ว่าด้วยปัญหาของฟิลเตอร์เฟ้นสุ่มบางปัญหาที่ใช้ในทางการเงิน

นางสาวธิดารัตน์ เปลี่ยนพานิช

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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ON SOME PROBLEMS OF STOCHASTIC FILTERING APPLIED TO FINANCE

Tidarut Plienpanich

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Thesis Examining Committee

____

(Asst. Prof. Dr. Eckart Schulz)

Chairperson

____

(Prof. Dr. Pairote Sattayatham)

Member (Thesis Advisor)

____

(Prof. Dr. Tran Hung Thao)

Member

____

(Prof. Aumnuay Kananthai)

Member

____

(Prof. Dr. Suthep Suantai)

Member

(Prof. Dr. Pairote Sattayatham)	(Assoc. Prof. Dr. Prapan Manyum)
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ธิดารัตน์ เปลี่ยนพานิช : ว่าด้วยปัญหาของฟิลเตอร์เฟ้นสุ่มบางปัญหาที่ใช้ในทางการเงิน (ON SOME PROBLEMS OF STOCHASTIC FILTERING APPLIED TO FINANCE) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร.ไพโรจน์ สัตยธรรม, 83 หน้า.

วิทยานิพนธ์ฉบับนี้ศึกษาเกี่ยวกับปัญหาฟิลเตอร์เฟ้นสุ่ม ฟิลเตอร์เฟ้นสุ่มใช้สำหรับประมาณกระบวนการสัญญาณด้วยกระบวนการสังเกตซึ่งขึ้นต่อกัน

ในลำดับแรกได้พิจารณาปัญหาฟิลเตอร์เฟ้นสุ่มซึ่งกระบวนการสังเกตเป็นกระบวนการแบบจุด ในขณะที่กระบวนการสัญญาณเป็นกระบวนการเซมิมาร์ติงเกล กระบวนการเฟลเลอร์ หรือกระบวนการออร์นสเตน-เออเลนเบค ตามลำดับ

ในลำดับต่อมา ได้นำเสนอวิธีการประมาณสำหรับปัญหาฟิลเตอร์เฟ้นสุ่มเศษส่วนที่มีกระบวนการสังเกตเป็นกระบวนการเศษส่วน โดยที่กระบวนการสัญญาณเป็นกระบวนการทั่วไป กระบวนการเซมิมาร์ติงเกล หรือกระบวนการเศษส่วนตามลำดับ ได้มีการสร้างสมการฟิลเตอร์แบบประมาณเมื่อฟิลเตอร์จริงเป็นลิมิตของฟิลเตอร์แบบประมาณ

ท้ายที่สุด ได้นำผลลัพธ์ที่ได้ก่อนหน้านี้มาประยุกต์ใช้ในแบบจำลองทางการเงิน เช่น แบบจำลองอัตราดอกเบี้ย และแบบจำลองของความผันผวน

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา________________________

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TIDARUT PLIENPANICH : ON SOME PROBLEMS OF STOCHASTIC FILTERING APPLIED TO FINANCE. THESIS ADVISOR : PROF. PAIROTE SATTAYATHAM, Ph.D. 83 PP.

STOCHASTIC FILTERING / POINT PROCESS / FELLER PROCESS / ORNSTEIN-UHLENBECK PROCESS / FRACTIONAL PROCESS / SEMIMARTINGALE

In this thesis, some stochastic filtering problems are studied. Stochastic filtering is used to estimate a signal process from an observation process depending on it.

First, the filtering problem with point process observation is considered, where the signal process is either a semimartingale process or a Feller process or an Ornstein-Uhlenbeck process, respectively.

Next, an approximate approach to fractional stochastic filtering problems with fractional observation process is introduced, where the signal process can be either a general process or a semimartingale process or a fractional process. An approximate filtering equation is established where the real filter is a limit case of approximate filters.

Finally, these results are applied to some financial models, such as interest rate model and volatility model.

School of Mathematics	Student's Signature ________________
Academic Year 2007	Advisor's Signature ________________
	Co-advisor's Signature ______________

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Tidarut Plienpanich

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CHAPTER I

INTRODUCTION

Mathematical finance is the important discipline of applied mathematics concerned with financial markets. It appeared for the first time in 1900 with the contribution by Louis Bachelier on speculation in markets. More than one century has passed since then and many substantial achievements on mathematical finance have been achieved, among them there are some important turning points such as the discovery of the Black-Scholes Theory of European Options in 1973, Arbitrage Pricing Theory, Hedging Theory and Term Structures Theory for interest rates and credit spreads. These achievements play a crucial role in giving decisions for investing in financial markets such as stock markets, bond markets, currency markets, derivatives markets, etc. Strong and continuous requirements of real markets are motivations of mathematical research for establishing suitable financial models and methods that could be put to practice in more and more efficient ways.

Filtering problems involve the estimation of some quantities that cannot be observed directly (the signal process or the state process) throughout other quantities that depend on them and can be observed directly (the observation process).

In financial modeling it is sometimes the case that not all quantities, which determine the dynamics of security prices, can be fully observed. Some of the factors that characterize the evolution of the market are hidden. However, these unobserved factors may be essential to reflect in a market model the type of dynamics that one empirically observes. This leads naturally to filtering methods.

These methods determine the distribution and allow then to compute the expectation of quantities that are dependent on unobserved factors, for instance, derivative prices.

On the other hand, when specifying a financial market model, one has also to specify the model coefficients. The latter may however be only partially known or depend on stochastic factors that in turn may not be observable. When solving problems related to financial markets, like portfolio optimization or derivative pricing and hedging, it is therefore appropriate to exploit all information coming from the market itself to continuously update the knowledge of the not fully known coefficients or parameters in the model, and this is where stochastic filtering proves itself as a very useful technique. In fact, in stochastic filtering, which can be viewed as a dynamic extension of Bayesian statistics, all not fully known quantities are considered as random variables or stochastic processes and their distribution is continuously updated on the basis of currently available information.

The main actors in a financial market are the various assets that may be classified into two main categories: primary assets (underlying assets) and derivative assets, where the prices of the latter are "derived" from those of the primary assets and can be expressed as expectations under a so-called martingale measure. In a complete market there exists only one martingale measure and so all prices are fully specified within the model. If however the market is incomplete, and this corresponds to essentially all practical situations, then there exist more martingale measures and so, in order to perform the pricing of derivatives that are not already traded on the market, one has first to infer the prevailing martingale measure or, equivalently, the so-called market price of risk. This market price of risk cannot be directly observed on the market so that, again, filtering techniques may be used to continuously update its knowledge.

The prices of the primary assets as well as those of derivative assets that are liquidly traded constitute the main information available on a given market and thus also the basic ingredient of filtering. In this context, the fact that the prices of the derivative assets, also of those that are liquidly traded, are specified as expectations under a martingale measure become a major problem since the actual observations take place under the real world probability measure, under which the dynamics of the observable in a stochastic dynamic filtering model have thus to be specified.

The estimation of some financial factors that cannot be observed directly (for instance the volatility or parameters of some financial models) has to based on some direct observation process such as stock price S_t depending on time $t,\ 0 \leq t \leq T$. But in reality, the observation can be made only at discrete times $t_n,\ n = 0, 1, 2, ...$ so the observation process is a stochastic process of discrete times. More general, the observation can be made at random times $T_0(\omega), T_1(\omega), ..., T_n(\omega), ...$ So it is natural to use a point process to express such an observation. There are three ways to introduce a point process:

- by a sequence of random variables,

- by a discrete random measure,

- and by a counting process.

The first major part of this thesis is reserved to the study of filtering problems based on observation given by a point process introduced by the third way mentioned above.

One has realized also that various evolutions of many financial factors can be perturbed not only by white noise as Brownian motion W_t, but also by a fractional process such as fractional Brownian motion W_t^H, where H is the Hurst index, $0 \leq H \leq 1$.

Thus, it is very natural to consider fractional filtering problems, where either the signal process or observation process or both can be perturbed by fractional Brownian motion. Many authors have made some attempts to solve those problems (refer to Decreusefond, Oksendal, etc) but it seems that their approaches are too complicated to be applied to the practice of financial markets.

Thus, another major part of this thesis is the study of fractional filtering problems from an approximation point-of-view that can be more easily applied to finance than other academic approaches.

CHAPTER II

INTRODUCTION TO STOCHASTIC FILTERING THEORY

In this chapter, we introduce the background of stochastic filtering theory. Most of these results can be found in Chiganski (2005).

2.1 Problems Setting and Definition

2.1.1 Problems Setting

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. We shall consider two processes:

1. A signal process $\{X_t\}_{t\geq 0}$, which is not directly observable
2. An observation process $\{Y_t\}_{t\geq 0}$, whose value depend on the signal process and can be directly observed.

The signal process is described by a semimartingale

$$X_t = X_0 + \int_0^t H_s ds + W_t. \tag{2.1.1}$$

and the observation process is given by

$$Y_t = \int_0^t h_s ds + V_t. \tag{2.1.2}$$

where H_t is some stochastic process, h_t is a process such that $h_t = h(X_t)$, $E\int_0^t h_s^2 ds < \infty$ and W_t, V_t are independent Brownian motions. Denote by $\mathcal{F}_t^Y$ the

σ-algebra generated by all random variables $(Y_u, u \le t)$: $\mathcal{F}_t^Y = \sigma(Y_u, 0 \le u \le t)$. The filtering $\pi(X_t)$ of X_t based on information given by $\mathcal{F}_t^Y$ is defined by

$$\pi(X_t) := E[X_t|\mathcal{F}_t^Y], \tag{2.1.3}$$

More general, the filter can be defined via a function $f \in C^2$ by

$$\pi(f(X_t)) = E[f(X_t)|\mathcal{F}_t^Y]. \tag{2.1.4}$$

The problem now is how to find the filter $\pi(X_t)$ or $\pi(f(X_t))$. It is usually found as a solution of a stochastic differential equation that is called filtering equation.

2.2 Girsanov Theorem

Theorem 2.2.1. *(Girsanov Theorem). Let W_t be Brownian motion process and X_t be an $\mathcal{F}_t$-adapted process, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and satisfying*

$$\int_0^t X_t^2 dt < \infty \quad a.s. \tag{2.2.1}$$

and define

$$Z_t = \exp(\int_0^t X_s dW_s - \frac{1}{2}\int_0^t X_s^2 ds) \tag{2.2.2}$$

Assume that $EZ_t = 1$ holds for all t and define the probability measure Q by

$$\frac{dQ}{dP}(\omega)\bigg|_{\mathcal{F}_t} = Z_t(\omega) \tag{2.2.3}$$

Then

$$V_t = W_t - \int_0^t X_s ds \tag{2.2.4}$$

is a Brownian motion process with respect to $\mathcal{F}_t$ under Q.

Proof : Clearly V_t has continuous paths and $V_0 = 0$. Thus it is left to verify

$$E_Q[\exp\{i\lambda(V_t - V_s)\}|\mathcal{F}_s] = \exp\{-\frac{1}{2}\lambda^2(t-s)\}, \quad s \le t \tag{2.2.5}$$

Since $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = Z_t$ is the restriction of Randon-Nykodym derivative on $\mathcal{F}_t \subset \mathcal{F}$, then

$$\begin{aligned} E_Q[\exp\{i\lambda(V_t - V_s)\}|\mathcal{F}_s] &= \frac{E_P[\exp\{i\lambda(V_t - V_s)\}Z_t|\mathcal{F}_s]}{E_P[Z_t|\mathcal{F}_s]} \\ &= \exp\{-i\lambda V_s\}\frac{E_P[\exp\{i\lambda V_t\}Z_t|\mathcal{F}_s]}{E_P[Z_t|\mathcal{F}_s]} \\ &= \frac{E_P[\exp\{i\lambda V_t\}Z_t|\mathcal{F}_s]}{\exp\{i\lambda V_s\}E_P[Z_t|\mathcal{F}_s]} \end{aligned}$$

By the Ito formula Z_t satisfies

$$dZ_t = Z_t X_t dW_t \tag{2.2.6}$$

or

$$Z_t = Z_s + \int_s^t Z_u X_u dW_u \tag{2.2.7}$$

It follows from the martingale property of the Itô integral, that the process Z_t is a martingale, that is

$$E_P[Z_t|\mathcal{F}_s] = Z_s \tag{2.2.8}$$

The Itô formula applied to the process $Y_t := \exp\{i\lambda V_t\}Z_t$ yields

$$dY_t = -\frac{\lambda^2}{2}Y_t dt + (i\lambda Y_t + Y_t X_t)dW_t \tag{2.2.9}$$

which implies

$$Y_t = Y_s - \int_s^t \frac{\lambda^2}{2}Y_u du + \int_s^t Y_u(i\lambda + X_u)dW_u \tag{2.2.10}$$

and in turn

$$E_P[Y_t|\mathcal{F}_s] = E_P[Y_s|\mathcal{F}_s] - \int_s^t \frac{\lambda^2}{2}E_P[Y_u|\mathcal{F}_s]du \tag{2.2.11}$$

where the martingale property of the stochastic integral has been used. This linear equation is explicitly solved for $E[Y_t|\mathcal{F}_s]$

$$E_P[Y_t|\mathcal{F}_s] = Y_s \exp\{-\frac{1}{2}\lambda^2(t-s)\} \tag{2.2.12}$$

Hence

$$\begin{aligned} E_Q[\exp\{i\lambda(V_t - V_s)\}|\mathcal{F}_s] &= \frac{\exp\{i\lambda V_s\}Z_s \exp\{-\frac{1}{2}\lambda^2(t-s)\}}{\exp\{i\lambda V_s\}Z_s} \\ &= \exp\{-\frac{1}{2}\lambda^2(t-s)\} \end{aligned}$$

This proves that the process V_t is a Brownian motion process with respect to $\mathcal{F}_t$ under Q.

2.3 Innovation Processes

Definition 2.3.2. (Innovation process).

$$m_t = Y_t - \int_0^t \pi(h_s)ds \tag{2.3.1}$$

is called an innovation process of the observation process Y_t, where $\pi(h_s) = E[h_s|\mathcal{F}_s^Y]$.

Theorem 2.3.3. *m_t is a Brownian motion with respect to $\mathcal{F}_t^Y$.*

Proof : Substituting (2.1.2) into (2.3.1), we have

$$m_t = \int_0^t (h_s - \pi(h_s))ds + V_t. \tag{2.3.2}$$

For any $0 \leq s \leq t$

$$\begin{aligned} E[m_t|\mathcal{F}_s^Y] - m_s &= E[\int_s^t (h_u - \pi(h_u))du + (V_t - V_s)|\mathcal{F}_s^Y] \\ &= E[\int_s^t (h_u - \pi(h_u))du|\mathcal{F}_s^Y] + E[V_t - V_s|\mathcal{F}_s^Y] \\ &= E[\int_s^t \{E[h_u|\mathcal{F}_u^Y] - \pi(h_u)\}du|\mathcal{F}_s^Y] + E[E[V_t - V_s|\mathcal{F}_s]|\mathcal{F}_s^Y] \\ &= 0 \end{aligned}$$

by properties of conditional expectations and properties of Brownian motion process V_t. Therefore, m_t is a martingale with respect to $\mathcal{F}_t^Y$. And the quadratic variation

$$\langle m, m\rangle_t = \langle V, V\rangle_t = t. \tag{2.3.3}$$

By virtue of a Levy's Theorem on characterization of Brownian motions, m_t is a Brownian motion with respect to $\mathcal{F}_t^Y$.

Theorem 2.3.4. *Every martingale M_t with respect to the filtration $\{\mathcal{F}_t^Y\}$ admits a representation of the form*

$$M_t = M_0 + \int_0^t K_s dm_s \tag{2.3.4}$$

where K_t is $\mathcal{F}_t^Y$-measurable and satisfies $\int_0^t K_s^2 ds < \infty$ a.s.

Proof : Set

$$Z_t = \exp\Big(- \int_0^t \pi(h_s)dm_s - \frac{1}{2}\int_0^t \pi^2(h_s)ds\Big) \tag{2.3.5}$$

is $\mathcal{F}_t^Y$-martingale. According to the Girsanov Theorem, the process

$$Y_t = m_t + \int_0^t \pi(h_s)ds \tag{2.3.6}$$

is a Brownian motion with respect to $\mathcal{F}_t^Y$ under the new probability measure Q defined by

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^Y} = Z_t. \tag{2.3.7}$$

Next, we define

$$\begin{aligned} \Lambda_t &:= Z_t^{-1} \\ &= \exp\Big(\int_0^t \pi(h_s)dm_s + \frac{1}{2}\int_0^t \pi^2(h_s)ds\Big) \\ &= \exp\Big(\int_0^t \pi(h_s)dY_s - \frac{1}{2}\int_0^t \pi^2(h_s)ds\Big) \end{aligned}$$

and notice the "likelihood ratio"

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t^Y} = Z_t, \quad \frac{dP}{dQ}\Big|_{\mathcal{F}_t^Y} = \Lambda_t. \tag{2.3.8}$$

And the processes Z_t and Λ_t satisfy the equations

$$\begin{aligned} Z_t &= 1 - \int_0^t Z_s \pi(h_s) dm_s \\ \Lambda_t &= 1 + \int_0^t \Lambda_s \pi(h_s) dY_s \end{aligned}$$

Because of Bayes formula and M_t, Z_t are $\mathcal{F}_t^Y$-martingales. We found that, for any $s < t$

$$E_Q[\Lambda_t M_t | \mathcal{F}_s^Y] = \frac{E_P[\Lambda_t M_t Z_t | \mathcal{F}_s^Y]}{E_P[Z_t | \mathcal{F}_s^Y]} = \Lambda_s M_s, \tag{2.3.9}$$

that is $\Lambda_t M_t$ is $\mathcal{F}_t^Y$-martingale under probability Q. Then there exists the process Ψ_t that

$$\begin{aligned} \Lambda_t M_t &= \Lambda_0 M_0 + \int_0^t \Psi_s dY_s \\ &= M_0 + \int_0^t \Psi_s (dm_s + \pi(h_s) ds) \end{aligned}$$

Now from integration by parts formula, we obtain

$$\begin{aligned} M_t &= (\Lambda_t M_t) Z_t \\ &= \Lambda_0 M_0 Z_0 + \int_0^t \Lambda_s M_s dZ_s + \int_0^t Z_s d(\Lambda_s M_s) + \langle \Lambda M, Z \rangle_t \\ &= M_0 - \int_0^t M_s \pi(h_s) dm_s + \int_0^t Z_s \Psi_s (dm_s + \pi(h_s) ds) + \langle \Lambda M, Z \rangle_t \\ &= M_0 + \int_0^t (Z_s \Psi_s - M_s \pi(h_s)) dm_s \\ &= M_0 + \int_0^t K_s dm_s \end{aligned}$$

where $K_t = Z_t \Psi_t - M_t \pi(h_t)$. This proves the theorem.

2.4 Fujisaki-Kallianpur-Kunita Theorem

Theorem 2.4.5. $M_t = E[X_0|\mathcal{F}_t^Y] - \pi(X_0) + E[\int_0^t H_s ds | \mathcal{F}_t^Y] - \int_0^t \pi(H_s) ds + E[W_t | \mathcal{F}_t^Y]$ *is a martingale with respect to* $\mathcal{F}_t^Y$.

Proof : For any $0 \le s \le t$, by the rules of conditional expectation, we have

$$E[E[X_0|\mathcal{F}_t^Y] - \pi(X_0)|\mathcal{F}_s^Y] = E[X_0|\mathcal{F}_s^Y] - \pi(X_0). \tag{2.4.1}$$

And

$$\begin{aligned}
E\ & [E[\int_0^t H_u du|\mathcal{F}_t^Y] - \int_0^t \pi(H_u)du|\mathcal{F}_s^Y] \\
= & \int_0^t E[H_u|\mathcal{F}_s^Y]du - \int_0^t E[\pi(H_u)|\mathcal{F}_s^Y]du \\
= & E[\int_0^s H_u du|\mathcal{F}_s^Y] - \int_0^s \pi(H_u)du + \int_s^t E[H_u|\mathcal{F}_s^Y]du - \int_s^t E[\pi(H_u)|\mathcal{F}_s^Y]du \\
= & E[\int_0^s H_u du|\mathcal{F}_s^Y] - \int_0^s \pi(H_u)du. \qquad (2.4.2)
\end{aligned}$$

Since W_t is a Brownian motion process, so it is a $\mathcal{F}_t$-martingale then

$$E[E[W_t|\mathcal{F}_t^Y]|\mathcal{F}_s^Y] = E[W_t|\mathcal{F}_s^Y] = E[E[W_t|\mathcal{F}_s]|\mathcal{F}_s^Y] = E[W_s|\mathcal{F}_s^Y]. \tag{2.4.3}$$

Combining equation (2.4.1)-(2.4.3) and definition of M_t yields

$$\begin{aligned}
E[M_t|\mathcal{F}_s^Y] &= E[X_0|\mathcal{F}_s^Y] - \pi(X_0) + E[\int_0^s H_u du|\mathcal{F}_s^Y] - \int_0^s \pi(H_u)du + E[W_s|\mathcal{F}_s^Y] \\
&= M_s. \qquad (2.4.4)
\end{aligned}$$

This shows that M_t is a $\mathcal{F}_t^Y$-martingale.

Theorem 2.4.6. *(Fujisaki-Kallianpur-Kunita Theorem). The filter $\pi(X_t)$ satisfies the Fujisaki-Kallianpur-Kunita equation*

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s)ds + \int_0^t \{\pi(X_s h_s) - \pi(X_s)\pi(h_s)\}dm_s \tag{2.4.5}$$

where m_t is the innovation process defined in (2.3.1)

Proof : By Theorem 2.3.4 and Theorem 2.4.5, there exists a process K_t such that

$$M_t = \int_0^t K_s dm_s.$$

We should show that

$$K_s = \pi(X_s h_s) - \pi(X_s)\pi(h_s), \tag{2.4.6}$$

which is equivalent to

$$\int_0^t E[\lambda_s(K_s - \pi(X_s h_s) + \pi(X_s)\pi(h_s))]ds = 0, \tag{2.4.7}$$

for any bounded $\mathcal{F}_t^Y$-adapted λ_t.

Put $z_t = \int_0^t \lambda_s dm_s$ and $\xi_t = \int_0^t K_s dm_s$, then

$$\int_0^t E[\lambda_s K_s]ds = E[z_t \xi_t]$$

On the other hand,

$$E[z_t\xi_t] = E[z_t(\pi(X_t) - \pi(X_0) - \int_0^t \pi(H_s)ds)] = E[z_t X_t - \int_0^t z_s H_s ds],$$

since $E[z_t\pi(X_0)] = E[\pi(X_0)]E[z_t|\mathcal{F}_0^Y] = 0$, $E[z_t\pi(X_t)] = E[z_t]E[X_t|\mathcal{F}_t^Y] = E[z_t X_t]$ and

$$\begin{aligned} E[z_t \int_0^t \pi(H_s)ds] &= E[\int_0^t E[z_t|\mathcal{F}_s^Y]\pi(H_s)ds] \\ &= \int_0^t z_s \pi(H_s)ds \\ &= \int_0^t E[z_s H_s|\mathcal{F}_s^Y]ds \\ &= E[\int_0^t z_s H_s ds]. \end{aligned}$$

Using the definition of the innovation process m_t (from (2.3.1) and (2.3.2)) we see that

$$z_t = \int_0^t \lambda_s dV_s + \int_0^t \lambda_s\{h_s - \pi(h_s)\}ds. \tag{2.4.8}$$

Then

$$\begin{aligned} E[z_t\xi_t] = \quad & E[X_t \int_0^t \lambda_s dV_s - \int_0^t (\int_0^s \lambda_u dV_u)H_u ds] + E[X_t \int_0^t \lambda_s\{h_s - \pi(h_s)\}ds \\ & - \int_0^t (\int_0^s \lambda_u\{h_u - \pi(h_u)\}du)H_s ds]. \end{aligned} \tag{2.4.9}$$

We claim that the first expectation vanishes: indeed

$$E[X_0 \int_0^t \lambda_s dV_s] = E[X_0]E[\int_0^t \lambda_s dV_s|\mathcal{F}_0] = 0$$

and

$$\begin{aligned} E[\int_0^t (\int_0^s \lambda_u dV_u) H_s ds] &= E[\int_0^t E[\int_0^t \lambda_u dV_u|\mathcal{F}_s] H_s ds] \\ &= E[\int_0^t E[H_s \int_0^t \lambda_u dV_u|\mathcal{F}_s] ds] \\ &= E[\int_0^t \lambda_u dV_u \int_0^t H_s ds] \end{aligned}$$

and hence

$$\begin{aligned} E[X_t \int_0^t \lambda_s dV_s - \int_0^t (\int_0^s \lambda_u dV_u) H_s ds] &= E[\int_0^t \lambda_s dV_s (X_t - X_0 - \int_0^t H_s ds)] \\ &= E[\int_0^t \lambda_s dV_s W_t] = 0, \end{aligned}$$

where the latter equality holds since the Brownian motion process W_t is independent of the Brownian motion process V_t. Next consider

$$\begin{aligned} E[X_t \int_0^t \lambda_s \{h_s - \pi(h_s)\} ds] &= E[\int_0^t \lambda_s \{X_s(h_s - \pi(h_s))\} ds] \\ &\quad + E[\int_0^t \lambda_s (X_t - X_s)\{h_s - \pi(h_s)\} ds] \\ &= E[\int_0^t \lambda_s \{\pi(X_s h_s) - \pi(X_s)\pi(h_s)\} ds] \\ &\quad + E[\int_0^t \lambda_s (Z_t - Z_s)\{h_s - \pi(h_s)\} ds] \\ &\quad + E[\int_0^t \lambda_s \int_s^t H_u du \{h_s - \pi(h_s)\} ds] \\ &= E[\int_0^t \lambda_s \{\pi(X_s h_s) - \pi(X_s)\pi(h_s)\} ds] \\ &\quad + E[\int_0^t H_s (\int_0^s \lambda_u \{h_s - \pi(h_s)\} du) ds]. \end{aligned}$$

Assembling all parts together we obtain

$$E[z_t \xi_t] = \int_0^t E[\lambda_s \{\pi(X_s h_s) - \pi(X_s)\pi(h_s)\} ds]$$

This completes the proof.

2.5 General Filtering Equation

We consider a general model as follows:

Signal process:

$$X_t = X_0 + \int_0^t H_s ds + Z_t. \tag{2.5.1}$$

Observation process:

$$Y_t = \int_0^t h_s ds + BW_t. \tag{2.5.2}$$

where $E\int_0^t h_s^2 ds < \infty, h_t = h(X_t)$, $B > 0$ is a constant and Z_t is a $\mathcal{F}_t$ martingale independent of the Brownian motion process W_t.

Innovation process:

$$m_t = B^{-1}(Y_t - \int_0^t \pi(h_s)ds) \tag{2.5.3}$$

We can show that m_t is a Brownian motion process with respect to $\mathcal{F}_t^Y$ as in the proof of Theorem (2.3.3).

Theorem 2.5.7. $M_t = E[X_0|\mathcal{F}_t^Y] - \pi(X_0) + E[\int_0^t H_s ds|\mathcal{F}_t^Y] - \int_0^t \pi(H_s)ds + E[Z_t|\mathcal{F}_t^Y]$ *is a martingale with respect to* $\mathcal{F}_t^Y$

Proof : For any $0 \leq s \leq t$, by a property of conditional expectation, we have

$$E[E[X_0|\mathcal{F}_t^Y] - \pi(X_0)|\mathcal{F}_s^Y] = E[X_0|\mathcal{F}_s^Y] - \pi(X_0), \tag{2.5.4}$$

and

$$\begin{aligned}
E \quad & [E[\int_0^t H_u du|\mathcal{F}_t^Y] - \int_0^t \pi(H_u)du|\mathcal{F}_s^Y] \\
= & \int_0^t E[H_u|\mathcal{F}_s^Y]du - \int_0^t E[\pi(H_u)|\mathcal{F}_s^Y]du \\
= & E[\int_0^s H_u du|\mathcal{F}_s^Y] - \int_0^s \pi(H_u)du + \int_s^t E[H_u|\mathcal{F}_s^Y]du - \int_s^t E[\pi(H_u)|\mathcal{F}_s^Y]du \\
= & E[\int_0^s H_u du|\mathcal{F}_s^Y] - \int_0^s \pi(H_u)du. \qquad (2.5.5)
\end{aligned}$$

Since Z_t is a $\mathcal{F}_t$-martingale then

$$E[E[Z_t|\mathcal{F}_t^Y]|\mathcal{F}_s^Y] = E[Z_t|\mathcal{F}_s^Y] = E[E[Z_t|\mathcal{F}_s]|\mathcal{F}_s^Y] = E[Z_s|\mathcal{F}_s^Y]. \tag{2.5.6}$$

Combining equation (2.5.4)-(2.5.6) and definition of M_t yields

$$\begin{aligned} E[M_t|\mathcal{F}_s^Y] &= E[X_0|\mathcal{F}_s^Y] - \pi(X_0) + E[\int_0^s H_u du|\mathcal{F}_s^Y] - \int_0^s \pi(H_u)du + E[Z_s|\mathcal{F}_s^Y] \\ &= M_s. \end{aligned} \tag{2.5.7}$$

This shows that M_t is a $\mathcal{F}_t^Y$-martingale.

2.5.1 Fujisaki-Kallianpur-Kunita Filtering Equation

Theorem 2.5.8. *(Fujisaki-Kallianpur-Kunita Filtering Equation). The filter $\pi(X_t)$ satisfies the Fujisaki-Kallianpur-Kunita equation*

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s)ds + \int_0^t B^{-1}\{\pi(X_s h_s) - \pi(X_s)\pi(h_s)\}dm_s \tag{2.5.8}$$

where m_t is the innovation process defined in (2.3.1)

Proof : By Theorem 2.3.4 and Theorem 2.5.7, there exists some process K_t such that

$$M_t = \int_0^t K_s dm_s.$$

We should show that

$$K_s = \frac{\pi(X_s h_s) - \pi(X_s)\pi(h_s)}{B}, \tag{2.5.9}$$

which is equivalent to

$$\int_0^t E[\lambda_s(K_s - \frac{\pi(X_s h_s) - \pi(X_s)\pi(h_s)}{B})]ds = 0, \tag{2.5.10}$$

for any bounded $\mathcal{F}_t^Y$-adapted λ_t.

Let $z_t = \int_0^t \lambda_s dm_s$ and $\xi_t = \int_0^t K_s dm_s$, then

$$\int_0^t E[\lambda_s K_s]ds = E[z_t \xi_t]$$

On the other hand,

$$E[z_t\xi_t] = E[z_t(\pi(X_t) - \pi(X_0) - \int_0^t \pi(H_s)ds)] = E[z_tX_t - \int_0^t z_sH_sds],$$

since $E[z_t\pi(X_0)] = E[\pi(X_0)]E[z_t|\mathcal{F}_0^Y] = 0$, $E[z_t\pi(X_t)] = E[z_t]E[X_t|\mathcal{F}_t^Y] = E[z_tX_t]$ and

$$\begin{aligned} E[z_t\int_0^t \pi(H_s)ds] &= E[\int_0^t E[z_t|\mathcal{F}_s^Y]\pi(H_s)ds] \\ &= \int_0^t z_s\pi(H_s)ds \\ &= \int_0^t E[z_sH_s|\mathcal{F}_s^Y]ds \\ &= E[\int_0^t z_sH_sds]. \end{aligned}$$

It follows from the definition of m_t that

$$z_t = \int_0^t \lambda_s dW_s + \int_0^t \lambda_s \frac{h_s - \pi(h_s)}{B} ds, \tag{2.5.11}$$

and

$$\begin{aligned} E[z_t\xi_t] = \quad & E[X_t\int_0^t \lambda_s dW_s - \int_0^t (\int_0^s \lambda_u dW_u)H_u ds] + E[X_t\int_0^t \lambda_s \frac{h_s - \pi(h_s)}{B} ds \\ & - \int_0^t (\int_0^s \lambda_u \frac{h_u - \pi(h_u)}{B} du)H_s ds]. \end{aligned} \tag{2.5.12}$$

We claim that the first expectation vanishes: indeed

$$E[X_0\int_0^t \lambda_s dW_s] = E[X_0]E[\int_0^t \lambda_s dW_s|\mathcal{F}_0] = 0$$

and

$$\begin{aligned} E[\int_0^t (\int_0^s \lambda_u dW_u)H_s ds] &= E[\int_0^t E[\int_0^t \lambda_u dW_u|\mathcal{F}_s]H_s ds] \\ &= E[\int_0^t E[H_s\int_0^t \lambda_u dW_u|\mathcal{F}_s]ds] \\ &= E[\int_0^t \lambda_u dW_u \int_0^t H_s ds] \end{aligned}$$

and hence

$$\begin{aligned} E[X_t \int_0^t \lambda_s dW_s - \int_0^t (\int_0^s \lambda_u dW_u) H_s ds] &= E[\int_0^t \lambda_s dW_s (X_t - X_0 - \int_0^t H_s ds)] \\ &= E[\int_0^t \lambda_s dW_s Z_t] = 0, \end{aligned}$$

where the latter equality holds since the martingale Z_t is independent of W_t. Next consider

$$\begin{aligned} E[X_t \int_0^t \lambda_s \frac{h_s - \pi(h_s)}{B} ds] &= E[\int_0^t \lambda_s \frac{X_s(h_s - \pi(h_s))}{B} ds] \\ &\quad + E[\int_0^t \lambda_s (X_t - X_s) \frac{h_s - \pi(h_s)}{B} ds] \\ &= E[\int_0^t \lambda_s \frac{\pi(X_s h_s) - \pi(X_s)\pi(h_s)}{B} ds] \\ &\quad + E[\int_0^t \lambda_s (Z_t - Z_s) \frac{h_s - \pi(h_s)}{B} ds] \\ &\quad + E[\int_0^t \lambda_s \int_s^t H_u du \frac{h_s - \pi(h_s)}{B} ds] \\ &= E[\int_0^t \lambda_s \frac{\pi(X_s h_s) - \pi(X_s)\pi(h_s)}{B} ds] \\ &\quad + E[\int_0^t H_s (\int_0^s \lambda_u \frac{h_s - \pi(h_s)}{B} du) ds]. \end{aligned}$$

Assembling all parts together we obtain

$$E[z_t \xi_t] = \int_0^t E[\lambda_s \frac{\pi(X_s h_s) - \pi(X_s)\pi(h_s)}{B} ds]$$

The proof is thus complete.

2.6 Kushner Equation

The Fujisaki-Kallianpur-Kunita equation takes a somewhat more concrete form in the case when (X_t, Y_t) are diffusion process, namely the solution of

$$\begin{aligned} dX_t &= a(X_t)dt + b(X_t)dW_t \qquad X_0 = \xi \\ dY_t &= A(X_t)dt + dV_t \qquad\qquad Y_0 = 0 \end{aligned}$$

where ξ is a random variable with probability density $p_0(x)$, independent of Brownian motion process W_t and V_t.

2.6.1 Kushner Theorem

Theorem 2.6.9. *Assume there is an $\mathcal{F}_t^Y$-adapted random process $q_t(x)$, satisfying the Kushner-Stratonovich stochastic partial integral-differential equation*

$$q_t(x) = p_0(x) + \int_0^t (\mathcal{L}^* q_s)(x)ds + \int_0^t q_s(x)(A(x) - \pi_s(A))dm_s \tag{2.6.1}$$

where

$$(\mathcal{L}^* f)(x) = -\frac{\partial}{\partial x}(a(x)f(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(b^2(x)f(x)) \tag{2.6.2}$$

and

$$\pi_t(A) = \int_{\mathbb{R}} A(x)q_t(x)dx \tag{2.6.3}$$

Then $q_t(x)$ is a version of the conditional density of X_t given $\mathcal{F}_t^Y$, i.e. for any bounded function f

$$E[f(X_t)|\mathcal{F}_t^Y] = \int_{\mathbb{R}} f(x)q_t(x)dx \tag{2.6.4}$$

Proof : An application of the Itô formula to the function $f(X_t)$ gives us:

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= [f'(X_t)a(X_t) + \frac{1}{2}f''(X_t)b^2(X_t)]dt + f'(X_t)b(X_t)dW_t, \end{aligned}$$

or equivalently,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t [a(X_s)\frac{\partial}{\partial X_s}f(X_s) + \frac{1}{2}b^2(X_s)\frac{\partial^2}{\partial X_s^2}f(X_s)]ds \\ &\quad + \int_0^t b(X_s)f'(X_s)dW_s \\ &= f(X_0) + \int_0^t (\mathcal{L}f)(X_s)ds + \int_0^t b(X_s)f'(X_s)dW_s, \end{aligned}$$

where

$$(\mathcal{L}f)(x) = a(x)\frac{\partial}{\partial x}f(x) + \frac{1}{2}b^2(x)\frac{\partial^2}{\partial x^2}f(x). \tag{2.6.5}$$

Next, consider

$$\begin{aligned}\pi_0(f) &= \int_{\mathbb{R}} f(x)q_0(x)dx \\ &= \int_{\mathbb{R}} f(x)p_0(x)dx\end{aligned}$$

and

$$\begin{aligned}\pi_s(\mathcal{L}f) &= \int_{\mathbb{R}} (\mathcal{L}f)(x)q_s(x)dx \\ &= \int_{\mathbb{R}} \left(a(x)\frac{\partial}{\partial x}f(x) + \frac{b^2(x)}{2}\frac{\partial^2}{\partial x^2}f(x)\right)q_s(x)dx \\ &= \int_{\mathbb{R}} \left(-\frac{\partial}{\partial x}a(x)q_s(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}b^2(x)q_s(x)\right)f(x)dx \\ &= \int_{\mathbb{R}} (\mathcal{L}^*q_s)(x)f(x)dx\end{aligned}$$

and

$$\begin{aligned}\pi_s(fA) - \pi_s(f)\pi_s(A) &= \int_{\mathbb{R}} f(x)A(x)q_s(x)dx - \pi_s(A)\int_{\mathbb{R}} f(x)q_s(x)dx \\ &= \int_{\mathbb{R}} f(x)q_s(x)[A(x) - \pi_s(A)]dx\end{aligned}$$

Then the right hand side of Fujisaki-Kallianpur-Kunita equation reads

$$\begin{aligned}\pi_t(f) &= \pi_0(f) + \int_0^t \pi(\mathcal{L}f)ds + \int_0^t \{\pi_s(fA) - \pi_s(f)\pi_s(A)\}dm_s \\ &= \int_{\mathbb{R}} f(x)\left(p_0(x) + \int_0^t (\mathcal{L}^*q_s)(x)ds + \int_0^t q_s(x)(A(x) - \pi_s(A))dm_s\right)dx \\ &= \int_{\mathbb{R}} f(x)q_t(x)dx.\end{aligned}$$

This proves the theorem.

2.7 Zakai Equation

2.7.1 Quasi-filtering

In this section we consider a transformation of the probability P into another probability Q and denote by L_t the restriction to $\mathcal{F}_t^Y$ of the Radon-Nykodym derivative $\frac{dP}{dQ}$:

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t^Y} = L_t \tag{2.7.1}$$

and define a stochastic process $\sigma(X_t)$ as follows

$$\sigma(X_t) := E_Q[X_t L_t | \mathcal{F}_t^Y], \tag{2.7.2}$$

where E_Q is denoted the expectation under the new probability Q. This process is called from now on the quasi-filter of X_t based on the information $\mathcal{F}_t^Y$ given by the observation Y_t. Now the relation between the filter $\pi(X_t)$ and the quasi-filter $\sigma(X_t)$ can be expressed as

$$\pi(X_t) = \frac{\sigma(X_t)}{\sigma(1_t)}. \tag{2.7.3}$$

2.7.2 Zakai Equation

Theorem 2.7.10. *The quasi-filter $\sigma(X_t)$ satisfies the following equation*

$$d\sigma(X_t) = \sigma(H_t)dt + \sigma(X_t h_t)dY_t. \tag{2.7.4}$$

This equation is called Zakai filtering equation.

Proof : We have by the formula (2.2.7) in the proof of the Girsanov Theorem:

$$L_t = 1 + \int_0^t L_s \pi(h_s) dY_s, \tag{2.7.5}$$

and

$$L_t \pi(X_t) = E_Q[X_t L_t | \mathcal{F}_t^Y] = \sigma(X_t). \tag{2.7.6}$$

Now we see that

$$\begin{aligned}
\sigma(X_t) &= L_t\pi(X_t) = L_0\pi(X_0) + \int_0^t L_s d\pi(X_s) + \int_0^t \pi(X_s)dL_s + \langle L, \pi(X)\rangle_t \\
&= L_0\pi(X_0) + \int_0^t L_s[\pi(H_s)ds + (\pi(h_sX_s) - \pi(h_s)\pi(X_s))]dm_s \\
&\quad + \int_0^t \pi(X_s)L_s\pi(h_s)dY_s + \langle L, \pi(X)\rangle_t \\
&= \sigma(X_0) + \int_0^t \sigma(H_s)ds + \int_0^t L_s[\pi(h_sX_s) - \pi(h_s)\pi(X_s)](dY_s - \pi(h_s)ds) \\
&\quad + \int_0^t \pi(X_s)L_s\pi(h_s)dY_s + \langle L, \pi(X)\rangle_t.
\end{aligned}$$

Hence

$$\sigma(X_t) = \sigma(X_0) + \int_0^t \sigma(H_s)ds + \int_0^t \sigma(h_sX_s)dY_s,$$

where integration by parts has been used. This equation is equivalent to:

$$d\sigma(X_t) = \sigma(X_th_t)dY_t + \sigma(H_t)dt \tag{2.7.7}$$

Remark It follows from the proof of the previous theorem that if $\pi(X_t)$ satisfies (2.4.5), then $\sigma(X_t)$ satisfies (2.7.4).

Theorem 2.7.11. *If $\sigma(X_t)$ is a solution of the Zakai equation, then the process $\pi_t = \pi(X_t)$ defined by (2.7.3) is a solution of the Fujisaki-Kallianpur-Kunita equation*

Proof : Consider

$$\begin{aligned}
d\pi(X_t) &= d\left(\frac{\sigma(X_t)}{\sigma(1_t)}\right) \\
&= \frac{1}{\sigma(1_t)}d\sigma(X_t) - \frac{\sigma(X_t)}{\sigma^2(1_t)}d\sigma(1_t) + \frac{\sigma(X_t)}{\sigma^3(1_t)}(d\sigma(1_t))^2 - \frac{1}{\sigma^2(1_t)}d\sigma(X_t)d\sigma(1_t) \\
&= \frac{1}{\sigma(1_t)}[\sigma(H_t)dt + \sigma(X_th_t)dY_t] - \frac{\sigma(X_t)}{\sigma^2(1_t)}[\sigma(h_t)dY_t] + \frac{\sigma(X_t)}{\sigma^3(1_t)}\sigma^2(h_t)dt \\
&\quad - \frac{\sigma(h_t)\sigma(X_th_t)}{\sigma^2(1_t)}dt.
\end{aligned}$$

Next, we see that

$$\begin{aligned}
d\pi(X_t) &= \frac{\sigma(H_t)}{\sigma(1_t)}dt + \frac{\sigma(X_t h_t)}{\sigma(1_t)}dY_t - \frac{\sigma(X_t)\sigma(h_t)}{\sigma^2(1_t)}dY_t + \frac{\sigma(X_t)\sigma^2(h_t)}{\sigma^3(1_t)}dt \\
&\quad -\frac{\sigma(h_t)\sigma(X_t h_t)}{\sigma^2(1_t)}dt \\
&= \pi(H_t)dt + \pi(X_t h_t)dY_t - \pi(X_t)\pi(h_t)dY_t + \pi(X_t)\pi^2(h_t)dt \\
&\quad -\pi(h_t)\pi(X_t h_t)dt \\
&= \pi(H_t)dt + [\pi(X_t h_t) - \pi(X_t)\pi(h_t)]dY_t \\
&\quad +[\pi(X_t)\pi^2(h_t)dt - \pi(h_t)\pi(X_t h_t)dt] \\
&= \pi(H_t)dt + [\pi(X_t h_t) - \pi(X_t)\pi(h_t)](dY_t - \pi(h_t)dt) \\
&= \pi(H_t)dt + [\pi(X_t h_t) - \pi(X_t)\pi(h_t)]dm_t.
\end{aligned}$$

Then we have finally,

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s)ds + \int_0^t (\pi(X_s h_s) - \pi(X_s)\pi(h_s))dm_s \tag{2.7.8}$$

This proves the theorem.

The Zakai equation takes a somewhat more concrete form in the case when (X_t) and (Y_t) are diffusion process, i.e. the process (X_t, Y_t) the solution of the system:

$$\begin{aligned}
dX_t &= a(t, X_t)dt + b(t, X_t)dW_t \qquad X_0 = \eta \\
dY_t &= g(t, X_t)dt + dV_t
\end{aligned}$$

where W_t and V_t are two independent Brownian motions and η is a random variable with probability density $p_0(x)$ with $\int_{\mathbb{R}} x^2 p_0(x)dx < \infty$.

Theorem 2.7.12. *Assume that there is an $\mathcal{F}_t^Y$-adapted nonnegative random process $\rho_t(x)$, satisfying the Zakai PDE*

$$d\rho_t(x) = (\mathcal{L}^*\rho_t)(x)dt + g(t,x)\rho_t(x)dY_t, \qquad \rho_0(x) = p_0(x), \tag{2.7.9}$$

where

$$(\mathcal{L}^* f)(x) = -\frac{\partial}{\partial x}(a(t,x)f(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(b^2(t,x)f(x)). \tag{2.7.10}$$

Then $\rho_t(x)$ is a version of the unnormalized conditional density of X_t given $\mathcal{F}_t^Y$, so that for any measurable function f, such that $Ef^2(X_t) < \infty$

$$E[f(X_t)|\mathcal{F}_t^Y] = \frac{\int_{\mathbb{R}} f(x)\rho_t(x)dx}{\int_{\mathbb{R}} \rho_t(x)dx}. \tag{2.7.11}$$

Proof : The Itô formula applied to the function $f(X_t)$ gives us:

$$f(X_t) = f(X_0) + \int_0^t (\mathcal{L}f)(X_s)ds + \int_0^t b(X_s)f'(X_s)dW_s,$$

where

$$(\mathcal{L}f)(x) = a(t,x)\frac{\partial}{\partial x}f(x) + \frac{1}{2}b^2(t,x)\frac{\partial^2}{\partial x^2}f(x) \tag{2.7.12}$$

Next, we see that

$$\begin{aligned}
\sigma_0(f) &= \int_{\mathbb{R}} f(x)\rho_0(x)dx \\
&= \int_{\mathbb{R}} f(x)p_0(x)dx,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_s(\mathcal{L}f) &= \int_{\mathbb{R}} (\mathcal{L}f)(x)\rho_s(x)dx \\
&= \int_{\mathbb{R}} \left(a(t,x)\frac{\partial}{\partial x}f(x) + \frac{b^2(t,x)}{2}\frac{\partial^2}{\partial x^2}f(x)\right)\rho_s(x)dx \\
&= \int_{\mathbb{R}} \left(-\frac{\partial}{\partial x}a(t,x)\rho_s(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}b^2(t,x)\rho_s(x)\right)f(x)dx \\
&= \int_{\mathbb{R}} (\mathcal{L}^*\rho_s)(x)f(x)dx,
\end{aligned}$$

and

$$\sigma_s(fh) = \int_{\mathbb{R}} f(x)h(s,x)\rho_s(x)dx.$$

Then the right hand side of Zakai equation reads

$$\begin{aligned}\sigma_t(f) &= \sigma_0(f) + \int_0^t \sigma_s(\mathcal{L}f)ds + \int_0^t \sigma_s(fh)dY_s \\ &= \int_{\mathbb{R}} f(x)\left(p_0(x) + \int_0^t (\mathcal{L}^*\rho_s)(x)ds + \int_0^t h(s,x)\rho_s(x)dY_s\right)dx \\ &= \int_{\mathbb{R}} f(x)\rho_t(x)dx.\end{aligned}$$

The proof is thus complete.

CHAPTER III

FILTERING PROBLEM WITH POINT PROCESS OBSERVATION

In this chapter, after establishing the filtering equation and quasi-filtering equation with point process observation, we study the case of a Markov-Feller signal process and we prove some theorems of filtering for Ornstein-Uhlenbeck processes.

3.1 Introduction

In financial filtering, we want to estimate some financial factors through some direct observation process depending on time. But in practice, this observation process can be observe only at discrete times, so in this Chapter we consider a point process as an observation. For Definitions and Theorems in this section one can refer to Brémaud (1981).

A point process over $[0, \infty)$ can be introduced into three different ways: as a sequence of nonnegative random variables, as a discrete random measure, or via its associated counting process. In this Chapter, we use the last way to study financial filtering problems with point process observation.

3.1.1 Point Processes

Definition 3.1.1. (Simple Univariate Point Processes). A realization of a point process over $[0, \infty)$ can be described by a sequence T_n in $[0, \infty]$ such that

$$\begin{aligned} T_0 &= 0 \\ T_n < \infty &\Rightarrow T_n < T_{n+1}. \end{aligned}$$

This realization is nonexplosive, i.e.

$$T_\infty = \lim_{n \to \infty} T_n = +\infty.$$

To each realization T_n corresponds a counting function N_t defined by

$$N_t = \begin{cases} n, & \text{if } t \in [T_n, T_{n+1}); \\ +\infty, & \text{if } t \geq T_\infty. \end{cases}$$

N_t is therefore a right-continuous step function such that $N_0 = 0$ and its jumps are upward jumps of magnitude 1.

If the above T_n's are random variable, defined on some probability space $(\Omega, \mathcal{F}, P)$, one then calls the sequence T_n a point process. The associated counting process N_t is also called a point process. Henceforward, unless explicitly mentioned, attention will be restricted to P-nonexplosive point process, that is to say point processes such that, P-a.s.,

$$N_t < \infty, \quad t \geq 0 \quad (\text{or equivalently } \; T_\infty \equiv \infty)$$

Moreover, if the condition

$$E[N_t] < \infty, \quad t \geq 0$$

holds, the point process N_t is said to be integrable.

Definition 3.1.2. (Multivariate Point Processes). Let T_n be a point process defined on $(\Omega, \mathcal{F}, P)$, and let $(Z_n, n \geq 1)$ be a sequence of $\{1, 2, ..., k\}$-valued random variables, also defined on $(\Omega, \mathcal{F}, P)$. Define for all $i, 1 \leq i \leq k$ and all $t \geq 0$

$$N_t(i) = \sum_{n \geq 1} \mathbf{1}(T_n \leq t)\mathbf{1}(Z_n = i)$$

Both the k-vector process $N_t = (N_t(1), ..., N_t(k))$ and the double sequence $(T_n, Z_n, n \geq 1)$ are called k-variate point processes.

Definition 3.1.3. (Doubly Stochastic Poisson Processes or Conditional Poisson Processes). Let N_t be a point process adapted to a history $\mathcal{F}_t$, and let λ_t be a nonnegative measurable process (all given on the same probability space $(\Omega, \mathcal{F}, P)$) Suppose that

$$\lambda_t \text{ is } \mathcal{F}_0 - \text{measurable}, \quad t \geq 0$$

and that

$$\int_0^t \lambda_s ds < \infty \quad P - a.s., \quad t \geq 0$$

If for all $0 \leq s \leq t$ and all $u \in \mathbb{R}$

$$E[e^{iu(N_t - N_s)}|\mathcal{F}_s] = \exp\left\{(e^{iu} - 1)\int_s^t \lambda_v dv\right\}$$

then N_t is called a $(P, \mathcal{F}_t)$-doubly stochastic Poisson process or a $(P, \mathcal{F}_t)$-conditional Poisson process with the (stochastic) intensity λ_t.

If λ_t is deterministic (the notation $\lambda(t)$ is used), then N_t is called a $(P, \mathcal{F}_t)$-Poisson process. If moreover $\mathcal{F}_t \equiv \mathcal{F}_t^N$, one simply says; N_t is a Poisson process with the intensity $\lambda(t)$. If $\mathcal{F}_t \equiv \mathcal{F}_t^N, \lambda(t) \equiv 1$, then N_t is the standard Poisson process.

Theorem 3.1.4. *(Characterization of Doubly Stochastic Poisson Processes or Conditional Poisson Processes). Let N_t be a point process adapted to some history $\mathcal{F}_t$, and let λ_t be a nonnegative measurable process such that for all $t \geq 0$*

(a) λ_t is $\mathcal{F}_0$-measurable,

(b) $\int_0^t \lambda_s ds < \infty, P - a.s.$

Then, if the equality

$$E\Big[\int_0^\infty C_s dN_s\Big] = E\Big[\int_0^\infty C_s \lambda_s ds\Big]$$

is verified for all nonnegative $\mathcal{F}_t$-predictable process C_t, N_t is a doubly stochastic Poisson process with the $\mathcal{F}_t$-intensity λ_t.

Proof : See Brémaud (1981).

Theorem 3.1.5. *(Watanabe Theorem). Let N_t be a point process adapted to the history $\mathcal{F}_t$, and let $\lambda(t)$ be a locally integrable nonnegative measurable function. Suppose that*

$$N_t - \int_0^t \lambda(s)ds \quad \textit{is an} \quad \mathcal{F}_t\textit{-martingale.}$$

Then N_t is an $\mathcal{F}_t$-Poisson process with the intensity $\lambda(t)$. (i.e., for all $0 \leq s \leq t, N_t - N_s$ is a Poisson random variable with parameter $\int_0^t \lambda(u)du$, independent of $\mathcal{F}_s$).

Proof : See Brémaud (1981).

Definition 3.1.6. (Progressive Process). The process X_t is said to be $\mathcal{F}_t$-progressive iff for all $t \geq 0$ the mapping $[0,t] \times \Omega$ into $\mathbb{R}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable.

Definition 3.1.7. (Stochastic Intensity). Let N_t be a point process adapted to some history $\mathcal{F}_t$, and let λ_t be a nonnegative $\mathcal{F}_t$-progressive process such that for all $n \geq 1$

$$\int_0^t \lambda_s ds < \infty \quad P - a.s.$$

If for all nonnegative $\mathcal{F}_t$-predictable processes C_t, the equality

$$E\left[\int_0^\infty C_s dN_s\right] = E\left[\int_0^\infty C_s \lambda_s ds\right]$$

is verified, then we say: N_t admits the $(P, \mathcal{F}_t)$-intensity (or $\mathcal{F}_t$-intensity) λ_t.

Theorem 3.1.8. *(Stochastic Intensity Martingale Characterization). Let N_t be a nonexplosive point process adapted to $\mathcal{F}_t$, and suppose that for some nonnegative $\mathcal{F}_t$-progressive process λ_t and for all $n \geq 1$*

$$N_{t\wedge T_n} - \int_0^{t\wedge T_n} \lambda_s ds \quad \textit{is a} \quad (P, \mathcal{F}_t) - \textit{martingale.}$$

Then λ_t is the $\mathcal{F}_t$-intensity of N_t.

Proof : See Brémaud (1981).

3.2 Filtering of a General Process from Point Process Observation

3.2.1 Problem Setting and Assumptions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which all processes are defined and adapted to a filtration $(\mathcal{F}_t, \ \ t \geq 0)$.

We consider a filtering problem where the signal processes is a semimartingale

$$X_t = X_0 + \int_0^t H_s ds + Z_t, \tag{3.2.1}$$

where Z_t is a $\mathcal{F}_t$-martingale, H_t is a bounded $\mathcal{F}_t$-progressive process and $E[\sup_{s\leq t}|X_s|] < \infty$ for every $t \geq 0$, X_0 is a random variable such that $E|X_0|^2 < \infty$; the observation is given by a point process $\mathcal{F}_t$-semimartingale of the form

$$Y_t = \int_0^t h_s ds + M_t, \tag{3.2.2}$$

where M_t is a $\mathcal{F}_t$-martingale with mean 0, $M_0 = 0$ such that the future σ- field $\sigma(M_u - M_t; u \geq t)$ is independent of the past one $\sigma(Y_u, h_u; u \leq t)$, $h_t = h(X_t)$ is a positive bounded $\mathcal{F}_t$- progressive process such that $E\int_0^t h_s^2 ds < \infty$ for every t. Moreover, we suppose that Z_t and M_t are independent.

Denote by $\mathcal{F}_t^Y$ the σ-algebra generated by all random variables $Y_s, s \leq t$. Thus $\mathcal{F}_t^Y$ records all information about the observation up to the time t.

Suppose that the process $u_s = \frac{d}{ds} < Z, M >_s$ is $\mathcal{F}_s$- predictable $(s \leq t)$ where $<, >$ stands for the quadratic variation of Z_t and M_t. Denote also by $\hat{u}_s$ the $\mathcal{F}_t^Y$- predictable projection of u_s. By assumptions imposed on Z and M we see that $< Z, M >= 0$, so $u_s = 0$.

The filter of (X_t) based on information given by (Y_t) is defined as the conditional expectation

$$\pi(X_t) := E[X_t|\mathcal{F}_t^Y], \tag{3.2.3}$$

or more general

$$\pi_t(f) := E[f(X_t)|\mathcal{F}_t^Y], \tag{3.2.4}$$

where f is a bounded continuous function or $f \in C_b(\mathbb{R})$.

Denote by $\pi(h_t)$ the filtering process corresponding to the process h_t in (3.2.2).

3.2.2 Innovation Process

Definition 3.2.9. Let m_t be a process defined by

$$m_t := Y_t - \int_0^t \pi(h_s) ds. \tag{3.2.5}$$

The process m_t is called the innovation from the observation process Y_t.

Lemma 3.2.10. *m_t is a point process $\mathcal{F}_t^Y$-martingale and for any t, the future σ-field $\sigma(m_t - m_s\ ;\ t \geq s)$ is independent of $\mathcal{F}_s^Y$.*

Proof : We have by definition of m_t in (3.2.5) and Y_t in (3.2.2) that, for any $t \geq s > 0$,

$$\begin{aligned} m_t - m_s &= Y_t - Y_s - \int_s^t \pi(h_u)du \\ &= M_t - M_s + \int_s^t \{h_u - \pi(h_u)\}du. \end{aligned} \tag{3.2.6}$$

Since $\mathcal{F}_s^Y \subset \mathcal{F}_s$ for any $s \geq 0$ and M_t is $\mathcal{F}_t$-martingale that

$$E[M_t - M_s|\mathcal{F}_s^Y] = E\big[E[M_t - M_s|\mathcal{F}_s]\big|\mathcal{F}_s^Y\big] = 0. \tag{3.2.7}$$

It follow from $\mathcal{F}_u^Y \supset \mathcal{F}_s^Y$ whenever $u \geq s > 0$ and definition of $\pi(h_u)$ in (3.2.4) that

$$E[h_u|\mathcal{F}_s^Y] = E\big[E[h_u|\mathcal{F}_u^Y]\big|\mathcal{F}_s^Y\big] = E[\pi(h_u)|\mathcal{F}_s^Y]. \tag{3.2.8}$$

From (3.2.8). Hence

$$\int_s^t E[h_u - \pi(h_u)|\mathcal{F}_s^Y]du = 0. \tag{3.2.9}$$

Fubini's Theorem implies

$$E\left[\int_s^t \{h_u - \pi(h_u)\}du\middle|\mathcal{F}_s^Y\right] = 0. \tag{3.2.10}$$

Thus, for any $t \geq s > 0$, we get

$$E[m_t - m_s|\mathcal{F}_s^Y] = E[M_t - M_s|\mathcal{F}_s^Y] + E\left[\int_s^t \{h_u - \pi(h_u)\}du\middle|\mathcal{F}_s^Y\right] = 0, \tag{3.2.11}$$

and therefore the process m_t is $\mathcal{F}_t^Y$-martingale.

Now for any s, t such that $0 \leq s \leq t$ we consider two families $\mathcal{C}_t$ and $\mathcal{D}_t$ of sets of random variables defined as follows:

$$\mathcal{C}_{s,t} = \{\text{sets } \mathcal{C}_a, s \leq a \leq t\}, \text{where } \mathcal{C}_a = \{m_t - m_\alpha; a \leq \alpha \leq t\}$$

$$\mathcal{D}_s = \{\text{sets } \mathcal{D}_b, 0 \leq b \leq t\}, \text{where } \mathcal{D}_b = \{Y_\beta; b \leq \beta \leq s\}.$$

It is easy to check that $\mathcal{C}_{s,t}$ and $\mathcal{D}_s$ are π-system, i.e. they are closed with respect to finite intersection. Also they are independent each of other by (3.2.11). It follows that (refer to Kallenberg (2002)) the σ-algebra $\sigma(\mathcal{C}_{s,t}) = \sigma(m_t - m_s, s \leq t)$ generated by $\mathcal{C}_{s,t}$ is independent of the σ-algebra $\sigma(\mathcal{D}_s) = \mathcal{F}_s^Y$ generated by $\mathcal{D}_s$. The second assertion of this Lemma as thus established.

Lemma 3.2.11. *Let R_t be a $\mathcal{F}_t^Y$-martingale. Then there exists a $\mathcal{F}_t^Y$-predictable process K_t such that for all $t \geq 0$,*

$$\int_0^t K_s \pi(h_s) ds < \infty \quad P - a.s, \tag{3.2.12}$$

and such that R_t has the following representation:

$$R_t = R_0 + \int_0^t K_s dm_s. \tag{3.2.13}$$

Proof : See Brémaud (1981).

3.2.3 General Filtering Equation Theorem

Theorem 3.2.12. *The filtering equation for the filtering problem (3.2.1)- (3.2.2) is given by:*

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s) ds + \int_0^t \{\pi(h_s)\}^{-1} \{\pi(X_{s^-} h_s) - \pi(X_{s^-}) \pi(h_s)\} dm_s. \tag{3.2.14}$$

Proof : Define

$$\bar{M}_t := \pi(X_t) - \pi(X_0) - \int_0^t \pi(H_s) ds. \tag{3.2.15}$$

First, we aim to prove that $\bar{M}_t$ is a $\mathcal{F}_t^Y$-martingale. To see this, we note from the definition of $\bar{M}_t$ in (3.2.15) that, for any $t \geq s > 0$,

$$\bar{M}_t - \bar{M}_s = \pi(X_t) - \pi(X_s) - \int_s^t \pi(H_u) du.$$

Moreover, by the rules for calculation of conditional expectation, we have

$$E[\pi(X_t)|\mathcal{F}_s^Y] = E\big[E[X_t|\mathcal{F}_t^Y]\big|\mathcal{F}_s^Y\big] = E[X_t|\mathcal{F}_s^Y] \qquad (s \leq t)$$

and

$$E[\pi(X_s)|\mathcal{F}_s^Y] = \pi(X_s) = E[X_s|\mathcal{F}_s^Y].$$

Thus

$$\begin{aligned} E[\bar{M}_t - \bar{M}_s|\mathcal{F}_s^Y] &= E\left[\pi(X_t) - \pi(X_s) - \int_s^t \pi(H_u)du\middle|\mathcal{F}_s^Y\right] \\ &= E[X_t|\mathcal{F}_s^Y] - E[X_s|\mathcal{F}_s^Y] - E\left[\int_s^t \pi(H_u)du\middle|\mathcal{F}_s^Y\right] \\ &= E\left[X_t - X_s - \int_s^t \pi(H_u)du\middle|\mathcal{F}_s^Y\right]. \end{aligned} \tag{3.2.16}$$

Substituting the process X_t from (3.2.1) into (3.2.16), we get

$$E[\bar{M}_t - \bar{M}_s|\mathcal{F}_s^Y] = E\left[Z_t - Z_s + \int_s^t \{H_u - \pi(H_u)\}du\middle|\mathcal{F}_s^Y\right]. \tag{3.2.17}$$

Since Z_t is a $\mathcal{F}_t$-martingale then

$$E[Z_t - Z_s|\mathcal{F}_s^Y] = E\big[E[Z_t - Z_s|\mathcal{F}_s]\big|\mathcal{F}_s^Y\big] = 0.$$

On the other hand, for any $u \in (s, t)$,

$$E[H_u|\mathcal{F}_s^Y] = E\big[E[H_u|\mathcal{F}_u^Y]\big|\mathcal{F}_s^Y\big] = E[\pi(H_u)|\mathcal{F}_s^Y].$$

Thus

$$E[H_u - \pi(H_u)|\mathcal{F}_s^Y] = 0,$$

and hence

$$\int_s^t E[H_u - \pi(H_u)|\mathcal{F}_s^Y]du = 0.$$

Fubini's Theorem implies

$$E\left[\int_s^t H_u - \pi(H_u)du\middle|\mathcal{F}_s^Y\right] = 0.$$

We summarize the above results into (3.2.17)

$$E[\bar{M}_t - \bar{M}_s|\mathcal{F}_s^Y] = E[Z_t - Z_s|\mathcal{F}_s^Y] + E[\int_s^t \{H_u - \pi(H_u)\}du|\mathcal{F}_s^Y] = 0.$$

This proves $\bar{M}_t$ is a $\mathcal{F}_t^Y$-martingale.

Now we can utilize Lemma 3.2.11 to assert that there exists a $\mathcal{F}_t^Y$-predictable process K_t such that $\int_0^t K_s\pi(h_s)ds < \infty$ P-a.s., $\forall t \leq 0$ and

$$\bar{M}_t = \bar{M}_0 + \int_0^t K_s dm_s. \tag{3.2.18}$$

Equating (3.2.15) and (3.2.18) gives

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s)ds + \int_0^t K_s dm_s. \tag{3.2.19}$$

Lemma 3.2.10 shows that m_t is a $\mathcal{F}_t^Y$-martingale and by Lemma 3.2.11, there exists a $\mathcal{F}_t^Y$-predictable process U_t such that $\int_0^t U_s\pi(h_s)ds < \infty$ P-a.s., $\forall t \leq 0$ and

$$m_t = m_0 + \int_0^t U_s dm_s. \tag{3.2.20}$$

Substituting Y_t from (3.2.2) into (3.2.5), we see that m_t can be expressed as

$$m_t = \int_0^t \{h_s - \pi(h_s)\}ds + M_t. \tag{3.2.21}$$

Equating (3.2.20) and (3.2.5), we get

$$Y_t = \int_0^t U_s dm_s + \int_0^t \pi(h_s)ds. \tag{3.2.22}$$

Finally, we shall show that

$$K_s = \{\pi(h_s)\}^{-1}\{\pi(X_{s-}h_s) - \pi(X_{s-})\pi(h_s)\}.$$

By definition of $\pi(X_t)$ and properties of conditional expectation, we have

$$E[\pi(X_t)Y_t] = E\big[E[X_t|\mathcal{F}_t^Y]Y_t\big] = E\big[E[X_tY_t|\mathcal{F}_t^Y]\big] = E[X_tY_t]. \tag{3.2.23}$$

We know that the integration by parts formula applied to the processes X_t and Y_t has the form:

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s + \langle X, Y\rangle_t, \tag{3.2.24}$$

where $X_{s-} = \lim_{u\to s, u<s} X_u$ and $\langle X, Y\rangle_t$ stands for the quadratic covariation of X_t and Y_t. Now substituting Y_t from (3.2.22) into the second term of the right hand side of (3.2.24) we get

$$\int_0^t X_{s-}dY_s = \int_0^t X_{s-}\{U_s dm_s + \pi(h_s)ds\}. \tag{3.2.25}$$

Next, substituting m_t from (3.2.21) into (3.2.25), we get

$$\begin{aligned}\int_0^t X_{s-}dY_s &= \int_0^t X_{s-}U_s\big\{dM_s + \{h_s - \pi(h_s)\}ds\big\} + \int_0^t X_{s-}\pi(h_s)ds \\ &= \int_0^t X_{s-}U_s dM_s + \int_0^t X_{s-}U_s\{h_s - \pi(h_s)\}ds \\ &\quad + \int_0^t X_{s-}\pi(h_s)ds. \end{aligned} \tag{3.2.26}$$

Substituting X_t from (3.2.1) into the third term on the right hand side of (3.2.24), we get

$$\int_0^t Y_{s-}dX_s = \int_0^t Y_{s-}\{H_s ds + dZ_s\} = \int_0^t Y_{s-}H_s ds + \int_0^t Y_{s-}dZ_s. \tag{3.2.27}$$

It follows from the definition of X_t in (3.2.1) and Y_t in (3.2.2) that

$$\langle X, Y\rangle_t = \langle Z, M\rangle_t = 0. \tag{3.2.28}$$

Combining (3.2.26)-(3.2.28) and (3.2.24) yields

$$\begin{aligned} X_tY_t &= X_0Y_0 + \int_0^t X_{s-}U_s dM_s + \int_0^t X_{s-}U_s\{h_s - \pi(h_s)\}ds \\ &\quad + \int_0^t X_{s-}\pi(h_s)ds + \int_0^t Y_{s-}H_s ds + \int_0^t Y_{s-}dZ_s. \end{aligned} \tag{3.2.29}$$

Because the expectations of the second and sixth terms of the right hand side of (3.2.29) are equal to 0, then

$$\begin{aligned} E[X_tY_t] &= E[X_0Y_0] + E\bigg[\int_0^t X_{s-}U_s\{h_s - \pi(h_s)\}ds\bigg] \\ &\quad + E\bigg[\int_0^t X_{s-}\pi(h_s)ds\bigg] + E\bigg[\int_0^t Y_{s-}H_sds\bigg]. \end{aligned}$$

The rules for calculation of the conditional expectation show that

$$\begin{aligned} E[X_tY_t] &= E[X_0Y_0] + E\bigg[\int_0^t U_s\{\pi(X_{s-}h_s) - \pi(X_{s-})\pi(h_s)\}ds\bigg] \\ &\quad + E\bigg[\int_0^t \{X_{s-}\pi(h_s) + Y_{s-}H_s\}ds\bigg]. \end{aligned} \tag{3.2.30}$$

On the other hand, integration by parts gives

$$\pi(X_t)Y_t = \pi(X_0)Y_0 + \int_0^t \pi(X_{s-})dY_s + \int_0^t Y_{s-}d\pi(X_s) + \langle \pi(X), Y\rangle_t. \tag{3.2.31}$$

Substituting Y_t from (3.2.22) into the second term of (3.2.31), we get

$$\int_0^t \pi(X_{s-})dY_s = \int_0^t \pi(X_{s-})\{U_sdm_s + \pi(h_s)ds\}. \tag{3.2.32}$$

Next, substituting m_t from (3.2.21) into (3.2.32), we obtain

$$\begin{aligned} \int_0^t \pi(X_{s-})dY_s &= \int_0^t \pi(X_{s-})U_s\big\{dM_s + \{h_s - \pi(h_s)\}ds\big\} + \int_0^t \pi(X_{s-})\pi(h_s)ds \\ &= \int_0^t \pi(X_{s-})U_sdM_s + \int_0^t \pi(X_{s-})U_s\{h_s - \pi(h_s)\}ds \\ &\quad + \int_0^t \pi(X_{s-})\pi(h_s)ds. \end{aligned} \tag{3.2.33}$$

Substituting $\pi(X_t)$ from (3.2.19) into the third term of (3.2.31), we get

$$\int_0^t Y_{s-}d\pi(X_s) = \int_0^t Y_{s-}\{\pi(H_s)ds + K_sdm_s\}. \tag{3.2.34}$$

Next, substituting m_t from (3.2.21) into (3.2.34), we get

$$\begin{aligned} \int_0^t Y_{s-}d\pi(X_s) &= \int_0^t Y_{s-}\pi(H_s)ds + \int_0^t Y_{s-}K_s\big\{dM_s + \{h_s - \pi(h_s)\}ds\big\} \\ &= \int_0^t Y_{s-}\pi(H_s)ds + \int_0^t Y_{s-}K_sdM_s \\ &\quad + \int_0^t Y_{s-}K_s\{h_s - \pi(h_s)\}ds. \end{aligned} \tag{3.2.35}$$

By using the expressing of $\pi(X_t)$ from (3.2.19) and that of Y_t from (3.2.22), we have

$$\langle \pi(X), Y\rangle_t = \langle \int_0^t K_s dm_s, \int_0^t U_s dm_s\rangle_t = \int_0^t U_s K_s d\langle m, m\rangle_s = \int_0^t U_s K_s h_s ds. \tag{3.2.36}$$

Combining (3.2.33), (3.2.35), (3.2.36) and (3.2.31), we obtain

$$\begin{aligned} \pi(X_t)Y_t &= \pi(X_0)Y_0 + \int_0^t \pi(X_{s-})U_s dM_s + \int_0^t \pi(X_{s-})U_s\{h_s - \pi(h_s)\}ds \\ &\quad + \int_0^t \pi(X_{s-})\pi(h_s)ds + \int_0^t Y_{s-}K_s dM_s + \int_0^t Y_{s-}\pi(H_s)ds \\ &\quad + \int_0^t Y_{s-}K_s\{h_s - \pi(h_s)\}ds + \int_0^t U_s K_s h_s ds. \end{aligned} \tag{3.2.37}$$

The expectations of the second and fifth terms of the right hand side of (3.2.37) are equal to 0, so

$$\begin{aligned} E[\pi(X_t)Y_t] &= E[\pi(X_0)Y_0] + E\Big[\int_0^t \pi(X_{s-})U_s\{h_s - \pi(h_s)\}ds\Big] \\ &\quad + E\Big[\int_0^t \pi(X_{s-})\pi(h_s)ds\Big] + E\Big[\int_0^t Y_{s-}\pi(H_s)ds\Big] \\ &\quad + E\Big[\int_0^t Y_{s-}K_s\{h_s - \pi(h_s)\}ds\Big] + E\Big[\int_0^t U_s K_s h_s ds\Big]. \end{aligned}$$

The properties of conditional expectation reveal that

$$E[\pi(X_t)Y_t] = E[X_0Y_0] + E\Big[\int_0^t \{Y_{s-}H_s + X_{s-}\pi(h_s)\}ds\Big] + E\Big[\int_0^t U_s K_s \pi(h_s)ds\Big]. \tag{3.2.38}$$

It follows from (3.2.23), (3.2.30) and (3.2.38) that

$$E\Big[\int_0^t U_s\{K_s\pi(h_s) - \pi(X_{s-}h_s) + \pi(X_{s-})\pi(h_s)\}ds\Big] = 0.$$

For all $t \geq 0$ and all $\mathcal{F}_t^Y$-predictable processes U_t such that $\int_0^t U_s\pi(h_s)ds < \infty$, P-a.s., $\forall t \geq 0$, if C_t is any nonnegative bounded $\mathcal{F}_t^Y$-predictable process satisfying the same requirement as U_t, then

$$E\Big[\int_0^t C_s\{K_s\pi(h_s) - \pi(X_{s-}h_s) + \pi(X_{s-})\pi(h_s)\}ds\Big] = 0,$$

the latter equality being valid for all nonnegative bounded $\mathcal{F}_t^Y$-predictable processes C_t, that is

$$K_s = \frac{\pi(X_{s-}h_s) - \pi(X_{s-})\pi(h_s)}{\pi(h_s)} \quad \text{a.s.}$$

Substituting K_s into (3.2.18), we get

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s)ds + \int_0^t \{\pi(h_s)\}^{-1}\{\pi(X_{s-}h_s) - \pi(X_{s-})\pi(h_s)\}dm_s.$$

The proof of this theorem is thus complete.

3.2.4 Quasi-filtering

There is some inconvenience in application of (3.2.14) because the appearance of the factor $\{\pi(h_s)\}^{-1}$. To avoid this difficulty we introduce the unnormalized conditional filtering or quasi-filtering in other terms.

As we know in the method of reference probability, the probability P actually governing the statistics of the observation Y_t is obtained from a probability Q by an absolutely continuous change $P \to Q$. We assume that Q is the reference probability such that Y is a $(Q, \mathcal{F}_t)$- Poisson process of intensity 1, where $\mathcal{F}_t = \mathcal{F}_t^Y \vee \mathcal{F}_\infty^X$.

Denoting for every $t \geq 0$ by P_t and Q_t the restrictions of P and Q respectively to $(\Omega, \mathcal{F}_t)$ we have $P_t << Q_t$. It is known that the corresponding Radon-Nykodym derivative is the unique solution of a Doleans-Dade equation of the form:

$$L_t = 1 + \int_0^t L_{s-}(h_s - 1)d(Y_s - s), \tag{3.2.39}$$

where h_t and Y_t are given in (3.2.2).

The explicit solution of (3.2.39) is

$$L_t = \frac{dP_t}{dQ_t} = \prod_{0\leq s\leq t} h_s \Delta Y_s \exp\left\{\int_0^t (1 - h_s)ds\right\}. \tag{3.2.40}$$

Let Z_t be a real valued and bounded process adapted to $\mathcal{F}_t$, then for every history $\mathcal{G}_t$ such that $\mathcal{G}_t \subseteq \mathcal{F}_t,\ t \geq 0$ we have a Bayes formula

$$E_P[Z_t|\mathcal{G}_t] = \frac{E_Q[Z_t L_t|\mathcal{G}_t]}{E_Q[L_t|\mathcal{G}_t]}, \tag{3.2.41}$$

where $E_P(.|\mathcal{G}_t)$ and $E_Q(.|\mathcal{G}_t)$ are conditional expectations under probabilities P and Q respectively.

Definition 3.2.13. The process $\sigma(X_t)$ defined by

$$\sigma(X_t) = E_Q[L_t X_t|\mathcal{F}_t^Y] \tag{3.2.42}$$

is called the optimal quasi-filter (or quasi-filter) of X_t based on data $\mathcal{F}_t^Y$. It is in fact an unnormalized filter of X_t.

Then the filter of the process X_t can be written as

$$\pi(X_t) = \frac{\sigma(X_t)}{\sigma(1_t)}, \tag{3.2.43}$$

or in more general

$$\pi(f(X_t)) = \frac{\sigma(f(X_t))}{\sigma(1_t)}. \tag{3.2.44}$$

Theorem 3.2.14. *The assumptions are those prevailing in Theorem 3.2.12. Moreover, assume that Z_t and M_t have no common jumps. Then the quasi-filter $\sigma(X_t)$ satisfies the following equation*

$$\sigma(X_t) = \sigma(X_0) + \int_0^t \sigma(H_s)ds + \int_0^t \{\sigma(X_{s^-}h_s) - \sigma(X_{s^-})\}d\mu_s, \tag{3.2.45}$$

where

$$\mu_t = Y_t - t. \tag{3.2.46}$$

$$1_s = 1 \quad \textit{for every} \quad s \quad \textit{and} \quad \sigma(1_s) = E_Q(L_s|\mathcal{F}_s^Y).$$

Proof : It is known that $E_Q[L_t|\mathcal{F}_t^Y]$ satisfies the equation

$$E_Q[L_t|\mathcal{F}_t^Y] = 1 + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y](h_s - 1)d(Y_s - s). \qquad (3.2.47)$$

An application of the integration by parts formula gives

$$\begin{aligned} E_Q[L_t|\mathcal{F}_t^Y]\pi(X_t) &= E_Q[L_0|\mathcal{F}_0^Y]\pi(X_0) + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]d\pi(X_s) \\ &\quad + \int_0^t \pi(X_{s-})dE_Q[L_s|\mathcal{F}_s^Y] \\ &\quad + \langle E_Q[L|\mathcal{F}^Y], \pi(X)\rangle_t. \qquad (3.2.48) \end{aligned}$$

Next we shall compute the second term on the right hand side of (3.2.48). Substituting $\pi(X_t)$ by its expression from (3.2.14), we get

$$\begin{aligned} \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]d\pi(X_s) &= \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\big\{\pi(H_s)ds + K_s dm_s\big\} \\ &= \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]K_s\{dY_s - \pi(h_s)ds\} \\ &\quad + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\pi(H_s)ds, \qquad (3.2.49) \end{aligned}$$

where

$$K_t = \{\pi(h_t)\}^{-1}\{\pi(X_{t-}h_t) - \pi(X_{t-})\pi(h_t)\}. \qquad (3.2.50)$$

By (3.2.47), the third term on the right hand side of (3.2.48) becomes

$$\int_0^t \pi(X_{s-})dE_Q[L_s|\mathcal{F}_s^Y] = \int_0^t \pi(X_{s-})\Big\{E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\{\pi(h_s)-1\}d(Y_s-s)\Big\}. \quad (3.2.51)$$

It follows from the definition of $E_Q[L_t|\mathcal{F}_t^Y]$ in (3.2.47) and $\pi(X_t)$ in (3.2.14) that

$$\langle E_Q[L|\mathcal{F}^Y], \pi(X)\rangle_t = \int_0^t K_s E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\{\pi(h_s) - 1\}dY_s. \qquad (3.2.52)$$

Substituting (3.2.49), (3.2.51) and (3.2.52) into (3.2.48), we get

$$\begin{aligned} E_Q[L_t|\mathcal{F}_t^Y]\pi(X_t) &= E_Q[L_0|\mathcal{F}_0^Y]\pi(X_0) + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\pi(H_s)ds \\ &\quad + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]K_s\{dY_s - \pi(h_s)ds\} \\ &\quad + \int_0^t \pi(X_{s-})\Big\{E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\{\pi(h_s) - 1\}d(Y_s - s)\Big\} \\ &\quad + \int_0^t K_s E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\{\pi(h_s) - 1\}dY_s. \qquad (3.2.53) \end{aligned}$$

Combining the third and fifth terms on the right hand side of (3.2.53), we have

$$\begin{aligned} E_Q[L_t|\mathcal{F}_t^Y]\pi(X_t) &= E_Q[L_0|\mathcal{F}_0^Y]\pi(X_0) + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\pi(H_s)ds \\ &\quad + \int_0^t \pi(X_{s-})\Big\{E_Q[L_{s-}|\mathcal{F}_{s-}^Y](\pi(h_s)-1)d(Y_s-s)\Big\} \\ &\quad + \int_0^t K_s\pi(h_s)E_Q[L_{s-}|\mathcal{F}_{s-}^Y]d(Y_s-s). \end{aligned} \tag{3.2.54}$$

Substituting K_t from (3.2.50) into (3.2.54), we obtain

$$\begin{aligned} E_Q[L_t|\mathcal{F}_t^Y]\pi(X_t) &= E_Q[L_0|\mathcal{F}_0^Y]\pi(X_0) + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\pi(H_s)ds \\ &\quad + \int_0^t \pi(X_{s-})\Big\{E_Q[L_{s-}|\mathcal{F}_{s-}^Y](\pi(h_s)-1)d(Y_s-s)\Big\} \\ &\quad + \int_0^t \{\pi(X_{s-}h_s)-\pi(X_{s-})\pi(h_s)\}E_Q[L_{s-}|\mathcal{F}_{s-}^Y]d(Y_s-s). \end{aligned} \tag{3.2.55}$$

Combining the third and fourth terms on the right hand side of (3.2.55) gives

$$\begin{aligned} E_Q[L_t|\mathcal{F}_t^Y]\pi(X_t) &= E_Q[L_0|\mathcal{F}_0^Y]\pi(X_0) + \int_0^t E_Q[L_{s-}|\mathcal{F}_{s-}^Y]\pi(H_s)ds \\ &\quad + \int_0^t \{\pi(X_{s-}h_s)-\pi(X_{s-})\}E_Q[L_{s-}|\mathcal{F}_{s-}^Y]d(Y_s-s). \end{aligned} \tag{3.2.56}$$

We note from (3.2.42) and (3.2.44) that

$$\sigma(f(X_t)) = \pi(f(X_t))\sigma(1_t) = \pi(f(X_t))E_Q[L_t|\mathcal{F}_t^Y], \quad \forall f \in C_b(\mathbb{R}). \tag{3.2.57}$$

By choosing suitable functions $f \in C_b(\mathbb{R})$ and substituting (3.2.57) into (3.2.56), we get

$$\begin{aligned} \sigma(X_t) &= \sigma(X_0) + \int_0^t \sigma(H_s)ds + \int_0^t \{\sigma(X_{s^-}h_s)-\sigma(X_{s^-})\}d(Y_s-s) \\ &= \sigma(X_0) + \int_0^t \sigma(H_s)ds + \int_0^t \{\sigma(X_{s^-}h_s)-\sigma(X_{s^-})\}d\mu_s, \end{aligned}$$

where $\mu_t = Y_t - t$. The proof is now complete.

Theorem 3.2.15. *If $\sigma(X_t)$ satisfies (3.2.45), then the process $\pi(X_t)$ satisfies (3.2.14).*

Proof : We assume that Q is the probability such that Y_t is a $(Q, \mathcal{F}_t)$-Poisson process of intensity 1 (i.e. $h_s = 1$). Then

$$\pi(h_s) \quad = \quad \frac{\sigma(h_t)}{\sigma(1_t)} = \frac{E_Q[h_s L_s|\mathcal{F}_s^Y]}{E_Q[L_s|\mathcal{F}_s^Y]} = \frac{E_Q[L_s|\mathcal{F}_s^Y]}{E_Q[L_s|\mathcal{F}_s^Y]} = 1$$

and $\pi^{-1}(h_s) = 1$. Consider

$$\begin{aligned}
d\pi(X_t) &= d\left(\frac{\sigma(X_t)}{\sigma(1_t)}\right) \\
&= \frac{1}{\sigma(1_t)}d\sigma(X_t) - \frac{\sigma(X_t)}{\sigma^2(1_t)}d\sigma(1_t) + \frac{\sigma(X_t)}{\sigma^3(1_t)}(d\sigma(1_t))^2 - \frac{1}{\sigma^2(1_t)}d\sigma(X_t)d\sigma(1_t) \\
&= \frac{1}{\sigma(1_t)}[\sigma(H_t)dt + \sigma(X_{t-}h_t)dm_t] - \frac{\sigma(X_t)}{\sigma^2(1_t)}[(\sigma(h_t) - \sigma(1_t))dM_t] \\
&\quad + \frac{\sigma(X_t)}{\sigma^3(1_t)}[(\sigma(h)_t - \sigma(1_t))dM_t]^2 \\
&\quad - \frac{1}{\sigma^2(1_t)}[\sigma(H_t)dt + \sigma(X_{t-}h_t)dm_t][(\sigma(h_t) - \sigma(1_t))dM_t] \\
&= \frac{\sigma(H_t)}{\sigma(1_t)}dt + \frac{\sigma(X_{t-}h_t) - \sigma(X_{t-})}{\sigma(1_t)}dm_t \\
&= \frac{\sigma(H_t)}{\sigma(1_t)}dt + \frac{\sigma(1_t)}{\sigma(h_t)}\left[\frac{\sigma(X_{t-}h_t)}{\sigma(1_t)} - \frac{\sigma(X_{t-})}{\sigma(1_t)}\frac{\sigma(h_t)}{\sigma(1_t)}\right]dm_t \\
&= \pi(H_t)dt + \pi^{-1}(h_s)[\pi(X_{t-}h_t) - \pi(X_{t-})\pi(h_t)]dm_t.
\end{aligned}$$

Then we have finally,

$$\pi(X_t) \quad = \quad \pi(X_0) + \int_0^t \pi(H_s)ds + \int_0^t \{\pi(h_s)\}^{-1}\{\pi(X_{s-}h_s) - \pi(X_{s-})\pi(h_s)\}dm_s$$

This proves the theorem.

3.3 Filtering for a Fellerian System

3.3.1 Filtering for a Feller Process with Point Process Observation

Suppose that X_t is a Markov process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ taking values in $\mathbb{R}$ and that the semigroup $(P_t\ ,\ t \geq 0)$ associated with the transition probability $P_t(x, E)$ is a Feller semigroup, that is

$$P_t f(x) = \int_0^t P_t(x, dy) f(y), \tag{3.3.1}$$

maps $C(\mathbb{R})$ into itself for all $t \geq 0$ and satisfies the following relation

$$\lim_{t \downarrow 0} P_t f(x) = f(x), \tag{3.3.2}$$

uniformly in $\mathbb{R}$ for all $f \in C(\mathbb{R})$, where $C(\mathbb{R})$ is the space of all real continuous function over $\mathbb{R}$. Assume that the observation Y_t is a Poisson process of intensity $h_t = h(X_t) \in C(\mathbb{R})$.

As before the filter π_t is defined as:

$$\pi_t(f) := \pi(f(X_t)) = E_P[f(X_t)|\mathcal{F}_t^Y]. \tag{3.3.3}$$

Also we have

$$\sigma_t(f) := \sigma(f(X_t)) = E_Q[L_t f(X_t)|\mathcal{F}_t^Y], \tag{3.3.4}$$

where the probability Q and the likelihood ratio are defined as before.

Denote by m_t the innovation process of Y_t:

$$m_t := Y_t - \int_0^t \pi(h_s) ds = Y_t - \int_0^t \frac{\sigma(h_s)}{\sigma(1_s)} ds. \tag{3.3.5}$$

Theorem 3.3.16. *If $\mathcal{A}$ is the infinitesimal generator of the semigroup P_t of the signal process, then the optimal filter $\pi_t(f) = \pi(f(X_t))$ satisfies the two following* *equations:*

$$
\begin{aligned}
(a)\quad \pi_t(f) &= \pi_0(f) + \int_0^t \pi_s(\mathcal{A}f)ds \\
&\quad + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(fh) - \pi_{s^-}(f)\pi_s(h)\}dm_s \qquad (3.3.6)\\
(b)\quad \pi_t(f) &= \pi_0(P_tf) + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(hP_{t-s}f) \\
&\quad -\pi_{s^-}(P_{t-s}f)\pi_s(h)\}dm_s\ . \qquad (3.3.7)
\end{aligned}
$$

where $f \in C_b(\mathbb{R})$ *and* $\pi_{s-}(f) = \pi(f(X_{s-}))$.

Proof :

(a) First, we prove that $C_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds$ is a $\mathcal{F}_t$-martingale, where $(\mathcal{F}_t)$ is the filtration to which (X_t) is adapted. To do that, for any $t \geq s > 0$,

$$
\begin{aligned}
E[C_t^f - C_s^f|\mathcal{F}_s] &= E\left[f(X_t) - f(X_s) - \int_s^t \mathcal{A}f(X_u)du\middle|\mathcal{F}_s\right] \\
&= E[f(X_t)|\mathcal{F}_s] - E[f(X_s)|\mathcal{F}_s] - E\left[\int_s^t \mathcal{A}f(X_u)du)\middle|\mathcal{F}_s\right] \\
&= E[f(X_t)|\mathcal{F}_s] - f(X_s) - E\left[\int_s^t \mathcal{A}f(X_u)du)\middle|\mathcal{F}_s\right] \\
&= P_{t-s}f(X_s) - f(X_s) - E\left[\int_s^t \mathcal{A}f(X_u)du)\middle|\mathcal{F}_s\right] \\
&= P_{t-s}f(X_s) - f(X_s) - \int_s^t \mathcal{A}E[f(X_u)du|\mathcal{F}_s]du \\
&= P_{t-s}f(X_s) - f(X_s) - \int_s^t \mathcal{A}P_{u-s}f(X_s)du \\
&= P_{t-s}f(X_s) - f(X_s) - \int_0^{t-s} \mathcal{A}P_uf(X_s)du. \qquad (3.3.8)
\end{aligned}
$$

Recall the property of Markov processes that

$$
P_tf - f = \int_0^t P_s\mathcal{A}fds = \int_0^t \mathcal{A}P_sfds. \qquad (3.3.9)
$$

It follows from (3.3.8) and (3.3.9), that

$$
E[C_t^f - C_s^f|\mathcal{F}_s] = 0.
$$

This proves C_t^f is a $\mathcal{F}_t$-martingale. Next we consider the signal process

$$f(X_t) = f(X_0) + \int_0^t \mathcal{A}f(X_s)ds + C_t^f$$

with the observation process

$$Y_t = \int_0^t h_s ds + M_t.$$

By using Theorem 3.2.12, we obtain

$$\begin{aligned}\pi(f(X_t)) &= \pi(f(X_0)) + \int_0^t \pi(\mathcal{A}f(X_s))ds \\ &\quad \int_0^t \{\pi(h_s)\}^{-1}\{\pi(f(X_{s-})h_s) - \pi(f(X_{s-}))\pi(h_s)\}dm_s.\end{aligned}$$

(b) For $f \in C_b(\mathbb{R})$ we put

$$Q_s^t = \begin{cases} f(X_t), & \text{if } t < s; \ (1) \\ P_{t-s}f(X_s), & \text{if } t \geq s. \ (2) \end{cases}$$

First, we prove that $(Q_s^t)_s$ is a $\mathcal{F}_s$-martingale. We have to prove that $E[Q_s^t|\mathcal{F}_u] = Q_u^t$ for any $u < s$.

case 1: if $u \leq s \leq t$

$$\begin{aligned}E[Q_s^t|\mathcal{F}_u] &= E[P_{t-s}f(X_s)|\mathcal{F}_u] \\ &= E\big[E[f(X_t)|\mathcal{F}_s]\big|\mathcal{F}_u\big] \\ &= E[f(X_t)|\mathcal{F}_u] \\ &= P_{t-u}f(X_u) \ \text{ (by definition of the operator } P_t) \\ &= Q_u^t \ \text{ (by definition (2) of } Q_s^t)\end{aligned}$$

case 2: if $u \leq t \leq s$

$$\begin{aligned}E[Q_s^t|\mathcal{F}_u] &= E[f(X_t)|\mathcal{F}_u] \\ &= P_{t-u}f(X_u) \ \text{ (by definition of the operator } P_t) \\ &= Q_u^t \ \text{ (by definition (2) of } Q_s^t)\end{aligned}$$

case 3: if $t \le u \le s$

$$\begin{aligned} E[Q_s^t|\mathcal{F}_u] &= E[f(X_t)|\mathcal{F}_u] \\ &= f(X_t) \ \ \text{(because } f(X_t) \text{ is measurable w.r.t. } \mathcal{F}_u, u \ge t) \\ &= Q_u^t \ \ \text{(by definition (1) of } Q_s^t) \end{aligned}$$

Next we consider $X_s = Q_s^t$ as a signal process with the observation process

$$Y_s = \int_0^s h_u du + M_s.$$

By using Theorem 3.2.12, we obtain

$$\pi(X_t) = \pi(X_0) + \int_0^t \{\pi(h_s)\}^{-1}\{\pi(X_{s-}h_s) - \pi(X_{s-})\pi(h_s)\}dm_s.$$

It follows from the definition of Q_s^t in (1) and (2), we obtain

$$\begin{aligned} \pi(f(X_t)) &= \pi(P_t f(X_0)) + \int_0^t \{\pi(h_s)\}^{-1}\{\pi(P_{t-s}f(X_{s-})h_s) \\ &\quad -\pi(P_{t-s}f(X_{s-}))\pi(h_s)\}dm_s. \end{aligned}$$

Theorem 3.3.17. *The quasi-filter σ_t satisfies the two following equations:*

$$\begin{aligned} \text{(a)} \ \ \sigma_t(f) &= \sigma_0(f) + \int_0^t \sigma_s(\mathcal{A}f)ds + \int_0^t \{\sigma_{s-}(hf) - \sigma_{s-}(f)\}d\mu_s \\ \text{(b)} \ \ \sigma_t(f) &= \sigma_0(P_t f) + \int_0^t \{\sigma_{s-}(hP_{t-s}f) - \sigma_{s-}(P_{t-s}f)\}d\mu_s \end{aligned}$$

where $f \in C_b(\mathbb{R})$, $\sigma_{s-}(f) = \sigma(f(X_{s-}))$.and $\mu_t = Y_t - t$.

Proof :

(a) Recall that $C_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds$ is a $\mathcal{F}_t$-martingale. Consider the signal process

$$f(X_t) = f(X_0) + \int_0^t \mathcal{A}f(X_s)ds + C_t^f$$

with the observation process

$$Y_t = \int_0^t h_s ds + M_t.$$

By using Theorem 3.2.14, we obtain

$$\sigma(f(X_t)) = \sigma(f(X_0)) + \int_0^t \sigma(\mathcal{A}f(X_s))ds + \int_0^t \{\sigma(f(X_{s-})h_s) - \sigma(f(X_{s-}))\}d\mu_s$$

(b) Put

$$Q_s^t = \begin{cases} f(X_t), & \text{if } t < s; \\ P_{t-s}f(X_s), & \text{if } t \geq s. \end{cases}$$

We can see that Q_s^t is a $\mathcal{F}_s$-martingale. Next we consider $X_s = Q_s^t$ as a signal process with the observation process

$$Y_s = \int_0^s h_u du + M_s.$$

By using Theorem 3.2.14, we obtain

$$\sigma(f(X_t)) = \sigma(P_t f(X_0)) + \int_0^t \{\sigma(P_{t-s}f(X_{s-})h_s) - \sigma(P_{t-s}f(X_{s-}))\}d\mu_s.$$

3.4 Filtering for Ornstein-Uhlenbeck Process

Let X_t be stochastic process with initial value X_0 of standard normal distribution $X_0 \sim \mathcal{N}(0,1)$. X_t is called an Ornstein-Uhlenbeck process if it satisfies one of seven definitions below.

Definition 3.4.18. X_t is a solution of SDE

$$\begin{aligned} dX_t &= -\alpha X_t dt + \gamma dW_t, \qquad (3.4.1) \\ X_0 &\sim \mathcal{N}(0,1). \end{aligned}$$

Definition 3.4.19. X_t satisfies

$$\begin{aligned} X_t &= X_0 e^{-\alpha t} + \gamma \int_0^t e^{-\alpha(t-u)} dW_u, \\ X_0 &\sim \mathcal{N}(0,1). \end{aligned}$$

Definition 3.4.20. X_t is a Gaussian process with

(a) $EX_t = 0 \;\; \forall t$

(b) $R(s,t) = E(X_t X_s) = \dfrac{\gamma^2}{2\alpha} e^{-\alpha|t-s|}$.

Definition 3.4.21. X_t is a stationary Markov process with the density of the transition probability is

$$p_t(x,y) \;\; = \;\; \frac{1}{\sqrt{\gamma\pi(1-e^{-2\alpha t})}} \exp\left\{ -\frac{(y-xe^{-2\alpha t})^2}{\gamma(1-2e^{-2\alpha t})} \right\}.$$

In general

$$P(x,s;y,t) \;\; = \;\; \frac{1}{\sqrt{\gamma\pi(1-e^{-2\alpha(t-s)})}} \exp\left\{ -\frac{(y-xe^{-2\alpha(t-s)})^2}{\gamma(1-2e^{-2\alpha(t-s)})} \right\}.$$

Definition 3.4.22. X_t is a Feller process with semigroup $(P_t, t \geq 0)$ defined as

$$P_t f(x) \;\; = \;\; \int_{\mathbb{R}} f\left(e^{-\alpha t} x + \frac{\gamma^2}{2\alpha} \sqrt{1-e^{-2\alpha t} y} \right) \mu(dy) \tag{3.4.2}$$

where μ is Gaussian measure on $\mathbb{R}$

$$\mu(dx) \;\; = \;\; \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

and

$$\lim_{t \to 0} P_t f(x) \;\; = \;\; f(x).$$

Definition 3.4.23. X_t is a Feller process with $(P_t, t \geq 0)$ defined as

$$\begin{aligned} P_t f(x) \;\; &= \;\; E\Big[f\big(e^{-\alpha t} x + \frac{\gamma^2}{2\alpha} \sqrt{1-e^{-2\alpha t} Y} \big) \Big], \\ Y \;\; &\sim \;\; \mathcal{N}(0,1). \end{aligned}$$

Definition 3.4.24. X_t is expressed by

$$X_t \;\; = \;\; c_t X + s_t Y$$

where

$$\begin{aligned} c_t &= e^{-\alpha t} \\ s_t &= \frac{\gamma^2}{2\alpha}\sqrt{1-e^{-2\alpha t}} \end{aligned}$$

X, Y are two independent standard Gaussian random variables (i.e., $X, Y \sim \mathcal{N}(0,1)$).

3.4.1 Filtering for Ornstein-Uhlenbeck Process from Point Process Observation

We recall in this section some facts on Ornstein- Uhlenbeck processes and show how to use them to our filtering problems. This process is of importance in studies in finance. It has various ' good properties ' to describe many elements in financial models such as that of interest rate (Vasiček, Ho-Lee, Hull-White, etc.) or stochastic volatility of asset pricing.

We will apply results of the previous section to the following filtering problem:

• Signal process: An Ornstein-Uhlenbeck process X_t that is solution of the equation (3.4.1).

• Observation process: A point process N_t of intensity $\lambda_t > 0$.

So the signal and observation processes (X_t, N_t) can be expressed in the form

$$dX_t = -\alpha X_t dt + \gamma dW_t \ , X_0 \sim \mathcal{N}(0,1), \tag{3.4.3}$$

$$dN_t = \lambda_t dt + dM_t, \tag{3.4.4}$$

where $\alpha, \gamma > 0$, λ_t is a $\mathcal{F}_t$-adapted process, M_t is a point process martingale independent of W_t.

Denote by $\mathcal{F}_t^N$ the σ-algebra of observation that is generated by $(N_s, s \leq t)$.

The filter of (X_t) based on data given by $(\mathcal{F}_t^N)$ is denoted now by $\hat{X}_t$:

$$\hat{X}_t = \pi_t(X) = E(X_t|\mathcal{F}_t^N)$$

and also $\pi_t(f) = \hat{f}(X_t) = E(f(X_t)|\mathcal{F}_t^N)$, $f \in C_b(\mathbb{R})$.

The innovation process m_t is given by

$$m_t = N_t - \int_0^t \pi(\lambda_s)ds, \tag{3.4.5}$$

and $dm_t = dN_t - \pi(\lambda_t)dt$.

Since the semigroup $(P_t\ , t \geq 0)$ for X_t is defined by (3.4.2), the infinitesimal operator $\mathcal{A}_t$ is given by

$$\mathcal{A}_t f = \lim_{t\to 0} \frac{1}{t}(P_t f - f) = -\alpha x f'(x) + \frac{1}{2\alpha}\gamma^2 f''(x). \tag{3.4.6}$$

On the other hand, $P_t f$ can be expressed under the form:

$$(P_t f)(x) = E\Big[f\big(e^{-\alpha t}x + \frac{\gamma^2}{2\alpha}\sqrt{1-e^{-2\alpha t}}Y\big)\Big], \tag{3.4.7}$$

where Y is a standard Gaussian variable, $Y \sim \mathcal{N}(0,1)$.

Then from Theorem 3.3.16 and 3.3.17 we can get:

Theorem 3.4.25. *The filter $\pi_t(f)$ for the filtering problem (3.4.3)- (3.4.4) is given by one of two following equations:*

$$\begin{aligned}
\text{(a)}\ \ \pi_t(f) &= \pi_0(f) + \int_0^t \pi_s\Big(-\alpha X f'(X) + \frac{\gamma^2}{2\alpha} f''(X)\Big)ds \\
&\quad + \int_0^t \pi_s^{-1}(\lambda)\{\pi_{s^-}(\lambda f) - \pi_{s-}(f)\pi_s(\lambda)\}dm_s, \\
\text{(b)}\ \ \pi_t(f) &= \pi_0(P_t f) + \int_0^t \pi_s^{-1}(\lambda)\{\pi_{s^-}(\lambda P_{t-s} f) - \pi_{s-}(P_{t-s} f)\pi_s(\lambda)\}dm_s,
\end{aligned}$$

where $\pi_{s-}(f) = \pi(f(X_{s-}))$, $m_t = N_s - \int_0^t \pi_s(\lambda)ds$ and P_t is given by (3.4.7).

Theorem 3.4.26. *The quasi-filter $\sigma_t(f)$ for the filtering problem (3.4.3)- (3.4.4) is given by one of two following equations:*

$$
\begin{aligned}
(a)\quad \sigma_t(f) &= \sigma_0(f) + \int_0^t \sigma_s\Big(-\alpha X f'(X) + \frac{\gamma^2}{2\alpha} f''(X)\Big) ds \\
&\quad + \int_0^t \{\sigma_{s-}(\lambda f) - \sigma_{s-}(f)\} d\mu_s, \\
(b)\quad \sigma_t(f) &= \sigma_0(P_t f) + \int_0^t \{\sigma_{s-}(\lambda P_{t-s} f) - \sigma_{s-}(P_{t-s} f)\} d\mu_s,
\end{aligned}
$$

where $\mu_t = N_t - t$, $\sigma_{s-}(f) = \sigma(f(X_{s-}))$, $f \in C_b(\mathbb{R})$ and P_t is given by (3.4.7).

CHAPTER IV

FRACTIONAL FILTERING THEORY

In this chapter, we consider a fractional filtering problem from an approximation approach. We prove that the limit of the approximate filters is the solution of the original fractional filtering problem. A general problem, where both signal and observation are fractional, is investigated as well.

4.1 Introduction to Fractional Brownian Motion

It is known that fractional Brownian motion (fBm) was introduced first by Mandelbrot and Van Nees (1968). This is a centered Gaussian process $B^H = \{B_t^H,\ t \geq 0\}$ with covariance

$$E(B_s^H B_t^H) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big), \tag{4.1.1}$$

where H is called the *Hurst parameter*, $0 < H < 1$.

In the case where $H = \frac{1}{2}$,

$$E(B_s^{1/2} B_t^{1/2}) = \frac{1}{2}(s + t - |t-s|), \tag{4.1.2}$$

we have an ordinary standard Brownian motion. This is in general neither a martingale nor a Markov process. In contrary, it exhibits a long-range dependence. Some approaches to fractional stochastic calculus have been introduced by Coutin and Decreusefond (2000), Dai and Heyde (1996), Decreusefond and Üstünel (1999).

Stochastic filtering problems in fractional stochastics were studied by various authors. The chief obstacle in the study of these problems is the fact that the signal process or the observation process is driven not by a martingale and powerful tools of martingale theory can not be applied as in traditional stochastic filtering theory. Some attempts have been made by Decreusefond and Üstünel (1999) to overcome this difficulty by invoking the Malliavin Calculus

We know that, the fBm $B^H = (B_t^H,\ t \geq 0)$ has the following representation

$$B_t^H = \frac{1}{\Gamma(1-\alpha)}\left\{Z_t + \int_0^t (t-s)^\alpha dW_s\right\}, \tag{4.1.3}$$

where $\{W_s,\ s \in \mathbb{R}\}$ is a standard Brownian motion, $\alpha = H - \frac{1}{2} \in \big(-\frac{1}{2}, \frac{1}{2}\big)$. Since the process $Z_t = \int_{-\infty}^0 \big[(t-s)^\alpha - (-s)^\alpha\big] dW_s$ has absolutely continuous trajectories, it suffices to consider the term

$$B_t = \int_0^t (t-s)^\alpha dW_s. \tag{4.1.4}$$

In fact, B_t is a fractional Brownian motion of the Liouville form.

4.2 Convergence of a Semimartingales B_t^ε

Let B_t^H be fractional Brownian motion and W_t be the corresponding Brownian motion in its representation (4.1.3). Suppose that $0 < \alpha < \frac{1}{2}$, where $\alpha = H - \frac{1}{2}$. Define

$$B_t = \int_0^t (t-s)^\alpha dW_s \tag{4.2.1}$$

and

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s \tag{4.2.2}$$

for every $\varepsilon > 0$. The Ito stochastic differential of B_t^ε is then

$$dB_t^\varepsilon = \Big(\int_0^t \alpha(t-s+\varepsilon)^{\alpha-1} dW_s\Big) dt + \varepsilon^\alpha dW_t. \tag{4.2.3}$$

Indeed by applying the stochastic theorem of Fubini, we have

$$\begin{aligned}\int_0^t\int_0^s (s-u+\varepsilon)^{\alpha-1}dW_u ds &= \int_0^t \Big[\int_u^t (s-u+\varepsilon)^{\alpha-1}ds\Big]dW_u \\ &= \frac{1}{\alpha}\int_0^t \Big[(t-u+\varepsilon)^{\alpha}-\varepsilon^{\alpha}\Big]dW_u \\ &= \frac{1}{\alpha}\Big[\int_0^t (t-u+\varepsilon)^{\alpha}dW_u - \varepsilon^{\alpha}W_t\Big] \quad (4.2.4)\end{aligned}$$

Substituting (4.2.2) into (4.2.4) then

$$\int_0^t\int_0^s (s-u+\varepsilon)^{\alpha-1}dW_u ds = \frac{1}{\alpha}(B_t^{\varepsilon}-\varepsilon^{\alpha}W_t). \quad (4.2.5)$$

We get B_t^{ε} by rearranging (4.2.5)

$$B_t^{\varepsilon} = \alpha\int_0^t\int_0^s (s-u+\varepsilon)^{\alpha-1}dW_u ds + \varepsilon^{\alpha}W_t. \quad (4.2.6)$$

Define

$$\varphi_t^{\varepsilon} = \int_0^t (t-u+\varepsilon)^{\alpha-1}dW_u. \quad (4.2.7)$$

It follows from definition of φ_t^{ε} in (4.2.7). Hence B_t^{ε} in (4.2.6) can be written as

$$B_t^{\varepsilon} = \int_0^t \alpha\varphi_s^{\varepsilon}ds + \varepsilon^{\alpha}W_t \quad (4.2.8)$$

or equivalently,

$$dB_t^{\varepsilon} = \alpha\varphi_t^{\varepsilon}dt + \varepsilon^{\alpha}dW_t. \quad (4.2.9)$$

So B_t^{ε} is a semimartingale.

We recall here a fundamental result given in Thao (2006).

Theorem 4.2.1. *B_t^{ε} converges to B_t in $L^2(\Omega,\mathcal{F},P)$ when $\varepsilon\to 0$. This convergence is uniform with respect to $t\in[0,T]$.*

Proof : See Thao (2003) and Sealim (2004).

Remarks

1. The σ-field generated by the random variables $\{B_s;\ 0 \leq s \leq t\}$ can be denoted by

$$\mathcal{F}_t^B = \sigma(B_s;\ 0 \leq s \leq t). \tag{4.2.10}$$

   In a similar way, the σ-field generated by the random variables $\{W_s;\ 0 \leq s \leq t\}$ can be denoted by

$$\mathcal{F}_t^W = \sigma(W_s;\ 0 \leq s \leq t) \tag{4.2.11}$$

   where W_t is the Brownian motion corresponding to fractional Brownian motion B_t.

2. Denote the σ-field generated by the random variables $\{B_{s+\varepsilon};\ s \leq t\}$ as

$$\mathcal{F}_t^{B_{\cdot+\varepsilon}} = \sigma(B_{s+\varepsilon};\ s \leq t). \tag{4.2.12}$$

   We see that

$$\begin{aligned} \mathcal{F}_t^{B_{\cdot+\varepsilon}} &= \sigma(B_{s+\varepsilon};\ s \leq t) \\ &= \sigma(B_{s+\varepsilon};\ s+\varepsilon \leq t+\varepsilon) \\ &= \sigma(B_u;\ u \leq t+\varepsilon) \\ &= \mathcal{F}_{t+\varepsilon}^B. \end{aligned} \tag{4.2.13}$$

3. We consider

$$\begin{aligned} \mathcal{F}_t^B &= \sigma(B_s;\ 0 < s \leq t) \\ &\subset \sigma(B_s;\ 0 < s \leq t+\varepsilon) \\ &= \mathcal{F}_{t+\varepsilon}^B \end{aligned} \tag{4.2.14}$$

and

$$\begin{aligned}\mathcal{F}_t^{B_{\cdot+\varepsilon}} &= \sigma(B_{s+\varepsilon};\ s \le t) \\ &= \sigma(W_s;\ 0 \le s \le t+\varepsilon) \\ &= \mathcal{F}_{t+\varepsilon}^W. \end{aligned} \tag{4.2.15}$$

Hence

$$\mathcal{F}_t^B \subset \mathcal{F}_{t+\varepsilon}^B = \mathcal{F}_t^{B_{\cdot+\varepsilon}} = \mathcal{F}_{t+\varepsilon}^W \tag{4.2.16}$$

4.3 Fractional Filtering for a General Signal Process

In this section, we consider a filtering problem where the signal process is a general stochastic process and the observation process is a fractional process.

Signal process:

$$X_t, \quad 0 \le t \le T, \tag{4.3.1}$$

where $E|X_t| < \infty$, $\forall t \in [0, T]$.

Observation process:

$$Y_t = \int_0^t h_s ds + B_t, \quad 0 \le t \le T, \tag{4.3.2}$$

where $h_t = h(X_t)$ is a continuous process with $E\int_0^t h_s^2 ds < \infty$ and B_t is the fractional process given by

$$B_t = \int_0^t (t-s)^\alpha dW_s. \tag{4.3.3}$$

For any $\varepsilon > 0$, we establish a new filtering problem (or an approximate filtering problem).

Signal process:

$$X_t, \quad 0 \le t \le T, \tag{4.3.4}$$

where $E|X_t| < \infty$, $\forall t \in [0, T]$.

Observation process:

$$Y_t^\varepsilon = \int_0^t h_s ds + B_t^\varepsilon, \quad 0 \le t \le T, \tag{4.3.5}$$

where $h_t = h(X_t)$ is a continuous process with $E\int_0^t h_s^2 ds < \infty$ and B_t^ε is given by

$$B_t^\varepsilon = \int_0^t (t - s + \varepsilon)^\alpha dW_s. \tag{4.3.6}$$

Define the filter of the process $(X_t, 0 \le t \le T)$ based on observation process $(Y_t, 0 \le t \le T)$ as the following conditional expectation

$$\pi(X_t) := E[X_t | \mathcal{F}_t^Y], \tag{4.3.7}$$

or more general

$$\pi_t(f) := E[f(X_t) | \mathcal{F}_t^Y], \tag{4.3.8}$$

where f is any continuous and bounded function on $\mathbb{R}$ (or $f \in C_b(\mathbb{R})$) and $\mathcal{F}_t^Y$ is a σ-algebra generated by $(Y_s, s \le t)$.

Also the filter of the process $(X_t, 0 \le t \le T)$ based on observation $(Y_t^\varepsilon, 0 \le t \le T)$ is

$$\pi^\varepsilon(X_t) := E[X_t | \mathcal{F}_t^{Y^\varepsilon}], \tag{4.3.9}$$

or in more general form

$$\pi_t^\varepsilon(f) := E[f(X_t) | \mathcal{F}_t^{Y^\varepsilon}], \tag{4.3.10}$$

where $f \in C_b(\mathbb{R})$ and $\mathcal{F}_t^{Y^\varepsilon}$ is the σ-algebra generated by $(Y_s^\varepsilon, s \le t)$.

Theorem 4.3.2. *The filter $\pi_t^\varepsilon(f)$ converges to $\pi_t(f)$ in $L^2(\Omega, \mathcal{F}, P)$ as $\varepsilon \to 0$.*

Proof : Consider the process Y_t^ε from (4.3.5). It follows from (4.3.2) and (4.5.7) that

$$
\begin{aligned}
\| Y_t - Y_t^\varepsilon \| &= \Big(E|Y_t - Y_t^\varepsilon|^2\Big)^{1/2} \\
&= \Big(E|(\int_0^t h_s ds + B_t) - (\int_0^t h_s ds + B_t^\varepsilon)|^2\Big)^{1/2} \\
&= \Big(E|B_t - B_t^\varepsilon|^2\Big)^{1/2} \\
&= \| B_t - B_t^\varepsilon \|
\end{aligned}
$$

Theorem 4.2.1 shows that $B_t^\varepsilon \to B_t$ in $L^2(\Omega, \mathcal{F}, P)$ as $\varepsilon \to 0$, then $Y_t^\varepsilon \to Y_t$ in $L^2(\Omega, \mathcal{F}, P)$ as $\varepsilon \to 0$. If we take $\varepsilon = \dfrac{1}{n}$, then $Y_t^{1/n} \to Y_t$ in $L^2(\Omega, \mathcal{F}, P)$ as $n \to \infty$.

On the other hand, we have

$$
\mathcal{F}_t^{Y^{1/n}} \subset \mathcal{F}_{t+\frac{1}{n}}^Y
$$

We have a non-increasing collection of σ-algebras $(\mathcal{F}_{t+\frac{1}{n}}^Y)$ such that $\cap_n \mathcal{F}_{t+1/n}^Y = \mathcal{F}_t^Y$ (i.e. $\mathcal{F}_t^{Y^{1/n}} \to \mathcal{F}_t^Y$ as $n \to \infty$). And by assumption $E|X_t| < \infty$, it follows from the Levy Theorem that

$$
E[f(X_t)|\mathcal{F}_t^{Y^{1/n}}] \to E[f(X_t)|\mathcal{F}_t^Y] \quad \text{as} \quad n \to \infty. \tag{4.3.11}
$$

It follows from definition of $\pi_t^{1/n}(f)$, $\pi_t(f)$ and (4.3.11), we obtain

$$
\pi_t^{1/n}(f) \to \pi_t(f) \quad \text{as} \quad n \to \infty \tag{4.3.12}
$$

Because we take $\varepsilon = \frac{1}{n}$, then

$$
\pi_t^\varepsilon(f) \to \pi_t(f) \quad \text{as} \quad \varepsilon \to 0 \tag{4.3.13}
$$

and the convergence holds in $L^2(\Omega, \mathcal{F}, P)$ and almost surely as $\varepsilon \to 0$.

4.4 Fractional Filtering for a Semimartingale Signal Process

In this section, we consider a filtering problem where the signal process is a semimartingale process and the observation process is a fractional process.

Signal process:

$$X_t = X_0 + \int_0^t H_s ds + V_t, \quad 0 \leq t \leq T, \tag{4.4.1}$$

where V_t is a Brownian motion and H_t is a stochastic process such that $E \int_0^t H_s^2 ds < \infty$.

Observation process:

$$Y_t = \int_0^t h_s ds + B_t, \quad 0 \leq t \leq T, \tag{4.4.2}$$

where $h_t = h(X_t)$ is a process with $E \int_0^t h_s^2 ds < \infty$ and B_t is a fractional Brownian motion defined by

$$B_t = \int_0^t (t-s)^{\alpha} dW_s, \tag{4.4.3}$$

where Brownian motion process W_t in this expression is independent of V_t.

As in the last section, we can consider the new problem (an approximate filtering problem).

Signal process:

$$X_t = X_0 + \int_0^t H_s ds + V_t, \quad 0 \leq t \leq T, \tag{4.4.4}$$

where V_t is a Brownian motion and H_t is a stochastic process such that $E \int_0^t H_s^2 ds < \infty$.

Observation process:

$$Y_t^{\varepsilon} = \int_0^t h_s ds + B_t^{\varepsilon}, \quad 0 \leq t \leq T, \tag{4.4.5}$$

where $h_t = h(X_t)$ with $E\int_0^t h_s^2 ds < \infty$ and B_t^ε is given by

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s. \tag{4.4.6}$$

As before, we define the filter for an exact filtering problem as

$$\pi_t(f) := E[f(X_t)|\mathcal{F}_t^Y], \tag{4.4.7}$$

where $\mathcal{F}_t^Y = \sigma(Y_s, s \le t)$ and $f \in C_b(\mathbb{R})$. And also, define the filter for an approximate filtering problem as

$$\pi_t^\varepsilon(f) := E[f(X_t)|\mathcal{F}_t^{Y^\varepsilon}], \tag{4.4.8}$$

where $\mathcal{F}_t^{Y^\varepsilon} = \sigma(Y_s^\varepsilon, s \le t)$ and $f \in C_b(\mathbb{R})$. And define the innovation process:

$$\nu_t^\varepsilon = \frac{1}{\varepsilon^\alpha}[Y_t^\varepsilon - \int_0^t \pi_s^\varepsilon(\bar{h})ds], \tag{4.4.9}$$

then ν_t^ε is a $\mathcal{F}_t^{Y^\varepsilon}$- martingale.

Theorem 4.4.3. *The filter $\pi_t(f) = E[f(X_t)|\mathcal{F}_t^Y]$ is written by*

$$\pi_t(f) = L^2 - \lim_{\varepsilon \to 0} \pi_t^\varepsilon(f), \tag{4.4.10}$$

where $\pi_t^\varepsilon(f)$ satisfies the equation

$$\pi_t^\varepsilon(f) = \pi_0^\varepsilon(f) + \int_0^t \pi_s^\varepsilon(\bar{H})ds + \int_0^t [\pi_s^\varepsilon(f(X)\bar{h}) - \pi_s^\varepsilon(f(X))\pi_s^\varepsilon(\bar{h})]\varepsilon^{-\alpha} d\nu_s^\varepsilon, \tag{4.4.11}$$

where

$$\bar{H}_t = f'(X_t)H_t + \frac{1}{2}f''(X_t) \tag{4.4.12}$$

$$\bar{h}_t = h_t + \alpha\varphi_t^\varepsilon, \quad \varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1}dW_t \tag{4.4.13}$$

$$\nu_t^\varepsilon = \frac{1}{\varepsilon^\alpha}[Y_t^\varepsilon - \int_0^t \pi_s^\varepsilon(\bar{h})ds] \tag{4.4.14}$$

Proof : It follows from (4.4.5) and (4.2.8)

$$\begin{aligned} Y_t^\varepsilon &= \int_0^t h_s ds + B_t^\varepsilon \\ &= \int_0^t h_s ds + \int_0^t \alpha \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t \\ &= \int_0^t \bar{h}_s ds + \varepsilon^\alpha W_t, \end{aligned} \tag{4.4.15}$$

where $\bar{h}_s = h_s + \alpha\varphi_s^\varepsilon$. So Y_t^ε is a $\mathcal{F}_t^W$- semimartingale.

Consider

$$\begin{aligned} \bar{h}_s^2 &= (h_s + \alpha\varphi_s^\varepsilon)^2 \\ &\leq 2(h_s^2 + \alpha^2(\varphi_s^\varepsilon)^2), \end{aligned} \tag{4.4.16}$$

then

$$\begin{aligned} E(\bar{h}_s^2) &\leq E[2(h_s^2 + \alpha^2(\varphi_s^\varepsilon)^2)] \\ &= 2E[h_s^2] + 2\alpha^2 E[(\varphi_s^\varepsilon)^2], \end{aligned} \tag{4.4.17}$$

i.e.

$$\int_0^t E(\bar{h}_s^2) ds \leq 2\int_0^t E(h_s^2) ds + 2\alpha^2 \int_0^t E[(\varphi_s^\varepsilon)^2] ds. \tag{4.4.18}$$

By definition of φ_s^ε from (4.2.7) and Itô Isometry property, we get

$$\begin{aligned} E[(\varphi_s^\varepsilon)^2] &= E\big[(\int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u)^2\big] \\ &= \int_0^s E[(s-u+\varepsilon)^{2(\alpha-1)}] du \\ &= \int_0^s (s-u+\varepsilon)^{2(\alpha-1)} du \\ &\leq \int_0^s \varepsilon^{2(\alpha-1)} du \\ &= s\varepsilon^{2(\alpha-1)} < \infty. \end{aligned} \tag{4.4.19}$$

It follows by Fubini's Theorem and (4.4.18) that

$$\begin{aligned} E\Big(\int_0^t \bar{h}_s^2 ds\Big) &= \int_0^t E(\bar{h}_s^2) ds \\ &\le 2\int_0^t E(h_s^2) ds + 2\alpha^2 \int_0^t E[(\varphi_s^\varepsilon)^2] ds \\ &= 2E\Big(\int_0^t h_s^2 ds\Big) + 2\alpha^2 \int_0^t E[(\varphi_s^\varepsilon)^2] ds. \end{aligned} \tag{4.4.20}$$

Then from assumption of h_t, (4.4.19) and (4.4.20), we can see that

$$E\int_0^t \bar{h}_s^2 ds < \infty. \tag{4.4.21}$$

We can write down the FKK (Fujisaki - Kallianpur - Kunita) equation for the filtering problem (4.4.4) and (4.4.5) by using general filtering problem:

$$\pi_t^\varepsilon(f) = \pi_0^\varepsilon(f) + \int_0^t \pi_s^\varepsilon(\bar{H}) ds + \int_0^t [\pi_s^\varepsilon(f(X)\bar{h}) - \pi_s^\varepsilon(f(X))\pi_s^\varepsilon(\bar{h})]\varepsilon^{-\alpha} d\nu_s^\varepsilon, \tag{4.4.22}$$

where $\bar{H}_t = f'(X_t)H_t + \frac{1}{2}f''(X_t)$, $f \in C_b(\mathbb{R})$ and $\pi_0^\varepsilon(f) = E[f(X_0)|\mathcal{F}_0^{Y^\varepsilon}]$. Notice that from (4.4.1), we have

$$\begin{aligned} E|X_t| &= E|X_0 + \int_0^t H_s ds + V_t| \\ &\le E\Big(|X_0| + |\int_0^t H_s ds| + |V_t|\Big) \\ &= E|X_0| + E|\int_0^t H_s ds| + E|V_t| \\ &\le E|X_0| + E\Big(\int_0^t |H_s| ds\Big) + E|V_t|. \end{aligned}$$

It follows from Cauchy-Schwarz inequality that

$$E|X_t| \le E|X_0| + T^{1/2}[E\int_0^t H_s^2 ds]^{1/2} + E|V_t|. \tag{4.4.23}$$

Notice that $EV_t = 0$ and $V_t = V_t^+ - V_t^-$ imply $EV_t^+ < \infty$ and $EV_t^- < \infty$. So

$$E|V_t| = E[V_t^+ + V_t^-] = EV_t^+ + EV_t^- < \infty. \tag{4.4.24}$$

By (4.4.23) and (4.4.24), then

$$E|X_t| \leq E|X_0| + T^{1/2}[E\int_0^t H_s^2 ds]^{1/2} + E|V_t| < \infty,$$

by the Levy Theorem we can see that $L^2 - \lim_{\varepsilon\to 0} \pi_t^\varepsilon(f)$ exists and by Theorem 4.3.2, then $\pi_t(f) = L^2 - \lim_{\varepsilon\to 0} \pi_t^\varepsilon(f)$.

4.5 General Fractional Filtering

In this section, we consider a filtering problem where the signal process and the observation process are fractional processes.

Signal process:

$$X_t = X_0 + \int_0^t H_s ds + B_t^{(1)}, \quad 0 \leq t \leq T, \tag{4.5.1}$$

where $E|X_t| < \infty$, H_t is $\mathcal{F}_t$-adapted process with $E\int_0^t H_s^2 ds < \infty$ and

$$B_t^{(1)} = \int_0^t (t-s)^\beta dU_s. \tag{4.5.2}$$

Observation process:

$$Y_t = \int_0^t h_s ds + B_t^{(2)}, \quad 0 \leq t \leq T, \tag{4.5.3}$$

where $h_t = h(X_t)$ is $\mathcal{F}_t$-adapted continuous process with $E\int_0^t h_s^2 ds < \infty$ and

$$B_t^{(2)} = \int_0^t (t-s)^\alpha dW_s, \tag{4.5.4}$$

where U_t and W_t are two independent standard Brownian motions. As before, we consider a new filtering problem (an approximate filtering problem).

Signal process:

$$X_t^\varepsilon = X_0 + \int_0^t H_s ds + B_t^{(1)\varepsilon}, \quad 0 \leq t \leq T, \tag{4.5.5}$$

where H_t satisfies $E\int_0^t H_s^2 ds < \infty$ and for every $\varepsilon > 0$,

$$B_t^{(1)\varepsilon} = \int_0^t (t-s+\varepsilon)^{\beta} dU_s. \tag{4.5.6}$$

Observation process:

$$Y_t^{\varepsilon} = \int_0^t h_s ds + B_t^{(2)\varepsilon}, \quad 0 \le t \le T, \tag{4.5.7}$$

where $h_t = h(X_t^{\varepsilon})$ and for every $\varepsilon > 0$,

$$B_t^{(2)\varepsilon} = \int_0^t (t-s+\varepsilon)^{\alpha} dW_s. \tag{4.5.8}$$

The filter for an exact problem is defined as

$$\pi_t(f) := E[f(X_t)|\mathcal{F}_t^Y], \tag{4.5.9}$$

where $\mathcal{F}_t^Y = \sigma(Y_s, s \le t)$ and $f \in C_b(\mathbb{R})$. And the filter for an approximate problem is defined as

$$\pi_t^{\varepsilon}(f) := E[f(X_t^{\varepsilon})|\mathcal{F}_t^{Y_t^{\varepsilon}}], \tag{4.5.10}$$

where $f \in C_b(\mathbb{R})$ and $\mathcal{F}_t^{Y^{\varepsilon}} = \sigma(Y_s^{\varepsilon}, s \le t)$.

Lemma 4.5.4. *Let X_n be a sequence of random variables converging to X and $|X_n| \le Y$ for all n, where Y is integrable. If $(\mathcal{F}_n)$ is an increasing (resp. decreasing) sequence of σ-algebras, then $E[X_n|\mathcal{F}_n]$ converges a.s to $E[X|\mathcal{F}]$ where $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$ (resp. $\mathcal{F} = \cap_n \mathcal{F}_n$).*

Proof : Take $\varepsilon > 0$ and put

$$A_m = \inf_{k \ge m} X_k, \quad B_m = \sup_{k \ge m} X_k, \tag{4.5.11}$$

where m is chosen such that

$$E[B_m - A_m] < \varepsilon \quad . \tag{4.5.12}$$

For any $n \geq m$ we have

$$\begin{aligned} E[A_m|\mathcal{F}_n] &= E[\inf_{k\geq m} X_k|\mathcal{F}_n] \\ &\leq E[X_n|\mathcal{F}_n] \\ &\leq E[\sup_{k\geq m} X_k|\mathcal{F}_n] \\ &= E[B_m|\mathcal{F}_n]. \end{aligned} \tag{4.5.13}$$

By Levy's Theorem, we get that $E[A_m|\mathcal{F}_n] \to E[A_m|\mathcal{F}]$ a.s. and $E[B_m|\mathcal{F}_n] \to E[B_m|\mathcal{F}]$ a.s.. Notice that, for any $n \geq m$, $A_m \leq X_n \leq B_m$ implies

$$\begin{aligned} E[A_m|\mathcal{F}] &= \lim_{n\to\infty} E[A_m|\mathcal{F}_n] \\ &= \liminf_{n\to\infty} E[A_m|\mathcal{F}_n] \\ &\leq \liminf_{n\to\infty} E[X_n|\mathcal{F}_n] \end{aligned} \tag{4.5.14}$$

$$\begin{aligned} E[B_m|\mathcal{F}] &= \lim_{n\to\infty} E[B_m|\mathcal{F}_n] \\ &= \limsup_{n\to\infty} E[B_m|\mathcal{F}_n] \\ &\geq \limsup_{n\to\infty} E[X_n|\mathcal{F}_n] \end{aligned} \tag{4.5.15}$$

By using (4.5.14)-(4.5.15), we obtain

$$E[A_m|\mathcal{F}] \leq \liminf_{n\to\infty} E[X_n|\mathcal{F}_n] \leq \limsup_{n\to\infty} E[X_n|\mathcal{F}_n] \leq E[B_m|\mathcal{F}]. \tag{4.5.16}$$

It follows from (4.5.12) that

$$\begin{aligned} E\big[E[B_m|\mathcal{F}] - E[A_m|\mathcal{F}]\big] &= E\big[E[B_m|\mathcal{F}]\big] - E\big[E[A_m|\mathcal{F}]\big] \\ &= E[B_m] - E[A_m] \\ &= E[B_m - A_m] < \varepsilon. \end{aligned} \tag{4.5.17}$$

Using (4.5.16) and (4.5.17), we get

$$E\big[\limsup_{n\to\infty} E[X_n|\mathcal{F}_n] - \liminf_{n\to\infty} E[X_n|\mathcal{F}_n]\big] \leq \varepsilon. \tag{4.5.18}$$

This shows that $\lim_{n\to\infty} E[X_n|\mathcal{F}_n]$ exist. So from the existent of this limit and (4.5.16), then

$$E[A_m|\mathcal{F}] \le \lim_{n\to\infty} E[X_n|\mathcal{F}_n] \le E[B_m|\mathcal{F}]. \tag{4.5.19}$$

It follows from (4.5.19) that

$$\lim_{m\to\infty} E[A_m|\mathcal{F}] \le \lim_{n\to\infty} E[X_n|\mathcal{F}_n] \le \lim_{m\to\infty} E[B_m|\mathcal{F}]. \tag{4.5.20}$$

Note that $\lim_{m\to\infty} A_m = \lim_{m\to\infty} B_m$ implies that

$$E[\lim_{m\to\infty} A_m|\mathcal{F}] = E[\lim_{m\to\infty} B_m|\mathcal{F}] \tag{4.5.21}$$

Using Fubini's theorem, we have

$$\lim_{m\to\infty} E[A_m|\mathcal{F}] = \lim_{m\to\infty} E[B_m|\mathcal{F}] \tag{4.5.22}$$

It follows from (4.5.20) and (4.5.22) that

$$\lim_{m\to\infty} E[A_m|\mathcal{F}] = \lim_{n\to\infty} E[X_n|\mathcal{F}_n] = \lim_{m\to\infty} E[B_m|\mathcal{F}]. \tag{4.5.23}$$

On the other hand, the inequality $A_m \le X \le B_m$ implies

$$E[A_m|\mathcal{F}] \le E[X|\mathcal{F}] \le E[B_m|\mathcal{F}] \tag{4.5.24}$$

And then

$$\lim_{m\to\infty} E[A_m|\mathcal{F}] \le E[X|\mathcal{F}] \le \lim_{m\to\infty} E[B_m|\mathcal{F}] \tag{4.5.25}$$

It follows from (4.5.22) and (4.5.25) that

$$\lim_{m\to\infty} E[A_m|\mathcal{F}] = E[X|\mathcal{F}] = \lim_{m\to\infty} E[B_m|\mathcal{F}]. \tag{4.5.26}$$

By (4.5.23) and (4.5.26), we have $\lim\limits_{n\to\infty} E[X_n|\mathcal{F}_n] = E[X|\mathcal{F}]$ or $E[X_n|\mathcal{F}_n] \to E[X|\mathcal{F}]$ a.s.

This Lemma still holds if we replace the a.s. convergence by the L^2- convergence.

Theorem 4.5.5. *The filter* $\pi_t(f) = E[f(X_t)|\mathcal{F}_t^Y]$ *is determined by*

$$\pi_t(f) = L^2 - \lim_{\varepsilon\to 0} \pi_t^\varepsilon(f), \;\; f \in C_b(\mathbb{R}) \tag{4.5.27}$$

where $\pi_t^\varepsilon(f)$ *satisfies the following filtering equation*

$$\pi_t^\varepsilon(f) = \pi_0^\varepsilon(f) + \int_0^t \pi_s^\varepsilon(\bar{\bar{H}})ds + \int_0^t [\pi_s^\varepsilon(f(X)\bar{h}) - \pi_s^\varepsilon(f(X))\pi_s^\varepsilon(\bar{h})]\varepsilon^{-\alpha}d\nu_s^\varepsilon, \tag{4.5.28}$$

where

$$\bar{\bar{H}}_t = f'(X_t^\varepsilon)\bar{H}_t + \frac{\varepsilon^{2\beta}}{2}f''(X_t^\varepsilon) \tag{4.5.29}$$

$$\bar{H}_t = H_t + \beta\psi_t^\varepsilon, \;\;\; \psi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\beta-1}dU_s \tag{4.5.30}$$

$$\bar{h}_t = h_t + \alpha\varphi_t^\varepsilon, \;\;\; \varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1}dW_s \tag{4.5.31}$$

$$\nu_t^\varepsilon = \frac{1}{\varepsilon^\alpha}[Y_t^\varepsilon - \int_0^t \pi_s^\varepsilon(\bar{h})ds], \tag{4.5.32}$$

Proof : It follows from the definition of X_t^ε in (4.5.5), X_t in (4.5.1) and from Theorem 4.2.1 that

$$\begin{aligned} X_t^\varepsilon &= X_0 + \int_0^t H_s ds + B_t^{(1)\varepsilon} \\ &\to X_0 + \int_0^t H_s ds + B_t^{(1)} \\ &= X_t, \end{aligned} \tag{4.5.33}$$

i.e. $X_t^\varepsilon \to X_t$ in $L^2(\Omega, \mathcal{F}, P)$ as $\varepsilon \to 0$. As for Y_t^ε, from (4.5.3) and (4.5.7) we can see that

$$\begin{aligned} Y_t^\varepsilon - Y_t &= \Big(\int_0^t h(X_s^\varepsilon)ds + B_t^{(2)\varepsilon}\Big) - \Big(\int_0^t h(X_s)ds + B_t^{(2)}\Big) \\ &= \int_0^t \big(h(X_s^\varepsilon) - h(X_s)\big)ds + \big(B_t^{(2)\varepsilon} - B_t^{(2)}\big), \end{aligned} \tag{4.5.34}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function by assumption. The $L^2(\Omega, \mathcal{F}, P)$-convergence of $B_t^{(2)\varepsilon}$ and X_t^ε respectively to $B_t^{(2)}$ and X_t respectively, imply that $Y_t^\varepsilon \to Y_t$ in $L^2(\Omega, \mathcal{F}, P)$ as $\varepsilon \to 0$.

It follows from (4.5.7) and (4.2.8)

$$\begin{aligned} Y_t^{\varepsilon} &= \int_0^t h_s ds + B_t^{(2)\varepsilon} \\ &= \int_0^t h_s ds + \int_0^t \alpha \varphi_s^{\varepsilon} ds + \varepsilon^{\alpha} W_t \\ &= \int_0^t \bar{h}_s ds + \varepsilon^{\alpha} W_t, \end{aligned} \tag{4.5.35}$$

where $\bar{h}_s = h_s + \alpha\varphi_s^{\varepsilon}$. So Y_t^{ε} is a $\mathcal{F}_t^W$- semimartingale.

Consider

$$\begin{aligned} \bar{h}_s^2 &= (h_s + \alpha\varphi_s^{\varepsilon})^2 \\ &\leq 2(h_s^2 + \alpha^2(\varphi_s^{\varepsilon})^2), \end{aligned} \tag{4.5.36}$$

then

$$\begin{aligned} E(\bar{h}_s^2) &\leq E[2(h_s^2 + \alpha^2(\varphi_s^{\varepsilon})^2)] \\ &= 2E[h_s^2] + 2\alpha^2 E[(\varphi_s^{\varepsilon})^2], \end{aligned} \tag{4.5.37}$$

i.e.

$$\int_0^t E(\bar{h}_s^2) ds \leq 2\int_0^t E(h_s^2) ds + 2\alpha^2 \int_0^t E[(\varphi_s^{\varepsilon})^2] ds. \tag{4.5.38}$$

By definition of φ_s^{ε} from (4.5.31) and Itô Isometry property, we get

$$\begin{aligned} E[(\varphi_s^{\varepsilon})^2] &= E\big[\Big(\int_0^s (s-u+\varepsilon)^{\alpha-1} dW_u\Big)^2\big] \\ &= \int_0^s E(s-u+\varepsilon)^{2(\alpha-1)} du \\ &= \int_0^s (s-u+\varepsilon)^{2(\alpha-1)} du \\ &\leq \int_0^s \varepsilon^{2(\alpha-1)} du \\ &= s\varepsilon^{2(\alpha-1)} < \infty. \end{aligned} \tag{4.5.39}$$

It follow by Fubini Theorem and (4.5.38) that

$$\begin{aligned} E\Big(\int_0^t \bar{h}_s^2 ds\Big) &= \int_0^t E(\bar{h}_s^2) ds \\ &\leq 2\int_0^t E(h_s^2) ds + 2\alpha^2 \int_0^t E[(\varphi_s^\varepsilon)^2] ds \\ &= 2E\Big(\int_0^t h_s^2 ds\Big) + 2\alpha^2 \int_0^t E[(\varphi_s^\varepsilon)^2] ds. \end{aligned} \tag{4.5.40}$$

Then from assumption of h_t, (4.5.39) and (4.5.40), we can see that

$$E\int_0^t \bar{h}_s^2 ds < \infty. \tag{4.5.41}$$

It follows from (4.5.5) and (4.2.8)

$$\begin{aligned} X_t^\varepsilon &= X_0 + \int_0^t H_s ds + B_t^{(1)\varepsilon} \\ &= X_0 + \int_0^t H_s ds + \int_0^t \beta\psi_s^\varepsilon ds + \varepsilon^\beta U_t \\ &= X_0 + \int_0^t \bar{H}_s ds + \varepsilon^\beta U_t, \end{aligned} \tag{4.5.42}$$

where $\bar{H}_s = H_s + \beta\psi_s^\varepsilon$. So X_t^ε is a $\mathcal{F}_t^W$- semimartingale.

Consider

$$\begin{aligned} \bar{H}_s^2 &= (H_s + \beta\psi_s^\varepsilon)^2 \\ &\leq 2(H_s^2 + \beta^2(\psi_s^\varepsilon)^2), \end{aligned} \tag{4.5.43}$$

then

$$\begin{aligned} E(\bar{H}_s^2) &\leq E[2(H_s^2 + \beta^2(\psi_s^\varepsilon)^2)] \\ &= 2E[H_s^2] + 2\beta^2 E[(\psi_s^\varepsilon)^2], \end{aligned} \tag{4.5.44}$$

i.e.

$$\int_0^t E(\bar{H}_s^2) ds \leq 2\int_0^t E(H_s^2) ds + 2\beta^2 \int_0^t E[(\psi_s^\varepsilon)^2] ds. \tag{4.5.45}$$

By definition of ψ_s^ε from (4.5.30) and Itô Isometry property, we get

$$\begin{aligned} E[(\psi_s^\varepsilon)^2] &= E\big[(\int_0^s (s-u+\varepsilon)^{\beta-1} dU_u)^2\big] \\ &= \int_0^s E(s-u+\varepsilon)^{2(\beta-1)} du \\ &= \int_0^s (s-u+\varepsilon)^{2(\beta-1)} du \\ &\le \int_0^s \varepsilon^{2(\beta-1)} du \\ &= s\varepsilon^{2(\beta-1)} < \infty. \end{aligned} \tag{4.5.46}$$

It follow by Fubini Theorem and (4.5.45) that

$$\begin{aligned} E\big(\int_0^t \bar{H}_s^2 ds\big) &= \int_0^t E(\bar{H}_s^2) ds \\ &\le 2\int_0^t E(H_s^2) ds + 2\beta^2 \int_0^t E[(\psi_s^\varepsilon)^2] ds \\ &= 2E(\int_0^t H_s^2 ds) + 2\beta^2 \int_0^t E[(\psi_s^\varepsilon)^2] ds. \end{aligned} \tag{4.5.47}$$

Then from assumption of H_t, (4.5.46) and (4.5.47), we can see that

$$E\int_0^t \bar{H}_s^2 ds < \infty. \tag{4.5.48}$$

We have a new approximate filtering problem:

Signal process:

$$X_t^\varepsilon = X_0 + \int_0^t \bar{H}_s ds + \varepsilon^\beta U_t. \tag{4.5.49}$$

Observation process:

$$Y_t^\varepsilon = \int_0^t \bar{h}_s ds + \varepsilon^\alpha W_t, \tag{4.5.50}$$

where

$$\bar{H}_t = H_t + \beta\psi_t^\varepsilon, \quad \psi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\beta-1} dU_s,$$

$$\bar{h}_t = h_t + \alpha\varphi_t^\varepsilon, \quad \varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s.$$

and the innovation process:

$$\nu_t^\varepsilon = \frac{1}{\varepsilon^\alpha}[Y_t^\varepsilon - \int_0^t \pi_s^\varepsilon(\bar{h})ds]. \tag{4.5.51}$$

We can write the FKK filtering equation for the approximate model (4.5.49) and (4.5.50) as

$$\pi_t^\varepsilon(f) = \pi_0^\varepsilon(f) + \int_0^t \pi_s^\varepsilon(\bar{\bar{H}})ds + \int_0^t [\pi_s^\varepsilon(f(X)\bar{h}) - \pi_s^\varepsilon(f(X))\pi_s^\varepsilon(\bar{h})]\varepsilon^{-\alpha}d\nu_s^\varepsilon. \tag{4.5.52}$$

where

$$\bar{\bar{H}}_t = f'(X_t^\varepsilon)\bar{H}_t + \frac{\varepsilon^{2\beta}}{2}f''(X_t^\varepsilon) \tag{4.5.53}$$

Because $X_t^\varepsilon \to X_t$ and $Y_t^\varepsilon \to Y_t$ in $L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t^{Y_t^\varepsilon} \searrow \mathcal{F}_t^Y$ as $\varepsilon \to 0$, then by virtue of Lemma 4.5.4 we have

$$\pi_t(f) = L^2 - \lim_{\varepsilon \to 0} \pi_t^\varepsilon(f) \tag{4.5.54}$$

CHAPTER V

APPLICATION FOR FINANCIAL MODEL OF ORNSTEIN-UHLENBECK PROCESS

In this chapter, some financial filtering models are studied. The results of filtering for Ornstein-Uhlenbeck process from point process observation from Chapter III are applied to the study of the volatility in asset pricing and term structure models for interest rates such as Vasiček model and Hull-White model.

5.1 A Filtering Problem for the Volatility Model

In this section, we consider filtering problem for the volatility Σ_t model which can be represented by

$$d(\ln \Sigma_t) = -\alpha(\ln \Sigma_t)dt + \gamma dW_t. \tag{5.1.1}$$

Set $X_t = \ln \Sigma_t$. Hence $f(X_t) = e^{X_t} = \Sigma_t$. Next, we consider the filtering problem.

Signal process:

$$dX_t = -\alpha X_t dt + \gamma dW_t. \tag{5.1.2}$$

Observation process:

$$dS_t = h(X_t)dt + dM_t. \tag{5.1.3}$$

From the results of Theorem 3.4.25 and Theorem 3.4.26, we obtain the following theorems.

Theorem 5.1.1. *The filter of the filtering problem for the volatility model in (5.1.2)-(5.1.3)is given by one of two following equations:*

$$\begin{aligned}
(a)\ \ \pi_t(\Sigma) &= \pi_0(\Sigma) + \int_0^t \pi_s\Big(- \alpha\Sigma\ln(\Sigma) + \frac{\gamma^2}{2\alpha}\Sigma\Big) ds \\
&\quad + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(h\Sigma) - \pi_{s-}(\Sigma)\pi_s(h)\} dm_s, \\
(b)\ \ \pi_t(\Sigma) &= \pi_0(P_t\Sigma) + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(hP_{t-s}\Sigma) - \pi_{s-}(P_{t-s}\Sigma)\pi_s(h)\} dm_s,
\end{aligned}$$

where $m_t = S_t - \int_0^t \pi(h_s)ds$ *and* P_t *is given by*

$$(P_t\Sigma)(x) = E\Big[\exp\big(e^{-\alpha t}x + \frac{\gamma^2}{2\alpha}\sqrt{1 - e^{-2\alpha t}}Y\big)\Big].$$

Theorem 5.1.2. *The quasi-filter of the filtering problem for the volatility model in (5.1.2)-(5.1.3) is given by one of two following equations:*

$$\begin{aligned}
(a)\ \ \sigma_t(\Sigma) &= \sigma_0(\Sigma) + \int_0^t \sigma_s\Big(- \alpha\Sigma\ln(\Sigma) + \frac{\gamma^2}{2\alpha}\Sigma\Big) ds \\
&\quad + \int_0^t \{\sigma_{s-}(hf) - \sigma_{s-}(f)\} d\mu_s, \\
(b)\ \ \sigma_t(\Sigma) &= \sigma_0(P_t\Sigma) + \int_0^t \{\sigma_{s-}(hP_{t-s}\Sigma) - \sigma_{s-}(P_{t-s}\Sigma)\} d\mu_s.
\end{aligned}$$

where $\mu_t = S_t - t$ *and*

$$(P_t\Sigma)(x) = E\Big[\exp\big(e^{-\alpha t}x + \frac{\gamma^2}{2\alpha}\sqrt{1 - e^{-2\alpha t}}Y\big)\Big].$$

5.2 A Filtering Problem for the Vasiček Model

The term structure for the Vasiček model which given by the following equation

$$dr_t = (b - ar_t)dt + \gamma W_t,$$

where r_t is the interest rate, a, γ are positive constants and b is any real number.

Given $X_t = ar_t - b$, then $f(X_t) = \dfrac{X_t + b}{a} = r_t$. Now we study the following filtering problem:

Signal process:

$$dX_t = -aX_t dt + a\gamma dW_t. \tag{5.2.4}$$

Observation process:

$$dS_t = h(X_t)dt + dM_t. \tag{5.2.5}$$

It follows from Theorem 3.4.25 and Theorem 3.4.26 that

Theorem 5.2.3. *The filter of the filtering problem for the Vasiček model in (5.2.4)-(5.2.5) is given by one of two following equations:*

$$\begin{aligned}
(a)\quad \pi_t(r) &= \pi_0(r) + \int_0^t \pi_s(b - ar)ds \\
&\quad + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(hr) - \pi_{s-}(r)\pi_s(h)\}dm_s, \\
(b)\quad \pi_t(r) &= \pi_0(P_t r) + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(hP_{t-s}r) - \pi_{s-}(P_{t-s}r)\pi_s(h)\}dm_t,
\end{aligned}$$

where $m_t = S_t - \int_0^t \pi(h_s)ds$ *and* P_t *is given by*

$$(P_t r)(x) = E\Big[\exp\big(e^{-at}x + \frac{a\gamma^2}{2}\sqrt{1 - e^{-2at}}Y\big)\Big].$$

Theorem 5.2.4. *The quasi-filter of the filtering problem for the Vasiček model in (5.2.4)-(5.2.5) is given by one of two following equations:*

$$\begin{aligned}
(a)\quad \sigma_t(r) &= \sigma_0(r) + \int_0^t \sigma_s(b - ar)ds + \int_0^t \{\sigma_{s-}(hr) - \sigma_{s-}(r)\}d\mu_s, \\
(b)\quad \sigma_t(r) &= \sigma_0(P_t r) + \int_0^t \{\sigma_{s-}(hP_{t-s}r) - \sigma_{s-}(P_{t-s}r)\}d\mu_s.
\end{aligned}$$

where $\mu_t = S_t - t$ *and*

$$(P_t r)(x) = E\Big[\exp\big(e^{-at}x + \frac{a\gamma^2}{2}\sqrt{1 - e^{-2at}}Y\big)\Big].$$

5.3 A Filtering Problem for the Hull-White Model

Here we consider the Hull-White model for interest rate r_t given by

$$dr_t = (b(t) - a(t)r_t)dt + \gamma(t)dW_t, \tag{5.3.6}$$

where $a(t), b(t)$ and $\gamma(t)$ are deterministic continuous functions of t with $a(t) > 0$ and $\gamma(t) > 0$.

Let $X_t = a(t)r_t - b(t)$, then $f(X_t) = \dfrac{X_t + b(t)}{a(t)} = r_t$. Next we establish the following filtering problem.

Signal process:

$$dX_t = -a(t)X_t dt + a(t)\gamma(t)dW_t. \tag{5.3.7}$$

Observation process:

$$dS_t = h(X_t)dt + dM_t. \tag{5.3.8}$$

By using Theorem 3.4.25 and Theorem 3.4.26, we found the filtering and quasi-filtering equations for the Hull-White model as the following theorems.

Theorem 5.3.5. *The filter of the filtering problem for the Hull-White model in (5.3.7)-(5.3.8) is given by one of two following equations:*

$$\begin{aligned}
\text{(a)}\quad \pi_t(r) &= \pi_0(r) + \int_0^t \pi_s(b(t) - a(t)r)ds \\
&\quad + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(hr) - \pi_{s-}(r)\pi_s(h)\}dm_s, \\
\text{(b)}\quad \pi_t(r) &= \pi_0(P_t r) + \int_0^t \pi_s^{-1}(h)\{\pi_{s^-}(hP_{t-s}r) - \pi_{s-}(P_{t-s}r)\pi_s(h)\}dm_t,
\end{aligned}$$

where $m_t = S_t - \int_0^t \pi(h_s)ds$ *and* P_t *is given by*

$$(P_t r)(x) = E\Big[\exp\big(e^{-a(t)t}x + \frac{a(t)\gamma^2(t)}{2}\sqrt{1 - e^{-2a(t)t}}Y\big)\Big].$$

Theorem 5.3.6. *The quasi-filter of the filtering problem for the Hull-White model in (5.3.7)-(5.3.8) is given by one of two following equations:*

$$\begin{aligned}
(a)\ \ \sigma_t(r) &= \sigma_0(r) + \int_0^t \sigma_s(b(t) - a(t)r)ds + \int_0^t \{\sigma_{s-}(hr) - \sigma_{s-}(r)\}d\mu_s, \\
(b)\ \ \sigma_t(r) &= \sigma_0(P_t r) + \int_0^t \{\sigma_{s-}(hP_{t-s}r) - \sigma_{s-}(P_{t-s}r)\}d\mu_s,
\end{aligned}$$

where $\mu_t = S_t - t$ *and*

$$(P_t r)(x) = E\Big[\exp\big(e^{-a(t)t}x + \frac{a(t)\gamma^2(t)}{2}\sqrt{1 - e^{-2a(t)t}}Y\big)\Big].$$

CHAPTER VI

CONCLUSIONS

In this thesis, we have studied some stochastic filtering problems that can be applied to finance. The main results of this thesis are divided into two parts. The first part is the stochastic filtering problem with point process observation, While the second part is the stochastic fractional filtering problem.

An observation in reality can be made only at discrete times so the observation process is a stochastic process of discrete times. In general, the observation can be made at random times. So a point process is used as an observation process. In the first part, a stochastic filtering problem with semimartingale signal process and observation process given by a point process is studied. The advantage of the representation of a martingale as an integral with respect to the innovation process is that stochastic calculus can be used to attain the filtering equation. By using reference probability and Bayes formula, the quasi-filtering equation is obtained. After that a Feller process and an Ornstein-Uhlenbeck process are used as a signal processes.

Many financial processes can be perturbed not only by white noise as a Brownian motion but also by a fractional process such as a fractional Brownian motion. So fractional filtering is needed in finance. In the second part, a fractional filtering with fractional observation process is studied in three cases. First, a general signal process is considered. Second, a semimartingale signal process is studied. Finally, a fractional signal process is examined. The convergence of a semimartingale B_t^{ε} and general stochastic filtering theorem are used for the proof of the fractional filtering equation.

Apart from these results, the Thesis includes also some applications of filtering problem with point process observation to estimate the volatility in asset pricing models as well as in term structure models such as those of Vasiček and Hull-White.

The author hopes that various practical problems arising in financial markets can be found solutions via the methods and results presented in this Thesis.

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CURRICULUM VITAE

NAME: Tidarut Plienpanich. **GENDER:** Female.

NATIONALITY: Thai.

DATE OF BIRTH: August 30, 1980.

EDUCATIONAL BACKGROUND:

- B.Sc. in Mathematics, Chiang Mai University, Chiang Mai, Thailand, 2002.

- M.Sc. in Mathematics, Chiang Mai University, Chiang Mai, Thailand, 2004.

PUBLICATIONS:

- Plienpanich, T., Niamsup, P., and Lenbury, Y. (2005). Controllability and stability of the perturbed Chen chaotic dynamical system. **Applied Mathematics and Computation**. 171(2): 927-947.

- Plienpanich, T., Sattayatham, P. and Thao, T. H (2008). Nonlinear filtering from counting process observation. **The Pacific Journal of Pure and Applied Mathematics**. (Accepted).

- Plienpanich, T. (2008). Filtering for stochastic volatility from point process observation. **Studia Universitatis "Babes-Bolyai", Mathematica**. (Accepted).

- Thao, T. H., Sattayatham, P. and Plienpanich, T. (2008). On the fractional stochastic filtering. **Studia Universitatis "Babes-Bolyai", Mathematica**. (Accepted).

- Plienpanich, T. and Sattayatham, P. and Thao, T. H (2008). Fractional Integrated GARCH(1,1) Models of Continuous Time. (To appear).

SCHOLARSHIPS:

- Development and Promotion of Science and Technology Talents Project of Thailand (DPST), 1996-2005.

- Royal Golden Jubilee Ph.D. research scholarship, 2005-2007.