การประจำที่ของโฟตอนและการแผ่ในปริภูมิเวลาในทฤษฎีสนามควอนตัม

นายประสพชัย วิริยะศรีสุวัฒนา

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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LOCALIZATION OF PHOTONS AND PROPAGATION IN SPACETIME IN QUANTUM FIELD THEORY

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in Physics

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LOCALIZATION OF PHOTONS AND PROPAGATION IN SPACETIME IN QUANTUM FIELD THEORY

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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ประสพชัย วิริยะศรีสุวัฒนา : การประจำที่ของโฟตอนและการแผ่ในปริภูมิเวลาในทฤษฎีสนามควอนตัม (LOCALIZATION OF PHOTONS AND PROPAGATION IN SPACETIME IN QUANTUM FIELD THEORY) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร.เอ็ดเวิร์ด มานูเกียน, 196 หน้า.

การวิเคราะห์ส่วนมาก ที่เกี่ยวข้องในวิทยานิพนธ์นี้คือ การกำหนดสูตรที่ชัดเจนสำหรับอธิบายการแผ่ของโฟตอนในปริภูมิเวลา ตามกระบวนการวิวัฒน์แห่งเวลา ด้วยแอมพลิจูดที่สอดคล้องของการเปลี่ยนสถานะระหว่างจุดปริภูมิเวลาในทฤษฎีสนามควอนตัม หลังจากการวิเคราะห์อย่างละเอียดตามสถานการณ์ที่คล้องจองสำหรับอนุภาคที่ไม่ใช่อนุภาคเชิงสัมพัทธ์ในฟิสิกส์ควอนตัมต่อปัญหาที่น่าสนใจมากคือเรื่องการสะท้อนของอนุภาคดังกล่าวบนผิวสะท้อนตามทฤษฎีควอนตัม การวิเคราะห์ได้ทำการขยายไปถึงสถานการณ์ของโฟตอนที่เป็นแบบอนุภาคเชิงสัมพัทธ์ยิ่งยวดในปริภูมิเวลาในทฤษฎีสนามควอนตัม สูตรทางพลศาสตร์ไฟฟ้าควอนตัมได้รับพัฒนาอย่างมีระเบียบแบบแผนเพื่ออธิบายการแผ่ของโฟตอนในปริภูมิเวลาตามกระบวนการวิวัฒน์แห่งเวลาบนฐานแห่งกระบวนการเชิงกายภาพแท้จริงของการแผ่ระหว่างตัวส่งสัญญาณและตัวรับสัญญาณที่ใช้ในกระบวนการสะท้อนของโฟตอน กระบวนการศึกษาที่กล่าวมานี้และยังรวมทั้งการศึกษาของไฟยน์มันในช่วงเวลาก่อนหน้านี้ เป็นที่ประจักษ์ชัดว่า ในเชิงการคำนวณและในสูตรเชิงพลศาสตร์ของปริภูมิเวลาที่เหมาะสมที่ทำนายได้ยังไม่มีปรากฏให้เห็นการเสนอเชิงนัยทั่วไปของการทดลองต่างๆและทฤษฎีสนามที่เกี่ยวข้องอื่นๆการเน้นย้ำถึงความเหมาะสมของปัญหาได้อธิบายไว้ในการศึกษานี้ อันได้แก่ โดยการใช้การกระจายแบบยูนิทารีของแอมพลิจูดของการเปลี่ยนสถานะสุญญากาศถึงสุญญากาศ $\langle 0_+|0_-\rangle$ สำหรับการกระตุ้นโฟตอนซึ่งได้รวมกับนิพจน์สำหรับแอมพลิจูดของการส่งด้วยตัวส่งและแอมพลิจูดของการรับของตัวรับของการกระตุ้นโฟตอน แอมพลิจูดที่คล้องจองของการแผ่ของการกระตุ้นโฟตอนระหว่างจุดปริภูมิเวลาต่างๆกันในปริภูมิแผ่ขยายอย่างอนันต์ และยังรวมถึงในครึ่งปริภูมิตามกระบวนการวิวัฒน์แห่งเวลา แสดงให้เห็นว่าผลการศึกษาไม่ตรงกันกับสูตรที่เรียกกันว่าตัวแผ่ของไฟยน์มันด้วยเงื่อนไขขอบเขตที่คล้องจองกันในครึ่งปริภูมิ สูตรในทฤษฎีสนามควอนตัมแอมพลิจูดที่ได้มาจะสอดคล้องกับการประจำที่ของการกระตุ้นโฟตอนในปริภูมิโครงแบบ ด้วยเหตุผลเชิงความน่าจะเป็นเชิงควอนตัม ความน่าจะเป็นที่การกระตุ้นโฟตอนเหล่านี้เกิด ณ แห่งใดในปริภูมิในช่วงการแผ่ของเวลาที่กำหนดให้ โดยเฉพาะอย่างยิ่งแอมพลิจูดทั้งหลายปฏิบัติตาม

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สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา ______________________________

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PRASOPCHAI VIRIYASRISUWATTANA : LOCALIZATION OF PHOTONS AND PROPAGATION IN SPACETIME IN QUANTUM FIELD THEORY. THESIS ADVISOR : PROF. EDOUARD B. MANOUKIAN, Ph.D. 196 PP.

SPACETIME DESCRIPTION OF QUANTUM FIELD THEORY/ PROPAGATION OF PHOTONS IN SPACETIME AS TIME EVOLUTION PROCESSES/ QUANTUM FIELD THEORY OF REFLECTION OF PHOTONS IN SPACETIME.

The major analysis involved in this thesis is to provide a rigorous formalism for the propagation of photons in *spacetime* as a time evolution process with associated amplitudes of transitions between different spacetime points in quantum field theory. After a detailed analysis of the corresponding situation for non-relativistic particles in quantum physics dealing with the intriguing problem of reflections of such particles off a reflecting surface *according* to quantum theory, the analysis is extended to the situation of photons, as ultra-relativistic particles, in spacetime in quantum field theory. A QED formalism is systematically developed to describe photon propagation in *spacetime* as a time evolution process based on the actual *physical* process of propagation between emitters and detectors as applied, in particular to the reflection of photons. This development, as well as early studies by Feynman, clearly show that a practical, computational and predictive dynamical formalism in *spacetime* was lacking. The present one generalizes to different experimental situations and *other* interacting field theories as well emphasizing the practicality of the problem treated here. For example, by using a unitarity expansion of the vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle$, supplemented by the expressions for the amplitudes of emission, by an emitter, and detection, by a detector, of photon excitations, the corresponding amplitudes of propagation of photon excitations between different spacetime points in infinitely extended space as well as in half-space, as time evolution processes, and show that they do not coincide with the so-called Feynman propagators with the corresponding boundary conditions in half-space. In the quantum field theory formalism, derived amplitudes are associated with the localization of photon excitations in configuration space, that lead, in the quantum probabilistic sense, probabilities as to where these excitations were in space within given time spans. In particular, these amplitudes satisfy important completeness relations for the internal consistency of the formalism. As photon excitations travel from an emitter to a detector, they may have points of impact at any point on a reflecting surface. A key result is that the quantum field theory treatment via the derived amplitudes mentioned above, show that all amplitudes with points of impact at any point on the surface are exponentially damped relative to the classical point of impact. Finally in an Appendix, we have also derived a *closed* expression for the $\hbar$-quantum correction to the average *number* of photons emitted in synchrotron radiation.

School of Physics — Student's Signature ____________________

Academic Year 2007 — Advisor's Signature ____________________

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Prasopchai Viriyasrisuwattana

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CHAPTER I

INTRODUCTION

A non-speculative quantum mechanical treatment of the propagation of photons in *spacetime* has been an intriguing problem for years and is certainly a difficult one and far from trivial to develop. Several attempts have been made in recent years (e.g., Bialynicki-Birula, 1998; Allard et al., 1997) to describe the localization of photons in space (e.g., Hong and Mandel, 1986). It is fair to say, however, that there was still no explicit dynamical, non-heuristic, actual quantum (field) theory QED formalism worked out in spacetime as dictated by the latter. In spite of the spatial localization of photons in the laboratory (Hong and Mandel (1986), see also Ou, Hong and Mandel (1987); Hardy (1991)), as just mentioned, there were no completely satisfactory theoretical descriptions of their localization. The great difficulty of providing a theoretical framework of the localization of photons became clear through the ingenious work of Newton and Wigner (1948), see also Pryce (1948); Foldy and Woutheysen (1950); Tani (1951); Cini and Touschek (1958); Bose, Gamba and Sudarshan (1959); Wightman (1962); Bacry (1981), (1988a), (1988b); Manoukian (1990); Mourad (1993); Ingall (1996)) who tried to define position operators for quantum relativistic particles, and of course, the photon being massless is an ultra-relativistic particle. In general, a quantum relativistic particle of mass M cannot be localized within a volume of extension smaller than its Compton wavelength $\hbar/Mc$. As the photon is massless, one already sees the difficulty encountered in localizing it. The construction of position operators for quantum relativistic particles turned out to be a difficult one and depends heavily on the assumptions made in a theory to define them. For example, Newton and Wigner insisted on the commutativity of the three components $X_i, [X_i, X_j] = 0$ of the position operator but this condition was relaxed in several other attempts (Pryce (1948); Jauch and Piron (1967);

Amrein (1969); Bacry (1988); Mourad (1993); Brook and Schroeck Jr. (1996)) . The reconciliation of the definition of a position operator, defined as a three vector, with relativity was also investigated (McDonald (1970); Broyles (1970); Johnson (1971); Han, Kim and Noz (1987); Kim and Wigner (1987); Bacry (1993). A notable contribution was due to Acharya and Sudarshan (1960) who showed the possibility of plane-localization (in the form of a wave front) of light as opposed to a localization at a point. Many attempts were also done in constructing "wave function" as solutions extracted from Maxwell's equations (Pike and Sarkar (1987); Ziokowski (1989); Palmer and Donnelly (1993); Adlard, Pike and Sarkar (1987); see also Amrein (1969)) which have at least shown a rapid vanishing properties spatially of the solutions with a power law $|\mathbf{x}|^{-\alpha}$, $\alpha > 0$, for $|\mathbf{x}| \to 0$, The momentum description of a photon and its interaction with elementary particles such as QED (Schwinger (1948), (1949a), (1949b), (1951c); Feynman (1949a), (1949b), (1950); Tomonaga (1948); Dyson (1949a), (1949b)) and in the Electro-Weak theory (Salam (1980), Weinberg (1980) and Glashow (1980)) has been, however successfully handled. The spatial description of a photon, however, becomes important when describing its localization by a detector or a photon counter, or, in general, in describing its interaction with a macroscopic object such as a reflecting surface. The importance of investigating the role of a quantum mechanical particle, and in particular the photon, in typical classical situations was particularly emphasized by Feynman (1985) in his fascinating, though non-technical lectures on QED and matter, a topic on which much research was done before and since then (Ballian and Duplantier (1977), (1978); Schwinger De Raad Jr. and Milton (1978); Deutsch and Candelas (1979), Kenedy, Critchley and Dowker (1980); Manoukian (1987a), (1987b), (1989a), (1989b), (1989c), (1989d), (1990a), (1990b), (1991), (1992a), (1992b), (1993), (1996), (1997)). In particular, the correlation of photons and their tendency to travel in the same direction, as in beam formation, is worked out in Manoukian (1992b). Basic experiments involving photons interacting with macroscopic objects, in a momentum description, however, were also given (see also Manoukian, (1992a), (1993), (1996), (1997)). A spacetime description of massive quantum particles interacting with macroscopic objects were carried out (Manoukian (1987a), (1987b), (1990)) as well as for stimulated emissions (Manoukian (1988)). Some progress has been also made on the propagation of photons, in spacetime as formally propagation between an emitter and a detector (Manoukian (1989b), (1989c), (1990), (1991)). The quantum field theoretical description of localized photons in spacetime has remained, however, a formidable problem. Even a quantum field theory description in *spacetime* of a simplest experiment as the reflection of photons off a reflecting surface as a time evolution process was lacking. This is certainly remarkable in the progress of physics, knowing that QED has been around for sometime and as Feynman (1985, p. 3) puts it, it has been thoroughly analyzed, in his legendary Alix G, Mautner Memonal Lectures. The latter fascinating, though heuristic treatment (Feynman, 1985) in words is of course, far from a definite theoretical description but, in spite being addressed to non-specialists, the discussion clearly indicates, and as the present analysis shows, that a theoretical formalism, as stated above, to explain a simplest experiment in spacetime in a quantum (field) theory QED setting was lacking, For one thing, the amplitude of propagation of photons in spacetime, as a time evolution process in infinitely extended space. for example, from a point x_1^μ to a point x_2^μ turns out to be given by $(\mathrm{i}/(\pi)^2)(x_2^0 - x_1^0)^2/[(x_2 - x_1)^2]^2$ rather than by the familiar Feynman propagator $\mathrm{i}/(x_2 - x_1)^2$, with the former satisfying a key completeness relation for the internal consistency of the theory as formulated in spacetime. The purpose of this work is, in particular, to develop such formalism in detail based on the actual physical process of *the propagation of photons from emitters to detectors* obtained from the so-called vacuum-to-vacuum transition amplitude for the underlying theory. This method has been quite successful over the years in the easiness of momentum space computations of physical processes, avoiding of introducing so-called wavefunctions, not to mention of the elegance of the formalism as opposed to more standard techniques, and at the same time gaining much physical insight as particles propagate from emitters, interact, and finally particles reach the detectors as occurring in practice. The present analysis rests on three general key points: (i) By working directly in spacetime for the vacuum-to-vacuum transition amplitude, for given boundary conditions (B.C.), and from the expressions of the amplitudes for the emission and detection of photon excitations by the external sources, an amplitude of propagation between different spacetime points from emitters to detectors, causally arranged, is extracted and, as mentioned above, it does not coincide with the Feynman propagator for the corresponding B.C. This step already shows the power of determining amplitudes of propagation by introducing external sources. (ii) The amplitude of propagation is shown to satisfy a completeness relation as photons propagate between different points critical for the internal consistency of the theory in spacetime: (iii) Application of these amplitudes to describe in detail the experiment on reflection being sought by showing, in the process, very rapid exponential damping beyond the classical point of impact for the corresponding amplitude of occurrence. The reader will soon realize that our theoretical quantum (field) theory QED formalism is reduced to a non-operator approach and opens a way to describe, as a time evolution process, photon dynamics in spacetime and other field theory interactions in different experimental situations as well. Let $|0_\mp\rangle$ denote the vacuum states before/after the external current $J^\mu(x)$, coupled to the vector potential $A_\mu(x)$ in Maxwell's Lagrangian, is switched on/off. The boundary conditions taken are $\langle 0_+|\mathbf{E}_{||}(x)|0_-\rangle = 0$, $\langle 0_+|\mathbf{B}_\perp|(x)|0_-\rangle = 0$ for $z \to +0$, where the reflecting surface is taken to consist the $x^1 - x^2$ plane, with $x^3 \equiv z \geq 0$, and $\mathbf{E}_{||}/\mathbf{B}_\perp$ denote the components of the electric/magnetic fields parallel/perpendicular to the $x^1 - x^2$ plane. The vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle$ in particular, is derived. By carrying a unitarity expression of $\langle 0_+|0_-\rangle$ for photon excitations between emitters and detectors, coupled to the expressions for the emissions and detections by these sources, the amplitudes for propagation of photon excitations between different *spacetime* points in infinitely extended space as well as in half-space as time evolution processes are extracted. In particular, we show, that by an explicit derivation, that the photon excitations may reflect off the reflecting surface at *any* point. All such points are shown to be exponentially damped relative to the classical point of impact. By localization, it is understood that one may associate an *amplitude* for any pair of spacetime points $(x_1^0, \mathbf{x}_1)$, $(x_2^0, \mathbf{x}_2)$ for $x_2^0 > x_1^0$ for which $|\mathbf{x}_2 - \mathbf{x}_1|$ may be chosen arbitrarily small. In reference to a reflecting surface, the amplitude for reflecting off the classical point is exponentially dominating over any other point on the surface, thus attributing a high degree of localization of a photon excitation at classical pint of impact at a given time in the history of its time evolution process. In Chapter II, the *spacetime* propagation of a non-relativistic particle in half-space for application in the reflection process, *according* to quantum physics, is developed as a guide for the far more complex problem dealing with photons in quantum field theory analyzed in Chapter V. Chapter III deals with a momentum description of reflection and thus provides *no* information on the spacetime propagation of photon excitations. In Chapter IV, the spacetime description of photon excitations is developed first in infinitely extended space and then in half-space and the corresponding amplitudes of propagation between different spacetime points are derived. The intriguing application of the half-space formalism, as developed in Chapter IV, is then applied rigorously to the reflection process in Chapter V in a spacetime description as a time evolution process where photon excitations encounter an obstacle. Finally in an Appendix, a closed expression for the exact $\hbar$-quantum correction to the average *number* of photons emitted in synchrotron radiation is derived. Chapter VI summarizes our findings and emphasizes key steps in the development and the application of our investigations. In the quantum field theory analyses in this thesis, we choose units such that $\hbar = 1, \ c = 1$ as is customary done.

CHAPTER II

NON–RELATIVISTIC PARTICLE IN HALF–SPACE: RECONCILIATION WITH THE LAW OF REFLECTION

2.1 Introduction

Before discussing the propagation of photons in spacetime in Chapters III and IV, we consider first the propagation of a non-relativistic particle in space as a time evolution process and apply the study to investigate the law of reflection by restricting the particle to move in half-space. This will provide us guidelines to investigate the far more complex problem involving photons in quantum field theory.

We first recall that the amplitude of propagation of a particle of mass m from a space time point $(\mathbf{x}_1, t_1)$ to a space time point $(\mathbf{x}_2, t_2)$, in the infinitely extended space, in quantum physics is given by the well known formula:

$$\langle \mathbf{x}_2, t_2 \,|\, \mathbf{x}_1, t_1 \rangle = \left(\frac{m}{2\pi \mathrm{i}\hbar (t_2 - t_1)} \right)^{3/2} \exp \mathrm{i} \frac{m|\mathbf{x_2} - \mathbf{x_1}|^2}{2\hbar (t_2 - t_1)}. \tag{2.1.1}$$

We will, however, consider the propagation of such a particle in half-space as a time evolution process. The interesting question to consider here is what quantum mechanics says about the reflection of this particle off a reflecting surface. This question is answered through the following two sections.

2.2 Most Probable Detection Sight

The time-dependent Green's function $G(\mathbf{r},\mathbf{r'}z,z',T)$, with $\mathbf{r}=(x_1,x_2)$, $-\infty < x_1 < \infty, -\infty < x_2 < \infty, x_3 = z > 0, T = t - t' > 0$ and similarly for $\mathbf{r'}$ and z', is given by

$$(-\mathrm{i})G(\mathbf{r},\mathbf{r'}z,z';T) = \Big(\frac{m}{2\pi \mathrm{i}\hbar T}\Big)^{3/2} \exp\left[\frac{im|\mathbf{r}-\mathbf{r'}|^2}{2\hbar T}\right.\Bigg\{ \exp\left[\frac{\mathrm{i}m|z-z'|^2}{2\hbar T}\right]$$

$$\left. - \exp\left[\frac{\mathrm{i}m|z+z'|^2}{2\hbar T}\right]\Bigg\}\right], \tag{2.2.1}$$

where the second term ensures that $G(\mathbf{r},\mathbf{r'}z,z';T)|_{z=0}=0$ and $G(\mathbf{r},\mathbf{r'}z,z';T)|_{z'=0}=0$ so that the particle remains restricted in the upper half region of space. We consider the reflecting (infinite) surface to lie parallel to the (x,y)-plane with the surface extending above the z-axis by a small amount $z\sim\sigma$, where the order of magnitude of σ will be discussed later.

We consider the following experimental situation. Suppose that a particle start at space-time $(x_1,y_1,z_1,0)$ reaches, at a three-dimensional region encompassing a point $(x_0,y_0,0)$ and then "reflects off" somewhere (location unknown) to the $z>0$ region in an additional time T_2. *Given* that this has occurred, we determine the $conditional$ probability that it reflects off to a space-time point (x_2,y_2,z_2,T_1+T_2). To make the calculations easier, and hence more transparent, we consider a Gaussian (Feynman, 1965) region, centered about the point $(x_0,y_0,0)$, where σ_1,σ_2,σ in the (x,y,z)-directions, respectively, where σ_1 and σ_2 are arbitrary. (For simplicity of computations we integrate symmetrically about the point $z=0$, with σ small). Consider the Green functions in one dimension, with coordinate (x,y,z) ,

$$G_x(x_1,x_2) = \int_{-\infty}^{\infty} \mathrm{d}x' G_x(x_2,x')G_x(x',x_1) \equiv C_1' I_1, \tag{2.2.2}$$

$$G_y(y_1, y_2) = \int\limits_{-\infty}^{\infty} \mathrm{d}y' G_y(y_2, y') G_y(y', y_1) \equiv C_2' I_2, \tag{2.2.3}$$

$$G_z(z_1, z_2) = \int\limits_{0}^{\infty} \mathrm{d}z' G_z(z_2, z') G_z(z', z_1) \equiv C_3' I_3. \tag{2.2.4}$$

For a particle go from (x_1, y_1, z_1) to (x_2, y_2, z_2), The Green functions in 3 dimensions can be written as

$$G(\mathbf{R}_2, \mathbf{R}_1) = G_x(x_2, x_1) G_y(y_2, y_1) G_z(z_2, z_1) \equiv C_1' C_2' C_3' I_1 I_2 I_3 = I, \tag{2.2.5}$$

where $\mathbf{R}_1 = (x_1, y_1, z_1), \mathbf{R}_2 = (x_2, y_2, z_2)$, and the T_1, T_2 dependents were supposed. We are thus led to evaluate the integrals

$$I = \left(\frac{m}{2\pi \mathrm{i}\hbar T_1}\right)^{3/2} \left(\frac{m}{2\pi \mathrm{i}\hbar T_2}\right)^{3/2} (2\sigma_1 \sigma_2 \sqrt{2\pi}\sigma)^{-1} I_1 I_2 I_3, \tag{2.2.6}$$

where

$$I_1 = \int_{-\infty}^{\infty} \mathrm{d}x' \exp\left[\frac{\mathrm{i}m(x_1 - x')^2}{2\hbar T_1}\right] \exp\left[\frac{\mathrm{i}m(x_2 - x')^2}{2\hbar T_2}\right] \exp\left[\frac{-(x' - x_0)^2}{2\sigma_1^2}\right], \tag{2.2.7}$$

$$I_2 = \int_{-\infty}^{\infty} \mathrm{d}y' \exp\left[\frac{\mathrm{i}m(y_1 - y')^2}{2\hbar T_1}\right] \exp\left[\frac{\mathrm{i}m(y_2 - y')^2}{2\hbar T_2}\right] \exp\left[\frac{-(y' - y_0)^2}{2\sigma_2^2}\right], \tag{2.2.8}$$

$$I_3 = \int_{-\infty}^{\infty} \mathrm{d}z' \left[\exp\left[\frac{\mathrm{i}m(z_1 - z')^2}{2\hbar T_1}\right] - \exp\left[\frac{\mathrm{i}m(z_2 + z')^2}{2\hbar T_1}\right]\right]$$

$$\times [\exp\left[\frac{\mathrm{i}m(z_1 - z')^2}{2\hbar T_1}\right] - \exp\left[\frac{\mathrm{i}m(z_2 + z')^2}{2\hbar T_1}\right]] \exp\left[-\frac{z'^2}{2\sigma^2}\right]. \tag{2.2.9}$$

To evaluate these integrals, we use the following useful integral of the product of three Gaussian functions:

$$\int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{\exp[-(x_1-x)^2/2\sigma_1^2]}{\sqrt{2\pi}\sigma_1} \frac{\exp[-(x_2-x)^2/2\sigma_2^2]}{\sqrt{2\pi}\sigma_2} \frac{\exp[-(x_3-x)^2/2\sigma_1^2]}{\sqrt{2\pi}\sigma_3} \right]$$

$$= \frac{1}{2\pi\sqrt{A}} \exp\left[-\frac{B}{2A} \right], \tag{2.2.10}$$

where

$$A = \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2, \tag{2.2.11}$$

$$B = \sigma_3^2(x_1-x_2)^2 + \sigma_2^2(x_1-x_3)^2 + \sigma_1^2(x_2-x_3)^2. \tag{2.2.12}$$

We can use the integral of product of three Gaussian functions mentioned above for I_1

$$I_1 = 2\pi\sigma_1\sigma_2\sqrt{2\pi}\sigma_3 \int_{-\infty}^{\infty} \mathrm{d}x' \frac{\exp\left[\frac{\mathrm{i}m(x_1-x')^2}{2\hbar T_1}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{\mathrm{i}m(x_2-x')^2}{2\hbar T_2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(x_0-x')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_3}, \tag{2.2.13}$$

where

$$\frac{\mathrm{i}m}{2\hbar T_1} = -\frac{1}{2\sigma_1^2}; \qquad \sigma_1 = \sqrt{\frac{\mathrm{i}\hbar T_1}{m}};$$

$$\frac{\mathrm{i}m}{2\hbar T_2} = -\frac{1}{2\sigma_2^2}; \qquad \sigma_2 = \sqrt{\frac{\mathrm{i}\hbar T_2}{m}};$$

$$\sigma_3^2 = \sigma_1^2, \tag{2.2.14}$$

we then write I_1 as

$$I_1 = 2\pi \left(\sqrt{\frac{\mathrm{i}\hbar T_1}{m}} \right) \left(\sqrt{\frac{\mathrm{i}\hbar T_2}{m}} \right) \sqrt{2\pi}\sigma_3$$

$$\times \int_{-\infty}^{\infty} \mathrm{d}x' \, \frac{\exp\left[\frac{-(x_1-x')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{-(x_2-x')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(x_0-x')^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_3}, \tag{2.2.15}$$

from (2.2.10) we get the result of I_1 as

$$I_1 = 2\pi \left(\sqrt{\frac{\mathrm{i}\hbar T_1}{m}}\right) \left(\sqrt{\frac{\mathrm{i}\hbar T_2}{m}}\right) \sqrt{2\pi}\sigma_1 \frac{1}{2\pi\sqrt{A_1}} \exp\left[-\frac{B_1}{2A_1}\right]$$

$$= \sqrt{\frac{2\pi}{A_1} \frac{\mathrm{i}\,\hbar}{m} T_1 \frac{\mathrm{i}\,\hbar}{m} T_2}\, \sigma_1 \exp\left[-\frac{B_1}{2A_1}\right], \tag{2.2.16}$$

where

$$A_1 = \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2 = \sigma_1^2(\sigma_1^2 + \sigma_2^2) + \sigma_2^2\sigma_1^2; \quad \sigma_3^2 = \sigma_1^2$$

$$= \sigma_1^2\,(T_1 + T_2)\,\frac{\mathrm{i}\hbar}{m} - \frac{T_1 T_2 \hbar^2}{m^2}, \tag{2.2.17}$$

$$B = \sigma_1^2(x_1 - x_2)^2 + \frac{\mathrm{i}\hbar T_2}{m}(x_1 - x_0)^2 + \frac{\mathrm{i}\hbar T_1}{m}(x_2 - x_0)^2. \tag{2.2.18}$$

Also by using (2.2.10) we obtain I_2 as

$$I_2 = 2\pi \left(\sqrt{\frac{\mathrm{i}\hbar T_1}{m}}\right) \left(\sqrt{\frac{\mathrm{i}\hbar T_2}{m}}\right) \sqrt{2\pi}\sigma_1 \frac{1}{2\pi\sqrt{A_2}} \exp\left[-\frac{B_2}{2A_2}\right]$$

$$= \sqrt{\frac{2\pi}{A_2} \frac{\mathrm{i}\,\hbar}{m} T_1 \frac{\mathrm{i}\,\hbar}{m} T_2}\, \sigma_1 \exp\left[-\frac{B_2}{2A_2}\right], \tag{2.2.19}$$

where

$$A_2 = \sigma_2^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2 = \sigma_1^2(\sigma_1^2 + \sigma_2^2) + \sigma_2^2\sigma_1^2; \quad \sigma_3^2 = \sigma_1^2$$

$$= \sigma_1^2\,(T_1 + T_2)\,\frac{\mathrm{i}\hbar}{m} - \frac{T_1 T_2 \hbar^2}{m^2}, \tag{2.2.20}$$

$$B_2 = \sigma_2^2(y_1 - y_2)^2 + \frac{\mathrm{i}\hbar T_2}{m}(y_1 - y_0)^2 + \frac{\mathrm{i}\hbar T_1}{m}(y_2 - y_0)^2, \tag{2.2.21}$$

we can write I_1 and I_2 as

$$I_j = \sqrt{\frac{2\pi \mathrm{i}\hbar T_1 T_2 \mathrm{i}\hbar T_2}{m^2 A_j}}\sigma_j \exp\left[\frac{-B_j}{2A_j}\right], \qquad j = 1, 2, \tag{2.2.22}$$

$$A_j = \left[\sigma_j^2(T_1 + T_2)\frac{\mathrm{i}\hbar}{m} - \frac{T_1 T_2 \hbar^2}{m^2}\right], \tag{2.2.23}$$

$$B_1 = \left[\sigma_1^2(x_1 - x_2)^2 + \frac{\mathrm{i}\hbar T_2}{m}(x_1 - x_0)^2 - \frac{\mathrm{i}\hbar T_1}{m}(x_2 - x_0)^2\right], \tag{2.2.24}$$

$$B_2 = \left[\sigma_2^2(y_1 - y_2)^2 + \frac{\mathrm{i}\hbar T_2}{m}(y_1 - y_0)^2 - \frac{\mathrm{i}\hbar T_1}{m}(y_2 - y_0)^2\right]. \tag{2.2.25}$$

In particular, in absolute values squared we have

$$|I_j|^2 = \left|\sqrt{\frac{2\pi}{A_j}\frac{\mathrm{i}\,\hbar}{m}T_1\frac{\mathrm{i}\,\hbar}{m}T_2}\,\sigma_j \exp\left[-\frac{B_j}{2A_j}\right]\right|^2$$

$$= \frac{2\pi}{A_j}\frac{\hbar^2}{m^2}T_1 T_2\ \sigma_j^2 \exp\left[-\frac{B_j}{A_j}\right]$$

$$|I_j|^2 = 2\pi \frac{\hbar T_1 T_2 \sigma_2}{m\sqrt{C_j}} \exp\left[-\frac{1}{C_j}\{\cdot\}_j\right], \tag{2.2.26}$$

where

$$C_j = \left[\sigma_j^4 (T_1 + T_2)^2 + T_1^2 T_2^2 \frac{\hbar^2}{m^2}\right], \tag{2.2.27}$$

$$\{\cdot\}_1 = \sigma_1^2 (T_1 + T_2)\left[T_2(x_1 - x_0)^2 + T_1(x_2 - x_0)^2\right] - T_1 T_2 \sigma_1^2 (x_1 - x_2)^2, \tag{2.2.28}$$

$$\{\cdot\}_2 = \sigma_2^2 (T_1 + T_2)\left[T_2(y_1 - y_0)^2 + T_1(y_2 - y_0)^2\right] - T_1 T_2 \sigma_2^2 (y_1 - y_2)^2. \tag{2.2.29}$$

To establish the I_3 , we suppose that a particle emitted at $t' + 0$ from a point Q at a height $z' >> \delta$ above $z = 0$ plane , reaches the reflecting body (location within unknown), at some time, say T_2 . Given that this has occurred with probability one, we determine partial contribution amplitude of finding the reflected particle at any given z at time t.

We consider the reflecting Gaussian region along the z axes about the point $z = \delta/2$ with standard deviation, we integrate, for simplicity, symmetrically for the amplitude along the z-axes . we thus have the latter amplitude the expression

$$\left(\frac{m}{2\pi \mathrm{i}\hbar T_2}\right)^{1/2} \left(\frac{m}{2\pi \mathrm{i}\hbar T_1}\right)^{1/2} I_3, \tag{2.2.30}$$

and using the z-dependent part of Green function mentioned above,

$$I_3 = \int_{-\infty}^{\infty} \mathrm{d}z' \left\{\exp\left[\frac{im(z_1 - z')^2}{2\hbar T_1}\right] - \exp\left[\frac{im(z_1 + z')^2}{2\hbar T_1}\right]\right\}$$

$$\times \left\{\exp\left[\frac{\mathrm{i}m(z_2 - z')^2}{2\hbar T_2}\right] - \exp\left[\frac{\mathrm{i}m(z_2 + z')^2}{2\hbar T_2}\right]\right\} \exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right], \tag{2.2.31}$$

for small σ. We have extended the z' integration beyond the region $0 < z' < \delta$,which is justified provided the integral

$$\int_{-\infty}^{0} \mathrm{d}z' \, \exp - \frac{(z' - \delta/2)^2}{2\sigma^2} = \frac{1}{\sqrt{2\pi}\sigma} \int_{\delta}^{\infty} \mathrm{d}z' \, \exp - \frac{(z' - \delta/2)^2}{2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\delta/2\sigma}^{\infty} \mathrm{d}z' \, \exp - z'^2/2, \qquad (2.2.32)$$

is small, where the first equality follows from symmetry, An upper bound to integral above is given by

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \mathrm{d}z' \, \exp - \frac{z'^2}{2} \leqslant \sqrt{\frac{2}{\pi}} \frac{\exp - a^2/2}{a}, a = \delta/2\sigma > 0, \qquad (2.2.33)$$

giving the upper bound 16×10^{-24} in comparison to 1. I_3 can be carried out by using integral of the three Gaussian functions formula,

$$\int_{-\infty}^{\infty} \mathrm{d}z \frac{\exp\left[-(z_1 - z)^2/2\sigma_1^2\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[-(z_2 - z)^2/2\sigma_2^2\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[-(z_3 - z)^2/2\sigma_3^2\right]}{\sqrt{2\pi}\sigma_3}$$

$$= \frac{1}{2\pi\sqrt{A}} \exp\left[-\frac{B}{2A}\right], \qquad (2.2.34)$$

where

$$A = \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2, \qquad (2.2.35)$$

$$B = \sigma_1^2(z_2 - z_3)^2 + \sigma_2^2(z_1 - z_3)^2 + \sigma_3^2(z_2 - z_1)^2. \qquad (2.2.36)$$

For $\sigma_3 = \sigma$ and $z_3 = \delta/2$ we then write

$$I_3 = \int_{-\infty}^{\infty} \mathrm{d}z' \left\{ \exp\left[\frac{\mathrm{i}m(z_1 - z')^2}{2\hbar T_1}\right] - \exp\left[\frac{\mathrm{i}m(z_1 + z')^2}{2\hbar T_1}\right]\right\}$$

$$\times \left\{ \exp\left[\frac{\mathrm{i}m(z_2 - z')^2}{2\hbar T_2}\right] - \exp\left[\frac{\mathrm{i}m(z_2 + z')^2}{2\hbar T_2}\right]\right\} \exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right]$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \exp\left[\frac{\mathrm{i}m(z_1 - z')^2}{2\hbar T_1}\right] \exp\left[\frac{\mathrm{i}m(z_2 - z')^2}{2\hbar T_2}\right] \exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right]$$

$$- \int_{-\infty}^{\infty} \mathrm{d}z' \exp\left[\frac{\mathrm{i}m(z_1 + z')^2}{2\hbar T_1}\right] \exp\left[\frac{\mathrm{i}m(z_2 - z')^2}{2\hbar T_2}\right] \exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right]$$

$$- \int_{-\infty}^{\infty} \mathrm{d}z' \exp\left[\frac{\mathrm{i}m(z_1 - z')^2}{2\hbar T_1}\right] \exp\left[\frac{\mathrm{i}m(z_2 + z')^2}{2\hbar T_2}\right] \exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right]$$

$$+ \int_{-\infty}^{\infty} \mathrm{d}z' \exp\left[\frac{\mathrm{i}m(z_1 + z')^2}{2\hbar T_1}\right] \exp\left[\frac{\mathrm{i}m(z_2 + z')^2}{2\hbar T_2}\right] \exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right],$$

$$= \sqrt{2\pi}\sigma_1 \sqrt{2\pi}\sigma_2 \sqrt{2\pi}\sigma \left\{ \int_{-\infty}^{\infty} \mathrm{d}z' \, \frac{\exp\left[\frac{-(z_1 - z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \, \frac{\exp\left[\frac{-(z_2 - z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma} \right.$$

$$- \int_{-\infty}^{\infty} \mathrm{d}z' \, \frac{\exp\left[\frac{-(z_1 + z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{-(z_2 - z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma}$$

$$- \int_{-\infty}^{\infty} \mathrm{d}z' \, \frac{\exp\left[\frac{-(z_1 - z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{-(z_2 + z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(z' - \delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma}$$

$$+\int_{-\infty}^{\infty} \mathrm{d}z'\, \frac{\exp\left[\frac{-(z_1+z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{-(z_2+z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2}\; \frac{\exp\left[\frac{-(z'-\delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma} \Bigg\}. \tag{2.2.37}$$

Evaluate the first integral in (2.2.37),

$$\int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp\left[\frac{-(z_1-z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1^2} \frac{\exp\left[\frac{-(z_2-z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2^2} \frac{\exp\left[\frac{-(z'-\delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma^2}$$

$$= \frac{1}{2\pi\sqrt{A}} \exp\left[\frac{-B}{2A}\right], \tag{2.2.38}$$

where

$$A = \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma^2 + \sigma^2\sigma_1^2 = \sigma_1^2(\sigma_1^2+\sigma_2^2) + \sigma_2^2\sigma_1^2$$

$$= \sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}, \tag{2.2.39}$$

$$B = \sigma_1^2(z_2-z_3)^2 + \frac{\mathrm{i}\hbar T_2}{m}(z_1-\delta/2)^2 + \frac{\mathrm{i}\hbar T_1}{m}(z_2-\delta/2)^2. \tag{2.2.40}$$

We now prove the product of three Gaussian integral formula to use for integrating the other integral that like the product of three Gaussian integral.

$$\int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp\left[\frac{-(z_1-z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1^2} \frac{\exp\left[\frac{-(z_2-z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2^2} \frac{\exp\left[\frac{-(z_3-z')^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp-\left[\frac{z_1^2}{2\sigma_1^2} - \frac{2z_1}{2\sigma_1^2}z' + \frac{z'^2}{2\sigma_1^2} + \frac{z_2^2}{2\sigma_2^2} - \frac{2z_2}{2\sigma_2^2}z' + \frac{z'^2}{2\sigma_2^2} + \frac{z_3^2}{2\sigma_3^2} - \frac{2z_3}{2\sigma_3^2}z' + \frac{z'^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_1^2\,\sqrt{2\pi}\sigma_2^2\,\sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp - \left[\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_3^2} \right) z'^2 + \left(-\frac{2z_1}{2\sigma_1^2} - \frac{2z_2}{2\sigma_2^2} - \frac{2z_3}{2\sigma_3^2} \right) z' + \left(\frac{z_1^2}{2\sigma_1^2} + \frac{z_2^2}{2\sigma_2^2} + \frac{z_3^2}{2\sigma_3^2} \right) \right]}{\sqrt{2\pi}\sigma_1^2 \sqrt{2\pi}\sigma_2^2 \sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp - [Pz'^2 + Qz' + R]}{S}$$

$$= \frac{1}{S}\sqrt{\frac{\pi}{P}} \exp \frac{(Q)^2 - 4PR}{4P}, \tag{2.2.41}$$

where

$$(Q)^2 = \left\{ -\left(\frac{2z_1}{2\sigma_1^2} + \frac{2z_2}{2\sigma_2^2} + \frac{2z_3}{2\sigma_3^2} \right) \right\}^2$$

$$= \left(\frac{2z_1\sigma_2^2\sigma_3^2 + 2z_2\sigma_3^2\sigma_1^2 + 2z_3\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2} \right)^2$$

$$= \frac{\left(2z_1\sigma_2^2\sigma_3^2\right)^2 + \left(2z_2\sigma_3^2\sigma_1^2\right)^2 + \left(2z_3\sigma_1^2\sigma_2^2\right)^2 + 8z_1z_2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2}{\left(2\sigma_1^2\sigma_2^2\sigma_3^2\right)^2}$$

$$+ \frac{8z_1z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2 + 8z_2z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_1^2}{\left(2\sigma_1^2\sigma_2^2\sigma_3^2\right)^2}, \tag{2.2.42}$$

$$R = \frac{z_1^2}{2\sigma_1^2} + \frac{z_2^2}{2\sigma_2^2} + \frac{z_3^2}{2\sigma_3^2}$$

$$= \frac{z_1^2\sigma_2^2\sigma_3^2 + z_2^2\sigma_3^2\sigma_1^2 + z_3^2\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}, \tag{2.2.43}$$

$$4P = 4\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_3^2} \right)$$

$$= 4\left(\frac{\sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2 + \sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right), \tag{2.2.44}$$

$$4PR = \left(\frac{\sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2 + \sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right)\left(\frac{z_1^2\sigma_2^2\sigma_3^2 + z_2^2\sigma_3^2\sigma_1^2 + z_3^2\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right),$$

$$= \frac{4\left(z_1^2\sigma_2^2\sigma_3^2\sigma_2^2\sigma_3^2 + z_2^2\sigma_3^2\sigma_1^2\sigma_3^2\sigma_1^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2 + z_1^2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2 + z_2^2\sigma_3^2\sigma_1^2\sigma_2^2\sigma_3^2\right)}{\left(2\sigma_1^2\sigma_2^2\sigma_3^2\right)^2}$$

$$+ \frac{z_3^2\sigma_1^2\sigma_2^2\sigma_2^2\sigma_3^2 + z_1^2\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2 + z_2^2\sigma_3^2\sigma_1^2\sigma_1^2\sigma_2^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_3^2\sigma_1^2}{\left(2\sigma_1^2\sigma_2^2\sigma_3^2\right)^2}. \tag{2.2.45}$$

From integration of Gaussian function

$$\int \mathrm{d}z' \frac{\exp -(Pz'^2 - Qz' + R)}{S} = \frac{1}{S}\sqrt{\frac{\pi}{P}} \exp \frac{\left\{(-Q)^2 - 4PR\right\}}{4P},$$

$$= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2}}$$

$$\times \exp\left[-\frac{\sigma_1^2(z_2 - z_3)^2 + \sigma_2^2(z_3 - z_1)^2 + \sigma_3^2(z_1 - z_2)^2}{2(\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2)}\right]$$

$$= \frac{1}{2\pi\sqrt{A}} \exp\left[-\frac{B}{2A}\right], \tag{2.2.46}$$

where

$$A = \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2, \tag{2.2.47}$$

$$B = \sigma_1^2(z_2 - z_3)^2 + \sigma_2^2(z_3 - z_1)^2 + \sigma_3^2(z_1 - z_2)^2. \tag{2.2.48}$$

For integrating the second integral in (2.2.37),

$$\int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp\left[\frac{-(z_1+z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1^2} \frac{\exp\left[\frac{-(z_2-z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2^2} \frac{\exp\left[\frac{-(z_3-z')^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp - \left[\frac{z_1^2}{2\sigma_1^2} + \frac{2z_1}{2\sigma_1^2}z' + \frac{z'^2}{2\sigma_1^2} + \frac{z_2^2}{2\sigma_2^2} - \frac{2z_2}{2\sigma_2^2}z' + \frac{z'^2}{2\sigma_2^2} + \frac{z_3^2}{2\sigma_3^2} - \frac{2z_3}{2\sigma_3^2}z' + \frac{z'^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_1^2\sqrt{2\pi}\sigma_2^2\sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp - \left[\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_3^2}\right)z'^2 + \left(\frac{2z_1}{2\sigma_1^2} - \frac{2z_2}{2\sigma_2^2} - \frac{2z_3}{2\sigma_3^2}\right)z'\right]}{\sqrt{2\pi}\sigma_1^2\sqrt{2\pi}\sigma_2^2\sqrt{2\pi}\sigma_3^2}$$

$$+ \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp - \left[\left(\frac{z_1^2}{2\sigma_1^2} + \frac{z_2^2}{2\sigma_2^2} + \frac{z_3^2}{2\sigma_3^2}\right)\right]}{\sqrt{2\pi}\sigma_1^2\sqrt{2\pi}\sigma_2^2\sqrt{2\pi}\sigma_3^2}. \tag{2.2.49}$$

P and R are the same as that of the first integral, but Q^2 is different from that of the early case, we will have,

$$4PR = 4\left(\frac{z_1^2\sigma_2^2\sigma_3^2\sigma_2^2\sigma_3^2 + z_2^2\sigma_3^2\sigma_1^2\sigma_3^2\sigma_1^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2 + z_1^2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right.$$

$$\left. + \frac{z_2^2\sigma_3^2\sigma_1^2\sigma_2^2\sigma_3^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_2^2\sigma_3^2 + z_1^2\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2 + z_2^2\sigma_3^2\sigma_1^2\sigma_1^2\sigma_2^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_3^2\sigma_1^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right), \tag{2.2.50}$$

$$(Q)^2 = \left(\frac{2z_1}{2\sigma_1^2} - \frac{2z_2}{2\sigma_2^2} - \frac{2z_3}{2\sigma_3^2}\right)^2$$

$$= \left(\frac{2z_1\sigma_2^2\sigma_3^2 - 2z_2\sigma_3^2\sigma_1^2 - 2z_3\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right)^2$$

$$= \frac{(2z_1\sigma_2^2\sigma_3^2)^2 + (2z_2\sigma_3^2\sigma_1^2)^2 + (2z_3\sigma_1^2\sigma_2^2)^2 + 8z_1z_2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2}{(2\sigma_1^2\sigma_2^2\sigma_3^2)^2}$$

$$+\frac{8z_1z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2+8z_2z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_1^2}{\left(2\sigma_1^2\sigma_2^2\sigma_3^2\right)^2}. \tag{2.2.51}$$

From the Gaussian integration

$$\int\limits_{-\infty}^{\infty}\mathrm{d}z'\frac{\exp-(Pz'^2+Qz'+R)}{S}=\frac{1}{S}\sqrt{\frac{\pi}{P}}\exp\frac{\left\{(Q)^2-4PR\right\}}{4P},$$

$$=\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2}}\exp\left[-\frac{\sigma_1^2(z_2+z_3)^2+\sigma_2^2(z_3-z_1)^2+\sigma_3^2(z_1-z_2)^2}{2(\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2)}\right]$$

$$=\frac{1}{2\pi\sqrt{A}}\exp\left[-\frac{B}{2A}\right], \tag{2.2.52}$$

where

$$A=\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2, \tag{2.2.53}$$

$$B=\sigma_1^2(z_2+z_3)^2+\sigma_2^2(z_3-z_1)^2+\sigma_3^2(z_1-z_2)^2. \tag{2.2.54}$$

We now consider the third integral in (2.2.37),

$$\int\limits_{-\infty}^{\infty}\mathrm{d}z'\frac{\exp\left[\frac{-(z_1-z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1^2}\frac{\exp\left[\frac{-(z_2+z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2^2}\frac{\exp\left[\frac{-(z_3-z')^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_3^2}$$

$$=\int\limits_{-\infty}^{\infty}\mathrm{d}z'\frac{\exp-\left[\frac{z_1^2}{2\sigma_1^2}-\frac{2z_1}{2\sigma_1^2}z'+\frac{z'^2}{2\sigma_1^2}+\frac{z_2^2}{2\sigma_2^2}+\frac{2z_2}{2\sigma_2^2}z'+\frac{z'^2}{2\sigma_2^2}+\frac{z_3^2}{2\sigma_3^2}-\frac{2z_3}{2\sigma_3^2}z'+\frac{z'^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_1^2\sqrt{2\pi}\sigma_2^2\sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp - \left[\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_3^2} \right) z'^2 + \left(-\frac{2z_1}{2\sigma_1^2} + \frac{2z_2}{2\sigma_2^2} - \frac{2z_3}{2\sigma_3^2} \right) z' + \left(\frac{z_1^2}{2\sigma_1^2} + \frac{z_2^2}{2\sigma_2^2} + \frac{z_3^2}{2\sigma_3^2} \right) \right]}{\sqrt{2\pi}\sigma_1^2 \sqrt{2\pi}\sigma_2^2 \sqrt{2\pi}\sigma_3^2}. \tag{2.2.55}$$

P and R are the same as that of the first integral, but Q^2 is different from that of the early case, we will have,

$$\begin{aligned} 4PR =& 4 \left(\frac{z_1^2\sigma_2^2\sigma_3^2\sigma_2^2\sigma_3^2 + z_2^2\sigma_3^2\sigma_1^2\sigma_3^2\sigma_1^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2 + z_1^2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2}{2\sigma_1^2\sigma_2^2\sigma_3^2} + \right. \\ &+ \left. \frac{z_2^2\sigma_3^2\sigma_1^2\sigma_2^2\sigma_3^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_2^2\sigma_3^2 + z_1^2\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2 + z_2^2\sigma_3^2\sigma_1^2\sigma_1^2\sigma_2^2 + z_3^2\sigma_1^2\sigma_2^2\sigma_3^2\sigma_1^2}{2\sigma_1^2\sigma_2^2\sigma_3^2} \right), \end{aligned} \tag{2.2.56}$$

$$\begin{aligned} (Q)^2 &= \left(\frac{2z_1}{2\sigma_1^2} - \frac{2z_2}{2\sigma_2^2} - \frac{2z_3}{2\sigma_3^2} \right)^2 \\ &= \left(\frac{2z_1\sigma_2^2\sigma_3^2 - 2z_2\sigma_3^2\sigma_1^2 - 2z_3\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2} \right)^2 \\ &= \frac{(2z_1\sigma_2^2\sigma_3^2)^2 + (2z_2\sigma_3^2\sigma_1^2)^2 + (2z_3\sigma_1^2\sigma_2^2)^2 + 8z_1z_2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2}{(2\sigma_1^2\sigma_2^2\sigma_3^2)^2} \\ &+ \frac{8z_1z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2 + 8z_2z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_1^2}{(2\sigma_1^2\sigma_2^2\sigma_3^2)^2}. \end{aligned} \tag{2.2.57}$$

By using the Gaussian integral, i.e.,

$$\int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp -(Pz'^2 + Qz' + R)}{S} = \frac{1}{S}\sqrt{\frac{\pi}{P}} \exp \frac{\left\{(Q)^2 - 4PR\right\}}{4P}, \tag{2.2.58}$$

we get

$$(Q)^2 = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2}} \exp\left[-\frac{\sigma_1^2(z_2-z_3)^2+\sigma_2^2(z_3+z_1)^2+\sigma_3^2(z_1-z_2)^2}{2(\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2)}\right]$$

$$= \frac{1}{2\pi\sqrt{A}} \exp\left[-\frac{B}{2A}\right], \tag{2.2.59}$$

where

$$A = \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2, \tag{2.2.60}$$

$$B = \sigma_1^2(z_2-z_3)^2 + \sigma_2^2(z_3+z_1)^2 + \sigma_3^2(z_1-z_2)^2. \tag{2.2.61}$$

Considering the 4^{th} integral in (2.2.37),

$$\int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp\left[\frac{-(z_1+z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1^2} \frac{\exp\left[\frac{-(z_2+z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2^2} \frac{\exp\left[\frac{-(z_3-z')^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp-\left[\frac{z_1^2}{2\sigma_1^2}+\frac{2z_1}{2\sigma_1^2}z'+\frac{z'^2}{2\sigma_1^2}+\frac{z_2^2}{2\sigma_2^2}+\frac{2z_2}{2\sigma_2^2}z'+\frac{z'^2}{2\sigma_2^2}+\frac{z_3^2}{2\sigma_3^2}-\frac{2z_3}{2\sigma_3^2}z'+\frac{z'^2}{2\sigma_3^2}\right]}{\sqrt{2\pi}\sigma_1^2\sqrt{2\pi}\sigma_2^2\sqrt{2\pi}\sigma_3^2}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}z' \frac{\exp-\left[\left(\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_2^2}+\frac{1}{2\sigma_3^2}\right)z'^2+\left(\frac{2z_1}{2\sigma_1^2}+\frac{2z_2}{2\sigma_2^2}-\frac{2z_3}{2\sigma_3^2}\right)z'+\left(\frac{z_1^2}{2\sigma_1^2}+\frac{z_2^2}{2\sigma_2^2}+\frac{z_3^2}{2\sigma_3^2}\right)\right]}{\sqrt{2\pi}\sigma_1^2\sqrt{2\pi}\sigma_2^2\sqrt{2\pi}\sigma_3^2}. \tag{2.2.62}$$

P and R are the same as that of the first integral, but Q^2 is different from that of the early case, we will have,

$$4PR = 4\left(\frac{z_1^2\sigma_2^2\sigma_3^2\sigma_2^2\sigma_3^2+z_2^2\sigma_3^2\sigma_1^2\sigma_3^2\sigma_1^2+z_3^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2+z_1^2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}+\right.$$

$$+\frac{z_2^2\sigma_3^2\sigma_1^2\sigma_2^2\sigma_3^2+z_3^2\sigma_1^2\sigma_2^2\sigma_2^2\sigma_3^2+z_1^2\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2+z_2^2\sigma_3^2\sigma_1^2\sigma_1^2\sigma_2^2+z_3^2\sigma_1^2\sigma_2^2\sigma_3^2\sigma_1^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\Bigg),$$

(2.2.63)

$$(Q)^2=\left(\frac{2z_1}{2\sigma_1^2}+\frac{2z_2}{2\sigma_2^2}-\frac{2z_3}{2\sigma_3^2}\right)^2$$

$$=\left(\frac{2z_1\sigma_2^2\sigma_3^2+2z_2\sigma_3^2\sigma_1^2-2z_3\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2\sigma_3^2}\right)^2$$

$$=\frac{(2z_1\sigma_2^2\sigma_3^2)^2+(2z_2\sigma_3^2\sigma_1^2)^2+(2z_3\sigma_1^2\sigma_2^2)^2+8z_1z_2\sigma_2^2\sigma_3^2\sigma_3^2\sigma_1^2}{(2\sigma_1^2\sigma_2^2\sigma_3^2)^2}$$

$$+\frac{-8z_1z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_2^2-8z_2z_3\sigma_2^2\sigma_3^2\sigma_1^2\sigma_1^2}{(2\sigma_1^2\sigma_2^2\sigma_3^2)^2}. \tag{2.2.64}$$

By using Gaussian integration (2.2.62) becomes

$$=\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2}}\exp\left[-\frac{\sigma_1^2(z_2-z_3)^2+\sigma_2^2(z_3+z_1)^2+\sigma_3^2(z_1-z_2)^2}{2(\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2)}\right]$$

$$=\frac{1}{2\pi\sqrt{A}}\exp\left[-\frac{B}{2A}\right], \tag{2.2.65}$$

where

$$A=\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2, \tag{2.2.66}$$

$$B=\sigma_1^2(z_2-z_3)^2+\sigma_2^2(z_3+z_1)^2+\sigma_3^2(z_1-z_2)^2. \tag{2.2.67}$$

We now can write I_3 as

$$I_3 = \sqrt{2\pi}\sigma_1\sqrt{2\pi}\sigma_2\sqrt{2\pi}\sigma \left\{ \int_{-\infty}^{\infty} dz' \, \frac{\exp\left[\frac{-(z_1-z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \, \frac{\exp\left[\frac{-(z_2-z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(z'-\delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma} \right.$$

$$- \int_{-\infty}^{\infty} dz' \, \frac{\exp\left[\frac{-(z_1+z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{-(z_2-z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(z'-\delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma}$$

$$- \int_{-\infty}^{\infty} dz' \, \frac{\exp\left[\frac{-(z_1-z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{-(z_2+z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \frac{\exp\left[\frac{-(z'-\delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma}$$

$$\left. + \int_{-\infty}^{\infty} dz' \, \frac{\exp\left[\frac{-(z_1+z')^2}{2\sigma_1^2}\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[\frac{-(z_2+z')^2}{2\sigma_2^2}\right]}{\sqrt{2\pi}\sigma_2} \, \frac{\exp\left[\frac{-(z'-\delta/2)^2}{2\sigma^2}\right]}{\sqrt{2\pi}\sigma} \right\}$$

$$= \sqrt{2\pi}\sigma_1\sqrt{2\pi}\sigma_2 \left(\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2}} \right)$$

$$\times \left\{ \exp\left[-\frac{\sigma_1^2(z_2-z_3)^2 + \sigma_2^2(z_3-z_1)^2 + \sigma_3^2(z_1-z_2)^2}{2(\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2)} \right] \right.$$

$$- \exp\left[-\frac{\sigma_1^2(z_2+z_3)^2 + \sigma_2^2(z_3-z_1)^2 + \sigma_3^2(z_1-z_2)^2}{2(\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2)} \right]$$

$$- \exp\left[-\frac{\sigma_1^2(z_2-z_3)^2 + \sigma_2^2(z_3+z_1)^2 + \sigma_3^2(z_1-z_2)^2}{2(\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2)} \right]$$

$$\left. + \exp\left[-\frac{\sigma_1^2(z_2-z_3)^2 + \sigma_2^2(z_3-z_1)^2 + \sigma_3^2(z_1+z_2)^2}{2(\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2)} \right] \right\}. \tag{2.2.68}$$

Letting $\sigma_3 = \sigma$ and $z_3 = \delta/2$, so we have

$$A = \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma^2 + \sigma^2\sigma_1^2$$

$$= \sigma_1^2(\sigma_1^2 + \sigma_2^2) + \sigma_2^2\sigma_1^2$$

$$= \sigma^2\left(T_1 + T_2\right)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}$$

$$B = \sigma_1^2(z_2 - z_3)^2 + \frac{\mathrm{i}\hbar T_2}{m}(z_1 - \delta/2)^2 + \frac{\mathrm{i}\hbar T_1}{m}(z_2 - \delta/2)^2, \qquad (2.2.69)$$

we then have I_3 as

$$I_3 = \sqrt{2\pi}\sqrt{\frac{\mathrm{i}\hbar T_1}{m}}\sqrt{2\pi}\sqrt{\frac{\mathrm{i}\hbar T_2}{m}}\sqrt{2\pi}\sigma\left(\frac{1}{2\pi\sqrt{\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}}}\right)$$

$$\times\left\{\exp\left[-\frac{\frac{-\hbar^2T_1^2}{m^2}(z_2 - \frac{\delta}{2})^2 - \frac{\hbar^2T_2^2}{m^2}(\frac{\delta}{2} - z_1)^2 + \sigma^2(z_1 - z_2)^2}{2\left[\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}\right]}\right]\right.$$

$$- \exp\left[-\frac{(z_2 + \frac{\delta}{2})^2 + \sigma_2^2(\frac{\delta}{2} - z_1)^2 + \sigma^2(z_1 - z_2)^2}{2\left[\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}\right]}\right]$$

$$- \exp\left[-\frac{\frac{\mathrm{i}\hbar T_1}{m}(z_2 - \frac{\delta}{2})^2 + \frac{\mathrm{i}\hbar T_2}{m}(\frac{\delta}{2} + z_1)^2 + \sigma^2(z_1 - z_2)^2}{2\left[\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}\right]}\right]$$

$$\left. + \exp\left[-\frac{\sigma_1^2(z_2 - z_3)^2 + \sigma_2^2(z_3 - z_1)^2 + \sigma_3^2(z_1 + z_2)^2}{2(\sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_1^2)}\right]\right\}$$

$$I_3 = \frac{-\hbar\sqrt{T_1T_2}}{m}\sqrt{2\pi}\sigma\left(\frac{1}{\sqrt{\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}}}\right)$$

$$\times\left\{\exp\left[-\frac{\frac{-\hbar^2T_1^2}{m^2}(z_2-\frac{\delta}{2})^2-\frac{\hbar^2T_2^2}{m^2}(\frac{\delta}{2}-z_1)^2+\sigma^2(z_1-z_2)^2}{2\left[\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m}-\frac{T_1T_2\hbar^2}{m^2}\right]}\right]\right.$$

$$-\exp\left[-\frac{(z_2+\frac{\delta}{2})^2+\sigma_2^2(\frac{\delta}{2}-z_1)^2+\sigma^2(z_1-z_2)^2}{2\left[\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m}-\frac{T_1T_2\hbar^2}{m^2}\right]}\right]$$

$$-\exp\left[-\frac{\frac{\mathrm{i}\hbar T_1}{m}(z_2-\frac{\delta}{2})^2+\frac{\mathrm{i}\hbar T_2}{m}(\frac{\delta}{2}+z_1)^2+\sigma^2(z_1-z_2)^2}{2\left[\sigma^2\left(T_1+T_2\right)\frac{\mathrm{i}\hbar}{m}-\frac{T_1T_2\hbar^2}{m^2}\right]}\right]$$
$$\left.+\exp\left[-\frac{\sigma_1^2(z_2-z_3)^2+\sigma_2^2(z_3-z_1)^2+\sigma_3^2(z_1+z_2)^2}{2(\sigma_1^2\sigma_2^2+\sigma_2^2\sigma_3^2+\sigma_3^2\sigma_1^2)}\right]\right\}. \tag{2.2.70}$$

We now consider a case of delta is equal to zero we get constant A as

$$A=\sigma^2(T_1+T_2)\frac{\mathrm{i}\hbar}{m}-T_1T_2\frac{\hbar^2}{m^2}=v-w \tag{2.2.71}$$

$$A^*=-\sigma^2(T_1+T_2)\frac{\mathrm{i}\hbar}{m}-T_1T_2\frac{\hbar^2}{m^2}=-v-w \tag{2.2.72}$$

$$AA^*=\sigma^4(T_1+T_2)^2\frac{\hbar^2}{m^2}+T_1^2T_2^2\frac{\hbar^4}{m^4}$$

$$=\left(\sigma^4(T_1+T_2)^2+T_1^2T_2^2\frac{\hbar^2}{m^2}\right)\frac{\hbar^2}{m^2}$$

$$=C\frac{\hbar^2}{m^2}=C'. \tag{2.2.73}$$

In the case of $\delta=0$ we then write I_3 as

$$I_3=-\frac{\hbar\sqrt{T_1T_2}}{m}\sqrt{2\pi}\,\sigma\frac{1}{\sqrt{A}}\left\{\exp\left[-\frac{\frac{\mathrm{i}\hbar T_1}{m}(z_2^2)+\frac{\mathrm{i}\hbar T_2}{m}(z_1^2)+\sigma^2(z_1^2-2z_1z_2+z_2^2)}{2A}\right]\right.$$

$$- \exp\left[-\frac{\frac{\mathrm{i}\hbar T_1}{m}(z_2^2) + \frac{\mathrm{i}\hbar T_2}{m}(z_1^2) + \sigma^2(z_1^2 - 2z_1z_2 + z_2^2)}{2A}\right]$$

$$- \exp\left[-\frac{\frac{\mathrm{i}\hbar T_1}{m}(z_2^2) + \frac{\mathrm{i}\hbar T_2}{m}(z_1^2) + \sigma^2(z_1^2 - 2z_1z_2 + z_2^2)}{2A}\right]$$

$$+ \exp\left[-\frac{\frac{\mathrm{i}m}{\hbar T_1}(z_2^2) + \frac{\mathrm{i}m}{\hbar T_2}(z_1^2) + \sigma^2(z_1^2 + 2z_1z_2 + z_2^2)}{2A}\right] \Bigg\}, \tag{2.2.74}$$

let

$$a = \frac{\mathrm{i}\hbar T_1}{m}(z_2^2);\;\; b = \frac{\mathrm{i}\hbar T_2}{m}(z_1^2) \tag{2.2.75}$$

$$c = \sigma^2(z_1^2 - 2z_1z_2 + z_2^2); d = \sigma^2(z_1^2 + 2z_1z_2 + z_2^2), \tag{2.2.76}$$

we then can write I_3 as

$$I_3 = -\frac{\hbar\sqrt{T_1T_2}}{m}\sqrt{2\pi}\,\sigma\frac{1}{\sqrt{A}}\left\{\exp\left[-\frac{a+b+c}{2A}\right] - \exp\left[-\frac{a+b+c}{2A}\right]\right.$$

$$\left. - \exp\left[-\frac{a+b+c}{2A}\right] + \exp\left[-\frac{a+b+d}{2A}\right]\right\}$$

$$= -\frac{\hbar\sqrt{T_1T_2}}{m}\sqrt{2\pi}\,\sigma\frac{1}{\sqrt{A}}\left\{-\exp\left[-\frac{a+b+c}{2A}\right] + \exp\left[-\frac{a+b+d}{2A}\right]\right\}, \tag{2.2.77}$$

$$|I_3|^2 = I_3I_3^* = \left[-\frac{\hbar^2\sqrt{T_1T_2}}{m^2}\sqrt{2\pi}\,\sigma\right]^2\frac{1}{\sqrt{A}}\frac{1}{\sqrt{A^*}}\{\cdot\}\{\cdot\}^*$$

$$= \frac{\hbar T_1T_2}{m}2\pi\,\sigma^2\frac{1}{\sqrt{C}}\{\cdot\}\{\cdot\}^*, \tag{2.2.78}$$

where

$$\{\cdot\} = \left\{ -\exp\left[-\frac{a+b+c}{2A}\right] + \exp\left[-\frac{a+b+\sigma^2(z_1^2+2z_1z_2+z_2^2)}{2A}\right]\right\}$$

$$= -\exp\left[-\frac{a+b}{2A}\right]\left\{\exp\left[-\frac{c}{2A}\right] - \exp\left[-\frac{\sigma^2(z_1^2+2z_1z_2+z_2^2)}{2A}\right]\right\}. \tag{2.2.79}$$

Considering

$$\{\cdot\}\{\cdot\}^* = \left[-\exp\left[-\frac{a+b}{2A}\right]\left\{\exp\left[-\frac{c}{2A}\right] - \exp\left[-\frac{d}{2A}\right]\right\}\right]$$
$$\times\left[-\exp\left[-\frac{a^*+b^*}{2A^*}\right]\left\{\exp\left[-\frac{c}{2A^*}\right] - \exp\left[-\frac{d}{2A^*}\right]\right\}\right]$$

$$= \exp\left[-\frac{a+b}{2A} - \frac{a^*+b^*}{2A^*}\right]\left\{\exp\left[-\frac{c}{2A}\right] - \exp\left[-\frac{d}{2A}\right]\right\}$$

$$\times\left\{\exp\left[-\frac{c}{2A^*}\right] - \exp\left[-\frac{d}{2A^*}\right]\right\}$$

$$= \exp\left[-\frac{(a+b)\,A^* + (-a-b)\,A}{2AA^*}\right]$$

$$\times\left\{\exp\left[-\frac{c}{2A} - \frac{c}{2A^*}\right] - \exp\left[-\frac{d}{2A} - \frac{c}{2A^*}\right]\right.$$

$$\left. - \exp\left[-\frac{c}{2A} - \frac{d}{2A^*}\right] + \exp\left[-\frac{d}{2A} - \frac{d}{2A^*}\right]\right\}$$

$$= \exp\left[-\frac{(a+b)(-v-w) + (-a-b)(v-w)}{2AA^*}\right]$$

$$\times\left\{\exp\left[-\frac{c(-v-w)+c(v-w)}{2AA^*}\right]-\exp\left[-\frac{d(-v-w)+c(v-w)}{2AA^*}\right]\right.$$

$$\left.-\exp\left[-\frac{c(-v-w)+d(v-w)}{2AA^*}\right]+\exp\left[-\frac{d(-v-w)+d(v-w)}{2AA^*}\right]\right\}$$

$$=\exp\left[\frac{(a+b)\,v}{C'}\right]\ \left\{\exp\left[\frac{cw}{C'}\right]-\exp\left[\frac{(-c+d)v+(c+d)w}{2C'}\right]\right.$$

$$\left.-\exp\left[\frac{(c-d)v+(c+d)w}{2C'}\right]+\exp\left[\frac{dw}{C'}\right]\right\};\qquad C'=AA^*$$

$$=\exp\left[\frac{(a+b)\,v}{C'}\right]\ \left\{\exp\left[\frac{cw}{C'}\right]-\exp\left[\frac{(c+d)w}{2C'}\right]\right.$$

$$\left.\times\left\{\exp\left[\frac{(-c+d)v}{2C'}\right]+\exp\left[\frac{(c-d)v}{2C'}\right]\right\}+\exp\left[\frac{dw}{C'}\right]\right\}$$

$$=\exp\left[\frac{(a+b)\,v}{C'}\right]\ \left\{\exp\left[\frac{\sigma^2(z_1^2-2z_1z_2+z_2^2)w}{C'}\right]\right.$$

$$-\exp\left[\frac{(\sigma^2(z_1^2-2z_1z_2+z_2^2)+\sigma^2(z_1^2+2z_1z_2+z_2^2))w}{2C'}\right]$$

$$\times\left\{\exp\left[\frac{(-\sigma^2(z_1^2-2z_1z_2+z_2^2)+\sigma^2(z_1^2+2z_1z_2+z_2^2))v}{2C'}\right]\right.$$

$$\left.+\exp\left[\frac{(\sigma^2(z_1^2-2z_1z_2+z_2^2)-\sigma^2(z_1^2+2z_1z_2+z_2^2))v}{2C'}\right]\right\}$$

$$\left.+\exp\left[\frac{\sigma^2(z_1^2+2z_1z_2+z_2^2)w}{C'}\right]\right\}$$

$$= \exp\left[\frac{(a+b)\,v}{C'}\right]\left\{\exp\left[\frac{(\sigma^2 z_1^2+\sigma^2 z_2^2)w}{C'}\right]\right.$$

$$\times\exp\left[\frac{-\sigma^2 2z_1z_2w}{C'}\right]-\exp\left[\frac{(\sigma^2 z_1^2+\sigma^2 z_2^2)w}{C'}\right]$$

$$\times\left\{\exp\left[\frac{+\sigma^2 4z_1z_2v}{2C'}\right]+\exp\left[\frac{-\sigma^2 4z_1z_2v}{2C'}\right]\right\}$$

$$\left.+\exp\left[\frac{(\sigma^2 z_1^2+\sigma^2 z_2^2)w}{C'}\right]\exp\left[\frac{\sigma^2 2z_1z_2w}{C'}\right]\right\}$$

$$= \exp\left[\frac{(a+b)\,v}{C'}\right]\exp\left[\frac{(\sigma^2 z_1^2+\sigma^2 z_2^2)w}{C'}\right]\left\{\exp\left[\frac{-\sigma^2 2z_1z_2w}{C'}\right]\right.$$

$$\left.+\exp\left[\frac{+\sigma^2 2z_1z_2v}{C'}\right]+\exp\left[\frac{-\sigma^2 2z_1z_2v}{C'}\right]+\exp\left[\frac{\sigma^2 2z_1z_2w}{C'}\right]\right\}$$

$$= \exp\left[\frac{\left(\frac{\mathrm{i}\hbar T_1 z_2^2}{m}+\frac{\mathrm{i}\hbar T_2 z_1^2}{m}\right)\sigma^2(T_1+T_2)\frac{\mathrm{i}\hbar}{m}}{C\frac{\hbar^2}{m^2}}\right]\exp\left[\frac{(\sigma^2 z_1^2+\sigma^2 z_2^2)T_1T_2}{C}\right]$$

$$\times\left\{2\frac{\exp\left[\frac{\sigma^2 2z_1z_2T_1T_2}{C}\right]+\exp\left[\frac{-\sigma^2 2z_1z_2T_1T_2}{C}\right]}{2}-\right.$$

$$\left.-\frac{\exp\left[\frac{+\sigma^2 4z_1z_2\sigma^2(T_1+T_2)\frac{\mathrm{i}\hbar}{m}}{2C'}\right]+\exp\left[\frac{-\sigma^2 4z_1z_2\sigma^2(T_1+T_2)\frac{\mathrm{i}\hbar}{m}}{2C'}\right]}{2}\right\}$$

$$= \exp\left[-\frac{(T_1z_2^2+T_2z_1^2)\,\sigma^2(T_1+T_2)}{C}\right]\exp\left[\frac{(\sigma^2 z_1^2+\sigma^2 z_2^2)T_1T_2}{C}\right]$$

$$\times 2 \left\{ \frac{\exp\left[\frac{\sigma^2 2 z_1 z_2 T_1 T_2}{C}\right] + \exp\left[\frac{-\sigma^2 2 z_1 z_2 T_1 T_2}{C}\right]}{2} \right.$$

$$\left. - \frac{\exp\left[\frac{+2\mathrm{i}\sigma^4 z_1 z_2 (T_1+T_2)}{C\hbar/m}\right] + \exp\left[\frac{-2\mathrm{i}\sigma^4 z_1 z_2 (T_1+T_2)}{C\hbar/m}\right]}{2} \right\}. \tag{2.2.80}$$

Substituting (2.2.80) into (2.2.78) we then have

$$|I_3|^2 = \frac{4\pi\hbar T_1 T_2 \sigma^2}{m\sqrt{C}} \exp\left[-\frac{\sigma^2}{C}(T_2^2 z_1^2 + T_1^2 z_2^2)\right]$$

$$\times \left\{ \cosh \frac{2T_1 T_2 \sigma^2 z_1 z_2}{C} - \cos \frac{2\,(T_1+T_2)\,\sigma^4 z_1 z_2}{C\hbar/m} \right\}. \tag{2.2.81}$$

Given that the particle starts at space-time point $(x_1, y_1, z_1, 0)$, *reaches*, at a given time, the Gaussian region encompassing the point $(x_0, y_0, 0)$ in the reflecting body, and reflects somewhere (location unknown) to the $z > 0$ region in an additional time T_2, We determine the conditional probability that the particle reaches a specific point $(x_2, y_2, z_2)(z > 0)$. To do this we have to compute $\int_0^\infty \mathrm{d}z_2 \int_{-\infty}^\infty \mathrm{d}y_2 \int_{-\infty}^\infty \mathrm{d}x_2 |I|^2$ and then finally divide $|I|^2$ by the later. To evaluate the above-mentioned integral we use the integral of the product of two Gaussians obtained from (2.2.10) by multiplying both sides of the latter equation by $\sqrt{2\pi}\sigma_3$ and then taking the limit σ_3 go to infinity or $\sigma_3 \gg \sigma_1$, σ_2 we have,

$$\int_{-\infty}^{\infty} dx \frac{\exp\left[-(x_1-x)^2/2\sigma_1^2\right]}{\sqrt{2\pi}\sigma_1} \frac{\exp\left[-(x_2-x)^2/2\sigma_2^2\right]}{\sqrt{2\pi}\sigma_2}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_2^2+\sigma_1^2}} \exp\left[-\frac{(x_1-x_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right]. \tag{2.2.82}$$

To compute probability that the particle arrives point $x_2, y_2, z_2(z_2 > 0)$,

$$P(x_2, y_2, z_2) = P_1 P_2 P_3. \tag{2.2.83}$$

We first evaluate the probability of the particle in x-direction

$$P_1 = |I_1|^2 \Big/ \int_{-\infty}^{\infty} |I_1|^2 \, dx_2. \tag{2.2.84}$$

From (2.2.26) we rewrite $|I_1|^2$ in detail as

$$|I_1|^2 = \left| \frac{2\pi}{A_i} \frac{\mathrm{i}\,\hbar}{m} T_1 \frac{\mathrm{i}\,\hbar}{m} T_2 \; \sigma_1^2 \exp\left[-\frac{B_1}{2A_1} \right] \right|^2$$

$$= 2\,\pi \frac{\hbar}{m\sqrt{C_1}} T_1 T_2 \; \sigma_1^2 \exp\left[-\frac{1}{C_1} \left\{ \sigma_1^2 \left(T_1 + T_2\right) \left[T_2 (x_1 - x_0)^2 + T_1 (x_2 - x_0)^2 \right] \right\} \right]$$

$$\times \exp\left[-\frac{1}{C_1} \left\{ \sigma_1^2 T_1 T_2 (x_1 - x_2)^2 \right\} \right], \tag{2.2.85}$$

where

$$C_1 = \sigma_1^4 \left(T_1 + T_2\right)^2 + \frac{T_1^2 T_2^2 \hbar^2}{m^2}. \tag{2.2.86}$$

Considering the denominator of (2.2.84)

$$\int_{-\infty}^{\infty} |I_1|^2 \, \mathrm{d}x_2 = \int_{-\infty}^{\infty} \mathrm{d}x_2 \, 2\pi \frac{\hbar}{m\sqrt{C_1}} T_1 T_2 \; \sigma_1^2$$

$$\times \exp\left[-\frac{1}{C_1} \left\{ \sigma_1^2 \left(T_1 + T_2\right) \left[T_2 (x_1 - x_0)^2 + T_1 (x_2 - x_0)^2 \right] - \sigma_1^2 T_1 T_2 (x_1 - x_2)^2 \right\} \right]$$

$$\times \exp\left[-\frac{1}{C_1}\left\{\sigma_1^2\left(T_1+T_2\right)\left[T_2\left(x_1-x_0\right)^2\ +T_2\left(x_2-x_0\right)^2\right]-\sigma_1^2T_1T_2\left(x_1-x_2\right)^2\right]\right.$$

$$=C''\int_{-\infty}^{\infty}\mathrm{d}x_2\exp\left[-\frac{1}{C_1}\left\{\sigma_1^2\left(T_1+T_2\right)\left[T_2\left(x_1^2-2x_1x_0+x_0^2\right)\right.\right.\right.$$

$$\left.+\,T_2\left(x_2^2-2x_2x_0+x_0^2\right)\right]-\sigma_1^2T_1T_2\left(x_1^2-2x_1x_2+x_2^2\right)\Big]\,,$$

where C'' is a constant, and in turn the above integral is equal to

$$C''\int_{-\infty}^{\infty}\mathrm{d}x_2\exp\left[-\frac{1}{C_1}\left\{\sigma_1^2\left(T_1+T_2\right)\left[T_2\ \left(x_1^2-2x_1x_0+x_0^2+T_1x_0^2\right)\right]\ -\sigma_1^2T_1T_2x_1^2\right.\right.$$

$$+\,T_2\left(x_2^2-2x_2x_0+x_0^2\right)\Big]-\sigma_1^2T_1T_2\left(x_1^2-2x_1x_2+x_2^2\right)\Big]$$

$$+\left[\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2\right]x_2^2+\left[-2\sigma_1^2\left(T_1+T_2\right)T_1x_0+2\sigma_1^2T_1T_2x_1\right]x_2\Big\}\,. \tag{2.2.87}$$

By using the integral

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\exp-(Px^2+Qx+R)=\sqrt{\frac{\pi}{P}}\exp\frac{\{Q^2-4PR\}}{4P}. \tag{2.2.88}$$

$$\int_{-\infty}^{\infty}|I_1|^2\,\mathrm{d}x_2=C''\sqrt{\frac{\pi}{(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}}$$

$$\exp\left[\frac{((-2\sigma_1^2\left(T_1+T_2\right)T_1x_0+2\sigma_1^2T_1T_2x_1)/C_1)^2}{4(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}\right.$$

$$+ \frac{-4\left[(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1\right]}{4(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}$$

$$\times \left.\frac{\left[(\sigma_1^2\left(T_1+T_2\right)\left(T_2(x_1^2-2x_1x_0+x_0^2)+T_1x_0^2\right)-\sigma_1^2T_1T_2x_1^2\right]/C_1}{}\right]$$

$$= C''\sqrt{\frac{\pi}{(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}}$$

$$\times \exp\left[\frac{\left(-(2\sigma_1^2\left(T_1+T_2\right)T_1x_0+2\sigma_1^2T_1T_2x_1)/C_1\right)^2}{4(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}\right.$$

$$+ \frac{-\frac{4}{C_1^2}(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)}{4(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}$$

$$\times \left.\frac{\left[-\sigma_1^2\left(T_1+T_2\right)\left(T_2(x_1^2-2x_1x_0+x_0^2)+T_1x_0^2\right)-\sigma_1^2T_1T_2x_1^2\right]}{}\right]$$

$$= C''\sqrt{\frac{\pi}{(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}}$$

$$\times \exp\left[\frac{\left(-(2\sigma_1^2\left(T_1+T_2\right)T_1x_0+2\sigma_1^2T_1T_2x_1)/C_1\right)^2-\frac{4}{C_1^2}(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)}{4(\sigma_1^2\left(T_1+T_2\right)T_1-\sigma_1^2T_1T_2)/C_1}\right.$$

$$\left.\frac{\left[-\sigma_1^2\left(T_1+T_2\right)\left(T_2(x_1^2-2x_1x_0+x_0^2)+T_1x_0^2\right)-\sigma_1^2T_1T_2x_1^2\right]}{}\right]. \tag{2.2.89}$$

We can get P_1 as

$$P_1 = \frac{|I_1|^2}{\int\limits_{-\infty}^{\infty} |I_1|^2 \,\mathrm{d}x_2} = \frac{T_1}{\sqrt{\pi}}\sqrt{\frac{\sigma_1^2}{C_1}} \exp\left[-\frac{\sigma_1^2}{C_1}\left[x_0\left(T_1+T_2\right)-\left(x_1T_2+x_2T_1\right)\right]^2\right]. \tag{2.2.90}$$

And also for P_2

$$P_2 = |I_2|^2 \Bigg/ \int\limits_{-\infty}^{\infty} |I_2|^2 \,\mathrm{d}y_2$$

$$= \frac{T_2}{\sqrt{\pi}}\sqrt{\frac{\sigma_2^2}{C_2}} \exp\left[-\frac{\sigma_2^2}{C_2}\left[y_0\left(T_1+T_2\right)-\left(y_1T_2+y_2T_1\right)\right]^2\right]. \tag{2.2.91}$$

For P_3 we can get by start from

$$P_3 = |I_3|^2 \Bigg/ \int\limits_{-\infty}^{\infty} |I_3|^2 \, dx_2, \tag{2.2.92}$$

and we have to consider the integral term,

$$\int\limits_0^{\infty} |I_3|^2 \,\mathrm{d}z_2 = \int\limits_0^{\infty} \frac{4\pi\hbar T_1 T_2 \sigma^2}{m\sqrt{C}} \exp\left[-\frac{\sigma^2}{C}(T_2^2 z_1^2 + T_1^2 z_2^2)\right]$$

$$\times \left\{\cosh\frac{2T_1T_2\sigma^2 z_1 z_2}{C} - \cos\frac{2\left(T_1+T_2\right)\sigma^4 z_1 z_2}{C\hbar/m}\right\}\mathrm{d}z_2$$

$$= \frac{4\pi\hbar T_1 T_2 \sigma^2}{m\sqrt{C}} \exp -\frac{\sigma^2}{C}T_2^2 z_1^2 \left\{\int\limits_0^{\infty} \exp -\frac{\sigma^2}{C}T_1^2 z_2^2 \cosh\frac{2T_1T_2\sigma^2 z_1 z_2}{C}\mathrm{d}z_2\right.$$

$$-\int_0^\infty \exp -\frac{\sigma^2}{C}T_1^2 z_2^2 \cos\frac{2\left(T_1+T_2\right)\sigma^4 z_1 z_2}{C\hbar/m}\mathrm{d}z_2\Bigg\}. \tag{2.2.93}$$

We use the formulas,

$$\int_0^\infty \exp\left[-az^2\right]\cos\left(bz\right)\mathrm{d}z = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left[-\frac{b^2}{4a}\right]$$

$$\int_0^\infty \exp\left[-az^2\right]\cosh\left(bz\right)\mathrm{d}z = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2}{4a}\right]. \tag{2.2.94}$$

$$\int_0^\infty |I_3|^2\,\mathrm{d}z_2 = \frac{4\pi\hbar T_1 T_2\sigma^2}{m\sqrt{C}}\exp -\frac{\sigma^2}{C}T_2^2 z_1^2$$

$$\times\left\{\frac{1}{2}\sqrt{\frac{\pi C}{\sigma^2 T_1^2}}\exp\left[\frac{4T_1^2T_2^2\sigma^4 z_1^2}{C^2\left(4\frac{\sigma^2}{C}T_1^2\right)}\right]-\frac{1}{2}\sqrt{\frac{\pi C}{\sigma^2 T_1^2}}\exp\left[-\frac{4\left(T_1+T_2\right)^2\sigma^8 z_1^2}{C^2\frac{\hbar^2}{m^2}\left(4\frac{\sigma^2}{C}T_1^2\right)}\right]\right\}$$

$$=\frac{4\pi\hbar T_1 T_2\sigma^2}{m\sqrt{C}}\exp -\frac{\sigma^2}{C}T_2^2 z_1^2\frac{1}{2}\sqrt{\frac{\pi C}{\sigma^2 T_1^2}}$$

$$\times\left\{\exp\left[\frac{T_2^2\sigma^2 z_1^2}{C}\right]-\exp\left[-\frac{\left(T_1+T_2\right)^2\sigma^6 z_1^2}{C\frac{\hbar^2}{m^2}T_1^2}\right]\right\}. \tag{2.2.95}$$

We now can get for P_3

$$P_3 = |I_3|^2\Bigg/\int_0^\infty |I_3|^2\,dz_2. \tag{2.2.96}$$

$$=\left(\frac{4\pi\hbar T_1 T_2\sigma^2}{m\sqrt{C}}\exp\left[-\frac{\sigma^2}{C}(T_2^2 z_1^2+T_1^2 z_2^2)\right]\right.$$

$$\times \left\{ \cosh \frac{2T_1 T_2 \sigma^2 z_1 z_2}{C} - \cos \frac{2\,(T_1 + T_2)\,\sigma^4 z_1 z_2}{C\hbar/m} \right\} \Big) \Big/$$

$$\Big(\frac{4\pi\hbar T_1 T_2 \sigma^2}{m\sqrt{C}} \exp - \frac{\sigma^2}{C} T_2^2 z_1^2 \times \frac{1}{2}\sqrt{\frac{\pi C}{\sigma^2 T_1^2}}$$

$$\times \left\{ \exp\left[\frac{T_2^2 \sigma^2 z_1^2}{C}\right] - \exp\left[-\frac{(T_1+T_2)^2 \sigma^6 z_1^2}{C\frac{\hbar^2}{m^2} T_1^2}\right] \right\} \Big)$$

$$= \frac{\frac{2T_1}{\sqrt{\pi}}\sqrt{\frac{\sigma^2}{C}} \exp\left[-\frac{\sigma^2}{C} T_1^2 z_2^2\right] \left\{ \cosh \frac{2T_1 T_2 \sigma^2 z_1 z_2}{C} - \cos \frac{2(T_1+T_2)\sigma^4 z_1 z_2}{C\hbar/m} \right\}}{\exp\left[\frac{T_2^2 \sigma^2 z_1^2}{C}\right] - \exp\left[-\frac{(T_1+T_2)^2 \sigma^6 z_1^2}{C T_1^2 \frac{\hbar^2}{m^2}}\right]}. \quad (2.2.97)$$

So now we can calculate the probability that a particle reaches the point (x_2, y_2, z_2) as,

$$P(x_2, y_2, z_2) = P_1 P_2 P_3. \quad (2.2.98)$$

In particular, in the classical limit (classical particle),

$$\frac{m}{T_1} \min(\sigma^2, \sigma_1^2, \sigma_2^2) \gg \hbar \to \frac{m}{T_1}(\sigma^2 \texttt{or}\ \sigma_1^2 or\ \sigma_2^2) \gg \hbar, \quad (2.2.99)$$

with a macroscopic limit

$$\sigma \ll \frac{\sqrt{T_1 T_2}}{T_1 + T_2} \min(z_1, z_2), \to \sigma \ll \frac{\sqrt{T_1 T_2}}{T_1 + T_2}(z_1 \text{or}\ z_2). \quad (2.2.100)$$

In the classical limit, P_3, P_1, P_2 simplify, respectively, as shown below.

In particular,

$$P_3 = \frac{\frac{2T_1}{\sqrt{\pi}}\sqrt{\frac{\sigma^2}{C}}\exp\left[-\frac{\sigma^2}{C}T_1^2z_2^2\right]\left\{\cosh\frac{2T_1T_2\sigma^2z_1z_2}{C} - \cos\frac{2(T_1+T_2)\sigma^4z_1z_2}{C\hbar/m}\right\}}{\exp\left[\frac{T_2^2\sigma^2z_1^2}{C}\right] - \exp\left[-\frac{(T_1+T_2)^2\sigma^6z_1^2}{CT_1^2\frac{\hbar^2}{m^2}}\right]}. \tag{2.2.101}$$

We now consider only the exponential terms of (2.2.101)

$$\exp-\frac{\sigma^2T_1^2z_2^2}{C} = \exp\left(-\frac{T_1^2z_2^2}{\sigma^2(T_1+T_2)^2}\right), \tag{2.2.102}$$

where

$$\begin{aligned}\frac{\sigma^2}{C} &= \frac{\sigma^2}{\sigma^4(T_1+T_2)^2 + T_1^2T_2^2\frac{\hbar^2}{m^2}} = \frac{\sigma^2}{\sigma^4\left((T_1+T_2)^2 + \frac{T_1^2T_2^2}{\sigma^4}\frac{\hbar^2}{m^2}\right)} \\ &= \frac{1}{\sigma^2(T_1+T_2)^2}.\end{aligned} \tag{2.2.103}$$

Considering next the exponential term in (2.2.101)

$$\exp\frac{\sigma^2T_2^2z_1^2}{C} = \exp\left(\frac{T_2^2z_1^2}{\sigma^2(T_1+T_2)^2}\right)$$

$$\begin{aligned}\exp\left(-\frac{(T_1+T_2)^2\sigma^6z_1^2}{CT_1^2\frac{\hbar^2}{m^2}}\right) &= \exp\left(-\frac{(T_1+T_2)^2z_1^2}{T_1^2\frac{\hbar^2}{m^2}}\frac{\sigma^6}{C}\right) \\ &= \exp\left(-\frac{(T_1+T_2)^2z_1^2\sigma^4}{T_1^2\frac{\hbar^2}{m^2}}\frac{1}{(T_1+T_2)^2\sigma^2}\right) \\ &= \exp\left(-\frac{z_1^2\sigma^4}{T_1^2\frac{\hbar^2}{m^2}}\right) \to 0.\end{aligned} \tag{2.2.104}$$

And consider the hyperbolic cosine term

$$\cosh \frac{2T_1T_2\sigma^2 z_1 z_2}{C} = \frac{\exp \frac{2T_1T_2z_1z_2}{(T_1+T_2)^2\sigma^2} - \exp \frac{-2T_1T_2z_1z_2}{(T_1+T_2)^2\sigma^2}}{2}$$

$$= \frac{1}{2} \exp \frac{2T_1T_2z_1z_2}{(T_1+T_2)^2 \sigma^2}$$

$$\cos \frac{2(T_1+T_2)\sigma^4 z_1 z_2}{C\hbar/m} \simeq 1 - \frac{2(T_1+T_2)^2 \sigma^4 z_1^2 z_2^2}{\hbar^2/m^2} \frac{1}{(T_1+T_2)^4 \sigma^4}$$

$$\simeq 1 - \frac{2z_1^2z_2^2}{\hbar^2/m^2} \frac{1}{(T_1+T_2)^2} = 0. \quad (2.2.105)$$

Substituting above equations into P_3 in classical limit,

$$P_3^C = \frac{2T_1}{\sqrt{\pi}} \sqrt{\frac{1}{\sigma^2(T_1+T_2)^2}} \exp\left[-\left(\frac{T_1^2 z_2^2}{\sigma^2(T_1+T_2)^2} + \frac{T_2^2 z_1^2}{\sigma^2(T_1+T_2)^2}\right)\right]$$

$$\times \left\{\frac{1}{2} \exp \frac{2T_1T_2z_1z_2}{(T_1+T_2)^2\sigma^2}\right\}$$

$$= \frac{T_1}{\sqrt{\pi}} \frac{1}{\sigma(T_1+T_2)} \exp\left[-\left(\frac{T_1^2 z_2^2}{\sigma^2(T_1+T_2)^2} - \frac{2T_1T_2z_1z_2}{(T_1+T_2)^2\sigma^2} + \frac{T_2^2 z_1^2}{\sigma^2(T_1+T_2)^2}\right)\right]$$

$$= \frac{1}{\sqrt{\pi}\sigma} \frac{T_1}{(T_1+T_2)} \exp\left[-\left(\frac{T_1^2 z_2^2 - 2T_1T_2z_1z_2 + T_2^2 z_1^2}{\sigma^2(T_1+T_2)^2}\right)\right]$$

$$= \frac{1}{\sqrt{\pi}\sigma} \frac{T_1}{(T_1+T_2)} \exp\left[-\left(\frac{T_1^2}{\sigma^2(T_1+T_2)^2}(z_2 - z_1\frac{T_2^2}{T_1^2})^2\right)\right]. \quad (2.2.106)$$

For P_1^C

$$
\begin{aligned}
P_1^C =& \frac{T_1}{\sqrt{\pi}}\sqrt{\frac{\sigma_1^2}{C_1}} \exp\left[-\frac{\sigma_1^2}{C_1}\left[x_0\left(T_1+T_2\right)-\left(x_1T_2+x_2T_1\right)\right]^2\right] \\
=& \frac{T_1}{\sqrt{\pi}}\frac{1}{\sigma_1(T_1+T_2)} \exp\left[-\frac{1}{\sigma_1^2(T_1+T_2)^2}\left[x_0\left(T_1+T_2\right)-\left(x_1T_2+x_2T_1\right)\right]^2\right] \\
=& \frac{1}{\sqrt{\pi}\sigma_1}\frac{T_1}{(T_1+T_2)} \exp\left[-\frac{1}{\sigma_1^2(T_1+T_2)^2}T_1^2\left[x_0\left(1+\frac{T_2}{T_1}\right)-\left(x_1\frac{T_2}{T_1}+x_2\right)\right]^2\right] \\
=& \frac{1}{\sqrt{\pi}\sigma_1}\frac{T_1}{(T_1+T_2)} \exp\left[-\frac{1}{\sigma_1^2(T_1+T_2)^2}T_1^2\left[-x_0\left(1+\frac{T_2}{T_1}\right)+\left(x_1\frac{T_2}{T_1}+x_2\right)\right]^2\right] \\
=& \frac{1}{\sqrt{\pi}\sigma_1}\frac{T_1}{(T_1+T_2)} \exp\left[-\frac{1}{\sigma_1^2(T_1+T_2)^2}T_1^2\left[x_2+x_1\frac{T_2}{T_1}-x_0\left(1+\frac{T_2}{T_1}\right)\right]^2\right].
\end{aligned}
\tag{2.2.107}
$$

And similarly for P_2, we get

$$
\begin{aligned}
P_2^C =& \frac{T_2}{\sqrt{\pi}}\sqrt{\frac{\sigma_2^2}{C_2}} \exp\left[-\frac{\sigma_2^2}{C_2}\left[y_0\left(T_1+T_2\right)-\left(y_1T_2+y_2T_1\right)\right]^2\right] \\
=& \frac{1}{\sqrt{\pi}\sigma_2}\frac{T_1}{(T_1+T_2)} \exp\left[-\frac{1}{\sigma_2^2(T_2+T_2)^2}T_1^2\left[y_2+y_1\frac{T_2}{T_1}-y_0\left(1+\frac{T_2}{T_1}\right)\right]^2\right].
\end{aligned}
\tag{2.2.108}
$$

Consider P_1^C, P_2^C, P_3^C peak around the classical value,

$$
x_2T_1+x_1T_2-x_0(T_1+T_2)=0 \tag{2.2.109}
$$

$$y_2 T_1 + y_1 T_2 - y_0(T_1 + T_2) = 0 \tag{2.2.110}$$

$$z_2 T_1 - z_1 T_2 = 0, \tag{2.2.111}$$

For small $\sigma^2, \sigma_1^2, \sigma_2^2 \sim 0$, we have the classical result as a delta function,

$$P_1^C \simeq \delta\left(x_2 + x_1\frac{T_2}{T_1} - x_0(1 + \frac{T_2}{T_1})\right), \tag{2.2.112}$$

$$P_2^C \simeq \delta\left(y_2 + y_1\frac{T_2}{T_1} - y_0(1 + \frac{T_2}{T_1})\right), \tag{2.2.113}$$

$$P_3^C \simeq \delta\left(z_2 + z_1\frac{T_2}{T_1}\right). \tag{2.2.114}$$

Which

$$\int_{-\infty}^{\infty} P^C(x_2, y_2, z_2)\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}z_2 = 1 \tag{2.2.115}$$

$$P^C = P_1^C P_2^C P_3^C. \tag{2.2.116}$$

as expected.

2.3 Where Do the Reflections Actually Occur? : Law of Reflection

Now suppose that we have no information on where does the particle hit the (x, y)-plane in the reflecting body and we only know that it reaches the (x, y)-plane (location unknown) at time T_l within a Gaussian width of standard deviation a in the z-direction as discussed in section 2.2 If $(x_1, y_1, z_1, 0)$ denotes the initial space-time point of the particle, and the latter is found at x_2, y_2 at time $T_1 + T_2$ after reflection, at T_1 off the (x, y)-plane, then, given that this has occurred, the conditional probability of finding the particle at a z-value z_2 (at time T_1+T_2) is simply P_3 as given in (2.2.92). And in the classical limit $m\sigma^2/T_1 \gg \hbar$, with the macroscopic limit (2.2.100) satisfied, $P_3(= P_3^C)$ will peak around the value

$$z_2T_1 - z_1T_2 = 0, \tag{2.3.1}$$

$$z_2 = \frac{z_1T_2}{T_1}. \tag{2.3.2}$$

becoming precise for arbitrary small σ, not violating the classical limit $m\sigma^2/T_1 \gg \hbar$. Therefore, the theory predicts that after reflection the particle, in the classical limit, will have a z- value z_2 satisfying relation (2.3.1). Note that by a classical limit it is meant applicable to a classical particle.

In this section we consider the converse to the problem treated in Section 2.2. More precisely, we consider the following problem. Suppose we do an experiment E: A particle at space-time point $(x_1y_1, z_1, 0)$ reaches the reflecting body (location of impact unknown in the (x, y)-plane) within a Gaussian width of standard deviation a, about the surface of the reflecting body, in the z-direction, at time T_1 and finally reaches the space-time point $(x_2, y_2, z_2, T_1 + T_2)$. Here z_2 is the most probable value satisfying (2.3.1) becoming very precisely given by the latter equation in the classical limit for a sufficiently small as predicted by the theory.

Consider an *arbitrary* point $(\bar{x}, \bar{y}, 0)$ in the reflecting body in the (x, y)-plane. We consider a Gaussian region encompassing this point with standard deviations σ_1, σ_2 in the $x-, y-$directions, respectively. Given that the experiment E has been realized, the *contribution* to the corresponding *conditional* full amplitude coming from an integration over this Gaussian region may be inferred from (2.2.22), so I_1 an I_2 and then the full amplitude as

$$I_1 = \frac{\mathrm{i}\,\hbar}{m}\sqrt{\frac{2\pi}{A_1}T_1T_2}\,\sigma_1 \exp\left[-\frac{\sigma_1^2(x_1-x_2)^2 + \frac{\mathrm{i}\,\hbar T_2}{m}(x_1-\bar{x})^2 + \frac{\mathrm{i}\hbar T_1}{m}(x_2-\bar{x})^2}{2A_1}\right]. \tag{2.3.3}$$

$$I_2 = \frac{\mathrm{i}\,\hbar}{m}\sqrt{\frac{2\pi}{A_2}T_1T_2}\,\sigma_2 \exp\left[-\frac{\sigma_2^2(y_1-y_2)^2 + \frac{\mathrm{i}\,\hbar T_2}{m}(y_1-\bar{y})^2 + \frac{\mathrm{i}\hbar T_1}{m}(y_2-\bar{y})^2}{2A_2}\right]. \tag{2.3.4}$$

And then

$$I_1I_2 = \frac{\mathrm{i}\,\hbar}{m}\sqrt{\frac{2\pi}{A_1}T_1T_2}\,\sigma_1 \exp\left[-\frac{\sigma_1^2(x_1-x_2)^2 + \frac{\mathrm{i}\,\hbar T_2}{m}(x_1-\bar{x})^2 + \frac{\mathrm{i}\hbar T_1}{m}(x_2-\bar{x})^2}{2A_1}\right] \tag{2.3.5}$$

$$\times \frac{\mathrm{i}\,\hbar}{m}\sqrt{\frac{2\pi}{A_2}T_1T_2}\,\sigma_2 \exp\left[-\frac{\sigma_2^2(y_1-y_2)^2 + \frac{\mathrm{i}\,\hbar T_2}{m}(y_1-\bar{y})^2 + \frac{\mathrm{i}\hbar T_1}{m}(y_2-\bar{y})^2}{2A_2}\right]$$

$$= \frac{-\,\hbar^2}{m^2}\,\frac{2\pi}{\sqrt{A_1A_2}}T_1T_2\sigma_1\sigma_2$$

$$\times \exp\left\{-\frac{1}{2A_1}\left[\sigma_1^2(x_1-x_2)^2 + \frac{\mathrm{i}\,\hbar T_2}{m}(x_1-\bar{x})^2 + \frac{\mathrm{i}\hbar T_1}{m}(x_2-\bar{x})^2\right]\right\}$$

$$\times \exp\left\{-\frac{1}{2A_2}\left[\sigma_2^2(y_1-y_2)^2 + \frac{\mathrm{i}\,\hbar T_2}{m}(y_1-\bar{y})^2 + \frac{\mathrm{i}\hbar T_1}{m}(y_2-\bar{y})^2\right]\right\}$$

$$= \frac{-\,\hbar^2}{m^2}\,\frac{2\pi}{\sqrt{A_1A_2}}T_1T_2\sigma_1\sigma_2 \exp\left\{-\frac{1}{2\left(\frac{\mathrm{i}\hbar T_1}{m}(T_1-T_2)\frac{\mathrm{i}\hbar}{m} - \frac{T_1T_2\hbar^2}{m^2}\right)}\frac{\mathrm{i}\hbar T_1}{m}(x_1-x_2)^2\right\}$$

$$\times \exp\left\{-\frac{1}{2A_1}\left[\frac{\mathrm{i}\,\hbar T_2}{m}(x_1-\bar{x})^2+\frac{\mathrm{i}\hbar T_1}{m}(x_2-\bar{x})^2\right]\right\}$$

$$\times \exp\left\{-\frac{1}{2A_2}\left[\sigma_2^2(y_1-y_2)^2+\frac{\mathrm{i}\,\hbar T_2}{m}(y_1-\bar{y})^2+\frac{\mathrm{i}\hbar T_1}{m}(y_2-\bar{y})^2\right]\right\}$$

$$=\frac{-\hbar^2}{m^2}\,\frac{2\pi}{\sqrt{A_1A_2}}T_1T_2\sigma_1\sigma_2\exp\left\{-\frac{m}{2\mathrm{i}\hbar\left((T_1+T_2)+T_2\right)}(x_1-x_2)^2\right\}$$

$$\times \exp\left\{-\frac{1}{2A_1}\left[\frac{\mathrm{i}\,\hbar T_2}{m}(x_1-\bar{x})^2+\frac{\mathrm{i}\hbar T_1}{m}(x_2-\bar{x})^2\right]\right\}$$

$$\times \exp\left\{-\frac{m}{2\mathrm{i}\hbar\left((T_1+T_2)+T_2\right)}(y_1-y_2)^2\right\}$$

$$\times \exp\left\{-\frac{1}{2A_2}\left[\frac{\mathrm{i}\,\hbar T_2}{m}(y_1-\bar{y})^2+\frac{\mathrm{i}\hbar T_1}{m}(y_2-\bar{y})^2\right]\right\}. \tag{2.3.6}$$

where

$$\frac{-\hbar^2 2\pi}{m^2}T_1T_2\sigma_1\sigma_2=\frac{-\hbar^2 2\pi}{m^2}T_1T_2\sqrt{\frac{\mathrm{i}\hbar T_1}{m}}\sqrt{\frac{\mathrm{i}\hbar T_2}{m}}$$

$$=\frac{-\hbar^2 2\pi}{m^2}T_1T_2\frac{\mathrm{i}\hbar}{m}\sqrt{T_1T_2}. \tag{2.3.7}$$

$$N^{-1}=\frac{2\pi m}{\mathrm{i}\hbar\,(T_1+T_2)}\exp\left[-\frac{2\pi m}{2\mathrm{i}\hbar\,(T_1+T_2)}\,(x_1-x_2)^2\right]$$

$$\times \exp\left[-\frac{2\pi m}{2\mathrm{i}\hbar\,(T_1+T_2)}\,(y_1-y_2)^2\right] \tag{2.3.8}$$

$$N = \frac{1}{\frac{2\pi m}{\mathrm{i}\hbar(T_1+T_2)} \exp\left[-\frac{2\pi m}{2\mathrm{i}\hbar(T_1+T_2)}\left(x_1-x_2\right)^2\right] \exp\left[-\frac{2\pi m}{2\mathrm{i}\hbar(T_1+T_2)}\left(y_1-y_2\right)^2\right]} \tag{2.3.9}$$

where N is a normalization factor, the partial amplitude (2.3.5) can be simplified to

$$\frac{\mathrm{i}\,\hbar}{2\pi m}\frac{T_1+T_2}{\sqrt{A_1}A_2}\exp[\mathrm{i}\alpha]\exp\left[-\frac{\sigma_1^2}{C_1}(T_1+T_2)^2\left(\bar{x}-\frac{x_1T_2+x_2T_1}{T_1+T_2}\right)^2\right]$$

$$\times\exp\left[-\frac{\sigma_2^2}{C_2}(T_1+T_2)^2\left(\bar{y}-\frac{y_1T_2+y_2T_1}{T_1+T_2}\right)^2\right], \tag{2.3.10}$$

where C_1, C_2 are defined in (2.2.27) and

$$\alpha = \frac{\hbar}{2m}\left(T_1+T_2\right)T_1T\left[\frac{\left(\bar{x}-\left(x_1T_2+x_2T_1\right)/(T_1+T_2)\right)^2}{C_1}\right.$$

$$\left.+\frac{\left(\bar{y}-\left(y_1T_2+y_2T_1\right)/(T_1+T_2)\right)^2}{C_2}\right]. \tag{2.3.11}$$

The partial amplitude (2.3.10) is properly normalized. This is readily checked by setting $\sigma_1 = 0, \sigma_2 = 0$, and by integrating over $\bar{x}(-\infty < \bar{x} < \infty)$ and $\bar{y}(-\infty < \bar{y} < \infty)$ to get unity (the full amplitude)! On the other hand, upon multiplying both sides of (**??**) by $2\pi\sigma_1\sigma_2$ and taking the limits $\sigma_1 \to \infty, \sigma_2 \to \infty$ we again obtain unity as expected. For

$$\left|\bar{x}-\frac{x_1T_2+x_2T_1}{T_1+T_2}\right| \to \infty, \tag{2.3.12}$$

and/or

$$\left|\bar{y}-\frac{y_1T_2+y_2T_1}{T_1+T_2}\right| \to \infty, \tag{2.3.13}$$

the partial amplitudes (2.3.10) give negligible contributions to the full amplitude reminding us that most of the contribution to the full amplitude comes from an integration over a region not far from the classical point of impact defined by

$$\bar{x}(T_1-T_2)-(x_1T_2+x_2T_1)=0,$$

$$\bar{y}(T_1 - T_2) - (y_1 T_2 + y_2 T_1) = 0. \tag{2.3.14}$$

and relation (2.3.1).

Now we study the classical limit (2.2.99), that is for a classical particle,

$$m\frac{T_1 + T_2}{T_1 T_2} \min(\sigma_1^2 \sigma_2^2) \gg \hbar. \tag{2.3.15}$$

In this limit the partial amplitude (2.3.10) becomes

$$\frac{1}{2\pi\sigma_1\sigma_2} \exp\left[- \frac{1}{2\sigma_1^2}\left(\bar{x} - \frac{x_1 T_2 + x_2 T_1}{T_1 + T_2}\right)^2 \right]$$

$$\times \exp\left[-\frac{1}{2\sigma_2^2}\left(\bar{y} - \frac{y_1 T_2 + y_2 T_1}{T_1 + T_2}\right)^2 \right]. \tag{2.3.16}$$

And with σ_l and σ_2 arbitrarily small not violating (2.3.15) and (2.3.16) is appreciably nonzero for $\bar{x}$ and $\bar{y}$ close the classical point of impact and thus behaving like a delta-function around the classical point of impact in the limit of (macro-scopically) small σ_1 and σ_2. That is, in the classical limit (classical particle!) the entire contribution to the full amplitude comes from an integration over the classical point of impact (2.3.14) over a $\ll circular \gg$ region with a radius of the order $\sim max(\sigma_1, \sigma_2)$ with σ_1 and σ_2 arbitrarily small not violating (2.3.15). This, together with relation (2.3.1), establishes the law of reflection (angle of incidence = angle of reflection) in the classical limit. The situation with photons in quantum electrodynamics is developed in Chapter V.

CHAPTER III

MOMENTUM DESCRIPTION OF PHOTON PROPAGATION IN HALF–SPACE

3.1 Introduction

Before investigating the far complex of the propagation of photons in spacetime as a time evolution process in half-space, we first consider the simpler description in momentum space for orientation and as a preparation for the spacetime description, which however provides no information on the photon coordinates in configuration space and their relative localizations on a reflecting surface.

The actual demonstration of the reflection law of a quantum mechanical particle off a reflecting surface, in reconciliation with the classical result, turned out to be more complex than one would naively expect. In such investigations, the reflecting surface is replaced by approximate boundary conditions, as done in classical physics, rather than considering a quantum mechanical model for it as made up of atoms and so on. In our work we were much inspired by the fascinating, but non-technical, treatment of reflection given by Feynman and by the abundant literature dealing with the role of quantum mechanical particle in a typical classical every day situation. We extend our earlier analysis in Chapter II in non-relativistic quantum mechanics to the far more interesting situation with light in a quantum field theory (QFT) treatment. We work in half-space $z > 0$ and set up in this region, away from the reflecting surface at $z = 0$, an emitter and a detector of photons. We solve for the Green function in half-space with boundary conditions (Jackson, 1975; Manoukian, 1987) $\left\langle \mathbf{E}_{||} \right\rangle = 0, \langle \mathbf{B}_{\perp} \rangle = 0$, at $z = 0$. From the vacuum-to-vacuum transition amplitude (Manoukian, 1984, 1985, 1986, 1988; Schwinger, 1951, 1953, 1954, 1970, 1976, 1977), the transition amplitude for scattering is extracted. In the process it is then explicitly shown that a photon retains energy with the direction of its momentum change according to the classical law of reflection. The important positivity condition of the theory is established. As mentioned above the analysis in this chapter is also restricted to the momentum description. The more difficult problem with spacetime propagation of photons in half-space as an evolution process in time will carried out in Chapter V.

3.2 Reflection of light in Quantum Field Theory: Momentum Description

Maxwell's equations in vacuum in the presence of external current J^ν:$(\rho, \mathbf{J})$, $\mu, \nu = 0, 1, 2, 3$ may be written as

$$\partial_\mu F^{\mu\nu} = -J^\nu. \tag{3.2.1}$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{3.2.2}$$

$F^{\mu\nu}$ and A^μ , stand for the Faraday tensor and the vector potential, respectively, and we work in the temporal gauge $A^0 = 0$:

$$A^\mu = \left(A^0, \mathbf{A}\right) = (\phi, \mathbf{A}) = (0, \mathbf{A}), \tag{3.2.3}$$

and for the Lagrangian density of photons we have

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A^i J^i, \tag{3.2.4}$$

where $i = 1, 2, 3$. From the conservation of current,

$$\partial_\mu J^\mu = 0, \qquad \partial_0 J^0 = -\partial_i J^i \; J^0 = -\frac{1}{\partial_0}\left(\partial_i J^i\right), \tag{3.2.5}$$

from (3.2.1) and $A^0 = 0$, we have,

$$\partial_\mu F^{\mu i} = -J^i, \tag{3.2.6}$$

and

$$\partial_0 F^{\mu 0} = -J^0. \tag{3.2.7}$$

From (3.2.6), we obtain

$$\partial_0 F^{0i} + \partial_j F^{ji} = -J^i, \tag{3.2.8}$$

and multiplying (3.2.8) by $(\partial_0)^{-1}\,\partial_i$, we obtain

$$\partial_\mu F^{\mu i} = -J^i$$

$$\partial_\mu \left(\partial^\mu A^j - \partial^j A^\mu\right) = -J^j$$

$$\partial_\mu \partial^\mu A^j - \partial_i \partial^j A^i = -J^j, \tag{3.2.9}$$

$$\left(-\Box \delta^{ij} + \partial^i \partial^j\right) A^j = J^i. \tag{3.2.10}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}, \tag{3.2.11}$$

where $\Box = \partial_\mu \partial^\mu$. In detail,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu;$$

$$= \begin{pmatrix} 0 & \partial^0 A^1 - \partial^1 A^0 & \partial^0 A^2 - \partial^2 A^0 & \partial^0 A^3 - \partial^3 A^0 \\ \partial^1 A^0 - \partial^0 A^1 & 0 & \partial^1 A^2 - \partial^2 A^1 & \partial^1 A^3 - \partial^3 A^1 \\ \partial^2 A^0 - \partial^0 A^2 & \partial^2 A^1 - \partial^1 A^2 & 0 & \partial^2 A^3 - \partial^3 A^2 \\ \partial^3 A^0 - \partial^0 A^3 & \partial^3 A^1 - \partial^1 A^3 & \partial^3 A^2 - \partial^2 A^3 & 0 \end{pmatrix} \quad (3.2.12)$$

From $(3.2.10)$ and $(3.2.11)$ we can write electric and magnetic components as

$$E^1 = F^{01} = \partial^0 A^1 - \partial^1 A^0 = \partial^0 A^1$$

$$E^2 = F^{02} = \partial^0 A^2 - \partial^2 A^0 = \partial^0 A^2$$

$$E^3 = F^{03} = \partial^0 A^3 - \partial^3 A^0 = \partial^0 A^3. \quad (3.2.13)$$

We then have

$$E^i = F^{0i} = \partial^0 A^i. \quad (3.2.14)$$

Similarly, for the magnetic components we have

$$B^i = \frac{1}{2}\varepsilon^{ijk}(\partial^j A^k - \partial^k A^i) \equiv \varepsilon^{ijk}\partial^j A^k. \quad (3.2.15)$$

The interesting component of $\mathbf{E}$ and $\mathbf{B}$ are,

$$E^1 = \partial^0 A^1,$$

$$E^2 = \partial^0 A^2,$$

$$E^a = \partial^0 A^a; \qquad a = 1, 2. \quad (3.2.16)$$

$$B^3 = \partial^1 A^2 - \partial^2 A^1. \quad (3.2.17)$$

Due to current conservation,

$$\partial_\mu J^\mu = 0,$$

$$\partial^i J^i = -\partial_0 J^0,$$

$$J^0 = -\left(\partial^0\right)^{-1} \partial^i J^i. \tag{3.2.18}$$

One cannot vary all the components of J^μ independently. We may, however, vary the space components **J** independently since there are no restriction amongst them. This point cannot be overemphasized (Manoukian, 1986). Upon taking the vacuum expectation value of (3.2.10) and using Schwinger's action principle (Manoukian, 1986; Schwinger, 1951, 1953, 1954, 1970, 1977), we obtain for the current-free Green function

$$D^{ij}\left(x, x'\right) = \frac{\mathrm{i}\left\langle 0_+\right| \left(A^i\left(x\right) A^j\left(x'\right)\right)_+ \left|0_-\right\rangle}{\left\langle 0_+ \mid 0_-\right\rangle}. \tag{3.2.19}$$

By using (3.2.10) we have

$$\left(-\Box\delta^{ij} + \partial^i\partial^j\right)\left\langle 0_+\right| A^j\left(x\right)\left|0_-\right\rangle = J^i(x)\left\langle 0_+ \mid 0_-\right\rangle \tag{3.2.20}$$

$$\left\langle 0_+\right| A^j\left(x\right)\left|0_-\right\rangle = (-\mathrm{i})\frac{\delta}{\delta J^j(x)}\left\langle 0_+ \mid 0_-\right\rangle \tag{3.2.21}$$

$$\left(-\Box\delta^{ij} + \partial^i\partial^j\right)(-i)\frac{\delta}{\delta J^k(x')}(-\mathrm{i})\frac{\delta}{\delta J^j(x)}\left\langle 0_+ \mid 0_-\right\rangle$$

$$= (-\mathrm{i})\left(\frac{\delta}{\delta J^k(x')}J^i(x)\right)\left\langle 0_+ \mid 0_-\right\rangle + J^i(x)(-\mathrm{i})\frac{\delta}{\delta J^k(x')}\left\langle 0_+ \mid 0_-\right\rangle, \tag{3.2.22}$$

$$D^{jk}(x, x') = \left.\frac{(-\mathrm{i})\frac{\delta}{\delta J^j(x)}(-\mathrm{i})\frac{\delta}{\delta J^k(x')}\left\langle 0_+ \mid 0_-\right\rangle}{\left\langle 0_+ \mid 0_-\right\rangle}\right|_{J=0}, \tag{3.2.23}$$

and from

$$\frac{\delta}{\delta J^k(x')} J^i(x) = \delta^{ik}\delta(x - x'), \tag{3.2.24}$$

we get

$$\left(-\Box\delta^{ij} + \partial^i\partial^j\right) D^{jk}\left(x, x'\right) = \delta^{ik}\delta^{(4)}\left(x, x'\right). \tag{3.2.25}$$

From (3.2.19) we may write

$$D^{ji}\left(x', x\right) = \frac{\mathrm{i}\left\langle 0_+\right|\left(A^j\left(x'\right) A^i\left(x\right)\right)_+\left|0_-\right\rangle}{\left\langle 0_+ \mid 0_-\right\rangle}. \tag{3.2.26}$$

Since

$$\left(A^i\left(x\right) A^j\left(x'\right)\right)_+ = \Theta(x - x')A^i\left(x\right) A^j\left(x'\right) + \Theta(x' - x)A^j\left(x'\right) A^i\left(x\right), \tag{3.2.27}$$

and

$$\left(A^j\left(x'\right) A^i\left(x\right)\right)_+ = \Theta(x' - x)A^j\left(x'\right) A^i\left(x\right) + \Theta(x - x')A^i\left(x\right) A^j\left(x'\right) \tag{3.2.28}$$

we then have

$$D^{3b}\left(x, x'\right) = D^{b3}\left(x', x\right). \tag{3.2.29}$$

To apply the appropriate boundary conditions, we can write (3.2.25) corresponding to the problem of reflection at the $z = 0$ plane,

$$-\Box\delta^{ii}D^{ik}\left(x, x'\right) + \partial^i\partial^j D^{jk}\left(x, x'\right) = \delta^{ik}\delta^{(4)}\left(x, x'\right), \tag{3.2.30}$$

and for $i = a \;:\; j = c, 3 :\; k = b$ we get

$$-\Box\delta^{aa}D^{ab}\left(x, x'\right) + \partial^a\partial^c D^{cb}\left(x, x'\right) + \partial^a\partial^3 D^{3b}\left(x, x'\right)$$

$$= \delta^{ab}\delta\left(z, z'\right)\delta^2\left(\mathbf{x}_{||}, \mathbf{x}'_{||}\right)\delta\left(x^0, x'^0\right) \tag{3.2.31}$$

$$- \Box D^{ab}\left(x,x'\right) + \partial^a\partial^c D^{cb}\left(x,x'\right) + \partial^a\partial^3 D^{3b}\left(x,x'\right)$$

$$= \delta^{ab}\delta\left(z,z'\right)\delta^2\left(\mathbf{x}_{||},\mathbf{x}'_{||}\right)\delta\left(x^0,x'^0\right). \qquad (3.2.32)$$

$$-\Box D^{3a}\left(x,x'\right) + \partial^3\partial^c D^{ca}\left(x,x'\right) + \partial^3\partial^3 D^{3a}\left(x,x'\right) = 0. \qquad (3.2.33)$$

For $i = a \;:\; j = c, 3:\; k = 3$ we get

$$-\Box\delta^{aa} D^{a3}\left(x,x'\right) + \partial^a\partial^c D^{ca}\left(x,x'\right) + \partial^a\partial^3 D^{33}\left(x,x'\right) = \delta^{a3}\delta^{(4)}\left(x,x'\right) \qquad (3.2.34)$$

$$-\Box D^{a3}\left(x,x'\right) + \partial^a\partial^c D^{ca}\left(x,x'\right) + \partial^a\partial^3 D^{33}\left(x,x'\right) = 0 \qquad (3.2.35)$$

For $i = 3 \;:\; j = c, 3:\; k = 3$ we get

$$-\Box\delta^{33} D^{33}\left(x,x'\right) + \partial^3\partial^c D^{c3} + \partial^3\partial^3 D^{33}\left(x,x'\right) = \delta^{33}\delta^{(4)}\left(x,x'\right) \qquad (3.2.36)$$

$$-\Box D^{33}\left(x,x'\right) + \partial^3\partial^c D^{c3} + \partial^3\partial^3 D^{33}\left(x,x'\right)$$

$$= \delta\left(z,z'\right)\delta^2\left(\mathbf{x}_{||} - \mathbf{x}'_{||}\right)\delta\left(x^0 - x'^0\right) \qquad (3.2.37)$$

where $\mathbf{x}_{||} = \left(x^1, x^2\right)$, and the reflecting surface is the $z = 0$ plane. To satisfy the boundary conditions

$$\left\langle E^a\right\rangle = 0, \quad \left\langle B^3\right\rangle = 0 \quad \text{for } z \to 0, \qquad (3.2.38)$$

we develop a Fourier-sine transform for $\delta(z, z')$ in (3.2.32). That is, we write

$$\delta\left(z,z'\right) = \int\limits_0^\infty \frac{2}{\pi}\mathrm{d}q \sin qz \sin qz' = \int\limits_{-\infty}^\infty \frac{\mathrm{d}q}{\pi}\sin qz \sin qz'; \quad \text{for } z, z' > 0. \qquad (3.2.39)$$

We have to use a Delta function in (3.2.39) associating with the boundary conditions, by using

$$\delta^2\left(\mathbf{x}_{||}-\mathbf{x}'_{||}\right)=\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{||}-\mathbf{x}'_{||}\right)},\tag{3.2.40}$$

$$\delta\left(x^0-x'^0\right)=\int\frac{\mathrm{d}Q^0}{2\pi}\mathrm{e}^{\mathrm{i}Q^0\left(x^0-x'^0\right)},\tag{3.2.41}$$

to get for $D^{ij}(x,x')$

$$D^{ab}\left(x,x'\right)=\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{ab}-\frac{k^ak^b}{Q^{02}}\right)$$

$$\times\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{||}-\mathbf{x}'_{||}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)},\tag{3.2.42}$$

$$D^{3b}\left(x,x'\right)=\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{q\mathrm{i}k^b}{Q^{02}}\right)$$

$$\times\frac{\cos qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{||}-\mathbf{x}'_{||}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)},\tag{3.2.43}$$

where

$$Q=(Q^0,\mathbf{k},q),\ \ x=(x^0,\mathbf{x},z).\tag{3.2.44}$$

From $D^{3b}\left(x,x'\right)=D^{b3}\left(x',x\right)$ we have

$$D^{b3}\left(x,x'\right)=\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{-\mathrm{i}k^bq}{Q^{02}}\right)$$

$$\times\frac{\sin qz\cos qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{||}-\mathbf{x}'_{||}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)},\tag{3.2.45}$$

and we use the Fourier-cosine transformation

$$\delta\left(z,z'\right)=\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\cos qz\cos qz';\quad\text{for }\; z,z'>0, \tag{3.2.46}$$

for $z, z' > 0$. Equations (3.2.35), (3.2.37) and (3.2.45) then lead to

$$D^{33}\left(x,x'\right)=\int\frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(1-\frac{q^2}{Q^{02}}\right)$$

$$\times\;\frac{\cos qz\cos qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}. \tag{3.2.47}$$

We now verify that (3.2.42), (3.2.43), (3.2.45) and (3.2.47) correspond to (3.2.32), (3.2.33), (3.2.35) and (3.2.37). We verify that (3.2.42) is the solution of (3.2.32), by substituting D^{ab} from (3.2.42) in (3.2.32). From the left-hand side of (3.2.32), we have

$$-\left(\partial^{i2}-\partial^{02}\right)\Bigg[\int\frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{ab}-\frac{k^ak^b}{Q^{02}}\right)$$

$$\times\;\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$+\,\partial^a\partial^c\Bigg[\int\frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{cb}-\frac{k^ck^b}{Q^{02}}\right)\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$+\,\partial^a\partial^3\Bigg[\int\frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{qik^b}{Q^{02}}\right)\frac{\cos qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$=\delta^{ab}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\sin qz\sin qz'\int\frac{\mathrm{d}^2\mathbf{k}}{\left(2\pi\right)^2}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\int\frac{\mathrm{d}Q^0}{2\pi}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}, \tag{3.2.48}$$

Let (a) denote the first term in (3.2.48), (b) denote the second term in (3.2.48) and (c) denote the third term in (3.2.48)

$$(a) = -\left(\partial^{i2} - \partial^{02}\right)\Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{ab} - \frac{k^a k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|} - \mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg],$$

$$(b) = \partial^a \partial^c \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{cb} - \frac{k^c k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|} - \mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg],$$

$$(c) = \partial^a \partial^3 \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{q\mathrm{i}k^b}{Q^{02}}\right) \frac{\cos qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|} - \mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg].$$

Considering (a), (b) and (c) in turn

$$(a) = -\left(\left(\boldsymbol{\partial}^{\|2} + \partial^{32}\right) - \partial^{02}\right)\Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{ab} - \frac{k^a k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|} - \mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg]$$

$$(b) = \partial^a \partial^c \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{cb} - \frac{k^c k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$(c) = \partial^a \partial^3 \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{q\mathrm{i}k^b}{Q^{02}}\right)$$

$$\times \frac{\cos qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]. \tag{3.2.49}$$

Consider (a) first

$$(a) = -\left(\left(\boldsymbol{\partial}^{\|2} + \partial^{32}\right) - \partial^{02}\right) \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{ab} - \frac{k^a k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$= \left(-(\mathrm{i}k)^2 - \left(-q^2\right) + \left(-\mathrm{i}Q^0\right)^2\right) \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{ab} - \frac{k^a k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$= \overbrace{\left(k^2 + q^2 - Q^{02}\right)}^{Q^2} \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{ab} - \frac{k^a k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$(a) = \delta^{ab} \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \sin qz \sin qz' \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}$$

$$-\left(k^2+q^2-Q^{02}\right)\frac{k^ak^b}{Q^{02}}\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}$$

$$\times\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}. \tag{3.2.50}$$

Consider (b) through the equations

$$(b)=\partial^a\partial^c\left[\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{cb}-\frac{k^ck^b}{Q^{02}}\right)\right.$$

$$\left.\times\,\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\;\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\;\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\right]$$

$$=\partial^a\partial^c\left[\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{cb}-\frac{k^ck^b}{Q^{02}}\right)\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\right.$$

$$\left.\times\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\right]$$

$$=\left(\mathrm{i}k^a\right)\left(\mathrm{i}k^c\right)\left[\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{cb}-\frac{k^ck^b}{Q^{02}}\right)\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\right.$$

$$\left.\times\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\right]$$

$$=\left(-k^ak^c\right)\left[\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{cb}-\frac{k^ck^b}{Q^{02}}\right)\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\right.$$

$$\times\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$(b) = \left(-k^a k^b\right)\Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\;\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}$$

$$+\left(\frac{k^a k^b k^{c2}}{Q^{02}}\right)\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}$$

$$\times\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\;\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]. \tag{3.2.51}$$

Finally consider (c)

$$(c) = \partial^a\partial^3\Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{q\mathrm{i}k^b}{Q^{02}}\right)$$

$$\times\frac{\cos qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$= \partial^a\partial^3\Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{q\mathrm{i}k^b}{Q^{02}}\right)$$

$$\times\frac{\cos qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$= \mathrm{i}k^a\,(q)\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{q\mathrm{i}k^b}{Q^{02}}\right)$$

$$\times\frac{\sin qz\sin qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}$$

$$= \left(\frac{q^2 k^a k^b}{Q^{02}}\right) \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{q\mathrm{i}k^b}{Q^{02}}\right)$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - \mathbf{x}'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}. \tag{3.2.52}$$

In an obvious notation, L.H.S = (a)+(b)+(c)

$$(a) + (b) + (c) = \delta^{ab} \int\limits_{-\infty}^{\infty} \frac{dq}{\pi} \sin qz \sin qz' \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - \mathbf{x}'_\parallel\right)} \int \frac{\mathrm{d}Q^0}{2\pi} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}$$

$$+ \left[-\left(k^2 + q^2 - Q^{02}\right) \frac{k^a k^b}{Q^{02}} + \left(-k^a k^b\right) + \left(\frac{k^a k^b k^{c2}}{Q^{02}}\right) + \frac{q^2 k^a k^b}{Q^{02}} \right]$$

$$\times \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - \mathbf{x}'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}$$

$$= \delta^{ab} \int\limits_{-\infty}^{\infty} \frac{dq}{\pi} \sin qz \sin qz' \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - \mathbf{x}'_\parallel\right)} \int \frac{\mathrm{d}Q^0}{2\pi} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}$$

$$+ \left[\left(-k^2 \frac{k^a k^b}{Q^{02}} - q^2 \frac{k^a k^b}{Q^{02}} + Q^{02} \frac{k^a k^b}{Q^{02}} \right) \right.$$

$$\left. + \left(-k^a k^b\right) + \left(\frac{k^a k^b k^{c2}}{Q^{02}}\right) + \frac{q^2 k^a k^b}{Q^{02}} \right]$$

$$\times \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - \mathbf{x}'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)},$$

$$= \delta^{ab} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \sin qz \sin qz' \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)} \int \frac{\mathrm{d}Q^0}{2\pi} \mathrm{e}^{\text{ - iQ}^0\left(x^0-x'^0\right)}$$

$$+ \left[\left(-k^2\frac{k^a k^b}{Q^{02}} - q^2\frac{k^a k^b}{Q^{02}} + Q^{02}\frac{k^a k^b}{Q^{02}}\right)\right]$$

$$+ \left(-k^a k^b\right) + \left(\frac{k^a k^b k^{c2}}{Q^{02}}\right) + \frac{q^2 k^a k^b}{Q^{02}}$$

$$\times \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}$$

$$= \delta^{ab} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \sin qz \sin qz' \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)} \int \frac{\mathrm{d}Q^0}{2\pi} \mathrm{e}^{\text{ - iQ}^0\left(x^0-x'^0\right)}$$

$$+ \left[-k^2\frac{k^a k^b}{Q^{02}} - q^2\frac{k^a k^b}{Q^{02}} + k^a k^b - k^a k^b + \frac{k^a k^b k^{c2}}{Q^{02}} + \frac{q^2 k^a k^b}{Q^{02}}\right]$$

$$\times \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}$$

$$(a) + (b) + (c) = \delta^{ab} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \sin qz \sin qz' \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel}-\mathbf{x}'_{\parallel}\right)}$$

$$\times \int \frac{\mathrm{d}Q^0}{2\pi} \mathrm{e}^{\text{ - iQ}^0\left(x^0-x'^0\right)}$$

$$= \delta^{ab}\delta\left(z,z'\right)\delta^{2}\left(\mathbf{x}_{\|},\mathbf{x}'_{\|}\right)\delta\left(x^{0},x'^{0}\right).$$

(3.2.53)

The above showed that D^{ab} in (3.2.42) is the solution of (3.2.32), and similarly for D^{3b}, D^{b3}, D^{33} in (3.2.43), (3.2.45) and (3.2.47) are solutions of (3.2.33), (3.2.35) and (3.2.37) respectively. We prove that (3.2.43) corresponds to the solution of (3.2.33) , by substituting (3.2.42) and (3.2.43) into (3.2.33).

From (3.2.33), we get

$$\left[\left(-\left(\left(\boldsymbol{\partial}^{\|2}+\partial^{32}\right)-\partial^{02}\right)\right)\int\frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^{0}}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{q\mathrm{i}k^{a}}{Q^{02}}\right)\right.$$

$$\left.\times\ \frac{\cos qz\sin qz'}{Q^{2}-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-x'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^{0}\left(x^{0}-x'^{0}\right)}\right]$$

$$+\ \partial^{3}\partial^{c}\left[\int\frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^{0}}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\delta^{cb}-\frac{k^{c}k^{a}}{Q^{02}}\right)\right.$$

$$\left.\times\ \frac{\sin qz\sin qz'}{Q^{2}-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-x'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^{0}\left(x^{0}-x'^{0}\right)}\right]$$

$$+\ \partial^{3}\partial^{3}\left[\int\frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^{0}}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{q\mathrm{i}k^{a}}{Q^{02}}\right)\frac{\cos qz\sin qz'}{Q^{2}-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^{0}\left(x^{0}-x'^{0}\right)}\right]$$

$$=\left[-\left(\left(\mathrm{i}\mathbf{k}\right)^{2}+\left(-q^{2}\right)-\left(-\mathrm{i}Q^{0}\right)^{2}\right)\int\frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}Q^{0}}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(\frac{q\mathrm{i}k^{a}}{Q^{02}}\right)\right.$$

$$\times \frac{\cos qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - x'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg]$$

$$+ (q)\left(\mathrm{i}k^c\right)\left[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\delta^{cb} - \frac{k^c k^a}{Q^{02}}\right)\right.$$

$$\times \frac{\sin qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - x'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg]$$

$$+ \left(-q^2\right)\left[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{q\mathrm{i}k^a}{Q^{02}}\right) \frac{\cos qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - \mathbf{x}'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\right]$$

$$= \left[\left(\mathbf{k}^2 + q^2 - Q^{02}\right) \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{q\mathrm{i}k^a}{Q^{02}}\right)\right.$$

$$\times \frac{\cos qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - x'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg]$$

$$+ \left[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\mathrm{iq}k^a\right)\right.$$

$$\times \frac{\cos qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - x'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg]$$

$$- \left[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{\mathrm{iq}k^a k^{c2}}{Q^{02}}\right)\right.$$

$$\times \frac{\cos qz \sin qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\parallel - x'_\parallel\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)}\Bigg]$$

$$+\left(-q^{2}\right)\left[\int \frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^{0}}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{\mathrm{i}qk^{a}}{Q^{02}}\right)\right.$$

$$\left.\times \frac{\cos qz \sin qz'}{Q^{2}-\mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-x'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^{0}\left(x^{0}-x'^{0}\right)}\right]$$

$$=0. \tag{3.2.54}$$

That is to say equation (3.2.43) is the solution to equation (3.2.33). Next, we want to prove that (3.2.45) is the solution to (3.2.35). From (3.2.35), we get

$$\left[\left(-\left(\left(\boldsymbol{\partial}^{\|2}+\partial^{32}\right)-\partial^{02}\right)\right) \int \frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^{0}}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(-\frac{q\mathrm{i}k^{a}}{Q^{02}}\right)\right.$$

$$\left.\times \frac{\sin qz \sin qz'}{Q^{2}-\mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^{0}\left(x^{0}-x'^{0}\right)}\right]$$

$$+\partial^{a}\partial^{c}\left[\int \frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^{0}}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(-\frac{\mathrm{i}k^{c}q}{Q^{02}}\right)\right.$$

$$\left.\times \frac{\sin qz \cos qz'}{Q^{2}-\mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^{0}\left(x^{0}-x'^{0}\right)}\right]$$

$$+\partial^{a}\partial^{3}\left[\int \frac{\mathrm{d}^{2}\mathbf{k}}{(2\pi)^{2}} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^{0}}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(1-\frac{\mathrm{q}^{a}}{Q^{02}}\right)\right.$$

$$\left.\times \frac{\cos qz \sin qz'}{Q^{2}-\mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)} \mathrm{e}^{-\mathrm{i}Q^{0}\left(x^{0}-x'^{0}\right)}\right]$$

$$= \Bigg[-\Big((\mathrm{i}\mathbf{k})^2 + \left(-q^2\right) - \left(-\mathrm{i}Q^0\right)^2 \Big) \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{q\mathrm{i}k^a}{Q^{02}} \right)$$

$$\times \frac{\sin qz \cos qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)} \Bigg]$$

$$+ \left(\mathrm{i}k^a\right)\left(\mathrm{i}k^c\right) \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(-\frac{\mathrm{i}k^c q}{Q^{02}} \right)$$

$$\times \frac{\sin qz \cos qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)} \Bigg]$$

$$+ \left(-q\right)\left(\mathrm{i}k^a\right) \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(1 - \frac{q^2}{Q^{02}} \right)$$

$$\times \frac{\sin qz \cos qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)} \Bigg]$$

$$= \Bigg[\left(\mathbf{k}^2 + q^2 - Q^{02}\right) \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(\frac{-q\mathrm{i}k^a}{Q^{02}} \right)$$

$$\times \frac{\sin qz \cos qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel} - x'_{\parallel}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)} \Bigg]$$

$$+ \left(-k^a k^c\right) \Bigg[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \left(-\frac{\mathrm{i}k^c q}{Q^{02}} \right)$$

$$\times \frac{\sin qz \cos qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\parallel} - \mathbf{x}'_{\parallel}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0 - x'^0\right)} \Bigg]$$

$$+\left(-q\mathrm{i}k^a\right)\Bigg[\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(1-\frac{q^2}{Q^{02}}\right)$$

$$\times\frac{\sin qz\cos qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$=0. \tag{3.2.55}$$

That is, (3.2.45) is the solution of equation (3.2.35). Next, we prove that (3.2.47) is the solution to (3.2.37). By substituting D^{33}, D^{c3} into (3.2.37) we have

$$=\left(-\left(\left(\boldsymbol{\partial}^{\|2}+\partial^{32}\right)-\partial^{02}\right)+\partial^3\partial^3\right)\Bigg[\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(1-\frac{q^2}{Q^{02}}\right)$$

$$\times\frac{\cos qz\cos qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$+\partial^a\partial^c\Bigg[\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(-\frac{\mathrm{i}k^c q}{Q^{02}}\right)$$

$$\times\frac{\sin qz\cos qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$=\left(-(\mathrm{i}\mathbf{k})^2+\left(-Q^0\right)^2\right)\int\frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}Q^0}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{\pi}\left(1-\frac{q^2}{Q^{02}}\right)$$

$$\times\frac{\cos qz\cos qz'}{Q^2-\mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\Bigg]$$

$$+ \left(\mathrm{i}k^c\right)(q)\left[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}Q^0}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi}\left(-\frac{\mathrm{i}k^c q}{Q^{02}}\right)\right.$$

$$\left.\times\ \frac{\cos qz \cos qz'}{Q^2 - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{||}-\mathbf{x}'_{||}\right)} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}\right]$$

$$= \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{||}-\mathbf{x}'_{||}\right)} \int \frac{\mathrm{d}Q^0}{2\pi} \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi} \cos qz \cos qz'$$

$$= \delta(z, z')\delta\left(\mathbf{x}_{||} - \mathbf{x}'_{||}\right)\delta\left(x^0 - x'^0\right)$$

$$= \delta^4(x, x'). \tag{3.2.56}$$

The Green function corresponding to the B.C. of reflection at the plate at $z = 0$, is given in terms of the electric and magnetic field components of interest. The latter relevant are given by

$$\left\langle E^a(x)\right\rangle = \partial^0 \int (\mathrm{d}x')\, D^{aj}(x, x')\, J^j(x') \left\langle 0_+ \mid 0_-\right\rangle^J, \tag{3.2.57}$$

$$\left\langle B^3(x)\right\rangle = \int (\mathrm{d}x') \left(\partial^1 D^{2j}(x, x') - \partial^2 D^{1j}(x, x')\right) J^j(x') \left\langle 0_+ \mid 0_-\right\rangle^J, \tag{3.2.58}$$

where $\left\langle 0_+ \mid 0_-\right\rangle^J$ is the vacuum to vacuum transition amplitude in the presence of the external current J^μ :

$$\left\langle 0_+ \mid 0_-\right\rangle^J = \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x')\, J^i(x)\, D^{ij}(x, x')\, J^j(x')\right], \tag{3.2.59}$$

with D^{ij} now given in (3.2.42), (3.2.43), (3.2.45) and (3.2.47)

We now consider the aspect of fields at the boundary. Since the parameters of the reflected and transmitted waves in terms of those of the incident wave. The three waves, incident, reflected and refracted waves will satisfy Maxwell's equations in the uniform material, and Maxwell's equation must also satisfy at the boundary between the two different materials. So we must now look at what happens right at the boundary. We will find that Maxwell's equations demand that the three waves fit together in a certain way.

As an example we explain as to what we mean by stating that the y-component of the electric field E must be the same on both sides, in general of a boundary. This is required by Faraday's law.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.2.60}$$

as we can see in the following way. Consider a little rectangular loop Γ which straddles the boundary, as shown in Fig.3.1. Equation (3.2.60) says that the line integral of $\mathbf{E}$ around Γ is equal to the rate of change of the flux of $\mathbf{B}$ through the loop:

$$\oint_\Gamma \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot \mathbf{n} \mathrm{d}a \tag{3.2.61}$$

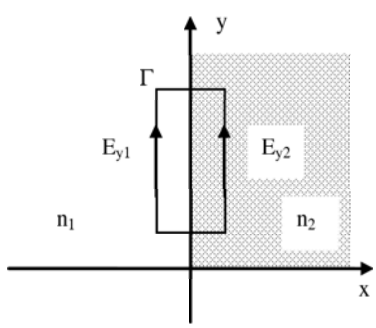

Figure 3.1 Boundary condition $E_{y2} = E_{y1}$ obtained from $\oint_\Gamma \mathbf{E} \cdot \mathrm{d}\mathbf{s} = 0$.

Now imagine that the rectangle is very narrow, so that the loop enclosed an infinitesimal area. If $\mathbf{B}$ remains the flux through the area is zero. So the line integral of

$\mathbf{E}$ must be zero. If E_{y1} and E_{y2} are the components of the field on the two sides of the boundary and if the length of the rectangle is l , we have

$$E_{y1}l - E_{y2}l = 0$$

$$E_{y1} = E_{y2}. \tag{3.2.62}$$

We now show the method that can be used for any problem-a general way of finding what happens at the boundary directly from the differential equations, begin with the Maxwell's equations for dielectric and write out explicitly all components

$$\boldsymbol{\nabla} \cdot \mathbf{E} = -\frac{\boldsymbol{\nabla} \cdot \mathbf{P}}{\varepsilon_0}, \tag{3.2.63}$$

$$\varepsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \right), \tag{3.2.64}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.2.65}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}, \tag{3.2.66}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}, \tag{3.2.67}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}. \tag{3.2.68}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{3.2.69}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \tag{3.2.70}$$

$$c^2\boldsymbol{\nabla}\times\mathbf{B}=\frac{1}{\varepsilon_0}\frac{\partial\mathbf{P}}{\partial t}+\frac{\partial\mathbf{E}}{\partial t}, \tag{3.2.71}$$

$$c^2\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)=\frac{1}{\varepsilon_0}\frac{\partial P_x}{\partial t}+\frac{\partial E_x}{\partial t}, \tag{3.2.72}$$

$$c^2\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)=\frac{1}{\varepsilon_0}\frac{\partial P_y}{\partial t}+\frac{\partial E_y}{\partial t}, \tag{3.2.73}$$

$$c^2\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)=\frac{1}{\varepsilon_0}\frac{\partial P_z}{\partial t}+\frac{\partial E_z}{\partial t}. \tag{3.2.74}$$

Now these equations must all hold in region 1 (to the left of the boundary) and region 2

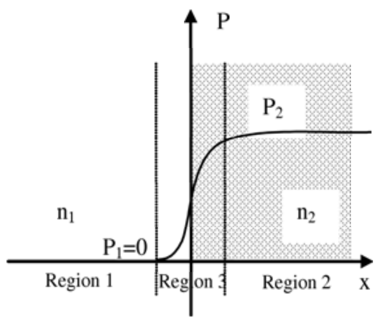

Figure 3.2 P_x varies from zero to very large value.

(to the right of the boundary). For instance, suppose that we have a boundary between vacuum (region 1) and glass (region 2). There is nothing to polarize in the vacuum, so $\mathbf{P}_1=0$, let's say there is a smooth, but rapid, transition. If we look at any component of $\mathbf{P}$, say P_x, it might vary from zero of P_{1x} to high of P_{2x}. Suppose now we take the first of our equations, (3.2.64). It involves derivatives of the components of $\mathbf{P}$ with respect to x, y, z. The y-and z-component are not interesting; nothing spectacular is happening in those directions. But the x-derivative of $\mathbf{P}$, will have some very large values in region 3 (between region 1 and 2), shown in Fig. 3.2 because of the tremendous slope of P_x. The derivative $\partial P_x/\partial x$ will have a sharp spike at the boundary as shown in Fig. 3.3. If we imagine squashing the boundary to an even thinner layer, the spike would get much higher. If the boundary is really sharp for the waves we are interested in, The magnitude of in region 3 will be much, much greater than any contributions we might have from the variation of P in the wave away from the boundary so we ignore any variations other than those due to the boundary. Now how can (3.2.64) be satisfied if there is a

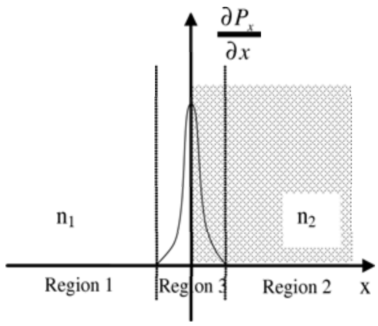

Figure 3.3 Polarization change between two regions.

whopping big spike on the right-hand side? Only if there is an equally whopping big spike on the other side. Something on the left-hand side must also be big. The only candidate is $\partial E_x/\partial x$, because the variations with y and z are only those small effects in the wave we just mentioned. So $\varepsilon_0\left(\partial E_x/\partial x\right)$ must be as drawn in Fig. 3.4 -just a copy of $\partial P_x/\partial x$. We have that

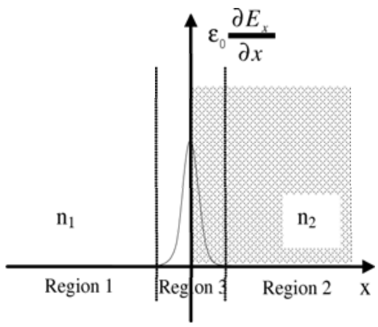

Figure 3.4 Electric field change rate.

$$\varepsilon_0 \frac{\partial E_x}{\partial x} = \frac{\partial P_x}{\partial x}, \tag{3.2.75}$$

If we integrate this equation with respect to x across region 3 , we conclude that

$$\varepsilon_0 \left(E_{x2} - E_{x1}\right) = -\left(P_{x2} - P_{x1}\right), \tag{3.2.76}$$

In other word, the jump in $\varepsilon_0 E_x$ in going from region 1 to region 2 must be equal to the jump in -P_x. We can rewrite (3.2.76) as

$$\varepsilon_0 E_{x2} - P_{x2} = \varepsilon_0 E_{x1} + P_{x1}, \tag{3.2.77}$$

which says that the quantity $\varepsilon_0 E_x - P_x$ has equal values in region 2 and region 1. One says: the quantity $\varepsilon_0 E_x - P_x$ in continuous across the boundary. We have, in this way, one of our boundary conditions. . Although we took as an illustration the case m which P_1, was zero because region 1 was a vacuum, it is clear that the same argument applies for any two materials in the two regions, so (3.2.77) is true in general. Let's now go through the rest of Maxwell's equations and see what each of them tells us. We take next (3.2.66). There are no x-derivatives, so it doesn't tell us anything. (Remember that the fields *themselves* do not get especially large at the boundary; only the derivatives with respect to x can become so large that they dominate the equation.) Next, we look at (3.2.67). There is an x-derivative, we have $\partial E_z/\partial x$ on the left-hand side. Suppose it has a huge derivative But there is nothing on the right-hand side to match it, with; therefore E_z cannot have any jump in going from region 1 to region 2 [If it did, there would be a spike on the left of (3.2.66) but none on the right, and the equation would be false.] So we have a new condition:

$$E_{z2} = E_{z1}, \tag{3.2.78}$$

By the same argument, (3.2.68) gives

$$E_{y2} = E_{y1}, \tag{3.2.79}$$

This last result is just what we got in (3.2.62) by a line integral argument.

We go on to (3.2.70) The only term that could have a spike is $\partial B_x/\partial x$ but there is nothing on the right to match it, so we conclude that

$$B_{x2} = B_{x1}, \tag{3.2.80}$$

On to the last of Maxwell's equations. Equation (3.2.72) gives nothing, because there are no x-derivatives. Equation (3.2.73) has one $-c^2\partial B_x/\partial x$, but there is nothing to match it with. We get

$$B_{z2} = B_{z1}, \tag{3.2.81}$$

The last equation is quite similar, and gives

$$B_{y2} = B_{y1}, \tag{3.2.82}$$

The last three equations gives us that $\mathbf{B}_2 = \mathbf{B}_1$. We want to emphasize, however, that we get this result only when the materials on both sides of the boundary are nonmagnetic-or rather, when we can neglect any magnetic effects of the materials. This can usually be done for most materials, except ferromagnetic

From the explicit expression for D^{aj} (and hence also of D^{a3}) we verify from (3.2.57) and (3.2.58)

$$\langle E^a(x)\rangle\big|_{z=+0} = 0, \quad \left\langle B^3(x)\right\rangle\big|_{z=+0} = 0. \tag{3.2.83}$$

We now prove that we can derive Maxwell's equation from equations of the half-space problem, since

$$\left\langle A^j\left(x\right)\right\rangle = \int (\mathrm{d}x')D^{jk}\left(x,x'\right)J^k\left(x'\right), \tag{3.2.84}$$

and

$$\left\langle E^i(x)\right\rangle = \left\langle \partial^0 A^i(x)\right\rangle = \int (\mathrm{d}x)\, \partial^0 D^{ik}(x,x')\, J^k(x'). \tag{3.2.85}$$

$$\left(-\Box\partial^j + \boldsymbol{\nabla}^2\partial^j\right) D^{jk}(x,x') = \partial^k \delta^{(4)}(x,x')$$

$$\left(\left(-\boldsymbol{\nabla}^2 + \partial^{02}\right)\partial^j + \boldsymbol{\nabla}^2\partial^j\right) D^{jk}(x,x') = \partial^k \delta^{(4)}(x,x')$$

$$\partial^{02}\partial^j D^{jk}(x,x') = \partial^k \delta^{(4)}(x,x')$$

$$\partial^0\partial^j D^{jk}(x,x') = \frac{\partial^k}{\partial^0}\delta^{(4)}(x,x'). \tag{3.2.86}$$

$$\partial^i\left\langle E^i(x)\right\rangle = \int (\mathrm{d}x)\, \partial^0\partial^i D^{ik}(x,x')\, J^k(x')$$

$$= \int (\mathrm{d}x) \left[\frac{\partial^k}{\partial^0}\delta^4(x,x')\right] J^k(x')$$

$$= \int (\mathrm{d}x) \left[\frac{-\delta^4(x,x')\,\partial'^k}{-\partial'^0}\right] J^k(x')$$

$$= \int (\mathrm{d}x)\, \delta^4(x,x') \frac{\partial'^k}{\partial'^0} J^k(x')$$

$$= \int (\mathrm{d}x)\, \delta^4(x,x') J^0(x')$$

$$\partial^i\left\langle E^i(x)\right\rangle = J^0(x), \tag{3.2.87}$$

The latter is one of Maxwell's equations which may be written in the differential form as

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \tag{3.2.88}$$

and is the famous Gauss's law. Now consider

$$\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \tag{3.2.89}$$

We can derive this law by considering

$$\left\langle B^i \right\rangle = \varepsilon^{ijk} \left\langle A^k (x) \right\rangle$$

$$= \int (\mathrm{d}x') \varepsilon^{ijk} \partial^j D^{kl} (x, x') J^l (x')$$

$$\boldsymbol{\nabla} \times \left\langle B^i \right\rangle = \varepsilon^{ijk} \left\langle \partial^v B^i (x) \right\rangle = \int (\mathrm{d}x') \underbrace{\left(\varepsilon^{iuv} \varepsilon^{ijk}\right)}_{\left(\delta^{uj}\delta^{vk} - \delta^{uk}\delta^{vj}\right)} \partial^v \partial^j D^{kl} (x, x') J^l (x')$$

$$\langle (\boldsymbol{\nabla} \times \mathbf{B}) \rangle^u = \int (\mathrm{d}x') \left(\delta^{uj}\delta^{vk}\partial^v\partial^j - \delta^{uk}\delta^{vj}\partial^v\partial^j\right) D^{kl} (x, x') J^l (x')$$

$$= \int (\mathrm{d}x') \left(\partial^k \partial^u D^{kl} (x, x') - \boldsymbol{\nabla}^2 D^{ul} (x, x')\right) J^l (x'), \tag{3.2.90}$$

$$\langle (\boldsymbol{\nabla} \times \mathbf{B})^u - \partial_0 E^u \rangle = \int (\mathrm{d}x') \left(\partial^k \partial^u D^{kl} (x, x') - \boldsymbol{\nabla}^2 D^{ul} (x, x')\right) J^l (x')$$

$$- \int (\mathrm{d}x') \partial^{02} D^{ul} (x, x') J^l (x')$$

$$= \int (\mathrm{d}x') \left(\partial^k \partial^u D^{kl} (x, x') - \left(\boldsymbol{\nabla}^2 + \partial^{02}\right) D^{ul} (x, x')\right) J^l (x')$$

$$= \int (\mathrm{d}x') \left(\underbrace{\partial^k \partial^u D^{kl}(x,x') - \Box D^{ul}(x,x')}_{\delta^{ul}\delta^4(x,x')} \right) J^l(x')$$

$$= \int (\mathrm{d}x') \delta^4(x,x') J^u(x')$$

$$\left\langle (\boldsymbol{\nabla} \times \mathbf{B}) - \frac{\partial \mathbf{E}}{\partial t} \right\rangle = \mathbf{J}(x). \tag{3.2.91}$$

The latter is the $4^t h$ law of Maxwell's equation. And the other two Maxwell's equations are derived as follow. Consider

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{3.2.92}$$

which may be derived from the expression

$$\langle B^i \rangle = \varepsilon^{ijk} \langle A^k(x) \rangle$$

$$= \int (\mathrm{d}x') \varepsilon^{ijk} \partial^j D^{kl}(x,x') J^l(x')$$

$$\partial^i \langle B^i \rangle = \int (\mathrm{d}x') \varepsilon^{ijk} \partial^i \partial^j D^{kl}(x,x') J^l(x')$$

$$= 0. \tag{3.2.93}$$

and

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{3.2.94}$$

Consider

$$\varepsilon^{ijk}\left\langle \partial^j E^k \right\rangle = \varepsilon^{ijk} \int \left(\mathrm{d}x'\right) \partial^0 \partial^j D^{kl}\left(x, x'\right) J^l\left(x'\right)$$

$$= \int \left(\mathrm{d}x'\right) \partial^0 \left(\varepsilon^{ijk} \partial^j D^{kl}\left(x, x'\right)\right) J^l\left(x'\right)$$

$$= -\left\langle \partial_0 B^i \right\rangle . \tag{3.2.95}$$

The positivity condition for the vacuum persistence probability from (3.2.59) will be established in (3.3.3)-(3.3.5). Since, by definition, the current J^μ is confined to the region $z > 0$ (that is, it is strictly zero for $z < 0$), we may, without loss of generality, extend the integrals in (3.2.59) over all z, z'. From (3.2.42), (3.2.43), (3.2.45) and (3.2.47), we then obtain after lengthy integrations and by using the identity $J^0(Q) = \frac{k^i J^i(Q)}{Q^0}$, the expression

$$\left\langle 0_+ \mid 0_- \right\rangle^J = \exp\left[\frac{\mathrm{i}}{2} \int (\mathrm{d}x)\left(\mathrm{d}x'\right) J^\mu\left(x\right) D'_{\mu\nu}\left(x, x'\right) J^\nu\left(x'\right)\right], \tag{3.2.96}$$

for (3.2.39), where $D'_{\mu\nu}(x, x')$ defines the photon propagator of the theory:

$$D'_{\mu\nu}\left(x, x'\right) = \int \frac{(\mathrm{d}Q)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}}{Q^2 - \mathrm{i}\varepsilon}$$

$$\times \left[g_{\mu\nu}\mathrm{e}^{\mathrm{i}q(z-z')} - g_{\mu\nu}\mathrm{e}^{-\mathrm{i}q(z+z')} + 2g_{\mu3}g_{3\nu}\mathrm{e}^{-\mathrm{i}q(z+z')}\right], \tag{3.2.97}$$

$Q = (Q^0, \mathbf{k}, q)$, $\varepsilon \to +0$, and

$$J^\mu(x) = \int \frac{(\mathrm{d}Q)}{(2\pi)^4} \mathrm{e}^{\mathrm{i}Qx} J^\mu(Q). \tag{3.2.98}$$

The presence of a non-covariant form $g_{\mu3}g_{3\nu}$ in (3.2.97) should not be surprising which is as a result of breaking translational invariance (along the z-axis). To describe the scattering process we write (Manoukian, 1984, 1985, 1986; Schwinger, 1951, 1953, 1954, 1970, 1976, 1977) $J^{\mu}(x) = J_1^{\mu}(x) + J_2^{\mu}(x)$, where $J_1^{\mu}(x), J_2^{\mu}(x)$ are causally arranged so that $J_2^{\mu}(x)$ is switched on after that $J_1^{\mu}(x)$ is switched off (Manoukian, 1984, 1985, 1986; Schwinger, 1951, 1953, 1954, 1970, 1976, 1977). $J_1^{\mu}(x)$ represents the emitter and $J_2^{\mu}(x)$ the detector of photons.

For $x^0 \neq x'^0$, (3.2.97) gives

$$D'_{\mu\nu}(x, x') = \mathrm{i} \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\mathrm{i}Q^0|x^0-x'^0|}}{2Q^0}$$

$$\times \left[g_{\mu\nu}\mathrm{e}^{\mathrm{i}q(z-z')} - g_{\mu\nu}\mathrm{e}^{-\mathrm{i}q(z+z')} + 2g_{\mu 3}g_{3\nu}\mathrm{e}^{-\mathrm{i}q(z+z')}\right], \qquad (3.2.99)$$

where $\mathbf{Q} = (\mathbf{k}, q)$, $Q^0 = \sqrt{\mathbf{k}^2 + q^2}$. we can then have, let

$$J^{\mu}(x) = J_1^{\mu}(x) + J_2^{\mu}(x). \qquad (3.2.100)$$

where $J_2^{\mu}(x)$is on after $J_1^{\mu}(x)$ is off,

$$D'_{\mu\nu}(x, x') = \mathrm{i} \int \frac{d^3\mathbf{Q}}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-iQ^0\left(x^0-x'^0\right)}}{2Q^0}$$

$$\times \left[g_{\mu\nu}\mathrm{e}^{\mathrm{i}q(z-z')} - g_{\mu\nu}\mathrm{e}^{-\mathrm{i}q(z+z')} + 2g_{\mu 3}g_{3\nu}\mathrm{e}^{-\mathrm{i}q(z+z')}\right] \qquad (3.2.101)$$

$$\langle 0_+ \mid 0_- \rangle^J = \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x')\left(J_1^{\mu}(x) + J_2^{\mu}(x)\right) D'_{\mu\nu}(x, x')\left(J_1^{\nu}(x') + J_2^{\nu}(x')\right)\right] \qquad (3.2.102)$$

$$= \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x')\left(J_2^{\mu}(x) D'_{\mu\nu}(x, x') J_2^{\nu}(x')\right.\right.$$

$$+ 2J_2^{\mu}(x)\, D'_{\mu\nu}(x,x')\, J_1^{\nu}(x') + \ J_1^{\mu}(x)\, D'_{\mu\nu}(x,x')\, J_1^{\nu}(x')\big)\Big] \tag{3.2.103}$$

$$\langle 0_+ \mid 0_- \rangle^J = \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)\,(\mathrm{d}x')\,\big(J_2^{\mu}(x)\, D'_{\mu\nu}(x,x')\, J_2^{\nu}(x')\right.$$

$$+ 2J_2^{\mu}(x)\, D'_{\mu\nu}(x,x')\, J_1^{\nu}(x')$$

$$\left. + \ J_1^{\mu}(x)\, D'_{\mu\nu}(x,x')\, J_1^{\nu}(x')\big)\right] \tag{3.2.104}$$

let

$$U = \mathrm{i}\int (\mathrm{d}x)\,(\mathrm{d}x') J_2^{\mu}(x)\, D'_{\mu\nu}(x,x')\, J_1^{\nu}(x') \tag{3.2.105}$$

$$U = \mathrm{i}\int (\mathrm{d}x)\,(\mathrm{d}x') J_2^{\mu}(x)\,\mathrm{i}\int \frac{\mathrm{d}^3Q}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\,\mathrm{i}Q^0\left(x^0-x'^0\right)}}{2Q^0}$$

$$\times\ \left[g_{\mu\nu}\mathrm{e}^{\mathrm{i}q(z-z')} - g_{\mu\nu}\mathrm{e}^{-\mathrm{i}q(z+z')} + 2g_{\mu 3}g_{3\nu}\mathrm{e}^{-\mathrm{i}q(z+z')}\right] J_1^{\nu}(x')$$

$$= \mathrm{i}\int (\mathrm{d}x)\,(\mathrm{d}x')\left[J_2^{\mu}(x)\,\mathrm{i}\int \frac{\mathrm{d}^3Q}{(2\pi)^3}\frac{\overbrace{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{\mathrm{i}q(z-z')}}^{\mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot(\mathbf{x}-\mathbf{x}')}}\,\mathrm{e}^{-\,\mathrm{i}Q^0\left(x^0-x'^0\right)}}{2Q^0} g_{\mu\nu}J_1^{\nu}(x')\right.$$

$$- J_2^{\mu}(x)\,\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3}\frac{\overbrace{\mathrm{e}^{\mathrm{i}(\mathbf{k}-q)\cdot\left(\left(\mathbf{x}_{\|}+z\right)-\left(\mathbf{x}'_{\|}+z'\right)\right)\ ;\ q=q\hat{z}}}^{\mathrm{i}\mathbf{Q}'\cdot(\mathbf{x}-\mathbf{x}')\ :\ \mathbf{Q}'=\mathbf{k}-\mathbf{q}}\ \overbrace{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{\|}-\mathbf{x}'_{\|}\right)}\mathrm{e}^{-\,\mathrm{i}q(z+z')}}\qquad \mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}}{2Q^0}$$

$$\times \left(g_{\mu\nu} - 2g_{\mu 3}g_{3\nu}\right) J_1^{\nu}\left(x'\right)\Big],$$

$$= \mathrm{i}\int (\mathrm{d}x)\,(\mathrm{d}x')\,\left[J_2^{\mu}\left(x\right)\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{Q}}{\left(2\pi\right)^3}\frac{\mathrm{e}^{iQ(x-x')}}{2Q^0}\,g_{\mu\nu}J_1^{\nu}\left(x'\right)\right.$$

$$\left. - J_2^{\mu}\left(x\right)\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{Q}}{\left(2\pi\right)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{Q'}\cdot(\mathbf{x}-\mathbf{x'})}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0-x'^0\right)}}{2Q^0}\left(g_{\mu\nu} - 2g_{\mu 3}g_{3\nu}\right) J_1^{\nu}\left(x'\right)\right]$$

$$= \int \frac{\mathrm{d}^3\mathbf{Q}}{\left(2\pi\right)^3 2Q^0}\,\left[\int (\mathrm{d}x)\left(\mathrm{i}J_2^{\mu}\left(x\right)\right)\mathrm{e}^{\mathrm{i}Qx}\,g_{\mu\nu}\int (\mathrm{d}x')\mathrm{e}^{-\mathrm{i}Qx'}\left(\mathrm{i}J_1^{\nu}\left(x'\right)\right)\right.$$

$$\left. - \int (\mathrm{d}x)\mathrm{i}J_2^{\mu}\left(x\right)\,\mathrm{e}^{\mathrm{i}Q'\cdot(x)}\mathrm{e}^{-\mathrm{i}Q^0\left(x^0\right)}\left(g_{\mu\nu} - 2g_{\mu 3}g_{3\nu}\right)\mathrm{e}^{\mathrm{i}Q'\cdot(-x')}\mathrm{e}^{-\mathrm{i}Q^0\left(-x'^0\right)}\left(\mathrm{i}J_1^{\nu}\left(x'\right)\right)\right]$$

$$= \int \frac{\mathrm{d}^3\mathbf{Q}}{\left(2\pi\right)^3 2Q^0}\,\left[\mathrm{i}\left(\int (\mathrm{d}x)\left(J_2^{\mu}\left(x\right)\right)\mathrm{e}^{-\mathrm{i}Qx}\right)^*\,g_{\mu\nu}\left(\mathrm{i}\int (\mathrm{d}x')\mathrm{e}^{-\mathrm{i}Qx'}J_1^{\nu}\left(x'\right)\right)\right.$$

$$\left. - \mathrm{i}\left(\int (\mathrm{d}x)J_2^{\mu}\left(x\right)\;\mathrm{e}^{-\mathrm{i}Q'x}\right)^*\left(g_{\mu\nu} - 2g_{\mu 3}g_{3\nu}\right)\int (\mathrm{d}x')\mathrm{e}^{-\mathrm{i}Qx'}\left(\mathrm{i}J_1^{\nu}\left(x'\right)\right)\right]$$

$$= \int \frac{\mathrm{d}^3\mathbf{Q}}{\left(2\pi\right)^3 2Q^0}\,\left[\left(\mathrm{i}J_2^{\mu}(Q^0,k,q)^*\right)\,g_{\mu\nu}\,\left(\mathrm{i}J_1^{\nu}(Q^0,k,q)\right)\right.$$

$$\left. - \mathrm{i}\;J_2^{\mu}\left(Q^0,k,-q\right)^*\left(g_{\mu\nu} - 2g_{\mu 3}g_{3\nu}\right)\left(\mathrm{i}J_1^{\nu}(Q^0,k,q)\right)\right], \qquad (3.2.106)$$

where

$$\int (\mathrm{d}x)\left(J_2^{\mu}\left(x\right)\right)\mathrm{e}^{-\mathrm{i}Qx} = J_2^{\mu}\left(Q\right)$$

$$J_2^{\mu}(Q^0,k,-q)^* = \left(\int (\mathrm{d}x)\left(J_2^{\mu}\left(x\right)\right)\mathrm{e}^{-\mathrm{i}Qx}\right)^*,$$

$$J_1^{\nu}(Q^0,k,q)=\int(\mathrm{d}x')\mathrm{e}^{-\mathrm{i}Qx'}J_1^{\nu}(x')\,, \tag{3.2.107}$$

using the conservation of current density $J^0(Q)=\frac{Q^iJ^i(Q)}{Q^0}$, (3.2.106) becomes

$$\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\,2Q^0}\,\left[\left(\mathrm{i}J_2^0(Q)^*\right)g_{00}\left(\mathrm{i}J_1^0(Q)\right)+\left(\mathrm{i}J_2^i(Q)^*\right)g_{ij}\left(\mathrm{i}J_1^i(Q)\right)\right.$$

$$-\left(\mathrm{i}J_2^{0*}(Q')\,g_{00}\left(\mathrm{i}J_1^0(Q)\right)+\mathrm{i}J_2^i(Q')^*\,g_{ij}\left(\mathrm{i}J_1^j(Q)\right)\right)$$

$$\left.-\left(\mathrm{i}J_2^0(Q')^*\,2g_{03}g_{30}\left(\mathrm{i}J_1^0(Q)\right)+\mathrm{i}J_2^{i*}(Q')\,2g_{i3}g_{3j}\left(\mathrm{i}J_1^j(Q)\right)\right)\right]$$

$$=\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\,2Q^0}\left[-\left(\mathrm{i}\frac{Q^iJ_2^i(Q)^*}{Q^0}\right)\left(\mathrm{i}\frac{Q^jJ_1^j(Q)}{Q^0}\right)+\left(\mathrm{i}J_2^i(Q)^*\right)\delta^{ij}\left(\mathrm{i}J_1^i(Q)\right)\right.$$

$$+\left(\mathrm{i}\frac{Q'^iJ_2^i(Q)^*}{Q^0}\left(\mathrm{i}\frac{Q^jJ_1^j(Q)}{Q^0}\right)+\mathrm{i}J_2^i(Q')^*\,\delta^{ij}\left(\mathrm{i}J_1^j(Q)\right)\right)$$

$$\left.+\left(\mathrm{i}J_2^i(Q')^*\right)2\delta^{i3}\delta^{3j}\left(\mathrm{i}J_1^j(Q)\right)\right],$$

$$=\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\,2Q^0}\,\left[-\left(\mathrm{i}J_2^i(Q)^*\right)\frac{Q^iQ^j}{Q^{02}}\left(\mathrm{i}J_1^j(Q)\right)+\left(\mathrm{i}J_2^i(Q)^*\right)\delta^{ij}\left(\mathrm{i}J_1^i(Q)\right)\right.$$

$$+\left(\left(\mathrm{i}J_2^i(Q')^*\right)\frac{Q'^iQ^j}{Q^{02}}\left(\mathrm{i}J_1^j(Q)\right)+\mathrm{i}J_2^i(Q')^*\,\delta^{ij}\left(\mathrm{i}J_1^j(Q)\right)\right)$$

$$\left.+\left(\left(\mathrm{i}J_2^i(Q')^*\right)2\delta^{i3}\delta^{3j}\left(\mathrm{i}J_1^j(Q)\right)\right)\right],$$

$$= \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\, 2Q^0} \left[\left(\mathrm{i}J_2^i\,(Q)^*\right) \left(\delta^{ij} - \frac{Q^i Q^j}{Q^{02}}\right) \left(\mathrm{i}J_1^j\,(Q)\right) \right.$$

$$\left. + \left(\mathrm{i}J_2^i\,(Q')^*\right) \left(-\delta^{ij} + \frac{Q'^i Q^j}{Q^{02}} + 2\delta^{i3}\delta^{3j}\right) \left(\mathrm{i}J_1^j\,(Q)\right) \right], \tag{3.2.108}$$

where

$$Q = (Q^0, \mathbf{k}, q),\ Q' = (Q^0, \mathbf{k}, -q),\ Q^0 = \sqrt{\mathbf{k}^2 + q^2}, \tag{3.2.109}$$

$$A^{ij} = \left(\delta^{ij} - \frac{Q^i Q^j}{|\mathbf{Q}|^2}\right), \tag{3.2.110}$$

$$B^{ij} = \left(-\delta^{ij} + \frac{Q'^i Q^j}{|\mathbf{Q}|^2} + 2\delta^{i3}\delta^{3j}\right). \tag{3.2.111}$$

We note the *transversality* conditions:

$$Q^i A^{ij} = 0, \quad A^{ij} Q^j = 0, \tag{3.2.112}$$

$$Q'^i B^{ij} = 0, \quad B^{ij} Q^j = 0. \tag{3.2.113}$$

The proof of above equations as shown in the following step:

$$A^{ij} Q^j = \left(\delta^{ij} - \frac{Q^i Q^j}{|\mathbf{Q}|^2}\right) Q^j = \ \delta^{ij} Q^j - \frac{Q^i Q^j Q^j}{|\mathbf{Q}|^2},$$

$$= Q^i - \frac{Q^i\,(Q^{j2})}{|\mathbf{Q}|^2} \qquad = Q^i - \frac{Q^i\,(|\mathbf{Q}^2|)}{|\mathbf{Q}|^2} = 0 \tag{3.2.114}$$

$$Q^i A^{ij} = Q^i \left(\delta^{ij} - \frac{Q^i Q^j}{|\mathbf{Q}|^2}\right) = Q^i \delta^{ij} - \frac{Q^i Q^i Q^j}{|\mathbf{Q}|^2}$$

$$= Q^j - \frac{(Q^{i2})\, Q^j}{|\mathbf{Q}|^2} \quad = \quad Q^j - \frac{\left(|\mathbf{Q}^2|\right) Q^j}{|\mathbf{Q}|^2} \quad = 0. \tag{3.2.115}$$

$$Q'^i B^{ij} = Q'^i \left(-\delta^{ij} + \frac{Q'^i Q^j}{|\mathbf{Q}|^2} + 2\delta^{i3}\delta^{3j} \right)$$

$$= -Q'^i \delta^{ij} + \frac{(Q'^i Q'^i)\, Q^j}{|\mathbf{Q}|^2} + 2Q'^i \delta^{i3}\delta^{3j}, \tag{3.2.116}$$

where

$$Q'^i = \left(Q^0, \mathbf{k}, -q\right) : - Q'^i = \left(Q^0, \mathbf{k}, q\right) = Q^i, \tag{3.2.117}$$

and

$$Q'^i Q'^i = \ |\mathbf{Q}'|^2 \ = \ |\mathbf{Q}|^2 = \ Q^i Q^i,$$

$$= Q^j + \frac{|\mathbf{Q}|^2\, Q^j}{Q^{02}} - 2Q^j = 0\ . \tag{3.2.118}$$

For any two three-dimensional unit vectors $\mathbf{n}, \mathbf{n}''$, we introduce polarization (unit) vectors as;

$$\mathbf{e}_1(n) = \frac{\mathbf{n}'' \times \mathbf{n}}{|\mathbf{n}'' \times \mathbf{n}|} = \mathbf{e}_1(n'') \tag{3.2.119}$$

$$\mathbf{e}_2(n) = \frac{\mathbf{n} \times (\mathbf{n}'' \times \mathbf{n})}{|\mathbf{n} \times (\mathbf{n}'' \times \mathbf{n})|} = \frac{\mathbf{n}''\,(\mathbf{n}\cdot\mathbf{n}) - \mathbf{n}\,(\mathbf{n}\cdot\mathbf{n}'')}{|\mathbf{n} \times (\mathbf{n}'' \times \mathbf{n})|}$$

$$= \frac{\mathbf{n}'' - (\mathbf{n}\cdot\mathbf{n}'')\,\mathbf{n}}{|\mathbf{n} \times (\mathbf{n}'' \times \mathbf{n})|} \tag{3.2.120}$$

$$\mathbf{e}_2(n'') = \frac{\mathbf{n}'' \times (\mathbf{n}'' \times \mathbf{n})}{|\mathbf{n}'' \times (\mathbf{n}'' \times \mathbf{n})|} = \frac{\mathbf{n}''(\mathbf{n}'' \cdot \mathbf{n}) - \mathbf{n}(\mathbf{n}'' \cdot \mathbf{n}'')}{|\mathbf{n}'' \times (\mathbf{n}'' \times \mathbf{n})|}$$

$$= \frac{(\mathbf{n} \cdot \mathbf{n}'')\,\mathbf{n}'' - \mathbf{n}}{|\mathbf{n}'' \times (\mathbf{n}'' \times \mathbf{n})|}. \tag{3.2.121}$$

satisfying the conditions:

$$\mathbf{e}_\lambda(n) \cdot \mathbf{e}_\alpha(n) = \delta_{\lambda\alpha} = \mathbf{e}_\lambda(n'') \cdot \mathbf{e}_\alpha(n'') \tag{3.2.122}$$

$$\mathbf{n} \cdot \mathbf{e}_\lambda(n) = 0, \;\; \mathbf{n}'' \cdot \mathbf{e}_\lambda(n'') = 0, \tag{3.2.123}$$

In particular for $\mathbf{n} = \frac{\mathbf{Q}}{|\mathbf{Q}|}, \;\; \mathbf{n}'' = \mathbf{n}' = \frac{\mathbf{Q}'}{|\mathbf{Q}|}$ we have

$$e^a_{{}_1}(n') = -e^a_2(n), \quad a = 1, 2 \tag{3.2.124}$$

$$e^3_{{}_2}(n') = e^3_1(n), \tag{3.2.125}$$

Upon using the completeness relations:

$$\delta^{ij} = n^i n^j + e^i_\lambda(n) e^j_\lambda(n), \tag{3.2.126}$$

$$\delta^{ij} = n'^i n'^j + e^i_\lambda(n') e^j_\lambda(n') \tag{3.2.127}$$

with summations over $\lambda = 1, 2$, and using the transversality conditions (3.2.112) and (3.2.113), we may rewrite (3.2.108) as

$$U = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\, 2Q^0} \left[\left(\mathrm{i}J^i_2\,(Q)^*\right) \left(\delta^{ij} - \frac{Q^i Q^j}{|\mathbf{Q}|^2}\right) \left(\mathrm{i}J^j_1\,(Q)\right) \right.$$

$$+\left(\mathrm{i}J_2^i\left(Q'\right)^*\right)\left(-\delta^{ij}+\frac{Q'^iQ^j}{|\mathbf{Q}|^2}+2\delta^{i3}\delta^{3j}\right)\left(\mathrm{i}J_1^j\left(Q\right)\right)\Big]$$

$$=\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\,2Q^0}\,\left[\left(\mathrm{i}J_2^i\left(Q\right)^*\right)\left(\delta^{ij}-n^in^j\right)\left(\mathrm{i}J_1^j\left(Q\right)\right)\;+\left(\mathrm{i}J_2^i\left(Q'\right)^*\right)\right.$$

$$\left.\times\left(-\delta^{ij}+n'^in^j+2\left(n'^in'^3+e^i_\lambda(n')e^3_\lambda(n')\right)\left(n^3n^j+e^3_\alpha(n)e^j_\alpha(n)\right)\right)\left(\mathrm{i}J_1^j\left(Q\right)\right)\right],$$

(3.2.128)

using (3.2.126) and (3.2.127) , we have

$$U=\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\,2Q^0}\,\left[\left(\mathrm{i}J_2^i\left(Q\right)^*\right)\left(e^i_\lambda(n)e^j_\lambda(n)\right)\left(\mathrm{i}J_1^j\left(Q\right)\right)\;+\left(\mathrm{i}J_2^i\left(Q'\right)^*\right)\right.$$

$$\left.\times\left(-\delta^{ij}+n'^in^j+2\left(n'^in'^3+e^i_\lambda(n')e^3_\lambda(n')\right)\left(n^3n^j+e^3_\alpha(n)e^j_\alpha(n)\right)\right)\left(\mathrm{i}J_1^j\left(Q\right)\right)\right],$$

$$=\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\,2Q^0}\,\left[\left(\mathrm{i}J_2^i\left(Q\right)^*\right)\left(e^i_\lambda(n)e^j_\lambda(n)\right)\left(\mathrm{i}J_1^j\left(Q\right)\right)\right.$$

$$\left.+\left(\mathrm{i}J_2^{i*}\left(Q'\right)e^i_\lambda(n')\right)\left(-e_\lambda(n')\cdot e_\alpha(n)+2e^3_\lambda(n')e^3_\alpha(n)\right)\left(\mathrm{i}J_1^j\left(Q\right)e^j_\alpha(n)\right)\right],\quad(3.2.129)$$

From (3.2.119) - (3.2.125) we readily derive

$$-\mathbf{e}_\lambda(n')\cdot\mathbf{e}_\alpha(n)+2e^3_\lambda(n')e^3_\alpha(n)=(-1)^\lambda\delta_{\lambda\alpha}\qquad(3.2.130)$$

$$\lambda,\,\alpha=1,\,2$$

or

$$U=\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\,2Q^0}\,\left[\left(\mathrm{i}J_2^{i*}\left(Q\right)e_\lambda^i(n)\right)\left(\mathrm{i}J_1^j\left(Q\right)e_\lambda^j(n)\right)\right.$$

$$\left.+\left(\mathrm{i}J_2^{i*}\left(Q'\right)e_\lambda^i(n')\right)\left(-1\right)^\lambda\delta_{\lambda\alpha}\left(\mathrm{i}J_1^j\left(Q\right)e_\alpha^j(n)\right)\right]$$

$$=\int\frac{\mathrm{d}^3\mathbf{Q}''}{\sqrt{(2\pi)^3\,2Q''^0}}\int\frac{\mathrm{d}^3\mathbf{Q}}{\sqrt{(2\pi)^3\,2Q^0}}\left[\left(\mathrm{i}J_2^{i*}\left(Q''\right)e_\lambda^i(n'')\right)\right.$$

$$\times\left[\delta^3\left(Q''-Q\right)\delta_{\lambda\alpha}+\delta_{\lambda\alpha}\left(-1\right)^\lambda\delta^3\left(Q''-Q'\right)\right]\left(\mathrm{i}J_1^j\left(Q\right)e_\alpha^j(n)\right),\tag{3.2.131}$$

where

$$\mathbf{Q}=(\mathbf{k},q),\qquad\mathbf{Q}'=(\mathbf{k},-q)\tag{3.2.132}$$

To obtain the transition amplitude in question, we consider the term:

$$\left\langle 0_+\mid\mathbf{Q}'',\lambda\right\rangle^{J_2}\left\langle\mathbf{Q}'',\lambda\mid\mathbf{Q},\alpha\right\rangle\left\langle\mathbf{Q},\alpha\mid 0_-\right\rangle^{J_1}\tag{3.2.133}$$

$$\left\langle\mathbf{Q},\alpha\mid 0_-\right\rangle^{J_1}\tag{3.2.134}$$

denotes the amplitude that a photon is emitted from J_1 with momentum $\mathbf{Q}$ and polarization α , and

$$\left\langle 0_+\mid\mathbf{Q}'',\lambda\right\rangle^{J_2}\tag{3.2.135}$$

denotes the amplitude that a photon is detected by J_2 with momentum $\mathbf{Q}''$ and polarization λ and

$$\langle \mathbf{Q}'', \lambda \mid \mathbf{Q},\alpha \rangle \tag{3.2.136}$$

denotes the amplitude in question where the emitted photon of momentum $\mathbf{Q}$ and polarization α ends up with a momentum $\mathbf{Q}''$ and polarization λ . Since formally,

$$\langle \mathbf{Q},\alpha \mid 0_- \rangle^{J_1} = \mathrm{i} J_1^j(Q) e_\alpha^j\left(n\right) \sqrt{\frac{\mathrm{d}^3\mathbf{Q}}{\left(2\pi\right)^3 2Q^0}} \tag{3.2.137}$$

$$\langle 0_+ \mid \mathbf{Q}'', \lambda \rangle^{J_2} = \mathrm{i} J_1^{j*}(Q'') e_\alpha^j\left(n''\right) \sqrt{\frac{\mathrm{d}^3\mathbf{Q}''}{\left(2\pi\right)^3 2Q''^0}} \tag{3.2.138}$$

for $J_1, J_2 \to 0$, we obtain from (3.2.104) and (3.2.131),

$$\langle \mathbf{Q}'', \lambda \mid \mathbf{Q},\alpha \rangle = \left[\delta^3\left(\mathbf{Q}'' - \mathbf{Q}\right)\delta_{\lambda\alpha} + \delta_{\lambda\alpha}\left(-1\right)^\lambda \delta^3\left(\mathbf{Q}'' - \mathbf{Q}'\right)\right] \mathrm{d}^3\mathbf{Q} \tag{3.2.139}$$

the first term of the right-hand side of (3.2.139) describes the non-scattering term. The second term deals with a scattering process where a scattered photon necessarily retains its energy and its polarization ($|\mathbf{Q}'| = |\mathbf{Q}|$) having the direction of its momentum charge according to the classical law of reflection.

3.3 The Positivity Condition

Finally we establish the positivity condition for the vacuum persistence probability:

$$\left|\langle 0_+ | 0_- \rangle^J\right|^2 \leqslant 1 \tag{3.3.1}$$

To this end we use

$$\operatorname{Im}\frac{1}{Q^2 - \mathrm{i}\varepsilon} = \pi\delta\left(Q^2\right) = \frac{\pi}{2\left|\mathbf{Q}\right|}\left[\delta\left(Q^0 - \left|\mathbf{Q}\right|\right) + \delta\left(Q^0 + \left|\mathbf{Q}\right|\right)\right]. \tag{3.3.2}$$

in (3.2.59) to obtain from (3.2.101)

$$\left|\langle 0_+ | 0_- \rangle\right|^2 = \exp -B, \tag{3.3.3}$$

where

$$B = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\, 2Q^0}\left[J^{i*}(Q)\, A^{ij} J^i(Q) + J^{i*}(Q')\, B^{ij} J^i(Q)\right]. \tag{3.3.4}$$

$\mathbf{Q} = \left(\sqrt{\mathbf{k}^2 + q^2}, \mathbf{k}, q\right)$, $\mathbf{Q}' = \left(\sqrt{\mathbf{k}^2 + q^2}, \mathbf{k}, -q\right)$; A^{ij}, B^{ij} are defined in (3.2.110) and (3.2.111), respectively. From the tranversality conditions (3.2.112) and (3.2.113) and the completeness relations (3.2.126) and (3.2.127) and the equality (3.2.130) , we may rewrite B as $Q^0 = \left(\sqrt{\mathbf{k}^2 + q^2}\right)$:

$$\begin{aligned} B = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\, 2Q^0} &\left[J^{i*}(Q)\, e^i_\lambda(n)\, J^j(Q)\, e^j_\lambda(n)\right. \\ &\left. + J^{i*}(Q')\, e^i_\lambda(n')\,(-1)^\lambda\, e^j_\lambda(n)\, J^j(Q)\right], \\ = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3\, 2Q^0} &\left[\left|J^i(Q)\, e^i_1(n) - J^i(Q')\, e^i_1(n')\right|^2\right. \\ &\left. + \left|J^i(Q)\, e^i_2(n) - J^i(Q')\, e^i_2(n')\right|^2\right] \geqslant 0. \end{aligned} \tag{3.3.5}$$

From this positivity restriction, the condition in (3.3.1) follows.

CHAPTER IV

PROPAGATION OF PHOTONS IN SPACETIME: INFINITELY EXTENDED SPACE AND HALF–SPACE

4.1 Introduction

The purpose of this chapter is to investigate rigorously the amplitude that a given current distribution $J_\mu(x)$ creates photon excitations, as well as investigate the *amplitude* that a photon excitation propagates from a point $(x_1^0, \mathbf{x}_1)$ to a point $(x_2^0, \mathbf{x}_2)$ as a time evolution process in infinitely extended space, and also derive various probabilities for detection of photon excitations in space after their emissions from sources. In particular we learn that the amplitude of propagation as a time evolution process in spacetime is not given by the so-called Feynman propagator.

There is by now an avalanche of experiments (Panarella, 1986; Sillitoe, 1972; Taylor, 1909; Dempster and Batho, 1927; Dontsov and Baz', 1967; Franson and Potochi, 1988; Gans and Miguez, 1917; Janossy and Naray, 1957, 1958; Grangier, Roger and Aspect,1986; Reynolds, Spartalian and Scarl, 1969; Griashaev, Naugol'nyi, Reprintsev, Tarasenko and Shenderovich, 1971) giving a clear indication that photons may be localized in space by detectors. Much theoretical effort (Ali, 1985; Amrein, 1969; Hegerfeldt, 1974; Kálnay, 1971; Kraus, 1970, 1971, 1977; Neumann, 1971, 1972; Price, 1948; Vries, 1970; Ali and Emch, 1974; Newton and Wigner, 1949; Yauch and Piron, 1967; de Azcárraga, Oliver and Pascual, 1973; Han, Kim and Noz, 1987) has been made in recent years to formulate the problem of localization of photons, with no success. Most of the attention in this effort has been given to defining a position operator and constructing wave functions, as done in non-relativistic quantum mechanics; this gives no hope whatsoever of formulating interacting theories, or asking probabilistic questions in configuration space, such as what is the probability that photons and/or other particles emerge spatially within cones after a collision or a decay process. The latter effort is also remote from actual physical situation, where photons travel between emitter and detectors in configuration space by the process of being created and destroyed, respectively, which is best described in the language of quantum field theory. And it has become urgent to extract the necessary information of the localization of photons directly from field theory. It is felt that any solution to this long-standing problem should have come by relying on the actual physical situation, mentioned above, of the propagation of photons from emitters to detectors, which is the starting point of modern formulations of field theory (Manoukian, 1984, 1985, 1986, 1988; Schwinger, 1951, 1953, 1954, 1970, 1972, 1973, 1977). We propose a solution to the problem by carrying out a unitarity expansion in configuration space based on the physical situation and verifying the associated completeness relation for a correct probabilistic interpretation. The basic tool to do this is the vacuum-to-vacuum transition amplitude due to Schwinger (Schwinger, 1951, 1953, 1954, 1970, 1972, 1973, 1977) in the presence of an external current (Manoukian, 1984, 1985, 1986, 1988; Schwinger, 1951, 1953, 1954, 1970, 1972, 1973, 1977). In Sect. 4.2, we carry out a space-time analysis of the propagation of photon excitations between emitters and detectors, based on the unitarity expansion in configuration space and its associated completeness relation. In Sect. 4.3, an expression is derived for the amplitude that a photon excitation travels from one time-space coordinate point to another. An explicit expression is obtained for the amplitude for a photon excitation detection at a given point in space after a given time when the latter is initially localized at its creation site. Detailed numerical and very precise estimates are carried out for the corresponding probabilities to interpret the problem of localization in terms of the constancy of the speed of light when photon excitations travel sufficiently large distances after their emission. Finally, an expression is obtained for the macroscopic amplitude of photon excitation propagation when it travels over macroscopic distances.

4.2 Space-time Analysis of the Propagation of Photon Excitations Between Emitters and Detectors: Unitary Expansion in Configuration Space

Our starting point is the vacuum-to-vacuum transition amplitude for photons in the presence of an external current J^μ. In the Coulomb gauge, the latter is given by the well-known expression

$$\langle 0_+ \mid 0_- \rangle = \mathrm{e}^{\mathrm{i}W}, \tag{4.2.1}$$

$$W = \frac{1}{2}\int (\mathrm{d}x)(\mathrm{d}x') \left[J^0(x) \frac{1}{\boldsymbol{\partial}^2} \delta\left(x - x'\right) J^0(x') \right.$$

$$\left. + J^i(x) \left(\delta^{ij} - \frac{\partial^i \partial^j}{\boldsymbol{\partial}^2} \right) D_+\left(x - x'\right) J^j(x') \right], \tag{4.2.2}$$

where

$$D_+\left(x - x'\right) = \int \frac{(\mathrm{d}k)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 - \mathrm{i}\varepsilon}, \qquad \varepsilon \to +0, \tag{4.2.3}$$

and

$$D_+\left(x - x'\right) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}k(x-x')} \int \frac{\mathrm{d}k^0}{2\pi} \frac{1}{k^2 - k^{02} - \mathrm{i}\varepsilon} \mathrm{e}^{\mathrm{i}k^0\left(x^0 - x'^0\right)}$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}k(x-x')} \int \frac{\mathrm{d}k^0}{2\pi} \frac{\mathrm{e}^{\mathrm{i}k^0\left(x^0 - x'^0\right)}}{\left[k^0 - (|k| - \mathrm{i}\varepsilon)\right]\left[k^0 + (|k| + \mathrm{i}\varepsilon)\right]}$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}k(x-x')} \frac{2\pi\mathrm{i}}{2\pi} \frac{\mathrm{e}^{\mathrm{i}k^0\left|x^0 - x'^0\right|}}{2k^0}$$

$$= \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}k(x-x')}\frac{\mathrm{e}^{\mathrm{i}k^0\left(x^0-x'^0\right)}}{2k^0}\,, \tag{4.2.4}$$

for $x^0 > x'^0,\ k^0 = |k|$. Note that by imposing the current conservation $\partial_\mu J^\mu = 0$, the exponent W in (4.2.2) may be written in the covariant form, from (4.2.2) we can write

$$\begin{aligned} W =& \frac{1}{2}\int (\mathrm{d}x)\,(\mathrm{d}x') \\ & \times \left(J^0(x)\,\frac{1}{\partial^2}\delta(x-x')\,J_0(x') + J^i(x)\left(\delta^{ij} - \frac{\partial^i\partial^j}{\partial^2}\right) D_+(x-x')\,J_j(x')\right) \end{aligned} \tag{4.2.5}$$

with

$$D_+(x-x') = \int \frac{(\mathrm{d}k)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2-\mathrm{i}\varepsilon} \tag{4.2.6}$$

and

$$D^{ij}(x-x') = \int \frac{(\mathrm{d}k)}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-x')}\frac{1}{k^2-\mathrm{i}\varepsilon}\left(\delta^{ij} - \frac{k^ik^j}{k^2}\right) \tag{4.2.7}$$

$$W = \frac{1}{2}\int (\mathrm{d}x)\,(\mathrm{d}x')\left(J^0(x)\overbrace{\int \frac{(\mathrm{d}k)}{(2\pi)^4}\mathrm{e}^{\mathrm{i}k(x-x')}}^{\delta(x-x')}\left(\frac{1}{\partial^2}\right)J_0(x')\right.$$

$$\left. + J^i(x)\overbrace{\int \frac{(\mathrm{d}k)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2-\mathrm{i}\varepsilon}}^{D_+(x-x')}\left(\delta^{ij} - \frac{\partial^i\partial^j}{\partial^2}\right)J_j(x')\right)$$

$$= \frac{1}{2} \int (\mathrm{d}x)\,(\mathrm{d}x') \left(J^0\,(x) \int \frac{(\mathrm{d}k)}{(2\pi)^4} \mathrm{e}^{\mathrm{i}k(x-x')} \frac{1}{k^2} J_0\,(x') \right.$$

$$\left. + \; J^i\,(x) \int \frac{(\mathrm{d}k)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 - \mathrm{i}\varepsilon} \left(\delta^{ij} - \frac{\partial^i \partial^j}{\partial^2} \right) J_j\,(x') \right)$$

$$= \frac{1}{2} \int (\mathrm{d}x)\,(\mathrm{d}x') \left(J^0\,(x) \int \frac{(\mathrm{d}k)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 - \mathrm{i}\varepsilon} J_0\,(x') \right.$$

$$\left. + \; J^i\,(x) \int \frac{(\mathrm{d}k)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 - \mathrm{i}\varepsilon} \left(\delta^{ij} - \frac{k^i k^j}{k^2} \right) J_j\,(x') \right) \tag{4.2.8}$$

$$W = \frac{1}{2} \int (\mathrm{d}x)\,(\mathrm{d}x') \left(J^0\,(x)\, D_+\,(x - x')\, J_0\,(x') + J^i\,(x)\, D_+\,(x - x')\, J_j\,(x') \right), \tag{4.2.9}$$

$$W = \frac{1}{2} \int (\mathrm{d}x)\,(\mathrm{d}x')\, J^\mu\,(x)\, D_+\,(x - x')\, J_\mu\,(x'). \tag{4.2.10}$$

We set

$$J_T^i\,(x) = \left(\delta^{ij} - \frac{\partial^i \partial^j}{\partial^2} \right) J^j\,(x)\,, \tag{4.2.11}$$

and note that obviously

$$\partial_i J_T^i\,(x) = \partial_i \left(\delta^{ij} - \frac{\partial^i \partial^j}{\partial^2} \right) J^j\,(x)$$

$$\partial_i J_T^i\,(x) = \delta^{ij} \partial_i J^j\,(x) - \frac{\partial_i \partial^i \partial^j}{\partial^2} J^j\,(x)$$

$$= \partial_j J^j(x) - \frac{(\partial_i \partial^i) \partial^j}{\partial^2} J^j(x)$$

$$\partial_i J_T^i(x) = 0. \tag{4.2.12}$$

We write

$$J^\mu(x) = J_1^\mu(x) + J_2^\mu(x), \tag{4.2.13}$$

where the current J_2^μ is switched on after that the current J_1^μ is switched off . Let x^0 denote any time in the intermediate range of values between the time J_1^μ is switched off and J_2^μ is switched on. That is, at time x^0, in particular, both currents J_1^μ, J_2^μ are zero. We note the relations

$$-\mathrm{i}D_+(x - x') = \int \mathrm{d}^3\mathbf{y} D(x - y) D(y - x'), \quad x'^0 < y^0 < x^0, \tag{4.2.14}$$

where

$$D(x) = \int \frac{\mathrm{d}^3\mathbf{k}\; \mathrm{e}^{\mathrm{i}kx}}{(2\pi)^3 \sqrt{2k^0}}. \tag{4.2.15}$$

Let $\mathbf{n}$ be *any* (three-dimensional) unit vector. Then a straightforward analysis shows that we may rewrite (4.2.1) as

$$\langle 0_+ \mid 0_- \rangle^J = \langle 0_+ \mid 0_- \rangle^{J_2} \exp\left[\int \mathrm{d}^3\mathbf{x}\; \mathrm{i}\mathbf{j}_2(x)^* \cdot \mathrm{i}\mathbf{j}_1(x)\right] \langle 0_+ \mid 0_- \rangle^{J_1}, \tag{4.2.16}$$

where

$$\mathbf{j}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \sqrt{2k^0}} \left[\mathbf{J}_T(k) - \left(\frac{\mathbf{k} + k^0\mathbf{n}}{k^0 + \mathbf{n}\cdot\mathbf{k}}\right) \mathbf{n}\cdot\mathbf{J}_T(k)\right] \mathrm{e}^{\mathrm{i}kx}. \tag{4.2.17}$$

To prove (4.2.11), substituting (4.2.12) in to U

$$U = \int \mathrm{d}^3\mathbf{x}\ \mathrm{i}\mathbf{j}_2(x)^* \cdot \mathrm{i}\mathbf{j}_1(x)$$

$$= \int \mathrm{d}^3\mathbf{x}\,\mathrm{i}\left(\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\left[\mathbf{J}_{T2}(k) - \left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_{T2}(k)\right]\mathrm{e}^{\mathrm{i}kx}\right)^* \cdot \mathrm{i}\mathbf{j}_1(x),$$

where $\mathbf{n}\cdot\mathbf{j} = 0$, as will be shown later in $(4.2.21)$. So the latter equation is equal to:

$$\int \mathrm{d}^3\mathbf{x}\,\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\mathrm{e}^{-\mathrm{i}kx}\mathbf{J}^*_{T2}(k)\cdot\mathrm{i}\mathbf{j}_1(x) - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}$$

$$\times\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}^*_{T2}(k)\,\mathrm{e}^{-\mathrm{i}kx}\cdot\mathrm{i}\mathbf{j}_1(x)$$

$$= \int \mathrm{d}^3\mathbf{x}\,\mathrm{i}^2\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\mathrm{e}^{-\mathrm{i}kx}\mathbf{J}^*_{T2}(k)\cdot\mathbf{j}_1(x) - \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}$$

$$\times(\mathrm{i})\left(\frac{\mathbf{k}\cdot\mathbf{j}_1(x) + k^0\overbrace{\mathbf{n}\cdot\mathbf{j}_1(x)}^{0}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}^*_{T2}(k)\,\mathrm{e}^{-\mathrm{i}kx},$$

$$= \int \mathrm{d}^3\mathbf{x}\ \mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{\mathrm{i}k'x}\mathrm{i}\,(\mathrm{i})\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\mathbf{J}^*_{T2}(k)$$

$$\times\left(\int \frac{\mathrm{d}^3\mathbf{k'}}{(2\pi)^3\sqrt{2k'^0}}\left[\mathbf{J}_{T1}(k') - \left(\frac{\mathbf{k'}+k'^0\mathbf{n}}{k'^0+\mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\right]\right)$$

$$-\int \mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{\mathrm{i}k'x}(\mathrm{i})\int \frac{\mathrm{d}^3\mathbf{k'}}{(2\pi)^3\sqrt{2k^0}}(\mathrm{i})$$

$$\times \frac{1}{k^0 + \mathbf{n}\cdot\mathbf{k}}\mathbf{k}\Bigg(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\Bigg[\mathbf{J}_{T1}(k')$$

$$-\left(\frac{\mathbf{k}' + k'^0\mathbf{n}}{k'^0 + \mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\Bigg]\Bigg)\mathbf{n}\cdot\mathbf{J}^*_{T2}(k),$$

$$= \underbrace{\int \mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')x}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}\mathrm{e}^{-\mathrm{i}\left(k^0-k'^0\right)x^0}\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\mathbf{J}^*_{T2}(k)$$

$$\times\left(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\left[\mathbf{J}_{T1}(k') - \left(\frac{\mathbf{k}' + k'^0\mathbf{n}}{k'^0 + \mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\right]\right)$$

$$-\underbrace{\int \mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')x}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}\mathrm{e}^{-i\left(k^0-k'^0\right)x^0}\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}(\mathrm{i})$$

$$\times\Bigg(\frac{1}{k^0 + \mathbf{n}\cdot\mathbf{k}}\mathbf{k}\cdot\Bigg(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\Bigg[\mathbf{J}_{T1}(k')$$

$$-\left(\frac{\mathbf{k}' + k'^0\mathbf{n}}{k'^0 + \mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\Bigg]\Bigg)\Bigg)\mathbf{n}\cdot\mathbf{J}^*_{T2}(k)$$

$$\times\left(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\left[\mathbf{J}_{T1}(k') - \left(\frac{\mathbf{k}' + k'^0\mathbf{n}}{k'^0 + \mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\right]\right)$$

$$-\underbrace{\int \mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')x}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}\mathrm{e}^{-\mathrm{i}\left(k^0-k'^0\right)x^0}\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}(\mathrm{i})$$

$$\times\Bigg\{\frac{1}{k^0 + \mathbf{n}\cdot\mathbf{k}}\mathbf{k}\Bigg(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}$$

$$\times \left[\mathbf{J}_{T1}(k') - \left(\frac{\mathbf{k}' + k'^0\mathbf{n}}{k'^0 + \mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\right]\Bigg)\Bigg\}\mathbf{n}\cdot\mathbf{J}^*_{T2}(k)\,,$$

$$= \underbrace{\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')x}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}\,\mathrm{e}^{-\mathrm{i}\left(k^0-k'^0\right)x^0}\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\mathbf{J}^*_{T2}(k)$$

$$\times \left(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\left[\mathbf{J}_{T1}(k') - \left(\frac{\mathbf{k}' + k'^0\mathbf{n}}{k'^0 + \mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\right]\right)$$

$$- \Bigg[\underbrace{\int \mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')x}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}\,\mathrm{e}^{-\mathrm{i}\left(k^0-k'^0\right)x^0}\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}(\mathrm{i})$$

$$\times \left(\frac{1}{k^0 + \mathbf{n}\cdot\mathbf{k}}\mathbf{k}\cdot\Bigg(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\Bigg[\mathbf{J}_{T1}(k')\right.$$

$$\left. - \left(\frac{\mathbf{k}' + k'^0\mathbf{n}}{k'^0 + \mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k')\Bigg]\Bigg)\right)\mathbf{n}\cdot\mathbf{J}^*_{T2}(k)\Bigg]$$

$$= \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\mathbf{J}^*_{T2}(k)\cdot(\mathrm{i})$$

$$\times \left(\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\left[\mathbf{J}_{T1}(k) - \left(\frac{\mathbf{k} + k^0\mathbf{n}}{k^0 + \mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k)\right]\right)$$

$$- \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}(\mathrm{i})\frac{1}{k^0 + \mathbf{n}\cdot\mathbf{k}}\;\mathbf{k}\cdot$$

$$\times \left(\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\left[\mathbf{J}_{T1}(k) - \left(\frac{\mathbf{k} + k^0\mathbf{n}}{k^0 + \mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_{T1}(k)\right]\right)\mathbf{n}\cdot\mathbf{J}^*_{T2}(k)\,,$$

$$= \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} \Bigg(\Bigg[\mathbf{J}^*_{T2}\,(k) \cdot \mathbf{J}_{T1}\,(k)$$

$$- \left(\frac{\mathbf{J}^*_{T2}(k) \cdot \mathbf{k} + \mathbf{J}^*_{T2}(k) \cdot k^0 \mathbf{n}}{k^0 + \mathbf{n} \cdot \mathbf{k}} \right) \mathbf{n} \cdot \mathbf{J}_{T1}(k) \Bigg] \Bigg)$$

$$- \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} \Bigg(\frac{1}{k^0 + \mathbf{n} \cdot \mathbf{k}} \Bigg(\Bigg[\mathbf{k} \cdot \mathbf{J}_{T1}(k)$$

$$- \left(\frac{k \cdot k + \mathbf{k} \cdot k^0 \mathbf{n}}{k^0 + \mathbf{n} \cdot k} \right) \mathbf{n} \cdot \mathbf{J}_{T1}(k) \Bigg] \Bigg) \Bigg) \mathbf{n} \cdot \mathbf{J}^*_{T2},$$

$$= \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} \Bigg\{ \Bigg[J^*_{T2}(k) \cdot \mathbf{J}_{T1}(k)$$

$$- \left(\frac{\overbrace{\mathbf{J}^*_{T2}\,(k) \cdot k}^{0} + \mathbf{J}^*_{T2}\,(k) \cdot k^0 n}{k^0 + \mathbf{n} \cdot k} \right) \mathbf{n} \cdot \mathbf{J}_{T1}(k) \Bigg] \Bigg\}$$

$$- \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k^0} \frac{1}{k^0 + \mathbf{n} \cdot \mathbf{k}} \Bigg\{ \overbrace{k \cdot \mathbf{J}_{T1}(k)}^{=0}$$

$$- \left(\frac{\overbrace{\mathbf{k} \cdot \mathbf{k} + \mathbf{k} \cdot k^0 \mathbf{n}}^{k^0\left(k^0 + \mathbf{n}\cdot\mathbf{k}\right)}}{k^0 + \mathbf{n} \cdot \mathbf{k}} \right) \mathbf{n} \cdot \mathbf{J}_{T1}(k) \Bigg\} \mathbf{n} \cdot \mathbf{J}^*_{T2}\,(k)$$

$$= \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} \left[\mathbf{J}^*_{T2}\,(k) \cdot \mathbf{J}_{T1}\,(k) - \left(\frac{\mathbf{J}^*_{T2}\,(k) \cdot k^0 \mathbf{n}\, (\mathbf{n} \cdot \mathbf{J}_{T1}\,(k))}{k^0 + \mathbf{n} \cdot \mathbf{k}} \right) \right]$$

$$- \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} \left(\frac{[-k^0 \mathbf{n} \cdot \mathbf{J}_{T1}\,(k)]\, \mathbf{n} \cdot \mathbf{J}^*_{T2}\,(k)}{k^0 + \mathbf{n} \cdot \mathbf{k}} \right),$$

$$= \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \, 2k^0} \left[\mathbf{J}^*_{T2}(k) \cdot \mathbf{J}_{T1}(k) - \left(\frac{\mathbf{J}^*_{T2}(k) \cdot k^0 \mathbf{n} \left(\mathbf{n} \cdot \mathbf{J}_{T1}(k) \right)}{k^0 + \mathbf{n} \cdot \mathbf{k}} \right) \right]$$

$$+ \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \, 2k^0} \left(\frac{k^0 \mathbf{n} \cdot \mathbf{J}_{T1}(k) \left(\mathbf{n} \cdot \mathbf{J}^*_{T2}(k) \right)}{k^0 + \mathbf{n} \cdot \mathbf{k}} \right)$$

$$= \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \, 2k^0} [\mathbf{J}^*_{T2}(k) \cdot \mathbf{J}_{T1}(k)],$$

$$= \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 \, 2k^0} [\mathbf{J}^*_{T2}(k) \cdot \mathbf{J}_{T1}(k)]$$

From $J^i_T(x) = \left(\delta^{ij} - \frac{\partial^i \partial^j}{\partial^2} \right) J^j(x)$

$$J^i_T(k) = \left(J^i(k) - \frac{k^i \left(k_i J^i(k) \right)}{k^{02}} \right)$$

$$= \left(J^i(k) - \frac{k^i J^0(k)}{k^0} \right)$$

$$J^i_T(k) = \left(J^i(k) - \frac{k^i J^0(k)}{k^0} \right)$$

$$J^i_T(k)^* \, J_{Ti}(k) = \left(J^i(k) - \frac{k^i J^0(k)}{k^0} \right)^* \left(J_i(k) - \frac{k^i J_0(k)}{k^0} \right)$$

$$= \left(J^i(k)^* J_i(k) - \frac{k^i J^0(k)^*}{k^0} J_i(k) \right)$$

$$- \left(J^i(k)^* \frac{k^i J_0(k)}{k^0} - \frac{k^i J^0(k)^*}{k^0} \frac{k^i J_0(k)}{k^0} \right)$$

$$= \left(J^i(k)^* J_i(k) - J^0(k)^* J_0(k)\right)$$

$$= J^\mu(k)^* J_\mu(k)$$

$$J^\mu(k)^* = \int (\mathrm{d}x)\, \mathrm{e}^{-\mathrm{i}kx} J^\mu(x)^*$$

$$J^\mu(k)^* J_\mu(k) \;= J^\mu(-k) J_\mu(k)$$

$$= \int (\mathrm{d}x)\, \mathrm{e}^{-\mathrm{i}kx} J^\mu(x) \int (\mathrm{d}x')\, \mathrm{e}^{-\mathrm{i}kx'} J_\mu(x')$$

$$= \int (\mathrm{d}x)\,(\mathrm{d}x')\, \mathrm{e}^{\mathrm{i}k(x-x')} J^\mu(x) J_\mu(x'),$$

$$U = \mathrm{i}^2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} \left[\mathbf{J}^*_{T2}(k)\, \mathbf{J}_{T1}(k)\right]$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} J^\mu(k)^* J_\mu(k)$$

$$= \mathrm{i} \int (\mathrm{d}x)\,(\mathrm{d}x')\, \mathrm{e}^{\mathrm{i}k(x-x')} J^\mu(x)\,(\mathrm{i}) \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} J_\mu(x')$$

$$= \mathrm{i} \int (\mathrm{d}x)\,(\mathrm{d}x') J^\mu(x)\,(\mathrm{i}) \int \frac{\mathrm{d}^3k\, \mathrm{e}^{\mathrm{i}k(x-x')}}{(2\pi)^3\, 2k^0} J_\mu(x')$$

$$U \;= \mathrm{i} \int (\mathrm{d}x)\,(\mathrm{d}x') J^\mu(x)\, D_+(x-x')\, J_\mu(x') \tag{4.2.18}$$

where for $x^0 > x'^0$

$$D_+(x-x') = \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}\mathrm{e}^{\mathrm{i}k(x-x')}}{(2\pi)^3 2k^0}, \tag{4.2.19}$$

$x = (x^0, \mathbf{x})\,, k^0 = |\mathbf{k}|$. It is easily checked that $\mathbf{j}(x)$ satisfies the following two important relations:

$$\mathbf{n}\cdot\mathbf{j}(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}} \times \left[\mathbf{n}\cdot\mathbf{J}_T(k) - \overbrace{\left(\frac{\mathbf{n}\cdot\mathbf{k}+k^0\mathbf{n}\cdot\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)}^{=1}\mathbf{n}\cdot\mathbf{J}_T\left(k\right)\right]\mathrm{e}^{\mathrm{i}kx}. \tag{4.2.20}$$

Hence

$$\mathbf{n}\cdot\mathbf{j}(x) = 0, \tag{4.2.21}$$

and as shown below,

$$\int \mathrm{d}^3\mathbf{x}\left|\mathbf{j}\left(x\right)\right|^2 = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0}\left|\mathbf{J}_T\left(k\right)\right|^2 > 0. \tag{4.2.22}$$

To prove (4.2.22) consider

$$\int \mathrm{d}^3\mathbf{x}\left|\mathbf{j}\left(x\right)\right|^2 = \int \mathrm{d}^3\mathbf{x}\left(\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\left[\mathbf{J}_T\left(k\right) - \left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_T\left(k\right)\right]\mathrm{e}^{\mathrm{i}kx}\right)^*$$

$$\times\left(\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\left[\mathbf{J}_T\left(k'\right) - \left(\frac{k'+k'^0\mathbf{n}}{k'^0+\mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_T\left(k'\right)\right]\mathrm{e}^{\mathrm{i}k'x}\right)$$

$$= \int \mathrm{d}^3\mathbf{x}\left(\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\left[\mathbf{J}^*_T\left(k\right) - \left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}^*_T\left(k\right)\right]\mathrm{e}^{-\mathrm{i}kx}\right)$$

$$\times\left(\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\left[\mathbf{J}_T(k')-\left(\frac{k'+k'^0n}{k'^0+\mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_T(k')\right]\mathrm{e}^{\mathrm{i}k'x}\right)$$

$$=\underbrace{\int\mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}x(\mathbf{k}-\mathbf{k}')}}_{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}\left(\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2k^0}}\left[\mathbf{J}_T^*(k)-\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_T^*(k)\right]\right)$$

$$\times\left(\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3\sqrt{2k'^0}}\left[\mathbf{J}_T(k')-\left(\frac{\mathbf{k}'+k'^0\mathbf{n}}{k'^0+\mathbf{n}\cdot\mathbf{k}'}\right)\mathbf{n}\cdot\mathbf{J}_T(k')\right]\right)$$

$$=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2k^0}\left[\mathbf{J}_T^*(k)-\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_T^*(k)\right]$$

$$\times\left[\mathbf{J}_T(k)-\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_T(k)\right]$$

$$=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2k^0}\left(\mathbf{J}_T^*(k)\,\mathbf{J}_T(k)-\mathbf{J}_T(k)\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_T^*(k)\right.$$

$$-\mathbf{J}_T^*(k)\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_T(k)+\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)$$

$$\left.\times\,\mathbf{n}\cdot\mathbf{J}_T^*(k)\left(\frac{\mathbf{k}+k^0\mathbf{n}}{k^0+\mathbf{n}\cdot\mathbf{k}}\right)\mathbf{n}\cdot\mathbf{J}_T(k)\right)$$

$$=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2k^0}\left|\mathbf{J}_T(k)\right|^2>0. \tag{4.2.23}$$

The unit vector $\mathbf{n}$ is arbitrary, and one choice for this orientation vector arises in the following. If, for example, the current J_1 emits photons with a nonzero average direction of propagation, then we may choose $\mathbf{n}$ to be in the direction of the three-vector

$$\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2k^0} \left(\frac{\mathbf{k}}{k^0}\right) |\mathbf{J}_{1T}(k)|^2, \tag{4.2.24}$$

defining an average direction of propagation of photon emitted by J_1. Let $\mathbf{J}_T^{\perp}$ perpendicular to $\mathbf{n}$ in the $(\mathbf{J}_T, \mathbf{n})$ plane. We may write

$$\mathbf{n} \cdot \mathbf{J}_T = -\frac{\mathbf{k} \cdot \mathbf{J}_T^{\perp}}{\mathbf{n} \cdot \mathbf{k}}, \tag{4.2.25}$$

since $\mathbf{k} \cdot \mathbf{J}_T = 0$. It is then easily checked that a value $\mathbf{k} = -k^0\mathbf{n}$ provides only an apparent singularity in (4.2.13), since in this case $\mathbf{k} \cdot \mathbf{J}_T = 0$.

Upon writing in coordinate space a completeness relation

$$\delta^{ij} = n^i n^j + \sum_{\lambda} e^i_{\lambda} e^j_{\lambda}, \tag{4.2.26}$$

where $\lambda = 1, 2$ and $\mathbf{e}_1, \mathbf{e}_2$ are two unit vectors such that

$$\mathbf{e}_{\lambda} \cdot \mathbf{e}_{\lambda'} = \delta_{\lambda\lambda'},\ \mathbf{n} \cdot \mathbf{e}_{\lambda} = 0, \tag{4.2.27}$$

we may write

$$\mathbf{j}_2(x)^* \cdot \mathbf{j}_1(x) = \sum_{\lambda} a_{2\lambda}(x)^* a_{1\lambda}(x), \tag{4.2.28}$$

where

$$a_{\lambda}(x) = \mathbf{e}_{\lambda} \cdot \mathbf{j}(x), \tag{4.2.29}$$

defining only two degrees of freedom for the photon excitations. We introduce a convenient discrete space variable notation (a lattice) by introducing in the process the notation (Manoukian, 1984, 1985; Schwinger, 1951, 1953, 1954, 1970, 1972, 1973,

1977) $\sigma \equiv (\lambda, x)$:

$$a_\sigma = \sqrt{\mathrm{d}^3\mathbf{x}} a_\lambda (x) \,, \tag{4.2.30}$$

at a fixed time x^0 For any nonnegative integer N , let $N_{\sigma 1}, N_{\sigma 2}, ...,$ denote the number of photon excitations at lattice sites and degree of freedom respectively, where $N = N_{\sigma 1} + N_{\sigma 2} + ...$ The expression in the square brackets in (4.2.13) may be then simply rewritten as

$$\sum_\sigma \mathrm{i}a^*_{2\sigma}\mathrm{i}a_{1\sigma}. \tag{4.2.31}$$

To prove (4.2.30), consider

$$\begin{aligned}\int \mathrm{d}^3\mathbf{x}\mathrm{i}\mathbf{j}_2 (x)^* \cdot \mathrm{i}\mathbf{j}_1 (x) &= \int \mathrm{d}^3\mathbf{x}\mathrm{i}\mathbf{j}_2 (x)^* \cdot \mathrm{i}\mathbf{j}_1 (x) \\ &= \int \mathrm{d}^3\mathbf{x} \, (\mathrm{i}) \, (\mathrm{i}) \sum_\lambda a_{2\lambda} (x)^* a_{1\lambda} (x) \\ &= \int \sqrt{\mathrm{d}^3\mathbf{x}}\sqrt{\mathrm{d}^3\mathbf{x}} \sum_\lambda \mathrm{i}a_{2\lambda} (x)^* \mathrm{i}a_{1\lambda} (x) \\ &= \sum_\sigma \mathrm{i}a^*_{2\sigma}\mathrm{i}a_{1\sigma},\end{aligned} \tag{4.2.32}$$

where $a_\sigma = \sqrt{\mathrm{d}^3 x} a_\lambda (x)$. We carry out unitarity expression of $\langle 0_+ \,|\, 0_- \rangle^J$ in configuration space:

$$\sum_N \sum_{N_{\sigma_1}+N_{\sigma_2}+...=N} \left\langle 0_+ | N; N_{\sigma_1}, N_{\sigma_2}, ..., x^0 \right\rangle^{J_2} \times \left\langle N; N_{\sigma_1}, N_{\sigma_2}, ..., x^0 | 0_- \right\rangle^{J_1}, \tag{4.2.33}$$

$$\left\langle N; N_{\sigma_1}, N_{\sigma_2}, ..., x^0|0_-\right\rangle^{J_1} \tag{4.2.34}$$

is the amplitude that the current J_1 emits N photon excitations, N_{σ_1} of which are found at a lattice site and of degree of freedom σ_1, N_{σ_2} of which are found at a lattice site and of degree of freedom σ_2 , and so on, at a time x^0 after the current J_1 ceases to operate. Similarly, $\langle 0_+|N; N_{\sigma_1}, N_{\sigma_2}, ..., x^0\rangle^{J_2}$ denotes the amplitude that N photon excitations are absorbed by J_2, N_{σ_1} of which were found at a lattice site and of degree of freedom σ_1, and so on, at a time x^0 before J_2 was switched on.

Upon using (4.2.29) and making a standard comparision (Manoukian, 1984, 1985; Schwinger, 1951, 1953, 1954, 1970, 1972, 1973, 1977) of the unitarity expansion in (4.2.30) with the expression for $\langle 0_+\,|\,0_-\rangle^J$ in (4.2.12), we may infer that

$$\left\langle N; N_{\sigma_1}, N_{\sigma_2}, ..., x^0|0_-\right\rangle^J = \langle 0_+|0_-\rangle^J \frac{(\mathrm{i}a_{\sigma_1})^{N_{\sigma_1}}}{\sqrt{N_{\sigma_1}!}} \frac{(\mathrm{i}a_{\sigma_2})^{N_{\sigma_2}}}{\sqrt{N_{\sigma_2}!}} ..., \tag{4.2.35}$$

$$\left\langle 0_+|N; N_{\sigma_1}, N_{\sigma_2}, ..., x^0\right\rangle^J = \langle 0_+|0_-\rangle^J \frac{(\mathrm{i}a^*_{\sigma_1})^{N_{\sigma_1}}}{\sqrt{N_{\sigma_1}!}} \frac{(\mathrm{i}a^*_{\sigma_2})^{N_{\sigma_2}}}{\sqrt{N_{\sigma_2}!}} ..., \tag{4.2.36}$$

for a given current J. Because of the indispensable property (4.2.20), one verifies from (4.2.32) the completeness relation in configuration space (valid for all n):

$$\sum_{N=0}^{\infty} \sum_{N_{\sigma_1}+N_{\sigma_2}+...=N} \left|\left\langle N : N_{\sigma_1}, N_{\sigma_2}, ..., x^0|0_-\right\rangle^J\right|^2 = 1, \tag{4.2.37}$$

verifying the consistency of the analysis (A similar treatment for $\langle 0_+|N; N_{\sigma_1}, N_{\sigma_2}, ..., x^0\rangle^{J_2}$ can be given.). It is precisely this point that has led us to the configuration-space analysis of photons.

The probability that a current J emits N photon excitations, which at a time x^0 after the current ceases to operate are found to be localized in a region $\triangle$ (such as with in a cone), N_1 with degrees of freedom $\lambda = 1$, and N_2 with degrees of freedom $\lambda = 2$, may be then directly inferred from our earlier analysis (see Manoukian, 1984) to be

$$\frac{\left(\int\limits_{\triangle} \mathrm{d}^3\mathbf{x}\, |a_1(x)|^2\right)^{N_1}}{N_1!} \frac{\left(\int\limits_{\triangle} \mathrm{d}^3\mathbf{x}\, |a_2(x)|^2\right)^{N_2}}{N_2!} \times \exp\left[-\int \mathrm{d}^3\mathbf{x}\, |\mathbf{j}(x)|^2\right]. \qquad (4.2.38)$$

and is obviously time x^0 dependent, where $N = N_1 + N_2$. In particular, for unpolarized photon excitations, the probability to find the N excitations in $\triangle$ at time x^0 is

$$\frac{\left(\int\limits_{\triangle} \mathrm{d}^3\mathbf{x}\, |\mathbf{j}(x)|^2\right)^{N}}{N!} \exp\left[-\int \mathrm{d}^3\mathbf{x}\, |\mathbf{j}(x)|^2\right]. \qquad (4.2.39)$$

Only when $\triangle$ do we obtain a Poisson distribution:

$$\frac{\left(\int \mathrm{d}^3\mathbf{x}\, |\mathbf{j}(x)|^2\right)}{N!} \exp\left[-\int \mathrm{d}^3\mathbf{x}\, |\mathbf{j}(x)|^2\right]. \qquad (4.2.40)$$

and the latter is time-independent.

The probability density that a current J emits one photon excitation and the latter is found at time-space coordinate $x = (x^0, \mathbf{x})$ with a degree of freedom after ceases to operate is

$$|a_\lambda(x)|^2 \exp\left[-\int \mathrm{d}^3\mathbf{x}\, |\mathbf{j}(x)|^2\right]. \qquad (4.2.41)$$

Hence, given that a current has emitted one photon excitation the (condition) probability it is found at time-space coordinate $x = (x^0, \mathbf{x})$ with a degree of freedom λ is $|\phi_\lambda(x)|^2$, where

$$\phi_\lambda(x) = \frac{a_\lambda(x)}{\left[\int d^3\mathbf{x}\, |\mathbf{j}(x)|^2\right]^{\frac{1}{2}}}, \qquad (4.2.42)$$

denotes the corresponding amplitude.

How we translate $\phi_\lambda(x)$ forward in time, consistent with the completeness relation (4.2.35) and hence with the fundamental probabilistic interpretation and in particular find the amplitude that a photon excitation travels from one time-space coordinate to another, is the subject of the next section.

4.3 Amplitude of Propagation From One Time-Space Coordinate to Another and Associated Probabilities of Detection

Upon using the relation

$$(-\mathrm{i})D_+(x-x') = \int \mathrm{d}^3\mathbf{x}_1\mathrm{d}^3\mathbf{x}_2\mathrm{d}^3\mathbf{y}D(x-x_2)D(x_2-y)$$

$$\times\, \mathrm{i}\frac{\overleftrightarrow{\partial}}{\partial y^0}D(y-x_1)D(x_1-x'), \qquad (4.3.1)$$

where $x^0 > x_2^0 > y^0 > x_1^0 > x'^0$ and $\overleftrightarrow{\partial}/\partial y^0 = \overrightarrow{\partial}/\partial y^0 - \overleftarrow{\partial}/\partial y^0$, we may write in reference to (4.2.12)

$$\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')\mathbf{J}_T^2(x)D_+(x-x')\cdot\mathbf{J}_T^1(x') = \sum_{\lambda_1,\lambda_2}\int \mathrm{d}^3\mathbf{x}_1\mathrm{d}^3\mathbf{x}_2\mathrm{i}a_{\lambda_2}^2(x_2)^*$$

$$\times\, \tilde{\delta}_{\lambda_2\lambda_1}(x_2-x_1)\mathrm{i}a_{\lambda_1}^1(x_1), \qquad (4.3.2)$$

where

$$\tilde{\delta}_{\lambda_2\lambda_1}(x_2-x_1) = \delta_{\lambda_2\lambda_1}\int \mathrm{d}^3\mathbf{y}\left[D(x_2-y)\mathrm{i}\frac{\overleftrightarrow{\partial}}{\partial y^0}D(y-x_1)\right]. \qquad (4.3.3)$$

From Eqs. (4.2.11), (4.2.25) and (4.3.2) we infer the *rule* for translating $\phi_{\lambda_1}(x_1)$ forward in time to x_2^0, consistent with the completeness relation (4.2.29), to be given by

$$\int \mathrm{d}^3\mathbf{x}_1\tilde{\delta}_{\lambda_2\lambda_1}(x_2-x_1)\phi_{\lambda_1}(x_1), \qquad (4.3.4)$$

coinciding with $\phi_{\lambda_2}(x_1)$ as expected. In particular this also shows that the amplitude that a photon excitation travels from time-space coordinate $x_1 = (x_1^0, \mathbf{x}_1)$ to time-space coordinate $x_2 = (x_2^0, \mathbf{x}_2)$ is given by (4.3.3). The latter is explicitly worked out to be

$$\tilde{\delta}_{\lambda_2\lambda_1}(x_2 - x_1) = \frac{\mathrm{i}\delta_{\lambda_2\lambda_1}}{\pi^2}\left(\frac{x_2^0 - x_1^0}{[(x_2 - x_1)^2]^2}\right), \tag{4.3.5}$$

which may be rigorously rewritten using the Schwinger representation (see (5.2.52), (6.0.4)), and does *not* coincide with the so-called Feynman propagator as one might guess.

Consider a fixed value for the degree of freedom λ of a photon excitation, say, $\lambda = 1$. Suppose that at time $x'^0 = 0$, the photon excitation is initially created in a region of space described by the initial configuration$(r' = |\mathbf{x}'|)$

$$\phi_1(x') = \frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4} \exp\left(\frac{-r'^2}{2\sigma^2}\right), \tag{4.3.6}$$

where σ is a scale parameter and may be taken for example to denote the Bohr radius (see below) if the photon excitation is emitted from a hydrogen-atom site. We note that $\phi_1(x')$ is properly normalized:

$$\int \mathrm{d}^3\mathbf{x}'|\phi_1(x')|^2 = 1. \tag{4.3.7}$$

The amplitude to find the photon excitation at time-space coordinate $x = (x^0, \mathbf{x})$ is then given from Eqs. (4.3.4) and (4.3.5) as $(x'^0 = 0)$

$$\phi_1(x) = \frac{\mathrm{i}}{\pi^2}\int \mathrm{d}^3\mathbf{x}\tilde{\delta}_{11}(x - x')\phi_1(x). \tag{4.3.8}$$

The evaluation of (4.3.8) is tedious and the details are given in the appendix. It is explicitly given by

$$\phi_1(x) = \frac{1}{2r\sigma^{1/2}} \left(\frac{1}{\pi}\right)^{3/4} [A(x) + \mathrm{i}B(x)], \tag{4.3.9}$$

where

$$A(x) = z \exp\left(-\frac{z^2}{2}\right) - (z+\lambda) \exp\left(-\frac{(z+\lambda)^2}{2}\right), \tag{4.3.10}$$

$$B(x) = \sqrt{\frac{2\pi}{z^2}} \int_0^1 \mathrm{d}u \exp\left[-\frac{z^2}{2}(1-u^2)\right] - \sqrt{\frac{2\pi}{(z+\lambda)^2}}$$

$$\times \int_0^1 \mathrm{d}u \exp\left[-\frac{(z^2+\lambda)^2}{2}(1-u^2)\right], \tag{4.3.11}$$

$$z = \frac{r - ct}{\sigma} \tag{4.3.12}$$

$$\lambda = \frac{2ct}{\sigma} \tag{4.3.13}$$

For $t = 0$, $\lambda = 0$, we have $B(x) = 0$, and we check that $\phi_1(x)$ reduces to the expression in (4.3.6). Also $\phi_1(x)$ is properly normalized, that is,

$$\int \mathrm{d}^3\mathbf{x} |\phi_1(x)|^2 = 1. \tag{4.3.14}$$

for *all* x^0. The probability density of finding the photon excitation at a radial distance r at time x^0 is then

$$f(r) = \frac{1}{\sqrt{\pi}\sigma} \left[|A(x)|^2 + |B(x)|^2\right] \tag{4.3.15}$$

where

$$\int_{0}^{\infty} \mathrm{d}r f(r) = 1. \tag{4.3.16}$$

We also introduce the general probability expression

$$\text{Prob}[z_1 \leq z \leq z_2] = \sigma \int_{z_1}^{z_2} \mathrm{d}z f(r), \tag{4.3.17}$$

where

$$-\frac{\lambda}{2} \leq z_1 < z_2 < \infty, \tag{4.3.18}$$

where z and λ are defined in Eqs. (4.3.12) and (4.3.13), respectively.

4.4 The Half–Space Description of photon propagation in space-time

Consider the expression

$$\phi\left(x'\right) = \int_{-\infty}^{\infty} \mathrm{d}^2\mathbf{x}_{||} \int_{0}^{\infty} \mathrm{d}z D_{>}\left(x', x\right)\phi\left(x\right), \tag{4.4.1}$$

$$D_{>}\left(x', x\right) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^2\mathbf{Q}_{||}}{(2\pi)^2}\frac{\mathrm{d}q}{(2\pi)} \mathrm{e}^{\mathrm{i}\mathbf{Q}_{||}\cdot(\mathbf{x}'-\mathbf{x})}\left[\mathrm{e}^{\mathrm{i}q(z'-z)} - \mathrm{e}^{-\mathrm{i}q(z'+z)}\right]\mathrm{e}^{-\mathrm{i}Q^0 T}, \tag{4.4.2}$$

where

$$T = \left(x'^0 - x^0\right) = x'^0; x^0 = 0 \tag{4.4.3}$$

$$\phi\left(x\right)=\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}},\tag{4.4.4}$$

$$\phi\left(x'\right)=\int\limits_{-\infty}^{\infty}\mathrm{d}^2x_{||}\int\limits_{0}^{\infty}\mathrm{d}z\Bigg[\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}^2\mathbf{Q}_{||}}{(2\pi)^2}\frac{dq}{(2\pi)}\mathrm{e}^{\mathrm{i}\mathbf{Q}_{||}\cdot(\mathbf{x}'-\mathbf{x})}\left[\mathrm{e}^{\mathrm{i}q(z'-z)}-\mathrm{e}^{-\mathrm{i}q(z'+z)}\right]\mathrm{e}^{-\mathrm{i}Q^0T}\Bigg]$$

$$\times\left[\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}}\right].\tag{4.4.5}$$

Or

$$\phi\left(x'\right)\equiv\phi_1\left(x'\right)-\phi_2\left(x'\right)$$

$$=\int\limits_{-\infty}^{\infty}\mathrm{d}^2\mathbf{x}_{||}\int\limits_{0}^{\infty}\mathrm{d}z\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}^2\mathbf{Q}_{||}}{(2\pi)^2}\frac{\mathrm{d}q}{(2\pi)}\mathrm{e}^{\mathrm{i}\mathbf{Q}_{||}\cdot(\mathbf{x}'-\mathbf{x})}\mathrm{e}^{\mathrm{i}q(z'-z)}\mathrm{e}^{-\mathrm{i}Q^0T}\left[\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}}\right]$$

$$-\int\limits_{-\infty}^{\infty}\mathrm{d}^2\mathbf{x}_{||}\int\limits_{0}^{\infty}\mathrm{d}z\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}^2\mathbf{Q}_{||}}{(2\pi)^2}\frac{\mathrm{d}q}{(2\pi)}\mathrm{e}^{\mathrm{i}\mathbf{Q}_{||}\cdot(\mathbf{x}'-\mathbf{x})}\mathrm{e}^{-\mathrm{i}q(z'+z)}\mathrm{e}^{-\mathrm{i}Q^0T}$$

$$\times\left[\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}}\right],\tag{4.4.6}$$

with

$$\phi_1\left(x'\right)=\frac{1}{2}\int\limits_{-\infty}^{\infty}\mathrm{d}^3\mathbf{x}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot(\mathbf{R}_1'-\mathbf{x})}\mathrm{e}^{-\mathrm{i}Q^0T}\left[\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}}\right],\tag{4.4.7}$$

where

$$\mathbf{R}_1'=\left(\mathbf{x}_{||}',z'\right),\quad\mathbf{x}=\left(\mathbf{x}_{||},z\right).$$

Similarly for $\phi_2(x')$ we will get,

$$\phi_2(x') = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \int_{-\infty}^{\infty} \frac{\mathrm{d}^3 Q}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot(\mathbf{R}_2'-\mathbf{x})} \mathrm{e}^{-\mathrm{i}Q^0 T} \left[\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4} \mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}}\right], \qquad (4.4.8)$$

where

$$\mathbf{R}_2' = \left(\mathbf{x}_{||}', -z'\right), \quad \mathbf{x} = \left(\mathbf{x}_{||}, z\right).$$

Considering only the integral over $\mathbf{x}$ in equation (4.4.7)

$$\int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot(\mathbf{R}_1'-\mathbf{x})} \mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}} = \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1' - \mathrm{i}\mathbf{Q}\cdot\mathbf{x}} \mathrm{e}^{-\frac{\left(x^2+2\mathbf{x}\cdot\mathbf{a}-a^2\right)}{2\sigma^2}}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \mathrm{e}^{-\frac{1}{2\sigma^2}\left(x^2-2\mathbf{x}\cdot\mathbf{a}+2\sigma^2\mathrm{i}\mathbf{Q}\cdot\mathbf{x}\right)} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \mathrm{e}^{-\frac{1}{2\sigma^2}\left(x^2+2\mathbf{x}\cdot(\sigma^2\mathrm{i}\mathbf{Q}-a)\right)} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \mathrm{e}^{-\frac{1}{2\sigma^2}\left(x^2+2\mathbf{x}\cdot(\sigma^2\mathrm{i}\mathbf{Q}-a)+(\sigma^2\mathrm{i}Q-a)^2\right)} \mathrm{e}^{\frac{1}{2\sigma^2}(\sigma^2\mathrm{i}Q-a)^2} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \mathrm{e}^{-\frac{1}{2\sigma^2}\left(x+(\sigma^2\mathrm{i}Q-a)\right)^2} \mathrm{e}^{\frac{1}{2\sigma^2}(\sigma^2\mathrm{i}Q-a)^2} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'}. \qquad (4.4.9)$$

$$\int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot(\mathbf{R}_1'-\mathbf{x})} \mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$= \int_{-\infty}^{\infty} 2\pi x^2 \mathrm{d}x \int_{-1}^{1} \mathrm{d}\cos\theta \mathrm{e}^{-\frac{1}{2\sigma^2}\left(x+(\sigma^2 \mathrm{i}Q-a)\right)^2} \mathrm{e}^{\frac{1}{2\sigma^2}(\sigma^2 \mathrm{i}Q-a)^2}$$

$$\times \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'}. \tag{4.4.10}$$

By using the formula

$$\int_0^{2\pi} x^{2n} \mathrm{e}^{-px^2} \mathrm{d}x = \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}};\ p > 0,\ n = 0, 1, 2, ... \tag{4.4.11}$$

(4.4.10) becomes

$$= 4\pi \frac{1}{2(2\frac{1}{2\sigma^2})} \sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}} \mathrm{e}^{\frac{1}{2\sigma^2}(\sigma^2 \mathrm{i}Q-a)^2} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'},$$

$$= 2\pi\sigma^3 \sqrt{2\pi} \mathrm{e}^{\frac{1}{2\sigma^2}(\sigma^2 \mathrm{i}Q-a)^2} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'}, \tag{4.4.12}$$

and from (4.4.7) we get

$$\phi_1(x') = \frac{1}{2} \frac{1}{\sigma^{3/2}} \left(\frac{1}{\pi}\right)^{3/4} \left(2\pi\sigma^3 \sqrt{2\pi}\right)$$

$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} \mathrm{e}^{-\frac{\sigma^2}{2}(Q^2 + 2\mathbf{Q}\cdot\frac{\mathrm{i}\mathbf{a}}{\sigma^2} - \frac{a^2}{\sigma^4})} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'} \mathrm{e}^{-\mathrm{i}Q^0 T}$$

$$= \frac{1}{2} \frac{1}{\sigma^{3/2}} \left(\frac{1}{\pi}\right)^{3/4} \left(2\pi\sigma^3 \sqrt{2\pi}\right)$$

$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3} \mathrm{e}^{-\frac{\sigma^2 Q^2}{2}} \mathrm{e}^{-\frac{\sigma^2}{2} 2\mathbf{Q}\cdot\frac{\mathrm{i}\mathbf{a}}{\sigma^2}} \mathrm{e}^{\frac{a^2}{2\sigma^2}} \mathrm{e}^{-\frac{a^2}{2\sigma^2}} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'} \mathrm{e}^{-\mathrm{i}Q^0 T}$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3}\mathrm{e}^{-\frac{\sigma^2Q^2}{2}}\mathrm{e}^{-\mathrm{i}\mathbf{Q}\cdot\mathbf{a}}\mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}_1'}\mathrm{e}^{-\mathrm{i}Q^0T}$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\int_{-\infty}^{\infty}\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3}\mathrm{e}^{-\frac{\sigma^2Q^2}{2}}\mathrm{e}^{-\mathrm{i}\mathbf{Q}\cdot(\mathbf{a}-\mathbf{R}_1')}\mathrm{e}^{-\mathrm{i}Q^0T}$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)$$

$$\times\int_{0}^{\infty}\frac{2\pi\mathrm{Q}^2\mathrm{d}Q}{(2\pi)^3}\int_{-1}^{1}\mathrm{dcos}\theta\mathrm{e}^{-\frac{\sigma^2Q^2}{2}}\mathrm{e}^{-\mathrm{iQ}\left|R_1'-a\right|\cos\theta}\mathrm{e}^{-\mathrm{i}Q^0x'^0},$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)$$

$$\times\int_{0}^{\infty}\frac{2\pi\mathrm{Q}^2\mathrm{d}Q}{(2\pi)^3}\mathrm{e}^{-\frac{\sigma^2Q^2}{2}}\frac{1}{\mathrm{iQ}\left|R_1'-a\right|}\left[\mathrm{e}^{\mathrm{iQ}\left|R_1'-a\right|}-\mathrm{e}^{-\mathrm{iQ}\left|\mathbf{R_1'}-\mathbf{a}\right|}\right]\mathrm{e}^{-\mathrm{i}Q^0x'^0}$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)$$

$$\times\int_{0}^{\infty}\frac{2\pi\mathrm{Q}\mathrm{d}Q}{(2\pi)^3}\mathrm{e}^{-\frac{\sigma^2Q^2}{2}}\frac{1}{\mathrm{i}\left|\mathbf{R_1'}-\mathbf{a}\right|}\left[\mathrm{e}^{\mathrm{i}\left(\mathrm{Q}\left|\mathbf{R_1'}-\mathbf{a}\right|-Q^0x'^0\right)}-\mathrm{e}^{-\mathrm{i}\left(\mathrm{Q}\left|\mathbf{R_1'}-\mathbf{a}\right|+Q^0x'^0\right)}\right],$$

(4.4.13)

by letting $\sigma Q = v \to Q = v/\sigma, \quad \mathrm{d}Q = \mathrm{d}v/\sigma,$ (4.4.13) becomes

$$\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\frac{1}{(2\pi)^3}$$

$$\times\int_0^\infty 2\pi\frac{v}{\sigma}\frac{\mathrm{d}v}{\sigma}\mathrm{e}^{-\frac{v^2}{2}}\frac{1}{\mathrm{i}\left|\mathbf{R_1'}-\mathbf{a}\right|}\left[\mathrm{e}^{\mathrm{i}\left(\frac{v}{\sigma}\left|\mathbf{R_1'}-\mathbf{a}\right|-Q^0x'^0\right)} - \mathrm{e}^{-\mathrm{i}\left(\frac{v}{\sigma}\left|\mathbf{R_1'}-\mathbf{a}\right|+Q^0x'^0\right)}\right]$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\frac{1}{(2\pi)^3}$$

$$\times\int_0^\infty 2\pi\frac{v}{\sigma}\frac{\mathrm{d}v}{\sigma}\mathrm{e}^{-\frac{v^2}{2}}\frac{1}{\mathrm{i}\left|\mathbf{R_1'}-\mathbf{a}\right|}\left[\mathrm{e}^{\mathrm{i}v\left(\frac{\left|\mathbf{R_1'}-\mathbf{a}\right|}{\sigma}-\frac{Q^0x'^0}{v}\right)} - \mathrm{e}^{-\mathrm{i}v\left(\frac{\left|\mathbf{R_1'}-\mathbf{a}\right|}{\sigma}+\frac{Q^0x'^0}{v}\right)}\right], \quad (4.4.14)$$

and by letting

$$\omega_1 = \frac{\left|\mathbf{R_1'}-\mathbf{a}\right|}{\sigma} - \frac{Q^0x'^0}{v}$$

$$= \frac{|r_1|}{\sigma} - \frac{Qt}{Q\sigma}$$

$$= \frac{r_1}{\sigma} - \frac{\lambda}{2}; \qquad \lambda = \frac{2t}{\sigma} \quad (4.4.15)$$

then (4.4.14) becomes

$$\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\frac{1}{(2\pi)^3}\frac{1}{\sigma^2}$$

$$\times\int_0^\infty 2\pi v\mathrm{d}v\,\mathrm{e}^{-\frac{v^2}{2}}\frac{1}{\mathrm{i}r_1}\left[\mathrm{e}^{\mathrm{i}v\left(\frac{r_1}{\sigma}-\frac{\lambda}{2}\right)} - \mathrm{e}^{-\mathrm{i}v\left(\frac{r_1}{\sigma}+\frac{\lambda}{2}\right)}\right]$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\frac{1}{(2\pi)^3}\frac{1}{\sigma^2}$$

$$\times\int_0^\infty 2\pi v \mathrm{d}v\, \mathrm{e}^{-\frac{v^2}{2}}\frac{1}{\mathrm{i}r_1}\left[\mathrm{e}^{\mathrm{i}v\omega_1} - \mathrm{e}^{-\mathrm{i}v(\omega_1+\lambda)}\right]$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\frac{1}{(2\pi)^3}\frac{1}{\sigma^2}$$

$$\times\int_0^\infty 2\pi v \mathrm{d}v\, \mathrm{e}^{-\frac{v^2}{2}}\frac{1}{\mathrm{i}r_1}\left[\mathrm{e}^{\mathrm{i}v\left(\frac{r_1}{\sigma}-\frac{\lambda}{2}\right)} - \mathrm{e}^{-\mathrm{i}v\left(\frac{r_1}{\sigma}+\frac{\lambda}{2}\right)}\right]$$

$$=\frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}\left(2\pi\sigma^3\sqrt{2\pi}\right)\frac{1}{(2\pi)^3}\frac{1}{\sigma^2}$$

$$\times\int_0^\infty 2\pi v \mathrm{d}v \mathrm{e}^{-\frac{v^2}{2}}\frac{1}{\mathrm{i}r_1}\left[\mathrm{e}^{\mathrm{i}v\omega_1} - \mathrm{e}^{-\mathrm{i}v(\omega_1+\lambda)}\right]$$

$$=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma^{1/2}}\left(\frac{1}{\pi}\right)^{3/4}\int_0^\infty v\mathrm{d}v\mathrm{e}^{-\frac{v^2}{2}}$$

$$\times\frac{1}{r_1}\left[\left(\sin v\omega_1+\sin v\left(\omega_1+\lambda\right)\right)-\mathrm{i}\left(\cos v\omega_1-\cos v\left(\omega_1+\lambda\right)\right)\right]. \tag{4.4.16}$$

so we can write $\phi_1(x')$ as

$$\phi_1\left(x'\right)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma^{1/2}}\left(\frac{1}{\pi}\right)^{3/4}\int\limits_0^\infty v\mathrm{d}v\,\mathrm{e}^{-\frac{v^2}{2}}\frac{1}{r_1}\left[U_1(\omega_1)-\mathrm{i}V_1(\omega_1)\right]$$

$$U_1(\omega_1)=\int\limits_0^\infty v\mathrm{d}v\,\mathrm{e}^{-\frac{v^2}{2}}\left(\sin v\omega_1+\sin v\left(\omega_1+\lambda\right)\right)$$

$$V_1(\omega_1)=\int\limits_0^\infty v\mathrm{d}v\,\mathrm{e}^{-\frac{v^2}{2}}\left(\cos v\omega_1-\cos v\left(\omega_1+\lambda\right)\right), \tag{4.4.17}$$

To solve for $U_1\left(\omega_1\right)$ and $V_1\left(\omega_1\right)$, consider the integrals

$$I\left(\omega_1,b\right)=\int\limits_0^\infty \mathrm{d}\nu\cos\nu\omega_1\mathrm{e}^{-\nu^2 b}=\sqrt{\frac{\pi}{4b}}\mathrm{e}^{-\omega_1^2/4\mathrm{b}}$$

$$I\left(\omega_1,\frac{1}{2}\right)=\int\limits_0^\infty \mathrm{d}\nu\cos\nu\omega_1\mathrm{e}^{-\nu^2\frac{1}{2}}=\sqrt{\frac{\pi}{2}}\mathrm{e}^{-\omega_1^2/2}$$

$$-\frac{\partial}{\partial\omega_1}I\left(\omega_1,\frac{1}{2}\right)=\int\limits_0^\infty \mathrm{d}\nu\sin\nu\omega_1\mathrm{e}^{-\nu^2\frac{1}{2}}=-\frac{\partial}{\partial\omega_1}\sqrt{\frac{\pi}{2}}\mathrm{e}^{-\omega_1^2/2},$$

$$-\frac{\partial}{\partial\omega_1}I\left(\omega_1,\frac{1}{2}\right)=-\frac{\partial}{\partial\omega_1}\sqrt{\frac{\pi}{2}}\mathrm{e}^{-\omega_1^2/2}=\sqrt{\frac{\pi}{2}}\left(\omega_1\right)\mathrm{e}^{-\omega_1^2/2}$$

$$I\left(\omega_1+\lambda,\frac{1}{2}\right)=\int\limits_0^\infty d\nu\cos\nu\left(\omega_1+\lambda\right)\mathrm{e}^{-\nu^2\frac{1}{2}}=\sqrt{\frac{\pi}{2}}\mathrm{e}^{-(\omega_1+\lambda)^2/2}$$

$$-\frac{\partial}{\partial\left(\omega_1+\lambda\right)}I\left(\omega_1+\lambda,\frac{1}{2}\right)=-\frac{\partial}{\partial\left(\omega_1+\lambda\right)}\sqrt{\frac{\pi}{2}}\mathrm{e}^{-(\omega_1+\lambda)^2/2}$$

$$= \sqrt{\frac{\pi}{2}}\left(\omega_1 + \lambda\right) \mathrm{e}^{-\left(\omega_1+\lambda\right)^2/2}, \tag{4.4.18}$$

$$U_1\left(\omega_1\right) = -\frac{\partial}{\partial\omega_1} I\left(\omega_1, \frac{1}{2}\right) - \frac{\partial}{\partial\left(\omega_1+\lambda\right)} I\left(\omega_1+\lambda, \frac{1}{2}\right)$$

$$= \sqrt{\frac{\pi}{2}}\omega_1 \mathrm{e}^{-z^2/2} + \sqrt{\frac{\pi}{2}}\left(\omega_1+\lambda\right) \mathrm{e}^{-\left(\omega_1+\lambda\right)^2/2}$$

$$= \sqrt{\frac{\pi}{2}}\left(\omega_1 \mathrm{e}^{-\left(\omega_1+\lambda\right)^2/2} + \left(\omega_1+\lambda\right) \mathrm{e}^{-\left(\omega_1+\lambda\right)^2/2}\right)$$

$$U_1(\omega_1) = \sqrt{\frac{\pi}{2}}\left(\omega_1 \mathrm{e}^{-\frac{\omega_1^2}{2}} + \left(\omega_1+\lambda\right) \mathrm{e}^{-\frac{\left(\omega_1+\lambda\right)^2}{2}}\right), \tag{4.4.19}$$

$$V_1(\omega_1) = \int_0^\infty \nu \cos \nu\omega_1 \mathrm{d}\nu\, \mathrm{e}^{-\frac{\nu^2}{2}} - \int_0^\infty \nu \cos \nu\left(\omega_1+\lambda\right) \mathrm{d}\nu\, \mathrm{e}^{-\frac{\nu^2}{2}}$$

$$= V_1'(\omega_1) - V_1''(\omega_1)$$

$$V_1'(\omega_1) = \int_0^\infty \nu \cos \nu\omega_1 \mathrm{d}\nu\, \mathrm{e}^{-\frac{\nu^2}{2}},$$

$$V_1''(\omega_1) = \int_0^\infty \nu \cos \nu\left(\omega_1+\lambda\right) \mathrm{d}\nu\, \mathrm{e}^{-\frac{\nu^2}{2}}, \tag{4.4.20}$$

Using the Gaussian integral,

$$\sqrt{\frac{2}{\pi}}\nu\int_{0}^{\infty}\mathrm{e}^{-\frac{\nu^2y^2}{2}}\mathrm{d}y = 1, \tag{4.4.21}$$

and setting

$$u = \frac{y}{\sqrt{y^2+1}}$$

$$u^2 = \frac{y^2}{y^2+1}$$

$$u^2\left(y^2+1\right) = y^2$$

$$u^2y^2+u^2 = y^2$$

$$\left(u^2-1\right)y^2 = -u^2,$$

$$y^2 = \frac{u^2}{(1-u^2)}$$

$$2y\mathrm{d}y = -\frac{(u^2-1)\,2u\mathrm{d}u - u^2 2u\mathrm{d}u}{(1-u^2)^2}$$

$$= -\frac{(u^2 2u\mathrm{d}u - 2u\mathrm{d}u) - u^2 2u\mathrm{d}u}{(1-u^2)^2}$$

$$2y\mathrm{d}y = \frac{2u\mathrm{d}u}{(1-u^2)^2}$$

$$\mathrm{d}y = \frac{u\mathrm{d}u}{y\,(1-u^2)^2}$$

$$= \frac{u\mathrm{d}u}{\frac{u}{\sqrt{1-u^2}}\left(1-u^2\right)^2}$$

$$= \frac{\mathrm{d}u}{\left(1-u^2\right)^{3/2}}, \tag{4.4.22}$$

we have from (4.4.21)

$$\sqrt{\frac{2}{\pi}}\nu\int_0^1 \mathrm{e}^{-\frac{\nu^2}{2}\frac{u^2}{(1-u^2)}}\frac{\mathrm{d}u}{\left(1-u^2\right)^{3/2}} = 1 \tag{4.4.23}$$

From $V_1'(\omega_1)$ in (4.4.20) we may write

$$V_1'(\omega_1) = \int_0^\infty \nu\cos\nu\omega_1\mathrm{d}\nu\,\mathrm{e}^{-\frac{\nu^2}{2}}\sqrt{\frac{2}{\pi}}\nu\int_0^1 \mathrm{e}^{-\frac{\nu^2}{2}\frac{u^2}{(1-u^2)}}\frac{\mathrm{d}u}{\left(1-u^2\right)^{3/2}}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \frac{\mathrm{d}u}{\left(1-u^2\right)^{3/2}}\int_0^\infty \nu^2\cos\nu\omega_1\mathrm{d}\nu\,\mathrm{e}^{-\frac{\nu^2}{2(1-u^2)}}, \tag{4.4.24}$$

since

$$I\left(\omega_1,b\right) = \int_0^\infty \mathrm{d}\nu\cos\nu\omega_1\mathrm{e}^{-\nu^2 b} = \sqrt{\frac{\pi}{4b}}\mathrm{e}^{-\omega_1^2/4\mathrm{b}}$$

$$\left(\frac{\partial}{\partial\omega_1}\right)^2\int_0^\infty \mathrm{d}\nu\cos\nu\omega_1\mathrm{e}^{-\nu^2 b} = \frac{\partial}{\partial\omega_1}\int_0^\infty d\nu\frac{\partial}{\partial\omega_1}\cos\nu\omega_1\mathrm{e}^{-\nu^2 b}$$

$$= \frac{\partial}{\partial\omega_1}\int_0^\infty \mathrm{d}\nu\left(-\nu\sin\nu\omega_1\right)\mathrm{e}^{-\nu^2 b}$$

$$= -\int_0^\infty \mathrm{d}\nu \left(\nu^2 \cos \nu\omega_1\right) \mathrm{e}^{-\nu^2 b}. \tag{4.4.25}$$

Consider the integral

$$\int_0^\infty \mathrm{d}\nu \left(\nu^2 \cos \nu\omega_1\right) \mathrm{e}^{-\nu^2 b} = -\left(\frac{\partial}{\partial\omega_1}\right)^2 \sqrt{\frac{\pi}{4b}} \mathrm{e}^{-\omega_1^2/4b}$$

$$= -\left(\frac{\partial}{\partial\omega_1}\right) \sqrt{\frac{\pi}{4b}} \mathrm{e}^{-\omega_1^2/4b} \left(-\frac{2\omega_1}{4b}\right)$$

$$= \left(\frac{\partial}{\partial\omega_1}\right) \sqrt{\frac{\pi}{4b}} \mathrm{e}^{-\omega_1^2/4b} \left(\frac{\omega_1}{2b}\right)$$

$$= \sqrt{\frac{\pi}{4b}} \left(\mathrm{e}^{-\omega_1^2/4b} \left(\frac{1}{2b}\right) + \left(\frac{\omega_1}{2b}\right) \mathrm{e}^{-\omega_1^2/4b} \left(-\frac{\omega_1}{2b}\right)\right)$$

$$= \sqrt{\frac{\pi}{4b}} \mathrm{e}^{-\omega_1^2/4b} \frac{1}{2b} \left(1 - \frac{\omega_1^2}{2b}\right), \tag{4.4.26}$$

so we have

$$V_1'(\omega_1) = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{\mathrm{d}u}{(1-u^2)^{3/2}} \int_0^\infty \nu^2 \cos \nu\omega_1 \mathrm{d}\nu\, \mathrm{e}^{-\frac{\nu^2}{2(1-u^2)}}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \frac{\mathrm{d}u}{(1-u^2)^{3/2}} \left\{ -\left(\frac{\partial}{\partial\omega_1}\right)^2 \sqrt{\frac{\pi}{4\frac{1}{2(1-u^2)}}} \mathrm{e}^{-\omega_1^2 / 4\frac{1}{2(1-u^2)}} \right\}$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^1 \frac{\mathrm{d}u}{(1-u^2)^{3/2}} \left(\frac{\partial}{\partial\omega_1}\right)^2 \sqrt{\frac{\pi}{2}(1-u^2)} \mathrm{e}^{-\omega_1^2(1-u^2)/2}, \tag{4.4.27}$$

By using

$$\left(\frac{\partial}{\partial\omega_1}\right)^2\sqrt{\frac{\pi}{4b}}\mathrm{e}^{-\omega_1^2/4b} = -\sqrt{\frac{\pi}{4b}}\mathrm{e}^{-\omega_1^2/4b}\frac{1}{2b}\left(1-\frac{\omega_1^2}{2b}\right). \tag{4.4.28}$$

we get

$$V_1'(\omega_1) = \sqrt{\frac{2}{\pi}}\int_0^1 \frac{\mathrm{d}u}{(1-u^2)^{3/2}}$$

$$\times\sqrt{\frac{\pi}{2}(1-u^2)}\mathrm{e}^{-\omega_1^2\left(1-u^2\right)/2}\left(1-u^2\right)\left(1-\left(1-u^2\right)\omega_1^2\right),$$

$$= \int_0^1 \mathrm{d}u\mathrm{e}^{-\omega_1^2\left(1-u^2\right)/2}\left(1-\omega_1^2+u^2\omega_1^2\right)$$

$$= -\omega_1^2\int_0^1 \mathrm{d}u\mathrm{e}^{-\omega_1^2\left(1-u^2\right)/2} + \int_0^1 \mathrm{d}u\mathrm{e}^{-\omega_1^2\left(1-u^2\right)/2}\left(1+u^2\omega_1^2\right)$$

$$= -\omega_1^2\int_0^1 \mathrm{d}u\mathrm{e}^{-\omega_1^2\left(1-u^2\right)/2} + \int_0^1 \mathrm{d}u\mathrm{e}^{-\omega_1^2\left(1-u^2\right)/2}\left(1+u^2\omega_1^2\right) \tag{4.4.29}$$

thus

$$V_1'(\omega_1) = -\omega_1^2\int_0^1 \mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}\,\mathrm{d}u + \int_0^1 \mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}\,\mathrm{d}u\left[1+u^2\omega_1^2\right]. \tag{4.4.30}$$

The second integral of (4.4.30) may be written as

$$\int_0^1 \mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}\,\mathrm{d}u\left[1+u^2\omega_1^2\right] = \int_0^1 \mathrm{d}u\left[1+u\frac{\partial}{\partial u}\right]\mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}, \tag{4.4.31}$$

where

$$u\frac{\partial}{\partial u}\,\mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)} = \mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}\left(\omega_1^2 u^2\right).$$

$$\int_0^1 \mathrm{d}u\left[1+u\frac{\partial}{\partial u}\right]\mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)} = \int_0^1 \mathrm{d}u\,\mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)} + \int_0^1 \mathrm{d}u\,u\frac{\partial}{\partial u}\,\mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}$$

$$= u\exp\left[-\frac{\omega_1^2}{2}\left(1-u^2\right)\right]\Bigg|_0^1$$

$$= 1 \tag{4.4.32}$$

$$V_1'(\omega_1) = -\omega_1^2\int_0^1 \mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}\,\mathrm{d}u + 1$$

$$V_1''(\omega_1) = -\left(\omega_1+\lambda\right)\int_0^1 \mathrm{e}^{-\frac{(\omega_1+\lambda)^2}{2}\left(1-u^2\right)}\,\mathrm{d}u + 1. \tag{4.4.33}$$

Since

$$V_1(\omega_1) = V_1'(\omega_1) - V_1''(\omega_1)$$

$$= \int_0^\infty \nu\cos\nu\omega_1\mathrm{d}\nu\,\mathrm{e}^{-\frac{\nu^2}{2}} - \int_0^\infty \nu\cos\nu\left(\omega_1+\lambda\right)\mathrm{d}\nu\,\mathrm{e}^{-\frac{\nu^2}{2}}, \tag{4.4.34}$$

$$= -\omega_1^2\int_0^1 \mathrm{d}u\exp\left[-\frac{\omega_1^2}{2}\left(1-u^2\right)\right] + 1$$

$$+\left(\omega_1+\lambda\right)^2\int_0^1 \mathrm{d}u\exp\left[-\frac{\left(\omega_1+\lambda\right)^2}{2}\left(1-u^2\right)\right]-1$$

$$=-\omega_1^2\int_0^1 \mathrm{d}u\exp\left[-\frac{\omega_1^2}{2}\left(1-u^2\right)\right]$$

$$+\left(\omega_1+\lambda\right)^2\int_0^1 \mathrm{d}u\exp\left[-\frac{\left(\omega_1+\lambda\right)^2}{2}\left(1-u^2\right)\right], \tag{4.4.35}$$

$$V_1(\omega_1)=-\omega_{_1}^2\int_0^1 \mathrm{d}u\mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)}+\left(\omega_1+\lambda\right)^2\int_0^1 \mathrm{d}u\mathrm{e}^{-\frac{\left(\omega_1+\lambda\right)^2}{2}\left(1-u^2\right)}. \tag{4.4.36}$$

For $\phi_2\left(x'\right)$ we can then write as,

$$\phi_2\left(x'\right)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma^{1/2}}\left(\frac{1}{\pi}\right)^{3/4}\int_0^\infty v\mathrm{d}v\mathrm{e}^{-\frac{v^2}{2}}\frac{1}{r_2}\left[U_2(\omega_2)-\mathrm{i}V_2(\omega_2)\right], \tag{4.4.37}$$

where

$$U_2(\omega_2)=\int_0^\infty v\mathrm{d}v\left(\sin v\omega_2+\sin v\left(\omega_2+\lambda\right)\right) \tag{4.4.38}$$

$$V_2(\omega_2)=\int_0^\infty v\mathrm{d}v\left(\cos v\omega_2-\cos v\left(\omega_2+\lambda\right)\right),, \tag{4.4.39}$$

and finally we have $U_2(\omega_2)$ and $V_2(\omega_2)$

$$U_2(\omega_2)=\sqrt{\frac{\pi}{2}}\left(\omega_2\mathrm{e}^{-\frac{\omega_{12}^2}{2}}+\left(\omega_2+\lambda\right)\mathrm{e}^{-\frac{\left(\omega_2+\lambda\right)^2}{2}}\right), \tag{4.4.40}$$

$$V_2(\omega_2) = -\omega_2^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{\omega_2^2}{2}\left(1-u^2\right)} + (\omega_2+\lambda)^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{(\omega_2+\lambda)^2}{2}\left(1-u^2\right)}, \tag{4.4.41}$$

From $\phi\left(x'\right) \equiv \phi_1\left(x'\right) - \phi_2\left(x'\right)$

$$\phi\left(x'\right) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma^{1/2}}\left(\frac{1}{\pi}\right)^{3/4}$$

$$\times\left[\frac{U_1\left(\omega_1\right) - \mathrm{i}V_1\left(\omega_1\right)}{r_1} - \frac{U_2\left(\omega_1\right) - \mathrm{i}V_2\left(\omega_1\right)}{r_2}\right], \tag{4.4.42}$$

$$\phi\left(x'\right) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma^{1/2}}\left(\frac{1}{\pi}\right)^{3/4}$$

$$\times\left[\left(\frac{U_1\left(\omega_1\right)}{\sigma\left(\omega_1+\frac{\lambda}{2}\right)} - \frac{U_2\left(\omega_2\right)}{\sigma\left(\omega_2+\frac{\lambda}{2}\right)}\right) + \mathrm{i}\left(\frac{V_1\left(\omega_1\right)}{\sigma\left(\omega_1+\frac{\lambda}{2}\right)} - \frac{V_2\left(\omega_2\right)}{\sigma\left(\omega_2+\frac{\lambda}{2}\right)}\right)\right],$$

$$\phi\left(x'\right) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}$$

$$\times\left[\left(\frac{U_1\left(\omega_1\right)}{\left(\omega_1+\frac{\lambda}{2}\right)} - \frac{U_2\left(\omega_2\right)}{\left(\omega_2+\frac{\lambda}{2}\right)}\right) + \mathrm{i}\left(\frac{V_1\left(\omega_1\right)}{\left(\omega_1+\frac{\lambda}{2}\right)} - \frac{V_2\left(\omega_2\right)}{\left(\omega_2+\frac{\lambda}{2}\right)}\right)\right],$$

$$\phi\left(x'\right) = \frac{1}{2}\frac{1}{\sigma^{3/2}}\left(\frac{1}{\pi}\right)^{3/4}$$

$$\times\left[\left(\frac{A_1\left(\omega_1\right)}{\left(\omega_1+\frac{\lambda}{2}\right)} - \frac{A_2\left(\omega_2\right)}{\left(\omega_2+\frac{\lambda}{2}\right)}\right) + \mathrm{i}\left(\frac{B_1\left(\omega_1\right)}{\left(\omega_1+\frac{\lambda}{2}\right)} - \frac{B_2\left(\omega_2\right)}{\left(\omega_2+\frac{\lambda}{2}\right)}\right)\right]$$

$$= \frac{G\left(\omega_1, \omega_2, \lambda\right)}{\sigma^{3/2}}, \tag{4.4.43}$$

where

$$A_1(\omega_1) = \sqrt{\frac{2}{\pi}} U_1(\omega_1)\,,\; A_2(\omega_1) = \sqrt{\frac{2}{\pi}} U_2(\omega_2)\,,$$

$$B_1(\omega_1) = -\sqrt{\frac{2}{\pi}} V_1(\omega_1)\,,\; B_2(\omega_2) = -\sqrt{\frac{2}{\pi}} V_2(\omega_2)\,, \tag{4.4.44}$$

Since

$$U_1(\omega_1) = \sqrt{\frac{\pi}{2}} \left(\omega_1 \mathrm{e}^{-\frac{\omega_1^2}{2}} + (\omega_1 + \lambda)\, \mathrm{e}^{-\frac{(\omega_1+\lambda)^2}{2}} \right)$$

$$A_1(\omega_1) = \sqrt{\frac{\pi}{2}} U_1(\omega_1) = \omega_1 \mathrm{e}^{-\frac{\omega_1^2}{2}} + (\omega_1 + \lambda)\, \mathrm{e}^{-\frac{(\omega_1+\lambda)^2}{2}}, \tag{4.4.45}$$

$$A_1 = \left(\sqrt{X_{\|}'^2 + Z'^2} - \frac{\lambda}{2} \right) \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X'_{\|^2} + Z'^2} - \frac{\lambda}{2} \right)^2}$$

$$+ \left(\sqrt{X_{\|}'^2 + Z'^2} + \frac{\lambda}{2} \right) \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2 + Z'^2} + \frac{\lambda}{2} \right)^2}, \tag{4.4.46}$$

$$A_2 = \left(\sqrt{X_{\|}'^2 + \left(Z' + \frac{2H}{\sigma} \right)^2} - \frac{\lambda}{2} \right) \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2 + \left(Z' + \frac{2H}{\sigma} \right)^2} - \frac{\lambda}{2} \right)^2}$$

$$+ \left(\sqrt{X_{\|}'^2 + \left(Z' + \frac{2H}{\sigma} \right)^2} + \frac{\lambda}{2} \right) \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2 + \left(Z' + \frac{2H}{\sigma} \right)^2} + \frac{\lambda}{2} \right)^2}, \tag{4.4.47}$$

$$V_1(\omega_1) = -\omega_1^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)} + (\omega_1 + \lambda)^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{(\omega_1+\lambda)^2}{2}\left(1-u^2\right)}$$

$$B_1(\omega_1) = -\sqrt{\frac{2}{\pi}} V_1(\omega_1)$$

$$= \sqrt{\frac{2}{\pi}} \omega_1^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{\omega_1^2}{2}\left(1-u^2\right)} - \sqrt{\frac{2}{\pi}} (\omega_1 + \lambda)^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{(\omega_1+\lambda)^2}{2}\left(1-u^2\right)}$$

$$B_1 \quad = \sqrt{\frac{2}{\pi}} \left(\sqrt{X_{||}'^2 + Z'^2} - \frac{\lambda}{2}\right)^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{||}'^2+Z'^2}-\frac{\lambda}{2}\right)^2\left(1-u^2\right)}$$

$$- \sqrt{\frac{2}{\pi}} \left(\sqrt{X_{||}'^2 + Z'^2} + \frac{\lambda}{2}\right)^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{||}'^2+Z'^2}+\frac{\lambda}{2}\right)^2\left(1-u^2\right)}, \qquad (4.4.48)$$

$$B_2 = \sqrt{\frac{2}{\pi}} \left(\sqrt{X_{||}'^2 + \left(Z' + \frac{2H}{\sigma}\right)^2} - \frac{\lambda}{2}\right)^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{||}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}-\frac{\lambda}{2}\right)^2\left(1-u^2\right)}$$

$$- \sqrt{\frac{2}{\pi}} \left(\sqrt{X_{||}'^2 + \left(Z' + \frac{2H}{\sigma}\right)^2} + \frac{\lambda}{2}\right)^2 \int_0^1 \mathrm{d}u \mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{||}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}+\frac{\lambda}{2}\right)^2\left(1-u^2\right)},$$

(4.4.49)

where

$$\mathbf{r}_1 = \mathbf{R}_1' - \mathbf{a}$$

$$\mathbf{R}_1' = \left(\mathbf{x}_{||}', z'\right) \text{ and } \quad \mathbf{a} = \left(\mathbf{a}_{||}, H\right)$$

$$\mathbf{r}_1 = \left(\mathbf{x}_{||}' - \mathbf{a}_{||}, z' - H\right)$$

$$\mathbf{r}_2 = \mathbf{R}_2' - \mathbf{a}$$

$$\mathbf{R}_2' = \left(\mathbf{x}_{||}', -z'\right)$$

$$\mathbf{r}_2 = \left(\mathbf{x}'_{\|} - \mathbf{a}_{\|}, -z' - H\right), \tag{4.4.50}$$

let

$$\frac{\mathbf{x}'_{\|} - \mathbf{a}}{\sigma} = \mathbf{X}'_{\|}, \qquad \frac{z' - H}{\sigma} = Z'$$

$$\frac{-z' - H}{\sigma} = -\left(Z' + \frac{2H}{\sigma}\right). \tag{4.4.51}$$

From

$$\omega_1 = \frac{r_1}{\sigma} - \frac{\lambda}{2} = \sqrt{\frac{\left(x'_{\|} - a\right)^2 + (z' - H)^2}{\sigma^2}} - \frac{\lambda}{2}$$

$$= \sqrt{X'^2_{\|} + Z'^2} - \frac{\lambda}{2}$$

$$= R - \frac{\lambda}{2}, \tag{4.4.52}$$

$$\omega_2 = \frac{r_2}{\sigma} - \frac{\lambda}{2} = \sqrt{\frac{\left(x'_{\|} - a\right)^2 + (-z' - H)^2}{\sigma^2}} - \frac{\lambda}{2}$$

$$= \sqrt{X'^2_{\|} + \left(Z' + \frac{2H}{\sigma}\right)^2} - \frac{\lambda}{2}$$

$$= \sqrt{R^2 + 4\frac{H}{\sigma}\left(Z' + \frac{H}{\sigma}\right)} - \frac{\lambda}{2}; \qquad R^2 = X'^2_{\|} + Z'^2$$

$$= \sqrt{R^2 + 4\frac{H}{\sigma}\left(R\cos\vartheta + \frac{H}{\sigma}\right)} - \frac{\lambda}{2}$$

$$= \sqrt{R^2 + 4\frac{H}{\sigma}R\alpha + 4\frac{H^2}{\sigma^2}} - \frac{\lambda}{2} \qquad\qquad \alpha = \cos\vartheta. \tag{4.4.53}$$

Let

$$G_{\mathtt{NEW}} = G\left(R, \alpha, \lambda, \frac{H}{\sigma}\right) \tag{4.4.54}$$

$$G\left(R, \alpha, \lambda, \frac{H}{\sigma}\right) = \frac{1}{2}\left(\frac{1}{\pi}\right)^{3/4}\left\{\left[\frac{A_1}{R} - \frac{A_2}{\sqrt{R^2 + 4\frac{H}{\sigma}R\alpha + \frac{H^2}{\sigma^2}}}\right]\right.$$

$$\left. - \mathrm{i}\left[\frac{B_1}{R} - \frac{B_2}{\sqrt{R^2 + 4\frac{H}{\sigma}R\alpha + \frac{H^2}{\sigma^2}}}\right]\right\}, \tag{4.4.55}$$

Probability of finding photon excitation in half-space must be equal to one. So, in the old variables, we have

$$\int\limits_{-\infty}^{\infty} \mathrm{d}^2\mathbf{x}'_{||} \int\limits_{0}^{\infty} \mathrm{d}z' \left|\phi\left(x'\right)\right|^2 = 1. \tag{4.4.56}$$

Also in new variable,

$$\int\limits_{-\infty}^{\infty} \mathrm{d}^2\mathbf{X}'_{||} \int\limits_{-\frac{H}{\sigma}}^{\infty} \mathrm{d}Z' \left|G_{\mathtt{NEW}}\right|^2 = 1. \tag{4.4.57}$$

and

$$\left|G\left(R, \alpha, \lambda, \frac{H}{\sigma}\right)\right|^2 = \left|\frac{1}{2}\left(\frac{1}{\pi}\right)^{3/4}\right|^2 \left\{\left[\frac{A_1}{R} - \frac{A_2}{\sqrt{R^2 + 4\frac{H}{\sigma}R\alpha + 4\frac{H^2}{\sigma^2}}}\right]^2\right.$$

$$+\left[\frac{B_1}{R}-\frac{B_2}{\sqrt{R^2+4\frac{H}{\sigma}R\alpha+4\frac{H^2}{\sigma^2}}}\right]^2\Bigg\},\tag{4.4.58}$$

$$I=\left|G\left(\sqrt{X_{\|}'^2+Z'^2},Z',\frac{H}{\sigma}\right)\right|^2$$

$$=\frac{1}{4\pi^{\frac{3}{2}}}\Bigg\{\Bigg[\frac{\left(\sqrt{X_{\|}'^2+Z'^2}-\frac{\lambda}{2}\right)\mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+Z'^2}-\frac{\lambda}{2}\right)^2}}{\sqrt{X_{\|}'^2+Z'^2}}$$

$$+\left(\frac{\left(\sqrt{X_{\|}'^2+Z'^2}+\frac{\lambda}{2}\right)\mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+Z'^2}+\frac{\lambda}{2}\right)^2}}{\sqrt{X_{\|}'^2+Z'^2}}\right)$$

$$\times\left(\frac{\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}-\frac{\lambda}{2}\right)\mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}-\frac{\lambda}{2}\right)^2}}{\sqrt{X_{\|}'^2+Z'^2}}\right)$$

$$-\frac{\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}+\frac{\lambda}{2}\right)\mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}+\frac{\lambda}{2}\right)^2}}{\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}}\Bigg]^2$$

$$+\Bigg[\frac{\sqrt{\frac{2}{\pi}}\left(\sqrt{X_{\|}'^2+Z'^2}-\frac{\lambda}{2}\right)^2\int\limits_0^1\mathrm{d}u\mathrm{e}^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+Z'^2}-\frac{\lambda}{2}\right)^2\left(1-u^2\right)}}{\sqrt{X_{\|}'^2+Z'^2}}$$

$$-\frac{\sqrt{\frac{2}{\pi}}\left(\sqrt{X_{\|}'^2+Z'^2}+\frac{\lambda}{2}\right)^2\int\limits_0^1 \mathrm{d}u\,e^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+Z'^2}+\frac{\lambda}{2}\right)^2\left(1-u^2\right)}}{\sqrt{X_{\|}'^2+Z'^2}}$$

$$-\frac{\sqrt{\frac{2}{\pi}}\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}-\frac{\lambda}{2}\right)^2\int\limits_0^1 \mathrm{d}u\,e^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}-\frac{\lambda}{2}\right)^2\left(1-u^2\right)}}{\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}}$$

$$-\left(\frac{\sqrt{\frac{2}{\pi}}\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}+\frac{\lambda}{2}\right)^2}{\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}}\right)$$

$$\times\left.\frac{\int\limits_0^1 \mathrm{d}u\,e^{-\frac{1}{2}\left(\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}+\frac{\lambda}{2}\right)^2\left(1-u^2\right)}}{\sqrt{X_{\|}'^2+\left(Z'+\frac{2H}{\sigma}\right)^2}}\right]^2\Bigg\} \tag{4.4.59}$$

4.5 Graphs of Probabilities of Photon Detection

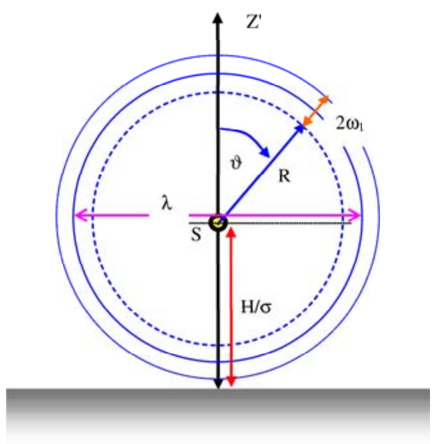

Figure 4.1 Position of source relative to reflecting plane, for H/σ larger than R.

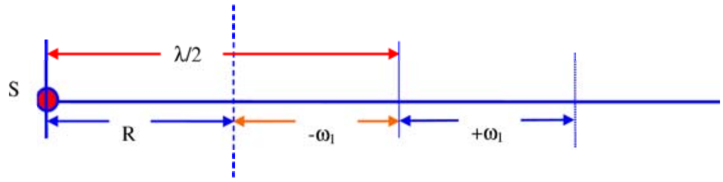

Figure 4.2 Relation of distances from a source.

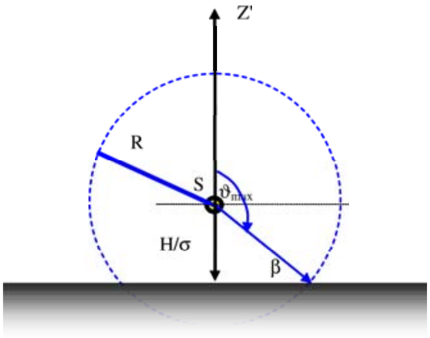

Figure 4.3 Position of source relative to reflecting plane, for H/σ less than R.

Table 4.1 Comparison of probabilities at different angles $\arccos\alpha$ specifying the location of the detector.

λ	H	σ	R_{min}	R_{max}	α_{min}	α_{max}	Prob.
100	10	0.01	0.1	0.9	−0.3	1	0.649999
100	10	0.01	0.1	0.9	−0.6	1	0.799999
100	10	0.01	0.1	0.9	−1	1	0.999998

Table 4.2 Comparison of probabilities of detection at different separation distances H of the emitter from the reflecting surface.

λ	H	σ	R_{min}	R_{max}	α_{min}	α_{max}	Prob.
100	10	0.01	0.4	0.6	−1	1	0.999766
60	10	0.01	0.4	0.6	−1	1	0.999766
10	10	0.01	0.4	0.6	−1	1	0.999766

Table 4.3 Comparison of probabilities of detection at different distances, R of the emitter from the detector.

λ	H	σ	R_{min}	R_{max}	α_{min}	α_{max}	Prob.
100	10	0.01	0.45	0.55	−1	1	0.997769
100	10	0.01	0.44	0.56	−1	1	0.998788
100	10	0.01	0.43	0.57	−1	1	0.999266
100	10	0.01	0.40	0.60	−1	1	0.999766
100	10	0.01	0.35	0.65	−1	1	0.999937
100	10	0.01	0.30	0.70	−1	1	0.999976
100	10	0.01	0.25	0.75	−1	1	0.999989
100	10	0.01	0.20	0.80	−1	1	0.999995
100	10	0.01	0.15	0.85	−1	1	0.999997
100	10	0.01	0.10	0.90	−1	1	0.999999
100	10	0.01	0.05	0.95	−1	1	0.999999

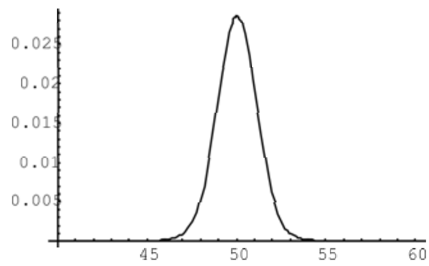

Figure 4.4 Plot of Probability of detection with $\lambda = 100, H = 10, \sigma = 0.01, \alpha = 0, R = 0.4\lambda$ to 0.6λ.

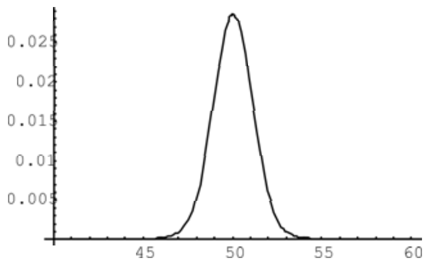

Figure 4.5 Plot of Probability of detection with $\lambda = 100, H = 10, \sigma = 0.01, \alpha = 1, R = 0.4\lambda$ to 0.6λ.

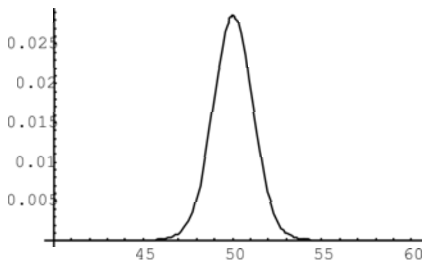

Figure 4.6 Plot of Probability of detection with $\lambda = 100, H = 100, \sigma = 0.01, \alpha = -1, R = 0.4\lambda$ to 0.6λ.

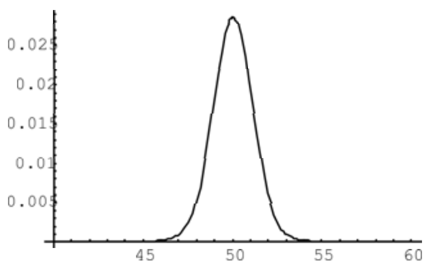

Figure 4.7 Plot of Probability of detection with $\lambda = 100, H = 10, \sigma = 0.01, \alpha = 0.3, R = 0.4\lambda$ to 0.6λ.

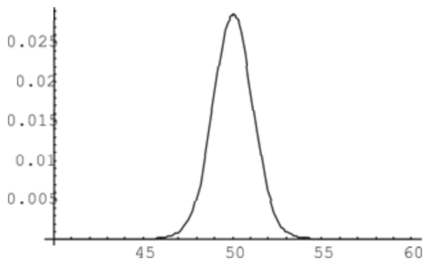

Figure 4.8 Plot of Probability of detection with $\lambda = 100, H = 10, \sigma = 0.01, \alpha = 0.4, R = 0.4\lambda$ to 0.6λ.

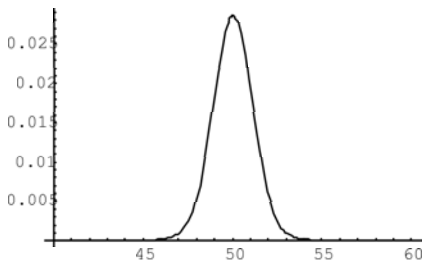

Figure 4.9 Plot of Probability of detection with $\lambda = 100, H = 10, \sigma = 0.01, \alpha = 0.5, R = 0.4\lambda$ to 0.6λ.

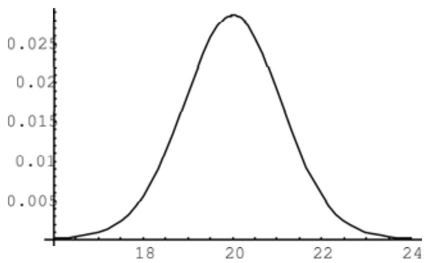

Figure 4.10 Plot of Probability of detection with $\lambda = 40, H = 10, \sigma = 0.01, \alpha = 1, R = 0.4\lambda$ to 0.6λ.

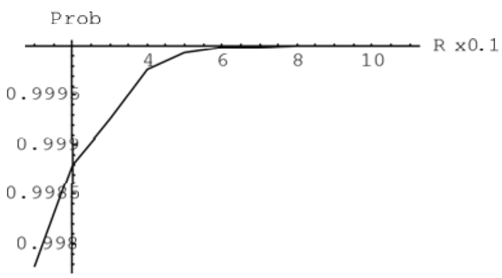

Figure 4.11 Plot of probability of detection at all points from the detector.

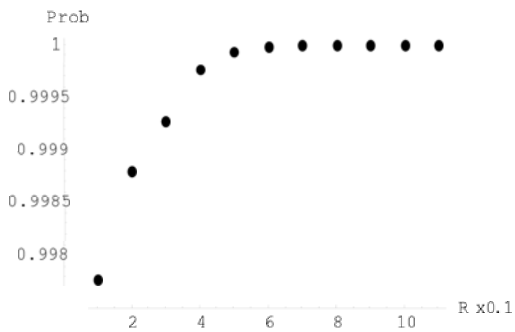

Figure 4.12 Plot of probability of detection at all points from the detector.

CHAPTER V

PROPAGATION OF PHOTONS IN SPACETIME AS A TIME EVOLUTION PROCESS IN HALF–SPACE: QUANTUM ELECTRODYNAMICS DERIVATION OF THE LAW OF REFLECTION

5.1 Introduction

Much progress has been done over the years (Bialynicki-Birula, 1998; Allard, Pike and Sakar, 1997) to describe, especially quantum theoretically, the localization of photons in space (Hong and Mandel, 1986). It is fair to say, however, that there was still no explicit dynamical, non-heuristic, actual quantum (field) theory QED formalism worked out, as dictated by the latter, to describe the propagation of photons in *spacetime* in explaining even a simplest experiment as the reflection of photons off a reflecting surface as a time evolution process. This is certainly remarkable in the progress of physics, knowing that QED has been around for sometime and, as Feynman (Feynman, 1985) puts it, it has been thoroughly analyzed, in his legendary Alix G. Mautner Memorial Lectures. The latter fascinating, though heuristic treatment (Feynman, 1985) in words is, of course, far from a definite theoretical description but, in spite being addressed to non-specialists, the discussion clearly indicates, and as our present analysis shows, that a theoretical formalism, as stated above, to explain a simplest experiment in *spacetime* in a quantum (field) theory QED setting is lacking. For one thing, the amplitude of propagation of photons in *spacetime*, as a time evolution process, in infinitely extended space, for example, from a point x_1^μ to a point x_2^μ turns out to be given by

$$\frac{\mathrm{i}}{(\pi)^2}\frac{(x_2^0 - x_1^0)^2}{[(x_2 - x_1)^2]^2}. \tag{5.1.1}$$

as shown in Chapter IV rather than by the familiar Feynman propagator

$$\frac{\mathrm{i}}{(x_2 - x_1)^2}. \tag{5.1.2}$$

with the former satisfying a key completeness relation for the internal consistency of the theory as formulated in spacetime. The purpose of this chapter is to develop such a formalism in detail based on the actual *physical process of the propagation of photons from emitters to detectors* as obtained from the so-called vacuum-to-vacuum transition amplitude (Schwinger, 1951, 1954, 1969, 1971, 1973; Manoukia, 1986, 1991, 1992) for the underlying theory. This method has been quite successful over the years in the easiness of momentum space computations of physical processes, avoiding of introducing so-called wave functions, not to mention of the elegance of the formalism as opposed to more standard techniques, and at the same time gaining much physical insight as particles propagate from emitters, interact, and finally particles reach the detectors as occurring in practice. The present analysis rests on three general key points:

(i.) By working directly in spacetime for the vacuum-to-vacuum transition amplitude, for given boundary conditions (B.C.), and from the expressions of the amplitudes for the emission and detection of photon excitations by the external sources, an amplitude of propagation between different spacetime points from emitters to detectors, causally arranged, is extracted and, as mentioned above, it does *not* coincide with the Feynman propagator for the corresponding B.C.. This step already shows the power of determining amplitudes of propagation by introducing external sources.

(ii.) The amplitude of propagation is shown to satisfy a *completeness* relation as photons propagate between different points critical for the internal consistency of the theory in spacetime.

(iii.) Application of these amplitudes to describe in detail the experiment being sought by showing, in the process, very rapid exponential damping beyond the classical point of impact for the corresponding amplitude of occurrence. One soon realizes that our theoretical quantum (field) theory QED formalism is reduced to a non-operator approach and opens a way to describe, as a time evolution process, photon dynamics in *spacetime* and other field theory interactions in different experimental situations as well.

5.2 The Time Evolution Process in Half-Space

Let $|0_\mp\rangle$ denote the vacuum states before/after the external current $J^\mu(x)$, coupled to the vector potential $A_\mu(x)$ in Maxwell's Lagrangian, is switched on/off. The boundary conditions, as discuses in Chapter IV, are taken to be

$$\left\langle 0_+ \middle| \mathbf{E}_{||} \middle| 0_- \right\rangle = 0, \tag{5.2.1}$$

$$\langle 0_+ | \mathbf{B}_\perp | 0_- \rangle = 0. \tag{5.2.2}$$

For $z \to +0$, where the reflecting surface is taken to consist the $x^1 - x^2$ plane, with $x^3 \equiv z \geq 0$, and $\mathbf{E}_{||}/\mathbf{B}_\perp$ denote the components of the electric/magnetic fields parallel/perpendicular to the $x^1 - x^2$ plane. The vacuum-to-vacuum transition amplitude $\langle 0_+ \,|\, 0_- \rangle^J$ is then given by (4.2.12).

$$\langle 0_+ \mid 0_- \rangle = \mathrm{e}^{\frac{\mathrm{i}}{2}\int (\mathrm{d}x_1)(\mathrm{d}x_2) J^\mu(x_2) D'_{\mu\nu}(x_2 - x_1) J^\nu(x_1)}, \tag{5.2.3}$$

where invoking the conservation law $\partial_\mu J^\mu = 0$, the photon propagator in half-space was derived in (3.2.97) in chapter 3 and is given by

$$D'_{\mu\nu}(x_2,x_1) = \int \frac{(\mathrm{d}Q)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}\mathrm{e}^{-\mathrm{i}Q^0\left(x_2^0-x_1^0\right)}}{Q^2-\mathrm{i}\varepsilon} \tag{5.2.4}$$

$$\times\left[g_{\mu\nu}\mathrm{e}^{\mathrm{i}q(z_2-z_1)} + (-g_{\mu\nu}+2g_{\mu3}g_{\nu3})\,\mathrm{e}^{-\mathrm{i}q(z_2+z_1)}\right],$$

$\varepsilon \to +0, x = (x^0, \mathbf{r}, z), Q = (Q^0, \mathbf{k}, q)$ with $\mathbf{r}$ lying in the $x^1 - x^2$ plane. Since $J^\mu(x)$, by definition, vanishes for $z \leq 0$, we may integrate over all spacetime points in (5.2.3). Gauge invariance of the theory as well as the positivity condition $|\left\langle 0_+ | 0_-\right\rangle^J|^2 \leq 1$ are readily established in (3.3.1) in Chapter III. We consider a causal arrangement, $J^\mu(x) = J_1^\mu(x) + J_2^\mu(x)$, of two currents with $J_2^\mu(x)$, the detector, switched on after $J_1^\mu(x)$, the emitter, is switched off. By invoking the condition $\partial_\mu J^\mu = 0$, we may then write

$$\left\langle 0_+ \mid 0_-\right\rangle^J = \left\langle 0_+ \mid 0_-\right\rangle^{J_2}\mathrm{e}^{\mathrm{i}\Omega}\left\langle 0_+ \mid 0_-\right\rangle^{J_1}. \tag{5.2.5}$$

$$\Omega = \int (\mathrm{d}x_1)(\mathrm{d}x_2)\,\mathrm{i}J^i_{2T}(x_2)\left[-\mathrm{i}\Delta_+(x_2,x_1)\,\delta^{ij}\right.$$

$$\left.-\,\mathrm{i}\Delta_+(x_2',x_1)\left(-\delta^{ij}+2\delta^{i3}\delta^{i3}\right)\right]\mathrm{i}J^j_{1T}(x_1). \tag{5.2.6}$$

For $x_2^0 > x_1^0$,

$$-\mathrm{i}\,\triangle_+(x_2,x_1) = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3 2Q^0}\mathrm{e}^{\mathrm{i}Q(x_2-x_1)}, \qquad Q^0 = |\mathbf{Q}|, \tag{5.2.7}$$

$$J^i_T(x) = \int \frac{(\mathrm{d}Q)}{(2\pi)^4}\mathrm{e}^{\mathrm{i}Qx}J^i_T(Q) \tag{5.2.8}$$

and for $Q^0 = |\mathbf{Q}|, Q^i J_T^i = 0$. The second term within the square brackets in (5.2.6) corresponds to a non-trivial transition.

Here we have used the identities:

$$\text{I.} \qquad Q^i\left[\delta^{ij} - \frac{Q^iQ^j}{\mathbf{Q}^2}\right] = 0. \tag{5.2.9}$$

Proof:

$$\begin{aligned}
Q^i\left[\delta^{ij} - \frac{Q^iQ^j}{\mathbf{Q}^2}\right] &= \left[Q^i\delta^{ij} - \frac{Q^iQ^iQ^j}{\mathbf{Q}^2}\right] \\
&= \left[Q^j - \frac{Q^{i2}Q^j}{\mathbf{Q}^2}\right] \\
&= \left[Q^j - Q^j\right] \\
&= 0.
\end{aligned} \tag{5.2.10}$$

$$\text{II.} \qquad \left[\delta^{ij} - \frac{Q^iQ^j}{\mathbf{Q}^2}\right]\left[\delta^{jk} - \frac{Q^jQ^k}{\mathbf{Q}^2}\right] = \left[\delta^{ik} - \frac{Q^iQ^k}{\mathbf{Q}^2}\right]. \tag{5.2.11}$$

Proof:

$$\begin{aligned}
\left[\delta^{ij} - \frac{Q^iQ^j}{\mathbf{Q}^2}\right]\left[\delta^{jk} - \frac{Q^jQ^k}{\mathbf{Q}^2}\right] &= \left[\delta^{ij}\delta^{jk} - \frac{Q^iQ^j\delta^{jk}}{\mathbf{Q}^2} - \delta^{ij}\frac{Q^jQ^k}{\mathbf{Q}^2} + \frac{Q^iQ^j}{\mathbf{Q}^2}\frac{Q^jQ^k}{\mathbf{Q}^2}\right] \\
&= \left[\delta^{ik} - \frac{Q^iQ^k}{\mathbf{Q}^2} - \frac{Q^iQ^k}{\mathbf{Q}^2} + \frac{Q^iQ^k}{\mathbf{Q}^2}\right] \\
&= \left[\delta^{ik} - \frac{Q^iQ^k}{\mathbf{Q}^2}\right].
\end{aligned} \tag{5.2.12}$$

$$\text{III.} \qquad J^{i*}(Q)\left[\delta^{ij} - \frac{Q^iQ^j}{\mathbf{Q}^2}\right]J^j(Q) = J_T^{i*}(Q)\delta^{ij}J_T^j(Q). \tag{5.2.13}$$

where

$$J_T^i(Q) = \left(\delta^{ij} - \frac{Q^iQ^j}{\mathbf{Q}^2}\right)J^j(Q), \qquad \text{i.e. } Q_iJ_T^i(Q) = 0. \tag{5.2.14}$$

Proof: from the right-hand side of (5.2.13)

$$J_T^{i*}(Q)\delta^{ij}J_T^j(Q) = \left(\delta^{ik} - \frac{Q^iQ^k}{\mathbf{Q}^2}\right)J^{k*}(Q)\delta^{ij}\left(\delta^{jl} - \frac{Q^jQ^l}{\mathbf{Q}^2}\right)J^l(Q)$$

$$= \left(J^{i*}(Q) - \frac{Q^iQ^k}{\mathbf{Q}^2}J^{k*}(Q)\right)\delta^{ij}\left(J^i(Q) - \frac{Q^jQ^l}{\mathbf{Q}^2}J^l(Q)\right)$$

$$= J^{i*}(Q)J^i(Q) - J^{i*}(Q)\frac{Q^iQ^l}{\mathbf{Q}^2}J^l(Q)$$

$$- \frac{Q^iQ^k}{\mathbf{Q}^2}J^{k*}(Q)J^i(Q) + \frac{Q^iQ^k}{\mathbf{Q}^2}J^{k*}(Q)\frac{Q^iQ^l}{\mathbf{Q}^2}J^l(Q)$$

$$= J^{i*}(Q)J^i(Q) - J^{i*}\frac{Q^iQ^l}{\mathbf{Q}^2}J^l(Q)$$

$$- \frac{Q^kJ^{k*}(Q)}{\mathbf{Q}^2}\mathbf{Q}\cdot\mathbf{J} + \frac{Q^kJ^{k*}(Q)}{\mathbf{Q}^2}\mathbf{Q}\cdot\mathbf{J}$$

$$= J^{i*}(Q)J^i(Q) - J^{i*}(Q)\frac{Q^iQ^l}{\mathbf{Q}^2}J^l(Q)$$

$$= J^{i*}(Q)\left(J^i(Q) - \frac{Q^iQ^j}{\mathbf{Q}^2}J^j(Q)\right)$$

$$= J^{i*}(Q)\left(\delta^{ij} - \frac{Q^i Q^j}{\mathbf{Q}^2}\right) J^j(Q). \tag{5.2.15}$$

$$\text{IV.} \qquad Q'^i\left[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\right] = 0, \tag{5.2.16}$$

where

$$\mathbf{Q} = (\mathbf{k}, q),\ \mathbf{Q'} = (\mathbf{k}, -q). \tag{5.2.17}$$

Proof:

$$Q'^i\left[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\right] = -Q'^j + 2Q'^3\delta^{j3} + \frac{Q'^{i2} Q^j}{\mathbf{Q}^2}$$

$$= -Q'^j - 2q\delta^{j3} + Q^j$$

$$= -(k^1 + k^2 - q) - 2q + (k^1 + k^2 - q)$$

$$= 0. \tag{5.2.18}$$

$$\text{V.} \qquad \left[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\right] Q^j = 0. \tag{5.2.19}$$

Proof:

$$\left[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\right] Q^j = -Q^i + 2\delta^{i3} Q^3 + \frac{Q'^i Q^{2j}}{\mathbf{Q}^2}$$

$$= -Q^i + 2\delta^{j3} q + Q^j$$

$$= -(k^1 + k^2 + q) + 2q + (k^1 + k^2 - q)$$

$$= 0. \tag{5.2.20}$$

VI. $$J^{i*}(Q')\Big[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\Big]J^j(Q) = J_T^{i*}(Q')\Big[-\delta^{ij} + 2\delta^{i3}\delta^{j3}\Big]J_T^j(Q). \tag{5.2.21}$$

Proof: using the identities in (5.2.19) and (5.2.20) we can rewrite (5.2.21) as

$$\Big(J^{i*}(Q') - \frac{Q'^i}{\mathbf{Q}^2}\mathbf{Q}\cdot\mathbf{J}^*\Big)\Big[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\Big]\Big(J^j(Q) - \frac{Q^j}{\mathbf{Q}^2}\mathbf{Q}\cdot\mathbf{J}\Big)$$

$$= \Big(J^{j*}(Q')\delta^{ij} - \frac{Q'^i}{\mathbf{Q}^2}Q^j J^{j*}(Q')\Big)\Big[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\Big]$$

$$\times\Big(\delta^{ij}J^i(Q) - \frac{Q^j}{\mathbf{Q}^2}Q^i J^i(Q)\Big)$$

$$= J^{j*}(Q')\Big(\delta^{ij} - \frac{Q'^i}{\mathbf{Q}^2}Q^j\Big)\Big[-\delta^{ij} + 2\delta^{i3}\delta^{j3} + \frac{Q'^i Q^j}{\mathbf{Q}^2}\Big]$$

$$\times\Big(\delta^{ij} - \frac{Q^j}{\mathbf{Q}^2}Q^i\Big)J^i(Q)$$

$$= J_T^{i*}(Q')\Big[-\delta^{ij} + 2\delta^{i3}\delta^{j3}\Big]J_T^j(Q), \tag{5.2.22}$$

where

$$J_T^{i*}(Q') = J^{j*}(Q')\Big(\delta^{ij} - \frac{Q'^i}{\mathbf{Q}^2}Q^j\Big),$$

$$J_T^j(Q) = \Big(\delta^{ij} - \frac{Q^j}{\mathbf{Q}^2}Q^i\Big)J^i(Q),$$

$$Q^i J_T^i(Q) = 0.$$

Now we use the identity

$$-\mathrm{i} \triangle_+ (x_4, x_1) = \int' \mathrm{d}^3\mathbf{x}_2 \int' \mathrm{d}^3\mathbf{x}_3 \int' \mathrm{d}^3\mathbf{x} D_>(x_4, x_3)$$

$$\times [D(x_3, x)\mathrm{i}\overleftrightarrow{\partial_0} D(x, x_2)]D_<(x_2, x_3). \tag{5.2.23}$$

in (5.2.6), where $\overleftrightarrow{\partial_0} = \overrightarrow{\partial_0} - \overleftarrow{\partial_0},\ x_4^0 > x_3^0 > x_2^0 > x_1^0, \int' \mathrm{d}^3\mathbf{x} = \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{r} \int_0^\infty \mathrm{d}z$, and

$$D_>(x_4, x_3) = \int \frac{\mathrm{d}^3\mathbf{Q}}{4\pi^3\sqrt{2Q^0}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_4 - \mathbf{r}_3)} \mathrm{e}^{-\mathrm{i}Q(x_4^0 - x_3^0)} \mathrm{e}^{\mathrm{i}qz_4} \sin qz_3, \tag{5.2.24}$$

$$D_<(x_2, x_1) = \int \frac{\mathrm{d}^3\mathbf{Q}}{4\pi^3\sqrt{2Q^0}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2 - \mathbf{r}_1)} \mathrm{e}^{-\mathrm{i}Q(x_2^0 - x_1^0)} \sin qz_2 \mathrm{e}^{-\mathrm{i}qz_1}, \tag{5.2.25}$$

$$D(x_2, x_1) = \int \frac{\mathrm{d}^3\mathbf{Q}}{2\pi^3\sqrt{2Q^0}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2 - \mathbf{r}_1)} \mathrm{e}^{-\mathrm{i}Q(x_2^0 - x_1^0)} \sin qz_2 \sin qz_1. \tag{5.2.26}$$

To demonstrate the correction of (5.2.23), we note that

$$-\mathrm{i}\Delta_+ (x_4, x_1) = \int \mathrm{d}^2\mathbf{x}_{2\|} \int_0^\infty \mathrm{d}z_2 \int \mathrm{d}^2 x_{3\|} \int_0^\infty \mathrm{d}z_3 \int \mathrm{d}^2\mathbf{x}_\| \int_0^\infty \mathrm{d}z$$

$$\times D(x_4, x_3) \left[D(x_3, x)\, \mathrm{i}\frac{\overleftrightarrow{\partial}}{\partial y^0} D(x, x_2) \right] D(x_2, x_1)$$

$$\int \mathrm{d}^2\mathbf{x}_\| \int_0^\infty \mathrm{d}z \left[D(x_3, x)\, \mathrm{i}\frac{\overleftrightarrow{\partial}}{\partial x^0} D(x, x_2) \right]$$

$$= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} dq \frac{\sin qz_3 \sin qz}{\sqrt{2|Q|}} e^{i\mathbf{k}\cdot(\mathbf{x}_{3\|}-\mathbf{x}_{\|})} e^{-i|Q|(x_3^0 - x^0)}$$

$$\times\, i \left\{ \frac{\overrightarrow{\partial}}{\partial x^0} - \frac{\overleftarrow{\partial}}{\partial x^0} \right\} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} dq' \frac{\sin q'z \sin q'z_2}{\sqrt{2|Q|'}} e^{i\mathbf{k}'\cdot(\mathbf{x}_{\|}-\mathbf{x}_{2\|})} e^{-i|Q|'(x^0 - x_2^0)}$$

$$= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\sqrt{2|\mathbf{Q}|}} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \frac{1}{\sqrt{2|\mathbf{Q}'|}}$$

$$\times \int_0^\infty \frac{2}{\pi} dq \int_0^\infty \frac{2}{\pi} dq' \int d^2\mathbf{x}_{\|} e^{i\mathbf{k}'\cdot(\mathbf{x}'_{3\|}-\mathbf{x}_{\|})} e^{i\mathbf{k}'\cdot(\mathbf{x}_{\|}-\mathbf{x}'_{2\|})} e^{-i|Q|(x_3^0 - x^0)} e^{-i|Q'|(x^0 - x_2^0)}$$

$$\times \sin qz_3 \sin q'z_2 \int_0^\infty dz \sin qz \sin q'z \; (i) \left[-i|\mathbf{Q}'| - i|\mathbf{Q}| \right].$$

From

$$\int_0^\infty dz \sin qz \sin q'z = \frac{\pi}{2} \left[\delta(q - q') - \delta(q + q') \right].$$

the above equation integral becomes

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} dq \frac{\sqrt{2|\mathbf{Q}|}}{\sqrt{2|\mathbf{Q}|}} \sin qz_3 \sin q'z_2 e^{i\mathbf{k}\cdot(\mathbf{x}_{3\|}-\mathbf{x}_{2\|})} e^{-i|\mathbf{Q}|(x_3^0 - x_2^0)}$$

$$= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} dq \sin qz_3 \sin q'z_2 e^{i\mathbf{k}\cdot(\mathbf{x}_{3\|}-\mathbf{x}_{2\|})} e^{-i|\mathbf{Q}|(x_3^0 - x_2^0)}$$

we then have

$$\int \mathrm{d}^2\mathbf{x}_{2\|} \int_0^\infty \mathrm{d}z_2 \left[\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} \mathrm{d}q \sin qz_3 \sin q' z_2 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_{3\|}-\mathbf{x}_{2\|})} \mathrm{e}^{-\mathrm{i}|\mathbf{Q}|(x_3^0-x_2^0)} \right] D(x_2, x_1)$$

$$= \int \mathrm{d}^2\mathbf{x}_{2\|} \int_0^\infty \mathrm{d}z_2 \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} \mathrm{d}q \sin qz_3 \sin qz_2 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_{3\|}-\mathbf{x}_{2\|})} \mathrm{e}^{-\mathrm{i}|\mathbf{Q}|(x_3^0-x_2^0)}$$

$$\times \int \frac{\mathrm{d}^2\mathbf{k}'''}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} \mathrm{d}q''' \left(\frac{\sin q''' z_2 \sin q''' z_1}{\sqrt{2\,|\mathbf{Q}|'''}} \right) \mathrm{e}^{\mathrm{i}\mathbf{k}'''\cdot(\mathbf{x}_{2\|}-\mathbf{x}_{1\|})} \mathrm{e}^{-\mathrm{i}|\mathbf{Q}'''|(x_2^0-x_1^0)}.$$

Since

$$\int_0^\infty \mathrm{d}z_2 \sin qz_2 \sin q''' z_2 = \frac{\pi}{2} \left[\delta(q - q''') - \delta(q + q''') \right],$$

so the above integral simplifies to

$$\int \mathrm{d}^2\mathbf{x}_{2\|} \int_0^\infty \mathrm{d}z_2 \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} \mathrm{d}q \sin qz_3 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_{3\|}-\mathbf{x}_{2\|})} \mathrm{e}^{-\mathrm{i}|\mathbf{Q}|(x_3^0-x_2^0)}$$

$$\times \int \frac{\mathrm{d}^2\mathbf{k}'''}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} \mathrm{d}q''' \left(\frac{\sin q''' z_1}{\sqrt{2\,|\mathbf{Q}'''|}} \right) \mathrm{e}^{\mathrm{i}\mathbf{k}'''\cdot(\mathbf{x}_{2\|}-\mathbf{x}_{1\|})} \mathrm{e}^{-\mathrm{i}|\mathbf{Q}'''|(x_2^0-x_1^0)}$$

$$\times \frac{\pi}{2} \left[\delta(q - q''') - \delta(q + q''') \right]$$

$$= \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \frac{2}{\pi} \mathrm{d}q \sin qz_3 \left(\frac{\sin qz_1}{\sqrt{2\,|\mathbf{Q}|}} \right) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_{2\|}-\mathbf{x}_{1\|})} \mathrm{e}^{-\mathrm{i}|\mathbf{Q}|(\mathbf{x}_2^0-x_1^0)}$$

$$D\left(x_4, x_3\right) \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_0^\infty \frac{2}{\pi} \mathrm{d}q \sin qz_3 \left(\frac{\sin qz_1}{\sqrt{2\left|\mathbf{Q}\right|}}\right) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{2\|}-x_{1\|}\right)} \mathrm{e}^{-\mathrm{i}\left|\mathbf{Q}\right|\left(x_2^0-x_1^0\right)}$$

$$= \int \mathrm{d}^2x_{3\|} \int\limits_0^\infty \mathrm{d}z_3 \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_0^\infty \frac{2}{\pi} \mathrm{d}q \sin qz_4 \left(\frac{\sin qz_3}{\sqrt{2\left|\mathbf{Q}\right|}}\right)$$

$$\times \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{4\|}-\mathbf{x}_{3\|}\right)} \mathrm{e}^{-\mathrm{i}\left|\mathbf{Q}\right|\left(x_4^0-x_3^0\right)} \int \frac{\mathrm{d}^2k'''}{(2\pi)^2} \int\limits_0^\infty \frac{2}{\pi} \mathrm{d}q''' \sin q''' z_3$$

$$\times \left(\frac{\sin q''' z_1}{\sqrt{2\left|\mathbf{Q}'''\right|}}\right) \mathrm{e}^{\mathrm{i}\mathbf{k}'''\cdot\left(\mathbf{x}_{2\|}-x_{1\|}\right)} \mathrm{e}^{-\mathrm{i}\left|\mathbf{Q}'''\right|\left(x_2^0-x_1^0\right)}.$$

But

$$\int\limits_0^\infty \mathrm{d}z_3 \sin qz_3 \sin q''' z_3 = \frac{\pi}{2}\left[\delta\left(q-q'''\right)-\delta\left(q+q'''\right)\right]$$

therefore, for the above integral we obtain

$$\int \mathrm{d}^2\mathbf{x}_{3\|} \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_0^\infty \frac{2}{\pi} \mathrm{d}q \sin qz_4 \left(\frac{1}{\sqrt{2\left|\mathbf{Q}\right|}}\right) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_{4\|}-\mathbf{x}_{3\|}\right)} \mathrm{e}^{-\mathrm{i}\left|Q\right|\left(x_4^0-x_3^0\right)}$$

$$\times \int \frac{\mathrm{d}^2\mathbf{k}'''}{(2\pi)^2} \int\limits_0^\infty \frac{2}{\pi} \mathrm{d}q''' \left(\frac{\sin q''' z_1}{\sqrt{2\left|\mathbf{Q}'''\right|}}\right) \mathrm{e}^{\mathrm{i}\mathbf{k}'''\cdot\left(\mathbf{x}_{2\|}-\mathbf{x}_{1\|}\right)} \mathrm{e}^{-\mathrm{i}\left|\mathbf{Q}'''\right|\left(x_2^0-x_1^0\right)}$$

$$\times \frac{\pi}{2}\left[\delta\left(q-q'''\right)-\delta\left(q+q'''\right)\right].$$

We finally get for $x^0 > x'^0$

$$(-\mathrm{i})\Delta_+\left(x, x'\right) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int\limits_0^\infty \frac{2}{\pi} \mathrm{d}q \frac{\sin qz \sin qz'}{2\left|\mathbf{Q}\right|} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\left(\mathbf{x}_\|-x'_\|\right)} \mathrm{e}^{-\mathrm{i}\left|\mathbf{Q}\right|\left(x^0-x'^0\right)}. \qquad (5.2.27)$$

Given two real unit 3-vectors

$$\mathbf{n} = (a, b, c) \equiv \mathbf{n}_+, \tag{5.2.28}$$

$$\mathbf{n}' = (a, b, -c) \equiv \mathbf{n}_-. \tag{5.2.29}$$

We introduce two sets of unit 3-vectors $(\mathbf{e}_1, \mathbf{e}_2), (\epsilon_1, \epsilon_2)$ by

$$\mathbf{e}_1 = \mathbf{n} \times \mathbf{n}'/|\mathbf{n} \times \mathbf{n}'| = \epsilon_1, \tag{5.2.30}$$

$$\mathbf{e}_2 = \mathbf{n} \times \mathbf{e}_1, \epsilon_2 = \mathbf{n}' \times \epsilon_1, \tag{5.2.31}$$

satisfying

$$\mathbf{n}_+ \cdot \mathbf{e}_\lambda = 0, \tag{5.2.32}$$

$$\mathbf{n}_- \cdot \epsilon_\lambda = 0, \tag{5.2.33}$$

for $\lambda = 1, 2$. We use the completeness relations

$$\delta^{ij} = n_+^i n_+^j + \Sigma_\lambda \epsilon_\lambda^i \epsilon_\lambda^j = n_-^i n_-^j + \Sigma_\lambda \epsilon_\lambda^i \epsilon_\lambda^j, \tag{5.2.34}$$

and also set

$$\mathbf{S}_\pm = \int \frac{\mathrm{d}^3\mathbf{Q}}{4\pi^3\sqrt{2Q^0}} \mathbf{S}_\pm(Q_\pm) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \mathrm{e}^{-\mathrm{i}Q^0 x^0} \sin(\pm qz), \tag{5.2.35}$$

with $Q_+ = Q, Q_- = Q' = (Q^0, \mathbf{k}, -q), Q^0 = |\mathbf{Q}|$,

$$\mathbf{S}_\pm(Q_\pm) = \mathbf{J}_T(Q_\pm) - \frac{\mathbf{Q}_\pm + Q^0\mathbf{n}_\pm}{Q^0 + \mathbf{n}\cdot\mathbf{Q}}\mathbf{n}_\pm \cdot \mathbf{J}_T(Q_\pm), \tag{5.2.36}$$

from which we have

$$S^{*i}_\pm(Q_\pm)\delta^{ij}S^{*j}_\pm(Q_\pm) = J^{*i}_T(Q_\pm)\delta^{ij}J^{*j}_T(Q_\pm), \tag{5.2.37}$$

$$S^{*i}_\mp(Q_\pm)[-\delta^{ij} + 2\delta^{i3}\delta^{j3}]S^{*j}_\mp(Q_\pm) = J^{*i}_T(Q_\mp)[-\delta^{ij} + 2\delta^{i3}\delta^{j3}]J^{*j}_T(Q_\pm). \tag{5.2.38}$$

Here we note that $\mathbf{n}_\pm \cdot \mathbf{Q}_\pm = \mathbf{n}\cdot\mathbf{Q}$, and that for the points $\mathbf{Q}_\pm = \mathbf{n}|\mathbf{Q}|$, not only the numerators in the second term in (5.2.35) vanish but also $\mathbf{n}_\pm \cdot \mathbf{J}_T(Q_\pm) = 0$. Hence these points are apparent singularities in (5.2.34) belonging to sets of measure zero. Note, in particular, that $\mathbf{n}_\pm \cdot \mathbf{S}_\pm(x) = 0$.

From (5.2.37) and (5.2.34)-(5.2.38), the following explicitly expression for Ω emerges

$$\Omega = \int' \mathrm{d}^3\mathbf{x}_1 \int' \mathrm{d}^3\mathbf{x}_2 \frac{\nabla_+(x_2,x_1)}{2}\sum_\lambda \Bigg[(\mathrm{i}\mathbf{S}^*_{2+}(x_2)\cdot\mathbf{e}_\lambda)(\mathrm{i}\mathbf{S}_{1+}(x_1)\cdot\mathbf{e}_\lambda)$$

$$+ (\mathrm{i}\mathbf{S}^*_{2-}(x_2)\cdot\boldsymbol{\epsilon}_\lambda)(\mathrm{i}\mathbf{S}_{1-}(x_1)\cdot\boldsymbol{\epsilon}_\lambda) + \Bigg[(-1)^\lambda(\mathrm{i}\mathbf{S}^*_{2-}(x_2)\cdot\boldsymbol{\epsilon}_\lambda)(\mathrm{i}\mathbf{S}_{1+}(x_1)\cdot\mathbf{e}_\lambda)$$

$$+ (-1)^\lambda(\mathrm{i}\mathbf{S}^*_{2+}(x_2)\cdot\mathbf{e}_\lambda)(\mathrm{i}\mathbf{S}_{1-}(x_1)\cdot\boldsymbol{\epsilon}_\lambda)\Bigg]\Bigg] \tag{5.2.39}$$

with $\nabla_+(x_2,x_1) = \int' \mathrm{d}^3\mathbf{x} D(x_3,x)\mathrm{i}\overleftrightarrow{\partial_0} D(x,x_1)$. Clearly, the last two terms in (5.2.39) correspond to non-trivial transitions.

Let $|\mathbf{e}_\lambda, \mathbf{n}_+, x\rangle \equiv |\lambda, +, x\rangle, |\boldsymbol{\epsilon}_\lambda, \mathbf{n}_-, x\rangle \equiv |\lambda, -, x\rangle$ denote photon excitation states emitted at spacetime point $x = (x_0, \mathbf{r}, z)$ with associated vectors

$(\mathbf{e}_\lambda, n_+), (\boldsymbol{\epsilon}_\lambda, n_-)$, respectively. The physical significance of these associated vectors will be discussed in the light of the experiment being sought. A unitarity expansion of $\langle 0_+ \,|\, 0_- \rangle^J$ will include, in particular, the following four terms describing the emission, propagation and detection of photon excitations:

$$\langle 0_+ \,|\, \lambda, +, x_2 \rangle^{J_2} \langle \lambda, +, x_2 | \alpha, +, x_1 \rangle \langle \alpha, +, x_1 \,|\, 0_- \rangle^{J_1},$$

$$\langle 0_+ \,|\, \lambda, -, x_2 \rangle^{J_2} \langle \lambda, -, x_2 | \alpha, +, x_1 \rangle \langle \alpha, +, x_1 \,|\, 0_- \rangle^{J_1},$$

$$\langle 0_+ \,|\, \lambda, -, x_2 \rangle^{J_2} \langle \lambda, -, x_2 | \alpha, -, x_1 \rangle \langle \alpha, -, x_1 \,|\, 0_- \rangle^{J_1},$$

$$\langle 0_+ \,|\, \lambda, +, x_2 \rangle^{J_2} \langle \lambda, +, x_2 | \alpha, -, x_1 \rangle \langle \alpha, -, x_1 \,|\, 0_- \rangle^{J_1}. \qquad (5.2.40)$$

Here, for example, $\langle \alpha, +, x_1 \,|\, 0_- \rangle^{J_1}$ denotes the amplitude for the emission of a photon excitation in state $|\alpha, +, x_1\rangle$, with associated vectors $\mathbf{e}_\alpha, \mathbf{n}_+$, and $\langle 0_+ \,|\, \lambda, -, x_2 \rangle^{J_2}$ denotes the amplitude for the detection of a photon excitation in state $|\lambda, -, x_2\rangle$ with associated vectors $\boldsymbol{\epsilon}_\lambda, \mathbf{n}_-$. Most importantly $\langle \lambda, -, x_2 | \alpha, +, x_1 \rangle$, for example, denotes the amplitude of propagation of a photon excitation from spacetime point x_1 and associated vectors $\mathbf{e}_\alpha, \mathbf{n}_+$, to a spacetime point x_2 and ending up with associated vectors $\boldsymbol{\epsilon}_\lambda, \mathbf{n}_-$. Upon comparing the four terms in (5.2.39) with the corresponding ones in Ω given in (5.2.38), *and* using the completeness relation

$$\sum_{\delta=\pm} \int' \mathrm{d}^3\mathbf{x} \langle \lambda, \delta_2, x_2 | \lambda, \delta, x \rangle \langle \lambda, \delta, x | \lambda, \delta_1, x_1 \rangle = \langle \lambda, \delta_2, x_2 | \lambda, \delta_1, x_1 \rangle. \qquad (5.2.41)$$

with $\delta_1, \delta_2 = \pm$, we obtain

$$\langle 0_+ \,|\, \lambda, +, x \rangle^{J_2} = (\mathrm{i}\mathbf{S}^*_{2+}(x) \cdot \mathbf{e}_\lambda) \langle 0_+ \,|\, 0_- \rangle^{J_2}, \qquad (5.2.42)$$

$$\langle 0_+ \,|\, \lambda, -, x \rangle^{J_2} = (\mathrm{i}\mathbf{S}^*_{2-}(x) \cdot \boldsymbol{\epsilon}_\lambda) \, \langle 0_+ \,|\, 0_- \rangle^{J_2} \,, \tag{5.2.43}$$

$$\langle \lambda, +, x \,|\, 0_- \rangle^{J_1} = (\mathrm{i}\mathbf{S}_{1+}(x) \cdot \mathbf{e}_\lambda) \, \langle 0_+ \,|\, 0_- \rangle^{J_1} \,, \tag{5.2.44}$$

$$\langle \lambda, -, x \,|\, 0_- \rangle^{J_1} = (\mathrm{i}\mathbf{S}_{1-}(x) \cdot \boldsymbol{\epsilon}_\lambda) \, \langle 0_+ \,|\, 0_- \rangle^{J_1} \,. \tag{5.2.45}$$

$$\langle \lambda, \pm, x_2 | \alpha, \pm, x_1 \rangle = \frac{1}{2} \delta_{\lambda\alpha} \nabla_+(x_2, x_1), \tag{5.2.46}$$

$$\langle \lambda, \pm, x_2 | \alpha, \mp, x_1 \rangle = \frac{(-1)^\lambda}{2} \delta_{\lambda\alpha} \nabla_+(x_2, x_1). \tag{5.2.47}$$

where $a_\lambda(x) = \boldsymbol{\epsilon}_\lambda \cdot \mathbf{j}(x)$ and $a_\sigma = \sqrt{\mathrm{d}^3\mathbf{x}}\, a_\lambda(x)$ as shown in (4.2.30). Note the factor 1/2 in (5.2.46) and (5.2.47) which is essential to satisfy the completeness relation (5.2.41). $\nabla_+(x_2, x_1)$ works out to be

$$\nabla_+(x_2, x_1) = \frac{\mathrm{i}}{\pi^2} \sum_{\kappa=\pm} \frac{\kappa}{[(\mathbf{r_2} - \mathbf{r_1})^2 + (z_2 - \kappa z_1)^2 - (x_2^0 - x_1^0)^2]^2}, \tag{5.2.48}$$

not coinciding with the Feynman propagator for the corresponding B.C.. To simplify the expression in (5.2.48), we use the Schwinger representation

$$\frac{1}{A^2} = -\int_0^\infty s \; \mathrm{d}s \mathrm{e}^{-\mathrm{i}s(A - \mathrm{i}\varepsilon)}, \qquad \varepsilon \to +0 \tag{5.2.49}$$

for any given A.

To prove the Schwinger representation, we explicitly integrate over s on the right-hand side of (5.2.49)

$$-\int_0^\infty s \mathrm{d}s \mathrm{e}^{-\mathrm{i}s(A - \mathrm{i}\varepsilon)} = \frac{1}{\mathrm{i}(A - \mathrm{i}\varepsilon)} \int_0^\infty s[-\mathrm{i}(A - \mathrm{i}\varepsilon)] \mathrm{e}^{-\mathrm{i}s(A - \mathrm{i}\varepsilon)} \mathrm{d}s$$

$$= \frac{1}{\mathrm{i}(A - \mathrm{i}\varepsilon)} \left[s\mathrm{e}^{-\mathrm{i}s(A-\mathrm{i}\varepsilon)} \Big|_0^\infty - \int_0^\infty \mathrm{e}^{-\mathrm{i}s(A-\mathrm{i}\varepsilon)} \mathrm{d}s \right]$$

$$= \frac{1}{\mathrm{i}(A - \mathrm{i}\varepsilon)} \left[0 + \frac{1}{-\mathrm{i}(A - \mathrm{i}\varepsilon)} \right]$$

$$= \frac{1}{A^2}; \qquad \varepsilon \to 0. \tag{5.2.50}$$

Accordingly, (5.2.48) may be written as

$$\nabla_+(x_2, x_1) = \frac{(x_2^0 - x_1^0)}{\mathrm{i}\pi^2} \sum_{\kappa=\pm} \kappa \int_0^\infty s\mathrm{d}s \; \mathrm{e}^{-\mathrm{i}s[(\mathbf{r_2}-\mathbf{r_1})^2 + (z_2 - \kappa z_1)^2 - (x_2^0 - x_1^0)^2 - \mathrm{i}\varepsilon]} \tag{5.2.51}$$

In the sequel we suppress the $\mathrm{i}\varepsilon$ factor to simplify the notation.

The expression in (5.2.51) follows from (5.2.49), by setting first $A = [(\mathbf{r_2} - \mathbf{r_1})^2 + (z_2 - \kappa z_1)^2 - (x_2^0 - x_1^0)^2]^2$ to get, in the process,

$$\frac{1}{[(\mathbf{r_2} - \mathbf{r_1})^2 + (z_2 - \kappa z_1)^2 - (x_2^0 - x_1^0)^2]^2} = -\int_0^\infty s\mathrm{d}s\mathrm{e}^{-\mathrm{i}s[(\mathbf{r_2}-\mathbf{r_1})^2 + (z_2 - \kappa z_1)^2 - (x_2^0 - x_1^0)^2 - \mathrm{i}\varepsilon]}. \tag{5.2.52}$$

Upon substituting (5.2.52) in (5.2.49) we obtain (5.2.51) as expected.

5.3 Transition Probabilities: Rigorous Analysis of The Reflection Process

The transition amplitude that a photon excitation in a state$|\lambda_1, \delta_1, x_1\rangle$ propagates from $x_1 = (x_1^0, \mathbf{r}_1, z_1)$, reaches the reflecting surface within a skin depth, specified by a scale parameter σ and described by a Gaussian density distribution $\mathrm{e}^{-z^2/\sigma}/2\sqrt{\pi}\sigma, 0 \leq z$, and ends up in a state $|\lambda_2, -\delta_1, x_2\rangle$ at $x_2 = (x_2^0, \mathbf{r}_2, z_2)$ is given from (5.2.39) and (5.2.41) to be $(x_1^0 < x^0 < x_2^0)$

$$\mathcal{A}(x_2, x_1) = \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{r} \int_0^\infty \mathrm{d}z \, \frac{\mathrm{e}^{-z^2/\sigma^2}}{2\sqrt{\pi}\sigma}$$

$$\times \sum_{\delta=\pm} \delta_{\lambda_1\lambda_2} \langle \lambda_2, -\delta_1, x_2 | \lambda_1, \delta, x \rangle \langle \lambda_1, \delta, x | \lambda_1, \delta_1, x_1 \rangle, \qquad (5.3.1)$$

suppressing, for the moment, the indices λ_1, δ_1 in $\mathcal{A}(x_2, x_1)$ to which we will return later. We note from (5.2.45), (5.2.46) and (5.2.50), that the z-integrand in (5.3.1) is even in z. We may also introduce a surface density amplitude $f(\mathbf{R})$ by

$$\mathcal{A}(x_2, x_1) = \int \mathrm{d}^2\mathbf{R} f(\mathbf{R}). \qquad (5.3.2)$$

By multiplying the $\mathbf{r}$-integrand in (5.3.1) by the identity

$$\int \frac{\mathrm{d}^2\mathbf{R}}{\pi\sigma_0^2} \mathrm{e}^{-(\mathbf{R}-\mathbf{r})^2/\sigma_0^2} = 1, \qquad (5.3.3)$$

valid for any $\sigma_0^2 > 0$, and *any* $\mathbf{r}$, giving from (5.2.45), (5.2.46), (5.3.1) and (5.2.50)

$$f(\mathbf{R}) = \frac{\delta_{\lambda_1\lambda_2}(-1)^{\lambda_1}}{8} \int \frac{\mathrm{d}^2\mathbf{r}}{\pi\sigma_0^2} \mathrm{e}^{-(\mathbf{R}-\mathbf{r})^2/\sigma_0^2}$$

$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{\pi}\sigma} \mathrm{e}^{-z^2/\sigma^2} \nabla_+(x_2, x) \nabla_+(x, x_1). \qquad (5.3.4)$$

Given that a photon excitation was emitted in state $|\lambda_1, \delta_1, x_1\rangle$, *reaching* the reflecting surface within a skin depth, and ending up in state $|\lambda_2, -\delta_1, x_2\rangle, \lambda_2 = \lambda_1$, the conditional amplitude density for the process is then given by

$$F(\mathbf{R}) = f(\mathbf{R}) / \mathcal{A}(x_2, x_1),$$

with

$$\int \mathrm{d}^2\mathbf{R}F(\mathbf{R}) = 1,$$

as a *summation* over impact centers on the reflecting surface whose nature will be now investigated. Let $T_1 = x^0 - x_1^0, T_2 = x_2^0 - x^0$. The $\mathbf{r}-, z-$ integrals in (5.3.4) may be explicitly carried out yielding

$$F(\mathbf{R}) = \frac{1}{N}\sum_{\kappa_1,\kappa_2}\kappa_1\kappa_2\int_0^\infty \mathrm{d}u_1\int_0^\infty \mathrm{d}u_2\frac{u_1u_2}{\sigma\sqrt{1+\mathrm{i}\,(u_1+u_2)}}$$

$$\times \exp\left[-\frac{z_1^2}{\sigma^2}\frac{(u_1+\kappa_1\kappa_2u_2z_2/z_1)^2}{1+\mathrm{i}\,(u_1+u_2)}\right]\left[1+\mathrm{i}\frac{\sigma_0^2}{\sigma^2}\,(u_1+u_2)\right]^{-1}$$

$$\times \exp\left[-\frac{\sigma_0^2}{\sigma^4}\frac{[u_1(\mathbf{r}_1-\mathbf{R})+u_2(\mathbf{r}_2-\mathbf{R})]^2}{1+\mathrm{i}\sigma_0^2\,(u_1+u_2)\,/\sigma^2}\right]\mathrm{e}^{-\mathrm{i}G(u_1,u_2,\mathbf{R})/\sigma^2}, \qquad (5.3.5)$$

$$N = \sum_{\kappa_1,\kappa_2}\kappa_1\kappa_2\int_0^\infty \mathrm{d}u_1\int_0^\infty \mathrm{d}u_2\frac{u_1u_2}{\sigma\sqrt{1+\mathrm{i}\,(u_1+u_2)}}$$

$$\times \exp\left[-\frac{z_1^2}{\sigma^2}\frac{(u_1+\kappa_1\kappa_2u_2z_2/z_1)^2}{1+\mathrm{i}\,(u_1+u_2)}\right]\int \mathrm{d}^2\mathbf{r}\mathrm{e}^{-\mathrm{i}G(u_1,u_2,\mathbf{r})/\sigma^2}, \qquad (5.3.6)$$

with

$$G(u_1,u_2,\mathbf{r}) = u_1[(\mathbf{r_1}-\mathbf{r})^2+z_1^2-T_1^2]+u_2[(\mathbf{r_2}-\mathbf{r})^2+z_2^2-T_2^2]. \qquad (5.3.7)$$

For the practical case $\sigma \ll z_1$, with a given initially chosen macroscopic value z_1, i.e., for $z_1^2/\sigma^2 \gg 1$, we may use the distributional limit

$$\frac{z_1/\sigma}{\sqrt{1+\mathrm{i}\left(u_1+u_2\right)}}\exp\left[-\frac{z_1^2}{\sigma^2}\frac{\left(u_1+\kappa_1\kappa_2u_2z_2/z_1\right)^2}{1+\mathrm{i}\left(u_1+u_2\right)}\right]\to\sqrt{\pi}\delta\left(u_1+\kappa_1\kappa_2u_2z_2/z_1\right) \tag{5.3.8}$$

in (5.3.5) as obtained, for example, by Fourier transform techniques and shown below, to obtain $u_1 = -\kappa_1\kappa_2u_2z_2/z_1$, with the necessary restrictions $\kappa_1 = \pm1, \kappa_2 = \mp1$, giving for $\sigma \ll z_1$,

$$F\left(\mathbf{R}\right)=\frac{\mathrm{i}}{\pi\sigma_0^2C}\int_0^\infty\frac{w^2\mathrm{d}w}{\sigma_0+\mathrm{i}w}\exp\left[-\frac{\mathrm{i}w\mathbf{G}\left(z_2,z_1,\mathbf{R}_0\right)}{\sigma_0^3\left(z_1+z_2\right)}\right]$$

$$\times\exp\left[-\frac{\mathrm{i}w\left(\mathbf{R}-\mathbf{R}_0\right)^2}{\sigma_0^2\left(\sigma_0+\mathrm{i}\omega\right)}\right], \tag{5.3.9}$$

where

$$C=\int_0^\infty w\mathrm{d}w\exp\left[-\frac{\mathrm{i}w\mathbf{G}\left(z_2,z_1,\mathbf{R}_0\right)}{\sigma_0^3\left(z_1+z_2\right)}\right] \tag{5.3.10}$$

$$\mathbf{R}_0=\left(z_2\mathbf{r}_1+z_1\mathbf{r}_2\right)/\left(z_1+z_2\right). \tag{5.3.11}$$

Here we note in reference to (5.3.8), that the Fourier transform of a Gaussian in one dimension:

$$\frac{1}{\sqrt{2\pi}\sigma}\exp-\frac{x^2}{2\sigma^2}, \tag{5.3.12}$$

is given by the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma} \exp -\frac{x^2}{2\sigma^2} \exp \mathrm{i}kx \equiv f(k), \tag{5.3.13}$$

which explicitly integrates out to

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma} \exp -\frac{x^2}{2\sigma^2} \exp \mathrm{i}kx$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma} \exp -\frac{1}{2\sigma^2}\left[x^2 - 2\mathrm{i}\sigma^2 kx + (\mathrm{i}\sigma^2 k)^2 - (\mathrm{i}\sigma^2 k)^2\right]$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma} \exp -\frac{1}{2\sigma^2}\left[(x - \mathrm{i}\sigma^2 k)^2 + \sigma^4 k^2\right]$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \mathrm{i}\sigma^2 k)^2}{2\sigma^2} - \frac{\sigma^2 k^2}{2}\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{\sigma^2 k^2}{2}} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{e}^{-(\frac{x - \mathrm{i}\sigma^2 k}{\sqrt{2\sigma^2}})^2}$$

$$= \mathrm{e}^{-\frac{\sigma^2 k^2}{2}} \equiv f(k) \tag{5.3.14}$$

For a function $g(x)$ in the scalar product:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma} \exp\left[-\frac{x^2}{2\sigma^2}\right] g(x) = \int_{-\infty}^{\infty} f(k)\tilde{g}(k)\frac{\mathrm{d}k}{2\pi}, \tag{5.3.15}$$

$\tilde{g}(k)$ is the Fourier transform of $g(x)$, and

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{x^2}{2\sigma^2}}\, \mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{-\frac{\sigma^2 k^2}{2}}, \tag{5.3.16}$$

as shown in (5.3.14)

Therefore, for $\sigma \to 0$, the right-hand side of (5.3.15) is equal to

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \tilde{g}(x) = g(0). \tag{5.3.17}$$

Since

$$g(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k\mathrm{e}^{\mathrm{i}kx}}{2\pi} \tilde{g}(x), \tag{5.3.18}$$

$$\lim_{\sigma \to 0} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}\sigma} \exp\left[-\frac{x^2}{2\sigma^2}\right] g(x) = g(0), \tag{5.3.19}$$

and the Gaussian behaves as a delta function for $\sigma \to 0$:

$$\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \to \delta(x) \tag{5.3.20}$$

For an effective area of impact $\pi\sigma_0^2$ about a point $\mathbf{R}$ for $\sigma_0 \to 0$, we obtain

$$\pi\sigma_0^2 F\left(\mathbf{R}\right) \to \exp\left[-\left(\mathbf{R}-\mathbf{R}_0\right)^2/\sigma_0^2\right] \frac{\int_0^\infty w\mathrm{d}w \exp\left[-\frac{\mathrm{i}w\mathrm{G}(z_2,z_1,\mathbf{R}_0)}{\sigma_0^3(z_1+z_2)}\right]}{\int_0^\infty w\mathrm{d}w \exp\left[-\frac{\mathrm{i}w\mathrm{G}(z_2,z_1,\mathbf{R}_0)}{\sigma_0^3(z_1+z_2)}\right]}, \tag{5.3.21}$$

with the second factor thus being independent of σ_0, giving the remarkably simple expression

$$\pi\sigma_0^2 F\left(\mathbf{R}\right) \rightarrow \mathrm{e}^{-(\mathbf{R}-\mathbf{R}_0)^2/\sigma_0^2}\,. \tag{5.3.22}$$

For $\sigma_0^2 \rightarrow 0$. Accordingly, for an arbitrary small σ_0, giving a point-like area of impact, about the point $\mathbf{R}$, the partial amplitude $\pi\sigma_0^2 F(\mathbf{R})$ vanishes *exponentially* for $\mathbf{R} \neq \mathbf{R_0}$, i.e., for the non-classical point of impact. On the other hand for $\mathbf{R} = \mathbf{R}_0$, we have

$$\pi\sigma_0^2 F(\mathbf{R}) \rightarrow 1, \tag{5.3.23}$$

for $\sigma_0^2 \rightarrow 0$.

The condition $\mathbf{R} = \mathbf{R}_0$, translates from (5.3.11) to $(\mathbf{r}_1 - \mathbf{R}_0)/z_1 = -(\mathbf{r}_2 - \mathbf{R}_0)/z_2$ which is nothing but the law of reflection with $\mathbf{R}_0$ denoting the classical point of impact. For given $(\mathbf{r}_1, z_1), (\mathbf{r}_2, z_2)$, leading from (5.3.11) to a fixed value of $\mathbf{R}_0$, we may choose the unit vectors $\mathbf{n}_+, \mathbf{n}_-$ in (5.2.34), to be directed along the vectors $(\mathbf{R}_0 - \mathbf{r}_1, -z_1), (\mathbf{r}_2 - \mathbf{R}_0, z_2) = z_2(\mathbf{R}_0 - \mathbf{r}_1, z_1)/z_1$, respectively, corresponding to classical rays, with the vectors $(\mathbf{e}_1, \mathbf{e}_2), (\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$ having the well know interpretations of polarization vectors perpendicular, respectively, to $\mathbf{n}_+, \mathbf{n}_-$, with *transformations* $\mathbf{e}_1 \leftrightarrow \boldsymbol{\epsilon}_1, \mathbf{e}_2 \leftrightarrow \boldsymbol{\epsilon}_2$ upon scattering.

Our formalism clearly opens the way for practical spacetime analyses of photon dynamics and *other* interacting field theories by using, in the process, functional differential techniques (Manoukian, 1986, 1991) in different experimental situations. It shows, in particular, how amplitudes of propagation are determined from the knowledge of amplitudes of emissions and absorption of particle excitations by emitters and detectors, respectively, signaling the power of the present method of analysis. These further developments in general field theories emphasize the practicality and generality of the problem treated here not just being restricted to it.

CHAPTER VI
CONCLUSION

This thesis was involved with a careful analysis of the propagation of photons in spacetime as a time evolution process dealing with *amplitudes* of transitions of photon excitations between different points $(x_1^0, \mathbf{x}_1), (x_2^0, \mathbf{x}_2)$ in the spacetime continuum. These amplitudes describe, as time evolution process, in spacetime, as photon excitations are emitted and absorbed (detected) by various sources. the quantum physics treatment of a non-relativistic particle provided a guide for the far more complex problem dealing with photons, as ultra-relativistic particles, in the quantum field theory analysis. For a non-relativistic particle of mass m, the amplitude of propagation from a point $(x_1^0, \mathbf{x}_1)$ to $(x_2^0, \mathbf{x}_2)$, for $x_2^0 > x_1^0$, is well known and is given by the expression

$$\left\langle x_2^0, \mathbf{x}_2 \,\middle|\, x_1^0, \mathbf{x}_1 \right\rangle = \left(\frac{m}{2\pi \mathrm{i}\hbar (x_2^0 - x_1^0)} \right)^{3/2} \exp\left[\frac{\mathrm{i}m|\mathbf{x_2} - \mathbf{x_1}|^2}{2\hbar (x_2^0 - x_1^0)} \right]. \tag{6.0.1}$$

The amplitude satisfies a very important completeness relation

$$\left\langle x_2^0, \mathbf{x}_2 \,\middle|\, x_1^0, \mathbf{x}_1 \right\rangle = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x} \left\langle x_2^0, \mathbf{x}_2 \,\middle|\, x^0, \mathbf{x} \right\rangle \left\langle x^0, \mathbf{x} \,\middle|\, x_1^0, \mathbf{x}_1 \right\rangle. \tag{6.0.2}$$

for any $x_2^0 > x^0 > x_1^0$. This completeness relation allows a systematic analysis of the propagation of non-relativistic particles in configuration space as a time evolution process in quantum physics. In the functional differential treatment of quantum field theory dealing with relativistic particles one is dealing with the so-called vacuum-to-vacuum transition amplitude $\langle 0_+ | 0_- \rangle$ in the presence of external sources. From the expression of the amplitude that a given some $J_\mu^1(x_1)$ in spacetime emits a photon excitation and is then absorbed by a source $J_\mu^2(x_2)$, representing a detector, the *amplitude* of a photon excitation from a point $x_1 = (x_1^0, \mathbf{x}_1)$ to a point $x_2 = (x_2^0, \mathbf{x}_2)$, as a time evolution process, is obtained from a unitarity expansion of $\langle 0_+ | 0_- \rangle$ to be

$$\left\langle x_2^0, \mathbf{x}_2 \,\middle|\, x_1^0, \mathbf{x}_1 \right\rangle = \frac{\mathrm{i}\delta_{\lambda_1\lambda_2}}{\pi^2} \frac{(x_2^0 - x_1^0)^2}{\left[(x_2 - x_1)^2\right]^2}. \tag{6.0.3}$$

when $\lambda_1 \lambda_2$ are associated with polarization vectors $\mathbf{e}_{\lambda_1},\ \mathbf{e}_{\lambda_2}$, which using the Schwinger representation may be written as

$$\frac{(x_2^0 - x_1^0)^2}{\mathrm{i}\pi^2} \delta_{\lambda_1\lambda_2} \int_0^\infty s \mathrm{d}s \mathrm{e}^{-\mathrm{i}s[(\mathbf{x}_2 - \mathbf{x}_1)^2 - (x_2^0 - x_1^0)^2 - \mathrm{i}\varepsilon]}, \tag{6.0.4}$$

for $\varepsilon \to +0$. This expression is different from the so-called Feynman propagator

$$\frac{\mathrm{i}}{(x_2 - x_1)^2}. \tag{6.0.5}$$

The amplitude in (6.0.3) satisfies a completeness relation as in (6.0.2) while (6.0.5) does not, showing the internal consistency of the analysis as a time evolution process in spacetime. The major problem of this project was to develop in detail this formalism for the propagation of photons in spacetime as a time evolution process based on the actual physical process of the propagation of photons excitations from emitters to detectors as obtained from the so-called vacuum-to-vacuum transition amplitude for the underlying theory when photon excitations encounter an obstacle - a reflecting surface. This method has been quite successful over the years in the easiness of momentum space computations, of physical processes, avoiding of introducing so-called wavefunctions, not to mention of the elegance of the formalism as opposed to more standard techniques, and at the same time gaining much physical insight as particles propagate from emitters, interact, and finally particles reach the detectors as occurring in practice. The analysis is applied not only to infinitely extended space but also in half-space dealing, rigorously, with the reflection process where photon excitations may encounter an obstacle. The present analysis rests on three general key points: (i) By working directly in spacetime for the vacuum-to-vacuum transition amplitude, for given boundary conditions (B.C.), and from the expressions of the amplitudes for the emission and detection of photon excitations by the external sources, an amplitude of propagation between different spacetime points from emitters to detectors, causally arranged, is extracted and, as mentioned above, it does not coincide with the Feynman propagator for the corresponding B.C. This step already shows the *power of determining amplitudes of propagation by introducing external sources*. (ii) The amplitude of propagation is shown to satisfy a completeness relation as photons propagate between different points critical for the internal consistency of the theory in spacetime. (iii) Application of these amplitudes to describe in detail the experiment being sought dealing with reflections off a surface by showing, in the process, very rapid exponential damping beyond the classical point of impact for the corresponding amplitude of occurrence. The reader will soon realize that our theoretical quantum (field) theory QED formalism is reduced to a non-operator approach and opens a way to describe, as a time evolution process, photon dynamics in spacetime and other field theory interactions in different experimental situations as well. The amplitude of photon propagation from an emitter to a detector, without reaching the reflecting surface, as well as the amplitude of propagation from as emitter to a detector while reaching the reflecting surface are derived in Chapter V by a unitarity expansion of the vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle$ supplemented by the expressions for the amplitudes of emissions and detections of photon excitations. By a very detailed analysis it is shown, by *explicit derivations*, that photon excitations may reflect off the reflecting surface at *any* point. All such points are shown to be *exponentially* damped relative to the classical point of impact. The derivation rests entirely on a quantum field theory derivation and involves intricate details described in Chapter V. In quantum field theory, (derived) amplitudes are associated with the localization of photon excitations in configuration space, that lead, in quantum probabilistic sense, probabilities as to where these excitations were in space within given time spans. In reference to the reflecting surface, at some time, a photon excitation, is found to reach the surface with an amplitude to go through the classical point of impact with an amplitude which exponentially dominates other possible points of impact with corresponding amplitudes. As a further analysis, we have carried out in an Appendix, the exact-$\hbar$-quantum correction for the average *number* of photons emitted in synchrotron radiation per revolution with our original contribution in obtaining and evaluating the corresponding integral for this number in closed form.

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APPENDICES

APPENDIX A

COUNTING NUMBER OF PHOTONS IN SYNCHROTRON RADIATION: EXACT $\hbar$–QUANTUM CORRECTION

The purpose of this appendix is to obtain the explicit closed $\hbar$-quantum correction for the number of photons in synchrotron radiation as they become localized in detectors. The historical development of the theory of synchrotron radiation and the fascinating story behind it are well documented in the literature (see, e.g., Pollock, 1983) and relatively recent theoretical progress in the field as well as extensive references may be found in Bordovitsyn (1999). Although many features of synchrotron radiation have been well known for a long time, there is room for further developments and certainly for improvements. For example, in a recent investigation (Manoukian and Jearnkulprasert, 2000), an explicit expression for the mean number $\langle N \rangle_C$ of photons emitted per revolution was derived, based on the classical analysis, involving a remarkably simple one-dimensional integral, with C, here, standing for classical. The latter is given by

$$\langle N \rangle_C = 2\alpha\beta^2 \int_0^\infty \frac{\mathrm{d}x}{x^2} \left[\frac{\left(\frac{\sin x}{x}\right)^2 - \cos(2x)}{1 - \beta^2 \left(\frac{\sin x}{x}\right)^2} \right] \tag{A.1}$$

where $\beta = v/c$, v is the speed of the charged particle, c is the speed of light and a is the fine-structure constant. For high energetic charged particles, (A.1) gives (Manoukian and Jearnkulprasert, 2000):

$$\langle N \rangle_C \simeq \frac{5\pi\alpha}{\sqrt{3(1-\beta^2)}} + a_0\alpha + O(\sqrt{1-\beta^2}) \tag{A.2}$$

where the constant a_0 is overwhelmingly large in magnitude and is given by

$$a_0 = 2 \int_0^\infty \frac{\mathrm{d}x}{x^2} \left[\frac{6\left(\frac{\sin x}{x}\right)^2 - \cos(2x) - 5}{1 - \left(\frac{\sin x}{x}\right)^2} \right] = -9.55797 \tag{A.3}$$

and the second term on the right hand of (A.2) gives an important contribution for high energetic particles and was unfortunately missing in the earlier investigations (see e.g., Particle Data Group, 2004).

For example, the relative errors in (A.2) are quite satisfactory with 4.11%, 1.34%, 0.063% for $\beta = 0.8$, 0.9, 0.99, respectively, to be compared with the relative errors of 160%, 82%, 17% of the well known expression tabulated earlier (Particle Data Group, 2004) involving only the first term on the right-hand side of (A.2). A systematic asymptotic analysis for high energetic relativistic particles has been also carried out more recently by Manoukian et al. (2004), based on (A.1), providing additional corrections to the ones on the right-hand side of (A.2) as functions of $\sqrt{1-\beta^2}$.

The purpose of this communication is to derive the quantum correction $\langle N \rangle_Q$ in closed form to the mean *number* $\langle N \rangle$ of photons emitted per revolution, to the order $\hbar$, to supplement our explicit expression in (A.1), which was based on the classical analysis, giving the final result $\langle N \rangle = \langle N \rangle_C + \langle N \rangle_Q$ where $\langle N \rangle_Q$ is given in (A.26).

The Quantum Correction

The explicit integral expression for the quantum correction, to the order $\hbar$, to the mean *number* $\langle N \rangle$ may be obtained from that of the formula of the power of (Schwinger, 1949, Eqs. (III), (6), (7); 1954, (24)) and Schwinger and Tsai (1978, (C.11)) given by

$$\langle N \rangle_Q = \alpha \hbar \left(\frac{\delta}{\delta \hbar} F(h) \right)_{\hbar \to 0} \tag{A.4}$$

where

$$F(h) = \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}\omega(1+\hbar\omega/E)\tau} \frac{\beta^2 \cos\omega_0\tau - 1}{\beta \sin\frac{\omega_0\tau}{2}} \sin\left(2\beta\frac{\omega}{\omega_0}\left(1+\frac{\hbar\omega}{E}\right)\sin\frac{\omega_0\tau}{2}\right), \tag{A.5}$$

and, in our notation,

$$\frac{E}{mc^2} = \frac{1}{\sqrt{1-\beta^2}}, \qquad \omega_0 = \frac{\beta c}{R} = \frac{cqB\sqrt{1-\beta^2}}{mc^2} \tag{A.6}$$

with q, R and B denoting, respectively, the magnitude of the charge of the particle, the radius of the classical circular motion, and the magnetic field in question (A.5) gives

$$\left.\frac{\delta}{\delta\hbar}F(h)\right|_{\hbar\to 0} = \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}\omega\tau} \frac{\beta^2\cos\omega_0\tau - 1}{\beta\sin\frac{\omega_0\tau}{2}}$$

$$\times \left[-\mathrm{i}\frac{\omega^2}{E}\tau\sin\left(2\beta\frac{\omega}{\omega_0}\sin\frac{\omega_0\tau}{2}\right) + \cos\left(2\beta\frac{\omega}{\omega_0}\sin\frac{\omega_0\tau}{2}\right)\left(2\beta\frac{\omega}{\omega_0}\frac{\omega}{E}\sin\frac{\omega_0\tau}{2}\right)\right]. \tag{A.7}$$

Let

$$\frac{\omega}{\omega_0} = z, \qquad \omega_0\tau = x, \qquad 2\beta\sin\frac{\omega_0\tau}{2} = a(x) \tag{A.8}$$

in (A.7) to rewrite the latter in the more convenient form:

$$\left.\frac{\delta}{\delta\hbar}F(h)\right|_{\hbar\to 0} = \frac{\omega_0}{E}\int_{-\infty}^\infty \mathrm{d}x \left(\beta^2\cos x - 1\right)\int_0^\infty \mathrm{d}z\ z^2$$

$$\times \left[\mathrm{e}^{-\mathrm{i}z(x+a(x))}\frac{x+a(x)}{a(x)} + \mathrm{e}^{-\mathrm{i}z(x-a(x))}\frac{x-a(x)}{-a(x)}\right]. \tag{A.9}$$

This seemingly complicated double integral may be computed in closed form. To this end, let

$$x + a(x) = \xi, \tag{A.10}$$

and note that

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = \frac{1}{(1 + \beta \cos \frac{x}{2})}. \tag{A.11}$$

Also for $x \to \pm\infty,\ \xi \to \pm\infty$, for $\xi = 0, x = 0$

Accordingly, $\langle N \rangle_Q$ in (A.4) may be obtained by taking the real part of (A.9) giving

$$\langle N \rangle_Q = \alpha\pi \left(\frac{\hbar\omega_0}{E}\right) \int\limits_{-\infty}^{\infty} \mathrm{d}\xi [I(\xi, \beta) + I(\xi, -\beta)], \tag{A.12}$$

where

$$I(\xi, \beta) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\delta(\xi)\right) \left(\frac{\beta^2 \cos x - 1}{2\beta \sin \frac{x}{2}}\right) \left(\frac{x + 2\beta \sin \frac{x}{2}}{1 + \beta \cos \frac{x}{2}}\right). \tag{A.13}$$

and we have used the integral

$$\mathrm{Re} \int\limits_{0}^{\infty} \mathrm{d}z \mathrm{e}^{-\mathrm{i}z\xi} = \pi\delta(\xi). \tag{A.14}$$

To evaluate the integral in (A.12), we use the relations

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} = \frac{1}{(1 + \beta \cos \frac{x}{2})^2} \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\frac{\beta}{2} \sin \frac{x}{2}}{1 + \beta \cos \frac{x}{2}} \frac{\mathrm{d}}{\mathrm{d}x}\right], \tag{A.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sin \frac{x}{2}}\right)\Bigg|_{x=0} = 0, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{x}{\sin \frac{x}{2}}\right)\Bigg|_{x=0} = \frac{1}{6}, \tag{A.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta\cos x-1}{1+\beta\cos\frac{x}{2}}\right)\Big|_{x=0}=0,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\beta\cos x-1}{1+\beta\cos\frac{x}{2}}\right)\Big|_{x=0}=\frac{-\beta(1+3\beta)}{4(1+\beta)}. \tag{A.17}$$

We first evaluate the integral

$$\int_{-\infty}^{\infty}\mathrm{d}\xi I(\xi,\beta)=\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\beta^2\cos x-1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)\Big|_{x=0} \tag{A.18}$$

where we have used the property of the delta function $\delta(\xi)$ and the fact that $\xi=0$ implies that $x=0$. In order to carry out the differentiation $\mathrm{d}^2/\mathrm{d}\xi^2$, we use (A.15). In detail, from (A.13) we have

$$\int_{-\infty}^{\infty}\mathrm{d}\xi I(\xi,\beta)=\int_{-\infty}^{\infty}\mathrm{d}\xi\left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\delta(\xi)\right)\left(\frac{\beta^2\cos x-1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)$$

$$=-\int_{-\infty}^{\infty}\mathrm{d}\xi\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\delta(\xi)\right)F(x), \tag{A.19}$$

where

$$F(x)=\left(\frac{\beta^2\cos x-1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right). \tag{A.20}$$

Integrating by parts (A.19) becomes

$$\int_{-\infty}^{\infty}\mathrm{d}\xi I(\xi,\beta)=-\int_{-\infty}^{\infty}\mathrm{d}\xi\delta(\xi)\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\beta^2\cos x-1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)\right),$$

$$=\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\beta^2\cos x-1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)\right)\Big|_{x=0}. \tag{A.21}$$

Using (A.15), (A.21) become

$$\int\limits_{-\infty}^{\infty} \mathrm{d}\xi I(\xi,\beta) = -\left(\frac{1}{(1+\beta\cos\frac{x}{2})^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\frac{\beta}{2}\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\right]\right.$$

$$\left.\times\left(\frac{\beta^2\cos x - 1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)\right)\Bigg|_{x=0}$$

$$= -\left(\frac{1}{(1+\beta\cos\frac{x}{2})^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\beta^2\cos x - 1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)\right.\right.$$

$$\left.\left.+\frac{\frac{\beta}{2}\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x - 1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)\right]\right)\Bigg|_{x=0}. \quad \text{(A.22)}$$

considering the first term in the brackets, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\beta^2\cos x - 1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x - 1}{1+\beta\cos\frac{x}{2}}\right)\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\beta^2\cos x - 1}{1+\beta\cos\frac{x}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)+\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x - 1}{1+\beta\cos\frac{x}{2}}\right)\right]$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\beta^2\cos x - 1}{1+\beta\cos\frac{x}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\right]$$

$$+\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x - 1}{1+\beta\cos\frac{x}{2}}\right)\right]$$

$$=\left(\frac{\beta^2\cos x - 1}{1+\beta\cos\frac{x}{2}}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x - 1}{1+\beta\cos\frac{x}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)$$

$$+\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\beta^2\cos x-1}{1+\beta\cos\frac{x}{2}}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x-1}{1+\beta\cos\frac{x}{2}}\right)$$

$$=\left(\frac{\beta^2\cos x-1}{1+\beta\cos\frac{x}{2}}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)+\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\beta^2\cos x-1}{1+\beta\cos\frac{x}{2}}\right)$$

$$=\left(\frac{\beta^2\cos x-1}{1+\beta\cos\frac{x}{2}}\right)\frac{1}{12\beta}+\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\frac{-\beta(1+3\beta)}{4(1+\beta)}$$

$$=\left(\frac{\beta^2-1}{1+\beta}\right)\frac{1}{12\beta}+\frac{-\beta(1+3\beta)}{4(1+\beta)}$$

$$=\frac{\beta-1}{12\beta}+\frac{-\beta(1+3\beta)}{4(1+\beta)}$$

$$=-\left(\frac{1-\beta}{12\beta}+\frac{\beta(1+3\beta)}{4(1+\beta)}\right), \tag{A.23}$$

where we have used the condition at $x=0$, by using (A.16), (A.17). For the second term in the bracket of (A.21), we similarly have

$$\frac{\frac{\beta}{2}\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x-1}{2\beta\sin\frac{x}{2}}\right)\left(\frac{x+2\beta\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\right)\Bigg|_{x=0}$$

$$=\frac{\frac{\beta}{2}\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\left(\frac{\beta^2\cos x-1}{1+\beta\cos\frac{x}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\Bigg|_{x=0}$$

$$+\frac{\frac{\beta}{2}\sin\frac{x}{2}}{1+\beta\cos\frac{x}{2}}\left(\frac{x}{2\beta\sin\frac{x}{2}}+1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2\cos x-1}{1+\beta\cos\frac{x}{2}}\right)\Bigg|_{x=0}$$

$$=0\,. \tag{A.24}$$

From the above we obtain

$$\int_{-\infty}^{\infty} \mathrm{d}\xi I(\xi,\beta) = \frac{1}{(1+\beta)^2}\left[\frac{(1-\beta)}{12\beta} + \frac{1+3\beta}{4}\right]. \tag{A.25}$$

The quantum correction $\langle N\rangle_Q$ then emerges from (A.12) and (A.18) to be (Manoukian and Viriyasrisuwattana, 2006):

$$\langle N\rangle_Q = \frac{8\pi}{3}\alpha\left(\frac{\hbar\omega_0}{mc^2}\right)\left[\left(\frac{E}{mc^2}\right) - \left(\frac{E}{mc^2}\right)^3\right]. \tag{A.26}$$

For $\beta \to 1$, this gives the truly asymptotic formula

$$\langle N\rangle_Q \to -\frac{8\pi}{3}\alpha\left(\frac{\hbar\omega_0}{mc^2}\right)\left(\frac{E}{mc^2}\right)^3. \tag{A.27}$$

The expression in (A.19) supplements the explicit form $\langle N\rangle_C$ in (A.1) in a quantum mechanical setting.

For the synchrotron in our institution, $R = 2.78$ m and $E = 1.2$ GeV. This gives the estimates $\langle N\rangle_C \sim \alpha 2.14\times 10^4$ and $\langle N\rangle_Q \sim \alpha 1.51\times 10^{-2}$, in magnitudes, as based on (A.1)/(A.2) and (A.19)/(A.20), respectively. The latter is indeed relatively small but may be, however, significant for several revolutions in the magnetic field. This small quantum correction is not necessarily to be dismissed on practical grounds and may be reminiscent of small radiative corrections such as the Lamb shift contribution to the spectrum of the hydrogen atom which has been measured with very high accuracy and has led to much new physics. The quantum correction given in this work may be equally challenging to detect experimentally. It is interesting to note that a singularity in β for $\beta \to 1$ in $\langle N\rangle_Q$ arises as in the classical treatment. Our quantum correction is based on a leading $\hbar$-contribution. At present it is not clear what would be the expression for $\langle N\rangle_Q$ in an exact $\hbar$-all order treatment. Would such an expression compete with its classical counterpart and would it be practically relevant? Would it be singular in β for $\beta \to 1$? The exact $\hbar$-all order treatment of $\langle N\rangle_Q$ as well as its experimental detection remain formidable problems and will be hopefully confronted with in the near future.

APPENDIX B

PUBLICATION PAPER

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Technical note

Quantum correction to the photon number emission in synchrotron radiation

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Abstract

The quantum correction to the mean *number* $\langle N \rangle$ of photons emitted per revolution, to the order $\hbar$, is derived in closed form in synchrotron radiation which supplements our explicit expression obtained earlier for $\langle N \rangle$ which was based on the classical analysis.

1. Introduction

The historical development of the theory of synchrotron radiation and the fascinating story behind it are well documented in the literature (see, e.g., Pollock, 1983) and relatively recent theoretical progress in the field as well as extensive references may be found in Bordovitsyn (1999). Although many features of synchrotron radiation have been well known for a long time, there is room for further developments and certainly for improvements. For example, in a recent investigation (Manoukian and Jearnkulprasert, 2000), an explicit expression for the mean *number* $\langle N \rangle_C$ of photons emitted per revolution was derived, based on the classical analysis, involving a remarkably simple one-dimensional integral, with C, here, standing for classical. The latter is given by

$$\langle N \rangle_C = 2\alpha\beta^2 \int_0^\infty \frac{dx}{x^2}\left[\frac{(\sin x/x)^2 - \cos(2x)}{1-\beta^2(\sin x/x)^2}\right], \qquad (1.1)$$

where $\beta = v/c$, v is the speed of the charged particle, c is the speed of light and α is the fine-structure constant. For high energetic charged particles, Eq. (1.1) gives (Manoukian and Jearnkulprasert, 2000)

$$\langle N \rangle_C \simeq \frac{5\pi\alpha}{\sqrt{3(1-\beta^2)}} + a_0\alpha + \mathcal{O}\left(\sqrt{1-\beta^2}\right), \qquad (1.2)$$

where the constant a_0 is overwhelmingly large in magnitude and is given by

$$a_0 = 2\int_0^\infty \frac{dx}{x^2}\left[\frac{6(\sin x/x)^2 - \cos(2x) - 5}{1-(\sin x/x)^2}\right] = -9.55797 \qquad (1.3)$$

and the second term on the right hand of (1.2) gives an important contribution for high energetic particles and was unfortunately missing in the earlier investigations (see, e.g., Particle Data Group, 2004).

For example, the relative errors in (1.2) are quite satisfactory with 4.11%, 1.34%, 0.063% for $\beta = 0.8, 0.9, 0.99$, respectively, to be compared with the relative errors of 160%, 82%, 17% of the well known expression tabulated earlier (Particle Data Group, 2004) involving only the first term on the right-hand side of (1.2). A systematic asymptotic analysis for high energetic

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relativistic particles has been also carried out more recently by Manoukian et al. (2004), based on (1.1), providing additional corrections to the ones on the right-hand side of (1.2) as functions of $\sqrt{1-\beta^2}$.

The purpose of this communication is to derive the quantum correction $\langle N\rangle_Q$ in closed form to the mean *number* $\langle N\rangle$ of photons emitted per revolution, to the order $\hbar$, to supplement our explicit expression in (1.1), which was based on the classical analysis, giving the final result $\langle N\rangle = \langle N\rangle_C + \langle N\rangle_Q$, where $\langle N\rangle_Q$ is given in (2.16).

2. Quantum correction

The explicit integral expression for the quantum correction, to the order $\hbar$, to the mean *number* $\langle N\rangle$ may be obtained from that of the formula of the power of (Schwinger, 1949, Eqs. (III), (6), (7); 1954, Eq. (24)) and Schwinger and Tsai (1978, Eq. (C.11)) given by

$$\langle N\rangle_Q = \alpha\hbar\left(\frac{\delta}{\delta\hbar}F(\hbar)\right)_{\hbar\to 0}, \tag{2.1}$$

where

$$F(\hbar) = \int_0^\infty d\omega \int_{-\infty}^{\infty} d\tau\, e^{-i\omega(1+\hbar\omega/E)\tau}\frac{(\beta^2\cos\omega_0\tau - 1)}{\beta\sin(\omega_0\tau/2)} \times \sin\left(2\beta\frac{\omega}{\omega_0}\left(1+\frac{\hbar\omega}{E}\right)\sin\frac{\omega_0\tau}{2}\right) \tag{2.2}$$

and, in our notation,

$$\frac{E}{mc^2} = \frac{1}{\sqrt{1-\beta^2}}, \quad \omega_0 = \frac{\beta c}{R} = \frac{cqB\sqrt{1-\beta^2}}{mc^2}, \tag{2.3}$$

with q, R and B denoting, respectively, the magnitude of the charge of the particle, the radius of the classical circular motion, and the magnetic field in question. Eq. (2.2) gives

$$\frac{\delta}{\delta\hbar}F(\hbar)\bigg|_{\hbar\to 0} = \int_0^\infty d\omega \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\frac{(\beta^2\cos\omega_0\tau - 1)}{\beta\sin(\omega_0\tau/2)} \times\left[-i\frac{\omega^2}{E}\tau\sin\left(2\beta\frac{\omega}{\omega_0}\sin\frac{\omega_0\tau}{2}\right) + \cos\left(2\beta\frac{\omega}{\omega_0}\sin\frac{\omega_0\tau}{2}\right)\times\left(2\beta\frac{\omega}{\omega_0}\frac{\omega}{E}\sin\frac{\omega_0\tau}{2}\right)\right]. \tag{2.4}$$

Let

$$\frac{\omega}{\omega_0} = z, \quad \omega_0\tau = x, \quad 2\beta\sin\frac{\omega_0\tau}{2} = a(x) \tag{2.5}$$

in (2.4) to rewrite the latter in the more convenient form:

$$\frac{\delta}{\delta\hbar}F(\hbar)|_{\hbar\to 0} = \left(\frac{\omega_0}{E}\right)\int_{-\infty}^{\infty} dx(\beta^2\cos x - 1)\int_0^\infty dz\, z^2 \times\left[e^{-iz(x+a(x))}\frac{x+a(x)}{a(x)} + e^{-iz(x-a(x))}\frac{x-a(x)}{-a(x)}\right]. \tag{2.6}$$

This seemingly complicated double integral may be computed in closed form. To this end, let

$$x + a(x) = \xi \tag{2.7}$$

and note that

$$\frac{dx}{d\xi} = \frac{1}{(1+\beta\cos(x/2))}. \tag{2.8}$$

Also for $x\to\pm\infty$, $\xi\to\pm\infty$, and for $\xi = 0$, $x = 0$.

Accordingly, $\langle N\rangle_Q$ in (2.1) may be obtained by taking the real part of (2.6) giving

$$\langle N\rangle_Q = \alpha\pi\left(\frac{\hbar\omega_0}{E}\right)\int_{-\infty}^{\infty} d\xi[I(\xi,\beta) + I(\xi,-\beta)], \tag{2.9}$$

where

$$I(\xi,\beta) = \left(-\frac{d^2}{d\xi^2}\delta(\xi)\right)\left(\frac{\beta^2\cos x - 1}{2\beta\sin(x/2)}\right)\left(\frac{x+2\beta\sin(x/2)}{1+\beta\cos(x/2)}\right) \tag{2.10}$$

and we have used the integral

$$\mathrm{Re}\int_0^\infty dz\, e^{-iz\xi} = \pi\delta(\xi). \tag{2.11}$$

To evaluate the integral in (2.9), we use the relations

$$\frac{d^2}{d\xi^2} = \frac{1}{(1+\beta\cos(x/2))^2}\left[\frac{d^2}{dx^2} + \frac{(\beta/2)\sin(x/2)}{1+\beta\cos(x/2)}\frac{d}{dx}\right], \tag{2.12}$$

$$\frac{d}{dx}\left(\frac{x}{\sin(x/2)}\right)\bigg|_{x=0} = 0, \quad \frac{d^2}{dx^2}\left(\frac{x}{\sin(x/2)}\right)\bigg|_{x=0} = \frac{1}{6}, \tag{2.13}$$

$$\frac{d}{dx}\left(\frac{\beta^2\cos x - 1}{1+\beta\cos(x/2)}\right)\bigg|_{x=0} = 0, \quad \frac{d^2}{dx^2}\left(\frac{\beta^2\cos x - 1}{1+\beta\cos(x/2)}\right)\bigg|_{x=0} = \frac{-\beta(1+3\beta)}{4(1+\beta)}, \tag{2.14}$$

from which we obtain

$$\int_{-\infty}^{\infty} d\xi I(\xi,\beta) = \frac{1}{(1+\beta)^2}\left[\frac{(1-\beta)}{12\beta} + \frac{1+3\beta}{4}\right]. \tag{2.15}$$

The quantum correction $\langle N\rangle_{\mathrm{Q}}$ then emerges from (2.9) and (2.15) to be

$$\langle N\rangle_{\mathrm{Q}} = \frac{8\pi}{3}\alpha\left(\frac{\hbar\omega_0}{mc^2}\right)\left[\left(\frac{E}{mc^2}\right) - \left(\frac{E}{mc^2}\right)^3\right]. \qquad (2.16)$$

For $\beta \to 1$, this gives the truly asymptotic formula

$$\langle N\rangle_{\mathrm{Q}} \to -\frac{8\pi}{3}\alpha\left(\frac{\hbar\omega_0}{mc^2}\right)\left(\frac{E}{mc^2}\right)^3. \qquad (2.17)$$

The expression in (2.16) supplements our explicit form $\langle N\rangle_{\mathrm{C}}$ in (1.1) in a quantum mechanical setting.

For the synchrotron in our institution, $R = 2.78\,\mathrm{m}$ and $E = 1.2\,\mathrm{GeV}$. This gives the estimates $\langle N\rangle_{\mathrm{C}} \sim \alpha 2.14 \times 10^4$ and $\langle N\rangle_{\mathrm{Q}} \sim \alpha 1.51 \times 10^{-2}$, in magnitudes, as based on (1.1)/(1.2) and (2.16)/(2.17), respectively. The latter is indeed relatively small but may be, however, significant for several revolutions in the magnetic field. This small quantum correction is not necessarily to be dismissed on practical grounds and may be reminiscent of small radiative corrections such as the Lamb shift contribution to the spectrum of the hydrogen atom which has been measured with very high accuracy and has led to much new physics. The quantum correction given in this work may be equally challenging to detect experimentally. It is interesting to note that a singularity in β for $\beta \to 1$ in $\langle N\rangle_{\mathrm{Q}}$ arises as in the classical treatment. Our quantum correction is based on a leading $\hbar$-contribution. At present it is not clear what would be the expression for $\langle N\rangle_{\mathrm{Q}}$ in an exact $\hbar$-treatment. Would such an expression compete with its classical counterpart and would it be practically relevant? Would it be singular in β for $\beta \to 1$? The exact $\hbar$-treatment of $\langle N\rangle_{\mathrm{Q}}$ as well as its experimental detection remain formidable problems and will be hopefully confronted with in the near future.

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APPENDIX C

PUBLICATION PAPER

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Propagation of Photons in Spacetime

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Abstract A quantum field theory QED formalism is systematically developed to describe photon propagation in *spacetime* as a time evolution process based on the actual *physical* process of propagation between emitters and detectors as applied to the reflection of photons. This development, as well as early studies by Feynman, clearly show that a practical, computational and predictive dynamical formalism in *spacetime* was lacking. The present one *generalizes* to different experimental situations and *other* interacting field theories as well emphasizing the practicality of the problem treated here.

Keywords Photon dynamics in spacetime and time evolution · QED and field theories in spacetime

Much progress has been done over the years [1, 2] to describe, especially quantum theoretically, the localization of photons in space [3]. It is fair to say, however, that there is still no explicit dynamical, non-heuristic, actual quantum (field) theory QED formalism worked out, as dictated by the latter, to describe the propagation of photons in *spacetime* in explaining even a simplest experiment as the reflection of photons off a reflecting surface as a time evolution process. This is certainly remarkable in the progress of physics, knowing that QED has been around for sometime and, as Feynman ([4], p. 3) puts it, it has been thoroughly analyzed, in his legendary Alix G. Mautner Memorial Lectures. The latter fascinating, though heuristic treatment [4] in words is, of course, far from a definite theoretical description but, in spite being addressed to non-specialists, the discussion clearly indicates, and as the present analysis shows, that a theoretical formalism, as stated above, to explain a simplest experiment in *spacetime* in a quantum (field) theory QED setting is lacking. For one thing, the amplitude of propagation of photons in *spacetime*, as a time evolution process, in infinitely extended space, for example, from a point x_1^μ to a point x_2^μ turns out to be given by $(\mathrm{i}/(\pi)^2)(x_2^0 - x_1^0)/[(x_2 - x_1)^2]^2$ rather than by the familiar Feynman propagator

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$i/(x_2 - x_1)^2$, with the former satisfying a key completeness relation for the internal consistency of the theory as formulated in spacetime. The purpose of this work is to develop such a formalism in detail based on the actual *physical process of the propagation of photons from emitters to detectors* as obtained from the so-called vacuum-to-vacuum transition amplitude [5–12] for the underlying theory. This method has been quite successful over the years in the easiness of momentum space computations of physical processes, avoiding of introducing so-called wavefunctions, not to mention of the elegance of the formalism as opposed to more standard techniques, and at the same time gaining much physical insight as particles propagate from emitters, interact, and finally particles reach the detectors as occurring in practice. The present analysis rests on three general key points: (i) By working directly in spacetime for the vacuum-to-vacuum transition amplitude, for given boundary conditions (B.C.), and from the expressions of the amplitudes for the emission and detection of photon excitations by the external sources, an amplitude of propagation between different spacetime points from emitters to detectors, causally arranged, is extracted and, as mentioned above, it does *not* coincide with the Feynman propagator for the corresponding B.C. This step already shows the power of determining amplitudes of propagation by introducing external sources. (ii) The amplitude of propagation is shown to satisfy a *completeness* relation as photons propagate between different points critical for the internal consistency of the theory in spacetime. (iii) Application of these amplitudes to describe in detail the experiment being sought by showing, in the process, very rapid exponential damping beyond the classical point of impact for the corresponding amplitude of occurrence. The reader will soon realize that our theoretical quantum (field) theory QED formalism is reduced to a non-operator approach and opens a way to describe, as a time evolution process, photon dynamics in *spacetime* and *other* field theory interactions in different experimental situations as well.

Let $|0_\mp\rangle$ denote the vacuum states before/after the external current $J^\mu(x)$, coupled to the vector potential $A_\mu(x)$ in Maxwell's Lagrangian, is switched on/off. The boundary conditions taken are $\langle 0_+ |\mathbf{E}_\parallel(x)|0_-\rangle = \mathbf{0}$, $\langle 0_+ |\mathbf{B}_\perp(x)|0_-\rangle = \mathbf{0}$ for $z \to +0$, where the reflecting surface is taken to consist the $x^1 - x^2$ plane, with $x^3 \equiv z \geq 0$, and $\mathbf{E}_\parallel/\mathbf{B}_\perp$ denote the components of the electric/magnetic fields parallel/perpendicular to the $x^1 - x^2$ plane. The vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle^J$ is then given by [11]

$$\langle 0_+|0_-\rangle^J = \mathrm{e}^{\frac{\mathrm{i}}{2}\int(\mathrm{d}x_1)(\mathrm{d}x_2)J^\mu(x_2)D'_{\mu\nu}(x_2,x_1)J^\nu(x_1)} \tag{1}$$

where invoking the conservation law $\partial_\mu J^\mu = 0$, the photon propagator in half-space may be written as

$$D'_{\mu\nu}(x_2, x_1) = \int \frac{(\mathrm{d}Q)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}\mathrm{e}^{-\mathrm{i}Q^0(x_2^0-x_1^0)}}{Q^2 - \mathrm{i}\varepsilon} \times [g_{\mu\nu}\mathrm{e}^{\mathrm{i}q(z_2-z_1)} + (-g_{\mu\nu} + 2g_{\mu 3}g_{\nu 3})\mathrm{e}^{-\mathrm{i}q(z_2+z_1)}] \tag{2}$$

$\varepsilon \to +0$, $x = (x^0, \mathbf{r}, z)$, $Q = (Q^0, \mathbf{k}, q)$ with $\mathbf{r}$ lying in the $x^1 - x^2$ plane. Since $J^\mu(x)$, by definition, vanishes for $z \leq 0$, we may integrate over all spacetime points in (1). Gauge invariance of the theory as well as the positivity condition $|\langle 0_+|0_-\rangle^J|^2 \leq 1$ are readily established [11]. We consider a causal arrangement, $J^\mu(x) = J_1^\mu(x) + J_2^\mu(x)$, of two currents with $J_2^\mu(x)$, the detector, switched on after $J_1^\mu(x)$, the emitter, is switched off. By invoking the condition $\partial_\mu J^\mu = 0$, we may then write

$$\langle 0_+|0_-\rangle^J = \langle 0_+|0_-\rangle^{J_2}\,\mathrm{e}^{\mathrm{i}\Omega}\,\langle 0_+|0_-\rangle^{J_1} \tag{3}$$

where, with $i, j = 1, 2, 3$, $x = (x^0, \mathbf{r}, z)$, $x' = (x^0, \mathbf{r}, -z)$,

$$\begin{aligned}\Omega = \int (\mathrm{d}x_1)(\mathrm{d}x_2)\mathrm{i}J^i_{2\mathrm{T}}(x_2)[-\mathrm{i}\Delta_+(x_2, x_1)\delta^{ij} \\ - \mathrm{i}\Delta_+(x'_2, x_1)(-\delta^{ij} + 2\delta^{i3}\delta^{j3})]\mathrm{i}J^j_{1\mathrm{T}}(x_1)\end{aligned} \tag{4}$$

and for $x^0_2 > x^0_1$,

$$-\mathrm{i}\Delta_+(x_2, x_1) = \int \frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3 2Q^0}\, \mathrm{e}^{\mathrm{i}Q(x_2-x_1)}, \qquad Q^0 = |\mathbf{Q}|, \tag{5}$$

$$J^{\mathrm{i}}_{\mathrm{T}}(x) = \int \frac{(\mathrm{d}Q)}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}Qx} J^i_{\mathrm{T}}(Q) \tag{6}$$

and for $Q^0 = |\mathbf{Q}|$, $Q^i J^i_{\mathrm{T}}(Q) = 0$. The second term within the square brackets in (4) corresponds to a non-trivial transition.

Now we use the identity

$$\begin{aligned}-\mathrm{i}\Delta_+(x_4, x_1) = \int' \mathrm{d}^3\mathbf{x}_2 \int' \mathrm{d}^3\mathbf{x}_3 \int' \mathrm{d}^3\mathbf{x} D_>(x_4, x_3) \\ \times [D(x_3, x)\mathrm{i}\overleftrightarrow{\partial_0} D(x, x_2)]D_<(x_2, x_1)\end{aligned} \tag{7}$$

in (4), where $\overleftrightarrow{\partial_0} = \overrightarrow{\partial_0} - \overleftarrow{\partial_0}$, $x^0_4 > x^0_3 > x^0_2 > x^0_1$, $\int' \mathrm{d}^3\mathbf{x} = \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{r} \int_0^\infty \mathrm{d}z$, and

$$D_>(x_4, x_3) = \int \frac{\mathrm{d}^3\mathbf{Q}}{4\pi^3\sqrt{2Q^0}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_4-\mathbf{r}_3)}\mathrm{e}^{-\mathrm{i}Q^0(x^0_4-x^0_3)}\mathrm{e}^{\mathrm{i}qz_4}\sin qz_3, \tag{8}$$

$$D_<(x_2, x_1) = \int \frac{\mathrm{d}^3\mathbf{Q}}{4\pi^3\sqrt{2Q^0}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}\mathrm{e}^{-\mathrm{i}Q^0(x^0_2-x^0_1)}\sin qz_2\mathrm{e}^{-\mathrm{i}qz_1}, \tag{9}$$

$$D(x_2, x_1) = \int \frac{\mathrm{d}^3\mathbf{Q}}{2\pi^3\sqrt{2Q^0}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}\mathrm{e}^{-\mathrm{i}Q^0(x^0_2-x^0_1)}\sin qz_2 \sin qz_1. \tag{10}$$

Given two real unit 3-vectors $\mathbf{n} = (a, b, c) \equiv \mathbf{n}_+$, $\mathbf{n}' = (a, b, -c) \equiv \mathbf{n}_-$, we introduce two sets of unit 3-vectors $(\mathbf{e}_1, \mathbf{e}_2)$, $(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$ by $\mathbf{e}_1 = \mathbf{n}\times\mathbf{n}'/|\mathbf{n}\times\mathbf{n}'| = \boldsymbol{\epsilon}_1$, $\mathbf{e}_2 = \mathbf{n}\times\mathbf{e}_1$, $\boldsymbol{\epsilon}_2 = \mathbf{n}'\times\boldsymbol{\epsilon}_1$, satisfying $\mathbf{n}_+\cdot\mathbf{e}_\lambda = 0$, $\mathbf{n}_-\cdot\boldsymbol{\epsilon}_\lambda = 0$ for $\lambda = 1, 2$. We use the completeness relations

$$\delta^{ij} = n^i_+ n^j_+ + \Sigma_\lambda e^i_\lambda e^j_\lambda = n^i_- n^j_- + \Sigma_\lambda \epsilon^i_\lambda \epsilon^j_\lambda \tag{11}$$

and also set

$$\mathbf{S}_\pm(x) = \int \frac{\mathrm{d}^3\mathbf{Q}}{4\pi^3\sqrt{2Q^0}} \mathbf{S}_\pm(Q_\pm)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{e}^{-\mathrm{i}Q^0x^0}\sin(\pm qz) \tag{12}$$

with $Q_+ = Q$, $Q_- = Q' = (Q^0, \mathbf{k}, -q)$, $Q^0 = |\mathbf{Q}|$,

$$\mathbf{S}_\pm(Q_\pm) = \mathbf{J}_{\mathrm{T}}(Q_\pm) - \frac{\mathbf{Q}_\pm + Q^0\mathbf{n}_\pm}{Q^0 + \mathbf{n}\cdot\mathbf{Q}}\, \mathbf{n}_\pm \cdot \mathbf{J}_{\mathrm{T}}(Q_\pm) \tag{13}$$

from which we have

$$S^{*i}_{\mp}(Q_\pm)\delta^{ij}S^{j}_{\pm}(Q_\pm) = J^{*i}_{\mathrm{T}}(Q_\pm)\delta^{ij}J^{j}_{\mathrm{T}}(Q_\pm), \tag{14}$$

$$S^{*i}_{\mp}(Q_\mp)[-\delta^{ij} + 2\delta^{i3}\delta^{j3}]S^{j}_{\pm}(Q_\pm) = J^{*i}_{\mathrm{T}}(Q_\mp)[-\delta^{ij} + 2\delta^{i3}\delta^{j3}]J^{j}_{\mathrm{T}}(Q_\pm). \tag{15}$$

Here we note that $\mathbf{n}_\pm \cdot \mathbf{Q}_\pm = \mathbf{n} \cdot \mathbf{Q}$, and that for the points $\mathbf{Q}_\pm = -\mathbf{n}_\pm|\mathbf{Q}|$, not only the numerators in the second term in (13) vanish but also $\mathbf{n}_\pm \cdot \mathbf{J}_{\mathrm{T}}(Q_\pm) = 0$. Hence these points are apparent singularities in (12) belonging to sets of measure zero. Note, in particular, that $\mathbf{n}_\pm \cdot \mathbf{S}_\pm(x) = 0$.

From (4), (11–15), the following explicit expression for Ω emerges

$$\begin{aligned}\Omega = \int' \mathrm{d}^3\mathbf{x}_1 \int' \mathrm{d}^3 x_2 \frac{\nabla_+(x_2, x_1)}{2} \sum_\lambda [&(\mathrm{i}\mathbf{S}^*_+(x_2)\cdot\mathbf{e}_\lambda)(\mathrm{i}\mathbf{S}_+(x_1)\cdot\mathbf{e}_\lambda) \\ &+ (\mathrm{i}\mathbf{S}^*_-(x_2)\cdot\boldsymbol{\epsilon}_\lambda)(\mathrm{i}\mathbf{S}_-(x_1)\cdot\boldsymbol{\epsilon}_\lambda) + (-1)^\lambda(\mathrm{i}\mathbf{S}^*_-(x_2)\cdot\boldsymbol{\epsilon}_\lambda)(\mathrm{i}\mathbf{S}_+(x_1)\cdot\mathbf{e}_\lambda) \\ &+ (-1)^\lambda(\mathrm{i}\mathbf{S}^*_+(x_2)\cdot\mathbf{e}_\lambda)(\mathrm{i}\mathbf{S}_-(x_1)\cdot\boldsymbol{\epsilon}_\lambda)]\end{aligned} \tag{16}$$

with $\nabla_+(x_2, x_1) = \int' \mathrm{d}^3\mathbf{x} D(x_3, x)\mathrm{i}\overleftrightarrow{\partial_0} D(x, x_1)$. Clearly, the last two terms in (16) correspond to non-trivial transitions.

Let $|\mathbf{e}_\lambda, \mathbf{n}_+, x\rangle \equiv |\lambda, +, x\rangle$, $|\boldsymbol{\epsilon}_\lambda, \mathbf{n}_-, x\rangle \equiv |\lambda, -, x\rangle$ denote photon excitation states emitted at spacetime point $x = (x^0, \mathbf{r}, z)$ with associated vectors $(\mathbf{e}_\lambda, \mathbf{n}_+)$, $(\boldsymbol{\epsilon}_\lambda, \mathbf{n}_-)$, respectively. The physical significance of these associated vectors will be discussed in the light of the experiment being sought. A unitarity expansion of $\langle 0_+ | 0_- \rangle^J$ will include, in particular, the following four terms describing the emission, propagation and detection of photon excitations:

$$\begin{aligned}&\langle 0_+ | \lambda, +, x_2\rangle^{J_2} \langle \lambda, +, x_2 | \alpha, +, x_1\rangle \langle \alpha, +, x_1 | 0_-\rangle^{J_1}, \\ &\langle 0_+ | \lambda, -, x_2\rangle^{J_2} \langle \lambda, -, x_2 | \alpha, +, x_1\rangle \langle \alpha, +, x_1 | 0_-\rangle^{J_1}, \\ &\langle 0_+ | \lambda, -, x_2\rangle^{J_2} \langle \lambda, -, x_2 | \alpha, -, x_1\rangle \langle \alpha, -, x_1 | 0_-\rangle^{J_1}, \\ &\langle 0_+ | \lambda, +, x_2\rangle^{J_2} \langle \lambda, +, x_2 | \alpha, -, x_1\rangle \langle \alpha, -, x_1 | 0_-\rangle^{J_1}.\end{aligned} \tag{17}$$

Here, for example, $\langle \alpha, +, x_1 | 0_-\rangle^{J_1}$ denotes the amplitude for the emission of a photon excitation in state $|\alpha, +, x_1\rangle$, with associated vectors $\mathbf{e}_\alpha, \mathbf{n}_+$, and $\langle 0_+ | \lambda, -, x_2\rangle^{J_2}$ denotes the amplitude for the detection of a photon excitation in state $|\lambda, -, x_2\rangle$ with associated vectors $\boldsymbol{\epsilon}_\lambda, \mathbf{n}_-$. Most importantly $\langle \lambda, -, x_2 | \alpha, +, x_1\rangle$, for example, denotes the amplitude of propagation of a photon excitation from spacetime point x_1 and associated vectors $\mathbf{e}_\alpha, \mathbf{n}_+$, to a spacetime point x_2 and ending up with associated vectors $\boldsymbol{\epsilon}_\lambda, \mathbf{n}_-$.

Upon comparing the four terms in (17) with the corresponding ones in Ω given in (16), *and* using the completeness relation

$$\sum_{\delta=\pm} \int' \mathrm{d}^3\mathbf{x} \langle \lambda, \delta_2, x_2 | \lambda, \delta, x\rangle \langle \lambda, \delta, x | \lambda, \delta_1, x_1\rangle = \langle \lambda, \delta_2, x_2 | \lambda, \delta_1, x_1\rangle \tag{18}$$

with $\delta_1, \delta_2 = \pm$, we obtain

$$\langle 0_+ | \lambda, +, x\rangle^{J_2} = (\mathrm{i}\mathbf{S}^*_+(x)\cdot\mathbf{e}_\lambda)\langle 0_+ | 0_-\rangle^{J_2}, \tag{19}$$

$$\langle 0_+ | \lambda, -, x\rangle^{J_2} = (\mathrm{i}\mathbf{S}^*_-(x)\cdot\boldsymbol{\epsilon}_\lambda)\langle 0_+ | 0_-\rangle^{J_2}, \tag{20}$$

$$\langle \lambda, +, x | 0_-\rangle^{J_1} = (\mathrm{i}\mathbf{S}_+(x)\cdot\mathbf{e}_\lambda)\langle 0_+ | 0_-\rangle^{J_1}, \tag{21}$$

$$\langle \lambda, -, x | 0_-\rangle^{J_1} = (\mathrm{i}\mathbf{S}_-(x)\cdot\boldsymbol{\epsilon}_\lambda)\langle 0_+ | 0_-\rangle^{J_1}, \tag{22}$$

$$\langle \lambda, \pm, x_2 | \alpha, \pm, x_1 \rangle = \frac{1}{2}\, \delta_{\lambda\alpha} \nabla_+(x_2, x_1), \tag{23}$$

$$\langle \lambda, \pm, x_2 | \alpha, \mp, x_1 \rangle = \frac{(-1)^\lambda}{2}\, \delta_{\lambda\alpha} \nabla_+(x_2, x_1). \tag{24}$$

See also Eqs. (95), (96) in [12] for definitions of absorption and emission amplitudes of photon excitations corresponding to the ones in (19–22). Note the factor 1/2 in (23), (24) which is essential to satisfy the completeness relation (18). $\nabla_+(x_2, x_1)$ works out to be

$$\nabla_+(x_2, x_1) = \frac{\mathrm{i}}{\pi^2}\,(x_2^0 - x_1^0) \sum_{\kappa=\pm 1} \frac{\kappa}{[(\mathbf{r}_2 - \mathbf{r}_1)^2 + (z_2 - \kappa z_1)^2 - (x_2^0 - x_1^0)^2]^2} \tag{25}$$

not coinciding with the Feynman propagator for the corresponding B.C., which upon using the Schwinger representation

$$\frac{1}{A^2} = -\int_0^\infty s\,\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}s(A - \mathrm{i}\varepsilon)}, \qquad \varepsilon \to +0 \tag{26}$$

is conveniently expressed as

$$\nabla_+(x_2, x_1) = \frac{(x_2^0 - x_1^0)}{\mathrm{i}\pi^2} \sum_{\kappa=\pm 1} \kappa \int_0^\infty s\,\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}s[(\mathbf{r}_2 - \mathbf{r}_1)^2 + (z_2 - \kappa z_1)^2 - (x_2^0 - x_1^0)^2 - \mathrm{i}\varepsilon]}. \tag{27}$$

In the sequel we suppress the $\mathrm{i}\varepsilon$ factor to simplify the notation.

The transition amplitude that a photon excitation in a state $|\lambda_1, \delta_1, x_1\rangle$ propagates from $x_1 = (x_1^0, \mathbf{r}_1, z_1)$, reaches the reflecting surface within a skin depth, specified by a scale parameter σ and described by a Gaussian density distribution $\mathrm{e}^{-z^2/\sigma^2}/2\sqrt{\pi}\sigma$, $0 \le z$, and ends up in a state $|\lambda_2, -\delta_1, x_2\rangle$ at $x_2 = (x_2^0, \mathbf{r}_2, z_2)$ is given from (16), (18) to be ($x_1^0 < x^0 < x_2^0$)

$$\begin{aligned} \mathcal{A}(x_2, x_1) = {} & \int_{R^2} \mathrm{d}^2\mathbf{r} \int_0^\infty \mathrm{d}z \frac{\mathrm{e}^{-z^2/\sigma^2}}{2\sqrt{\pi}\sigma} \\ & \times \sum_{\delta=\pm} \delta_{\lambda_1\lambda_2} \langle \lambda_2, -\delta_1, x_2 | \lambda_1, \delta, x \rangle \langle \lambda_1, \delta, x | \lambda_1, \delta_1, x_1 \rangle \end{aligned} \tag{28}$$

suppressing, for the moment, the indices λ_1, δ_1 in $\mathcal{A}(x_2, x_1)$ to which we will return later. We note from (23), (24), (27), that the z-integrand in (28) is even in z. We may also introduce a surface *density* amplitude $f(\mathbf{R})$ by

$$\mathcal{A}(x_2, x_1) = \int \mathrm{d}^2\mathbf{R}\, f(\mathbf{R}) \tag{29}$$

by multiplying the $\mathbf{r}$-integrand in (28) by the identity

$$\int \frac{\mathrm{d}^2\mathbf{R}}{\pi\sigma_o^2} \mathrm{e}^{-(\mathbf{R}-\mathbf{r})^2/\sigma_o^2} = 1 \tag{30}$$

valid for any $\sigma_o^2 > 0$, and any $\mathbf{r}$, giving from (23), (24), (28), (27)

$$f(\mathbf{R}) = \frac{\delta_{\lambda_1\lambda_2}(-1)^{\lambda_1}}{8} \int \frac{\mathrm{d}^2\mathbf{r}}{\pi\sigma_o^2} \mathrm{e}^{-(\mathbf{r}-\mathbf{R})^2/\sigma_o^2}$$

$$\times \int_{-\infty}^{\infty} \frac{dz}{\sqrt{\pi}\sigma} e^{-z^2/\sigma^2} \nabla_+(x_2, x)\nabla_+(x, x_1). \tag{31}$$

Given that a photon excitation was emitted in state $|\lambda_1, \delta_1, x_1\rangle$, *reaching* the reflecting surface within a skin depth, and ending up in state $|\lambda_2, -\delta_1, x_2\rangle$, $\lambda_2 = \lambda_1$, the conditional amplitude density for the process is then given by $F(\mathbf{R}) = f(\mathbf{R})/\mathcal{A}(x_2, x_1)$, with $\int d^2\mathbf{R} F(\mathbf{R}) = 1$, as a "summation" over impact centers whose nature will be now investigated.

Let $T_1 = x^0 - x_1^0$, $T_2 = x_2^0 - x^0$. The $\mathbf{r}-$, $z-$ integrals in (31) may be explicitly carried out yielding

$$\begin{aligned} F(\mathbf{R}) = \frac{1}{N} \sum_{\kappa_1,\kappa_2} \kappa_1\kappa_2 \int_0^\infty du_1 \int_0^\infty du_2 \frac{u_1 u_2}{\sigma\sqrt{1 + i(u_1 + u_2)}} \\ \times \exp\left[-\frac{z_1^2}{\sigma^2} \frac{(u_1 + \kappa_1\kappa_2 u_2 z_2/z_1)^2}{1 + i(u_1 + u_2)}\right]\left[1 + i\frac{\sigma_o^2}{\sigma^2}(u_1 + u_2)\right]^{-1} \\ \times \exp\left[-\frac{\sigma_o^2}{\sigma^4} \frac{[u_1(\mathbf{r}_1 - \mathbf{R}) + u_2(\mathbf{r}_2 - \mathbf{R})]^2}{1 + i\sigma_o^2(u_1 + u_2)/\sigma^2}\right] e^{-iG(u_1,u_2,\mathbf{R})/\sigma^2}, \end{aligned} \tag{32}$$

$$\begin{aligned} N = \sum_{\kappa_1,\kappa_2} \kappa_1\kappa_2 \int_0^\infty du_1 \int_0^\infty du_2 \frac{u_1 u_2}{\sigma\sqrt{1 + i(u_1 + u_2)}} \\ \times \exp\left[-\frac{z_1^2}{\sigma^2} \frac{(u_1 + \kappa_1\kappa_2 u_2 z_2/z_1)^2}{1 + i(u_1 + u_2)}\right] \int d^2\mathbf{r} e^{-iG(u_1,u_2,\mathbf{r})/\sigma^2} \end{aligned} \tag{33}$$

with

$$G(u_1, u_2, \mathbf{r}) = u_1[(\mathbf{r}_1 - \mathbf{r})^2 + z_1^2 - T_1^2] + u_2[(\mathbf{r}_2 - \mathbf{r})^2 + z_2^2 - T_2^2]. \tag{34}$$

For the practical case $\sigma \ll z_1$, with a given initially chosen macroscopic value z_1, i.e., for $z_1^2/\sigma^2 \gg 1$, we may use the distributional limit

$$\frac{z_1/\sigma}{\sqrt{1 + i(u_1 + u_2)}} \exp -\frac{z_1^2}{\sigma^2} \frac{(u_1 + \kappa_1\kappa_2 u_2 \frac{z_2}{z_1})^2}{1 + i(u_1 + u_2)} \to \sqrt{\pi}\delta\left(u_1 + \kappa_1\kappa_2 u_2 \frac{z_2}{z_1}\right) \tag{35}$$

in (32) as obtained, for example, by Fourier transform techniques, to obtain $u_1 = -\kappa_1\kappa_2 u_2 z_2/z_1$, with the necessary restrictions $\kappa_1 = \pm 1$, $\kappa_2 = \mp 1$, giving for $\sigma \ll z_1$,

$$\begin{aligned} F(\mathbf{R}) = \frac{i}{\pi\sigma_o^2 C} \int_0^\infty \frac{w^2 dw}{\sigma_o + iw} \exp\left[\frac{-iwG(z_2, z_1, \mathbf{R}_o)}{\sigma_o^3(z_1 + z_2)}\right] \\ \times \exp -i\frac{w}{\sigma_o^2} \frac{(\mathbf{R} - \mathbf{R}_o)^2}{(\sigma_o + iw)} \end{aligned} \tag{36}$$

where

$$C = \int_0^\infty w\,dw \exp\left[\frac{-iwG(z_2, z_1, \mathbf{R}_o)}{\sigma_o^3(z_1 + z_2)}\right], \tag{37}$$

$$\mathbf{R}_o = (z_2\mathbf{r}_1 + z_1\mathbf{r}_2)/(z_1 + z_2). \tag{38}$$

For an effective area of impact $\pi\sigma_o^2$ about a point $\mathbf{R}$ for $\sigma_o \to 0$, we obtain

$$\pi\sigma_o^2 F(\mathbf{R}) \to \exp -(\mathbf{R}-\mathbf{R}_o)^2/\sigma_o^2 \frac{\int_0^\infty w\mathrm{d}w \exp[\frac{-\mathrm{i}wG(z_2,z_1,\mathbf{R}_o)}{\sigma_o^3(z_1+z_2)}]}{\int_0^\infty w\mathrm{d}w \exp[\frac{-\mathrm{i}wG(z_2,z_1,\mathbf{R}_o)}{\sigma_o^3(z_1+z_2)}]} \tag{39}$$

with the second factor independent of σ_o, giving the remarkably simple expression

$$\pi\sigma_o^2\, F(\mathbf{R}) \to \mathrm{e}^{-(\mathbf{R}-\mathbf{R}_o)^2/\sigma_o^2} \tag{40}$$

for $\sigma_o^2 \to 0$.

Accordingly, for an arbitrary small σ_o, giving a point-like area of impact, about the point $\mathbf{R}$, the partial amplitude $\pi\sigma_o^2 F(\mathbf{R})$ vanishes exponentially for $\mathbf{R} \neq \mathbf{R}_o$, i.e., for the non-classical point of impact. On the other hand for $\mathbf{R} = \mathbf{R}_o$, we have $\pi\sigma_o^2 F(\mathbf{R}) \to 1$ for $\sigma_o^2 \to 0$.

The condition $\mathbf{R} = \mathbf{R}_o$, translates from (38) to $(\mathbf{r}_1 - \mathbf{R}_o)/z_1 = -(\mathbf{r}_2 - \mathbf{R}_o)/z_2$ which is nothing but the law of reflection with $\mathbf{R}_o$ denoting the classical point of impact. For given $(\mathbf{r}_1, z_1)$, $(\mathbf{r}_2, z_2)$, leading from (38) to a fixed value of $\mathbf{R}_o$, we may choose the unit vectors $\mathbf{n}_+$, $\mathbf{n}_-$ in (11), to be directed along the vectors $(\mathbf{R}_o - \mathbf{r}_1, -z_1)$, $(\mathbf{r}_2 - \mathbf{R}_o, z_2) = z_2(\mathbf{R}_o - \mathbf{r}_1, z_1)/z_1$, respectively, corresponding to classical rays, with the vectors $(\mathbf{e}_1, \mathbf{e}_2)$, $(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$ having the well know interpretations of polarization vectors perpendicular, respectively, to $\mathbf{n}_+$, $\mathbf{n}_-$, with transformations $\mathbf{e}_1 \leftrightarrow \boldsymbol{\epsilon}_1$, $\mathbf{e}_2 \leftrightarrow \boldsymbol{\epsilon}_2$ upon scattering.

Our formalism clearly opens the way for practical spacetime analyses of photon dynamics and *other* interacting field theories by using, in the process, functional differential techniques [10, 12] in different experimental situations. It shows, in particular, how amplitudes of propagation are determined from the knowledge of amplitudes of emissions and absorption of particle excitations by emitters and detectors, respectively, signaling the power of the present method of analysis. The time slicing procedure will also allow to derive path integrals for such amplitudes of propagation in spacetime. These further developments in general field theories emphasize the practicality and generality of the problem treated here not just being restricted to it. Such a program will be taken up in subsequent reports.

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