การจำแนกประเภทกลุ่มของสมการของไหลสามมิติ
ซึ่งมีความเฉื่อยภายใน

นางปิยะนุช ศิริวัฒน์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์
มหาวิทยาลัยเทคโนโลยีสุรนารี
ปีการศึกษา 2551

GROUP CLASSIFICATION OF THE THREE-DIMENSIONAL EQUATIONS OF FLUIDS WITH INTERNAL INERTIA

Piyanuch Siriwat

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

Suranaree University of Technology

Academic Year 2008

GROUP CLASSIFICATION OF THE THREE-DIMENSIONAL EQUATIONS OF FLUIDS WITH INTERNAL INERTIA

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for a Doctoral Degree.

Thesis Examining Committee

____

(Assoc. Prof. Dr. Prapasri Asawakun)

Chairperson

____

(Prof. Dr. Sergey Meleshko)

Member (Thesis Advisor)

____

(Dr. Kuntima Khomrod)

Member

____

(Assoc. Prof. Dr. Nikolay Moshkin)

Member

____

(Asst. Prof. Dr. Eckart Schulz)

Member

____

(Prof. Dr. Pairote Sattayatham)

Vice Rector for Academic Affairs

____

(Assoc. Prof. Dr. Prapan Manyum)

Dean of Institute of Science

ปิยะนุช ศิริวัฒน์ : การจำแนกประเภทกลุ่มของสมการของไหลสามมิติซึ่งมีความเฉื่อยภายใน (GROUP CLASSIFICATION OF THREE-DIMENSIONAL EQUATIONS OF FLUIDS WITH INTERNAL INERTIA) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร. เซอร์เก เมเลชโก, 85 หน้า.

วิทยานิพนธ์นี้ศึกษาเกี่ยวกับการประยุกต์การจำแนกประเภทเชิงกลุ่มของสมการของไหลสามมิติซึ่งมีความเฉื่อยภายใน โดยมีฟังก์ชันศักย์ขึ้นอยู่กับ $W(\rho,\dot{\rho})$ ในส่วนแรกกล่าวถึงผลของการจำแนกประเภทเชิงกลุ่มของสมการของไหลสามมิติ โดยได้กลุ่มลีสมมูลการจำแนกเชิงกลุ่ม ทำให้สามารถจำแนกของไหลที่มีความเฉื่อยภายในโดยเทียบกลุ่มลีแอดมีตเตดใน 15 กรณีที่แตกต่างกัน ในส่วนที่สองของวิทยานิพนธ์ศึกษาคำตอบของความวนเวียนพิเศษที่ตรงกันข้ามกับกรณีของสมการนาเวียร์-สโตค ซึ่งแสดงการมีอยู่ของผลเฉลยสำหรับกรณีของไหลที่มีความเฉื่อยภายใน ในกรณีนี้ระบบสมการสามมิติถูกลดรูปเป็นระบบที่มีตัวแปรอิสระสองตัวทำให้ได้การจำแนกประเภทเชิงกลุ่มของระบบลดรูปนี้ ต่อไปจะศึกษาผลเฉลยยืนยงทั้งหมดของระบบลดรูปนี้โดยมีฟังก์ชันศักย์ในรูป $W=-q_0\rho^{-5/3}\dot{\rho}^2+\beta\rho^{5/3}$ ในส่วนที่สามของวิทยานิพนธ์ได้หาผลเฉลยยืนยงของของไหลซึ่งมีฟังก์ชันศักย์ในรูป $W=-a\rho^{-3}\dot{\rho}^2+\beta\rho^3$ และได้แสดงการวิเคราะห์ระบบลดรูปนี้ด้วย

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา ________________________

ปีการศึกษา 2551 ลายมือชื่ออาจารย์ที่ปรึกษา __________________

PIYANUCH SIRIWAT : GROUP CLASSIFICATION OF THE THREE-DIMENSIONAL EQUATIONS OF FLUIDS WITH INTERNALINERTIA THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D. 85 PP.

GROUP CLASSIFICATION / EQUIVALENCE LIE GROUP / ADMITTED LIE GROUP / OPTIMAL SYSTEM OF SUBALGEBRAS / INVARIANT AND PARTIALLY INVARIANT SOLUTIONS.

This thesis is devoted to applications of the group analysis method to the equations of fluids with internal inertia where the potential function is of the form $W(\rho, \dot{\rho})$. The first result of the thesis is group classification of the three-dimensional equations. The equivalence Lie group is obtained at this step. Group classification separates fluids with internal inertia with respect to the admitted Lie group into 15 different cases. The second part of the thesis is devoted to the special vortex solution. In contrast to the Navier-Stokes equations, the existence of solutions for this class of equations and fluids with internal inertia has been shown. For this class the original three-dimensional system of equations is reduced to a system with two independent variables. Group classification of the reduced system is obtained. All invariant solutions of the reduced system with the potential function $W = -q_0\rho^{-5/3}\dot{\rho}^2 + \beta\rho^{5/3}$ are studied. In the third part of the thesis all one-dimensional invariant solutions of fluids with the potential function $W = -a\rho^{-3}\dot{\rho}^2 + \beta\rho^3$ are obtained. Analysis of the reduced equations is provided.

School of Mathematics — Student's Signature ________________

Academic Year 2008 — Advisor's Signature ________________

ACKNOWLEDGEMENT

I am profoundly grateful to my thesis advisor Prof. Dr. Sergey Meleshko for his support, patient help and offering many useful suggestions.

I would like to acknowledge the personal and professional support received from the faculty of the School of Mathematics, Suranaree University of Technology: Assoc. Prof. Dr. Prapasri Assawakun, Assoc. Prof. Dr. Nikolay Moshkin, Asst. Prof. Dr. Eckart Schulz.

I would like to thank Asst. Prof. Dr. Apichai Hematulin and Dr. Kuntima Khomrod and my friends who always have given their friendly help.

I acknowledge the financial support of the Ministry of University Affairs of Thailand (MUA). I am indebted to Mae Fah Luang University for grants to support my studies.

Finally, I am deeply grateful to my only son, Paphawin for his cheerfulness, and love.

Piyanuch Siriwat

CONTENTS

CONTENTS (Continued)

CONTENTS (Continued)

LIST OF TABLES

CHAPTER I

INTRODUCTION

Mathematical modelling is a basis for analyzing physical phenomena by partial differential equations. Almost all fundamental equations of physics are nonlinear, and in general, are very difficult to solve explicitly. Numerical methods are often used with much success for obtaining approximate, not exact solutions. Hence, there is interest in obtaining exact solutions of nonlinear equations. Each solution has value, firstly, as an exact description of a real process in the framework of a given model; secondly, as a model to compare with various numerical methods; thirdly, as a basis to improve the models used.

Group analysis is one of the methods for constructing particular exact solutions of partial differential equations. This method makes use of symmetry properties of differential equations. Symmetry means that any solution of a given system of partial differential equations is transformed by a Lie group of transformations to a solution of the same system. Moreover, a symmetry allows finding new solutions of the system. There are two types of solutions which can be obtained by group analysis: invariant and partially invariant solutions. Many applications of group analysis to partial differential equations are collected in Handbook (1994-1996). Group analysis, besides constructing exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements.

In the thesis, the group analysis method is applied to one class of dispersive models (Gavrilyuk and Shugrin, 1996), (Anderson, McFadden and Wheeler, 1998), (Gavrilyuk and Teshukov, 2001). See also references therein. The equations describing the behavior of a dispersive continuum are obtained as an Euler-Lagrange equation for the Lagrangian of the form

$$L = L(\rho, \rho_t, \nabla\rho, u)$$

where t is time, ∇ is the gradient operator with respect to space variables, ρ is the fluid density, u is the velocity field. The density ρ and the velocity satisfy the mass conservation equation

$$\rho_t + div(\rho u) = 0 \tag{1.1}$$

and the equation of conservation of linear momentum

$$\rho\dot{u} + \nabla p = 0, \tag{1.2}$$

where p is the presssure, and "dot" denotes the material time derivative: $\dot{f} = f_t + u\nabla f$. In the literature, there are two classes of dispersive models. One class of models, describing diffusive-interface behavior of fluids, is obtained for the internal energy $\varepsilon(\rho, \nabla\rho)$. The thesis is focused on group classification of another class of dispersive models. These models are defined by $L = \frac{1}{2}\rho|u|^2 - W(\rho, \dot{\rho})$, where $W(\rho, \dot{\rho})$ is a potential function. For this class of models the pressure is

$$p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\dot{\rho}}\right) - div\left(\frac{\partial W}{\partial\dot{\rho}}u\right)\right) - W, \tag{1.3}$$

where $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u. These models include the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski, 1960), (Kogarko, 1961), (Wijngaarden , 1968) and the dispersive shallow water model (Green and Naghdi, 1975), (Salmon, 1998).

The thesis is devoted to group classification of three-dimensional equations (1.1), (1.2), (1.3), where the function $W_{\dot{\rho}\dot{\rho}}$ satisfies the condition $W_{\dot{\rho}\dot{\rho}} \neq 0$. Notice that for $W_{\dot{\rho}\dot{\rho}} = 0$ or $W(\rho, \dot{\rho}) = \dot{\rho}\varphi(\rho) + \psi(\rho)$ the momentum equation becomes

$$\dot{u} + \psi'' \rho_x = 0.$$

Hence, in the case $W_{\dot{\rho}\dot{\rho}} = 0$ equations (1.1), (1.2), (1.3) are similar to the gas dynamics equations. This case was completely studied in (Chirkunov, 1989). the one-dimensional case of equations (1.1), (1.2), (1.3) was studied in (Hematulin, Meleshko and Gavrilyuk, 2007). Similar to the gas dynamics equations there are differences in the group classifications of one-dimensional and three-dimensional equations. Part of the thesis is devoted to the analysis of invariant and partially invariant solutions.

Applications of the group analysis method require to carry out a lot of complicated symbolic manipulations. Because this is a very laborious part, a computer was used for these tasks. All calculations were done with the REDUCE program (Hearn, 1999).

The thesis is organized as follows. Chapter II introduces equations describing the behavior of fluids with internal inertia. The representations of the function $W(\rho, \dot{\rho})$ for the Iordanski-Kogarko-Wijngaarden and Green-Naghdi models are given in this chapter. Chapter III gives notations of group analysis and provides references to known facts on application of group analysis to partial differential equations. Chapter IV is devoted to group classification of the three-dimensional equations (1.1), (1.2), (1.3). Group classification separates all set of models into 15 classes. Chapter V studies one class of partially invariant solutions: special vortex. Complete group classification of the reduced system of equations for invariant functions is given. Chapter VI studies invariant solutions of the one-dimensional case of the model with $W(\rho, \dot{\rho}) = -a\rho^{-3}\dot{\rho}^2 + \beta\rho^3$: optimal systems of subalgebras are constructed, representations of all invariant solutions are obtained. All invariant solutions for the reduced system of partial differential equations are presented.

CHAPTER II

FLUIDS WITH INTERNAL INERTIA

Equations of fluids with internal inertia are obtained on the base of the Euler-Lagrange principle with the Lagrangian

$$L = L(\rho, \rho_t, \nabla\rho, u),$$

where t is time, ∇ is the gradient operator with respect to the space variables x_1, x_2, x_3, ρ is the fluid density, $u = (u_1, u_2, u_3)$ is the velocity field. The density ρ and the velocity u satisfy the mass conservation equation and the equation of conservation of linear momentum

$$\dot{\rho} + \rho\, div(u) = 0, \quad \rho\dot{u} + \nabla p = 0, \tag{2.1}$$

where $\dot{()} = \partial/\partial t + u\nabla$ is the material derivative.

Among fluids with internal inertia two classes of models have been intensively studied. One class of models is constructed, assuming that the internal energy ε depends on the density ρ and the gradient of the density $|\nabla\rho|$. Review of these models can be found in (Gavrilyuk and Shugrin, 1996), (Anderson, McFadden and Wheeler, 1998) and references therein. The thesis is devoted to the study of another class of models. These models are obtained by assuming that the Lagrangian is of the form (Gavrilyuk and Teshukov, 2001):

$$L = \frac{1}{2}\rho|u|^2 - W(\rho, \dot{\rho}), \tag{2.2}$$

where $W(\rho, \dot{\rho})$ is a given potential. In this case the pressure p is given by the formula

$$p = \rho\left(\frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t}(\frac{\partial W}{\partial \dot{\rho}}) - div(\frac{\partial W}{\partial \dot{\rho}}u)\right) - W. \tag{2.3}$$

Notice that if W is a linear function with respect to $\dot{\rho}$, then these equations are reduced to the classical Euler equations of a barotropic gas.

In the next sections we give examples of two the most well-known models.

2.1 Iordanski-Kogarko-Wijngaarden model

The Iordanski-Kogarko-Wijngaarden model describes a bubbly fluid with incompressible liquid phase and small volume concentration of gas bubbles. This type of model was proposed by Iordanski (1960), Kogarko (1961) and Wijngaarden (1968).

This mathematical model can be written in the form

$$\begin{aligned} &\frac{\partial \rho_1}{\partial t} + div(\rho_1 u) = 0, \\ &\frac{\partial \rho_2}{\partial t} + div(\rho_2 u) = 0, \\ &\frac{\partial N}{\partial t} + div(N u) = 0, \\ &\dot{u} + \frac{1}{\rho}\nabla p = 0, \\ &R\ddot{R} + \frac{3}{2\dot{R}^2} = \frac{1}{\rho_{10}}(p_2 - p), \end{aligned} \tag{2.4}$$

where

$$\rho_1 = \alpha_1 \rho_{10}, \quad \rho_2 = \alpha_2 \rho_{20}, \tag{2.5}$$

$\rho_{10} = const$ is the physical density of the liquid, ρ_{20} is the physical density of the gas, $\alpha_i, \ (i = 1, 2)$ are the volume fractions: $\alpha_1 + \alpha_2 = 1$, N is the bubble number density, R is the bubble radius,

$$\rho = \rho_1 + \rho_2. \tag{2.6}$$

The volume fraction of the gas phase α_2 is defined by the formula

$$\alpha_2 = \frac{4}{3}\pi R^3 N. \tag{2.7}$$

The gas pressure p_2 is a given function of ρ_{20}:

$$p_2 = \rho_{20}^2 \varepsilon_{20}'(\rho_{20}),$$

where $\varepsilon_{20}(\rho_{20})$ is the internal energy of the gas phase. It is assumed that the mass concentrations $c_i = \rho_i/\rho,\ (i = 1, 2)$, and the number of bubbles per unit mass $n = N/\rho$ are constant. From (2.5)-(2.7) one obtains

$$R^3 = \frac{3}{4\pi n}(\frac{1}{\rho} - \frac{c_1}{\rho_{10}}),\ \ \rho_{20} = c_2(\frac{1}{\rho} - \frac{c_1}{\rho_{10}})^{-1}.$$

Bedford and Drumheller (1978) proved that equations (2.1), (2.3) can be obtained by using the potential function

$$W = \rho(C_2\varepsilon_{20}(\rho_{20}) - 2\pi n\rho_{10}R^3\dot{R}^2).$$

Replacing R and ρ_{20} in the potential function, one obtains that system of partial differential equations (2.4) is equivalent to (2.1) and (2.3) with the potential function

$$W(\rho, \dot{\rho}) = \psi(\rho) - k\dot{\rho}^2\rho^{\frac{8}{3}}(\frac{1}{\alpha - \rho})^{1/3},$$

where

$$\alpha = \frac{\rho_{10}}{c_1}, \qquad k = \frac{\rho_{10}}{8\pi n}(\frac{4\pi n\rho_{10}}{3c_1})^{1/3}.$$

2.2 Green-Naghdi model

Consider the dispersive shallow water equations of Green and Naghdi (1975)

$$\frac{\partial h}{\partial t} + div(hu) = 0 \tag{2.8}$$

$$\dot{u} + g\nabla h + \frac{\varepsilon^2}{(3h)}\nabla(h^2\ddot{h}) = 0, \tag{2.9}$$

where h is the water depth, u is the horizontal velocity, g is the gravity, ε is the ratio of the vertical length scale to the horizontal length scale. Replacing h by ρ, equations (2.8) take the form

$$\begin{aligned} &\frac{d\rho}{dt} + \rho div(u) = 0 \\ &\dot{u} + g\nabla\rho + \frac{\varepsilon^2}{3\rho}\nabla(\rho^2\ddot{\rho}) = 0. \end{aligned} \tag{2.10}$$

The last equation of (2.10) can be rewritten as

$$\rho\dot{u} + \nabla p = 0 \tag{2.11}$$

where

$$p = \frac{g}{2}\rho^2 + \frac{\varepsilon^2}{3}\rho^2\ddot{\rho}. \tag{2.12}$$

Introducing the potential function

$$W = \frac{g}{2}\rho^2 - \frac{\varepsilon^2}{6}\rho\dot{\rho}^2 \tag{2.13}$$

and substituting it into (2.3), one arrives at the Green-Naghdi model which is presented in the form (2.1) and (2.3) with the potential function (2.12).

The group analysis method was applied to one-dimensional equations (2.8) and (2.10) in Bagderina and Chupakhin (2005).

CHAPTER III

GROUP ANALYSIS METHOD

In this chapter, the group analysis method is discussed. An introduction to this method can be found in various textbooks (cf. Ovsiannikov 1978), (Olver, 1986), (Ibragimov, 1999), (Meleshko, 2005).

3.1 Lie Groups

Consider a set of invertible point transformations

$$\bar{z}^i = \varphi^i(z; a), \ a \in \Delta, \ z \in V, \tag{3.1}$$

where $i = 1, 2, ..., N$, a is a parameter, and Δ is a symmetric interval in R^1. The set V is an open set in R^N.

If $z = (x, u)$, then we use the notation $\varphi = (f, g)$. Here $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the vector of the independent variables, and $u = (u^1, u^2, ..., u^m) \in \mathbb{R}^m$ is the vector of the dependent variables. The transformation of the independent variables x, and the dependent variables u has the form

$$\bar{x}_i = f^i(x, u; a), \ \bar{u}^j = g^j(x, u; a), \tag{3.2}$$

where $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, $(x, u) \in V \subset R^n \times R^m$, and the set V is open in $R^n \times R^m$.

3.1.1 One-Parameter Lie-Group of Transformations

Definition 1. A set of transformations (3.1) is called a local one-parameter Lie group if it has the following properties

1. $\varphi(z;0) = z$ for all $z \in V$;
2. $\varphi(\varphi(z;a),b) = \varphi(z;a+b)$ for all $a, b, a+b \in \Delta, z \in V$;
3. If for $a \in \Delta$ one has $\varphi(z;a) = z$ for all $z \in V$, then $a = 0$;
4. $\varphi \in C^\infty(V,\Delta)$.

The Lie group of transformations (3.2) is called a one-parameter Lie group of point transformations. For a Lie group of point transformations, the functions f^i and g^j can be written by Taylor series expansion with respect to the group parameter a in a neighborhood of $a = 0$

$$\begin{aligned} \bar{x}_i &= x_i + a\left.\frac{\partial f^i}{\partial a}\right|_{a=\,0} + O(a^2), \\ \bar{u}^j &= u^j + a\left.\frac{\partial g^j}{\partial a}\right|_{a=\,0} + O(a^2). \end{aligned} \tag{3.3}$$

The transformations $\tilde{x_i} \approx x_i + a\xi^{x_i}(x,u)$ and $\tilde{u^j} \approx u^j + a\zeta^{u^j}(x,u)$ are called infinitesimal transformations of the Lie group of transformations (3.2), where

$$\xi^{x_i}(x,u) = \left.\frac{\partial f^i(x,u;a)}{\partial a}\right|_{a=\,0}, \zeta^{u^j}(x,u) = \left.\frac{\partial g^j(x,u;a)}{\partial a}\right|_{a=\,0}.$$

The components $\xi = (\xi^{x_1}, \xi^{x_2}, ..., \xi^{x_n})$, $\zeta = (\zeta^{u^1}, \zeta^{u^2}, ..., \zeta^{u^m})$ are called the infinitesimal representation of (3.2). This can be written in terms of the first-order differential operator

$$X = \xi^{x_i}(x,u)\partial_{x_i} + \zeta^{u^j}(x,u)\partial_{u^j}. \tag{3.4}$$

This operator X is called an infinitesimal generator.

There is a theorem, which relates a one-parameter Lie group G with its infinitesimal generator.

Theorem 1 (Lie). Let functions $f^i(x,u;a)$, $i = 1,...,n$ and $g^j(x,u;a)$, $j = 1,...,m$ satisfy the group properties and have the expansion

$$\bar{x}_i = f^i(x,u;a) \approx x_i + \xi^{x_i}(x,u)a,$$

$$\bar{u}^j = g^j(x,u;a) \approx u^j + \zeta^{u^j}(x,u)a$$

where

$$\xi^{x_i}(x,u) = \left.\frac{\partial f^i(x,u;a)}{\partial a}\right|_{a=0}, \zeta^{u^j}(x,u) = \left.\frac{\partial g^j(x,u;a)}{\partial a}\right|_{a=0}.$$

Then it solves the Cauchy problem

$$\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}a} = \xi^{x_i}(\bar{x},\bar{u}), \ \frac{\mathrm{d}\bar{u}^j}{\mathrm{d}a} = \zeta^{u^j}(\bar{x},\bar{u}) \tag{3.5}$$

with the initial data

$$\bar{x}_i|_{a=0} = x_i, \ \bar{u}^j\big|_{a=0} = u^j. \tag{3.6}$$

Conversely, given $\xi^{x_i}(x,u)$ and $\zeta^{u^j}(x,u)$, the solution of the Cauchy problem (3.5), (3.6) forms a Lie group.

Equations (3.5) are called the Lie equations.

To apply a Lie group of transformations (3.2) for studying differential equations one needs to know how this group acts on the functions $u^j(x)$ and their derivatives. For the sake of simplicity, let us explain the basic idea for the case $n = 1$ and $m = 1$. Assume that $u_0(x)$ is a given known function, and the transformation is

$$\begin{aligned} \bar{x} &= f(x,u;a) \approx x + a\xi^x(x,u) \\ \bar{u} &= g(x,u;a) \approx u + a\zeta^u(x,u). \end{aligned} \tag{3.7}$$

Substituting $u_0(x)$ into the first equation (3.7), one obtains

$$\bar{x} = f(x, u_0(x); a).$$

Since $f(x,u_0(x);0) = x$, the Jacobian at $a = 0$ is

$$\left.\frac{\partial \bar{x}}{\partial x}\right|_{a=0} = \left.\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x}\right)\right|_{a=0} = 1.$$

Thus, by virtue of the inverse function theorem, in some neighborhood of $a = 0$ one can express x as a function of $\bar{x}$ and a,

$$x = \theta(\bar{x}, a). \tag{3.8}$$

Note that after substituting (3.8) into the first equation (3.7), one has the identity

$$\bar{x} = f(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{3.9}$$

Substituting (3.8) into the second equation (3.7), one obtains the transformed function

$$u_a(\bar{x}) = g(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{3.10}$$

Differentiating equation (3.10) with respect to $\bar{x}$, one gets

$$\bar{u}_{\bar{x}} = \frac{\partial u_a(\bar{x})}{\partial \bar{x}} = \frac{\partial g}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial g}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}u_0'(x)\right)\frac{\partial \theta}{\partial \bar{x}},$$

where the derivative $\frac{\partial \theta}{\partial \bar{x}}$ can be found by differentiating equation (3.9) with respect to $\bar{x}$,

$$1 = \frac{\partial f}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right)\frac{\partial \theta}{\partial \bar{x}}.$$

Since

$$\frac{\partial f}{\partial x}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 1, \ \frac{\partial f}{\partial u}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 0, \tag{3.11}$$

one has $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x) \neq 0$ in some neighborhood of $a = 0$. Thus,

$$\frac{\partial \theta}{\partial \bar{x}} = \frac{1}{\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right)},$$

and

$$\bar{u}_{\bar{x}} = \frac{\frac{\partial g(x,u_0;a)}{\partial x} + \frac{\partial g(x,u_0;a)}{\partial u}u_0'(x)}{\frac{\partial f(x,u_0;a)}{\partial x} + \frac{\partial f(x,u_0;a)}{\partial u}u_0'(x)} = h(x, u_0(x), u_0'(x); a).$$

Transformation (3.2) together with

$$\bar{u}_{\bar{x}} = h(x, u, u_x; a) \tag{3.12}$$

is called the prolongation of (3.2).

As before, the function h can be written by Taylor series expansion with respect to the parameter a in a neighborhood of the point $a = 0$:

$$\bar{u}_{\bar{x}} = h(x, u, u_x; a) \approx u_x + a\zeta^{u_x}(x, u, u_x), \tag{3.13}$$

where

$$\zeta^{u_x}(x,u,u_x) = \left.\frac{\partial h(x,u,u_x;a)}{\partial a}\right|_{a=0}, \quad h|_{a=0} = u_x.$$

Equation (3.12) can be rewritten as

$$h(x,u,u_x;a)\left(\frac{\partial f(x,u;a)}{\partial x} + u_x\frac{\partial f(x,u;a)}{\partial u}\right) = \left(\frac{\partial g(x,u;a)}{\partial x} + u_x\frac{\partial g(x,u;a)}{\partial u}\right).$$

Differentiating this equation with respect to the group parameter a and substituting $a = 0$, one finds

$$\left.\left(\frac{\partial h}{\partial a}\left(\frac{\partial f}{\partial x} + u_x\frac{\partial f}{\partial u}\right) + h\left(\frac{\partial^2 f}{\partial x\partial a} + u_x\frac{\partial^2 f}{\partial u\partial a}\right)\right)\right|_{a=0} = \left.\left(\frac{\partial^2 g}{\partial x\partial a} + u_x\frac{\partial^2 g}{\partial u\partial a}\right)\right|_{a=0}$$

or

$$\begin{aligned}
\zeta^{u_x}(x,u,u_x) &= \left.\frac{\partial h}{\partial a}\right|_{a=0}\left.\left(\frac{\partial f}{\partial x} + u_x\frac{\partial f}{\partial u}\right)\right|_{a=0} \\
&= \left.\left(\frac{\partial^2 g}{\partial x\partial a} + u_x\frac{\partial^2 g}{\partial u\partial a}\right)\right|_{a=0} - h|_{a=0}\left.\left(\frac{\partial^2 f}{\partial x\partial a} + u_x\frac{\partial^2 f}{\partial u\partial a}\right)\right|_{a=0} \\
&= \left(\frac{\partial\zeta^u}{\partial x} + u_x\frac{\partial\zeta^u}{\partial u}\right) - u_x\left(\frac{\partial\xi^x}{\partial x} + u_x\frac{\partial\xi^x}{\partial u}\right) \\
&= D_x(\zeta^u) - u_x D_x(\xi^x)
\end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \ldots, \quad \xi^x = \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad \zeta^u = \left.\frac{\partial g}{\partial a}\right|_{a=0}, \quad \zeta^{u_x} = \left.\frac{\partial h}{\partial a}\right|_{a=0}.$$

The first prolongation of the generator (3.4) is given by

$$X^{(1)} = X + \zeta^{u_x}(x,u,u_x)\partial_{u_x}.$$

In the same way, one obtains the infinitesimal transformation of the second derivative

$$\bar{u}_{\bar{x}\bar{x}} \approx u_{xx} + a\zeta^{u_{xx}}(x,u,u_x,u_{xx}),$$

where $\zeta^{u_{xx}} = D_x(\zeta^{u_x}) - u_{xx}D_x(\xi^x)$, and the second prolongation of the generator (3.4) is

$$X^{(2)} = X^{(1)} + \zeta^{u_{xx}}(x,u,u_x,u_{xx})\partial_{u_{xx}}.$$

For constructing prolongations of an infinitesimal generator in case $n, m \geq 2$ one proceeds similarly.

Let $x = \{x_i\}$ be the set of independent variables and $u = \{u^j\}$ the set of dependent variables. The derivatives of the dependent variables are given by the sets $u_{(1)} = \left\{u_i^j\right\},\ u_{(2)} = \left\{u_{is}^j\right\}, \ldots$, where $j = 1, \ldots, m$ and $i, s = 1, \ldots, n$. The derivatives of the differentiable functions u^j can be written in terms of the total differentiation operator D_i :

$$\begin{aligned} u_i^j &= D_i(u^j), \\ u_{is}^j &= D_s(u_i^j), \end{aligned}$$

where

$$D_i = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + u_{is}^j \frac{\partial}{\partial u_s^j} + \ldots, \quad (i, s = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \quad (3.14)$$

The formula of the first prolongation of the generator $X = \xi^{x_i}(x,u)\partial_{x_i} + \zeta^{u^j}(x,u)\partial_{u^j}$ is

$$X^{(1)} = X + \zeta^{u_i^j}(x, u, u_{(1)})\partial_{u_i^j},$$

where

$$\zeta^{u_i^j} = D_i\left(\zeta^{u^j}\right) - u_s^j D_i\left(\xi^{x_s}\right) \quad ;\ i, s = 1, ..., n \ \ ;\ j = 1, ..., m.$$

The second prolongation of the generator X is

$$X^{(2)} = X^{(1)} + \zeta^{u_{i_1,i_2}^j}(x, u, u_{(1)}, u_{(2)})\partial_{u_{i_1,i_2}^j},$$

where

$$\zeta^{u_{i_1,i_2}^j} = D_{i_2}\left(\zeta^{u_{i_1}^j}\right) - u_{i_1,s}^j D_{i_2}\left(\xi^{x_s}\right) \quad ;\ i_1, i_2, s = 1, ..., n \ \ ;\ j = 1, ..., m. \quad (3.15)$$

In the general case, the k-th prolongation of the generator X is

$$X^{(k)} = X^{(k-1)} + \zeta^{u_{i_1,\ldots,i_k}^j}(x, u, u_{(1)}, ..., u_{(k)})\partial_{u_{i_1,\ldots,i_k}^j}$$

where

$$\zeta^{u^j_{i_1,\ldots,i_k}} = D_{i_k}\left(\zeta^{u^j_{i_1,\ldots,i_{k-1}}}\right) - u^j_{i_1,\ldots,i_{k-1},s} D_{i_k}\left(\xi^{x_s}\right);\ i_1,\ldots,i_k,s = 1,\ldots,n;\ j = 1,\ldots,m.$$

Lie groups of transformations are related with differential equations by the following.

Definition 2. Given a partial differential equation, a Lie group of transformations, which transforms a solution $u_0(x)$ into a solution $u_a(x)$ of the same equation is called an admitted Lie group of transformations.

Let $\mathcal{F} = (F^1, \ldots, F^k),\ k = 1, \ldots, N$ be differential functions of order p. The equations

$$F^k\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(p)}\right) = 0,\ k = 1, \ldots, N \tag{3.16}$$

compose a manifold $[\mathcal{F} = 0]$ in the space of the variables $x, u, u_{(1)}, u_{(2)}, \ldots, u_{(p)}$.

After applying an admitted Lie group of transformations to a solution $u(x)$, one has

$$F^k\left(\bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \ldots, \bar{u}_{(p)}\right) = 0,\ (k = 1, \ldots, N). \tag{3.17}$$

Differentiating these equations with respect to the group parameter a, and substituting $a = 0$, one finds

$$\left.\left(\frac{\partial F^k}{\partial x_i}\frac{\partial \bar{x}_i}{\partial a} + \frac{\partial F^k}{\partial u^j}\frac{\partial \bar{u}^j}{\partial a} + \frac{\partial F^k}{\partial u^j_{i_1}}\frac{\partial \bar{u}^j_{i_1}}{\partial a} + \ldots + \frac{\partial F^k}{\partial u^j_{i_1,i_2,\ldots,i_p}}\frac{\partial \bar{u}^j_{i_1,i_2,\ldots,i_p}}{\partial a}\right)\right|_{a=0} = 0$$

or

$$\xi^{x_i}\frac{\partial F^k}{\partial x_i} + \zeta^{u^j}\frac{\partial F^k}{\partial u^j} + \zeta^{u^j_{i_1}}\frac{\partial F^k}{\partial u^j_{i_1}} + \zeta^{u^j_{i_1,i_2}}\frac{\partial F^k}{\partial u^j_{i_1,i_2}} + \ldots + \zeta^{u^j_{i_1,i_2,\ldots,i_p}}\frac{\partial F^k}{\partial u^j_{i_1,i_2,\ldots,i_p}} = 0,$$

where

$$\xi^{x_i} = \left.\frac{\partial \bar{x}_i}{\partial a}\right|_{a=0},\quad \zeta^{u^j} = \left.\frac{\partial \bar{u}^j}{\partial a}\right|_{a=0},\quad \zeta^{u^j_{i_1}} = \left.\frac{\partial \bar{u}^j_{i_1}}{\partial a}\right|_{a=0},\ \ldots,\quad \zeta^{u^j_{i_1,\ldots,i_p}} = \left.\frac{\partial \bar{u}^j_{i_1,\ldots,i_p}}{\partial a}\right|_{a=0}.$$

The last equation can be expressed as an action of the prolonged infinitesimal generator

$$X^{(p)}F^k\Big|_{[\mathcal{F}=0]} = 0,\ \ (k = 1, ..., N), \tag{3.18}$$

where

$$X^{(p)} = \xi^{x_i}\frac{\partial}{\partial x_i} + \zeta^{u^j}\frac{\partial}{\partial u^j} + \zeta^{u^j_{i_1}}\frac{\partial}{\partial u^j_{i_1}} + \zeta^{u^j_{i_1,i_2}}\frac{\partial}{\partial u^j_{i_1,i_2}} + ... + \zeta^{u^j_{i_1,i_2,\dots,i_p}}\frac{\partial}{\partial u^j_{i_1,i_2,\dots,i_p}}.$$

Hence, in order to find the infinitesimal generator of the Lie group admitted by differential equations (3.16) one can use the following theorem.

Theorem 2. The differential equations (3.16) admits the group G with the generator X, if and only if, the following equations hold:

$$X^{(p)}F^k\Big|_{[\mathcal{F}=0]} = 0,\ \ (k = 1, ..., N). \tag{3.19}$$

Equations (3.19) are called the determining equations.

3.1.2 Multi-Parameter Lie-Group of Transformations

Let O be a ball in the space R^r with a center at the origin. Assume that ψ is a mapping, $\psi : O \times O \longrightarrow R^r$. The pair (O, ψ) is called a local multi-parameter Lie group with the multiplication law ψ if it has the following properties:

1. $\psi(a, 0) = \psi(0, a) = a$ for all $a \in O$;
2. $\psi(\psi(a, b), c) = \psi(a, \psi(b, c))$ for all $a, b, c \in O$ for which $\psi(a, b), \psi(b, c) \in O$;
3. $\psi \in C^\infty(O, O)$.

Let V be an open set in R^N. Consider transformations

$$\bar{z}^i = \varphi^i(z; a), \tag{3.20}$$

where $i = 1, 2, ..., N$, $z \in V$, and $a \in O$ is a vector-parameter.

Definition 3. The set of transformations (3.20) is called a local r-parameter Lie group G^r if it has the following properties:

1. $\varphi(z, 0) = z$ for all $z \in V$.

2. $\varphi(\varphi(z, a), b) = \varphi(z, \psi(a, b))$ for all $a, b, \psi(a, b) \in O,\ z \in V$.

3. If for $a \in O$ one has $\varphi(z, a) = z$ for all $z \in V$, then $a = 0$.

Note that if one fixes all parameters except one, for example a_k, then the multi-parameter Lie group of transformations (3.20) composes a one-parameter Lie group. Conversely, in group analysis it is proven that any r-parameter group is the union of one-parameter subgroups belonging to it.

Let G^r be a Lie group admitted by the system of partial differential equations

$$F^k(x, u, p) = 0,\ k = 1, ..., s.$$

Assume that $\{X_1, X_2, ..., X_r\}$ is a basis of the Lie algebra L^r, which corresponds to the Lie group G^r.

Definition 4. A function $\Phi(x, u)$ is called invariant of a Lie group G^r if

$$\Phi(\bar{x}, \bar{u}) = \Phi(x, u).$$

Theorem 3. A function $\Phi(x, u)$ is an invariant of the group G^r with the generators $X_i,\ (i = 1, ..., r)$ if and only if,

$$X_i\Phi(x, u) = 0,\ (i = 1, ..., r). \tag{3.21}$$

In order to find an invariant, one needs to solve the overdetermined system of linear equations (3.21). A set of functionally independent solutions of (3.21)

$$J = \left(\ J^1(x, u), J^2(x, u), ..., J^{m+n-r_*}(x, u)\ \right)$$

is called an universal invariant. Any invariant Φ can be expressed through this set

$$\Phi = \phi\left(\ J^1(x, u), J^2(x, u), ..., J^{m+n-r_*}(x, u)\ \right).$$

Here n, m is the numbers of independent and dependent variables, respectively and r_* is the total rank of the matrix composed by the coefficients of the generators $X_i,\ (i = 1, 2, ..., r)$.

Definition 5. A set M is said to be invariant with respect to the group G^r, if the transformation (3.20) carries every point z of M to a point of M.

Definition 6. Let V be an open subset of R^N, and $\Psi : V \longrightarrow R^t,\ t \leq N$ a mapping belonging to the class $C^1(V)$. The system of equations $\Psi(z) = 0$ is called regular, if for any point $z \in V$:

$$rank\ \left(\frac{\partial(\ \psi^1, ..., \psi^t)}{\partial\ (\ z_1, ..., z_N\ )}\right) = t$$

where $\Psi = (\psi^1, ..., \psi^t)$.

If a system $\Psi(z) = 0$ is regular, then for each $z_0 \in V$ with $\Psi(z_0) = 0$ there exists a neighborhood U of z_0 in V such that

$$M = \{\ z \in U\ :\ \Psi(z) = 0\ \}$$

is a manifold. Such a manifold is called a regularly assigned manifold.

Theorem 4. A regularly assigned manifold M is an invariant manifold with respect to a Lie group G^r with the generator $X_i,\ (i = 1, ..., r)$, if

$$X_i\psi^k(z)\big|_M = 0,\ (i = 1, ..., r),\ k = 1, ..., t.$$

3.2 Lie algebra

Before giving the definition of a Lie algebra, one needs to introduce the commutator. Let $X_1 = \xi_1^i\partial_{x_i} + \zeta_1^j\partial_{u_j},\ X_2 = \xi_2^i\partial_{x_i} + \zeta_2^j\partial_{u_j}$ be two generators. Let us define a new generator X, denoted by $[X_1, X_2]$, by the following formula

$$X = [X_1, X_2] = \left(X_1\xi_2^i - X_2\xi_1^i\right)\partial_{x_i} + \left(X_1\zeta_2^j - X_2\zeta_1^j\right)\partial_{u_j}.$$

The generator X is called the commutator of the generators X_1, X_2.

Definition 7. A vector space L over the field of real numbers with the operation of commutation $[\,\cdot\,,\,\cdot\,]$ is called a Lie algebra if $[X_1, X_2] \in L$ for any $X_1,\ X_2\ \in L$, and if the operation $[\,\cdot\,,\,\cdot\,]$ satisfies the axioms:

a.1 (bilinearity) : for any $X_1,\ X_2,\ X_3\ \in L$ and $a,\ b\ \in R$

$$\begin{aligned} [aX_1 + bX_2, X_3] &= a\,[X_1, X_3] + b\,[X_2, X_3] \\ [X_1, aX_2 + bX_3] &= a\,[X_1, X_2] + b\,[X_1, X_3] \end{aligned}$$

a.2 (antisymmetry) : for any $X_1,\ X_2\ \in L$

$$[X_1, X_2] = -\,[X_2, X_1]$$

a.3 (the Jacobi identity) : for any $X_1,\ X_2,\ X_3\ \in L$

$$[[X_1, X_2]\,, X_3] + [[X_2, X_3]\,, X_1] + [[X_3, X_1]\,, X_2] = 0.$$

Let L^r be an r-dimensional Lie algebra with basis $X_1, X_2, \ldots, X_r$: i.e., any vector $X \in L^r$ can be decomposed as

$$X = \sum_{k=1}^{r} x_k X_k$$

where x_k are the coordinates of the vector X in the basis $\{X_1, \ldots, X_r\}$. Then

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k; \quad i,\, j = 1, 2, \ldots, r$$

with real constants c_{ij}^k. The numbers c_{ij}^k are called the structural constants of the Lie algebra L^r for the basis $\{X_1, \ldots, X_r\}$.

Definition 8. A vector space $H \subset L$ is called a subalgebra of the Lie algebra L, if $[Y_1, Y_2] \in H$ for any $Y_1, Y_2\ \in H$.

Definition 9. A subalgebra $I \subset L$ is called an ideal of the Lie algebra L if for any $X \in L,\ Y \in I$ it is also true that $[X, Y] \in I$.

3.3 Classification of subalgebras

One of the main aims of group analysis is to construct exact solutions of differential equations. The set of all solutions can be divided into equivalence classes of solutions:

Definition 10. Two solutions u_1 and u_2 of a differential equation are said to be equivalent with respect to a Lie group G, if one of the solutions can be transformed into the other by a transformation belonging to the group G.

The problem of classification of exact solutions is equivalent to the classification of subgroups (or subalgebras) of the group G (or the subalgebra L). Because there is a one-to-one correspondence between Lie groups and Lie algebras let us explain here the classification of subalgebras. For this purpose, one needs the following definitions.

Definition 11. Let L and $\overline{L}$ be Lie algebras. A linear one-to-one map f of L onto $\overline{L}$ is called an isomorphism if it satisfies the equation

$$f\left([X_1, X_2]_L\right) = [f(X_1), f(X_2)]_{\overline{L}}\,, \ \forall\ X_1,\ X_2\ \in L$$

where the indices L and $\overline{L}$ denote the commutators in the corresponding algebras. An isomorphism of L onto itself is called an automorphism of the Lie algebra L. This mapping will be denoted by the symbol $A : L \to L$.

In the finite-dimensional case, isomorphic Lie algebras have the same dimensions. The criterion for two Lie algebras to be isomorphic can be stated in terms of their structural constants. Two Lie algebras L and $\overline{L}$ are isomorphic, if and only if there exist bases for each of them in which their structural constants are equal.

Let L be a Lie algebra with basis $\{X_1, X_2, \ldots, X_n\}$. Then one has

$$[X_i, X_j] = \sum_{\alpha=1}^{n} c_{ij}^{\alpha} X_\alpha; \quad (i,\, j = 1, 2, \ldots, n),$$

where c_{ij}^{α} are the structural constants. One constructs a one-parameter family of automorphism, A_i, $(i = 1, ..., n)$ on L,

$$A_i : \sum_{i=1}^{n} x_i X_i \to \sum_{i=1}^{n} \bar{x}_i X_i$$

where $\bar{x}_i = \bar{x}_i(a)$, as follows. Consider the system

$$\frac{\mathrm{d}\overline{x}_j}{\mathrm{d}a} = \sum_{\beta=1}^{n} c_{\beta i}^{j} \overline{x}_\beta, \quad (j = 1, 2, \ldots, n)\,. \tag{3.22}$$

Initial values for this system are $\overline{x}_j = x_j$ at $a = 0$. The set of solutions of these equations determines the set of automorphisms $\{A_i\}$.

The set of all subalgebras is divided into equivalence classes with respect to these automorphisms. A list of representatives, where each element of this list is one representative from every class, is called an optimal system of subalgebras.

Because of the difficulties in constructing the optimal system of subalgebras for Lie algebras of large dimension, there is a two-step algorithm (Ovsiannikov, 1994), which reduces this problem to the problem for constructing an optimal system of algebras of lower dimensions. In brief, let us consider an algebra L^r with basis $\{X_1, X_2, \ldots, X_r\}$. According to the algorithm, the algebra L^r is decomposed as $I_1 \oplus N_1$, where I_1 is an ideal of L^r and N_1 is a subalgebra of the algebra L^r. In the same way, the subalgebra N_1 can also be decomposed as $N_1 = I_2 \oplus N_2$. Repeat is the same process $(\alpha - 1)$ times one ends up with an algebra N_α, for which an optimal system of subalgebras can be easily constructed. By gluing the ideals I_l and subalgebras N_l starting from $l = \alpha$ to $l = 1$, together one constructs the optimal system of subalgebras for the algebra L^r. Note that for every subalgebra N_l one needs to check the subalgebra conditions and use the automorphisms to simplify the coefficients of these systems. Therefore, the problem for constructing an optimal system of subalgebras of the algebra L^r by this method is reduced to the problem of classification of algebras of lower dimensions.

After constructing the optimal system, one can start seeking invariant and partially invariant solutions of subalgebras from the optimal system.

3.4 Equivalence group of transformations

A system of PDES can be classified by the symbol $E(m, n, s, l)$, where m is the number of the dependent variables, n is the number of the independent variable, s is the order of the highest derivative and l is the number of differential equations. Normally the differential equations include arbitrary elements (θ). For searching Lie groups which are admitted by the original system, one needs to determine a group of transformations that changes arbitrary elements but does not change the differential structure. An infinitesimal approach (Meleshko, 1996) was applied for finding this group.

A nondegenerate change of dependent, independent variables and arbitrary elements which transfers any system of the differential equations of the given class

$$F_l(x, u, p, \theta) = 0 \tag{3.23}$$

to the system of the equations of the same class but with different arbitrary elements is called an equivalence transformation. Here p defines the partial derivatives $(u_{(1)}, u_{(2)}, ..., u_{(s)})$.

A Lie group of equivalence transformations with parameter a can be written as follows

$$\bar{x}_i = \phi^i(x, u, \theta; a), \;\; \bar{u}_j = \psi^j(x, u, \theta; a), \;\; \bar{\theta}_k = \Pi^k(x, u, \theta; a), \tag{3.24}$$

where $\theta_k = (\theta_{k^1}, \theta_{k^2}, ..., \theta_{k^\gamma})$ is the set of arbitrary elements. The generator of this group has the form

$$X^e = \xi^i \partial_{x_i} + \zeta^j \partial_{u_j} + \zeta^{\theta^k} \partial_{\theta_k},$$

where

$$\xi^i = \xi^i(x, u, \theta) = \frac{\partial \phi^i}{\partial a}|_{a=0}, \ \zeta^j = \zeta^j(x, u, \theta) = \frac{\partial \psi^j}{\partial a}|_{a=0}, \ \zeta^{\theta^k} = \zeta^{\theta^k}(x, u, \theta) = \frac{\partial \Pi^k}{\partial a}|_{a=0},$$

Transformations of arbitrary elements are obtained in the following way. Let $\theta_0(\theta, u)$ be given. By the inverse function theorem with equation (3.23), we can find $x = f(\bar{x}, \bar{u}; a)$ and $u = g(\bar{x}, \bar{u}; a)$. The transformed vector of arbitrary elements is

$$\theta_0(\bar{x}, \bar{u}) = \Pi(f(\bar{x}, \bar{u}; a), g(\bar{x}, \bar{u}; a), \theta_0(f(\bar{x}, \bar{u}; a), g(\bar{x}, \bar{u}; a)).$$

If $u_0(x)$ is solution of system (3.22) and $\theta_0(x, u)$ is a concrete value of the arbitrary elements, then we have

$$\bar{x} = \Phi(x, u_0(x), \theta_0(x, u_0(x)); a).$$

By the inverse function theorem, we can find

$$x = f(\bar{x}; a)$$

and we also obtain the transformed function

$$u_a(\bar{x}) = \Psi(f(\bar{x}, a), u_0(f(\bar{x}, a)), \theta_0(f(\bar{x}, a), u_0(f(\bar{x}, a)); a)). \tag{3.25}$$

Differentiating (3.24) with respect to $\bar{x}$, we get the transformation of derivatives $\bar{p}$. Since $u_a(\bar{x})$ is a solution of the same system with transformed arbitrary elements $\theta_a(\bar{x}, \bar{u})$ then

$$F_l(\bar{x}, u_a(\bar{x}), \bar{p_a}(\bar{x}), \theta_a(\bar{x}, u_a(\bar{x}))) = 0, \quad l = 1, 2, ...$$

The s-th prolongation of the infinitesimal generator X^e is

$$\bar{X}_e^{[s]} = X^e + \zeta_i^j \partial_{u_i^j} + \zeta_{x_i}^{\theta^k} \partial_{\theta^k_{x_i}} + \zeta_{u_j}^{\theta^k} \partial_{\theta^k_{u_j}} \tag{3.26}$$

where

$$\zeta_i^j = D_{x_i}\zeta^j - u_\alpha^j D_{x_i}\xi^\alpha,$$

$$\zeta_{x_i}^{\theta^k} = D_{x_i}^e \zeta^{\theta^k} - \theta^k_{x_\alpha} D^e_{x_i}\zeta^\alpha - \theta^k_{u^\beta} D^e_{x_i}\zeta^\beta,$$

$$\zeta_{u_j}^{\theta^k} = D_{x_i}^e \zeta^{\theta^k} - \theta^k_{x_\alpha} D^e_{u_j}\zeta^\alpha - \theta^k_{u^\beta} D^e_{u_j}\zeta^\beta.$$

Here

$$D_{x_i} = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u_j} + (\theta^k_{x_i} + \theta^k_{u_j} u_i^j)\frac{\partial}{\partial \theta_k} + ...,$$

$$D^e_{x_i} = \frac{\partial}{\partial x_i} + \theta^k_{x_i}\frac{\partial}{\partial \theta^k} + ..., \quad D^e_{u^j} = \frac{\partial}{\partial u^j} + \theta^k_{u^j}\frac{\partial}{\partial \theta^k} + ...$$

By the same way as for the admitted Lie group, one can obtain the determining equations for the equivalence Lie group.

Let $G(\theta)$ be admitted by the equations for all arbitrary elements. The group $G(\theta)$ is called a kernel of groups. The corresponding Lie-algebra is called a kernel of Lie algebras.

3.5 Invariant and partially invariant solutions

Let G be a Lie group admitted by a system of differential equation (S) and $H \subset G$ be a subgroup.

Definition 12. A solution $u = U(x)$ of the system (S) is called an H invariant solution if the manifold $u = U(x)$ is an invariant manifold with respect to any transformation of the group H.

The notion of invariant solution was introduced by Sophus Lie (1895). The notion of a partially invariant solution was introduced by Ovsiannikov (1958).

The notion of partially invariant solutions generalizes the notion of an invariant solution, and extends the scope of applications of group analysis for constructing exact solutions of partial differential equations. The algorithm of finding invariant and partially invariant solutions consists of the following steps.

Let L^r be a Lie algebra with the basis $X_1, ..., X_r$. The universal invariant J consists of $s = m + n - r_*$ functionally independent invariants

$$J = \left(\ J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u)\ \right),$$

where n, m is the numbers of independent and dependent variables, respectively and r_* is the total rank of the matrix composed by the coefficients of the generators $X_i,\ (i = 1, 2, ..., r)$. If the rank of the Jacobi matrix $\dfrac{\partial(J^1, ..., J^{m+n-r_*})}{\partial(u^1, ..., u^m)}$ is equal to q, then one can choose the first $q \leq m$ invariants $J^1, ..., J^q$ such that the rank of the Jacobi matrix $\dfrac{\partial(J^1, ..., J^q)}{\partial(u_1, ..., u_m)}$ is equal to q. A partially invariant solution is characterized by two integers: $\sigma \geq 0$ and $\delta \geq 0$. These solutions are also called $H(\sigma, \delta)$-solutions. The number σ is called the rank of a partially invariant solution. This number gives the number of the independent variables in the representation of the partially invariant solution. The number δ is called the defect of a partially invariant solution. The defect is the number of the dependent functions which can not be found from the representation of partially invariant solution. The rank σ and the defect δ must satisfy the conditions

$$\sigma = \delta + n - r_* \geq 0,\ \ \delta \geq 0,$$

$$\rho \leq \sigma < n,\ \max\{r_* - n, m - q, 0\} \leq \delta \leq \min\{r_* - 1, m - 1\},$$

where ρ is the maximum number of invariants which depend on the independent variables only. Note that for invariant solutions, $\delta = 0$ and $q = m$.

For constructing a representation of a $H(\sigma, \delta)$-solution one needs to choose $l = m - \delta$ invariants and separate the universal invariant in two parts:

$$\overline{J} = (J^1, ..., J^l),\ \overline{\overline{J}} = (J^{l+1}, J^{l+2}, ..., J^{m+n-r_*}).$$

The number l satisfies the inequality $1 \leq l \leq q \leq m$. The representation of the $H(\sigma, \delta)$-solution is obtained by assuming that the first l coordinates $\overline{J}$ of the universal invariant are functions of the invariants $\overline{\overline{J}}$:

$$\overline{J} = W(\overline{\overline{J}}). \tag{3.27}$$

Equation (3.27) forms the invariant part of the representation of a solution. The next assumption about a partially invariant solution is that equation (3.27) can be solved for the first l dependent functions, for example,

$$u^i = \phi^i(u^{l+1}, u^{l+2}, ..., u^m, x), \ (i = 1, ..., l). \tag{3.28}$$

It is important to note that the functions W^i, $(i = 1, ..., l)$ are involved in the expressions for the functions ϕ^i, $(i = 1, ..., l)$. The functions $u^{l+1}, u^{l+2}, ..., u^m$ are called superfluous. The rank and the defect of the $H(\sigma, \delta)$-solution are $\delta = m - l$ and $\sigma = m + n - r_* - l = \delta + n - r_*$, respectively.

Note that if $\delta = 0$, the above algorithm is the algorithm for finding a representation of an invariant solution. If $\delta \neq 0$, then equations (3.28) do not define all dependent functions. Since a partially invariant solution satisfies the restrictions (3.27), this algorithm cuts out some particular solutions from the set of all solutions.

After constructing the representation of an invariant or partially invariant solution (3.28), it has to be substituted into the original system of equations. The system of equations obtained for the functions W and superfluous functions u^k, $(k = l + 1, 2, ..., m)$ is called the reduced system. This system is overdetermined and requires an analysis of compatibility. Compatibility analysis for invariant solutions is easier than for partially invariant solutions. Another case of partially invariant solutions which is easier than the general case occurs when $\overline{\overline{J}}$

only depends on the independent variables

$$J^{l+1} = J^{l+1}(x), J^{l+2} = J^{l+2}(x), ..., J^{m+n-r_*} = J^{m+n-r_*}(x).$$

In this case, a partially invariant solution is called regular, otherwise it is irregular (Ovsiannikov, 1995). The number $\sigma - \rho$ is called the measure of irregularity.

CHAPTER IV

GROUP CLASSIFICATION OF THE THREE-DIMENSIONAL EQUATIONS

4.1 Introduction

This chapter is devoted to group classification of the three-dimensional equations describing flows of fluids with internal inertia. (Gavulyuk and Teshukof (2001)):

$$\begin{gathered}\dot{\rho} + \rho\, div(u) = 0, \qquad \rho\dot{u} + \nabla p = 0,\\ p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\dot{\rho}}\right) - div\left(\frac{\partial W}{\partial\dot{\rho}}u\right)\right) - W,\end{gathered} \tag{4.1}$$

where $W = W(\rho, \dot{\rho})$. The given equations include models as such the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski (1960), Kogarko (1961), Wijngaarden (1968)), and the dispersive shallow water model (Green and Naghdi (1976), Salmon (1988)). These models are obtained for special types of the function $W(\rho, \dot{\rho})$. In this chapter the admitted and equivalence Lie groups are found. Group classification separates out the function $W(\rho, \dot{\rho})$ out into different cases.

4.2 Equivalence Lie group

Since the function W depends on the derivatives of the dependent variables, and for the sake of simplicity of finding the equivalence Lie group, new dependent variables are introduced:

$$u_5 = \dot{\rho},\ \phi_1 = W,\ \phi_2 = W_\rho,\ \phi_3 = W_{\dot{\rho}},$$

where $u_4 = \rho$ and $x_4 = t$. An infinitesimal operator X^e of the equivalence Lie group is sought in the form (Meleshko, 2005),

$$X^e = \xi^i \partial_{x_i} + \zeta^{u_j} \partial_{u_j} + \zeta^{\phi_k} \partial_{\phi_k},$$

where all the coefficients ξ^i, ζ^{u_j} and ζ^{ϕ_k}, $(i = 1, 2;\ j = 1, 2, 3, 4, 5;\ k = 1, 2, 3)$ are functions of the variables x_i, u_j and ϕ_k. Hereafter a sum over repeated indices is implied.

The coefficients of the prolonged operator are obtained by using the prolongation formulae:

$$\zeta^{u_{\alpha,i}} = D_i^e \zeta^{u_\alpha} - u_{\alpha,j} D_i^e \xi^{x_j},\ (i = 1, 2, 3, 4),$$
$$D_i^e = \partial_{x_i} + u_{\alpha,i}\partial_{u_\alpha} + (\rho_{x_i} W_{\beta,1} + \dot\rho_{x_i} W_{\beta,2})\partial_{W_\beta},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\beta = (\beta_1, \beta_2)$ are multindices $(\alpha_i \geq 0,\ \ \beta_i \geq 0)$,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4), j = (\alpha_1 + \delta_{1j}, \alpha_2 + \delta_{2j}, \alpha_3 + \delta_{3j}, \alpha_4 + \delta_{4j}),$$
$$u_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\alpha_4} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \partial t^{\alpha_4}}, \quad W_{(\beta_1,\beta_2)} = \frac{\partial^{\beta_1+\beta_2} W}{\partial \rho^{\beta_1} \partial \dot\rho^{\beta_2}}.$$

The condition that W does not depend on t, x_i ,u_i, $(i = 1, 2, 3)$ give that

$$\zeta^{u_k}_{x_i} = 0,\ \zeta^{u_k}_{u_j} = 0,\ \zeta^{W}_{x_i} = 0,\ \zeta^{W}_{u_j} = 0,\ \ (i = 1, 2, 3, 4;\ j = 1, 2, 3;\ k = 4, 5).$$

Using these relations the prolongation formulae for the coefficients ζ^{W_β} become:

$$\zeta^{W_{\beta,i}} = \widetilde{D}_i^e \zeta^{W_\beta} - W_{\beta,1} \widetilde{D}_i^e \zeta^{u_4} - W_{\beta,2} \widetilde{D}_i^e \zeta^{u_5},\ (i = 1, 2),$$
$$\widetilde{D}_1^e = \partial_\rho + W_{\beta,1}\partial_{W_\beta}, \ \ \widetilde{D}_2^e = \partial_{\dot\rho} + W_{\beta,2}\partial_{W_\beta}.$$

For constructing the determining equations and solving them a symbolic computer system of calculations was applied. We used the system Reduce (Hearn, 1987). Calculations gave the following basis of generators of the equivalence Lie group

$$X_1^e = \partial_{x_1},\ X_2^e = \partial_{x_2},\ X_3^e = \partial_{x_3},$$

$$X_4^e = t\partial_{x_1} + \partial_{u_1},\ X_5^e = t\partial_{x_2} + \partial_{u_2},\ X_6^e = t\partial_{x_3} + \partial_{u_3},$$

$$X_7^e = u_2\partial_{u_2} - u_1\partial_{u_2} + x_2\partial_{x_1} - x_1\partial_{x_2},$$

$$X_8^e = u_3\partial_{u_1} - u_1\partial_{u_3} + x_3\partial_{x_1} - x_1\partial_{x_3},$$

$$X_9^e = u_3\partial_{u_2} - u_2\partial_{u_3} + x_3\partial_{x_2} - x_2\partial_{x_3},$$

$$X_{10}^e = \partial_t,\ X_{11}^e = t\partial_t + x_i\partial_{x_i},$$

$$X_{12}^e = \partial_W,\ X_{13}^e = \rho\partial_W,\ X_{14}^e = f(\rho)\dot{\rho}\partial_W,$$

$$X_{15}^e = \dot{\rho}\partial_{\dot{\rho}} + \rho\partial_\rho + W\partial_W,$$

$$X_{16}^e = x_i\partial_{x_i} + u_i\partial_{u_i} - 2\rho\partial_\rho.$$

Here only essential part of the operators X_i^e is written. For example, the operator X_{11}^e, found in the result of calculations, is

$$t\partial_t + x_i\partial_{x_i} - \dot{\rho}\partial_{\dot{\rho}}.$$

The part $-\dot{\rho}\partial_{\dot{\rho}}$ is obtained from X_{11}^e by using the prolongation formulae.

Since the equivalence transformations corresponding to the operators X_{11}^e, X_{12}^e,..., X_{16}^e are applied for simplifying the function W in the process of the classification, let us present these transformations. Because the function W depends on ρ and $\dot{\rho}$, only the transformations of these variables are presented:

$$\begin{array}{llll} X_{11}^e: & \rho' = \rho, & \dot{\rho}' = e^{-a}\dot{\rho}, & W' = W; \\ X_{12}^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & W' = W + a. \\ X_{13}^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & W' = W + a\rho; \\ X_{14}^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & W' = W + a\dot{\rho}f(\rho); \\ X_{15}^e: & \rho' = e^a\rho, & \dot{\rho}' = e^a\dot{\rho}, & W' = e^aW; \\ X_{16}^e: & \rho' = e^{-2a}\rho, & \dot{\rho}' = e^{-2a}\dot{\rho}, & W' = W; \end{array}$$

Here a is the group parameter.

4.3 Admitted Lie group of (4.1)

An admitted generator X of equations (4.1) is sought in the form

$$X = \xi^{x_1}\partial_{x_1} + \xi^{x_2}\partial_{x_2} + \xi^{x_3}\partial_{x_3} + \xi^{t}\partial_{t} + \zeta^{u_1}\partial_{u_1} + \zeta^{u_2}\partial_{u_2} + \zeta^{u_3}\partial_{u_3} + \zeta^{\rho}\partial_{\rho},$$

where the coefficients of the generator are functions of the variables $x_1, x_2, x_3, t, u_1, u_2, u_3, \rho$.

Calculations showed that

$$\xi^{x_1} = c_6x_1t + c_4t + c_3x_3 + c_2x_2 + x_1c_7 + x_1c_1 + c_5,$$

$$\xi^{x_2} = c_6x_2t + c_{12}t + x_3c_{11} + x_2c_7 + x_2c_1 - x_1c_2 + c_{13},$$

$$\xi^{t} = c_6t^2 + c_7t + c_8, \quad \zeta^{\rho} = (-3c_6t + c_{15})\rho,$$

$$\zeta^{u_1} = c_3u_3 + c_2u_2 - c_6u_1t + c_1u_1 + c_6x_1 + c_4,$$

$$\zeta^{u_2} = c_{11}u_3 - c_6u_2t + c_1u_2 - c_2u_1 + c_6x_2 + c_{12},$$

$$\zeta^{u_3} = -c_6u_3t + c_1u_3 - c_{11}u_2 - c_3u_1 + c_6x_3 + c_{16},$$

where the constants c_i, $(i = 1, 2, ..., 8, 11, 12, 13, 15)$ satisfy the conditions

$$\begin{gathered} 27c_6\rho^3(3W_{\dot\rho\rho\rho\rho}\dot\rho\rho + W_{\dot\rho\rho\rho}\dot\rho - 3W_{\rho\rho\rho}\rho - W_{\rho\rho}) + 600W_{\dot\rho\dot\rho}c_6\dot\rho^2\rho \\ +25\dot\rho^3(5W_{\dot\rho\dot\rho\dot\rho\dot\rho}\dot\rho^2(c_{15} - c_7) + 5W_{\dot\rho\dot\rho\dot\rho\rho}\dot\rho\rho c_{15} + 18W_{\dot\rho\dot\rho\rho}\rho c_{15} \\ +W_{\dot\rho\dot\rho\dot\rho}\dot\rho(28c_{15} - 33c_7 - 10c_1) + 18W_{\dot\rho\dot\rho}(c_{15} - 2c_7 - 2c_1)) = 0, \end{gathered} \tag{4.2}$$

$$W_{\dot\rho\dot\rho\dot\rho}\dot\rho(c_7 - c_{15}) - c_{15}\rho W_{\dot\rho\dot\rho\rho} + (2c_1 - c_{15} + 2c_7)W_{\dot\rho\dot\rho} + 3c_6W_{\dot\rho\dot\rho\dot\rho}\rho = 0, \tag{4.3}$$

$$\begin{gathered} 9W_{\dot\rho\rho\rho\rho}\dot\rho\rho^3c_{15} + 40W_{\dot\rho\dot\rho\dot\rho\dot\rho}\dot\rho^4(c_7 - c_{15}) + W_{\dot\rho\dot\rho\dot\rho\rho}\dot\rho^3\rho(9c_7 - 49c_{15}) - 9W_{\dot\rho\dot\rho\rho\rho}\dot\rho^2\rho^2c_7 \\ +8W_{\dot\rho\dot\rho\dot\rho}\dot\rho^3(10c_1 - 17c_{15} + 22c_7) + 2W_{\dot\rho\dot\rho\rho}\dot\rho^2\rho(9c_1 - 37c_{15} + 9c_7) - 9W_{\rho\rho\rho}\rho^3c_{15} \\ +9W_{\dot\rho\rho\rho}\dot\rho\rho^2(c_{15} - 2c_1) + 56W_{\dot\rho\dot\rho}\dot\rho^2(2c_1 - c_{15} + 2c_7) + 9W_{\rho\rho}\rho^2(2c_1 - c_{15}) = 0, \end{gathered} \tag{4.4}$$

$$c_6(5W_{\dot\rho\dot\rho\dot\rho}\dot\rho + 3W_{\dot\rho\dot\rho\rho}\rho + 5W_{\dot\rho\dot\rho}) = 0. \tag{4.5}$$

Let us consider equations (4.3)-(4.5). Since W is arbitrary, one can choose for example $W = \rho^2$. In this case these equations become $2c_1 - c_{15} + 2c_7 = 0, \;\; c_6 = 0$. After that equation (4.3) is reduced to the equation

$$W_{\dot\rho\dot\rho\dot\rho}\dot\rho(c_7 - c_{15}) - c_{15}\rho W_{\dot\rho\dot\rho\rho} = 0$$

Choosing $W = \dot\rho^2\rho$, the last equation gives $c_{15} = 0$. then $c_7 = 0$. Thus, $c_1 = 0, \;\; c_6 = 0, \;\; c_7 = 0, \;\; c_{15} = 0$. In this case all equations (4.2)-(4.5) are satisfied and the generator X is

$$X = c_2Y_9 + c_3Y_8 + c_4Y_4 + c_5Y_1 + c_8Y_{10} + c_{11}Y_7 + c_{12}Y_5 + c_{13}Y_2 + c_{16}Y_6 + c_{17}Y_3.$$

The kernel of admitted Lie algebras consists of the generators

$$Y_1 = \partial_{x_1}, \; Y_2 = \partial_{x_2}, \; Y_3 = \partial_{x_3}, \; Y_{10} = \partial_t,$$
$$Y_4 = t\partial_{x_1} + \partial_{u_1}, \; Y_5 = t\partial_{x_2} + \partial_{u_2}, \; Y_6 = t\partial_{x_3} + \partial_{u_3},$$
$$Y_7 = x_2\partial_{x_3} - x_3\partial_{x_2} + u_2\partial_{u_3} - u_3\partial_{u_2},$$
$$Y_8 = x_3\partial_{x_1} - x_1\partial_{x_3} + u_3\partial_{u_1} - u_1\partial_{u_3},$$
$$Y_9 = x_1\partial_{x_2} - x_2\partial_{x_1} + u_1\partial_{u_2} - u_2\partial_{u_1}.$$

Extensions of the kernel depend on the value of the function $W(\rho, \dot\rho)$. They can only be operators of the form

$$c_1X_1 + c_6X_6 + c_7X_7 + c_{15}X_{14},$$

where

$$X_1 = x_i\partial_{x_i} + u_i\partial_{u_i},$$
$$X_6 = t(t\partial_t + x_i\partial_{x_i} - u_i\partial_{u_i} - 3\rho\partial_\rho) + x_i\partial_{u_i}$$
$$X_7 = x_i\partial_{x_i} + t\partial t, \;\; X_9 = x_2\partial_{x_2} + u_2\partial_{u_2}, \;\; X_{14} = \rho\partial_\rho.$$

Relations between the constants c_1, c_6, c_7, c_{15} depend on the function $W(\rho, \dot\rho)$.

4.3.1 Case $c_6 \neq 0$

Let $c_6 \neq 0$, then equation (4.5) gives

$$5W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho} + 3W_{\dot{\rho}\dot{\rho}\rho}\rho + 5W_{\dot{\rho}\dot{\rho}} = 0.$$

The general solution of this equation is $W_{\dot{\rho}\dot{\rho}} = \rho^{-5/3}g(\dot{\rho}\rho^{-5/3})$, where the function g is an arbitrary function of integration. Substitution of $W_{\dot{\rho}\dot{\rho}}$ into equation (4.3) shows that the function $g = 2q_o$ is constant. Hence,

$$W = q_o\dot{\rho}^2\rho^{-5/3} + \varphi_1(\rho)\dot{\rho} + \varphi_2(\rho),$$

where the functions $\varphi_2(\rho)$ and $\varphi_1(\rho)$ are arbitrary. Substituting this potential function in the other equations (4.2)-(4.4), one obtains

$$3\rho\varphi_2''' + \varphi_2'' = 0, \quad (c_7 + 2c_1)\varphi_2'' = 0.$$

If $\varphi_2'' = 0$, then the extension of the kernel of admitted Lie algebras is given by the generators

$$X_6, \quad X_1 - 3X_{14}, \quad X_7 - 3X_{14}.$$

If $\varphi_2'' = C_2\rho^{-1/3} \neq 0$, then the extension of the kernel is given by the generators

$$X_6, \quad X_1 - 2X_7 + 3X_{14}.$$

4.3.2 Case $c_6 = 0$

Let $c_6 = 0$, then equation (4.3) becomes

$$-c_{15}a + (c_1 + c_7)b + c_7c = 0, \tag{4.6}$$

where

$$a = \dot{\rho}W_{\dot{\rho}\dot{\rho}\dot{\rho}} + \rho W_{\dot{\rho}\dot{\rho}\rho} + W_{\dot{\rho}\dot{\rho}}, \;\; b = 2W_{\dot{\rho}\dot{\rho}}, \quad c = \dot{\rho}W_{\dot{\rho}\dot{\rho}\dot{\rho}}.$$

Further analysis of the determining equations (4.2)-(4.4) is similar to the group classification of the gas dynamics equations (Ovsiannikov, 1978).

Let us analyze the vector space $Span(V)$, where the set V consists of vectors (a, b, c) with ρ and $\dot{\rho}$ are changed. If the function $W(\rho, \dot{\rho})$ is such that $\dim(Span(V)) = 3$, then equation (4.6) is only satisfied for

$$c_1 = 0, c_7 = 0, c_{15} = 0,$$

which does not give extensions of the kernel of admitted Lie algebras. Hence, one needs to study $\dim(Span(V)) \leq 2$.

Case $\dim(Span(V)) = 2$

Let $\dim(Span(V)) = 2$. There exists a constant vector $(\alpha, \beta, \gamma) \neq 0$, which is orthogonal to the set V:

$$\alpha a + \beta b + \gamma c = 0. \tag{4.7}$$

This means that the function $W(\rho, \dot{\rho})$ satisfies the equation

$$(\alpha + \gamma)\dot{\rho} W_{\dot{\rho}\dot{\rho}\dot{\rho}} + \alpha \rho W_{\rho\dot{\rho}\dot{\rho}} = -(\alpha + 2\beta) W_{\dot{\rho}\dot{\rho}}. \tag{4.8}$$

The characteristic system of this equation is

$$\frac{d\dot{\rho}}{(\alpha + \gamma)\dot{\rho}} = \frac{d\rho}{\alpha\rho} = \frac{dW_{\dot{\rho}\dot{\rho}}}{-(\alpha + 2\beta) W_{\dot{\rho}\dot{\rho}}}.$$

The general solution of equation (4.8) depends on the values of the constants α, β and γ.

Case $\alpha = 0$**.** Because of equation (4.7) and the condition $W_{\dot{\rho}\dot{\rho}} \neq 0$, one has $\gamma \neq 0$. The general solution of equation (4.8) is

$$W_{\dot{\rho}\dot{\rho}}(\rho, \dot{\rho}) = \tilde{\varphi} \dot{\rho}^k \tag{4.9}$$

where $k = -2\beta/\gamma$, and $\widetilde{\varphi}$ is an arbitrary function of integration. Substitution of (4.9) into (4.6) leads to

$$c_{15}\rho\tilde{\varphi}' - \tilde{\varphi}(\rho)\left(2c_1 - (k+1)c_{15} + (k+2)c_7\right) = 0. \tag{4.10}$$

If $c_{15} \neq 0$, the dimension $\dim(Span(V)) = 1$, which contradicts to the assumption. Hence, $c_{15} = 0$ and from (4.10) one obtains $\tilde{c_1} = -(k+2)c_7/2$. The extension of the kernel in this case is given by the generator

$$-pX_1 + 2X_7,$$

where $p = k + 2$.

If $(k+2)(k+1) \neq 0$, then integrating (4.9), one finds

$$W(\rho, \dot{\rho}) = \varphi(\rho)\dot{\rho}^p + \varphi_1(\rho)\dot{\rho} + \varphi_2(\rho),$$

where $\varphi_1(\rho)$ and $\varphi_2(\rho)$ are arbitrary functions. Substituting this function W into (4.2)-(4.4) one has $\varphi_2'' = 0$.

If $k = -2$, then

$$W(\rho, \dot{\rho}) = \varphi(\rho)\ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho),$$

and $\varphi_2'' = 0$, similar to the previous case.

If $k = -1$, then

$$W(\rho, \dot{\rho}) = \varphi(\rho)\dot{\rho}\ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho),$$

and also $\varphi_2'' = 0$.

Case $\alpha \neq 0$. The general solution of equation (4.9) is

$$W_{\dot{\rho}\dot{\rho}}(\rho, \dot{\rho}) = \varphi(\dot{\rho}\rho^k)\rho^\lambda, \tag{4.11}$$

where $k = -(1 + \gamma/\alpha)$, $\lambda = -(1 + 2\beta/\alpha)$ and φ is an arbitrary function. Substitution of this function into (4.6) leads to

$$k_0 \varphi' z + k_1 \varphi = 0,$$

where

$$z = \dot{\rho}\rho^k, \quad k_0 = c_7 - c_{15}(k+1), \quad k_1 = 2c_1 - c_{15}(\lambda + 1) + 2c_7.$$

Since $\dim(Span(V)) = 2$, one obtains that $k_0 = 0$ and $k_1 = 0$ or

$$c_7 = c_{15}(k+1), \quad c_1 = c_{15}(p-1)/2,$$

where $p = \lambda - 2k$. Integrating (4.11), one finds

$$W(\rho, \dot{\rho}) = \rho^p \varphi(\dot{\rho}\rho^k) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho). \tag{4.12}$$

Substitution of (4.12) into (4.2)-(4.4) gives

$$\rho\varphi_2''' + (2k - \lambda + 2)\varphi_2'' = 0. \tag{4.13}$$

Solving this equation, one has

$$\varphi_2'' = C_2 \rho^{p-2},$$

where C_2 is an arbitrary constant. The extension of the kernel is given by the generator

$$(p-1)X_1 + 2(k+1)X_7 + 2X_{14}.$$

Case $\dim(Span(V)) = 1$

Let $\dim(Span(V)) = 1$. There exists a constant vector $(\alpha, \beta, k) \neq 0$ such that

$$(a, b, c) = (\alpha, \beta, k)B$$

with some function $B(\rho,\dot\rho)\neq 0$. Because $W_{\dot\rho\dot\rho}\neq 0$, one has that $\beta\neq 0$. Hence, the function $W(\rho,\dot\rho)$ satisfies the equations

$$\dot\rho W_{\dot\rho\dot\rho\dot\rho}+\rho W_{\rho\dot\rho\dot\rho}+(1-2\tilde\alpha)W_{\dot\rho\dot\rho}=0,\ \ \dot\rho W_{\dot\rho\dot\rho\dot\rho}-2\gamma W_{\dot\rho\dot\rho}=0.$$

The general solution of the last equation is

$$W_{\dot\rho\dot\rho}(\rho,\dot\rho)=\varphi(\rho)\dot\rho^k$$

with arbitrary function $\varphi(\rho)$. Substituting this solution into the first equation, one obtains

$$\rho\varphi'(\rho)+(1-2\tilde\alpha+k)\varphi(\rho)=0.$$

Thus,

$$W_{\dot\rho\dot\rho}=-q_o\dot\rho^k\rho^\lambda, \tag{4.14}$$

where $\lambda=-(1-2\tilde\alpha+k)$, q_o is an arbitrary constant. Since $\dim(Span(V))=1$, then $q_o\neq 0$, λ and k are such that $\lambda^2+k^2\neq 0$.

Substituting (4.14) into (4.6), it becomes

$$-c_{15}(k+\lambda+1)+c_7(k+2)+2c_1=0. \tag{4.15}$$

Integration of (4.14) depends on the quantity of k.

If $(k+2)(k+1)\neq 0$, then integrating (4.14), one obtains

$$W(\rho,\dot\rho)=-q_o\rho^\lambda\dot\rho^p+\dot\rho\varphi_1(\rho)+\varphi_2(\rho),\ \ (p(p-1)\neq 0), \tag{4.16}$$

where $p=k+2$. Substituting this W into equations (4.2)-(4.4), one obtains

$$c_1=\left(c_{15}(p+\lambda-1)-c_7p)\right)/2,$$

with the function $\varphi_2(\rho)$ satisfying the condition

$$c_{15}\rho\varphi_2'''+\varphi_2''(-c_{15}(p+\lambda-2)+c_7p)=0.$$

If $\varphi_2'' = C_2\rho^{-\mu} \neq 0$, the extension of the kernel is given by the generator

$$(1-\mu)X_1 + 2(X_{14} + \phi X_7),$$

where $\phi = (\mu + \lambda + p - 2)/p$. If $\varphi_2'' = 0$, the extension is given by the generators

$$pX_1 - 2X_7, \quad (p + \lambda - 1)X_1 + 2X_{14}.$$

If $k = -2$, then integrating (4.14), one obtains

$$W(\rho, \dot{\rho}) = -q_o\rho^\lambda \ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho), \quad (q_o \neq 0).$$

Substituting this into equations (4.2)-(4.4), we obtain

$$c_1 = c_{15}(\lambda - 1)/2,$$

and the condition

$$c_{15}(\rho\varphi_2''' - \varphi_2''(\lambda + 2)) + q_o\lambda(\lambda - 1)(c_{15} - c_7)\rho^{\lambda-2} = 0.$$

If $\lambda(\lambda - 1) = 0$ and φ_2 is arbitrary, then the extension is given only by the generator

$$X_7.$$

If $\lambda(\lambda - 1) = 0$ and $\varphi_2'' = C_2\rho^{\lambda+2}$, then the extension of the kernel consists of the generators

$$(\lambda - 1)X_1 + 2X_{14}, \quad X_7.$$

If $\lambda(\lambda - 1) \neq 0$ and $\varphi_2'' = C_2\rho^{\lambda+2} - \frac{q_o}{4}\lambda(\lambda - 1)\mu\rho^{\lambda-2}$, then the extension is

$$(\lambda - 1)X_1 + 2(X_{14} + (\mu + 1)X_7)$$

where $c_7 = (\mu + 1)c_{15}$.

If $k = -1$, then integrating (4.14), one obtains

$$W(\rho, \dot{\rho}) = -q_o\rho^\lambda\dot{\rho}\ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho),$$

and substituting it into equations (4.2)-(4.4), we obtain

$$c_1 = (c_{15}\lambda - c_7)/2,$$

and the condition

$$c_{15}\rho\varphi_2''' + \varphi_2''(-c_{15}\lambda + c_{15} + c_7) = 0.$$

One needs to study two cases. If $\varphi_2'' \neq 0$, then the extension is possible only for $\varphi_2 = C_2\rho^{-\mu} \neq 0$, where $\mu = -\lambda + 1 + c_7/c_{15}$. The extension of the kernel is given by the generator

$$(1-\mu)X_1 + 2(\mu + \lambda - 1)X_7.$$

If $\varphi_2'' = 0$, then the extension of the kernel consists of the generators

$$X_1 - 2X_7, \quad X_{14} + \lambda X_7.$$

Case $\dim(Span(V)) = 0$

Let $\dim(Span(V)) = 0$. The vector (a, b, c) is constant:

$$(a, b, c) = (\alpha, \beta, k)$$

with some constant values α, β and k. This leads to

$$W_{\dot\rho\dot\rho} = -2q_o,$$

where $q_o \neq 0$ is constant. Integrating this equation, one obtains

$$W(\rho, \dot\rho) = -q_o\dot\rho^2 + \dot\rho\varphi_1(\rho) + \varphi_2(\rho). \tag{4.17}$$

Substituting (4.17) into equations (4.2)-(4.4), we obtain

$$c_1 = (c_{15} - 2c_7)/2,$$

and the condition

$$c_{15}\rho\varphi_2''' + 2c_7\varphi_2'' = 0.$$

If $\varphi_2'' \neq 0$, then $\varphi_2 = C_2\rho^{-\mu}$, where $\mu = 2c_7/c_{15}$. The extension of the kernel consists of the generator

$$(1-\mu)X_1 + 2X_{14} + \mu X_7.$$

If $\varphi_2'' = 0$, then the extension of the kernel is given by the generators

$$X_1 + 2X_{14}, \quad X_1 - X_7.$$

The result of group classification of equations (4.1) is summarized in Table 4.1. The linear part with respect to $\dot{\rho}$ of the function $W(\rho, \dot{\rho})$ is omitted. Notice also that the change $t \to -t$ has to conserve the potential function W, this leads to $\varphi_1(\rho) = 0$.

Remark. The Green-Naghdi model belongs to the class M_7 in Table 4.1 with $\lambda = 1,\ p = 2$ and $\mu = 0$. Invariant solutions of the one-dimensional Green-Naghdi model completely studied in (Bagderina and Chupakhin, 2005).

Remark. The one-velocity dissipation-free Iordanski-Kogarko-Wijngaarden model has an extension of the kernel of admitted Lie algebras only for a special internal energy of the gas phase (class M_3 $(p = 2)$ in Table 4.1), which corresponds to a Chaplygin gas $\varepsilon_{20}\left(\rho_{20}\right) = \gamma_1/\rho_{20} + \gamma_0$, where γ_1 and γ_0 are constants.

Table 4.1 Group classification of equations (4.1)

	$W(\rho,\dot\rho)$	Extensions	Remarks
M_1	$-q_o\rho^{-5/3}\dot\rho^2+\varphi_2(\rho)$	$X_6,\ \ X_1-2X_7+3X_{14}$	$\varphi_2''=C_2\rho^{-1/3}\neq 0$
M_2	$-q_o\rho^{-5/3}\dot\rho^2$	$X_6,\ \ X_1-3X_{14},\ X_7-3X_{14}$	
M_3	$\varphi(\rho)\dot\rho^p$	$-pX_1+2X_7$	
M_4	$\varphi(\rho)\ln\dot\rho$	X_7	
M_5	$\dot\rho\varphi(\rho)\ln\dot\rho$	X_1-2X_7	
M_6	$\rho^p\varphi(\dot\rho\rho^k)+\varphi_2$	$(p-1)X_1+2(X_7(k+1)+X_{14})$	$\varphi_2''=C_2\rho^{p-2}$
M_7	$-q_o\rho^\lambda\dot\rho^p+\varphi_2$	$(1-\mu)X_1+2(X_{14}+\phi X_7)$	$\varphi_2''=C_2\rho^{-\mu}\neq 0,$
			$p(p-1)\neq 0,$
			$\phi=(\mu+\lambda+p-2)/p$
M_8	$-q_o\rho^\lambda\dot\rho^p$	$pX_1-2X_7,$	,
		$(p+\lambda-1)X_1+2X_{14}$	$p(p-1)\neq 0$
M_9	$-q_o\rho^\lambda\ln\dot\rho+\varphi_2$	X_7	$\varphi_2(\rho)$ arbitrary,
			$\lambda(\lambda-1)=0$
M_{10}	$-q_o\rho^\lambda\ln\dot\rho+\varphi_2$	$(\lambda-1)X_1+2X_{14},$	$\varphi_2''=C_2\rho^{\lambda+2},$
		X_7	$\lambda(\lambda-1)=0$
M_{11}	$-q_o\rho^\lambda\ln\dot\rho+\varphi_2$	$(\lambda-1)X_1$	$\varphi_2''=C_2\rho^{\lambda+2}$
		$+2(X_{14}+(\mu+1)X_7)$	$-\frac{q_o}{4}\lambda(\lambda-1)\mu\rho^{\lambda-2},$
			$\lambda(\lambda-1)\neq 0$
M_{12}	$-q_o\rho^\lambda\dot\rho\ln\dot\rho+\varphi_2$	$(1-\mu)X_1+2(\mu+\lambda-1)X_7$	$\varphi_2=C_2\rho^{-\mu}\neq 0$
M_{13}	$-q_o\rho^\lambda\dot\rho\ln\dot\rho$	$X_1-2X_7,\ X_{14}+\lambda X_7$	
M_{14}	$-q_o\dot{\rho}^2+\varphi_2$	$(1-\mu)X_1+2X_{14}+\mu X_7$	$\varphi_2=C_2\rho^{-\mu}\neq 0$
M_{15}	$-q_o\dot{\rho}^2$	$X_1+2X_{14},\ \ X_1-X_7$	

CHAPTER V

SPECIAL VORTEX SOLUTION OF FLUIDS WITH INTERNAL INERTIA

This chapter is devoted to the special vortex solution. This solution was introduced by L.V.Ovsiannikov (Ovsiannikov, 1995) for ideal compressible and incompressible fluids. This is a partially invariant solution, generated by the Lie group of all rotations. L.V.Ovsyannikov called it a "singular vortex". It is related with the special choice of non-invariant function. He also gave complete analysis of the overdetermined system corresponding to this type of partially invariant solutions: all invariant functions satisfy the well-defined system of partial differential equations with two independent variables. The main features of the fluid flow, governed by the obtained solution, were pointed out in (Ovsiannikov, 1995). It was shown that trajectories of particles are flat curves in three-dimensional space. The position and orientation of the plane which contains the trajectory depends on the particle's initial location. Later particular solutions of this system of partial differential equations for invariant functions were studied in (Popovych, 2000), (Chupakhin, 2003), (Cherevko and Chupakhin, 2004), (Pavlenko, 2005) . For some other models this type of partially invariant solutions was considered in (Hematulin and Meleshko, 2002), (Golovin, 2005). Exact solutions in fluid dynamics generated by a rotation group are of great interest by virtue of their high symmetry. The classical spherically symmetric solution is one of particular cases of such solutions.

In this chapter a singular vortex of the mathematical model of fluids with internal inertia is studied. Complete group classification of the system of equations for invariant functions is given. All invariant solutions for this system are presented.

5.1 Equations of a special vortex solution

Using spherical coordinates

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta,$$

$$U = u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta,$$

$$U_2 = u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta,$$

$$U_3 = -u\sin\varphi + v\cos\varphi,$$

the generators X_7, X_8, X_9 are

$$X_7 = -\sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi + \cos\varphi(\sin\theta)^{-1}(U_2\partial_{U_3} - U_3\partial_{U_2})$$

$$X_8 = -\cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi + \sin\varphi(\sin\theta)^{-1}(U_2\partial_{U_3} - U_3\partial_{U_2}),$$

$$X_9 = \partial_\varphi.$$

Introducing cylindrical coordinates (H, ω) into the two-dimensional space of vectors (U_2, U_3)

$$U_2 = H\cos\omega,\ \ U_3 = H\sin\omega,$$

the first two generators become

$$X_7 = -\sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi + \cos\varphi(\sin\theta)^{-1}\partial_\omega,$$

$$X_8 = -\cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi + \sin\varphi(\sin\theta)^{-1}\partial_\omega.$$

The singular vortex solution (Ovsiannikov, 1995) is defined by the representation

$$U = U(t,r),\ H = H(t,r),\ \rho = \rho(t,r),\ \omega = \omega(t,r,\theta,\varphi). \qquad (5.1)$$

The function $\omega(t,r,\theta,\varphi)$ is "superfluous": it depends on all independent variables. If $H = 0$, then the tangent component of the velocity vector is equal to zero.

This corresponds to the spherically symmetric flows. For a singular vortex, it is assumed that $H \neq 0$.

In a manner similar to (Ovsiannikov, 1995) one finds that for system (1.1)-(1.4), the invariant functions $U(t,r)$, $H(t,r)$ and $\rho(t,r)$ have to satisfy the system of partial differential equations with the two independent variables t and r:

$$\begin{gathered} r^2 D_o\rho + \rho(r^2U)_r = \rho\alpha h, \ D_oU + \rho^{-1}p_r = r^{-3}\alpha^2, \\ D_oh = r^{-2}\alpha(h^2+1), \quad D_o\alpha = 0, \\ p = \rho(W_\rho - \dot\rho W_{\rho\dot\rho} - W_{\dot\rho\dot\rho}D_o\dot\rho) + W_{\dot\rho}\dot\rho - W, \end{gathered} \tag{5.2}$$

where $\alpha = rH$, $D_o = \partial_t + U\partial_r$, and the function $h(t,r)$ is introduced for convenience during the compatibility analysis.

5.2 Admitted Lie group of equation

The equivalence Lie group of equations (5.2) corresponds to the generator

$$X_0^e = \partial_t, \quad X_3^e = 2t\partial_t - U\partial_U - 3\rho\partial_\rho - 5\dot\rho\partial_{\dot\rho} - 3W\partial_W,$$

$$X_4^e = \dot\rho\partial_{\dot\rho} + \rho\partial_\rho + W\partial_W, \quad X_5^e = x\partial_x + U\partial_U + 2\alpha\partial_\alpha + 2W\partial_W.$$

An admitted generator X of equations (5.2) is sought in the form

$$X = \xi^{x_1}\partial_{x_1} + \xi^t\partial_t + \zeta^{u_1}\partial_{u_1} + \zeta^{u_2}\partial_{u_2} + \zeta^{u_3}\partial_{u_3} + \zeta^{u_4}\partial_{u_4} + \zeta^{u_5}\partial_{u_5},$$

where the coefficients of the generator are functions of the variables $r, t, \rho, v, \dot\rho, h, \alpha$.

Extensions of the kernel depend on the value of the function $W(\rho, \dot\rho)$. They can only be operators of the form

$$X = k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4$$

where

$$X_1 = -\alpha\partial_\alpha - \dot\rho\partial_{\dot\rho} - v\partial_v + t\partial_t,$$

$$X_2 = -5\dot{\rho}t\partial_{\dot{\rho}} - vt\partial_v - 3\rho t\partial_\rho - 3\rho\partial_{\dot{\rho}} + t^2\partial_t + tx\partial_x + x\partial_v,$$

$$X_3 = -5\dot{\rho}\partial_{\dot{\rho}} - v\partial_v + 2t\partial_t + x\partial_x - 3\rho\partial_\rho,$$

$$X_4 = \dot{\rho}\partial_{\dot{\rho}} + \rho\partial_\rho$$

satisfy the conditions

$$k_2(5\dot{\rho}W_{\dot{\rho}\dot{\rho}\dot{\rho}} + 3\rho W_{\dot{\rho}\dot{\rho}\rho} + 5W_{\dot{\rho}\dot{\rho}}) = 0, \tag{5.3}$$

$$\begin{gathered} 5(k_0 - k_{10})\dot{\rho}^2 W_{\dot{\rho}\dot{\rho}\dot{\rho}} + (-5\rho\dot{\rho}k_{10} - 9\rho^2 k_2)W_{\dot{\rho}\dot{\rho}\rho} \\ +5(2k_3 - k_{10})\dot{\rho}W_{\dot{\rho}\dot{\rho}} - 15k_2\rho W_{\dot{\rho}\dot{\rho}}W_{\dot{\rho}\dot{\rho}\dot{\rho}} = 0, \end{gathered} \tag{5.4}$$

$$\begin{gathered} 25k_{10}\dot{\rho}^3(-3\dot{\rho}\rho W_{\dot{\rho}\dot{\rho}\dot{\rho}\rho} + 11\dot{\rho}W_{\dot{\rho}\dot{\rho}\dot{\rho}} - 3\rho^2 W_{\dot{\rho}\dot{\rho}\rho\rho} + 5\rho W_{\dot{\rho}\dot{\rho}\rho} + 11W_{\dot{\rho}\dot{\rho}}) \\ +3k_2\rho(27\dot{\rho}\rho^3 W_{\dot{\rho}\rho\rho\rho} + 9\dot{\rho}\rho^2 W_{\dot{\rho}\rho\rho} + 200\dot{\rho}^2 W_{\dot{\rho}\dot{\rho}} - 27\rho^3 W_{\rho\rho\rho} - 9\rho^2 W_{\rho\rho}) \\ +50k_3\dot{\rho}^3(3\rho W_{\dot{\rho}\dot{\rho}\rho} + 11W_{\dot{\rho}\dot{\rho}}) + 25k_0\dot{\rho}^4(3\rho W_{\dot{\rho}\dot{\rho}\dot{\rho}\rho} - 11W_{\dot{\rho}\dot{\rho}\dot{\rho}}) = 0, \end{gathered} \tag{5.5}$$

$$\begin{gathered} k_{10}(3\dot{\rho}\rho^2 W_{\dot{\rho}\rho\rho\rho} + 3\dot{\rho}\rho W_{\dot{\rho}\rho\rho} + 5\dot{\rho}^3 W_{\dot{\rho}\dot{\rho}\dot{\rho}\rho} + 8\dot{\rho}^2\rho W_{\dot{\rho}\dot{\rho}\rho\rho} + 10\dot{\rho}^2 W_{\dot{\rho}\dot{\rho}\rho} \\ -3\rho^2 W_{\rho\rho\rho} - 3\rho W_{\rho\rho}) + 2k_3(-3\dot{\rho}\rho W_{\dot{\rho}\rho\rho} - 5\dot{\rho}^2 W_{\dot{\rho}\dot{\rho}\rho} + 3\rho W_{\rho\rho}) \\ +k_0(6\dot{\rho}\rho W_{\dot{\rho}\rho\rho} - 5\dot{\rho}^3 W_{\dot{\rho}\dot{\rho}\dot{\rho}\rho} - 3\dot{\rho}^2\rho W_{\dot{\rho}\dot{\rho}\rho\rho} - 6\rho W_{\rho\rho}) = 0. \end{gathered} \tag{5.6}$$

where $k_0 = k_1 + 2k_3$ and $k_{10} = k_4 - 3k - 3$.

5.2.1 Case $k_2 \neq 0$

Let $k_2 \neq 0$, then equation (5.3) gives

$$5W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho} + 3W_{\dot{\rho}\dot{\rho}\rho} + 5W_{\dot{\rho}\dot{\rho}} = 0.$$

The general solution of this equation is

$$W = q_0\rho^{-5/3} + \varphi_1(\rho)\dot{\rho} + \varphi_2(\rho),$$

If $\varphi_2'' = 0$, then the extension of the kernel of admitted Lie algebras is given by the generators

$$X_1, X_2, X_3.$$

If $\varphi_2'' \neq 0$, then the extension of the kernel of admitted Lie algebras is given by the generators

$$X_2, X_3.$$

5.2.2 Case $k_2 = 0$

Let $k_2 = 0$, then equation (5.3) becomes

$$k_1 a + k_3 b + k_4 c = 0, \tag{5.7}$$

where

$$a = \dot\rho W_{\dot\rho\dot\rho\dot\rho},$$

$$b = 5\dot\rho W_{\dot\rho\dot\rho\dot\rho} + 3\rho W_{\dot\rho\dot\rho\rho} + 5W_{\dot\rho\dot\rho},$$

$$c = -(\dot\rho W_{\dot\rho\dot\rho\dot\rho} + \rho W_{\dot\rho\dot\rho\rho} + W_{\dot\rho\dot\rho}).$$

Let us analyze the vector space $Span(V)$, where the set V consists of vectors (a, b, c) with ρ and $\dot\rho$ are changed. If the function $W(\rho, \dot\rho)$ is such that $\dim(Span(V)) = 3$, then equation (5.4) is only satisfied for

$$k_1 = 0, k_3 = 0, k_4 = 0$$

which does not give extensions of the kernel of admitted subalgebras. Hence, one needs to study $\dim(Span(V)) \leq 2$.

Case $\dim(Span(V)) = 2$

Let $\dim(Span(V)) = 2$. There exists a constant vector $(\alpha, \beta, \gamma) \neq 0$, which is orthogonal to the set V

$$\alpha a + \beta b + \gamma c = 0. \tag{5.8}$$

This means that the function $W(\rho, \dot\rho)$ satisfies the equation

$$(\alpha + 5\beta + \gamma)\dot\rho W_{\dot\rho\dot\rho\dot\rho} + (3\beta + \gamma)\rho W_{\rho\dot\rho\dot\rho} = -(5\beta + \gamma)W_{\dot\rho\dot\rho}. \tag{5.9}$$

The characteristic system of this equation is

$$\frac{d\dot{\rho}}{(\tilde{\alpha}+\tilde{\gamma})\dot{\rho}} = \frac{d\rho}{\tilde{\alpha}\rho} = \frac{dW_{\dot{\rho}\dot{\rho}}}{-(\tilde{\alpha}+2\tilde{\beta})W_{\dot{\rho}\dot{\rho}}}, \tag{5.10}$$

where

$$\tilde{\alpha} = 3\beta+\gamma, \ \ \tilde{\beta} = \beta, \quad \tilde{\gamma} = \alpha+2\beta.$$

Case $\tilde{\alpha} = 0$. Assume that $\tilde{\alpha} = 0$. Because of equation (5.10) and the condition $W_{\dot{\rho}\dot{\rho}} \neq 0$, one has $\tilde{\gamma} \neq 0$. The general solution of equation

$$\frac{d\dot{\rho}}{\tilde{\gamma}\dot{\rho}} = \frac{dW_{\dot{\rho}\dot{\rho}}}{-2\tilde{\beta}W_{\dot{\rho}\dot{\rho}}} \tag{5.11}$$

is

$$W_{\dot{\rho}\dot{\rho}} = \tilde{\varphi}(\rho)^{\dot{k}}, \tag{5.12}$$

where $k = -2\tilde{\beta}/\tilde{\gamma}$, and $\widetilde{\varphi}(\rho)$ is an arbitrary function. Substitution of (5.12) into (5.4) leads to

$$(3k_3-k_4)\rho\tilde{\varphi}' + \tilde{\varphi}[k(k_1+5k_3-k_4)+k_3-k_4] \tag{5.13}$$

If $(3k_3-k_4) \neq 0$ the dimension $\dim(Span(V)) = 1$, it contradicts to the assumption. Hence, $(3k_3-k_4) = 0$ and from (5.13) one obtains $3k_3 = k_4$.

If $(k+2)(k+1) \neq 0$, then integrating (5.12), one finds

$$W(\rho,\dot{\rho}) = \tilde{\varphi}(\rho)\dot{\rho}^p + \varphi_1(\rho)\dot{\rho} + \varphi_2(\rho), \tag{5.14}$$

where $p = k+2$, $\varphi_1(\rho)$ and $\varphi_2(\rho)$ are arbitrary functions. Substituting (5.14) into (5.3)-(5.6), one obtains $\varphi_2'' = 0$ and $k_3 = \frac{-k_1(p-2)}{2(p-1)}$ and $k_4 = \frac{-3k_1(p-2)}{2(p-1)}$. An extension of the kernel in this case is given by the generator

$$2(p-1)X_1 - (p-2)(X_3+3X_4).$$

where $p(p-1) \neq 0$.

If $k = -2$, then

$$W(\rho, \dot{\rho}) = \tilde{\varphi}(\rho)\ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho), \tag{5.15}$$

and similar to the previous case $\varphi_2'' = 0$. The extension of the kernel is

$$-X_1 + X_3 + 3X_4.$$

If $k = -1$, then

$$W(\rho, \dot{\rho}) = \varphi(\rho)\dot{\rho}\ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho), \tag{5.16}$$

and $\varphi_2'' = 0$. The extension of the kernel is

$$X_3 + 3X_4.$$

Assume that $\tilde{\alpha} \neq 0$. The general solution of equation (5.10) is

$$W_{\dot{\rho}\dot{\rho}}(\rho, \dot{\rho}) = \varphi(\dot{\rho}\rho^k)\rho^\lambda, \tag{5.17}$$

where $k = -(1 + \tilde{\gamma}/\tilde{\alpha})$, $\lambda = -(1 + 2\tilde{\beta}/\tilde{\alpha})$, and φ is an arbitrary function. Substitution of the function W into (5.8) leads to

$$\rho^k\dot{\rho}\varphi' k_0 + \varphi\tilde{k} = 0, \tag{5.18}$$

where

$$k_0 = k_1 + k_3(5 + 3k) - k_4(1 + k), \quad \tilde{k} = \lambda(3k_3 - k_4) - k_4 + 5k_3.$$

If $k_0 \neq 0$, then the dimension $\dim(Span(V)) = 1$. It contradicts to the assumption. Hence, $k_0 = 0$, which gives $k_3(6k + 3p + 5) - k_4(2k + 2p + 1) = 0$, and $k_1 = (k + 1)k_4 - (3k + 5)k_3$, where $p = 2k + 5/3$.

If $6k + 3p + 5 = 0$, then $k_4 = 0$, and

$$3\varphi_2'''\rho + \varphi_2''(6k + 11) = 0.$$

The general solution of this equation is $\varphi_2'' = C\rho^{-\mu}$, where $\mu = 2k + 11/3$. The extension of the kernel is defined by the generator

$$-(3k+5)X_1 + X_3.$$

If $6k + 3p + 5 \neq 0$, then $k_3 = (\frac{2kp+1}{6k+3p+5})k_4$, and

$$\varphi_2'''\rho - \varphi_2''(p-2) = 0.$$

The general solution of the last equation is $\varphi_2'' = C\rho^{p-2}$. The extension of the kernel is given by the generator

$$-2(k+p)X_1 + (2k+p+1)X_3 + (6k+3p+5)X_4.$$

Case $\dim(Span(V)) = 1$

There exists a constant vector $(\alpha, \beta, k) \neq 0$ such that

$$(a, b, c) = (\alpha, \beta, \gamma)B \tag{5.19}$$

Equation (5.19) are linear algebraic equations with respect to $W_{\rho\dot{\rho}\dot{\rho}}, W_{\dot{\rho}\dot{\rho}\dot{\rho}}$ and $W_{\dot{\rho}\dot{\rho}}$. These equations give that $(\beta + 2\gamma - 2\alpha) \neq 0$ and

$$\dot{\rho}W_{\dot{\rho}\dot{\rho}\dot{\rho}} - kW_{\dot{\rho}\dot{\rho}} = 0,$$

$$\rho W_{\rho\dot{\rho}\dot{\rho}} - \lambda W_{\dot{\rho}\dot{\rho}} = 0,$$

where

$$k = \frac{2\alpha}{\beta + 3\gamma - 2\alpha}, \quad \lambda = -\frac{\beta + 5\gamma}{\beta + 3\gamma - 2\alpha}.$$

The general solution of these equations is

$$W_{\dot{\rho}\dot{\rho}} = C\dot{\rho}^k\rho^\lambda, \tag{5.20}$$

where $C \neq 0$ is constant.

If $(k+2)(k+1) \neq 0$, then integrating (5.20) one obtains

$$W(\rho, \dot{\rho}) = -q_0\rho^{\lambda}\dot{\rho}^p + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho), \tag{5.21}$$

where $p(p-1) \neq 0$, $p = k+2$, and $q_0 = \frac{C}{p(p-1)}$. Substituting (5.21) into equations (5.3)-(5.6), one has

$$k_1 = \frac{k_3(-3\lambda - 5p + 5) + k_4(\lambda + p - 1)}{p-2}.$$

and

$$\varphi_2'''\rho(p-2)(3k_3 - k_4) + \varphi_2''((3k_3 - k_4)(2\lambda - 4 + 3p) + 2pk_3) = 0.$$

Consider $p \neq 2$.

If $\varphi_2'' = 0$, then k_3, k_4 are arbitrary constants. The extensions of the kernel are given by the generators.

$$(-3\lambda - 5p + 5)X_1 + (p-2)X_3, \quad (\lambda + p - 1)X_1 + (p-2)X_4.$$

If $\varphi_2'' = C\rho^{-\mu} \neq 0$, then in this case $3k_3 - k_4 \neq 0$. Let us introduce

$$\mu = [(2\lambda + 3p - 4) - \frac{2pk_3}{3k_3 - k_4}]\frac{1}{(p-2)}$$

The last relation can be rewritten as

$$\frac{2pk_3}{(3k_3 - k_4)} = (2\lambda + 3p - 4) - \mu(p-2). \tag{5.22}$$

We will solve this equation either with respect to k_3 or k_4. This leads us to the study of two cases. In the first case we assume that

$$(2\lambda + 3p - 4) - \mu(p-2) = 0. \tag{5.23}$$

Equation (5.22) gives that $k_3 = 0$. This defines the extension of the kernel, which is defined by the generator

$$(\lambda + p - 4)X_1 + (p-2)X_4. \tag{5.24}$$

In the second case, we assume that

$$(2\lambda + 3p - 4) - \mu(p - 2) \neq 0.$$

Equation (5.22) gives

$$k_4 = -(2p + 3((2\lambda + 3p - 4) - \mu(p - 2)))k_3.$$

The extension of the kernel becomes

$$2(-\lambda+(p-1)\mu-2p+2)X_1+(2\lambda+(p+2)\mu+3p-4)X_3+(6\lambda-(3p-6)\mu+11p-12)X_4. \tag{5.25}$$

Notice that (5.25) is reduced by (5.24) to (5.23). Thus these two cases can be defined in one case which is given by (5.25).

Consider $p = 2$. This case is separated into subcases, either $\lambda \neq -1$ or $\lambda = -1$.

Let us study $\lambda \neq -1$. Substituting (5.21) into equations (5.3)-(5.6) gives $k_4 = \frac{k_3(3\lambda+5)}{\lambda+1}$ and

$$\varphi_2''' \rho k_3 + [(k_1(\lambda + 1) + k_3(\lambda + 2)]\varphi_2'' = 0.$$

If $\varphi_2'' = 0$ then the extension of the kernel is combined by the generators

$$X_1, \ \ (3\lambda + 5)X_4 + (\lambda + 1)X_3.$$

Let us assume that $\lambda = -1$. Equations (5.3)-(5.6) give that $k_3 = 0$ and

$$k_4 \rho \varphi_2''' + (2k_1 + k_4)\varphi_2'' = 0.$$

If $\varphi_2'' = 0$, then the extension of kernel are given by the generators

$$X_1, \ \ X_4.$$

If $\varphi_2'' = C\rho^{-\mu} \neq 0$, we have $k_1 = (\frac{\mu-1}{2})k_4$ and the extension of the kernel is given by the generator

$$-(\mu+1)X_1 + 2X_4.$$

If $k = -2$, then integrating (5.20), one obtains

$$W(\rho, \dot{\rho}) = -q_0\rho^\lambda \ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho), \quad (q_0 \neq 0). \tag{5.26}$$

Substituting (5.26) into equations (5.3)-(5.6), we have

$$k_1 = \frac{k_3(3\lambda - 5) + k_4(-\lambda + 1)}{2},$$

and the condition

$$2(3k_3 - k_4)(\rho\varphi_2''' - (\lambda-2)\varphi_2'') = -q_0\rho^{\lambda-2}\lambda(\lambda-1)(\lambda(3k_3 - k_4) + 5k_3 - k_4). \tag{5.27}$$

Let $\lambda(\lambda - 1) = 0$. Equation (5.27) becomes

$$(3k_3 - k_4)(\rho\varphi_2''' - (\lambda-2)\varphi_2'') = 0.$$

If φ_2 is arbitrary, then $k_3 = 3k_4$, and the extension is only given by the generator

$$-X_1 + X_3 + 3X_4.$$

If $\varphi_2'' = C_2\rho^{\lambda-2} \neq 0$, then the extension of the kernel consists of the generators

$$(3\lambda - 5)X_1 + 2X_3, \quad (1-\lambda)X_1 + 2X_4.$$

Let $\lambda(\lambda - 1) \neq 0$, then $\varphi_2'' = \rho^{\lambda-2}(-q_0\frac{\lambda(\lambda-1)}{2}\mu \ln\rho + c_2)$, and the extension of the kernel is

$$(\mu - 2\lambda)X_1 + (\lambda - \mu + 1)X_3 + (3\lambda - 3\mu + 5)X_4.$$

If $k = -1$, then integrating (5.8), one finds

$$W(\rho, \dot{\rho}) = c_1\rho^\lambda\dot{\rho}\ln(\dot{\rho}) + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho), \tag{5.28}$$

and substituting (5.28) into equations (5.3)-(5.6), we obtain

$$k_1 = (3k_3 - k_4)\lambda,$$

and the condition

$$\rho\varphi_2'''(3k_3 - k_4) + \varphi_2''(2\lambda(-3k_3 + k_4) + k_3 - k_4) = 0.$$

One needs to study two cases. If $\varphi_2'' = 0$, then k_3, k_4 are arbitrary, and the extension of the kernel consists of the generator

$$3\lambda X_1 + X_3, \quad -\lambda X_1 + X_4.$$

If $\varphi_2'' = c_2\rho^{-\sigma} \neq 0$, then $3k_3 - k_4 \neq 0$, and substituting (5.28) into equations (5.3)-(5.6), we obtained

$$k_4 = \frac{k_3(6\lambda - 1 + 3\sigma)}{2\lambda - 1 + \sigma}.$$

In this case, the extension of kernel consists of the generator

$$-2\lambda X_1 + (2\lambda - 1 + \sigma)X_3 + (6\lambda - 1 + 3\sigma)X_4.$$

Case $\dim(Span(V)) = 0$

The vector (a, b, c) is constant:

$$(a, b, c) = (\alpha, \beta, k)$$

with some constant values α, β and k. This leads to

$$W_{\dot{\rho}\dot{\rho}} = -2q_0,$$

where $-2q_0 = \frac{1}{2\beta} - \alpha - \frac{3}{2k} \neq 0$ is constant. Integrating this equation, one obtains

$$W(\rho, \dot{\rho}) = -2q_0\dot{\rho}^2 + \dot{\rho}\varphi_1(\rho) + \varphi_2(\rho). \tag{5.29}$$

Substituting (5.29) into equations (5.3)-(5.6), we have

$$k_4 = 5k_3$$

and the condition

$$\varphi_2''' \rho k_3 + \varphi_2''(k_1 + 2k_3).$$

If $\varphi_2'' = 0$, then the extension of the kernel is given by the generators

$$X_1, \quad X_3 + 5X_4.$$

If $\varphi_2'' \neq 0$, then $k_3 \neq 0$ ans $\varphi_2 = C_2\rho^{-\mu}$, where $\mu = k_1 + \frac{2k_3}{k_3}$. The extension of the kernel consist of the generator

$$(\mu - 2)X_1 + X_3 + 5X_4.$$

These extensions are presented in Table 5.1.

5.2.3 Steady-state special vortex

Let us consider the invariant solution corresponding to the kernel $\{X_0\}$. This type of solution for the gas dynamics equations was studied in (Chupakhin, 2003). The representation of the solution is

$$\rho = \rho(r),\ U = U(r),\ h = h(r),\ \alpha = \alpha(r).$$

Equations (5.2) become

$$\begin{gathered} U\rho' + \rho(r^2U)' = \rho\alpha h,\ UU' + \rho^{-1}p' = r^{-3}\alpha^2, \\ Uh' = r^{-2}\alpha(h^2+1),\ \ U\alpha' = 0, \\ p = \rho(W_\rho - U\rho' W_{\rho\dot\rho} - W_{\dot\rho\dot\rho}U(U\rho')') + W_{\dot\rho}U\rho' - W, \\ \dot\rho = U\rho'. \end{gathered} \tag{5.30}$$

In (Chupakhin, 2003) it is shown that for the gas dynamics equations all dependent variables can be represented through the function $h(r)$, which satisfies a first-order

Table 5.1 Group classification of equations (5.2)

	$W(\rho,\dot\rho)$	Extensions	Remarks
M_1	$-q_o\dot\rho^2\rho^{-5/3}+\beta(\rho)^{5/3}$	$X_2,\ \ X_3$	$\beta\neq 0$
M_2	$-q_o\dot\rho^{-5/3}$	$X_1,\ \ X_2,\ X_3$	
M_3	$\varphi(\rho)\dot\rho\ln\dot\rho$	X_3+3X_4	
M_4	$\varphi(\rho)\dot\rho^p$	$2(p-1)X_1-(p-2)(X_3+3X_4)$	$p(p-1)\neq 0$
M_5	$\varphi(\rho)\ln\dot\rho$	$-X_1+X_3+3X_4$	
M_6	$\rho^p\varphi(\dot\rho\rho^k)+\varphi_2(\rho)$	$-2(k+p)X_1+(2k+p+1)X_3$	$\varphi_2''=c_2\rho^{p-2}$
		$+(6k+3p+5)X_4$	
M_7	$-q_0\rho^\lambda\dot\rho\ln\dot\rho$	$3\lambda X_1+X_3,-\lambda X_1+X_4$	
M_8	$-q_0\rho^\lambda\dot\rho\ln\dot\rho+\varphi_2(\rho)$	$-2X_1+(2\lambda-1+\sigma)X_3$	$\varphi_2''=c_0\rho^{-\sigma}\neq 0$
		$+(6\lambda-1+3\sigma)X_4$	
M_9	$-q_0\rho^\lambda\ln\dot\rho+\varphi_2(\rho)$	$-X_1+X_3+3X_4$	$\lambda(\lambda-1)=0$
M_{10}		$(3\lambda-5)X_1+2X_3,$	$\varphi_2''=c_2\rho^{\lambda-2},$
		$(1-\lambda)X_1+2X_4$	$\lambda(\lambda-1)=0$
M_{11}		$(\mu-2\lambda)X_1+(\lambda-\mu+1)X_3$	$\lambda(\lambda-1)\neq 0$
		$+(3\lambda-3\mu+5)X_4$	$\varphi=\varphi_*$
M_{12}	$-q_0\rho^\lambda\dot\rho^p$	$\frac{-3\lambda-5p+5}{p-2}X_1+X_3,\ \ \frac{\lambda+p-1}{p-2}X_1+X_4$	$p\neq 2,$
M_{13}	$-q_0\rho^\lambda\dot\rho^2$	$(\mu-\lambda-2)X_1+(\lambda+1)X_3$	$\varphi_2''=C_2\rho^{-\mu}\neq 0$
		$(3\lambda+5)X_4$	
M_{14}	$-q_0\dot\rho^2+\varphi_2(\rho)$	$(\mu-2)X_1+X_3+5X_4$	$\varphi_2''=C_2\rho^{-\mu}\neq 0$
M_{15}	$-q_0\dot\rho^2$	$X_1,\ \ X_3+5X_4$	

ordinary differential equation. Here also all dependent variables can be defined through the function $h(r)$, but the equation for $h(r)$ is a fourth-order ordinary differential equation. In fact, since $H \neq 0$, from (5.30) one obtains that $U \neq 0$. Hence, $\alpha = \alpha_o$, where α_o is constant. From the first and third equations of (5.30), one finds

$$\rho = R_o \frac{h'}{\sqrt{h^2+1}}, \quad U = \frac{\alpha_o(h^2+1)}{h'}.$$

In this case

$$\dot{\rho} = -\alpha_o R_o h' \left(\frac{\sqrt{h^2+1}}{h'} \right)'$$

and after substituting ρ and $\dot{\rho}$ into the formula for the pressure, one has

$$p = F(h, h', h'', h'''),$$

where the function F is defined by the potential function W. Substituting representations of ρ, U and p into the second equation of (5.2), one obtains the fourth-order ordinary differential equation for the function $h(r)$.

5.2.4 Invariant solutions of (5.2) with $W = -q_o\dot{\rho}^2\rho^{-5/3} + \beta\rho^{5/3}$

The system of equations (5.2) with the potential function

$$W = -q_o\dot{\rho}^2\rho^{-5/3} + \beta\rho^{5/3}$$

admit the Lie group corresponding to the Lie algebra $L_3 = \{X_0, X_2, X_3\}$.

If $\beta = 0$, then there is one more admitted generator X_1. The four-dimensional Lie algebra with the generators $\{X_0, X_1, X_2, X_3\}$ is denoted by L_4.

The structural constants of the Lie algebra L_4 are defined by the table of commutators:

	X_0	X_1	X_2	X_3
X_0	0	X_0	X_3	$2X_0$
X_1		0	X_2	0
X_2			0	$-2X_2$
X_3				0

Solving the Lie equations for the automorphisms, one obtains:

$$A_0 : \begin{cases} \widetilde{x}_0 &= x_0 + a_0(x_1 + 2x_3) + a_0^2 x_2, \\ \widetilde{x}_3 &= x_3 + a_0 x_2, \end{cases} \qquad A_1 : \begin{cases} \widetilde{x}_0 &= x_0 e^{-a_1}, \\ \widetilde{x}_2 &= x_2 e^{a_1}, \end{cases}$$

$$A_2 : \begin{cases} \widetilde{x}_2 &= x_2 + a_2(x_1 + 2x_3) + a_2^2 x_0, \\ \widetilde{x}_3 &= x_3 + a_2 x_0, \end{cases} \qquad A_3 : \begin{cases} \widetilde{x}_0 &= x_0 e^{a_3}, \\ \widetilde{x}_2 &= x_2 e^{a_3}. \end{cases}$$

Construction of the optimal system of one-dimensional admitted subalgebras consists of using the automorphisms A_i, $(i = 0, 1, 2, 3)$ for simplifications of the coordinates (x_0, x_1, x_2, x_3) of the generator

$$X = \sum_{j=0}^{3} x_j X_j$$

Here k is the dimension of the Lie algebra L_k, $(k = 3, 4)$. In the case L_3 one has to assume that the coordinate $x_2 = 0$.

Besides automorphisms for constructing optimal system of subalgebras one can use involutions. Equations (5.2) posses the involutions E, corresponding to the change $t \to -t$. The involution E acts on the generator

$$X = \sum_{j=0}^{3} x_j X_j.$$

by transforming the generator X into the generator $\widetilde{X}$ with the changed coordinates:

$$E : \begin{cases} \widetilde{x}_0 &= -x_0, \\ \widetilde{x}_2 &= -x_2. \end{cases}$$

Here only the changed coordinates are presented.

5.2.5 One-dimensional subalgebras

One can decompose the Lie algebra L_4 as $L_4 = I \oplus N$, where $I = L_3$ is an ideal and $N = \{X_1\}$ is a subalgebra of L_4. Classification of the subalgebra $N = \{X_1\}$ is simple: it consists of the subalgebras:

$$N_1 = \{0\}, \quad N_2 = \{X_1\}.$$

According to the algorithm (Ovsiannikov, 1993) for constructing an optimal system of one-dimensional subalgebras one has to consider two types of generators: (a) $X = x_0X_0 + x_2X_2 + x_3X_3$, (b) $X = X_1 + x_0X_0 + x_2X_2 + x_3X_3$. Notice that case (a) corresponds to the Lie algebra L_3. Hence, classifying the Lie algebra L_4, one also obtains classification of the Lie algebra L_3.

Case (a)

Assuming that $x_0 \neq 0$, choosing $a_2 = -x_3/x_0$, one maps x_3 into zero. This means that $\widetilde{x}_3 = 0$. For simplicity of explanation, we write it as $x_3(A_2) \to 0$. In this case $x_2(A_2) \to \widetilde{x}_2 = x_2 - {x_3}^2/x_0$. If $\widetilde{x}_2 \neq 0$, then applying $x_2(A_1) \to \pm 1$, hence, the generator X becomes

$$X_2 + \alpha X_0, \ (\alpha = \pm 1).$$

If $\widetilde{x}_2 = 0$, then one has the subalgebra: $\{X_0\}$.

In the case $x_0 = 0$, if $x_3 \neq 0$ or $x_2 \neq 0$, then applying A_0, one can obtain $x_0 \neq 0$, which leads to the previous case. Hence, without loss of generality one also assumes that $x_3 = 0$, $x_2 = 0$. Thus, the optimal system of one-dimensional subalgebras in case (a) consists of the subalgebras

$$\{X_2 \pm X_0\}, \quad \{X_0\}. \tag{5.31}$$

This set of subalgebras also composes an optimal system of one-dimensional subalgebras of the algebra L_3.

Case (b)

Assuming that $x_0 \neq 0$, choosing $a_2 = -x_3/x_0$, one maps x_3 into zero. In this case $x_2(A_2) \to \widetilde{x}_2 = x_2 - x_3(1 - x_3)/x_0$. If $\widetilde{x}_2 \neq 0$, then applying A_1, and E_2 (if necessary), one maps the generator X into

$$X_1 + X_2 + \gamma X_0,$$

where $\gamma \neq 0$ is an arbitrary constant. If $\widetilde{x}_2 = 0$, then $x_0(A_0) \to 0$, and the generator X becomes X_1.

In the case $x_0 = 0$, if $2x_3 + 1 \neq 0$ or $x_2 \neq 0$, then, applying A_0, one can obtain $x_0 \neq 0$, which leads to the previous case. Hence, without loss of generality one also assumes that $x_3 = -1/2$, $x_2 = 0$, and the generator X becomes $X_3 - 2X_1$.

Thus, the optimal system of one-dimensional subalgebras of the Lie algebra L_4 consists of the subalgebras

$$\{X_2 \pm X_0\}, \quad \{X_0\}, \quad \{X_1 + X_2 + \gamma X_0\}, \quad \{X_3 - 2X_1\}, \quad \{X_1\}, \tag{5.32}$$

where $\gamma \neq 0$ is an arbitrary constant.

Remark 1. An optimal system of subalgebras for $W = -q_o\rho^{-3}\dot{\rho}^2 + \beta\rho^3$ with arbitrary β consists of the subalgebras (5.31).

Remark 2. The subalgebra $\{X_2 - X_0\}$ is equivalent to the subalgebra $\{X_3\}$.

5.2.6 Invariant solutions of $X_1 + X_2 + \gamma X_0$

The generator of this Lie group is

$$X = \gamma X_0 + X_1 + X_2 = (t^2 + t + \gamma)\partial_t + tr\partial_r - 3t\rho\partial_\rho + (r - U(t+1))\partial_U - \alpha\partial_\alpha.$$

To find invariants, one needs to solve the equation

$$XJ = 0,$$

where $J = J(t, r, \rho, U, \alpha, h)$. The solution of this equation depends on the value of γ.

Let $\gamma = \mu^2 + 1/4$. In this case invariants of the Lie group are

$$y = rs, \quad V = s(((t+1/2)^2 + \mu^2)U - rt), \quad R = \rho s^{-3}, \quad \Lambda = \alpha e^{\frac{1}{\mu}\arctan(\frac{2t+1}{2\mu})}, \quad h,$$

where

$$s = \left((t+1/2)^2 + \mu^2\right)^{-1/2} e^{\frac{1}{2\mu}\arctan(\frac{2t+1}{2\mu})}.$$

The representation of an invariant solution is

$$s\left(\left((t+1/2)^2 + \mu^2\right)U - rt\right) = V(y), \quad \rho = s^3 R(y), \quad \alpha = \Lambda e^{-\frac{1}{\mu}\arctan(\frac{2t+1}{2\mu})}, \quad h = h(y).$$

Substituting the representation of a solution into (5.2), one obtains the system of four ordinary differential equations

$$V' = -\frac{R'}{R}V + (\Lambda h - 8Vy)/(4y^2), \quad h' = \frac{\Lambda}{V}\frac{(h^2+1)}{4y^2}, \quad \Lambda' = \frac{\Lambda}{V},$$

$$R''' = (-((8((3(4(44V + 5y)y - 19\Lambda h)R + 308R'Vy^2)R'^{\,2}$$
$$-3(88R'Vy^2 - 9\Lambda hR + 12(6V + y)Ry)R''R)Vq_oy - 9R^{2/3}(4(4(2V + y)V - (4\mu^2 + 1)y^2)y^2$$
$$+(\Lambda - 4hVy)\Lambda)R^3)y - 18(8(R^{2/3}Vy^3 + 4\Lambda hq_o)Vy - (2h^2 + 1)\Lambda^2 q_o$$
$$-4(8(5V + y)V - (4\mu^2 + 1)y^2)q_oy^2)R'R^2))/(288R^2V^2q_oy^4).$$

Let $\gamma = -\mu^2 + 1/4$. A representation of a solution is

$$s(((t+1/2)^2 - \mu^2)U - rt) = V(y), \; \alpha(t + 1/2 - \mu)^{\frac{1}{2\mu}}(t + 1/2 + \mu)^{-\frac{1}{2\mu}} = \Lambda(y),$$
$$\rho(t + 1/2 - \mu)^{3\alpha_1}(t + 1/2 + \mu)^{3\alpha_2} = R(y), \quad h = h(y),$$

where

$$y = rs, \quad s = (t + 1/2 - \mu)^{-\alpha_1}(t + 1/2 + \mu)^{-\alpha_2}, \quad \alpha_1 = \frac{2\mu - 1}{4\mu}, \; \alpha_2 = \frac{2\mu + 1}{4\mu}.$$

In this case

$$V' = -V\frac{R'}{R} + \frac{\Lambda h - 2Vy)}{y^2}, \quad h' = \Lambda\frac{(h^2+1)}{Vy^2}, \quad \Lambda' = \frac{\Lambda}{V},$$

$$R''' = (528R''R'RV^2q_oy^4 + 72R''R^2Vq_oy^2(-3\Lambda h + 6Vy + y^2) - 616R'\ ^3V^2q_oy^4$$
$$+24R'\ ^2RVq_oy^2(19\Lambda h - 44Vy - 5y^2) + 18R'R^2(2R^{2/3}V^2y^4 - 8\Lambda^2h^2q_o$$
$$-4\Lambda^2q_o + 32\Lambda hVq_oy - 40V^2q_oy^2 - 8Vq_oy^3 - 4\mu^2q_oy^4 + q_oy^4)$$
$$+9R^{2/3}R^3y(4\Lambda^2 - 4\Lambda hVy + 8V^2y^2 + 4Vy^3 + 4\mu^2y^4 - y^4))/(72R^2V^2q_oy^4).$$

Let $\gamma = 1/4$. A representation of an invariant solution is

$$s((t+1/2)^2U - rt) = V(y), \quad \rho = s^3R(y), \quad \alpha = e^{2/(2t+1)}\Lambda(y), \quad h = h(y),$$

where

$$y = rs, \quad s = \frac{1}{(t+1/2)}e^{-1/(2t+1)}.$$

In this case

$$V' = -V\frac{R'}{R} + \frac{(\Lambda h - 2Vy)}{y^2}, \quad h' = \frac{\Lambda}{V}\frac{(h^2+1)}{y^2}, \quad \Lambda' = \frac{\Lambda}{V},$$

$$R''' = (528R''R'RV^2q_oy^4 + 72R''R^2Vq_oy^2(-3\Lambda h + 6Vy + y^2) - 616R'\ ^3V^2q_oy^4$$
$$+24R'\ ^2RVq_oy^2(19\Lambda h - 44Vy - 5y^2) + 18R'R^2(2R^{2/3}V^2y^4 - 8\Lambda^2h^2q_o - 4\Lambda^2q_o$$
$$+32\Lambda hVq_oy - 40V^2q_oy^2 - 8Vq_oy^3 + q_oy^4) + 9R^{2/3}R^3y(4\Lambda^2 - 4\Lambda hVy + 8V^2y^2 + 4Vy^3$$
$$-y^4))/(72R^2V^2q_oy^4).$$

These equations were obtained assuming that $V \neq 0$. The case $V = 0$ leads to

$$\Lambda = 0, \quad 2q_oR' - yR^{5/3} = 0.$$

5.2.7 Invariant solutions of $X_3 - 2X_1$

Invariants of the generator

$$X_3 - 2X_1 = r\partial_r - 3\rho\partial_\rho + U\partial_U + 2\alpha\partial_\alpha$$

are

$$U = rV(y), \quad \rho = r^{-3}R(y), \quad \alpha = r^2\Lambda(y), \quad h = h(y),$$

where $y = t$. Substitution into equations (5.2) gives that the functions $V(y)$, $R(y)$, $\Lambda(y)$ and $h(y)$ have to satisfy the equations

$$h' = \Lambda(h^2+1), \quad \Lambda' = -2\Lambda V, \quad R' = \Lambda h R,$$

$$3(R^{2/3}+6q_o)(V'+V^2) = \Lambda^2\left(4q_o(h^2-3)+3(R^{2/3}+6q_o)\right).$$

5.2.8 Invariant solutions of X_1

Invariants of the generator X_1

$$X_1 = t\partial_t - U\partial_U - \alpha\partial_\alpha$$

are

$$x, \quad Ut, \quad \rho, \quad h, \quad \alpha t.$$

An invariant solution has the representation

$$U = t^{-1}V(y), \quad \rho = R(y), \quad \alpha = t^{-1}\alpha(y), \quad h = h(y)$$

where $y = x$. Substituting into equations (5.2), one obtains

$$V' = -V\frac{R'}{R} + \frac{\Lambda h - 2Vy}{y^2}, \quad h' = \frac{\Lambda}{V}\frac{(h^2+1)}{y^2}, \quad \Lambda' = \frac{\Lambda}{V},$$

$$R''' = (132R''R'RV^2q_oy^4 + 18R''R^2Vq_oy^2(-3\alpha h + 6Vy + y^2) - 154R'^3V^2q_oy^4$$
$$+6R'^2RVq_oy^2(19\alpha h - 44Vy - 5y^2) + 9R'R^2(R^{2/3}V^2y^4 - 4\alpha^2h^2q_o - 2\alpha^2q_o + 16\alpha hVq_oy$$
$$-20V^2q_oy^2 - 4Vq_oy^3) + 9R^{2/3}R^3y(\alpha^2 - \alpha hVy + 2V^2y^2 + Vy^3))/(18R^2V^2q_oy^4).$$

Here it is assumed that $V \neq 0$. The case $V = 0$ only leads to the condition $\Lambda = 0$.

5.2.9 Invariant solutions of $X_2 + X_0$

$$X_2 = t(t\partial_t + r\partial_r - U\partial_U - 3\rho\partial_\rho) + r\partial_U,$$

Invariants of the generator

$$X_2 + X_0 = (t^2+1)\partial_t + tr\partial_r - 3t\rho\partial_\rho + (r - tU)\partial_U$$

are

$$r(t^2+1)^{-1/2},\ \ U(t^2+1)^{1/2} - rt(t^2+1)^{-1/2},\ \ \rho(t^2+1)^{3/2},\ \ \alpha,\ \ h.$$

An invariant solution has the representation

$$U(t^2+1)^{1/2} - rt(t^2+1)^{-1/2} = V(y),\ \ \rho = (t^2+1)^{-3/2}R(y),\ \ \alpha = \alpha(y),\ \ h = h(y).$$

where $y = r(t^2+1)^{-1/2}$. Substituting into equations (5.2), one has to study two cases: (a) $V = 0$, and (b) $V \neq 0$.

Assuming $V = 0$, one obtains that $\Lambda = 0$, and the function R satisfies the equation

$$2(5\beta R^{4/3} - 9q_o)R' + 9yR^{5/3} = 0.$$

If $V \neq 0$, then one obtains

$$V' = -V\frac{R'}{R} + \frac{(\Lambda h - 2Vy)}{y^2},\ \ h' = \frac{\Lambda}{V}\frac{(h^2+1)}{y^2},\ \ \Lambda' = 0,$$

$$R''' = (132R''R'RV^2q_oy^4 + 54R''R^2Vq_oy^2(-\Lambda h + 2Vy) - 154R'^{\,3}V^2q_oy^4$$
$$+6R'^{\,2}RVq_oy^2(19\Lambda h - 44Vy) - 10R^{1/3}R'R^3\beta y^4 + 9R'R^2(R^{2/3}V^2y^4 - 4\Lambda^2h^2q_o$$
$$-2\Lambda^2q_o + 16\Lambda hVq_oy - 20V^2q_oy^2 + 2q_oy^4) + 9R^{2/3}R^3y(\Lambda^2 - \Lambda hVy + 2V^2y^2$$
$$-y^4))/(18R^2V^2q_oy^4)$$

5.2.10 Invariant solutions of $X_2 - X_0$

Since the Lie algebra $\{X_2 - X_0\}$ is equivalent to the Lie algebra with the generator $\{X_3\}$, then for the sake of simplicity an invariant solution with respect to

$$X_3 = 2t\partial_t + r\partial_r - U\partial_U - 3\rho\partial_\rho$$

is considered here. Invariants of the generator X_3 are

$$rt^{-1/2}, \;\; Ut^{1/2}, \;\; \rho t^{3/2}, \;\; h, \;\; \alpha.$$

An invariant solution has the representation

$$U = t^{-1/2}V(y), \;\; \rho = t^{-3/2}R(y), \;\; \alpha = \alpha(y), \;\; h = h(y)$$

where $y = rt^{-1/2}$.

Substituting into equations (5.2), one has to study two cases: (a) $V - y/2 = 0$, and (b) $V - y/2 \neq 0$.

Assuming $V - y/2 = 0$, one obtains that $\Lambda = 0$, and the function R satisfies the equation

$$2(20\beta R^{4/3} + 9q_o)R' - 9yR^{5/3} = 0.$$

If $V - y/2 \neq 0$, then one obtains

$$V' = (y/2 - V)\frac{R'}{R} + \frac{2\Lambda h - (4V - 3y)y}{2y^2}, \;\; h' = \frac{\Lambda}{(V - y/2)}\frac{(h^2 + 1)}{y^2}, \;\; \Lambda' = 0,$$

$$
\begin{aligned}
R''' = \; & (132R''R'Rq_oy^4(4V^2 - 4Vy + y^2) + 108R''\alpha R^2hq_oy^2(-2V + y) \\
& +108R''R^2q_oy^3(4V^2 - 4Vy + y^2) + 154R'^3q_oy^4(-4V^2 + 4Vy - y^2) \\
& +228R'^2\alpha Rhq_oy^2(2V - y) + 264R'^2Rq_oy^3(-4V^2 + 4Vy - y^2) \\
& +72R'\alpha^2R^2q_o(-2h^2 - 1) + 288R'\alpha R^2hq_oy(2V - y) \\
& -40R^{1/3}R'R^3bey^4 + 9R'R^2y^2(4R^{2/3}V^2y^2 - 4R^{2/3}Vy^3 + R^{2/3}y^4 \\
& -80V^2q_o + 80Vq_oy - 22q_oy^2) + 36R^{2/3}\alpha^2R^3y \\
& +18R^{2/3}\alpha R^3hy^2(-2V + y) + 9R^{2/3}R^3y^3(8V^2 - 8Vy + 3y^2)) \\
& /(18R^2q_oy^4(4V^2 - 4Vy + y^2))
\end{aligned}
$$

CHAPTER VI

INVARIANT SOLUTIONS OF ONE OF MODELS

This chapter is focused on obtaining invariant solutions of the equations of fluid with the potential function

$$W = -a\rho^{-3}\dot{\rho}^2 + \beta\rho^3. \tag{6.1}$$

where $a \neq 0$ and β are constant. The fluid dynamics equations corresponding to this model admit Lie algebra of the maximally dimension. These Lie algebras include the generator X corresponding to projective transformations. The motivation of the study of this model is that for the gas dynamics equations, existence of the projective transformation corresponds to a gas with a special structure.

6.1 Optimal system of admitted subalgebras

6.1.1 Admitted Lie algebra

The one-dimensional equations

$$\begin{gathered} \rho(u_t + uu_x) + P_x = 0, \quad \dot{\rho} + \rho u_x = 0, \\ P = 2a\rho^{-2}(\dot{\rho} + \ddot{\rho} + \rho^{-1}\dot{\rho^2}) + 2\beta\rho^3 \end{gathered} \tag{6.2}$$

with the potential function (6.1) admit the Lie group corresponding to the Lie algebra L_5 with the generators (Hematulin, Meleshko and Gavrilyuk, 2007)

$$Y_1 = \partial_t, \;\; Y_2 = \partial_x, \;\; Y_3 = t\partial_x + \partial_u, \;\; Y_4 = \rho\partial_\rho + u\partial_u - x\partial_x - 2t\partial_t,$$

$$Y_5 = t(x\partial_x + t\partial_t - \rho\partial_\rho - u\partial_u) + x\partial_u.$$

If $\beta = 0$, then there is one more admitted generator,

$$Y_6 = t\partial_t - u\partial_u.$$

The six-dimensional Lie algebra with the generators $\{Y_1, Y_2, ..., Y_6\}$ is denoted by L_6.

The structural constants of the Lie algebra are defined by the table of commutators:

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	0	0	Y_2	$-2Y_1$	$-Y_4$	Y_1
Y_2		0	0	$-Y_2$	Y_3	0
Y_3			0	Y_3	0	$-Y_3$
Y_4				0	$-2Y_5$	0
Y_5					0	$-Y_5$
Y_6						0

Solving the Lie equations (3.22) for the automorphisms, one obtains:

$$A_1 : \begin{cases} \widetilde{y_1} &= y_1 + \tau_1(y_6 - 2y_4) + \tau_1^2 y_5, \\ \widetilde{y_2} &= y_2 + \tau_1 y_3, \\ \widetilde{y_4} &= y_4 - \tau_1 y_5, \end{cases} \qquad A_5 : \begin{cases} \widetilde{y_3} &= y_3 - \tau_5 y_2, \\ \widetilde{y_4} &= y_4 + \tau_5 y_1, \\ \widetilde{y_5} &= y_5 + \tau_5(2y_4 - y_6) + \tau_5^2 y_1, \end{cases}$$

$$A_2 : \begin{cases} \widetilde{y_2} &= y_2 - \tau_2 y_4, \\ \widetilde{y_3} &= y_3 + \tau_2 y_5, \end{cases} \quad A_3 : \begin{cases} \widetilde{y_2} &= y_2 - \tau_3 y_1, \\ \widetilde{y_3} &= y_3 + \tau_3(y_4 - y_6), \end{cases}$$

$$A_4 : \begin{cases} \widetilde{y_1} &= y_1 e^{2\tau_4}, \\ \widetilde{y_2} &= y_2 e^{\tau_4}, \\ \widetilde{y_3} &= y_3 e^{-\tau_4}, \\ \widetilde{y_5} &= y_5 e^{-2\tau_4}, \end{cases} \quad A_6 : \begin{cases} \widetilde{y_1} &= y_1 e^{-\tau_6}, \\ \widetilde{y_3} &= y_3 e^{\tau_6}, \\ \widetilde{y_5} &= y_5 e^{\tau_6}. \end{cases}$$

Construction of the optimal system of one-dimensional admitted subalgebras consists of using the automorphisms $A_i,\ (i = 1, 2, ..., k)$ for simplifications coordinates

$(y_1, y_2, ..., y_k)$ of the generator

$$Y = \sum_{j=1}^{k} y_k Y_k.$$

Here k is the dimension of the Lie algebra L_k, $(k = 5, 6)$. In the case L_5 one has to assume that the coordinate $y_6 = 0$.

Besides automorphisms for constructing optimal system of subalgebras one can use involutions. Equations (6.2) posses two involutions. The first involution E_1 corresponds to the change $x \to -x$. The second involution E_2 is related with the change $t \to -t$. These involutions act on the generator

$$Y = \sum_{j=1}^{k} y_k Y_k$$

by transforming the generator Y into the generator $\widetilde{Y}$ with the changed coordinates:

$$E_1 : \begin{cases} \widetilde{y}_2 &= -y_2, \\ \widetilde{y}_3 &= -y_3, \\ & \end{cases} \quad E_2 : \begin{cases} \widetilde{y}_1 &= -y_1, \\ \widetilde{y}_3 &= -y_3, \\ \widetilde{y}_5 &= -y_5. \end{cases}$$

Here only the changed coordinates are presented.

6.1.2 One-dimensional subalgebras

One can decompose the Lie algebra L_6 as $L_6 = I \oplus N$, where $I = L_5$ is an ideal and $N = \{Y_6\}$ is a subalgebra of L_6. Classification of the subalgebra $N = \{Y_6\}$ is simple: it consists of the subalgebras:

$$N_1 = \{0\}, \quad N_2 = \{Y_6\}.$$

According to the algorithm (Ovsiannikov,1993) for constructing an optimal system of one-dimensional subalgebras one has to consider two types of generators: (a) $Y = \sum_{j=1}^{5} y_k Y_k$, (b) $X = Y_6 + \sum_{j=1}^{5} y_k Y_k$. Notice that case (a) corresponds to the

Lie algebra L_5. Hence, classifying the Lie algebra L_6, one also obtains classification of the Lie algebra L_5.

Case (a)

Assuming that $y_1 \neq 0$, choosing $\tau_5 = -y_4/y_1$, one maps y_4 into zero. This means that $\widetilde{y_4} = 0$. For simplicity of explanation we write it as $y_4(A_5) \to 0$. In this case $y_5(A_5) \to \widetilde{y_5} = y_5 - y_4^2/y_1$. If $\widetilde{y_5} \neq 0$, then sequentially applying A_3, A_2 and A_4: $y_2(A_3) \to 0$, $y_3(A_2) \to 0$, and then using A_4, the generator Y becomes

$$Y_5 + \alpha Y_1,\ (\alpha = \pm 1).$$

If $\widetilde{y_5} = 0$, then $y_2(A_3) \to 0$, and using A_4, one obtains two subalgebras:

$$\{Y_1 + Y_3\},\quad \{Y_1\}.$$

In the case $y_1 = 0$, if $y_4 \neq 0$ or $y_5 \neq 0$, then, applying A_1, one can obtain $y_1 \neq 0$, which leads to the previous case. Hence, without loss of generality one also assumes that $y_4 = 0$, $y_5 = 0$. The generator Y becomes $y_2Y_2 + y_3Y_3$. If $y_2 \neq 0$, then $y_3(A_5) \to 0$, and if $y_3 \neq 0$, then $y_2(A_1) \to 0$. One obtains the subalgebras

$$\{Y_2\},\quad \{Y_3\}.$$

Thus, the optimal system of one-dimensional subalgebras in case (a) consists of the subalgebras

$$\{Y_5 \pm Y_1\},\quad \{Y_1 + Y_3\},\quad \{Y_1\},\quad \{Y_2\},\quad \{Y_3\}. \tag{6.3}$$

This set of subalgebras also composes an optimal system of one-dimensional subalgebras of the algebra L_5.

Case (b)

Assuming that $y_1 \neq 0$, choosing $\tau_5 = -y_4/y_1$, one maps y_4 into zero. This means that $\widetilde{y_4} = 0$. which is written as $y_4(A_5) \to 0$. In this case $y_5(A_5) \to \widetilde{y_5} = y_5 + y_4(1 - y_4)/y_1$. If $\widetilde{y_5} \neq 0$, then sequentially applying A_3, A_2 and A_4: $y_2(A_3) \to 0$, $y_3(A_2) \to 0$, and then using A_4 and E_2 (if necessary), one maps the generator Y to

$$Y_6 + Y_5 + \alpha Y_1,$$

where $\alpha \neq 0$ is an arbitrary constant. If $\widetilde{y_5} = 0$, then, sequentially applying $y_1(A_1) \to 0$, and $y_3(A_3) \to 0$, the generator Y becomes $y_2 Y_2 + Y_6$. If $y_2 \neq 0$, then, using the automorphisms A_4 and the involution E_1, one comes to the subalgebra

$$\{Y_2 + Y_6\}.$$

The case $y_2 = 0$ gives the subalgebra:

$$\{Y_6\}.$$

In the case $y_1 = 0$, if $2y_4 - 1 \neq 0$ or $y_5 \neq 0$, then, applying A_1, one can obtain $y_1 \neq 0$, which leads to the previous case. Hence, without loss of generality one also assumes that $y_4 = 1/2$, $y_5 = 0$, and the generator Y becomes $y_2 Y_2 + y_3 Y_3 + Y_4 + 2Y_6$. Sequential application of A_3 and then A_2 leads to the subalgebra

$$\{Y_4 + 2Y_6\}.$$

Thus, the optimal system of one-dimensional subalgebras of the Lie algebra L_6 consists of the subalgebras

	Subalgebra		Subalgebra
1	$Y_5 \pm Y_1$		
2	$Y_1 + Y_3$	6	$\alpha Y_1 + Y_6 + Y_5$
3	Y_1	7	$Y_6 + \alpha Y_2$
4	Y_2	8	$2Y_6 + Y_4$
5	Y_3		

Here $\alpha \neq 0$ is an arbitrary constant.

Remark 1. Since the automorphism A_4 for $W = -a\rho^{-3}\dot{\rho}^2$ differs from the automorphism A_4 for the Green-Naghdi model, the subalgebras $Y_1 + \gamma Y_3$, $(\gamma \neq 0)$ considered in (Bagderina and Chupakhin, 2005) are equivalent here to $Y_1 + Y_3$.

Remark 2. Because of the automorphism A_4 the subalgebras $\{Y_5 + \beta Y_1\}$ are equivalent to one of the subalgebras: $\{Y_5 + Y_1\}$, $\{Y_5 - Y_1\}$ or $\{Y_5\}$. The subalgebra $\{Y_5 - Y_1\}$ is equivalent to $\{Y_4\}$. The subalgebra $\{Y_5\}$ is equivalent to $\{Y_1\}$. Notice also that the subalgebra $\{Y_6 + Y_5\}$ is equivalent to $\{Y_6\}$.

Remark 3. An optimal system of subalgebras for $W = -a\rho^{-3}\dot{\rho}^2 + \beta\rho^3$ with arbitrary β consists of the subalgebras (6.3).

6.2 Invariant solutions

6.2.1 Invariant solutions of $\alpha Y_1 + Y_5 + Y_6$

The generator of this Lie group is

$$Y = \alpha Y_1 + Y_5 + Y_6 = (t^2 + t + \alpha)\partial_t + tx\partial_x - t\rho\partial_\rho + (x - u(t+1))\partial_u.$$

For finding invariants one needs to solve the equation

$$YJ = 0,$$

where $J = J(t, x, \rho, u)$. A solution of this equation depends on the value of α.

Let $\alpha = 1/4 + \gamma^2$, $\gamma \neq 0$. In this case, invariants of the Lie group are

$$U = s\left(\left((t+1/2)^2+\gamma^2\right)u - xt\right), \quad R = x\rho,$$

where

$$s = \left((t+1/2)^2+\gamma^2\right)^{-1/2} e^{\frac{1}{2\gamma}\arctan\left(\frac{2t+1}{2\gamma}\right)}.$$

The representation of an invariant solution is

$$s\left(\left((t+1/2)^2+\gamma^2\right)u - xt\right) = U(y), \quad \rho = x^{-1}R(y), \quad y = xs.$$

Substituting the representation of a solution into (4.1), one obtains two ordinary differential equations. The general solution of the first equation (conservation of mass) is

$$U = kyR^{-1}.$$

The second equation becomes a third-order ordinary differential equation for the function R:

$$\begin{gathered} 8R'''ak^2R^2y^3 - 8R''akRy^2(10R'ky - R(7k-R)) + 120(R')^3ak^2y^3 \\ +8(R')^2akRy^2(3R-25k) + 2R'R^2y(R^2(a+4a\gamma^2-2k^2) + 4ak(11k-4R)) \\ -R^3((2k-R)^2+4\gamma^2R^2)(2a-R^2) = 0. \end{gathered} \tag{6.4}$$

If $\alpha = 1/4 - \gamma^2$, $\gamma \neq 0$. A representation of an invariant solution is

$$s\left(\left((t+1/2)^2-\gamma^2\right)u - xt\right) = U(y), \quad \rho = x^{-1}R(y),$$

where

$$s = (t+1/2-\gamma)^{-\gamma_1}(t+1/2+\gamma)^{-\gamma_2}, \quad y = xs, \ \gamma_1 = \frac{2\gamma-1}{4\gamma}, \ \gamma_2 = \frac{2\gamma+1}{4\gamma}.$$

In this case

$$U = kyR^{-1},$$

and the function R satisfies the third-order ordinary differential equation

$$\begin{gathered}8R'''ak^2R^2y^3 - 8R''akRy^2(10R'ky - R(7k-R)) + 120(R')^3ak^2y^3 \\ +8(R')^2akRy^2(3R-25k) + 2R'R^2y(R^2(a-4a\gamma^2-2k^2) + 4ak(11k-4R)) \\ -R^3((2k-R)^2 - 4\gamma^2R^2)(2a-R^2) = 0.\end{gathered} \tag{6.5}$$

Let $\alpha = 1/4$. A representation of an invariant solution is

$$s\left((t+1/2)^2u - xt\right) = U(y), \quad \rho = x^{-1}R(y),$$

where

$$s = (t+1/2)^{-1}e^{-1/(2t+1)}, \quad y = xs.$$

In this case

$$U = kyR^{-1},$$

and the function R satisfies the third-order ordinary differential equation

$$\begin{gathered}2R'''ak^2R^2y^3 - 2R''akRy^2(10R'ky - R(7k-4R)) + 30(R')^3ak^2y^3 \\ +2(R')^2akRy^2(12R-25k) + R'R^2y(R^2(8a-k^2) - 32akR + 22ak^2) \\ -R^3(2a-R^2)(k-2R)^2 = 0.\end{gathered} \tag{6.6}$$

Since equations (6.4), (6.5) and (6.6) are homogeneous, their order can be reduced. Using the substitution $R' = R(2a-R^2)f(R)/(2ay)$ these equations become

$$F + (f-1)a^2((R-2k)^2 + 4\gamma^2R^2) = 0,$$

$$F + (f-1)a^2((R-2k)^2 - 4\gamma^2R^2) = 0,$$

and

$$F + 4(f-1)a^2\left(6fkR + (2R-k)^2\right) = 0,$$

respectively. Here

$$\begin{gathered}F = (2a-R^2)kfR\left((2a-R^2)kR(ff''+f'^2) - 2f'(R(kfR+a) + 2ak(3f-2))\right) \\ +(f-1)a\left(4f^2k^2(6a-R^2) + 2fk(kR^2+4aR-10ak)\right).\end{gathered}$$

One can easily see that these equations have the constant solution $f = 1$. This solution corresponds to the relation

$$R' = \frac{R(2a - R^2)}{2ay}$$

or after integrating it

$$\frac{R}{\sqrt{R^2 - 2a}} = \frac{q}{y}, \tag{6.7}$$

where q is an arbitrary constant.

6.2.2 Invariant solutions of $Y_6 + \alpha Y_2,\ (\alpha = 0, 1)$

Invariants of the generator

$$Y_6 + \alpha Y_2 = t\partial_t + \alpha\partial_x - u\partial_u, \quad (\alpha = 0, 1)$$

are

$$x - \alpha \ln t, \quad ut, \quad \rho.$$

Hence, the invariant solution has the representation

$$u = t^{-1}U(y), \quad \rho = R(y), \quad y = x - \alpha \ln t.$$

Solving the conservation of mass equation, one obtains $U = \alpha + k/R$. Then the second equation becomes

$$2R'''ak^2R^2 - 2R''akR(10R'k + R^2) + 30(R')^3ak^2 + 6(R')^2akR^2$$
$$-R'k^2R^4 - R^6(k + \alpha R) = 0.$$

Using the substitution $R' = f(R)$, this equation is also can be reduced to a second-order ordinary differential equation.

6.2.3 Invariant solutions of $2Y_6 + Y_4$

Invariants of the generator

$$2Y_6 + Y_4 = -x\partial_x + \rho\partial_\rho - u\partial_u$$

are

$$t, \ \ u/x, \ \ \rho x.$$

Hence, the invariant solution has the representation

$$u = xU(t), \ \ \rho = x^{-1}R(t).$$

Substitution into equations (6.2) gives the trivial equations

$$R' = 0, \ \ (U' + U^2)(2a - R^2) = 0.$$

6.2.4 Invariant solutions of $Y_5 + Y_1$

Invariants of the generator

$$Y_5 + Y_1 = (t^2 + \alpha)\partial_t + tx\partial_x - t\rho\partial_\rho + (x - ut)\partial_u$$

are

$$x(t^2+1)^{-1/2}, \ \ u(t^2+1)^{1/2} - xt(t^2+1)^{-1/2}, \ \ \rho x.$$

An invariant solution has the representation

$$u(t^2+1)^{1/2} - xt(t^2+1)^{-1/2} = U(y), \ \ \rho = x^{-1}R(y),$$

where $y = x(t^2+1)^{-1/2}$. Substitution into equations (6.2) gives $U = ky/R$ and

$$2R'''ak^2R^2y^7 - 2R''ak^2Ry^6(10R'y - 7R) + 30(R')^3ak^2y^7$$
$$-50(R')^2ak^2Ry^6 + R'R^2y(y^4(22ak^2 + 2aR^2 - k^2R^2) + 6\beta R^6)$$
$$+R^3(y^4(k^2 + R^2)(R^2 - 2a) - 6\beta R^6) = 0$$

In the case $\beta = 0$ this equation is reduced by the substitution $R' = f(R)/y$ to the second-order ordinary differential equation

$$2ff''ak^2R^2 + 30f^3ak^2 - 20f^2f'ak^2R - 30f^2ak^2R + 2ff'^{\,2}ak^2R^2 + 10ff'ak^2R^2$$
$$+fR^2(10ak^2 + 2aR^2 - k^2R^2) + R^3(k^2 + R^2)(R^2 - 2a) = 0.$$

6.2.5 Invariant solutions of $Y_5 - Y_1$

Since the Lie algebra $\{Y_5 - Y_1\}$ is equivalent to the Lie algebra with the basis generator Y_4, then for the sake of simplicity an invariant solution with respect to

$$Y_4 = \rho\partial_\rho + u\partial_u - x\partial_x - 2t\partial_t$$

is considered here. Invariants of the generator Y_4 are

$$x/\sqrt{t}, \quad ux, \quad \rho x.$$

An invariant solution has the representation

$$u = x^{-1}U(y), \quad \rho = x^{-1}R(y),$$

where $y = x/\sqrt{t}$. Substitution into equations (6.2) gives $U = y^2(k/R + 1/2)$ and

$$8R'''ak^2R^2y^7 - 8ak^2Ry^6(10R''R'y - 7R) + 120(R')^3ak^2y^7$$
$$-200(R')^2ak^2Ry^6 + 2R'R^2y(y^4(44ak^2 - aR^2 - 2k^2R^2) + 12\beta R^6)$$
$$+R^3(y^4(4k^2 - R^2)(R^2 - 2a) - 24\beta R^6) = 0.$$

In the case $\beta = 0$ this equation is reduced by the substitution $R' = f(R)/y$ to the second-order ordinary differential equation

$$8ff''ak^2R^2 + 120f^3ak^2 - 80f^2f'ak^2R - 120f^2ak^2R + 8ff'^{\,2}ak^2R^2 + 40ff'ak^2R^2$$
$$+2fR^2(20ak^2 - aR^2 - 2k^2R^2) + R^3(4k^2 - R^2)(R^2 - 2a) = 0.$$

6.2.6 Invariant solutions of $Y_1 + Y_3$

Invariants of the generator

$$Y_1 + Y_3 = \partial_t + t\partial_x + \partial_u$$

are

$$x - t^2/2, \quad u - t, \quad \rho.$$

An invariant solution has the representation

$$u - t = U(y), \quad \rho = R(y),$$

where $y = x - t^2/2$. Substitution into equations (6.2) gives $U = k/R$ and

$$2ak^2(R'''R^2 - 10R''R'R + 15(R')^3) + R'R^4(6\beta R^4 - k^2) + R^7 = 0$$

This equation is also homogeneous, and, the it is easily reduced by the substitution $R' = f(R)$ to a second-order ordinary differential equation.

Remark 4. Using the criteria of linearization, obtained in (Ibragimov and Meleshko, 2005), (NeutPetitot, 2002), (Euler, WolfLeach and Euler, 2003), one can check that all previous third-order ordinary differential equations cannot be linearized by point, contact or the generalized Sundman transformations. The reduced second-order ordinary differential equations also do not satisfy the Lie criteria (Lie, 1883) of linearization.

6.2.7 Invariant solutions of Y_2

Invariants of the generator $Y_2 = \partial_x$ are

$$t, \quad u, \quad \rho.$$

An invariant solution has the representation

$$u = U(t), \quad \rho = R(t).$$

Substitution into equations (6.2) gives

$$R' = 0, \quad RU' = 0.$$

6.2.8 Invariant solutions of Y_3

Invariants of the generator $Y_3 = t\partial_x + \partial_u$ are

$$t, \quad u - x/t, \quad \rho.$$

An invariant solution has the representation

$$u - x/t = U(t), \quad \rho = R(t).$$

Substitution into equations (6.2) gives

$$R't + R = 0, \quad R\,(U't + U) = 0.$$

6.2.9 Invariant solutions of Y_1

Invariants of the generator $Y_1 = \partial_t$ are

$$x, \quad u, \quad \rho.$$

An invariant solution has the representation

$$u = U(x), \quad \rho = R(x).$$

Substitution into equations (6.2) gives $U = k/R$ and

$$2ak^2(R'''R^2 - 10R''R'R + 15(R')^3) + R'R^4(6\beta R^4 - k^2) = 0.$$

Assuming that $R' = f(R)$, the last equation is reduced to the second-order equation

$$f'' + \frac{1}{f}(f')^2 - \frac{10}{R}f' + \frac{15f}{R^2} + \frac{R^2}{2af}(6\frac{\beta}{k^2}R^4 - 1) = 0. \tag{6.8}$$

This equation is transformed to the free particle equation $z''(\tau) = 0$ by the change

$$z = ak^2 R^{-5} f^2(R) + \beta R^3 - \frac{k^2}{2} R^{-1}, \quad \tau = R.$$

Thus, one obtains the general solution of equation (6.8)

$$f^2(R) = \frac{R^4}{2ak^2} \left(k^2 + 2C_1 R + 2C_2 R^2 - 2\beta R^4 \right),$$

where C_1 and C_2 are arbitrary constants. After that the function $R(y)$ is found by quadrature $R' = f(R)$. Analysis of this solution is similar to the analysis of the soliton solution of the KdV equation.

CHAPTER VII

CONCLUSIONS

This thesis is devoted to an application of the group analysis method to the equations of fluids with internal inertia

$$\rho_t + div(\rho u) = 0,$$

$$\rho \dot{u} + \nabla p = 0,$$

where p is the pressure, and "dot" denotes the material time derivative: $\dot{f} = \frac{df}{dt} = f_t + u\nabla f$. The thesis is focused on group classification of a class of dispersive models which are defined by the Lagrangean $L = \frac{1}{2}\rho|u|^2 - W(\rho, \dot{\rho})$, where $W(\rho, \dot{\rho})$ is a potential function. For these models the pressure is

$$p = \rho\frac{\delta W}{\delta \rho} - W = \rho\left(\frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial \dot{\rho}}\right) - div\left(\frac{\partial W}{\partial \dot{\rho}}u\right)\right) - W.$$

This type of models includes Iordanski-Kogarko-Wijngaarden and Green-Naghdi models.

The first result of the thesis is group classification of the three-dimensional equations. Complete group classification of the equations is given. The group classification is considered with respect to the function $W(\rho, \dot{\rho})$. The equivalence Lie group is obtained on this step. The group classification separates all models with respect to admitted Lie group into 15 different classes. The result of this group classification is presented in Table (4.1).

The second part of the thesis is devoted to the special vortex solution. In contrast to the Navier-Stokes equations, the existence of solutions for this class of equations and fluids with internal inertia has been shown. For this class, the original three-dimensional system of equations is reduced to a system with two independent variables. Group classification of the reduced system is obtained. All invariant solutions of the reduced system with the potential function $W = -q_0\rho^{-5/3}\dot{\rho}^2 + \beta\rho^{5/3}$ are studied.

The last part of the thesis deals with the one-dimensional equations. All invariant solutions of fluids with the potential function $W = -a\rho^{-3}\dot{\rho}^2 + \beta\rho^3$ are considered in this part. All representations of invariant solutions are obtained by using the optimal system of admitted subalgebras. Analysis of the reduced equations is provided.

In the future work, analyze the fluids with internal inertia depending on $W(\rho, \dot{\rho}, S)$, where S is the entropy will be analyzed.

REFERENCES

REFERENCES

Bagderina, Yu. Yu. and Chupakhin A. P. (2005). Invariant and partially invariant solutions of the Green-Naghdi equations. **Journal of Applied Mechanics and Technical Physics**. 46: 791–799.

Cherevko A. A. and Chupakhin A. P. (2004). Homogeneous Singular Vortex. **J. Appl. Mech. Tech. Phys**. 45: 209–221.

Chupakhin A. P. (2003). Invariant submodels of special vortex. **Journal of Applied Mechanics and Technical Physics**. 67: 351–364.

Euler N. and Wolf T. and Leach P. G. L. and Euler M. (2003). Linearizable third-order ordinary differential equations and generalized Sundman transformations. **Acta Applicandae Mathematicae**. 76: 89–115.

Gavrilyuk, S. L. and Shugrin, S. M. (1996). Media with equations of state that depend on derivatives. **Communications in Nonlinear Science and Numerical Simulation**. 37: 177–189.

Gavrilyuk S. L. and Teshukov V.M. (2001). Generalized vorticity for bubbly liquid and dispersive shallow water equations. **Continuum Mech. Thermodyn.** 13: 365–382.

Golovin S. V. (2004). Applications of the differential invariants of infinite dimensional groups in hydrodynamics. **Communications in nonlinear science and numerical simulations**. 9: 35–51.

Green, A. E. and Naghdi, P. M. 1976. A derivation of equations for wave propagation in water of variable depth. **J. Fluid Mech**. 78: 237–246.

Hearn A. C.(1987). **REDUCE Users Manual**. Santa Monica: The Rand Corporation CP 78.

Hematulin A. and Meleshko S. V. and Gavrilyuk S. G. (2007). Group classification of one-dimensional equations of fluids with internal inertia. **Mathematical Methods in the Applied Sciences**. 30: 2101–2120.

Hematulin A. and Siriwat P. (2008).Invariant solutions of the special model of fluids with internal inertia. **Communications in Nonlinear Science and Numerical Simulation**. 4: 1–19.

Ibragimov N. H. (1994). Handbook of Lie Group Analysis of Differential Equations. **CRC Press**. 1.

Ibragimov N. H. (1995). Handbook of Lie Group Analysis of Differential Equations. **CRC Press**. 2.

Ibragimov N. H. (1996). Handbook of Lie Group Analysis of Differential Equations. **CRC Press**. 3.

Ibragimov N. H. and Meleshko S. V. (2005). Linearization of third-order ordinary differential equations by point and contact transformations. **J.Math.Anal.Appl**. vol 308: 266–289.

Iordanski S. V. (1960). On the equations of motion of the liquid containing gas bubbles. **Zhurnal Prikladnoj Mekhaniki i Tekhnitheskoj Fiziki**. 3: 102–111.

Kogarko B. S. (1961). On the model of cavitating liquid. **Dokl. AS USSR**. 137: 1331–1333.

Lie S. (1883). Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten. III. **Archiv for Matematik og Naturvidenskab**. 8: 371–427.

Neut S. and Petitot M. (2002). La géométrie de l'équation $y''' = f(x, y, y', y'')$. **C. R. Acad. Sci. Paris**. 335: 515–518.

Ovsiannikov L. V. (1964). Partially invariant solutions of the equations admitting a group. **Proceedings of 11-th Int. Congr. Appl. Mech**. 868–870.

Pavlenko A. S. (2005). Projective submodels of the Ovsyannikov vortex. **J. Appl. Mech. Tech. Phys**. 46: 459–470.

Popovych R. O. (1995). On Lie reduction of the Navier–Stokes equations. **Nonlinear Mathematical Physics**. 2: 301–311.

Salmon R. (1998). **Lectures on Geophysical Fluid Dynamics**. New York: Oxford University Press.

Wijngaarden van L. (1968). On the equations of motion for mixtures of liquid and gas bubbles. **J. Fluid Mech**. 33: 465–474.

CURRICULUM VITAE

NAME: Piyanuch Siriwat.

NATIONALITY: Thai. **GENDER:** Female.

DATE OF BIRTH: October 23, 1976.

EDUCATIONAL BACKGROUND

- B.Ed. in Mathematics, Chiang Mai University, Chiang Mai, Thailand, 1998.
- M.Sc. in Mathematics, Chiang Mai University, Chiang Mai, Thailand, 2001.

PUBLICATIONS:

- P. Siriwat and S.V. Meleshko. "Applications of Group Analysis to The Three-dimensional Equations of Fluids with Internal Inertia", Symmetry, Integrability and Geometry: Methods and Appications, (SIGMA) 4. (2008), 027, 19 pp.
- A. Hematulin and P. Siriwat. "Invariant Solutions of The Special Model of Fluids with Internal Inertia", Communications in Nonlinear Science and Numerical Simulations, Accepted 5 July 2008.
- Piyanuch Siriwat. "Application of Group Classification of Three-dimensional Equations of Fluids with Internal Inertia", The 8th National Graduate Research Conference 2007, Nakornpathom, Thailand.
- Piyanuch Siriwat. "Group Classification of Three-dimensional Equations of Fluids with Internal Inertia", The 1st SUT Graduate Conference 2007, Nakhon Ratchasima, Thailand.

SCHOLARSHIPS:

- The Ministry of University Affairs of Thailand (MUA), 2004-2007.