\ \dfrac{d\tilde{t}}{da} = 0 \\ \dfrac{d\tilde{\rho}}{da} = 0 \\ \dfrac{d\tilde{u}}{da} = 0 \\ \dfrac{d\tilde{\alpha}}{da} = 0 \\ \dfrac{d\tilde{\varepsilon}}{da} = f(\tilde{\rho})\sqrt{\tilde{\alpha}} \end{cases}$$

Solving these equations with internal conditions for $a = 0$:

$$\tilde{x} = x,\ \tilde{t} = t,\ \tilde{\rho} = \rho,\ \tilde{u} = u,\ \tilde{\alpha} = \alpha,\ \tilde{\varepsilon} = \varepsilon$$

we obtained the transformations:

$$\begin{aligned} X_7^e :&\quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{\varepsilon} = \varepsilon + \rho^{-1}a, \\ X_8^e :&\quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{\varepsilon} = \varepsilon + a, \\ X_9^e :&\quad \tilde{\rho} = \rho, \quad \tilde{\alpha} = \alpha, \quad \tilde{\varepsilon} = \varepsilon + \sqrt{\alpha} f(\rho) a; \end{aligned}$$

Because the function ε depends on ρ and α, only the transformations of these variables are presented. Here a is the group parameter.

Remark. Using the equivalence transformations corresponding to the generators X_7^e and X_8^e, a term $C_1\rho^{-1} + C_2$, appearing in the function $\varepsilon(\rho, \alpha)$ can be transformed to zero. Here C_1 and C_2 are arbitrary constants. By virtue of the equivalence transformations corresponding to the generator X_9^e, the function $\varepsilon(\rho, \alpha)$ is considered up to the term $\sqrt{\alpha} f(\rho)$ with arbitrary function $f(\rho)$.

4.2 Admitted Lie group

Let an infinitesimal generator of a one-parameter Lie group admitted by system of equations (1.3) be

$$X = \xi^x \partial_x + \xi^t \partial_t + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^\alpha \partial_\alpha,$$

where the coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^\alpha$ are functions of the variables (x, t, ρ, u, α).

Calculations have shown that

$$\xi^x = k_3 x + k_4 t x - k_5 t^3 - k_6 t^2 - k_7 t - k_8,$$

$$\xi^t = k_2 t - 2k_3 t - k_4 t^2 + k_9, \quad \zeta^\rho = k_1 \rho + k_4 \rho t,$$

$$\zeta^u = k_2 u - k_3 u - k_4(ut - x) - 3k_5 t^2 - 2k_6 t - k_7,$$

$$\zeta^\alpha = 2k_1 \alpha + 2k_3 \alpha + 4k_4 \alpha t,$$

$$\begin{gathered} k_1\left(2\alpha\rho^2\varepsilon_{\alpha\rho\rho\rho} + 10\alpha\rho\varepsilon_{\alpha\rho\rho} + 8\alpha\varepsilon_{\alpha\rho} + 4\alpha^2\rho\varepsilon_{\alpha\alpha\rho\rho} + 8\alpha^2\varepsilon_{\alpha\alpha\rho} - \rho^2\varepsilon_{\rho\rho\rho}\right. \\ \left. -4\rho\varepsilon_{\rho\rho} - 2\varepsilon_\rho\right) + 2k_2\left(2\alpha\rho\varepsilon_{\alpha\rho\rho} + 4\alpha\varepsilon_{\alpha\rho} - \rho\varepsilon_{\rho\rho} - 2\varepsilon_\rho\right) \\ +2k_3\left(-\alpha\rho\varepsilon_{\alpha\rho\rho} - 2\alpha\varepsilon_{\alpha\rho} + 2\alpha^2\rho\varepsilon_{\alpha\alpha\rho\rho} + 4\alpha^2\varepsilon_{\alpha\alpha\rho} + \rho\varepsilon_{\rho\rho} + 2\varepsilon_\rho\right) \\ -2k_6 q(\alpha) = 0, \end{gathered} \tag{4.1}$$

$$k_4\left(\rho\varepsilon_{\alpha\rho} + 8\alpha^2\varepsilon_{\alpha\alpha\alpha} + 2\alpha\rho\varepsilon_{\alpha\alpha\rho} + 16\alpha\varepsilon_{\alpha\alpha} + 2\varepsilon_\alpha\right) = 0, \tag{4.2}$$

$$\begin{gathered} k_1\left(\rho\varepsilon_{\alpha\rho} + 4\alpha^2\varepsilon_{\alpha\alpha\alpha} + 2\alpha\rho\varepsilon_{\alpha\alpha\rho} + 10\alpha\varepsilon_{\alpha\alpha} + 2\varepsilon_\alpha\right) + 2k_2\left(2\alpha\varepsilon_{\alpha\alpha} + \varepsilon_\alpha\right) \\ +2k_3\left(2\alpha^2\varepsilon_{\alpha\alpha\alpha} + 3\alpha\varepsilon_{\alpha\alpha}\right) = 0, \end{gathered} \tag{4.3}$$

$$\begin{gathered} k_4\left(2\alpha\rho^2\varepsilon_{\alpha\rho\rho\rho} - 8\alpha\rho\varepsilon_{\alpha\rho\rho} + 4\alpha\varepsilon_{\alpha\rho} + 8\alpha^2\rho\varepsilon_{\alpha\alpha\rho\rho} + 16\alpha^2\varepsilon_{\alpha\alpha\rho}\right. \\ \left. -\rho^2\varepsilon_{\rho\rho\rho} - 2\rho\varepsilon_{\rho\rho} + 2\varepsilon_\rho\right) - 6k_5 q(\alpha) = 0, \end{gathered} \tag{4.4}$$

where $k_i, (i = 1, 2, ..., 9)$ are constants, $q(\alpha) = a/\sqrt{\alpha}$ and $a^2 = 1$. The determining equations (4.1)-(4.4) define the kernel of admitted Lie algebras and its extensions.

The kernel of admitted Lie algebras consists of the generators which are admitted by equations (1.3) for any function $\varepsilon(\rho, \alpha)$. The kernel consists of the generators

$$Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = t\partial_x + \partial_u.$$

Extensions of the kernel of admitted Lie algebras depend on the value of the function $\varepsilon(\rho, \alpha)$. They can only be operators of the form

$$k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 + k_5X_5 + k_6X_6,$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ are constants, and

$$X_1 = \rho\partial_\rho + 2\alpha\partial_\alpha, \quad X_2 = t\partial_t - u\partial_u,$$

$$X_3 = u\partial_u + 2\alpha\partial_\alpha - x\partial_x - 2t\partial_t,$$

$$X_4 = \rho t\partial_\rho + (ut - x)\partial_u + 4\alpha t\partial_\alpha - tx\partial_x - t^2\partial_t,$$

$$X_5 = t^3\partial_x + 3t^2\partial_u, \quad X_6 = t^2\partial_x + 2t\partial_u.$$

Relations between the constants k_1, k_2, k_3, k_4, k_5, k_6 depend on the function $\varepsilon(\rho, \alpha)$.

For easily to solve the equations (4.1)-(4.4), first let us consider equation (4.2).

4.2.1 Case $k_4 \neq 0$

If $k_4 \neq 0$, then equation (4.2) gives

$$4\alpha g_\alpha + \rho g_\rho + 2g = 0,$$

where

$$g = 2\alpha\varepsilon_{\alpha\alpha} + \varepsilon_\alpha.$$

By using the equivalence transformation corresponding to X_9^e, the general solution of this equation is

$$\varepsilon(\rho, \alpha) = \rho^2\varphi(\alpha\rho^{-4}) + \varphi_2(\rho), \tag{4.5}$$

where $\varphi(z)$ and $\varphi_2(\rho)$ are arbitrary functions and $z = \alpha\rho^{-4}$.

Notice that if $(\sqrt{z}\varphi')' = 0$, then by virtue of the arbitrariness of the functions φ and φ_2, one can assume that $\varphi = 0$. Hence, we assume that $(\sqrt{z}\varphi')' \neq 0$.

For the function $\varepsilon(\rho, \alpha)$ having the representation (4.5), equation (4.4) becomes

$$k_4(\rho^2\varphi_2''' + 2\rho\varphi_2'' - 2\varphi_2') + 6k_5q(\alpha) = 0.$$

Splitting this equation with respect to α, one obtains

$$k_5 = 0$$

and

$$\rho^2\varphi_2^{'''} + 2\rho\varphi_2^{''} - 2\varphi_2^{'} = 0.$$

The general solution of the last equation is

$$\varphi_2 = C_1 + C_2\rho^2 + C_3\rho^{-1}.$$

Because of the equivalence transformation corresponding to X_7^e and X_8^e then

$$\varphi_2 = C_2\rho^2.$$

Substituting $\varepsilon(\rho,\alpha) = \rho^2\varphi(\alpha\rho^{-4}) + C_2\rho^2$ into (4.3), it gives

$$2(k_3 - k_1)z^2\varphi^{'''} + (2k_2 + 3(k_3 - k_1))z\varphi^{''} + k_2\varphi^{'} = 0. \tag{4.6}$$

For arbitrary function φ one has

$$(k_3 - k_1) = 0, k_2 = 0.$$

Then equation (4.1) gives

$$k_6 = 0.$$

In this case the extension of the kernel of admitted Lie algebras consists of the generators

$$X_1 + X_3,\ \ X_4. \tag{4.7}$$

Extensions of (4.7) can only occur for $(k_3 - k_1) \neq 0$. Equation (4.6) is an Euler ordinary differential equation. The general solution of (4.6) depends on $\mu = k_2(k_3 - k_1)^{-1}$.

If $\mu \neq 1/2$, then $\varphi^{'} = C_4 z^{-\mu}$, $C_4 \neq 0$. Equation (4.1) becomes

$$6(k_3 - k_1)\rho[(1 - \mu)(\varphi + C_2) + C_4 z^{1-\mu}] - k_6 q(\alpha) = 0.$$

Splitting this equation with respect to α, one obtains

$$k_6 = 0$$

and

$$(1 - \mu)(\varphi + C_2) + C_4 z^{1-\mu} = 0.$$

By virtue of $C_4 \neq 0$, one gets $\mu \neq 1$. Hence,

$$\varepsilon = C\alpha^{1-\mu}\rho^{2(2\mu-1)}, \ (\mu - 1/2)(\mu - 1) \neq 0.$$

and the extension of the kernel of admitted Lie algebras consists of the generators

$$X_1 - \mu X_2, \ \mu X_2 + X_3, \ X_4.$$

If $\mu = 1/2$, then $\varphi^{'} = C_4 \ln(z) z^{-1/2}$, and similar to the previous case, one obtains

$$k_6 = 0$$

and

$$\varepsilon = C\sqrt{\alpha}\ln(\alpha), \ (C \neq 0).$$

The extensions of the kernel of admitted Lie algebras are

$$2X_1 - X_2, \ X_2 + 2X_3, \ X_4.$$

4.2.2 Case $k_4 = 0$

If $k_4 = 0$, then equation (4.4) gives

$$k_5 = 0.$$

Equation (4.3) becomes

$$k_1 a + k_2 b + k_3 c = 0, \tag{4.8}$$

where

$$a = 2\alpha g_\alpha + \rho g_\rho + 2g, \ b = 2g, \ c = 2\alpha g_\alpha,$$

and

$$g = 2\alpha\varepsilon_{\alpha\alpha} + \varepsilon_\alpha.$$

Further analysis of the determining equations (4.1)-(4.4) is similar to the group classification of the gas dynamics equations (Ovsiannikov, 1978).

Let us analyze the vector space $Span(V)$, where the set V consists of vectors (a, b, c) with ρ and α varied. If the function $\varepsilon(\rho, \alpha)$ is such that $\dim(Span(V))$=3, then equation (4.8) is only satisfied for

$$k_1 = 0, \;\; k_2 = 0, \;\; k_3 = 0.$$

Substituting k_1, k_2, k_3 into (4.1) gives

$$k_6 = 0.$$

Hence, an extension of the kernel of admitted Lie algebras is only possible if $\dim(Span(V)) < 3$.

$\dim(Span(V)) = 2$

There exists a constant vector $(\lambda, \beta, \gamma) \neq 0$, which is orthogonal to the set V:

$$\lambda a + \beta b + \gamma c = 0.$$

This means that the function $g(\rho, \alpha)$ satisfies the equation

$$2\alpha(\lambda + \gamma)g_\alpha + \rho\lambda g_\rho + 2(\lambda + \beta)g = 0. \tag{4.9}$$

The characteristic system of this equation is

$$\frac{d\alpha}{-2\alpha(\lambda + \gamma)} = \frac{d\rho}{-\rho\lambda} = \frac{dg}{2(\lambda + \beta)g}.$$

The general solution of equation (4.9) depends on the constants λ, β and γ.

Case $\lambda = 0$. Since $\dim(Span(V)) = 2$, then $\gamma \neq 0$ and the general solution of equation (4.9) is

$$g = g_1(\rho)\alpha^k,$$

where $k = -\beta/\gamma$, $g_1(\rho)$ is an arbitrary function. Thus

$$2\alpha\varepsilon_{\alpha\alpha} + \varepsilon_\alpha = g_1(\rho)\alpha^k. \tag{4.10}$$

For obtaining the function ε one needs to integrate equation (4.10) with respect to α. This integration depends on the value of the exponent k.

If $(2k+1)(k+1) \neq 0$, then using the equivalence transformation corresponding to X_9^e,

$$\varepsilon(\rho,\alpha) = \varphi(\rho)\alpha^p + \varphi_2(\rho), \tag{4.11}$$

where $p = k + 1, \varphi(\rho) = g_1(\rho)/((2k+1)(k+1)), \varphi_2(\rho)$ are arbitrary functions. Substituting (4.11) into (4.8) leads to

$$k_1\rho\varphi^{'} + 2(k_2 + k_1p + k_3(p-1))\varphi = 0.$$

If $\varphi = C_1\rho^c$ then $\dim(Span(V)) = 1$. Hence,

$$k_1 = 0,\ k_2 = (1-p)k_3.$$

Equation (4.1) becomes

$$pk_3(\rho\varphi_2^{''} + 2\varphi_2^{'}) - k_6q(\alpha) = 0.$$

Splitting this equation with respect to α, one obtains $k_6 = 0$ and

$$k_3(\rho\varphi_2^{''} + 2\varphi_2^{'}) = 0.$$

If $k_3 = 0$, then $k_2 = 0$ and there is no extension of the kernel of admitted Lie algebras. Hence, an extension of the kernel of admitted Lie algebras occurs for the function φ_2 satisfying $\varphi_2 = C_1 + C_2\rho^{-1}$. According to the equivalence transformation corresponding to X_7^e and X_8^e, $\varphi_2 = 0$. In this case $\varepsilon = \varphi(\rho)\alpha^p$ and the extension of the kernel of admitted Lie algebras is given by the generator

$$(p-1)X_2 - X_3.$$

If $k = -1$, then according to the equivalence transformation corresponding to X_9^e,

$$\varepsilon(\rho,\alpha) = \varphi(\rho)\ln(\alpha) + \varphi_2(\rho). \tag{4.12}$$

Substituting (4.12) into (4.3) leads to

$$k_1\rho\varphi^{'} + 2(k_2 - k_3)\varphi = 0.$$

Since $\dim(Span(V)) = 2$, one has

$$k_1 = 0,\ k_2 = k_3.$$

Equation (4.1) becomes

$$k_3(\rho\varphi^{''} + \varphi^{'}) - 2k_6 q(\alpha) = 0.$$

Then

$$k_6 = 0$$

and

$$k_3(\rho\varphi^{''} + 2\varphi^{'}) = 0.$$

An extension of the kernel of admitted Lie algebras occurs for the function $\varphi = \beta + \gamma\rho^{-1}$. If $\beta\gamma = 0$, then $\dim(Span(V)) = 1$. Hence, $\beta\gamma \neq 0$,

$$\varepsilon = (\beta + \gamma\rho^{-1})\ln(\alpha) + \varphi_2(\rho)$$

and the extension of the kernel of admitted Lie algebras is

$$X_2 + X_3.$$

If $k = -1/2$, then according to the equivalence transformation corresponding to X_9^e,

$$\varepsilon(\rho, \alpha) = \varphi(\rho)\sqrt{\alpha}\ln(\alpha) + \varphi_2(\rho). \tag{4.13}$$

Substituting (4.13) into (4.8), it leads to

$$k_1\rho\varphi^{'} + (2k_2 + k_1 - k_3)\varphi = 0.$$

Since $\dim(Span(V)) = 2$, one obtains that

$$k_1 = 0,\ \ k_2 = \frac{k_3}{2}.$$

Equation (4.1) becomes

$$k_3(\rho\varphi_2^{''} + 2\varphi_2^{'}) - 2k_6 q(\alpha) = 0.$$

Then

$$k_6 = 0$$

and

$$k_3(\rho\varphi_2^{''} + 2\varphi_2^{'}) = 0.$$

Since for $k_3 = 0$, there is no extension of the kernel of admitted Lie algebras, then by using the equivalence transformation corresponding to X_7^e and X_8^e, $\varphi_2 = 0$. In this case

$$\varepsilon = \varphi(\rho)\sqrt{\alpha}\ln(\alpha).$$

The extension of the kernel of admitted Lie algebras is given by the generator

$$X_2 + 2X_3.$$

Case $\lambda \neq 0$. According to the equivalence transformation corresponding to X_9^e, the general solution of equation (4.9) is

$$\varepsilon(\rho, \alpha) = \rho^{\mu}\varphi(\alpha\rho^k) + \varphi_2(\rho), \tag{4.14}$$

where $k = -2(1 + \gamma/\lambda)$, $\mu = -2(\beta - \gamma)/\lambda$. Substituting (4.14) into (4.3) leads to

$$2((k+2)k_1 + 2k_3)z^2\varphi^{'''} + 2((k+\mu+5)k_1 + 2k_2 + 3k_3)z\varphi^{''} + ((k+\mu+2)k_1 + 2k_2)\varphi^{'} = 0,$$

where $z = \alpha\rho^k$. If $\varphi^{'} = C_1 z^c$ or $\varphi^{'} = C_1 z^c \ln(z)$ then $\dim(Span(V)) = 1$.
Hence

$$k_3 = -k_1(k+2)/2,\ k_2 = -k_1(k+\mu+2)/2.$$

Note that for extensions of the kernel of admitted Lie algebras, $k_1 \neq 0$. Substituting (4.14) into (4.1) one obtains

$$k_1(\rho^2\varphi_2^{'''} - (\mu-4)\rho\varphi_2^{''} - 2(\mu-1)\varphi_2^{'}) + 2k_6 q(\alpha) = 0.$$

Then

$$k_6 = 0$$

and

$$\rho^2\varphi_2^{'''} - (\mu-4)\rho\varphi_2^{''} - 2(\mu-1)\varphi_2^{'} = 0.$$

If $\mu \neq -1$, then $\varphi_2^{'} = C_1\rho^{\mu-1}$, the extension of the kernel of admitted Lie algebras is

$$2X_1 - (\mu + k + 2)X_2 - (k+2)X_3.$$

If $\mu = -1$, then $\varphi_2 = C_1\rho^{-1}\ln(\rho)$. Hence

$$\varepsilon = \rho^{\mu}\varphi(\alpha\rho^k) + C_1\rho^{-1}\ln(\rho)$$

and the extension of the kernel of admitted Lie algebras is

$$2X_1 - (k+1)X_2 - (k+2)X_3.$$

$\dim(Span(V)) = 1$

There exists a constant vector $(\lambda, \beta, k) \neq 0$ such that

$$(a, b, c) = (\lambda, \beta, k)B$$

for some function $B(\rho, \alpha) \neq 0$. Since $\dim(Span(V)) = 1$, then $g \neq 0$ and hence, $b \neq 0$. Without loss of generality one can assume that $\beta = 2$, which leads to $B = b/2$ and $a = (\lambda b)/2$, $c = (kb)/2$. Thus, the function $\varepsilon(\rho, \alpha)$ satisfies the equations

$$4\alpha^2\varepsilon_{\alpha\alpha\alpha} + 2\alpha\rho\varepsilon_{\alpha\alpha\rho} + \rho\varepsilon_{\alpha\rho} - (2\lambda - 10)\alpha\varepsilon_{\alpha\alpha} - (\lambda - 2)\varepsilon_\alpha = 0,$$
$$4\alpha^2\varepsilon_{\alpha\alpha\alpha} - 2(k-3)\alpha\varepsilon_{\alpha\alpha} - k\varepsilon_\alpha = 0.$$

According to the equivalence transformation corresponding to X_9^e, the general solution of the last equation is

$$\varepsilon_\alpha = g_2(\rho)\alpha^{k/2}, \ (k \neq -1).$$

Substituting this solution into the first equation, one obtains

$$\rho g_2' - (\lambda - k - 2)g_2 = 0.$$

Then

$$g_2 = C\rho^\gamma,$$

where $\gamma = \lambda - k - 2$ and C is an arbitrary constant. Thus,

$$\varepsilon_\alpha = C\rho^\gamma\alpha^{k/2}, \ (C \neq 0). \tag{4.15}$$

Since $\dim(Span(V)) = 1$, then γ and k are such that $\gamma^2 + k^2 \neq 0$. Substituting (4.15) into (4.8), one obtains

$$k_2 = \frac{-k_1(\gamma + k + 2) - k_3 k}{2}. \tag{4.16}$$

If $(k+2) \neq 0$, integrating (4.15), one obtains

$$\varepsilon(\rho, \alpha) = C\rho^\gamma\alpha^p + \varphi_2(\rho), \tag{4.17}$$

where $p = 1 + k/2 \neq 0$. Substituting (4.17) into (4.1) gives

$$k_1\rho^2\varphi_2''' + ((4-\gamma-2p)k_1 - 2pk_3)\rho\varphi_2'' + 2((1-\gamma-2p)k_1 - 2pk_3)\varphi_2' + 2k_6 q(\alpha) = 0.$$

Splitting this equation with respect to α, one has

$$k_6 = 0$$

and

$$k_1\rho^2\varphi_2''' + ((4-\gamma-2p)k_1 - 2pk_3)\rho\varphi_2'' + 2((1-\gamma-2p)k_1 - 2pk_3)\varphi_2' = 0. \tag{4.18}$$

For an arbitrary function φ_2 one obtains

$$k_1 = 0, k_3 = 0,$$

then there exists no extension of the kernel of admitted Lie algebras. Hence an extension of the kernel of admitted Lie algebras can only occur for special types of the function φ_2.

If $\varphi_2' = C_1\rho^{-2}$, then by using the equivalence transformation corresponding to X_7^e and X_8^e,

$$\varepsilon = C\rho^\gamma\alpha^p$$

and the admitted generators are

$$2X_1 - (\gamma + 2p)X_2,\ (p-1)X_2 - X_3.$$

If $\varphi_2' \neq C_1\rho^{-2}$, then $k_1 \neq 0$. Hence the general solution of (4.18) depends on $\mu = ((1-\gamma-2p)k_1 - 2pk_3)/k_1$.

If $\mu \neq 2$, then $\varphi_2' = C_1\rho^{-\mu} \neq 0$, and equation (4.1) gives

$$k_3 = \frac{(1-\mu-\gamma-2p)k_1}{2p}.$$

The extension of the kernel of admitted Lie algebras is given by the generator

$$2pX_1 + (p\mu - \gamma - 3p - \mu + 1)X_2 + (1-\gamma-2p-\mu)X_3.$$

If $\mu = 2$, then $\varphi_2' = C_1\rho^{-2}\ln(\rho) \neq 0$, hence

$$\varepsilon = C\rho^{\gamma}\alpha^{p} + C1\rho^{-1}\ln(\rho).$$

Equation (4.1) gives

$$k_3 = \frac{(-1-\gamma-2p)k_1}{2p}$$

and the admitted generator is

$$2pX_1 - (\gamma+p+1)X_2 - (\gamma+2p+1)X_3.$$

If $k = -2$, then

$$\varepsilon = C\rho^{\lambda}\ln(\alpha) + \varphi_2(\rho),\ C \neq 0. \tag{4.19}$$

Equation (4.16) gives

$$k_2 = \frac{2k_3 - \lambda k_1}{2}.$$

Substituting (4.19) into (4.1) and splitting it with respect to α one obtains

$$k_6 = 0$$

and

$$k_1(\rho^2\varphi_2^{'''} + (4-\lambda)\rho\varphi_2^{''} + 2(1-\lambda)\varphi_2^{'}) = -2C\lambda(\lambda+1)\rho^{\lambda-1}(k_1+k_3). \tag{4.20}$$

The solution of equation (4.20) depends on the value of λ.

Let $\lambda(\lambda+1) = 0$. In this case for arbitrary $\varphi_2^{'}$ one obtains

$$k_1 = 0,$$

hence, the extension of the kernel of admitted Lie algebras is only the generator

$$X_2 + X_3.$$

If $\lambda = 0$, and $\varphi_2^{'} = C_2\rho^{-1}$, then by using the equivalence transformation corresponding to X_7^e and X_8^e,

$$\varepsilon = C\rho^{\lambda}\ln(\alpha)$$

and there is one more admitted generator

$$X_1.$$

If $\lambda = -1$ and $\varphi_2^{'} = C_2\rho^{-2}\ln(\rho)$, then according to the equivalence transformation corresponding to X_7^e and X_8^e,

$$\varepsilon = C\rho^{\gamma}\alpha^{p} + C\rho^{-1}\ln(\rho)$$

and there is one more admitted generator

$$2X_1 + X_2.$$

If $\lambda(\lambda+1) \neq 0$, then from analysis of (4.20) one obtains that an extension of the kernel of admitted Lie algebras only occurs for

$$\varphi_2^{'} = \rho^{\lambda-1}(\gamma\ln(\rho) + l),$$

where γ and l are constants, and

$$k_2 = -(\frac{\gamma}{2C\lambda} + \frac{\lambda+2}{2})k_1,\ \ k_3 = -(1 + \frac{\gamma}{2C\lambda})k_1.$$

The extension of the kernel of admitted Lie algebras is given by the generator

$$X_1 - (\frac{\lambda+2}{2} + \frac{\gamma}{2C\lambda})X_2 - (\frac{\gamma}{2C\lambda} + 1)X_3.$$

$\dim(Span(V)) = 0$

In this case the vector (a, b, c) is constant:

$$(a, b, c) = (\lambda, \beta, k)$$

for some constant values λ, β and k. This leads to

$$\varepsilon(\rho, \alpha) = C\alpha + \varphi_2(\rho), \tag{4.21}$$

where $C \neq 0$. Substituting (4.21) into (4.3) gives

$$k_2 = -k_1.$$

After splitting Equation (4.1), one has

$$k_6 = 0$$

and

$$k_1\rho^2\varphi_2^{'''} + 2(k_1 - k_3)\rho\varphi_2^{''} - 2(k_1 + 2k_3)\varphi_2^{'} = 0.$$

Note that for extensions of the kernel, $k_1^2 + k_3^2 \neq 0$. Hence, for the particular choice of the function $\varphi_2 = 0$, the admitted generators are

$$X_1 - X_2,\ X_3.$$

For the choice of the function $\varphi_2^{'} = C_1\rho^{-\mu} \neq 0$, $\mu \neq 2$, the extension of the kernel of admitted Lie algebras only consists of the generator

$$2X_1 - 2X_2 - (\mu + 1)X_3.$$

For the function φ_2 satisfying $\varphi_2 = C_1\rho^{-1}\ln(\rho) \neq 0$, then the admitted generator is

$$2(X_1 - X_2) - 3X_3.$$

The result of the group classification of equation (1.3) is summarized in Table 1.

Table 1: Group classification

	$\varepsilon(\rho,\alpha)$	Extensions	Remarks
M_1	$\rho^2\varphi(\alpha\rho^{-4})$	$X_1+X_3,\ X_4$	φ arbitrary
M_2	$C\alpha^{1-\mu}\rho^{2(2\mu-1)}$	$X_1-\mu X_2,\ \mu X_2+X_3,\ X_4$	$(\mu-1/2)(\mu-1)\neq 0$
M_3	$C\sqrt{\alpha}\ln(\alpha)$	$2X_1-X_2,\ X_2+2X_3,\ X_4$	
M_4	$\varphi(\rho)\alpha^p$	$(p-1)X_2-X_3$	$p(2p-1)\neq 0$
M_5	$(\beta+\gamma\rho^{-1})\ln(\alpha)$ $+\varphi_2(\rho)$	X_2+X_3	$\beta\gamma\neq 0$ φ_2 arbitrary
M_6	$\varphi(\rho)\sqrt{\alpha}\ln(\alpha)$	X_2+2X_3	
M_7	$\rho^\mu\varphi(\alpha\rho^k)+C\rho^\mu$	$2X_1-(\mu+k+2)X_2$ $-(k+2)X_3$	$\mu(\mu+1)\neq 0$
M_8	$\rho^\mu\varphi(\alpha\rho^k)+C_2\ln(\rho)$	$2X_1-(k+2)(X_2+X_3)$	
M_9	$\rho^\mu\varphi(\alpha\rho^k)+C\rho^{-1}\ln(\rho)$	$2X_1-(k+1)X_2-(k+2)X_3$	
M_{10}	$C\rho^\gamma\alpha^p$	$2X_1-(\gamma+2p)X_2,$ $(p-1)X_2-X_3$	$p(2p-1)\neq 0$ $\gamma^2+(p-1)^2\neq 0$
M_{11}	$C\rho^\gamma\alpha^p+C_1\rho^\mu$	$2pX_1-(\gamma+2p-\mu)X_3$ $-(\gamma+p\mu+2p-\mu)X_2$	$C_1\mu p(2p-1)\neq 0$ $\gamma^2+(p-1)^2\neq 0$
M_{12}	$C\rho^\gamma\alpha^p+C_1\ln(\rho)$	$(\gamma+2p)(X_2+X_3)$ $-2pX_1$	$p(2p-1)\neq 0$ $\gamma^2+(p-1)^2\neq 0$
M_{13}	$C\rho^\gamma\alpha^p+C_1\ln(\rho)\rho^{-1}$	$2pX_1-(\gamma+p+1)X_2$ $-(\gamma+2p+1)X_3$	$p(2p-1)\neq 0$ $\gamma^2+(p-1)^2\neq 0$
M_{14}	$C\rho^\lambda\ln(\alpha)+\varphi_2(\rho)$	X_2+X_3	$\lambda(\lambda+1)=0$
M_{15}	$C\ln(\alpha)+C_2\ln(\rho)$	$X_1,\ X_2+X_3$	
M_{16}	$C\ln(\alpha)\rho^{-1}+C_2\ln(\rho)^2$	$2X_1+X_2,\ X_2+X_3$	
M_{17}	$C\rho^\lambda\ln(\alpha)+\varphi_2(\rho)$	$X_1-(\frac{\lambda+2}{2}+\frac{\gamma}{2C\lambda})X_2$ $-(\frac{\gamma}{2C\lambda}+1)X_3$	$\lambda(\lambda+1)\neq 0$ $\varphi_2'=\rho^{\lambda-1}(\gamma\ln(\rho)+l)$
M_{18}	$C\alpha$	$X_1-X_2,\ X_3$	
M_{19}	$C\alpha+C_1\rho^\mu$	$2(X_1-X_2)-(2-\mu)X_3$	$\mu\neq 0$
M_{20}	$C\alpha+C_1\ln(\rho)$	$X_1-X_2-X_3$	
M_{21}	$C\alpha+C_1\ln(\rho)\rho^{-1}$	$2(X_1-X_2)-3X_3$	

According to Table 1, the set of all models (1.3), equations of fluids with internal energy ϵ depending on the density ρ and the gradient of the density α, where $\epsilon \neq \sqrt{\alpha}\varphi_1(\rho)+\varphi_2(\alpha)$ can be separated into two types. The first type consists of models for which the admitted Lie algebra includes the generator X_4. The second type of models for which the admitted Lie algebra does not include the generator X_4. In the case of gas dynamics equations the generator X_4 corresponds to the mono-atomic gas.

CHAPTER V

INVARIANT SOLUTIONS OF MODEL M_4

Invariant solutions can be sought for a subalgebra of an admitted Lie algebra. Essentially different invariant solutions are obtained on the basis of an optimal system of admitted subalgebras. The set of all generators nonequivalent with respect to automorphisms composes an optimal system of one dimensional subalgebras (Ovsiannikov, 1978). This set is used for constructing nonequivalent invariant solutions. Equivalence of invariant solutions is considered with respect to an admitted Lie group.

Here we give examples of invariant solutions of equations (1.3) with

$$\varepsilon = \varphi(\rho)\alpha^p,$$

where $p(2p-1) \neq 0$ (model M_4). For the sake of simplicity it is also assumed that $(2p+1)(p+1) \neq 0$. In this case equations (1.3) admit the Lie algebra

$$L_4 = \{Y_1, Y_2, Y_3, Y_4\}, \tag{5.1}$$

where

$$Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = t\partial_x + \partial_u, \quad Y_4 = x\partial_x + (p+1)t\partial_t - pu\partial_u - 2\alpha\partial_\alpha.$$

5.1 Optimal system of subalgebras

In addition to automorphisms one has to use involutions for constructing an optimal system of subalgebras. Equations (1.3) posses two involutions. The first involution E_1 corresponds to the change $x \to -x$. The second involution E_2 is related with the change $t \to -t$. These involutions act on the generator

$$Y = y_1Y_1 + y_2Y_2 + y_3Y_3 + y_4Y_4$$

by transforming the generator Y into the generator $\widetilde{Y}$ with the changed coordinates:

$$E_1 : \begin{cases} \tilde{y}_1 = -y_2, \\ \tilde{y}_3 = -y_3, \end{cases} \qquad E_2 : \begin{cases} \tilde{y}_2 = -y_1, \\ \tilde{y}_3 = -y_3. \end{cases}$$

Here only changed coordinates are presented.

The table of commutators of (5.1) is

	Y_1	Y_2	Y_3	Y_4
Y_1	0	0	Y_2	$(p+1)Y_1$
Y_2		0	0	Y_2
Y_3			0	$-pY_3$
Y_4				0

The Lie equations for finding automorphisms are constructed according to the table of commutators. In particular, they are

$$A_1 : \begin{cases} \dfrac{d\tilde{y}_1}{da} = (p+1)\tilde{y}_4 \\ \dfrac{d\tilde{y}_2}{da} = \tilde{y}_3 \\ \dfrac{d\tilde{y}_3}{da} = 0 \\ \dfrac{d\tilde{y}_4}{da} = 0 \end{cases} \qquad A_2 : \begin{cases} \dfrac{d\tilde{y}_1}{da} = 0 \\ \dfrac{d\tilde{y}_2}{da} = \tilde{y}_4 \\ \dfrac{d\tilde{y}_3}{da} = 0 \\ \dfrac{d\tilde{y}_4}{da} = 0 \end{cases}$$

$$A_3 : \begin{cases} \dfrac{d\tilde{y}_1}{da} = 0 \\ \dfrac{d\tilde{y}_2}{da} = -\tilde{y}_1 \\ \dfrac{d\tilde{y}_3}{da} = -p\tilde{y}_4 \\ \dfrac{d\tilde{y}_4}{da} = 0 \end{cases} \qquad A_4 : \begin{cases} \dfrac{d\tilde{y}_1}{da} = -(p+1)\tilde{y}_1 \\ \dfrac{d\tilde{y}_2}{da} = \tilde{y}_2 \\ \dfrac{d\tilde{y}_3}{da} = -p\tilde{y}_3 \\ \dfrac{d\tilde{y}_4}{da} = 0 \end{cases}$$

Solving these equations with internal conditions for $a = 0$:

$$\tilde{y}_1 = y_1,\ \tilde{y}_2 = y_2,\ \tilde{y}_3 = y_3,\ \tilde{y}_4 = y_4,$$

we obtained the automorphisms:

$$A_1 : \begin{cases} \tilde{y}_1 = y_1 + a_1(p+1)y_4, \\ \tilde{y}_2 = y_2 + a_1 y_3, \end{cases} \quad A_2 : \left\{ \tilde{y}_2 = y_2 + a_2 y_4, \right. \quad A_3 : \begin{cases} \tilde{y}_2 = y_2 - a_3 y_1, \\ \tilde{y}_3 = y_3 - a_3 p y_4, \end{cases}$$

$$A_4 : \begin{cases} \tilde{y}_1 = y_1 e^{(p+1)a_4}, \\ \tilde{y}_2 = y_2 e^{a_4}, \\ \tilde{y}_3 = y_3 e^{-pa_4}, \end{cases}$$

In the following coordinates (y_1, y_2, y_3, y_4) of the generator Y are simplified by the automorphisms A_i, $(i = 1, 2, 3, 4)$.

The Lie algebra L_4 can be presented as the direct sum $L_4 = J \oplus N$, where $J = \{Y_1, Y_2, Y_3\}$ is an ideal and $N = Y_4$ is a subalgebra.* The classification of the subalgebra N is simple and consists of two elements

$$N_1 = \{0\}, \ N_2 = \{Y_4\}.$$

The next step consists of gluing the ideal J to the subalgebra N_i, (i=1,2) .

Let us consider the first case N_1. Assuming that $y_3 \neq 0$, and choosing $a_1 = -y_2/y_3$, one maps y_2 into zero. This means that $\tilde{y}_2 = 0$. For simplicity of explanation, we write it as $y_2 \to 0(A_1)$. If $\tilde{y}_1 \neq 0$, then applying A_4 and E_1, then $Y_1 + y_3 Y_3$ can be transform to $Y_1 + Y_3$. Hence, one has the subalgebra $\{Y_1 + Y_3\}$. If $\tilde{y}_1 = 0$, then one gets the subalgebra $\{Y_3\}$.

Let us assume that $y_3 = 0$. If $y_1 \neq 0$, then, $y_2 \to 0(A_3)$, and the subalgebra is $\{Y_1\}$. If $y_1 = 0$, one obtains the subalgebra $\{Y_2\}$.

One can study the case N_2 in a similar way. The optimal system of one dimensional subalgebras of the Lie algebra corresponding to L_4 consists of the subalgebras:

$$\{Y_1\}, \{Y_2\}, \ \{Y_3\}, \ \{Y_4\}, \ \{Y_1 + Y_3\}.$$

5.2 Invariant solutions of Y_1

Invariants of the generator

$$Y_1 = \partial_t$$

*For constructing optimal system of subalgebras we use the algorithm (Ovsiannikov, 1993).

are

$$x,\ \rho,\ u.$$

An invariant solution has the representation

$$u = U(x),\ \rho = R(x).$$

Substitution of this representation into equations (1.3) gives

$$U = k/R$$

and

$$2p(1-2p)R'''(R')^{2(p+2)}R^4\varphi + 4(1-2p)(p-1)(R'')^2(R')^{2p+3}R^4\varphi$$
$$+4p(1-2p)R''(R')^{2p+5}R^3(R\varphi' + \varphi)$$
$$+(1-2p)(R')^{7+2p}R^3(R\varphi'' + 2\varphi') - k^2(R')^7 = 0,$$

where k is a constant.

5.3 Invariant solutions of Y_2

Invariants of the generator

$$Y_2 = \partial_x$$

are

$$t,\ \rho,\ u.$$

An invariant solution has the representation

$$u = U(t),\ \rho = R(t).$$

Substitution of this representation into equations (1.3) gives

$$R = k, \quad U = C,$$

where k and c are constants.

5.4 Invariant solutions of Y_3

Invariants of the generator

$$Y_3 = t\partial_x + \partial_u$$

are

$$t,\ \rho,\ u - x/t.$$

An invariant solution has the representation

$$u = U(t) + x/t,\ \rho = R(t).$$

Substitution of this representation into equations (1.3) gives

$$R = k/t, \quad U = Ce^{-t},$$

where k and c are constants.

5.5 Invariant solutions of $Y_1 + Y_3$

Invariants of the generator

$$Y_1 + Y_3 = \partial_t + t\partial_x + \partial_u$$

are

$$x - t^2/2,\ \rho,\ u - t.$$

An invariant solution has the representation

$$u = U(z) + t,\ \rho = R(z),$$

where $z = x - t^2/2$. Substitution of this representation into equations (1.3) gives

$$U = k/R$$

and

$$\begin{gathered}
2p(1-2p)R^{'''}(R^{'})^{2p+4}R^4\varphi + 4p(1-2p)(p-1)(R^{''})^2(R^{'})^{2p+3}R^4\varphi \\
+4p(1-2p)R^{''}(R^{'})^{2p+5}R^3(R\varphi^{'} + \varphi) + (1-2p)(R^{'})^{2p+7}R^3(R\varphi^{''} + 2\varphi^{'}) \\
-k^2(R^{'})^7 + (R^{'})^6R^3 = 0,
\end{gathered}$$

where k is a constant.

5.6 Invariant solutions of Y_4

Invariants of the generator

$$Y_4 = x\partial_x + (p+1)t\partial_t - pu\partial_u - 2\alpha\partial_\alpha$$

are

$$xt^{-1/(p+1)}, \quad \rho, \quad x^p u.$$

An invariant solution has the representation

$$u = x^{-p}U(z), \quad \rho = R(z),$$

where $z = xt^{-1/(p+1)}$. Substitution of this representation into equations (1.3) gives

$$(p+1)zR^{'}U - z^{p+2}R^{'} + (p+1)zU^{'}R - p(p+1)UR = 0$$

and

$$\begin{gathered}
2p(1-2p^3-3p^2)z^{2p+1}\varphi R^{'''}(R^{'})^{2p+1}R^2 \\
+4p(\varphi^{'}R - 2\varphi^{'}Rp^3 - 3\varphi^{'}Rp^2 - 2\varphi p^3 - 3\varphi p^2 + \varphi)z^{2p+1}R^{''}(R^{'})^{2p+2}R \\
+4p(p-2p^4-p^3+3p^2-1)z^{2p+1}\varphi(R^{''})^2(R^{'})^{2p}R^2 \\
-(p+1)^2z(R^{'})^4U^2 + 2(p+1)z^{p+2}(R^{'})^4U + z^{2p+1}(R^{'})^4(-2(R^{'})^{2p}\varphi^{''}R^2p^3 \\
-3(R^{'})^{2p}\varphi^{''}R^2p^2 + (R^{'})^{2p}\varphi^{''}R^2 - 4(R^{'})^{2p}\varphi^{'}Rp^3 - 6(R^{'})^{2p}\varphi^{'}Rp^2 \\
+2(R^{'})^{2p}\varphi^{'}R - z^2) - z^{p+1}p(p+1)(R^{'})^3UR = 0.
\end{gathered}$$

CHAPTER VI

CONCLUSIONS

This thesis is devoted to an application of group analysis to the one-dimensional motion of fluids with internal energy depending on the density and the gradient of the density,

$$\rho_t + (\rho u)_x = 0, \ \ (\rho u)_t + (\rho u^2 + \Pi)_x = 0,$$

where

$$\Pi = P + \rho\lambda\rho_x^2, \ P = \rho^2 \frac{\partial \varepsilon}{\partial \rho} - \rho \frac{\partial}{\partial x}\left(\rho\lambda\rho_x\right), \ \lambda = 2\frac{\partial \varepsilon}{\partial \alpha}, \ \alpha = \rho_x^2,$$

t is time, ρ_x is the gradient of ρ with respect to the space variable x, P is the pressure and $\varepsilon(\rho, \alpha)$ is the internal energy. The case $\varepsilon = \sqrt{\alpha}\varphi_1(\rho) + \varphi_2(\rho)$, where φ_1 and φ_2 are arbitrary functions is excluded from the thesis consideration because it was studied easily.

Complete group classification of the equations is given. The classification is considered with respect to the function $\varepsilon(\rho, \alpha)$. The group classification separates all models of the one-dimensional motion of fluids with internal energy depending on the density and the gradient of the density into 21 classes, which are presented in Table 1. The set of all models is separated into two types. The first type consists of models for which the admitted Lie algebra includes the generator X_4. The second type of models for which the admitted Lie algebra does not include the generator X_4.

All invariant solutions of the particular model where the internal energy function $\varepsilon(\rho, \alpha) = \varphi(\rho)\alpha^p$ are obtained. For this we classified all subalgebras of the Lie group admitted the equations by constructed optimal system of admitted subalgebras found invariants of the subalgebra and then studied invariant solution.

In the future work I will analyze the three-dimensional motion of fluids with internal energy depending on the density and the gradient of the density, and I will also apply group analysis to the motion of fluids with internal energy depending on the density ρ, the gradient of the density $\nabla\rho$ and the entropy S, $\varepsilon = \varepsilon(\rho, \alpha, S)$.

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