เอกสารประกอบการสอน

Optical Electronics: 107611

Written By

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Topics to Be Covered:

1. Introduction of Laser
2. Nonlinear Optical Susceptibility: Introduction of Nonlinear Optics, Descriptions of Nonlinear Optics Interactions, Definitions of Properties of Nonlinear Susceptibility
3. Maxell's Equations and Wave Propagation in Nonlinear Media: Optical Harmonic Generation, Four-Wave Mixing, Phase Matching
4. Review of Density Matrix Formulation
5. Nonlinear Optics Effect in Quantized Media

(Used to Be)Topics to Be Covered:

6. Introduction of Atoms and Radiation
7. Review of Electromagnetics Theory: Gaussian Beam; Coherence
8. Gain and Optical Amplification
9. Optical Resonators
10. Laser Oscillations
11. Laser Pumping and Some Common Laser Systems, Q-switching, mode-locking
12. Diode Laser

Grade Breakdown: Problem set: 20%
Midterm Exam: 30%
Final Exam: 30%
Presentation: 20%

Text Books:

1. Robert W. Boyd, "Nonlinear Optics"
2. A. Yariv, "Quantum Electronics"
3. Pantell and Puthoff, "Fundamentals of Quantum Electronics"
4. Bloembergen, "Nonlinear Optics"
5. Zernike and Midwinter, "Applied Nonlinear Optics"
6. A. Yariv, "Introduction to Electronics"
7. Shen, "Nonlinear Infrared Generation"
8. Shen, "Principle of Nonlinear Optics"
9. Reintjes, "Nonlinear Optical Parametric Process in Liquids and Gases"
10. G. P. Agarwal and R. W. Boyd, "Contemporary Nonlinear Optics"
11. Siegman, "Laser"

Written By: Dr. Sukanya Tachatraiphop

School of Laser and Photonics 21/4/2008

1. Energy Levels in Atoms and Molecules

1.1 Introduction

- Laser is inherently by Q.M. device.
  - ✓ Einstein referred to Plank constant.
  - ✓ Spectral distribution
  - ✓ Stimulated emission
  - ✓ Amplification
  - ✓ Population inversion
- First Laser

  1960 : Ruby Laser
- Bt. 1927-1960, we needed

  - optics
  - combination of Q.M. and optics
  - technology problem
- Stationary state
- Absorption
- Spectral line
- Rate equation $\frac{dn}{dt} = C(N_2 - N_1)$

  $N_2 > N_1 \rightarrow n$ will grow with time

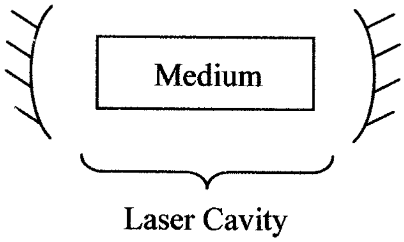
  i.e. the beam irradiance will be increased or amplified as it pass through the collection of atoms.

"Laser"

- Stimulated Emission Process
- Laser Cavity

  Single Pass: The beam irradiance will not be particular strong.

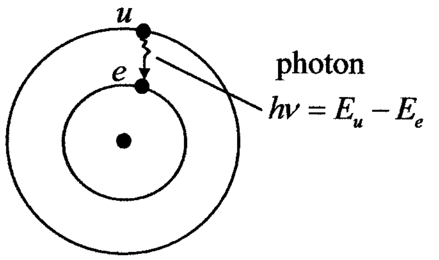
  Multi Passes : Using Mirrors



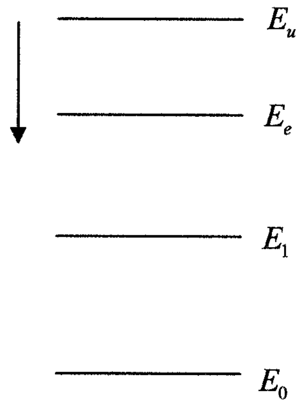
→ Reflectivity 100% at λ_{laser} i.e. no radiation from the cavity.

- Optical Feedback & wave front propagation.

Spontaneous Emission :

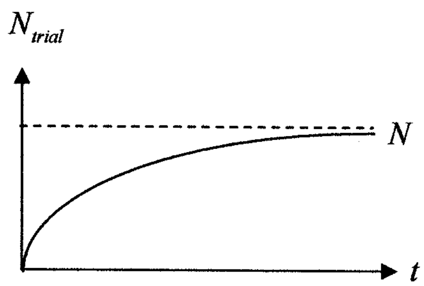


Bohr Atom



There is a finite probability per unit time A_{ul} that the e^- "jump" from $u \longrightarrow l$.

Repeatedly prepare a single atom in the upper state u. Do N trails to see if the photon is emitted before time t.



$N_{trail} = N\left(1 - e^{-A_{ul}t}\right)$

$\Downarrow$

# of atom at l states by spontaneous transition from $u \longrightarrow l$.

Probability that atom have made a spontaneous transition $u \longrightarrow l$ by elapse time t, $P_{ul}(t)$

$$P_{ul}(t) = \frac{N_{trail}}{N} = 1 - e^{-A_{ul}t}$$

A_{ul} $\downarrow$ Spontaneous transition probability "Just An Individual Atom"

Now expand this idea/concepts to many atoms. An average no. of atom remaining in state u after time $t, N_u(t)$ is described as

$$\begin{aligned} N_u(t) &= N_{u0} - N_{0u}P_{ul}(t) \\ &= N_{u0}e^{-A_{ul}t} \end{aligned}$$

Rate Equation:

$$\begin{aligned} \frac{dN_u(t)}{dt} &= -A_{ul}N_{u0}e^{-A_{ul}} \\ &= -A_{ul}N_u(t) \\ &= -\frac{N_u(t)}{(t_{sp})_{ul}}, \end{aligned}$$

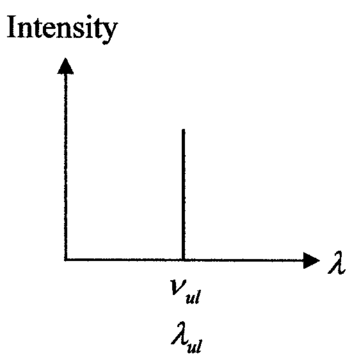
$(t_{sp})_{ul}$: Spontaneous life time

An average survival time in upper level before $u \longrightarrow l$ transition

where $(t_{sp})_{ul} = \frac{1}{A_{ul}}$.

2. Spectral Line Shape

Use spectrometer to observe the transition $u \longrightarrow l$.



$$hv_{ul} = E_u - E_l$$

$$I(\nu) \propto |E(\nu)|^2 \propto n(\nu)$$

$n(\nu)$ ↓ no. of photon

⇒ One would expect that the field has this frequency distribution given discrete transition.

t □ freq. :

$$E(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} e^{-i\omega t} \tilde{E}(w)$$

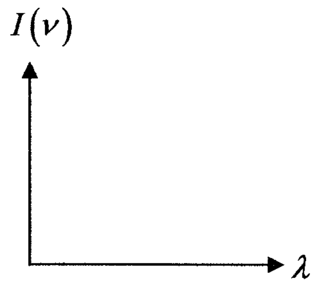
$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} E(t)$$

Here $\tilde{E}(\omega) \propto \delta(\omega - \omega_{ul}), \omega = 2\pi\nu_{ul}$

$$E(t) \propto \int \frac{d\omega}{\pi} e^{-i\omega t} \delta(\omega - \omega_{ul})$$

$$\propto e^{-i\omega_{ul} t}$$

Frequency distribution about ν_{ul} has a finite width (or non-zero bandwidth)



Finite bandwidth ?

is due to a number of different things

1) Finite duration of signal

2) Diff. broadening mechanism

Define: $g(\nu)d\nu \equiv$ Probability that a spontaneous emitted photon will appear at a frequency between ν and $\nu + d\nu$.

$g(\nu)$ is normalized so that

$$\int_0^{\infty} g(\nu) d\nu = 1$$

Origin of $g(\nu)$

a) Homogeneous Broadening

$g(\nu)$ is characteristic of a single emitter $\begin{pmatrix} \text{atom} \\ \text{molecule} \\ \text{ion} \end{pmatrix}$

"Natural Broadening"

The Q.M. of a single atom can be modeled as producing the field

$$E(t) = E_0 e^{-t/\tau} e^{-i\omega_{ul} t}.$$

Probability of measuring photon at time t,

$$P(t) \propto |E(t)|^2 = E_0^2 e^{-2t/\tau}$$

same way,

$$P(t) \propto e^{-A_{ul} t} = e^{-t/t_{sp}}$$

Prob. Atom made spontaneous $u \longrightarrow l$

$$P_{ul}(t) = 1 - e^{-A_{ul} t}$$

Prob. Still at the upper

$$= 1 - P_{ul}(t)$$
$$= e^{-A_{ul} t}$$

$$\frac{2}{\tau} \Rightarrow A_{ul}$$

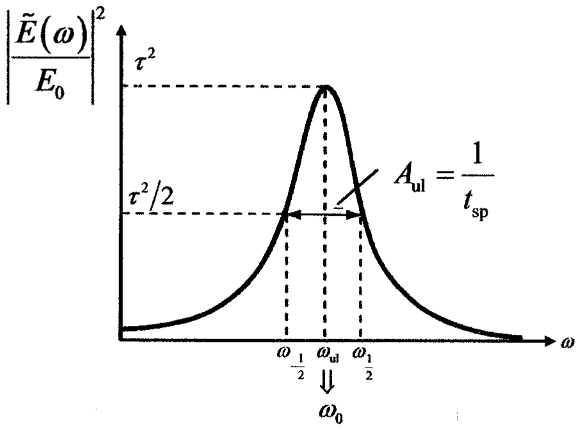
Spectral case:

$$\begin{aligned} \tilde{E}(\omega) &= \int dt e^{-i\omega t} E_0 e^{-t/\tau} e^{-\omega_{ul}} \\ &= E_0 \int_0^\infty dt e^{i\left(\omega - \omega_{ul} + \frac{i}{\tau}\right)t} \\ &= \frac{E_0}{i\left(\omega - \omega_{ul} + \frac{i}{\tau}\right)} \end{aligned}$$

From spectrometer, we see $\propto |\tilde{E}(\omega)|^2$

$$|\tilde{E}(\omega)|^2 \propto \frac{E_0^2}{(\omega - \omega_{ul})^2 + \frac{1}{\tau^2}}$$

i) Lorentzian Distribution



$$\frac{\tau^2}{2} = \frac{1}{(\omega-\omega_0)^2 + \frac{1}{\tau^2}} \quad \Rightarrow \quad \omega - \omega_0 = \pm\frac{1}{\tau}$$

$$\omega_{\pm 1/2} = \omega_0 \pm \frac{1}{\tau}$$

$$\Delta\omega = \omega_{+1/2} - \omega_{-1/2} = \frac{2}{\tau} = A_{ul} = \frac{1}{t_{sp}}$$

$$\Delta\nu = \frac{\Delta\omega}{2\pi} = \frac{A_{ul}}{2\pi} = \frac{1}{2\pi t_{sp}}$$

In general, for natural broadening

$$\Delta\nu = \frac{1}{2\pi}\left(\sum_k A_{uk} + \sum_m A_{lm}\right)$$

Homogeneous broadening = single emitter all emitter repeat single emission.

ii) Collision Broadening

During the emission of a single atom/molecule/ion often particles collide with it.

Imagine the duration of an individual collision to be short compare to $\frac{2\pi}{\omega}$

$$t_{\text{collision}} < t_{\text{period}}$$

$$\langle \Delta t \rangle_{\text{collision}} = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{N} \Delta t_i$$

or collision $\equiv \langle \Delta t_0 \rangle = \tau_0$

Recall:

$$
\begin{aligned}
\left|\tilde{E}(\omega)\right|^2 &= \frac{E_0^2}{(\omega-\omega_0)^2 + (1/\tau)^2} \\
\Delta\omega &= \frac{2}{\tau} \\
\tau &= \frac{2}{\Delta\omega} \\
\left|\tilde{E}(\omega)\right|^2 &\propto \frac{1}{(\omega-\omega_0)^2 + (\Delta\omega/2)^2} \\
&\propto \frac{1}{(\nu-\nu_0)^2 + (\Delta\nu/2)^2}
\end{aligned}
$$

where $\Delta\nu = \frac{1}{\tau_c}$.

Total homogeneous line width

$$\Delta\nu_{tot}^{h} = \frac{1}{2\pi}\underbrace{(\overset{\sum A_{uk}}{A_u} + \overset{\sum A_{lk}}{A_e})}_{\text{Natural broad}} + \underbrace{\nu_{cu} + \nu_{cl}}_{\text{collision}}$$

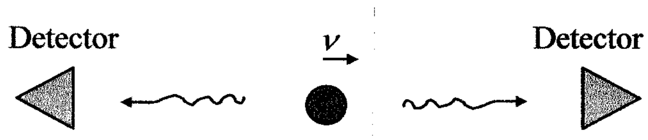
$$= \frac{1}{2\pi}\left(\underbrace{\frac{1}{\tau_u} + \frac{1}{\tau_l}}_{\text{spon.}} + \underbrace{\frac{1}{\tau_{ul}} + \frac{1}{\tau_{cl}}}_{\text{collision}}\right)$$

B) Inhomogeneous Broad ~ Doppler

Here $g(\nu)$ results from a distribution of emission frequency from a group of atoms.

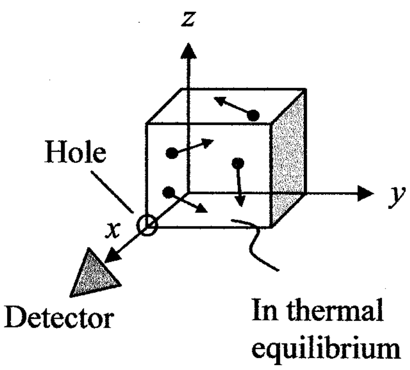
$\Rightarrow$ supports of the sample emit at diff. frequency giving rise to spectral distribution

i) Doppler Broad.



$\nu = \nu_0 - \frac{v}{c} \cdot \nu_0$	$\nu = \nu_0 + \frac{v}{c} \cdot \nu_0$
at frequency ν_0	$= \nu_0 \left(1 + \frac{v_x}{c}\right)$
$\nu = \nu_0 \left(1 - \frac{v_x}{c}\right)$	
red shift	blue shift
"Low frequency"	"High frequency"

Consider a box of gas at temperature T



The detector is sensitive only the x-component of atom velocity.

Atoms obeys Maxwell-Boltzmann velocity distribution of particle (atom)

$$f(v_x, v_y, v_z) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}(v_x^2 + v_y^2 + v_z^2)}$$

k — Boltzmann constant

$$T \uparrow \ \vec{v} \uparrow$$

Then, the probability distribution becomes

$$\iiint dv_x dv_y dv_z f\left(v_x, v_y, v_z\right) = 1$$

Probability of finding a particle with

$$v_x \text{ in } \left[v_x, v_x + dv_x\right]$$
$$v_y \text{ in } \left[v_y, v_y + dv_y\right]$$
$$v_z \text{ in } \left[v_z, v_z + dv_z\right]$$

Let relate it to $g(\nu)$:

Probability of increasing emission with frequency ν in

$$\left[\nu, \nu + d\nu\right] \equiv g(\nu) d\nu$$
$$\equiv f\left(v_x, v_y, v_z\right) dv_x dv_y dv_z$$

\ Probability of finding a particle with

appropriate velocity v_x so that ν is in the range

$[\nu, \nu + d\nu]$

(v_y, v_z are arbitrary)

B.T.W

$$\nu = \nu_0 + \frac{v}{c}\nu_0$$

only v_x

$$v_x = \left(\frac{\nu - \nu_0}{\nu_0}\right) c$$

$$dv_x = \frac{c}{\nu_0} d\nu$$

$$g(\nu)d\nu = \int_{-\infty}^{\infty} \iint \left[f(v_x, v_y, v_z) dv_y dv_z \right] dv_x$$

$$= \frac{c}{\nu_0} d\nu \iint dv_y dv_z f\left[\left(\frac{\nu - \nu_0}{\nu_0} \right) c, v_y, v_z \right]$$

$$= \left(\frac{m}{2\pi kT} \right)^{3/2} \left[\iint e^{-\frac{m}{2kT}\left(v_y^2 + v_z^2\right)} dv_y dv_z \right] \times \int e^{-\frac{mc^2}{2kT}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2} \frac{c}{\nu_0} d\nu$$

Note : $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$

$$g(\nu)d\nu = \int \frac{c}{\nu_0} \left(\frac{m}{2\pi kT} \right)^{1/2} e^{-\frac{mc^2}{2kT}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2} d\nu$$

Then, probability distribution of the Doppler broad:

$$g(\nu) = \frac{c}{\nu_0} \left(\frac{m}{2\pi kT} \right)^{1/2} e^{-\frac{mc^2}{2kT}\left(\frac{\nu-\nu_0}{\nu_0}\right)^2}$$

$$\frac{g(\nu \pm 1/2)}{g(\nu_0)} = \frac{1}{2}$$

$$= \exp\left(-\frac{mc^2}{2kT} \left(\frac{\nu \pm \nu_2 - \nu_0}{\nu_0} \right)^2 \right)$$

$$\Delta\nu_{\text{doppler}} = \nu_{+1/2} - \nu_{-1/2}$$

$$= \left(\frac{2kT}{mc^2} \ln 2 \right)^{1/2} 2\nu_0$$

$$= \Delta\nu_{\text{D}}$$

$$g_{\text{D}} = \frac{2(\ln 2)^{1/2}}{\sqrt{\pi}\Delta\nu_{\text{D}}} \exp\left[-4\ln\left(\frac{\nu - \nu_0}{\Delta\nu_{\text{D}}} \right)^2 \right]$$

Comparison of Different types of Line Broadenning :

Gas Laser Medium : $\Delta \nu_{\text{natural}} < \Delta \nu_{\text{doppler}}$

At Low Pressure : $\Delta \nu_{\text{D}} > \Delta \nu_{\text{collision}}$

High Pressure : $\Delta \nu_{\text{coll}} > \Delta \nu_{\text{D}}$

$$\Delta \nu_{\text{coll}} > \Delta \nu_{\text{nat}}$$

$$\Delta \nu_{\text{nat}} \square \frac{1}{2\pi} 2\Delta \nu_{\text{C}}$$

$$\square \frac{1}{2\pi} \Delta \nu_{\text{C}}$$

$\Rightarrow$ cooling

Ex : He-Ne laser

$$\lambda = 6328 \text{ A}^{0}$$

$$\Delta \nu_{\text{nat}} \square 3\times 10^{6} \sec^{-1}$$

$$t_{\text{sp}} = \frac{1}{A} = \frac{1}{\Delta \nu_{\text{nat}}} = \frac{10^{-6}}{3} \sec$$

$$= 300 \text{ ns}$$

At room temperature (300°K)

Atomic no.

$$m_{\text{Ne}} = 1.67\times 10^{-27} \text{ kg} \times 20 \text{ kg}$$

$$\Delta \nu_{\text{D}} = \left(\frac{2kT}{mc^{2}} \ln 2 \right)^{1/2} 2\nu_{0}$$

$$= \left(\frac{2\times 1.38\times 10^{-23} (300) \ln 2}{20\times 1.67\times 10^{-27} \left(3\times 10^{8}\right)^{2}} \right)^{1/2} \frac{2\times 3\times 10^{8}}{6328\times 10^{-10}}$$

$$= 1.3\times 10^{9} \sec^{-1}$$

$$\square \ \nu_{\text{nal}}$$

- Solid medium : Doppler

Typical order : $\Delta\nu_{col} > \Delta\nu_{nal} > \Delta\nu_{dop}$

vibration of atom called photon

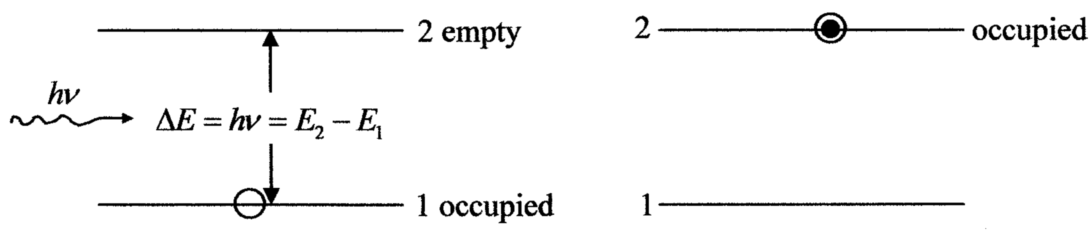
- Convolution

In gas, the net broadening is a convolution between natural / collision (Homogeneous or Lorentzian) and doppler broadening (Inhomogeneous or Gaussian)

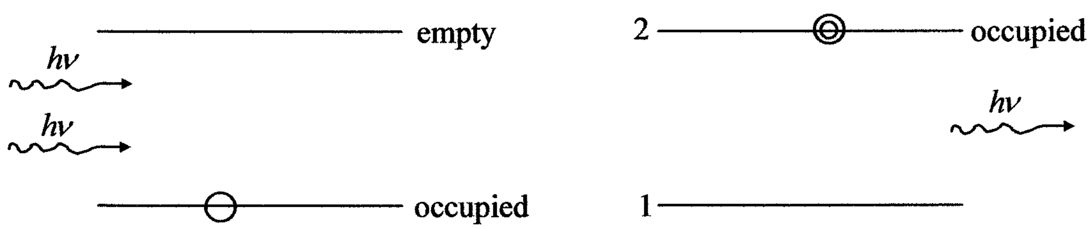
Absorption and Stimulated Emission :

Absorption : $$\frac{dN_2}{dt} = B_{12} N_1 \rho(\nu) = -\frac{dN_1}{dt}$$

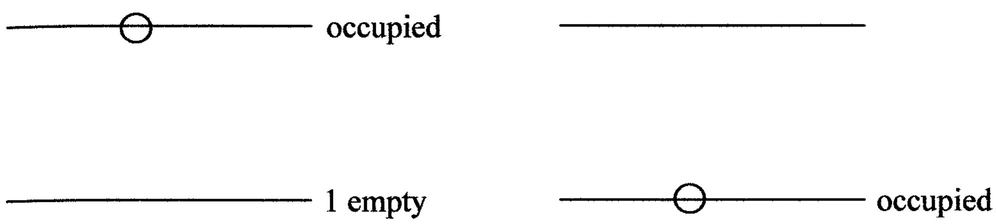
B_{12} → Absorb coef.; $\rho(\nu)$ → field; N_1 → No. of atoms at state1



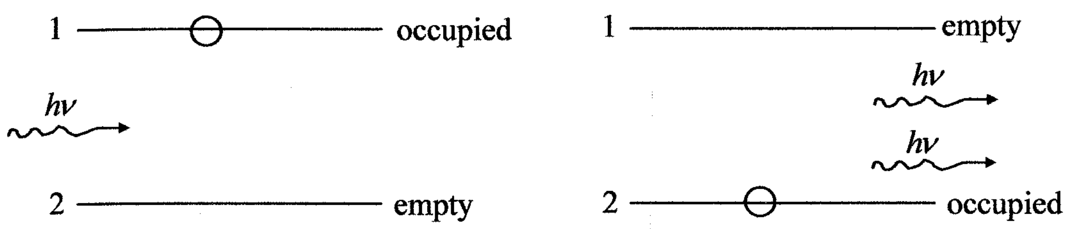
Initial state final state



Spontaneous Emission : $\frac{dN_2}{dt} = -A_{21} N_2$



Stimulated Emission :



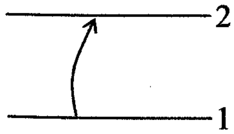
Interactions of atom and radiation

Consider the radiation field described by $\rho(\nu)$ where $\rho(\nu)d\nu$ is radiation field energy density in $\frac{\text{Joule}}{\text{m}^2}$ in $[\nu, \nu + d\nu]$

⇒ Absorption and stimulated emission rate for a single atom (or per atom) is proportional to $\rho(\nu)$.

⇒ The stimulated emission prob. or absorption prob. is proportional to the photon density or the number of photons.

Consider : The absorption rate for a single atom



$$W_{12} = B_{12}\rho(\nu) \qquad \left\langle \frac{1}{\text{sec}} \right\rangle$$

Transition rate :

$$W_{21} = \underbrace{B_{21}\rho(\nu)}_{\text{stimu.}} + \underbrace{A_{21}}_{\text{spon}}$$

N_1 = # of atoms/m^3 in state 1

N_2 = # of atoms/m^3 in state 2

For radiative process :

$$\text{Ⓐ} — \frac{dN_2}{dt}\text{total} = -\left(B_{21}\rho(\nu)+A_{21}\right)N_2 + B_{12}N_1\rho(\nu) = -\frac{dN_1}{dt} \qquad \left\langle \frac{1}{m^3 s} \right\rangle$$

$$\frac{d}{dt}\left(N_1+N_2\right)=0$$

Assume that the atom and the radiation field in equilibrium

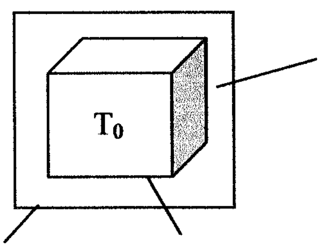
$$\boxed{\frac{dN_1}{dt}=\frac{dN_2}{dt}=0} \qquad \text{Steady State}$$

For steady state, $\frac{dN_2}{dt}$ in Ⓐ = 0, then

$$\text{Ⓑ} — \frac{N_2}{N_1}=\frac{B_{12}\rho(\nu)}{B_{21}\rho(\nu)+A_{21}}$$

Let consider "particular kind of equilibrium"

⇒ Thermal equilibrium



$$\text{Ⓒ} — \frac{N_2}{N_1}=e^{-h\nu/kT} \qquad \text{Boltzman's factor}$$

$T \; \square \; T_0$

final $T_0 \rightarrow T_0$

In thermal equilibrium more energetic level are less populated than lower level $\left(N_2 < N_1\right)$

Ⓑ = Ⓒ

$$e^{-h\nu/kT} = \frac{B_{12}\rho(\nu)}{B_{21}\rho(\nu)+A_{21}}$$

① — $$\rho(\nu) = \frac{A_{21}/B_{21}}{\left(\frac{B_{12}}{B_{21}}\right)e^{-h\nu/kT}-1}$$ "Thermal equilibrium"

We know that the energy density for EM field inside the cavity at the center frequency of interest is

② — $$\rho(\nu) = \frac{8\pi n^3 h\nu}{c^3}\left(\frac{1}{e^{-h\nu/kT}-1}\right)$$ n = index of refraction

"Plank Formular"

① = ②

$$\frac{A_{21}}{B_{21}} = \frac{8\pi n^3 h\nu^3}{c^3}$$

$$\frac{B_{12}}{B_{21}} = 1$$

These relationships are purely atomic in nature and independent of any assumption about "steady state" or "equilibrium"

Recall : $g(\nu)$ has finite width

$$W_{12}(\nu) = \int_{-\infty}^{\infty} W_{12}(\nu)d\nu$$

What is $W_{12}(\nu)$? Recall, absorp + spon emiss and inverse process

⇒ Absorp and spon emiss have frequency depended and this is given by $g(\nu)$

$$W_{12}(\nu) = B_{12} g(\nu)\rho(\nu)$$

$$W_{12}(\nu) = \int_{-\infty}^{\infty} B_{12} g(\nu)\rho(\nu)d\nu$$

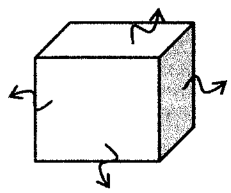
$$= B_{12}^{(0)} g(\nu_0) \underbrace{\int_{-\infty}^{\infty} g(\nu)d\nu}_{1}$$

$$W_{21}(\nu) = \int_{-\infty}^{\infty} B_{21} g(\nu)\rho(\nu)d\nu + A_{21}$$

$$= B_{21}\rho(\nu_0) + A_{21}$$

$$\neq W_{12}$$

Consider radiation distribution $\rho(\nu)$



Photon radiation in random direction

$$\frac{d\rho(\nu)}{dt} = W_{21}(\nu)N_2 h\nu - W_{12}(\nu)N_1 h\nu + N_2(\nu)A_{21}h\nu$$

$$W_{21}(\nu) = B_{21}^{(0)} g(\nu)\rho(\nu)$$

$$B_{21} = B_{12}$$

$$W_{12}(\nu) = B_{12} g(\nu)\rho(\nu)$$

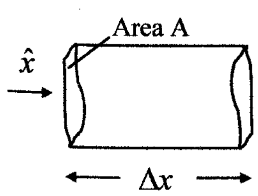
$W_{12}(\nu) = W_{21}$ $\Rightarrow$ rate of stim. emiss. per atom = rate of ab. per atom

$$\frac{d\rho(\nu)}{dt} = B_{21} g(\nu)\rho(\nu)h\nu(N_2 - N_1)$$

$$= \frac{c^3}{8\pi n^3 h\nu^3} A_{21}(\nu) g(\nu)\rho(\nu)h\nu(N_2 - N_1)$$

$$= \frac{c^3}{8\pi n^3 \nu^2 t_{sp}} g(\nu)\rho(\nu)(N_2 - N_1)$$

$I(\nu)d\nu =$ energy per unit area per unit time in $[\nu, \nu + d\nu]$



Relation between $\rho(\nu)+I(\nu)$ rate at which energy leaves vol.

$$= \mathrm{IA}$$

$$= \frac{\rho(\nu)\mathrm{A}_{\Delta x}}{\frac{\Delta x}{cn}}$$

$$= \frac{\rho c}{n}\mathrm{A}$$

Rate of change of energy density :

$$\frac{d\rho}{dt} \to \frac{\Delta\rho}{\Delta t} = \frac{\frac{n}{c}\Delta I}{\frac{n}{c}\Delta x} = \frac{\Delta I}{\Delta x} \to \frac{dI}{dx}$$

Therefore,

$$\frac{dI(\nu)}{dx} = \frac{c^2}{8\pi n^2 \nu^2 \mathrm{t}_{\mathrm{sp}}} \mathrm{g}(\nu)(\mathrm{N}_2 - \mathrm{N}_1) I(\nu)$$
$$= \gamma(\nu) I(\nu)$$

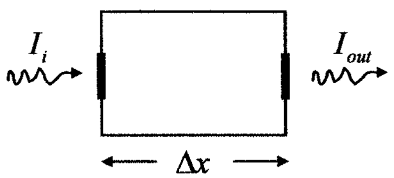
Where,

$$I_\nu(x) = I_\nu(0) e^{\gamma(\nu)x}$$

$\gamma(\nu)$ is called the "Gain" or "Loss"

Loss (or absorption) occurs for $\mathrm{N}_2 < \mathrm{N}_1$.

Gain (or amplification) occurs for $\mathrm{N}_2 > \mathrm{N}_1$.



Thermal Eq.

$\mathrm{N}_2 < \mathrm{N}_1$.

$\Rightarrow I_{\mathrm{out}} < I_i$.

$N_2 > N_1$ is not the natural situation, it's called "population inversion"

$$\boxed{\gamma(\nu) = \frac{c^2}{8\pi n^2 \left(\nu^2\right) t_{sp}} g(\nu)(N_2 - N_1) \qquad \left\langle \frac{1}{m} \right\rangle}$$

$$\nu \downarrow \gamma(\nu) \uparrow$$
$$\nu \uparrow \gamma(\nu) \downarrow \Rightarrow N_2 \square N_1$$

Like X-ray laser, it is hard to produce.

Again, at thermal equilibrium, there is

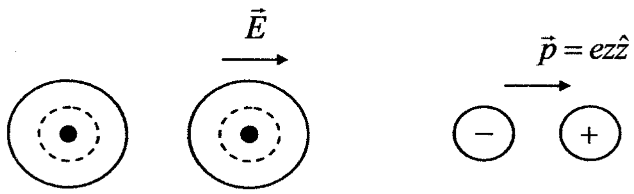
$$\frac{N_2}{N_1} = e^{-\Delta E/kT} < 1$$

"Non population Inversion"

<u>Simple model for atom :</u>

Some facts about atoms

1. fixed the nucleus surrounded by e^- charge
2. shape emission & absorption resonance
3. an external E field can introduced a dipole moment in the atom



Equate motion for electron

$$m\frac{d^2x}{dt^2} = -kx - eE$$

(CL) Resonance frequency of spring arrangement :

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Q.M.Correspondence with a real atom, it will be

$$\omega_0 = \frac{E_2 - E_1}{\hbar} \qquad \left(\hbar = \frac{h}{2\pi}\right)$$

We know that real Q.M. "oscillation" decays at a rate A. We also want our spring model to decay or damp

$$m\frac{d^2x}{dt^2} = -kx - eE - m\sigma\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \gamma\frac{dx}{dt} = -\frac{eE(t)}{m}$$

ω is important?

If $E(t)$ have ω not matching with spring resonance, spring doesn't affect much

$$E(t) \to \tilde{E}(\omega)$$

$$-\omega^2\tilde{\chi}(\omega) + \omega_0^2\tilde{\chi}(\omega) - i\omega\sigma\tilde{\chi}(\omega) = -\frac{e}{n}\tilde{E}(\omega)$$

$$\tilde{\chi}(\omega) = \frac{-\frac{e}{m}\tilde{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\sigma}$$

Dipole moment of a single oscillator

$$p(t) = e\underset{\sim}{\chi}(t) \;= -e\chi(t)$$

$$\tilde{p}(\omega) = e\underset{\sim}{\chi}(\omega) = -e\underset{\sim}{\chi}(\omega)\hat{x}$$

$$\tilde{p}(\omega) = \frac{+\frac{e^2}{m}\tilde{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\sigma}$$

Dipole moment per unit volume :

$$p(t) = \mathrm{N}p(t)$$

$$\tilde{p}(\omega) = \mathrm{N}\tilde{p}(\omega)$$

$$\tilde{p}(\omega) = \frac{\frac{\mathrm{N}e^2}{m}\tilde{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\sigma}$$

Recall :

$$p=\varepsilon_0\chi\mathrm{E}$$

$$\varepsilon_0=8.85\times10^{-12}$$

$$\tilde{p}(\omega)=\varepsilon_0\chi(\omega)\tilde{E}(\omega)$$

$$\chi(\omega)=\frac{\frac{\mathrm{Ne}^2}{m}}{\omega_0^2-\omega^2-i\omega\sigma}$$

Frequency dependent of susceptibility can invite

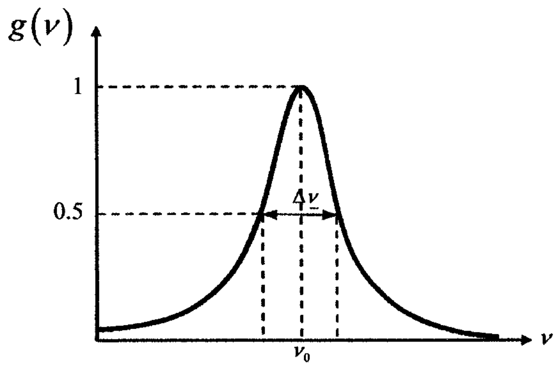
$$\begin{aligned}\chi(\omega)&=\frac{1}{\varepsilon_0}\frac{N_0^2}{m}\frac{\left(\omega_0^2-\omega^2\right)-i\omega\sigma}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\sigma^2}\\&=\chi'(\omega)+i\chi''(\omega)\\\chi'(\omega)&=\frac{1}{\varepsilon_0}\frac{Ne^2}{m}\frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\sigma^2}\\\chi''(\omega)&=\frac{1}{\varepsilon_0}\frac{Ne^2}{m}\frac{\omega\sigma}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\sigma^2}\end{aligned}$$

If the frequency of external field is near resonance i.e. $\omega=\omega_0,\ \tilde{\chi}(\omega)$ will become

$$\chi'(\omega)\cong\frac{Ne^2}{2\omega_0 m\varepsilon_0}\frac{\omega_0-\omega}{\left(\omega_0-\omega\right)^2+\frac{1}{4}\sigma^2}$$

$$\chi''(\omega)\cong\frac{Ne^2\sigma}{4\omega_0 m\varepsilon_0}\frac{1}{\left(\omega_0-\omega\right)^2+\frac{1}{4}\sigma^2}$$

Extra point to note



$$\int g(\nu)d\nu=1$$

If $g(\nu)$ were on stim, then

$$\int_{-\infty}^{\infty}g(\nu)d\nu=1=g(\nu_0)\Delta\nu_\nu$$

$$g(\nu_0)=\frac{1}{\Delta\nu}$$

For general $g(\nu)$, the proper expansion is

$$g(\nu) \cong \frac{1}{\Delta\nu} \qquad \langle \gamma \propto g(\nu)(N_2 - N_1) \rangle$$

"Wave number" units is cm^{-1}

$$\nu\lambda = \frac{c}{n}; \quad \nu = \frac{c}{n\lambda}$$

Since energy = $h\nu$; $\nu \propto$ Energy and $E \propto \frac{1}{\lambda}$

If the energy in wave number is "*y*" then the frequency corresponding to this "energy" is

$$\nu_1 = \frac{c}{n} \cdot y$$

Review Electromagnetic Theory :

Maxwell's equation :

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$\nabla \times \vec{H} = J_{\text{free}} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot D = \rho_{\text{free}}$$

$$\nabla \cdot B = 0$$

where

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} = \varepsilon_0 (1 + \chi) \underline{E}$$

$$\underline{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi_m) \underline{H} = \mu \underline{H}$$

$$\underline{M} = N \underline{m}$$

$\mu \approx \mu_{0,}$ non magnetic

$\Rightarrow \chi_n$ is very much

$$\nabla\times(\nabla\times\underline{\mathrm{E}})=\nabla\times-\frac{\partial B}{\partial t}$$

$$\nabla\cdot\underbrace{(\nabla\cdot\underline{\mathrm{E}})}_{\rho/\varepsilon_0}-\nabla^2\underline{\mathrm{E}}=-\frac{\partial}{\partial t}(\nabla\times\underline{\mathrm{B}})$$

$$\rho_{\text{free}}=0$$

$$-\nabla^2\underline{\mathrm{E}}=-\mu\frac{\partial}{\partial t}(\nabla\times\underline{\mathrm{H}})$$

$$=-\mu\left(\frac{\partial}{\partial t}J_{\mathrm{f}}+\frac{\partial^2 D}{\partial t^2}\right);\quad \text{no free current}$$

For most non metallic material $\mu\approx\mu_0$

$$-\nabla^2\underline{\mathrm{E}}\quad=-\mu_0\frac{\partial^2}{\partial t^2}(\varepsilon_0\underline{\mathrm{E}}\times\underline{\mathrm{P}})$$

$$\nabla^2\underline{\mathrm{E}}\quad=\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\bar{\mathrm{E}}=\mu\frac{\partial^2\vec{P}}{\partial t^2}(r,t)$$

Apply Fourier transform $\bar{E}(r,t)\rightarrow\tilde{E}(r,\omega)$

$$\nabla^2\tilde{E}(r,\omega)-\frac{\omega^2}{c^2}\tilde{E}(r,\omega)=-\mu_0\omega^2\tilde{P}(\vec{r},\omega)$$

$$=-\mu_0\omega^2\varepsilon_0\chi_T(\omega)\tilde{E}(r,\omega)$$

where

$$\chi_T(\omega)=\chi_0+\chi(\omega)$$

$$\nabla^2\tilde{E}(\vec{r},\omega)+\frac{\omega^2}{c^2}(1+(\chi_0+\chi))\tilde{E}(r,\omega)=0\qquad;\ v=\frac{c}{n}=\frac{1}{\sqrt{\varepsilon\mu}}$$

$$\nabla^2\tilde{E}+\frac{\omega^2}{c}(1+\chi_0)\left[1+\frac{\chi}{1+\chi_0}\right]\tilde{E}=0\qquad\qquad 1+\chi=\varepsilon$$

$$\nabla^2\tilde{E}+\frac{\omega^2}{c}n^2\left[1+\frac{\chi}{n^2}\right]\tilde{E}=0$$

For plane wave propagation along z : $\nabla^2 \rightarrow \left(\frac{\partial^2}{\partial z^2}\right)$

$$\frac{\partial^2}{\partial z^2}\tilde{E} + \frac{\omega^2}{c^2}n^2\left[1+\frac{\chi}{n^2}\right]\tilde{E} = 0$$

$$\tilde{E}(z,\omega) = \tilde{E}(0,\omega)e^{\pm i\beta(\omega)z}$$

$$\beta(\omega) = \frac{\omega}{c}n\left[1+\frac{\chi}{n^2}\right]^{1/2}$$

$$\frac{\chi}{n^2} = \frac{\chi}{1+\chi_0} = \frac{\chi/\chi_0}{1+1/\chi_0} < \frac{\chi}{\chi_0}; \text{ for } \chi \text{ small}$$

$$\propto \frac{\text{No. of active molecule}}{\text{No. of background molecule}}$$

□ 1 (order of %)

Using Taylor's expansion

$$\beta(\omega) \approx \frac{\omega}{c}n\left[1+\frac{1}{2}\frac{\chi}{n^2}\right] \quad : \quad (1+u)^q = 1 + qu + q\frac{(q-1)u^2}{2!}$$

Thus,

$$\tilde{E}(z,\omega) = \tilde{E}(0,\omega)e^{i\frac{\omega}{c}n\left[1+\frac{1}{2}\frac{\chi}{n^2}\right]z}$$

Recall for our medium of spring we have

$$\chi(\omega) = \chi'(\omega) + j\chi''(\omega)$$

$$\tilde{E}(z,\omega) = \tilde{E}(0,\omega)e^{i(k+\Delta k)z}e^{\frac{1}{2}\gamma(\omega)z}$$

wave no. in the host or background mat.

where $k = \frac{\omega}{c}n$; $\Delta k = \frac{\omega}{c}n\frac{\chi'}{2n^2} = \frac{k\chi'}{2n^2}$

$$\frac{1}{2}\gamma(\omega) = -\frac{k\chi''(\omega)}{2n^2}$$

gain or absorption coefficient

Note that !! $I(\omega,z) \propto \left|\tilde{E}(\omega,z)\right| \propto e^{\gamma(\omega)z}$

<u>Atomic susceptibility :</u>

Recall : Classical electron oscillator (when $\omega = \omega_0$)

$$\gamma(\omega) = \frac{-k\chi''(\omega)}{n^2}$$

$$① \text{———————} = \frac{-k}{n^2}\frac{Ne^2\sigma}{4\omega_0 m\varepsilon_0}\frac{1}{(\omega-\omega_0)^2+\frac{\sigma}{4}}$$

Let $\Delta\nu = \frac{\sigma}{2\pi} < 0,$ absorption

Lorenzian $\dfrac{1}{(\nu-\nu_0)^2+\dfrac{\sigma^2}{4\cdot 4\pi^2}} \longrightarrow \left(\dfrac{\sigma}{2\pi}\right)^2\dfrac{1}{4} \Rightarrow \dfrac{\Delta\nu^2}{4}$

Recall : Q.M. expression for γ

$$\gamma_{\text{Q.M.}}(\nu) = \frac{c^2}{8\pi n^2\nu^2 t_{sp}} g(\nu)(N_2 - N_1) \qquad \left\langle\frac{1}{\text{cm}}\right\rangle$$

Assume that we also have

$$② \text{———————} \quad \gamma_{\text{Q.M.}}(\nu) = \frac{-k}{n^2}\chi''_{\text{Q.M.}}(\nu)$$

(CL = Q.M)

① = ②

$$\chi''_{\text{Q.M.}}(\nu) = \frac{-n^2}{k}\gamma_{\text{Q.M.}}(\nu) = \frac{-n^2}{k}\frac{c^2}{8\pi n^2\nu^2 t_{sp}} g(\nu)(N_2 - N_1)$$

$$= -\frac{1}{16\pi^2}\frac{\lambda^3}{n t_{sp}} g(\nu)(N_2 - N_1)$$

If $g(\nu)$ is Lorntzian, then

$$\chi''_{\text{Q.M.}}(\nu) = -\frac{1}{16\pi^2}\frac{\lambda^3}{nt_{sp}}g(\nu)(N_1 - N_2)$$
$$= \frac{(N_1 - N_2)\lambda^3}{8\pi^3 t_{sp}\Delta\nu n}\frac{1}{1 + \frac{4(\nu - \nu_0)^2}{\Delta\nu^2}}$$

(What's about a Q.M. cally correct expression for χ' i.e. $\chi'_{Q.M.}$?)

For the spring model, note that

$$\frac{\chi'}{\chi''} = \frac{2}{\sigma}(\omega_0 - \omega) = \frac{4\pi}{\sigma}(\nu_0 - \nu)$$
$$= \frac{2}{\Delta\nu}(\nu_0 - \nu)$$

We will assume that also

$$\frac{\chi'_{\text{Q.M.}}}{\chi''_{\text{Q.M.}}} = \frac{2}{\Delta\nu}(\nu_0 - \nu)$$
$$\chi'_{\text{Q.M.}} = \chi''_{\text{Q.M.}}(\nu) * \frac{2}{\Delta\nu}(\nu_0 - \nu)$$

Cross section for Absorption or stimulated Emission

$$\gamma(\nu) = \frac{c^2}{8\pi n^2 \nu^2 t_{sp}}g(\nu)(N_2 - N_1)$$
$$\gamma(\nu) = \sigma(\nu)(N_2 - N_1)$$

↓

cross section for cm^2, Area

Gaussian Beams :

Laser system has been propagating that a beam implies that insufficient ray is far from the axis of prop. The field die away

$\longrightarrow$ It's not a plane wave :

Because properties ① finite in transverse extent

② prop. $\perp$ To transverse region

For free space or uniform medium,

$$\nabla \cdot \vec{E} = 0$$

$$\nabla = \nabla_t + \hat{z}\frac{d}{dz}$$

transverse $\quad \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

$$\nabla \cdot \vec{E} = \nabla_t \cdot E_t + \frac{\partial E_z}{\partial z} = 0$$

$$E_z \sim e^{ikz} E_{z0}$$

$$\frac{\partial E_z}{\partial z} = ikE_{z0}e^{ikz}$$

$$\tilde{E} = \vec{E}_t + \hat{z}\vec{E}_z$$

$\vec{E}_t$ → transverse, $\hat{z}\vec{E}_z$ → longitudinal

If the beam diameter is D then,

$$\nabla_t \cdot \vec{E}_t \sim \frac{E_t}{D}$$

$$\frac{\partial}{\partial z}E_z = \frac{E_z}{\lambda}$$

$$\frac{1}{D}E_t + ikE_{z0} \sim 0$$

$$\left|\frac{E_{z0}}{E_t}\right| \approx \frac{1}{kD} = \frac{\lambda}{2\pi D} \ll 1$$

He-Ne: $\lambda \sim 6000\ A^\circ = 6\mu = 0.6\times 10^{-3}$ mm

$D \sim 1$ mm

If D $\downarrow\downarrow$ $\rightarrow$ $\left|\frac{E_{z0}}{E_t}\right| \sim 1 \quad \left|\frac{E_z}{E_t}\right| = 10^{-3} \ll 1$

In the case of the beam (D finite), there is always a field component along the prop. direction.

$$\left(\frac{\partial}{\partial z}\right)_{\text{envel}} \sim \frac{1}{L} \qquad \text{(scale length)}^{-1} \text{ for envol.}$$

$$\left(\frac{\partial}{\partial z}\right)_{\text{oscill}} \sim k \qquad \text{(scale length)}^{-1} \text{ oscillator}$$

① ——— $\tilde{E}(\vec{r},\omega) = \tilde{E}_0 \underbrace{\psi(\vec{r}_t, z)}_{\text{slow space variation}} e^{jkz}$ → fast space variation

slow space variation → gives radial to fall off and slow variation along z

$\vec{r}_t =$ transverse coor.

Recall : $\nabla^2 \tilde{E} + \frac{\omega^2 n^2}{c^2} \tilde{E}(\omega) = 0$ ——— ②

Substitute $\tilde{E}$ into $\nabla^2 \tilde{E} + \frac{\omega^2 n^2}{j} \tilde{E}(\omega) = 0$,

then

③ ———
$$\nabla_t^2 E_t = E_0 e^{ikz} \nabla_t^2 \psi$$
$$\frac{\partial E_t}{\partial z} = E_0 \left(\frac{\partial \psi}{\partial z} e^{ikz} + ike^{ikz} \psi \right)$$
$$\frac{\partial^2 E_t}{\partial z^2} = E_0 \left(\frac{\partial^2 \psi}{\partial z^2} e^{ikz} + ike^{ikz} \frac{\partial \psi}{\partial z} - k^2 e^{ikz} \psi + ike^{ikz} \frac{\partial \psi}{\partial z} \right)$$

$$\nabla_t^2 \psi + 2ik \frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

(the $\frac{\partial^2 \psi}{\partial z^2}$ term is crossed out) ↗ Case has low divergence and k of optical wave is a large no.

$$\left(\nabla_t^2 E_t + \frac{\partial^2 E_t}{\partial z^2} + k^2 E_t = 0 \right); k = \frac{2\pi}{\lambda}$$

$$\boxed{\nabla_t^2 \psi + 2ik \frac{\partial \psi}{\partial z} = 0}$$

Paraxial wave Equation
SVEA

Intensity distribution

$$EE^* = |E|^2 |\psi|^2$$

$$\psi = e^{i\left[p(z)+\frac{kr^2}{2q(z)}\right]}$$

$$\frac{\partial \psi}{\partial r} = \frac{ikr}{q} e^{i\left[p(z)+\frac{kr^2}{2q(z)}\right]}$$

$$r\frac{\partial \psi}{\partial r} = \frac{ikr^2}{q} e^{i\left[p(z)+\frac{kr^2}{2q(z)}\right]}$$

In cylindrical coordinate : $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)$

$$= \frac{1}{r}\left[\frac{2ikr}{q} e^{\left[p(z)+\frac{kr^2}{2q(z)}\right]} + \frac{ikr^2}{q} e^{\left[p(z)+\frac{kr^2}{2q(z)}\right]} + \frac{ikr}{q} e^{i\left[p(z)+\frac{kr^2}{2q(z)}\right]}\right]$$

$$\frac{\partial \psi}{\partial z} = i\left[p' - \frac{kr^2}{2q} q'\right] e^{i\left[p(z)+\frac{kr^2}{2q(z)}\right]}$$

substitute back to SVEA, we get

$$\frac{2ik}{q} - \frac{k^2 r^2}{q^2} - 2k\left[p' - \frac{kr^2}{2q} q'\right] = 0$$

$$\frac{k^2}{q^2(z)}\left(q'(z)-1\right)r^2 - 2k\left(p'(z) - \frac{i}{q(z)}\right) = 0$$

Power function : $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$

$$q'(z) - 1 = 0 \Rightarrow q(z) = z + q_0$$

$$p'(z) - \frac{i}{q(z)} = 0$$

$\Rightarrow q(z) =$ "complex number"

$q(z) = z - iz_0$ ———— ⓘ

At $z = 0 : \psi(r, z=0) = e^{-\frac{kr^2}{2z_0}} e^{ip(z=0)}$

For $z \neq 0$: $$\frac{1}{q(z)} = \frac{1}{z - iz_0} = \frac{z + iz_0}{z^2 + z_0^2}$$

$$\psi \propto e^{\frac{-kz_0 r^2}{2\left(z^2+z_0^2\right)}} e^{i\frac{kzr^2}{2\left(z^2+z_0^2\right)}} e^{ip(z)}$$

Define,
$$\omega^2(z) = \frac{2}{kz_0}\left(z^2+z_0^2\right)$$
$$= \omega_0^2\left(1+z^2/z_0^2\right)$$

Define
$$R(z) = \frac{z^2+z_0^2}{z}$$
$$\psi \quad = e^{-r^2}/\omega^2(z) e^{ikr^2/2R(z)} e^{ip(z)}$$

Recall (ii)
$$p'(z) = i/q(z) = \frac{i}{z-iz_0}$$
$$p(z) = i\ln\left(z-iz_0\right)+A$$

But A just contribute to a constant phase, so we define

$$p(z=0)=0$$
$$p(z) \quad = i\ln\left(z-iz_0\right)-i\ln\left(iz_0\right)$$
$$= i\ln\left(1-\frac{z}{iz_0}\right)$$
$$= i\ln\left(1+\frac{iz}{z_0}\right)$$

$$\exp\left(iP(z)\right) = \frac{1}{1+iz/z_0} = \frac{1-iz/z_0}{1+\left(z/z_0\right)^2}$$

So,
$$= \frac{1}{1+\left(z/z_0\right)^2}\left[1+\left(z/z_0\right)^2\right]^{1/2} e^{-i\tan^{-1}z/z_0}$$
$$= \frac{1}{\sqrt{1+\left(z/z_0\right)^2}} e^{-i\tan^{-1}z/z_0}$$

Full sol :

$$E(r,z) = \psi(r,z)e^{ikz}$$
$$E(r,z) = E_0\frac{\omega_0}{\omega(z)} e^{-r^2/\omega(z)} e^{ikr^2/2R(z)} e^{i\left(kz-\tan^{-1}(z/z_0)\right)}$$

We get solution by using $\frac{1}{r}\left(r\frac{\partial\psi}{\partial r}\right)+2ik\frac{\partial\psi}{\partial z}=0$

(i) + (ii) | by assuming $\frac{\partial}{\partial\psi}=0$

to get this it is quite completely $e^{i\left(p(z)+\frac{kr^2}{2q(z)}\right)}$

& solve by B.C. where $E(z\to\infty)=0$

Since this solution satisfy the wave equation and B.C. it is the unique solution (for case $\frac{\partial}{\partial\psi}=0$).

Interpretation of Fundamental Gaussian Beam Solution :

i) Field Amplitude :
- Radial dependence
- z dependence

$$\omega^2(z) = \omega_0^2\left(1+\frac{z^2}{z_0}\right)$$

$$\frac{|E(z=z_0)|}{E_0^2}=\frac{1}{2}$$

$$\frac{r^2}{\omega^2(0)}=\frac{r^2}{\omega_0^2}$$

Interpretation :

$z=0$ is a place of min spot size ω_0.

What is z_0?

For $r=0$, $$\frac{|E(r=0,z)|}{E_0} \approx \frac{\omega_0}{\omega(z)} = \frac{1}{\left(1+z^2/z_0^2\right)^{1/2}}$$

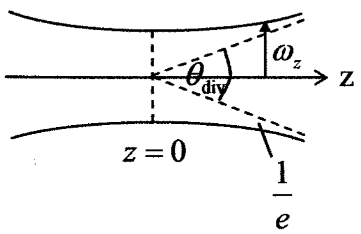
At $z=z_0$; $$\frac{|E(r=0,z=z_0)|^2}{E_0^2} = \frac{1}{2}$$

$\Rightarrow$ z_0 is the distance from the focus at which the intensity has dropped approx. by a factor of 2.

$$\left(\omega_0^2 = \frac{2z_0}{k} \;\; ; \;\; z_0 = \frac{k\omega_0^2}{2}\right)$$

ii) Beam Divergence

We can use the focal of $\frac{1}{e}$ field point to define a measurement of beam divergence.



$$\tan\left(\frac{1}{2}\theta_{\text{DIV}}\right) = \frac{\omega(z)}{z}$$

For small angle :

$$\frac{1}{2}\theta_{\text{DIV}} \approx \frac{\omega(z)}{z}$$

For $\frac{z}{z_0} \gg 1$,

$$\frac{\omega(z)}{z} = \frac{\omega_0\left(1+z^2/z_0^2\right)^{1/2}}{z}$$

$$\approx \frac{\omega_0}{z_0}$$

$$\Rightarrow \quad \theta_{\text{DIV}} \approx \frac{2\omega_0}{z_0} \quad ; \quad z_0 = \frac{k\omega_0^2}{2}$$

$$\approx \frac{2\omega_0}{\frac{\pi\omega_0^2}{2\pi/k}} = \frac{2\omega_0}{\frac{k\omega_0^2}{2}} = \frac{4}{k\omega_0}$$

$$\boxed{\theta_{\text{DIV}} = \frac{2\lambda}{\pi\omega_0}}$$

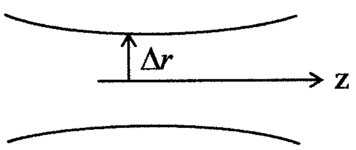
Since ω_0 for diff lasers : $\Rightarrow \quad \lambda\uparrow \quad \rightarrow \quad \theta_{\text{DIV}}\uparrow$

Same λ $\quad \Rightarrow \quad \omega_0\downarrow \quad \rightarrow \quad \theta_{\text{DIV}}\uparrow$

longer λ, the beam diverge faster (for fixed spot size)

Photon View of Beam Divergence :

It comes from uncertainly principle



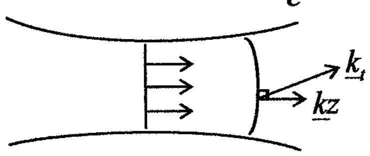
$\Delta p \Delta r \sim \hbar \quad ; \quad \Delta r \sim a$

$\Delta p \sim \frac{\hbar}{a}$ spread in radial photon momentum

Divergent angle : $\frac{\Delta p}{p_z} \sim \frac{\hbar/a}{\hbar k} \sim \frac{\lambda}{a}$

iii) Longitudinal Phase

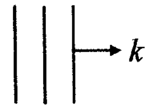
$e^{i\left(kz-\tan^{-1}(z/z_0)\right)} = e^{i\phi(z)}$



Behavior of plane wave :

Intensity never fall off

$$e^{i\underline{k}\cdot\underline{r}} = e^{ik\hat{z}\cdot r}$$
$$= e^{ikz}$$



* $\vec{k}$ is locally $\perp$ to $\underbrace{\text{phase surface}}_{\text{Surface of constant face}}$

for $z \to -\infty$

$$\tan^{-1}\frac{z}{z_0} \to -\pi/2$$

for $z \to +\infty$

$$\tan^{-1}\frac{z}{z_0} \to +\pi/2$$

$\Bigg\}$ phase shift by π

In going through the waist the wave picks up a phase change of

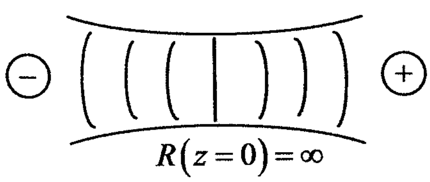
$$\pi/2-(-\pi/2)=\pi$$

- This phase change is due to change in direction of phase front curvature.
- $\tan\theta = z/z_0$, at $z \ll z_0$; $\theta \simeq z/z_0$
  $e^{i(kz+\tan^{-1} z/z_0)} \simeq e^{i(kz+z/z_0)}$

iv) Phase front curvature and radial dependence of phase

$$e^{ikr^2/2R(z)} \quad : \quad R(z)=z\left(1+z_0^2/z^2\right)$$

The phase fronts for the beam are curved. $R(z)$ is radius of curvature of these phase fronts.



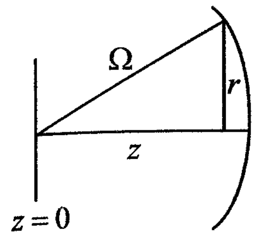
(+) For fixed z, as we move off axis phase increase.

(−) For fixed z, as we move off axis phase decrease.

$\ominus$: Phase $\downarrow$ $\rightarrow$ R"$-$"

$R(z<0)=$"$-$"

$\oplus / \ominus$ is the behavior of $e^{ikr^2/2R(z)}$



$$\Omega=\left(r^2+z^2\right)^{1/2}$$
$$=z\left(1+r^2/z^2\right)^{1/2}$$
$$\approx z\left(1+r^2/2z^2\right) \quad \text{for } \frac{r}{z}\ll 1$$

$$E(r,z)\propto \frac{1}{\omega(z)}e^{ikz}e^{ikr^2/2R(z)}e^{i\tan^{-1}(z/z_0)}$$

For large

$$z: E(r,z)\propto \frac{1}{z}e^{ikz}e^{ikr^2/2z_0}$$
$$\propto \frac{1}{\Omega}e^{ik\Omega} \quad ; \quad \omega(z)=\omega_0\left(1+z^2/z_0\right)^{1/2}$$
$$\Omega=1+\frac{r^2}{2z}$$

Far from beam waist, the phase fronts resemble those form a spherical wave.

$$\left(E(r,z)\propto \frac{1}{r}e^{ikr}\right)$$

Close to the beam waist, phase fronts are parabolic.

Ex : (i) He-Ne Laser $\lambda=6328A^{\circ}$ spot size 1 mm. Aim laser at the moon. How big is the beam at moon?

$$\omega_0=1\text{ mm}=0.1\text{ cm}$$
$$z_0=\frac{\pi n\omega_0^2}{\lambda}=\frac{\pi(0.1)^2}{6328\times10^{-8}}\frac{\text{cm}^2}{\text{cm}}=492\text{ cm}$$
$$\omega(z)=\omega_0\left(1+\frac{z^2}{z_0}\right)^{1/2} \quad \left(\text{at } z=z_0 \quad \omega(z_0)=\sqrt{2}\omega_0\right)$$
$$z_{\text{moon}}=240{,}000\times5280\times12\times2.84\text{ cm }=3.86\times10^{10}\text{ cm}$$
$$\omega_{\text{moon}}\left(z_{\text{moon}}\right)0.1\left(1+\left(\frac{3.86\times10^{10}}{496}\right)^2\right)^{1/2}=78\times10\text{ cm}=78\text{ km}$$

If we use lens or mirror to increase beam diameter to $\omega_0 = 5$ cm,

$$\Rightarrow z_0 \quad = \frac{\pi (5)^2}{6328 \times 10^8} = 1.29 \times 10^6 \text{ cm}$$

$$\omega(z) = 1.6 \text{ km}$$

ω_0	$\omega(z_{\text{moon}})$
1 mm	78 km
5 cm	1.6 km

(ii) Power in beam crossing a plane at z

$$\langle s \rangle_t = \frac{1}{2} c\varepsilon \left(\text{E} \times \text{H}^* \right)$$

Intensity :

$$I = \frac{1}{2} c\varepsilon \left| \tilde{E} \right|^2$$

Power :

$$P = \int_{\text{over plane}} I ds \qquad ds = dr 2\pi r$$

$$= \int_0^\infty dr 2\pi r \frac{1}{2} c\varepsilon E_0^2 \frac{\omega_0^2}{\omega^2(z)} e^{-2r^2/\omega^2(z)}$$

$$= \frac{1}{2} c\varepsilon E_0^2 \frac{\omega_0^2}{\omega^2(z)} 2\pi \underbrace{\int_0^\infty dr r e^{-2r^2/\omega^2(z)}}_{\frac{1}{2}\frac{\omega^2(z)}{2}}$$

$$= \frac{1}{2} c\varepsilon E_0^2 \frac{\omega_0^2}{\omega^2(z)} 2\pi \frac{\omega^2(z)}{4}$$

$$\text{Power} = \frac{1}{2} c\varepsilon E_0^2 \left(\frac{\pi \omega_0^2}{2}\right)$$

↓

independence of $z \Rightarrow$ since we must conserve the energy or power

Higher Order Gaussian Modes :

Our previous calculation assumed azimuthal sys $\left(\frac{\partial}{\partial \psi} = 0\right)$. Now, remove this requirement, and still require

$$E \to 0 \text{ as } r \to \infty$$

We get for

(i) Rectangular coordinate :

Hermit Polynomial represents modulation in transverse plane

$$\frac{E(x,y,z)}{E_0} = \underbrace{H_m}_{\text{of order m}} \left(\frac{\sqrt{2}n}{\omega(z)}\right) H_p \left(\frac{\sqrt{2}y}{\omega(z)}\right) \frac{\omega_0}{\omega(z)} e^{\left[-x^2+y^2/\omega^2(z)\right]} e^{i\left(kz-(m+p+1)\tan^{-1}(z/z_0)\right)} e^{ikr^2/2R(z)}$$

$$H_m(u) = (-1)^m e^{u^2} \frac{d^{2m}}{du^m} e^{-u^2}$$
$$H_0(u) = 1$$
$$H_1(u) = 2u$$
$$H_2(u) = 2(2u^2 - 1)$$

These mode are called TEM_{mp} modes :

$$\begin{pmatrix} E_z = H_z = 0 \\ \text{Actually,} \frac{E_z}{E_t} \Box 1 \end{pmatrix}$$

These is a total of $(m+1)(p+1)$ spot in the intensity distribution

(ii) Cylindrical Coordinate :

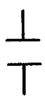
$$\frac{E(r,y,z)}{E_0} = \left(\frac{\sqrt{2}r}{\omega_0}\right)^{ilp} e^{ilp} L_p^l\left(\frac{2r^2}{\omega^2(z)}\right) e^{-r^2/\omega^2(z)} e^{i\left(kz - 2(p+l+1)\tan^{-1} z/z_0\right)} e^{ikr^2/2R(z)}$$

e^{ilp} ↓ get curl l

l = azimuthal index

p = radial index

θ_{BR} = Brewster angle

( hole) so we kill the outer 

Gaussian

Divergence of High Order Modes :

For higher modes, must consider extra beam radius due to Hermite Polynomial

Effect beam radius □ $\lambda_{\max} = \sqrt{m}\omega(z)$

$$\frac{1}{2}\theta_{\mathrm{DIV}} = \frac{x_{\max}}{2}$$

$$= \frac{\sqrt{m}\omega(z)}{z}$$

$$\frac{1}{2}\theta_{\mathrm{DIV}} \;\square\; \sqrt{m}\frac{\omega_0}{z_0} \qquad \text{for } \frac{z}{z_0} \;\square\; 1.$$

Divergence of n^{th} mode is $\sqrt{n}$ times larger than a zero-order mode.

$$\frac{1}{2}\theta_{\mathrm{DIV}} = \frac{\omega_0}{z_0}, \frac{z}{z_0} \;\square\; 1$$

Coherence :

Temporal or Longitudinal coherence : nice predicable wave with phase memory

$\tau_0 \equiv$ coherence time : "duration of phase memory" over which average time wave phase is predicable.

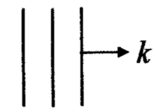
$\equiv$ average time between phase change

$l_c \equiv$ coherence length $= \frac{c}{n}\tau_c$

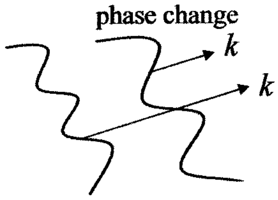
$\tau_c = \frac{2}{\Delta\omega}$: claim $\Delta\omega$ is FWHM due to any kind to line broadening.

Transverse coherence :

Degree to which the phase changes as you move transverse to propagate direction.



plane wave

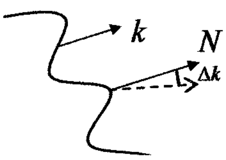


"inperfect transverse coherence"

has a "perfect" transverse coherence

- No phase change in $\perp$ direction

Divergence: Maxwell Eq. tell $\underline{k}$ vector is locally $\perp$ to phase fronts



$$\frac{1}{2}\theta_{\text{DIV}} \approx \frac{\Delta k}{k}$$

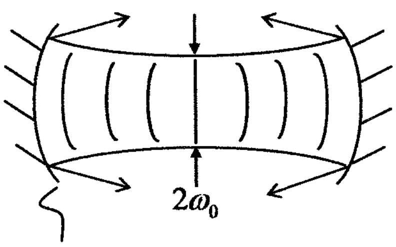
$$(\theta_{\text{DIV}})_{\max} \sim \frac{2\Delta k_{\max}}{k}$$

$\Delta k \uparrow$

The uglier the phase front

→ the more the divergence

Contrast laser beam vs. Flashlight beam :



$$\tau_c \sim \frac{2}{\Delta\omega}$$

phase front follows the contour of mirror

Flash Light: Temporal coherence is poor-random emission from incandescent filament.

$$\left(\frac{1}{2}\theta_{\text{DIV}}\right) = \left|\frac{\Delta k_{\max}}{k}\right|$$

large divergence

Measurement of coherent:

i) Longitudinal (or Temporal coherence)

Michelson Interferometer:

Fully Coherence: $\tau_c \to \infty$

Noisy coherence: $\tau_c \approx 0$

Max: $d_1 = d_2, \delta(d_1 - d_2)$

got peak

In case of partial coherence:

$$\left\langle (E_1 + E_2)^2 \right\rangle = \underbrace{\left\langle |\underline{E}_1|^2 \right\rangle + \left\langle |\underline{E}_2|^2 \right\rangle}_{\text{Background}} + \underbrace{\left\langle 2\underline{E}_1 \cdot \underline{E}_2 \right\rangle}_{\text{Fringe}}$$

$$\int_0^{L2\pi/\omega} dt \sin\left(\omega t + \varphi(t)\right) \sin \omega t$$

Outside curve

φ random $\to$ take time = 0

$\Rightarrow$ Flat background = 0

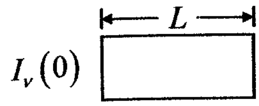
$\Rightarrow$ In side curve = 1

For $\dfrac{2|d_2 - d_1|}{c} > \tau_c$; fringes go away and just background is left

Measuring Transverse Coherence: Using Double Slits

$$\langle |E_1| \rangle^2 + \left\langle |E_2|^2 \right\rangle + 2\langle E_1 \cdot E_2 \rangle$$

Gain and Optical Amplification and Gain Saturation



$$I_\nu(L) = I_\nu(0)e^{\gamma(\nu)L}$$

$\pm$ depending on N_2, N_1

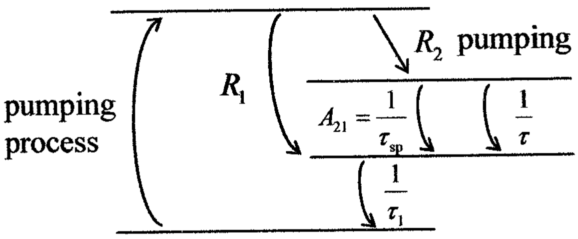
$$\gamma(\nu) = \frac{c^2}{8\pi n^2 \nu^2 t_{sp}} g(\nu)(N_2 - N_1)$$

Population N_1 and N_2 are affect by the photon field.

Rate Eq.:

$$\frac{dN_2}{dt} = -B_{21}^{(0)} g(\nu)\rho(\nu)N_2 - \frac{N_2}{t_{sp}} + B_{12}^{(0)} g(\nu)\rho(\nu)N_1$$

Consider a homogeneous broadened 4-level system:



$$\frac{1}{\tau} = \frac{1}{t_{sp}} + r_2$$

rate of depopulation method

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - (N_2 - N_1)\underbrace{B^{(0)} g(\nu)\rho(\nu)}_{\omega(\nu)} \qquad \left\langle \frac{1}{\text{sec}} \right\rangle$$

$\omega(\nu) \equiv$ rate per atom of stimulated absorption

where

$$\omega(\nu) = \frac{c^2 g(\nu) I_\nu}{8\pi n^2 h \nu^3 t_{sp}}$$

$$\frac{dN_1}{dt} = R_1 - \frac{N_1}{\tau_1} + \frac{N_2}{t_{sp}} + (N_2 - N_1)\omega(\nu)$$

pump

non-stimulated depopulation. rate

Spontaneous Fill up from level 2

S.S. approximation $\left(\frac{d}{dt} = 0\right)$ to describe behavior of N_1, N_2 for t □ time of slowest rate in system.

$\downarrow$ $R_1, R_2, \frac{1}{\tau ...}, ...$

Then,

$$0 = R_2 - \frac{N_2}{\tau_2} - (N_2 - N_1)\omega(\nu)$$

$$0 = R_1 - \frac{N_1}{\tau_1} - (N_2 - N_1)\omega(\nu)$$

Solve for N_1, N_2

$$0 \quad = (R_1 + R_2) + N_2\left(\frac{1}{t_{sp}} - \frac{1}{\tau_2}\right) - \frac{N_1}{\tau_1}$$

$$N_1 \quad = (R_1 + R_2) + N_2\left(\frac{1}{t_{sp}} - \frac{1}{\tau_2}\right)\tau_1$$

$$0 \quad = R_2 - \frac{N_2}{\tau_2} - N_2\omega(\nu) + \omega\tau_1\left[(R_1 + R_2) + N_2\left(\frac{1}{t_{sp}} - \frac{1}{\tau_2}\right)\right]$$

$$0 \quad = R_2 - N_2\left(\frac{1}{\tau_2} - \frac{\omega t_1}{t_{sp}} + \frac{\omega t_1}{\tau_2} + \omega\right)\omega t_1 (R_1 + R_2)$$

$$N_2 = \frac{R_2 + \omega\tau_1(R_1 + R_2)}{\frac{1}{\tau_2} - \omega\left(\frac{t_1}{t_{sp}} - \frac{t_1}{\tau_2} - 1\right)}$$

$$(N_2 - N_1) = \frac{R_2\tau_2 - (R_1 - \delta R_2)\tau_1}{1 + [\tau_2 + (1 - \delta)\tau_1]W(\nu)}; \qquad \delta = \frac{\tau_2}{t_{sp}}$$

In the absent of optical intensity : $I = 0$

$$\Rightarrow \omega(\nu) = 0$$

$$\begin{aligned} N_2 - N_1 &= R_2 t_2 - (R_1 - \delta R_2)\tau_1 \\ &= (N_2 - N_1)_{I=0} = \Delta N^\circ \\ N_2 - N_1 &= \frac{\Delta N^\circ}{1 + \phi t_{sp}\omega(\nu)}, \end{aligned}$$

where

$$\phi = \delta\left[1 + (1-\delta)\tau_1/\tau_2\right]$$

Now consider efficient laser system

①

$$r_2 \square \frac{1}{t_{sp}} \Rightarrow \frac{1}{\tau_2} \square \frac{1}{\tau_{sp}} \Rightarrow \delta \square 1$$

$$\Downarrow$$

t_{sp} small $\rightarrow$ fall fast :

No!!

②

$$t_1 \square\ t_{sp} \Rightarrow (N_2 - N_1)\uparrow\uparrow \text{ easier}$$

We also want to deplete level 1 as fast as possible in order to keep the inversion high (or to avoid absorption upper level)

$$\frac{1}{t_{sp}} \square \frac{1}{\tau_1}$$

$$\Rightarrow \frac{t_1}{\tau_2} \square \frac{t_1}{\tau_{sp}} \square\ 1$$

$$\Rightarrow (1-\delta)\frac{\tau_1}{\tau_2} \square\ 1 \text{ and so } \phi = 1$$

$$N_2 - N_1 = \frac{\Delta N^\circ}{1+\phi\tau_{sp}\omega(\nu)}$$

Recall:

$$\begin{aligned}\phi t_{sp}\omega(\nu) &= \phi\tau_{sp}B^{(0)}g(\nu)\rho(\nu)\\ &= \phi t_{sp}\frac{c^3}{8\pi n^3 h\nu^3}Ag(\nu)\frac{I}{c/n}\\ &= \phi\frac{c^2 g(\nu)I}{8\pi n^2 h\nu^2} = \frac{I}{I_s},\end{aligned}$$

where, $I_s = \dfrac{I8\pi n^2 h\nu^3}{\phi c^2 g(\nu)}$ "saturation intensity"

So

$$\boxed{N_2 - N_1 = \frac{\Delta N^\circ}{1+I/I_s}}$$

when $I = I_s$, the population inversion drops to $\frac{1}{2}$ its maximum value.

The gain is saturate when the rate of stimulated emission (per atom) is equal to the rate of spontaneous emission (per atom).

Gain becomes

$$\begin{aligned}\gamma(\nu) &= \frac{c^2 g(\nu)}{8\pi n^2 \nu^2 t_{sp}}(N_2 - N_1)\\ &= \frac{c^2 g(\nu)}{8\pi n^2 \nu^2 t_{sp}}\frac{\Delta N^\circ}{1+I(\nu)/I_s}\end{aligned}$$

Inhomogeneous : $\gamma(\nu) = \dfrac{\gamma_0(\nu)}{(1+I_\nu/I_s)^{1/2}}$ → saturation more weakly than homogeneous case

Homogeneous : $\gamma(\nu) = \dfrac{\gamma_0(\nu)}{(1+I_\nu/I_s)}$

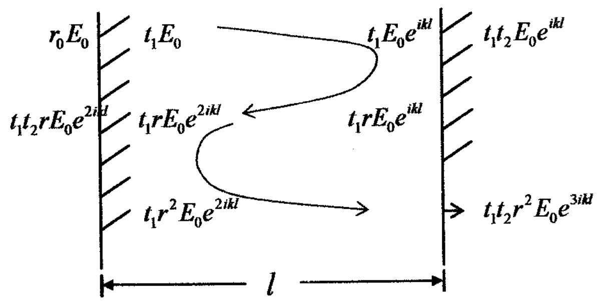
Doppler : If we operate at ν_0 it doesn't affect $\gamma(\nu)$ much.

Homogeneous : $g(\nu) \approx$ every atom

Affect one, affect all $\Rightarrow \gamma(\nu)$ change much

Optical Resonator

Single optical resonator : 2 flat, parallel optical surfaces



"Fabry-perot Etalon"

n' $\qquad\qquad$ n $\qquad\qquad$ n'

$$k_0 = \frac{\omega n'}{c} \qquad\qquad k = \frac{\omega n}{c}$$

Ratio of field not intensity

Transmission coefficient : from $n' \to n$ □ t_1

Reflection coefficient : from $n \to n$ □ r_0
(or $n \to n'$)

Transmission coefficient : from $n \to n'$ □ t_2

Reflection coefficient : from $n \to n$ □ r
(or $n' \to n$)

$$r = \frac{n-n'}{n'+n}$$

$$r_0 = \frac{n'-n}{n'+n} = -r$$

Reflected wave :

$$E_r = E_0\left(r_0 + \underbrace{t_1t_2re^{i2kl} + t_1t_2r^3e^{i4kl} + \ldots}_{re^{2ikl}t_1t_2\left(\frac{1}{1-re^{2ikl}}\right)}\right)$$

$$= E_0\left(r_0 + \frac{t_1t_2re^{i2kl}}{1-r^2e^{i2kl}}\right)$$

$$\frac{a}{1-x} = a + ax + ax^2 + ax^3 + \ldots; |x| < 1$$

or

$$(1+\mathrm{x})^{-1} = 1 - x + x^2 - x^3 + \ldots; -1 < x < 1$$

$$
\begin{aligned}
E_t &= \left(t_1 t_2 e^{ikl} + t_1 t_2 r^2 e^{3ikl} + t_1 t_2 r^4 e^{5ikl} + \ldots\right) E_0 \\
&= \left(\frac{t_1 t_2 e^{ikl}}{1 - r^2 e^{2ikl}}\right) E_0
\end{aligned}
$$

Recall :

$$
\begin{aligned}
t_1 &= \frac{2n'}{n+n'} && = 1 - r_0 \\
t_2 &= \frac{2n}{n+n'} && = 1 - r \\
r &= \frac{n-n'}{n'+n}, \quad r^2 = \frac{n^2 + (n')^2 - 2nn'}{(n+n')^2} \\
&\qquad = 1 - \frac{4nn'}{(n+n')^2} = 1 - t_1 t_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{E_r}{E_0} &= -r + \frac{\left(1-r^2\right) r e^{2ikl}}{1 - r^2 e^{i2kl}} \\
&= r \frac{\left[-1 + e^{2ikl}\right]}{1 - r^2 e^{i2kl}} \qquad r \equiv \text{real number, no absorption} \\
\frac{I_r}{I_0} = \left|\frac{E_r}{E_0}\right|^2 &= \frac{r^2\left(-1 + e^{2ikl}\right)\left(-1 + e^{-2ikl}\right)}{\left(1 - r^2 e^{i2kl}\right)\left(1 - r^2 e^{-i2kl}\right)} \\
&= \frac{r^2\left(1 - e^{2ikl} - e^{-2ikl}\right)}{1 + r^4 - r^2 e^{i2kl} + r^2 e^{-i2kl}} \\
&= \frac{r^2 (2 - 2\cos 2kl)}{1 + r^4 - 2r^2 \cos 2kl} \\
&= \frac{2R(1 - \cos\delta)}{1 - R^2 - 2R\cos\delta}
\end{aligned}
$$

$R = r^2$ "reflectivity"

$\delta = 2kl$ "round trip phase advance"

$\cos\delta = \cos^2 \delta/2 - \sin^2 \delta/2$

$$
\begin{aligned}
\frac{I_r}{I_0} &= \frac{2R\left(1 - \cos^2 \delta/2 + \sin^2 \delta/2\right)}{1 + R^2 - 2R\left(1 - 2\sin^2 \delta/2\right)} \\
&= \frac{4R \sin^2 \delta/2}{(1-R)^2 + 4R\sin^2 \delta/2}
\end{aligned}
$$

Similarity,

$$\frac{I_t}{I_0} = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2 \delta/2} = 1 - \frac{I_r}{I_0}$$

$$\frac{I_t}{I_0} + \frac{I_r}{I_0} = 1$$

"Fabry-Perot problem foe off-normal light"

Note : Laser is not a plane wave, but we use plane wave to understand Fabry-Perot.

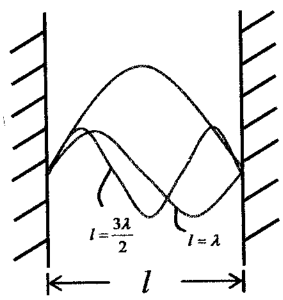
Transmission Characteristic of Fabry-Perot Etalon :

For $\left.\begin{array}{l}\sin^2 \delta/2 = 0 \\ \delta/2 = q\pi \text{ for } q = 1,2,3\end{array}\right\} \Rightarrow \frac{I_t}{I_0} = 1$

Round-trip phase advance : $2kl = 2q\pi$

$$l = \frac{q\pi}{k}\ ; \qquad k = nk_0 = \frac{n2\pi}{\lambda_0}$$

$$l = \frac{q\lambda}{2n}\ ; \qquad l \propto \text{ half wave length}$$



Separation l is integral # of half wavelength

"standing wave condition"

Frequency separation between resonances :

$$\left.\begin{aligned} kl &= q\pi \\ \frac{\omega nl}{c} &= q\pi \end{aligned}\right\} \quad \frac{2\pi\nu nl}{c} = q\pi$$

$$\boxed{\nu_q = \frac{qc}{2nl}}$$

"resonance frequency" within the resonator not the spring, atom,…

$$\Delta\nu = \nu_{q+1} - \nu_q = \frac{c}{2nl} \qquad \text{"Free spectral range"}$$

$$\nu_q = qc/2nl$$

$$d\nu_q = -\frac{qc}{2nl^2}dl$$

change of resonance frequency due to the change in *dl*

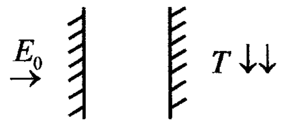
$$\frac{d\nu_q}{\Delta\nu} = \frac{-qc/2nl^2 dl}{\frac{c}{2nl}}$$

$$= -q\frac{dl}{l}$$

$$= \frac{-dl}{\lambda/2n}$$

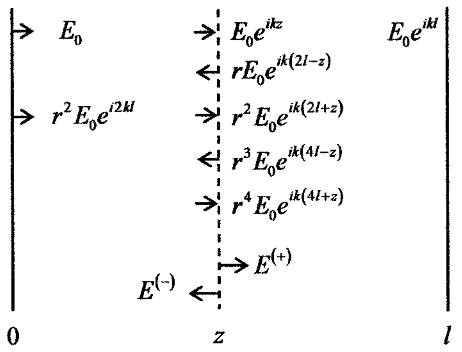
to change the resonant frequency by one free space range the etalon must be tuned through one-half wavelength

When $R \to 1$, the resonance sharper



We must get very close ν_q to get the signal or resonance pattern.

Field and Energy Inside Etalon :



Right going field :

$$
\begin{aligned}
E^{(+)} &= E_0 e^{ikz} + r^2 E_0 e^{ik(2l+z)} \\
&\quad + r^4 E_0 e^{ik(4l+z)} + \ldots \\
&= E_0 e^{ikz}\left(1 + r^2 e^{ik2l} + r^4 e^{i4kl} + \ldots\right) \\
&= \frac{E_0 e^{ikz}}{1 - r^2 e^{i2kl}}
\end{aligned}
$$

$$\frac{a}{1-x} = a + ax + ax^2 \quad : \quad \text{if } -1 < x < 1$$

Left going field :

$$\begin{aligned}
E^{(-)}(z) &= rE_0e^{ik(2l-z)} + r^3E_0e^{ik(4l-z)} + \ldots \\
&= rE_0e^{-ikz}e^{i2kl}\left[1 + r^2e^{i2kl} + r^4e^{i4kl} + \ldots\right] \\
&= \frac{rE_0e^{ikz}e^{i2kl}}{1 - r^2e^{i2kl}}
\end{aligned}$$

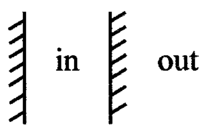
$$E_{\text{inside}}(z) = E^{+}(z) + E^{-}(z)$$

$$\begin{aligned}
I_{\text{inside}} &\propto \left|E_{\text{inside}}\right|^2 \\
&= \left|E^{+}(z) + E^{-}(z)\right|^2 \\
&= \left(E^{+}(z) + E^{-}(z)\right)\left(E^{+}(z)^{*} + E^{-}(z)^{*}\right) \\
&= \left|E^{+}(z)\right|^2 + \left|E^{-}(z)\right|^2 + E^{+}(z)^{*}E^{-}(z) + E^{+}(z)E^{-}(z)^{*} \\
&= \frac{E_0^2\left(1 + r^2 + re^{-2ikz}e^{2ikl} + re^{2ikz}e^{-2ikl}\right)}{1 + r^4 - 2r^2\cos 2kl}
\end{aligned}$$

$$\begin{aligned}
\frac{\left|E_{\text{inside}}\right|^2}{\left|E_0^{*}\right|^2} &= \frac{1 + r^2 + 2r\cos 2k(l - z)}{1 + r^4 - 2r^2\cos 2kl} \\
&= \frac{1 + R + 2\sqrt{R}\cos(\delta - 2kz)}{(1 - R)^2 + 4R\sin^2\delta/2}
\end{aligned}$$

$2kl$

The intensity inside resonator is also maximized for $\sin\delta/2 = 0$



E_{in} related to E_{out} via t.

Thus, E_{inside} max at $\sin\delta/2 = 0$

E_{out} max at $\sin\delta/2 = 0$ also.

Resonance Peak for Resonator : Outside resonator

$$\begin{aligned}
\frac{I_t}{I_0} &= \frac{(1 - R)^2}{(1 - R)^2 + 4R\sin^2\delta/2} \\
&= \frac{1}{1 + \frac{4R}{(1 - R)^2}\sin^2\delta/2}
\end{aligned}$$

At $\frac{I_t}{I_0} = 0.5$ if $\sin^2 \delta/2 = \frac{(1-R)^2}{4R}$

or $\sin^2 \delta/2 = \pm\frac{1-R}{2\sqrt{R}}$

(If know $\delta \to 2kl \Rightarrow$ so one can find $\Delta\delta_{1/2}$)

① $\sin\frac{\delta_{\text{right}}}{2} = +\frac{1-R}{2\sqrt{R}}$

$$\delta_{\text{right}} = 2q\pi + \frac{1}{2}\Delta\delta_{1/2}$$

$$\sin\frac{1}{2}\left(2q\pi + \frac{1}{2}\Delta\delta_{1/2}\right) = \frac{1-R}{2\sqrt{R}}$$

① ——— $\sin q\pi \cos\frac{\Delta\delta_{1/2}}{4} + \cos q\pi \sin\frac{\Delta\delta_{1/2}}{4} = \frac{1-R}{2\sqrt{R}}$

② $\delta_{\text{left}} = 2q\pi - \frac{1}{2}\Delta\delta_{1/2}$

$$\sin\frac{\delta_{\text{left}}}{2} = -\frac{1-R}{2\sqrt{R}} = \sin\left(q\pi - \frac{\Delta\delta_{1/2}}{4}\right)$$

$$= \sin q\pi \cos\frac{\Delta\delta_{1/2}}{4} - \cos q\pi \sin\frac{\Delta\delta_{1/2}}{4}$$

② ——— $= -\cos q\pi \sin\frac{\Delta\delta_{1/2}}{4}$

① - ② : $2\cos q\pi \sin\frac{\Delta\delta_{1/2}}{4} = \frac{1-R}{\sqrt{R}}$

$\frac{1-R}{\sqrt{R}} = 2\sin\frac{\Delta\delta_{1/2}}{4} \approx \frac{\Delta\delta_{1/2}}{2}$ for δ small

$$\boxed{\left|\Delta\delta_{1/2}\right| = \frac{2}{\sqrt{R}}(1-R)}$$

$R \to 1 \Rightarrow \Delta\delta_{1/2}$ small

But $\delta = 2kl = 2l\frac{2\pi n\nu}{c} =$

$$\Delta\delta_{1/2} = \frac{4\pi nl}{c}\Delta\nu_{1/2}$$

$$= \frac{2}{\sqrt{R}}(1-R)$$

$$\boxed{\Delta\nu_{1/2} = \frac{c}{2\pi nl}\frac{1-R}{\sqrt{R}}}$$

Measuring Quality of Resonator :

Resonator quality factor $Q \equiv \frac{\nu}{\Delta\nu_{1/2}}$

Better definition is "finesse"

$$F = \frac{\Delta\nu}{\Delta\nu_{1/2}}$$

⊔ free spectral range

⊔ FWHM of resonator

$$= \frac{\frac{c}{2nl}}{\frac{c}{2\pi nl}\frac{1-R}{\sqrt{R}}} = \frac{\pi\sqrt{R}}{1-R} \qquad R \to 1 \quad F \to \infty$$

Cavity Resonance for Gaussian Mode :

Previously analyze resonance for plane waves between 2 planar reflecting surfaces.

Now consider beams with curved phase fronts and cavities with curved mirror

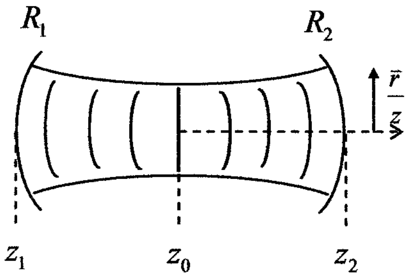
Recall :

$$\frac{E(x,y,z)}{E_0} = H_m\left(\frac{\sqrt{2}x}{\omega(z)}\right)H_p\left(\frac{\sqrt{2}y}{\omega(z)}\right)\frac{\omega_0}{\omega_z}e^{-r^2/\omega^2(z)} \times e^{ikz}e^{-i(m+p+1)\tan^{-1}\frac{z}{z_0}}e^{ikr^2/2R(z)}$$

Intrinsic property of plane wave

$$\left.\begin{aligned} R(z)\to\infty &: e^{ikr^2/2R(z)} \to 1 \\ z_0 \to \infty &: e^{-i\alpha \tan^{-1} z/z_0} \to 1 \end{aligned}\right\} e^{i\phi(z)}$$

$\Rightarrow$ because plane wave have $I(z)=\dfrac{I_0}{2}$ at $z\to\infty$, i.e. $z_0\to\infty$,



$$R(z)=z\left(1+z_0^2/z^2\right)$$

For $z^- \to R_1 = "-"$

$z^+ \to R_2 = "+"$

Plane back-forth :

Resonator condition = round trip phase delay is equal to integral $\times$ 2π

Phase shift : $2\left(\phi(z_r)-\phi(z_1)\right)=2q\pi$

$$\phi(z_r)-\phi(z_1) = q\pi$$

$$q\pi = k(z_2-z_1)-(m+p+1)\times \left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}\right\} + \frac{kr^2}{2}\left(\frac{1}{R(z_2)}-\frac{1}{R(z_1)}\right)$$

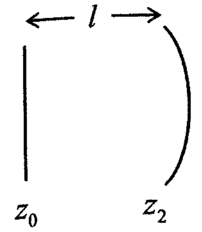
> Different value of q, m, and p satisfy resonance condition.
> $m+p \equiv$ transverse mode index
> $q \equiv$ longitudinal mode index

To simplify the problem, let $\boxed{r=0}$

$\Rightarrow$ this resonator is true every where on r.

$$q\pi = k(z_2-z_1)-(m+p+1)\left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}\right\}$$

ase: Plane-convex



$R \to \infty \qquad R_2$

$R = 2\pi\nu_{m,n,p}\frac{n}{c}$

$z_1 = 0$

$$q\pi = kz_2 - (m+p+1)\tan^{-1}\frac{z_2}{z_0}$$

From $R_2 = z_2\left[1+\frac{z_2^2}{z_0}\right]$

$$z_0 = \sqrt{lR_2}\left(1-\frac{l}{R_2}\right)^{1/2}$$

And,

$$\nu_{m,q,p} = \frac{c}{2nl}\left\{q + \frac{(m+p+1)}{\pi}\tan^{-1}\left[\frac{(l/R_2)^{1/2}}{(1-l/R_2)^{1/2}}\right]\right\}$$

$$= \frac{c}{2nl}\left\{q + \frac{(m+p+1)}{\pi}\cos^{-1}\left(1-\frac{l}{R_2}\right)^{1/2}\right\}$$

Case: Confocal Resonator

$R_1 \neq \infty$

$$\nu_{m,q,p} = \frac{c}{2nl}\left\{q + \frac{1+m+p}{\pi}\cos^{-1}\sqrt{g_1 g_2}\right\},$$

where $g_{1,2} = 1 - \frac{l}{R_{1,2}}$

General solution for ν is

$$\nu_{m,p,q} = \frac{qc}{2n(z_2 - z_1)} + \frac{c(m+p+1)}{2\pi n(z_2 - z_1)}\left\{\tan^{-1}\frac{z_2}{z_0} - \tan^{-1}\frac{z_1}{z_0}\right\}$$

(i) Longitudinal mode separation

For a given TEM_{mp} mode (m, p fixed), we have

① ——— $$k_{q+1}l-(m+p+1)\left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}\right\}=(q+1)\pi$$

② ——— $$k_q l-(m+p+1)\left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}\right\}=q\pi$$

① - ② $$(k_{q+1}-k_q)l=\pi$$

or $$\boxed{\Delta\nu_q=\nu_{q+1}-\nu_q=\frac{c}{2nl}}$$

(ii) Transverse mode separation

Here, q is fixed.

Ⓐ ——— $$k_2 l-(m+p+1)_2\left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}\right\}=q\pi$$

n charge + p fixed ⇒ like wise

Ⓑ ——— $$k_1 l-(m+p+1)_1\left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}\right\}=q\pi$$

Ⓐ - Ⓑ

$$(k_2-k_1)l=\Delta(m+p)\left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}-\right\}$$

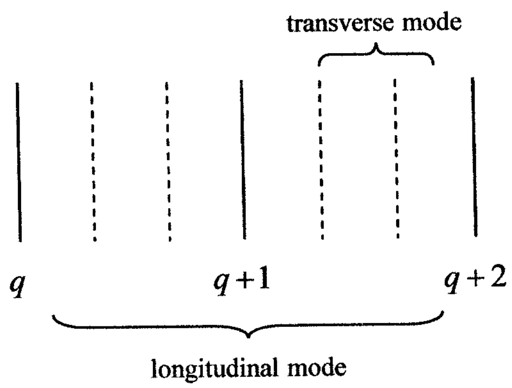
So,

$$\Delta\nu_{\text{mp}}=\nu_2-\nu_1$$

$$\Delta\nu_{\text{mp}}=\frac{c}{2\pi nl}\Delta(m+p)\left\{\tan^{-1}\frac{z_2}{z_0}-\tan^{-1}\frac{z_1}{z_0}-\right\}$$

Generally,

$$\Delta\nu_{mp} < \Delta\nu_q$$



$l = z_2 - z_1$	$z_0 = n\pi\omega_0^2/\lambda$
$R_1 = z_1\left(1+\frac{z_0^2}{z_1^2}\right)$	$R_2 = z_2\left(1+\frac{z_0^2}{z_2^2}\right)$

Solve for z_1, z_2 :

$$z_1 = \frac{R_1}{2} \pm \frac{1}{2}\sqrt{R_1^2 - 4z_0^2}$$
$$z_1 = \frac{R_2}{2} \pm \frac{1}{2}\sqrt{R_2^2 - 4z_0^2}$$

$\Rightarrow$ For given ω_0 and λ (and therefore z_0), the positions of mirrors and determined, (z_1, z_2).

Problem 2 : Given $R_1, R_2,$ and l what is $\omega_0(z_0)$?

Solve : Algebra problem

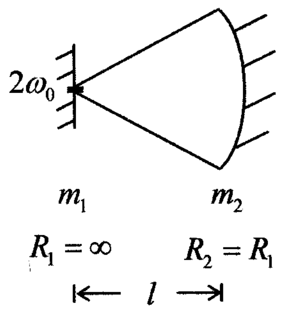
$$z_1 = \frac{l(l-R_2)}{R_2 - R_1 - 2l}$$
$$z_2 = \frac{-l(R_1+l)}{R_2 - R_1 - 2l}$$
$$z_0^2 = \frac{l(l-R_2)(l+R_1)(R_2-R_1-l)}{(R_2-R_1-2l)^2}$$

Examples :

1) Cavity with plan mirror

$R_1 = \infty,$

$R_2 = R,$ and $z_2 - l_1 = l$



Then,

$$z_2 = l$$
$$R(z_2) = R = l\left(1 + \frac{z_0^2}{l^2}\right)$$
$$\frac{R}{l} = 1 + \left(z_0^2 / l^2\right)$$
$$z_0 = l\left(\frac{R}{l} - 1\right)^{1/2}$$
$$= \frac{\pi \omega_0^2}{\lambda}$$
$$\Rightarrow \omega_0^2 = \frac{\lambda l}{\pi}\left(\frac{R}{l} - 1\right)^{1/2}$$

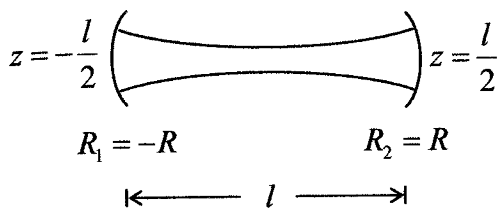
spot size at the flat mirror

$$\boxed{\omega_0 = \left(\frac{\lambda l}{\pi}\right)^{1/2} \left(\frac{R}{l} - 1\right)^{1/4}}$$

Note : If $\frac{R}{l} < 1$, ω_0 is complex number.

$\Rightarrow$ can't find a node to fit cavity.

2) Symmetric resonator : $|R_1| = |R_2| = R$



$$z_0^2 = \frac{l(l-R_2)(l+R_1)(R_2-R_1-l)}{(R_2-R_1-2l)^2}$$

$$= \frac{l(l-R)(l-R)(2R-l)}{(2R-2l)^2}$$

$$= \frac{l}{4}(2R-l)$$

$$\frac{\pi\omega_0^2}{\lambda} = z_0 = \left(\frac{l}{4}\right)^{1/2}(2R-l)^{1/2}$$

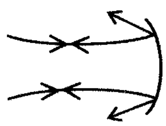
$$\omega_0 = \left(\frac{\lambda z_0}{\pi}\right)^{1/2} = \left(\frac{\lambda}{\pi}\right)^{1/2}\left(\frac{l}{4}\right)^{1/4}(2R-l)^{1/4}$$

$$\omega(z=\pm l/2) = \omega_0\left(1+\frac{l^2/4}{z_0^2}\right)^{1/2}$$

$$= \left(\frac{\lambda l}{2\pi}\right)^{1/2}\left(\frac{2R^2}{l(R-l/2)}\right)^{1/4}$$

$\Rightarrow$ Need $R > \frac{l}{2}$ for stability

If $R < \frac{l}{2}$, it means we can not find a Gaussian mode to fix into this resonator.



We want self-repetition or ability to "self-retrace"

$\Rightarrow$ i.e. modes don't exist in resonator.

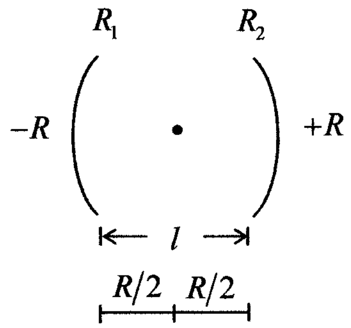
For nearly flat mirror $R \gg l$,

$$\omega_{\text{mirror}} = \omega(z \pm l/2) = \left(\frac{\lambda l}{\pi}\right)\left(\frac{2R}{l}\right)^{1/4} \approx \omega_0$$

$$\omega_0 = \left(\frac{\lambda}{\pi}\right)^{1/4}\left(\frac{l}{4}\right)(2R)^{1/4}$$

for nearly flat mirrors, there's very little beam divergence. Not surprise because flat mirror has beam divergence $\downarrow\downarrow$.

c) Confocal Resonator



$$
\begin{aligned}
z_0^2 &= \frac{l}{4}(2R-l) \\
&= \frac{R}{4}(2R-R) \\
&= \frac{1}{4}R^2; z_0 = \frac{R}{2}
\end{aligned}
$$

$$\omega_0 = \left(\frac{\lambda z_0}{\pi}\right)^{1/2} = \left(\frac{\lambda R}{2\pi}\right)^{1/2}$$

$$\omega(z = \pm l/2) = \sqrt{2}\omega_0$$

$\lambda = 6328\ A^\circ$

Example : He-Ne, $l = 50$ cm

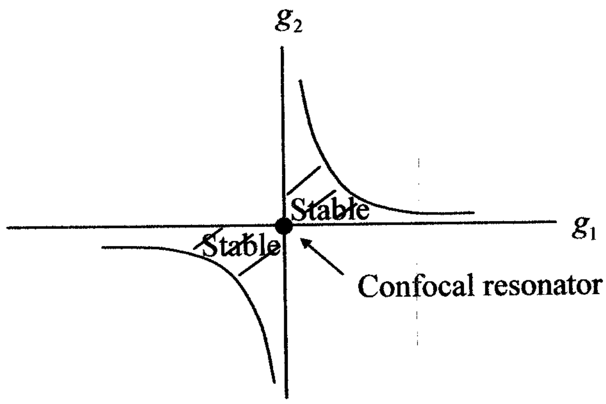
Confocal resonator $\Rightarrow R = 50$

$$
\begin{aligned}
\omega_{\text{at mirror}} &= \sqrt{2}\omega_0 = \sqrt{2}\left(\frac{\lambda R}{2\pi}\right)^{1/2} \\
&= 0.032 \text{ cm} \\
&= 350\ \mu\text{m}
\end{aligned}
$$

Stability Condition for Resonator

Suppose we have mirrors of radius R_1, R_2 separated by l. Stable or "self retracting" cavity modes can be supported if

$$0 \le \underbrace{\left(1-\frac{l}{|R_1|}\right)}_{g_1}\underbrace{\left(1-\frac{l}{|R_2|}\right)}_{g_2} \le 1$$



Example : Confocal resonator $|R_1| = |R_2| = R = l$

a) $g_1 g_2 = 0$ "At the edge of stability"

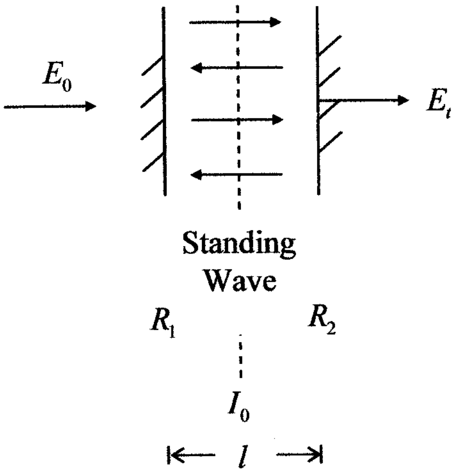
(In practical **l** is slightly adjusted below **R**)

b) Parallel flat mirror $|R_1| = |R_2| = \infty$

$g_1 g_2 = 1$ "At the edge of stability"

Unstable resonators are not necessarily to be avoid → They are useful in pulsed laser system when the stored energy is extracted after a few round trip.

Optical Resonator Losses :



Suppose we turn off :

How long does it take the field inside the resonator to decay?

$$E \propto e^{-t/\tau}$$

Consider "Bulk" Losses

- Attenuation Coefficient : α (unit = length)

$$I_0 \to R_1 R_2 e^{-2\alpha l} I_0$$

$$= I_0 e^{-2\beta l} \longrightarrow \text{ all losses impulsed into coeff. } \beta$$

$$e^{-2\beta l} = R_1 R_2 e^{-2\alpha l}$$

$$\beta = \alpha - \frac{1}{2l}\ln(R_1 R_2)$$

- Time

If x is the distance covered after many round trips after time $t, x = \frac{ct}{n}$

$$I(x) = I_0 e^{-\beta x}$$

$$I(t) = I_0 e^{-\beta \frac{c}{n} t}$$

$$= I_0 e^{-t/\tau},$$

where $\tau = \frac{n}{\beta c}$ "Photon Lifetime or Cavity energy decay time"

i.e. at $t = \tau \to$ The field inside the resonator disappears by going through mirror.

This can be related to Q and F ⟶ Finess $\frac{\Delta\nu}{\Delta\nu_{1/2}}$

↓

Quality factor $\frac{\nu}{\Delta\nu_{1/2}}$

Recall EM:

$$Q = \left|\left(\frac{\text{Cavity stored energy}}{\text{Energy dissipated per cycle}}\right)\right| 2\pi$$

$$= \left|\frac{2\pi\varepsilon}{p/f}\right|$$

(ε: Cavity stored energy; f: cycle; p: power dissipated)

$$= \left|\frac{\omega\varepsilon}{p}\right| * \frac{\text{Area}}{\text{Area}}$$

$$= \left|\frac{\omega I}{\frac{dI}{dt}}\right|$$

$$= \left|\frac{\omega I_0 e^{-t/\tau}}{-\frac{1}{\tau} I_0 e^{-t/\tau}}\right|$$

$$= \omega\tau$$

Power = $\int I ds$ (Power: $\frac{\text{Energy}}{\text{time}}$)

$$I = \frac{1}{2} c\varepsilon |E|^2$$

$$\frac{\varepsilon}{p} = \frac{\frac{\text{Energy}}{\text{area}}}{\frac{\text{Energy}}{\text{area time}}} = \frac{I}{\frac{dI}{dt}}$$

Recall : $Q = \frac{\nu}{\Delta\nu_{1/2}}$

Therefore,

$$\Delta\nu_{1/2} = \frac{\nu}{Q} = \frac{\nu}{\omega\tau} = \frac{1}{2\pi\tau}$$

$$\Delta\nu_{1/2} = \frac{c}{2\pi n}\left[\alpha - \frac{1}{2l}\ln(R_1 R_2)\right]$$

Laser Oscillator : □ **Put all the pieces to make a laser**

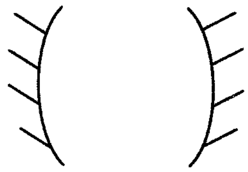
i) Medium with a population inversion, obtaining by pumping

Stimulated emission

Rate $\propto$ I inside resonator

$\propto |E_{\text{inside}}|^2$

ii) Resonant optical cavity



(i) + (ii) $\rightarrow$ (iii)

If have overlap between gain and cavity longitudinal modes get strong stimulated emission at cavity mode.

Recall : gain, $\gamma(\nu) = \frac{c^2}{8\pi n^2 \nu^2 t_{\text{sp}}} g(\nu)\Delta N,$

Where, $\Delta N = \frac{\Delta N^{(0)}}{(1 + I_\nu / I_s)^p}$

$p = 1$; homogeneous
$= 1/\sigma$; inhomogeneous

$\Delta N^{(0)} = (N_0 - N_1)_{I=0}$

(absent of optical field)

Implication :

1. $\Delta \nu_{\text{resonance}}$ □ $\Delta \nu_{\text{gain,medium}}$

2. Light generated is coherent

$\Rightarrow$ because it is consistant with cavity modes

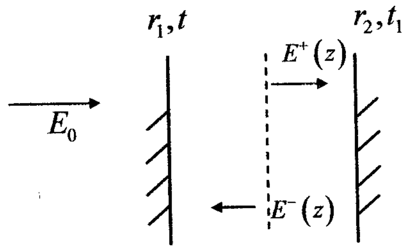
$$\text{Time} \propto \frac{1}{\Delta \nu_{\text{gain}}} \left| n \frac{\lambda}{2} \right|$$

3. Output has "nice" divergence properties due to transverse coherence of resonator modes

Oscillation Condition :

Put a medium with population inversion into an optical cavity what is the condition for cavity modes to build up near the gain frequency?

Recall : resonator



Prop. Const. in presence of gain and background\internal

$$\frac{E^+(z)}{E_0} = \frac{t e^{-i\beta z}}{1 - r_1 r_2 e^{-i2\beta l}}$$

$$\frac{E(z)}{E_0} = \frac{r_2 t_1 e^{-i\beta z} e^{i2\beta l}}{1 - r_1 r_2 e^{-i2\beta l}}$$

Use β instead of k because we have (gain materail)

Recall :

$$\beta \cong \frac{\omega n}{c}\left[1 + \frac{1}{2n^2}\chi(\omega)\right] + \frac{1}{2} i\alpha$$

Active susceptibility of active (Nd:) which are "enbedded" in background

$\alpha \equiv$ Bulk loss coefficien

$$\chi = \chi' + \chi''$$

χ': dispersion, χ'': gain/abs.

$$\beta = k + \Delta k - i\frac{1}{2}\gamma(\omega) + i\frac{1}{\sigma}\alpha$$

where $k = \frac{\omega n}{c}$, $\Delta k = \frac{\omega n}{c}\frac{\chi'}{2n^2} = \frac{k\chi'}{2n^2}$, $\left[\frac{1}{2}\gamma(\omega) = \frac{-k\chi''(\omega)}{2n^2}\right]$

$$\gamma(\nu) = \frac{c^2}{8\pi n^2 \nu^2 t_{sp}} g(\nu) \frac{\Delta N^{(0)}}{\left[1 + {I_\nu}/{I_s}\right]^p}$$

So, for the right-going wave :

$$\frac{E^+(z)}{E_0} = \frac{t_1 e^{i\beta z}}{1 - r_1 r_2 e^{2i(k+\Delta k)l} e^{(\gamma-\alpha)l}}$$

Spectral situation : If $\left(\frac{E^+}{E_0} \to \infty \right.$ Self regenerate because E_0 finite

$$0 = 1 - r_1 r_2 e^{2i(k+\Delta k)l} e^{(\gamma-\alpha)l} \Big)$$

$\Rightarrow$ Condition for S.S $\begin{pmatrix} E_0 = 0 \\ E_{inside,finite} \end{pmatrix}$

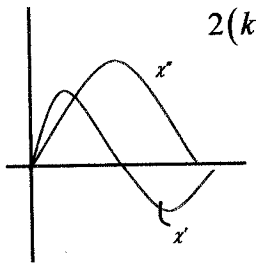
$$\Rightarrow \boxed{1 = r_1 r_2 e^{2i(k+\Delta k)l} e^{(\gamma-\alpha)l}}$$

A) Amp. Condition: $1 = r_1 r_2 e^{(\gamma-\alpha)l}$

i.e. $\gamma = \alpha$ (γ: gain, α: loss)

(If gain > loss, Intensity keeps going up
s.s $\Rightarrow I_{\text{cont}} \Rightarrow \gamma = \alpha$)

B) Phase condition:



$2(k + \Delta k)l = 2q\pi$ frequency res. condition cavity mode

Δk: Dispersion effect of active medium

$$\Delta k = \frac{\omega n}{c} \frac{\chi'(\omega)}{2n^2}$$

α fixed

γ is controlled by pumping.

$$0 = \ln r_1 r_2 + (\gamma - \alpha) l$$

$$\begin{aligned} \gamma_{\text{threshold}} &= \alpha - \frac{1}{l} \ln r_1 r_2 \\ &= \alpha - \frac{1}{2l} \ln R_1 R_2 \\ &= \frac{2\pi n}{c} \Delta \nu_{1/2} \\ &= \frac{n}{c\tau} \end{aligned}$$

For $\gamma < \gamma_{\text{threshold}}$

$$r_1 r_2 e^{(\gamma - \alpha) l} < 1$$

and there will be a decrease in field amplitude in each pass

$\Rightarrow$ Oscillation can't build up

Photon-lifetime

$$\tau = \frac{n}{c\left[\alpha - \frac{1}{2l}\ln R_1 R_2\right]}$$

$$\gamma_{\text{threshold}} = \frac{c^2}{8\pi n^2 \nu^2 t_{\text{sp}}} g(\nu)\Delta N_{\text{threshold}}$$

$$\Delta N_{\text{threshold}} = \frac{8\pi n^3 \nu^2 t_{\text{sp}}}{c^3 g(\nu)\tau}$$

Example : He-Ne laser $\lambda = 6328\ A^\circ$

$t_{\text{sp}} = 10^{-7}$ sec

$n = 1$

$R_1 = R_2 = 0.98$

$\alpha = 0$

$l = 10$ cm

$\Delta\nu = 10^9$ sec^{-1}

$$\tau = \frac{1}{3\times 10^{10}\left[-\frac{1}{20}\ln(0.98)^\circ\right]}$$

$$= 1.6\times 10^{-8}\ \text{sec}$$

$$\text{So } \Delta N_{\text{th}} = \frac{8\pi(1)^3\left(\frac{3\times 10^{10}}{6328\times 10^{-8}}\right)10^{-7}}{\left(3\times 10^{10}\right)^3 10^{-9} 1.6\times 10^{-8}}$$

$$= 1.3\times 10^{-9}\ \text{cm}^{-3}$$

Oscillation Frequency:

$$(k+\Delta k)l = q\pi$$

$$kl\left(1+\frac{\Delta k}{k}\right) = q\pi$$

of laser / \ cavity resonance

$$\nu\left(1+\frac{\chi'}{2n^2}\right) = \frac{qc}{2nl} = \nu_q$$

$\Rightarrow$ If $\chi' \to 0$ (no mat.) $\nu = \nu_q$

Recall : $\dfrac{\chi'(\omega)}{\chi''(\omega)} = \dfrac{2}{\Delta\nu}(\nu_0-\nu)$

$$\nu\left(1-\frac{1}{\Delta\nu}(\nu_0-\nu)\frac{c}{2\pi\nu n}\gamma(\nu)\right) = \nu_q$$

$$\chi''(\omega) = -\frac{n^2}{R}\gamma(\nu)$$

$$\nu_q = \nu\left[1-\left(\frac{\nu_0-\nu}{\Delta\nu}\right)\frac{\gamma(\nu)c}{2\pi nc}\right]$$

First, assume=me that $\dfrac{\nu_0-\nu}{\Delta\nu} \ll 1$ Oscillation that occur close to the peak of the gain

Then,

$$\begin{aligned}\nu &= \nu_0\left[1-\left(\frac{\nu_0-\nu}{\Delta\nu}\right)\frac{\gamma(\nu)c}{2\pi n\nu}\right]^{-1} \\ &\approx \nu_q\left[1+\left(\frac{\nu_0-\nu}{\Delta\nu}\right)\frac{\gamma(\nu)c}{2\pi n\nu}\right] \\ &= \nu_q + \frac{\nu_q}{\nu}\left(\frac{\nu_0-\nu}{\Delta\nu}\right)\frac{\gamma(\nu)c}{2\pi n}\end{aligned}$$

$[1+x]^{-1} = 1-x+x^2$

$[1-x]^{-1} = 1+x,$

when $|x| \le 1$

The oscillation occurs close to a cavity resonance, or $\nu = \nu_q$, so we put

$$\gamma(\nu) = \gamma(\nu_q)$$

$$\nu = \nu_q + \frac{(\nu_0-\nu_q)}{\Delta\nu}\frac{\gamma(\nu_q)c}{2\pi n}$$

Recall : in S.S Gain = Loss

We represented the loss in terms of photon lifetime τ :

$$\tau = \frac{n}{c\left[\alpha - \frac{1}{2l}\ln R_1 R_2\right]}$$

$$= \frac{n}{c\gamma_{\text{threshold}}},$$

where $$\gamma_{\text{threshold}} = \frac{n}{c\tau} = \frac{c^2}{8\pi n^2 \nu^2 t_{\text{sp}}} g(\nu)\Delta N_{\text{th}}$$

$$\Delta N_{\text{th}} = \frac{8\pi n^2 \nu^2 t_{\text{sp}}}{c^3 g(\nu)\tau}$$

At threshold :

$$\gamma(\nu_q) = \frac{n}{c\tau}$$

$$= \frac{n}{c} 2\pi\Delta\nu_{1/2}$$

This give $\nu = \nu_q + (\nu_0 - \nu_q)\frac{\Delta\nu_{1/2}}{\Delta\nu}$

Usually $\frac{\Delta\nu_{1/2}}{\Delta\nu} \ll 1$, so

Initially, we have $\gamma \gg \gamma_{\text{threshold}}$

Recall : $\gamma = \frac{\gamma_0}{(1 + I_r/I_s)^p}$

through this expression the gain will eventually adjust itself so that $\gamma = \gamma_{\text{th}}$

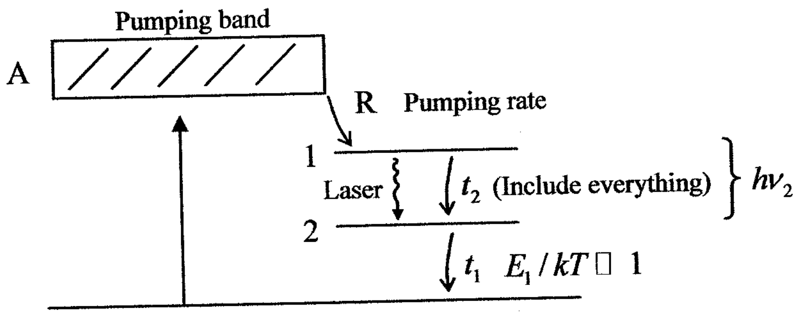
$\Rightarrow I_\nu$ is increasing then γ is $\downarrow\downarrow$ until $\gamma = \gamma_{\text{th}}$

Laser Pumping : What are the power requirements for the population inversion and gain?

Define : two kinds of laser systems

- 4 level
- 3 level

4 level system :



Old Tech : Pump to (A) by flash lamp, so $h\nu_R$ is high because big gap

→ it needs water cooling system.

New Tech : near to (2)

In case of thermal equilibrium :

$$\frac{N_1}{N_0} = e^{-E_1/kT}$$
$$E_1/kT \gg 1 \Rightarrow N_1 \ll N_0$$

1. If $E_1/kT \gg 1$

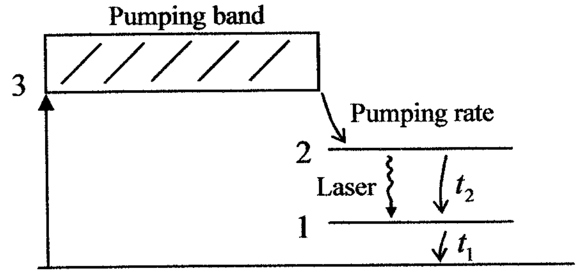
Lower level population is mainly determined via the pumping process and $\frac{N_1}{N_0} \ll 1$.

Lower level is far from ground state.

2. Also, want $t_1 \ll t_2$

$$\Rightarrow N_2 - N_1 \approx N_2 \text{ — check by rate equation}$$
$$\Delta N_{\text{thermal}} \approx N_0$$

3 level system



the lower level ① close to GND state.

Same as 4 level except

$$\left(\frac{N_1}{N_0}\right) = e^{-E_1/kT} \approx 1$$

$\Rightarrow -E_1/kT$ □ 1

Needs

1. $N_2 - N_1 = \Delta N^{th}$
   $N_2 + N_1 = N_0$

Then, $N_2 = \frac{1}{2}\left(N_0 + \Delta N^{th}\right)$

$$= \frac{1}{2} N_0 \left(1 + \frac{\Delta N^{th}}{N_0}\right)$$

2. $N_2 \geq \frac{1}{2} N_0$ at a minimum

At least 50% of GND state population must be "pumped" into the upper level.

$$\frac{\left(N_2^{\text{th}}\right)_{\text{3-level}}}{\left(N_2^{\text{th}}\right)_{\text{4-level}}} = \frac{\frac{1}{2} N_0}{\Delta N_{\text{th}}}$$ □ 1

Minimum power requires to achieve threshold

$p^{\text{th}} = N_2 \frac{h\nu_{\text{laser}}}{t_2} V$ — volume occupied by gain material

$$p_3^{\text{th}} = \frac{1}{2} N_0 V \frac{h\nu}{t_2}$$

$$p_4^{\text{th}} = N_2^{\text{th}} V \frac{h\nu}{t_2}$$

If level 2 depopulation is due to spontaneous emission alone, put $t_2 = t_{sp}$.

In general, t_2 includes a downward rates

In 4-level system :

$$\Delta N_{th} \approx N_2^{th} = \frac{8\pi n^3 \nu^2 t_{sp}}{c^3 g(\nu)\tau}$$

$$p_4^{th} = \frac{8\pi n^3 \nu^2 t_{sp}}{c^3 g(\nu)\tau} V \frac{h\nu}{t_2}$$

Ex : What is the minimum power required to set up steady state oscillation in a 4-level Nd^{3+} : glass laser system ?

Cavity length : $l = 10$ cm

$t_2 = t_{sp}$

$\lambda = 1.05$ μm

$R = R_1 = R_2 = 0.95$

$\alpha = 0 \Rightarrow$ so loss have only the end of the mirror

$V = 10$ cm^3

$n = 1.5$

$\Delta\nu = 3\times 10^{12}$ sec^{-1} $\Rightarrow$ phonon process not Doppler broadening

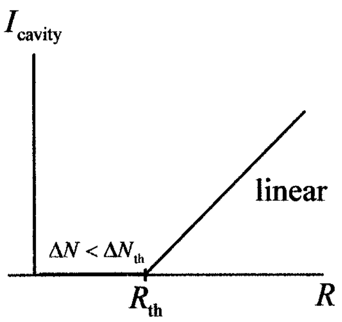
$$\tau = \frac{n}{c\left[\alpha - \frac{1}{2l}\ln R_1 R_2\right]}$$

$$= \frac{1.5}{3\left[0 - \frac{1}{2(10)}\ln(0.95)^2\right]} = 9.7\times 10^{-9} \text{ sec}$$

$$p_4^{th} = \frac{8\pi(1.5)^3 6.6\times 10^{-27}\times 3\times 10^{12}\times 10}{(1.5\times 10^{-4})^3(9.7\times 10^{-9})}$$

$$= 1.45\times 10^9 \frac{erg}{\text{sec}} = 145 \text{ W}$$

$\Delta N_{th}(\gamma_{th}) \rightarrow R_{th} : \Delta N < \Delta N_{th}$



Steady state under $\frac{dN}{dt} = 0$

Laser O/P power vs Pumping Power :

As we increase the pump power upto and beyond threshold, what happen to laser output power ?

R_1 we don't want $\Rightarrow$ "deterious pumping"

good pumping rate

① ——— $$\frac{dN_2}{dt} = R_2 - \frac{N_2}{t_2} - W(N_2 - N_1)$$

depopulation rate　　Stim. Emiss. Rate per atom

② ——— $$\frac{dN_1}{dt} = R_1 - \frac{N_1}{t_1} + W(N_2 - N_1) + N_2/t_2$$

$$B_g^{(0)}(\nu)\rho(\nu)$$

These equations describe a homogeneous broadened system.

We don't beak N_2 into subset $\Rightarrow$ all atom in upper level are the same

$\Rightarrow$ one represents all

Consider steady system situation : Transients have die down for

$$t > \max\left\{t_1, t_2, \frac{1}{W}, \frac{N_1}{R_1}, \frac{\langle N_2\rangle}{R_2}\right\}$$

$$\frac{dN_1}{dt} = 0 = \frac{dN_2}{dt}$$

Add ① + ② :

$$0 = R_1 + R_2 - \frac{N_1}{t_1}$$

Substitute $\quad 0 = R_1 - (R_1 + R_2)t_1$ into ②

$$0 = R_1 - (R_1 + R_2) + W\left(N_2 - (R_1 + R_2)t\right) + \frac{N_2}{t_2}$$

Or

$$N_2 = \frac{R_2 + (R_1 + R_2)Wt_1}{W + \frac{1}{t_2}}$$

$$N_2 - N_1 = \frac{R_2 + (R_1 + R_2)Wt_1 - (R_1 + R_2)t_1\left(W + \frac{1}{t_2}\right)}{W + \frac{1}{t_2}}$$

$$= \frac{R_2\left[1 - \left(1 + \frac{R_1}{R_2}\right)\frac{t_1}{t_2}\right]}{W + \frac{1}{t_2}}$$

$$N_2 - N_1 > 0 \Rightarrow 1 - \left(1 + \frac{R_1}{R_2}\right)\frac{t_1}{t_2} > 0$$

or

$$\frac{t_1}{t_2} < \frac{1}{1 + \frac{R_1}{R_2}} < 1$$

$\Rightarrow$ Because we want $t_1 \downarrow\downarrow$ when compare to t_2

Ideal Situation: $\frac{t_1}{t_2} \to 0$ and $N_1 \to 0$

$$N_2 - N_1 = \frac{R_2}{W + \frac{1}{t_2}}$$

So define effective pumping rate for non-ideal situation

$$R_{\text{eff}} = R_2\left[1 - \left(1 + \frac{R_1}{R_2}\right)\frac{t_1}{t_2}\right] \equiv R$$

So :

Ⓐ
$$\boxed{N_2 - N_1 = \frac{R}{W + \frac{1}{t_2}}}$$

Recall :

$$\Delta N_{\text{th}} = \left(N_0 - N_1\right)_{\text{th}} = \frac{8\pi n^3 \nu^2 t_{\text{sp}}}{c^3 g(\nu)\tau}$$

(τ: photon lifetime)

Basically, for pumping rates from zero ups to threshold

$\boxed{1}$ $0 \le R \le R_{\text{th}}$

Assume : R is " + "

& $0 < \Delta N \le N_{\text{th}}$

& $W = 0$ (No stim. emiss when)

$$\Rightarrow \quad \left(N_2 - N_1\right)_{\text{th}} = R_{\text{th}} t_2$$

$$R_{\text{th}} = \frac{\left(N_2 - N_1\right)_{\text{th}}}{t_2}$$

$$= \frac{8\pi n^3 \nu^2}{c^3 g(\nu)\tau}\left(\frac{t_{\text{sp}}}{t_2}\right)$$

(τ: cavity; $\frac{t_{\text{sp}}}{t_2}$: Atomic group)

2 $R \geq R_{th}$

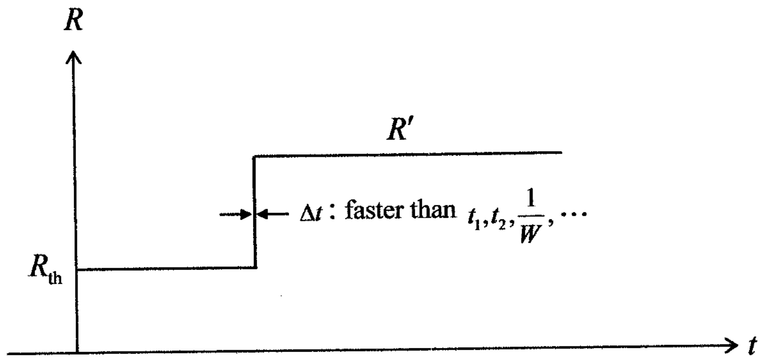
Then $W \neq 0$

And $\Delta N = \Delta N_{th}$ (steady state condition)

(A) becomes

$$\Delta N_{th} = \frac{R}{W + \frac{1}{t_2}}$$

$$W = \frac{R}{\Delta N_{th}} - \frac{1}{t_2}$$



1. The atom in upper level state is fallen to the lower level state. And then, the stimulated emission is happening. Thus ΔN decreases $\rightarrow \gamma \downarrow$

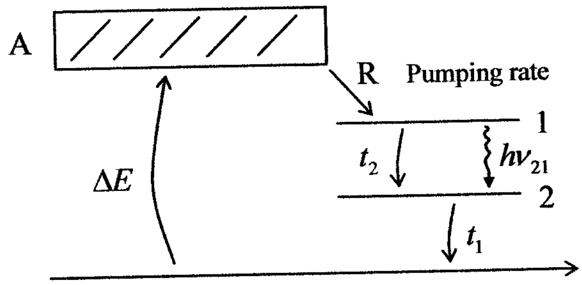
2. $\gamma < \gamma_{th}$: because the decrease # of atoms by stimulated emission is greater than the increase # of atoms by pumping conditions.

3. γ increase : because it has R', so γ is increased.

4. Same reason as " 1 "

5. After t^*; Atom is in the steady state which $\gamma = \gamma_{th}$. But it can reach γ_{th} suddenly.

Specific Laser System:



i) Quantum Efficiency

$$= \frac{h\nu_{21}}{\Delta E}$$

ii) Transfer Efficiency

$\Rightarrow$ Fraction of atoms pumped to 3 which makes transition to 2.

iii) Pump Efficiency :

$\equiv$ Fraction of pump power efficiency in populating level 3.

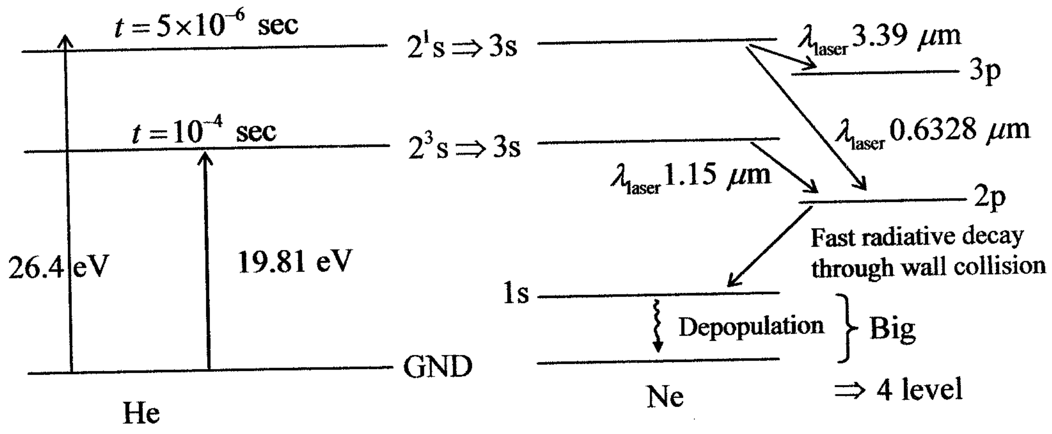
Over all laser efficient $= \eta =$ (i) (ii) (iii)

Example : CO_2 laser $\eta \approx 30\%$ (Quite high)

Nd^{3+} YAG $\eta \approx 1\%$ (for flash lamp)

Pump Visible & Laser IR $\Rightarrow$ Loss $\uparrow$

He-Ne Laser :



$\Rightarrow$ Smaller distance tubes show higher gain due to buffer wall collision of N_e is level

$$\gamma^{th} = \frac{c^2 g(\nu)}{8\pi n^2 \nu^2 t_{sp}} (N_2 - N_1)_{th}$$

$N_1 \downarrow\downarrow \quad \rightarrow \quad \gamma^{th} \uparrow\uparrow$

ϕ of tube $\downarrow$

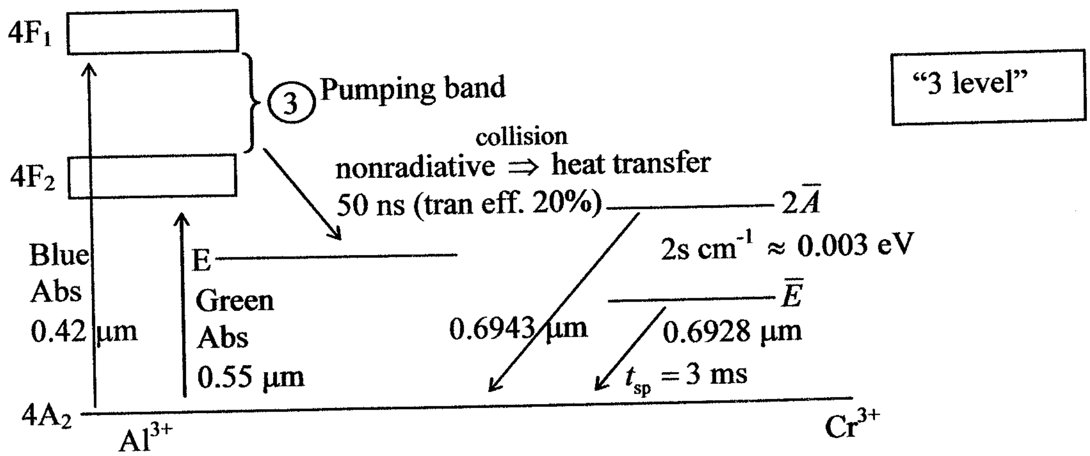
Normally $\gamma_{3.39}$ & $\gamma_{1.115} > \gamma_{0.6328}$

b) Ruby Laser – First demonstrated lose (1960)

Ruby : Sapphire Al_2O_3

$\Rightarrow$ Replace 1% of Al^{3+} ion with Cr^{3+} ions.

Cr^{3+} give ruby its pink / red colors

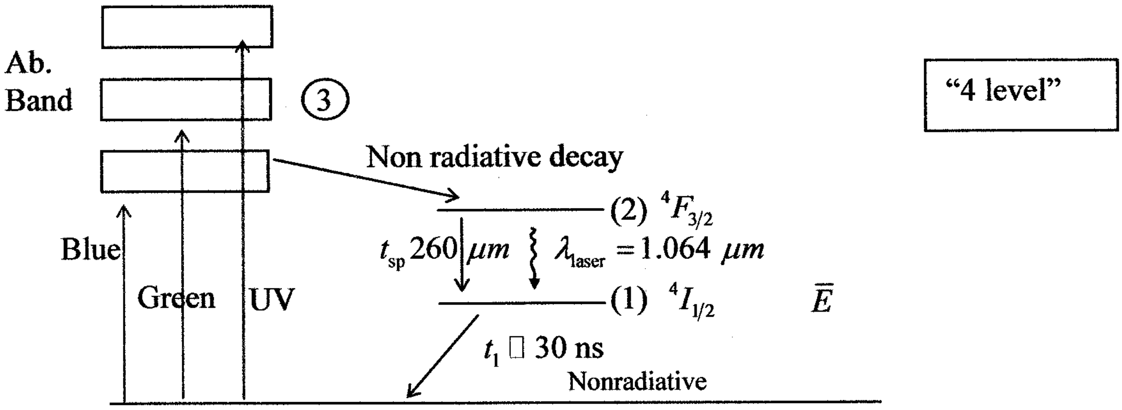


c) Nd : YAG

Neodenium Yttrium Aluminium Ganet

⇒ Al ions are replaced at a level of □ 1% by Nd.

Get higher than 1% ⇒
- Crystal strain
- Light donut
- Hand to grow x'tal



$$\frac{N_1}{N_0} = e^{-\Delta E/kT} \approx 10^{-10}$$

$$\gamma(\nu) = \sigma_{\text{stim}} N$$

		YAG		Ruby
$\frac{\sigma_{\text{YAG}}}{\sigma_{\text{ruby}}} \approx 70$	⇒	N_t	<	$\frac{N}{70}$
	Pumping	P_t	<	$\frac{P_t}{70}$

σ_{stim} is large, cw or s.s. operation of Nd: YAG is possible

Principle of Laser : Orazio Svelto

$$dI = \frac{c}{n}\left(\frac{\gamma L}{l} - \beta\right) I$$

$$= \frac{c}{n}\left(\frac{\gamma L}{l} - \frac{1}{\tau}\right) I \quad : \quad \frac{1}{\tau} = \frac{\beta c}{n}$$

$$\frac{dI}{d(t/\tau)} = \left(\frac{c}{n}\frac{\gamma L}{l}\tau - 1\right) : \quad t/\tau = T$$

$$\gamma_{th} = \frac{c}{n\tau}\frac{l}{L}$$

Method of Q-switching :

1) Mechanical

   i) Fast shutter in cavity

   ii) Rotating mirror

2) Saturable Absorber

   $\Rightarrow$ high Intensity $\Rightarrow$ Fraction absent $\downarrow$.

   Loss remain high (Low Q) until absorption is saturated $\overset{\text{cavity}}{\Rightarrow}$ become high Q