รหัสโครงการ SUT1-103-43-12-06

# รายงานการวิจัย

# Shape-Preserving Parametrization For Spline Interpolation

ได้รับเงินอุดหนุนการวิจัยจาก
มหาวิทยาลัยเทคโนโลยีสุรนารี



ผลงานวิจัยเป็นความรับผิดชอบของหัวหน้าโครงการวิจัยแต่เพียงผู้เดียว

# อภินันทนาการ

รหัสโครงการ SUT1-103-43-12-06

รายงานการวิจัย

# Shape-Preserving Parametrization
# For Spline Interpolation



ได้รับเงินอุดหนุนการวิจัยจาก
มหาวิทยาลัยเทคโนโลยีสุรนารี

ผลงานวิจัยเป็นความรับผิดชอบของหัวหน้าโครงการวิจัยแต่เพียงผู้เดียว

รหัสโครงการ SUT1–103-43-12-06

# รายงานการวิจัย

# Shape-Preserving Parametrization For Spline Interpolation

คณะผู้วิจัย

หัวหน้าโครงการ

**Assoc. Prof. Dr. Boris I. Kvasov**

สาขาวิชาคณิตศาสตร์

สำนักวิชาวิทยาศาสตร์

มหาวิทยาลัยเทคโนโลยีสุรนารี

**ได้รับเงินอุดหนุนการวิจัยจากมหาวิทยาลัยเทคโนโลยีสุรนารี ปีงบประมาณ พ.ศ. 2542**

**ผลงานวิจัยเป็นความรับผิดชอบของหัวหน้าโครงการวิจัยแต่เพียงผู้เดียว**

เมษายน 2544

## Acknowledgement

This is to acknowledge with many thanks the Suranaree University of Technology Research Fund Administration for financial support which made this project possible and led to its successful complition. The author also wishes to express his gratitude to his former M.Sc. student Opas Udomsomboonpon whose modified computer program was used to calculate 1-D parametric shape-preserving cubic splines.

# บทคัดย่อ

รายงานการวิจัยนี้เป็นการคำนวณตัวแปรเสริมโดยวิธีการแทรกข้อมูลด้วยสไปลน์ ผลการวิจัยพบว่า ถ้าใช้ขั้นตอนวิธีการเปลี่ยนแปลงตัวแปรเสริมจะให้รูปร่างของเส้นโค้งหรือพื้นผิวที่ดีกว่าการคำนวณแบบการสะสมความยาวคอร์ด จุดศูนย์กลางคอร์ด หรือการจัดตัวแปรสม่ำเสมอ ทำให้สามารถพัฒนาความสัมพันธ์ระหว่างเส้นโค้ง พื้นผิว และค่าเริ่มต้น และให้ขนาดของช่วงมีความสัมพันธ์กับอาณาบริเวณที่มีความชันเปลี่ยนแปลงมาก โดยทั่วไปขั้นตอนวิธีแบบตัวแปรเสริมนี้ สามารถนำไปใช้ได้กับการประมาณค่าในกรณีทั่วไปของพื้นผิวหลายตัวแปร ซึ่งได้เสนอผลการวิจัยโดยตัวอย่างการคำนวณเชิงตัวเลข

# Contents

## 1. Introduction

Let $P_i = (x_i, y_i)$, $i = 0, \ldots, N$ be a sequence of pairwise different data points in the $xy$-plane. In order to draw a curve passing through these points, it is in general necessary to construct a mesh $\Delta : a = t_0 < t_1 < \cdots < t_N = b$ and to define a continuous vector-function $C(t) = (C_x(t), C_y(t))$, $t \in [a, b]$, such that

$$C_x(t_i) = x_i, \quad C_y(t_i) = y_i, \quad i = 0, \ldots, N,$$

that is,

$$C(t_i) = P_i, \quad i = 0, \ldots, N.$$

The parameter values chosen are called the *interpolating nodes.* The shape of the curve is determined by the choice of the interpolating nodes as well as the method of interpolation used on this mesh. The choice of the nodes greatly influences the resulting curve. The problem of finding a good set of interpolating nodes is known as the *parametrization problem.*

The simplest and most widely used parametrization is the *uniform* parametrization, provided by

$$t_i = t_{i-1} + h, \quad h = (b-a)/N, \quad i = 1, \ldots, N.$$

This is generally unsatisfactory for the obvious reason that the nodes do not relate to the distribution of the data points. The choice of the interpolating nodes should be based on the behaviour of the data, giving a *data dependent* parametrization. It is generally accepted that a better choice is the *cumulative chord length* parametrization

$$t_i = t_{i-1} + \frac{|P_i - P_{i-1}|}{\sum_{j=1}^{N} |P_j - P_{j-1}|}(b-a), \quad i = 1, \ldots, N,$$

with $|\cdot|$ denoting the usual Euclidean distance. Here, the term "better" refers to a rather vague quality of the curve: its "fairness". There is no precise definition of this quality but it is customary to accept that a curve is fair if it reproduces the interpolation polygon well and has a high degree of smoothness (see Criterion A in [31]).

The theoretical foundation of parametrization by cumulative chord length for natural splines was laid by Epstein [8]. In this case the curve has no corners. But it was shown by Lee [25] that it may have cusps. Roughly, the distinction between a corner and a cusp (for a parametric curve) is this: the position of the tangent line changes discontinuously across a corner (at which the tangent line is undefined), whereas across a cusp, the tangent line varies continuously, but the unit tangent vector reverses its direction. Such a parametrization has sometimes been called the "natural parametrization". The main reason for this choice seems to be that it roughly approximates the *arc length* parametrization [6]:

$$t_i = t_{i-1} + \int_{t_{i-1}}^{t_i} \sqrt{[C'_x(t)]^2 + [C'_y(t)]^2}\, dt, \quad i = 1, \ldots, N,$$

which requires iterations [33]. However, here the aim is to create a curve through a given set of points, and it is not clear why one should strive for the arc length parametrization, nor is it clear that the suggested iterations should even converge.

One obtains the *exponential* parametrization [23] if

$$t_i = t_{i-1} + \frac{|P_i - P_{i-1}|^e}{\sum_{j=1}^{N} |P_j - P_{j-1}|^e}(b-a), \quad i = 1, \ldots, N, \quad 0 \le e \le 1.$$

As a particular case, for $e = 0, 0.5,$ and 1, this gives uniform, *centripetal* [23], and chord length parametrizations respectively. With nearly equally spaced points, these three parametrizations are roughly the same. In general, the centripetal parametrization gives better results than either the chord length or the uniform parametrizations.

The *affine invariant* parametrization of Foley and Nielson [12,29] takes the geometry of the control points into consideration and produces quality results for a wide variety of curve/surface fitting problems. The interpolating nodes can also be derived through optimization techniques [16,20,27]. The *intrinsic* parametrization by Hoschek [18] uses the minimization of the distance between the given points $P_i$ and an approximation curve, which is a nonlinear problem. But optimization methods are expensive, and moreover it is not entirely clear what objective function should be used.

The choice of the interpolating nodes can be based on the preservation of the data shape properties such as monotonicity, convexity, etc. We shall say that a curve $C$ is monotonicity preserving for the given data in the interval $[t_k, t_l]$, $l > k$ provided that the following conditions are fulfilled

$$C'_x(t)(x_{j+1} - x_j) > 0, \quad C'_y(t)(y_{j+1} - y_j) > 0 \quad \text{if} \quad t \in [t_j, t_{j+1}] \quad \text{for all} \quad j = k, \ldots, l-1. \tag{1}$$

## 2. Affine Invariance of Polynomials and Splines

Let $\mathbb{R} = (-\infty, \infty)$ be the real axis, and consider an affine transformation $\mathbb{R} \to \hat{\mathbb{R}} : \hat{t} = pt + q$, where $p \neq 0$ and $q$ are constant. Then the mesh $\Delta : t_0 < t_1 < \cdots < t_N$ is transformed into the mesh $\hat{\Delta} = \{ \hat{t}_i \mid \hat{t}_i = pt_i + q, \ i = 0, \ldots, N\}$. Let us show that interpolating Lagrange polynomials and interpolating polynomial splines are invariant with respect to such transformations.

**Lemma 1.** *The interpolating Lagrange polynomials are invariant with respect to affine transformations of the real line* $\mathbb{R}$.

**Proof:** The Lagrange polynomial of degree $n$ that interpolates the data $(t_j, f_j)$, $j = i, \ldots, i+n$, has the form

$$\mathbb{L}_{i,n}(t) = \sum_{j=i}^{i+n} f_j l_j(t), \quad l_j(t) = \prod_{\substack{k=i \\ k \neq j}}^{i+n} \frac{(t - t_k)}{(t_j - t_k)}.$$

Since here

$$l_j(t) = \prod_{\substack{k=i \\ k\neq j}}^{i+n} \frac{(t-t_k)}{(t_j-t_k)} = \prod_{\substack{k=i \\ k\neq j}}^{i+n} \frac{(\hat{t}-\hat{t}_k)}{(\hat{t}_j-\hat{t}_k)} = \hat{l}_j(\hat{t}),$$

then

$$\mathbb{L}_{i,n}(t) = \sum_{j=i}^{i+n} f_j l_j(t) = \sum_{j=i}^{i+n} f_j \hat{l}_j(\hat{t}) = \hat{\mathbb{L}}_{i,n}(\hat{t}).$$

This proves the lemma. □

Let $S_{n,\overline{\nu}}(\Delta)$ be a linear space of polynomial splines satisfying definition 3.1 in [22]. It was shown in [22, chapter 3] that using the extended mesh $\Delta$ one can construct a system of normalized B-splines $B_i$, $i = 1, \ldots, \rho$, such that any spline $S \in S_{n,\overline{\nu}}(\Delta)$ can be uniquely represented in the form

$$S(t) = \sum_{i=1}^{\rho} b_i B_{i,n}(t), \quad t \in [t_0, t_N].$$

Let $\hat{S}_{n,\overline{\nu}}(\hat{\Delta})$ be a set of polynomial splines on the mesh $\hat{\Delta}$ which is obtained from the linear space $S_{n,\overline{\nu}}(\Delta)$ by affine transformation of the variable $t$. Let $\hat{B}_i$, $i = 1, \ldots, \rho$, be a system of B-splines on the extended mesh $\hat{\Delta}$ forming a basis in $\hat{S}_{n,\overline{\nu}}(\hat{\Delta})$.

**Lemma 2.** *The interpolating splines in $S_{n,\overline{\nu}}(\Delta)$ are invariant with respect to affine transformations of the real line $\mathbb{R}$.*

**Proof:** Let us show that the equality $B_{i,n}(t) = \hat{B}_{i,n}(\hat{t})$, $i = 1, \ldots, \rho$, holds. If $n = 1$, then by definition (3.18) in [22]

$$B_{i,1}(t) = \begin{cases} 1 & \text{if } t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\hat{B}_{i,1}(\hat{t}) = B_{i,1}(t)$. Suppose the required equality is fulfilled for $n-1 = k$ $(k \geq 1)$. Then by virtue of the recurrence relation for B-splines (3.7) in [22] and by induction,

$$\begin{aligned} B_{i,n}(t) &= \frac{t-t_i}{t_{i+n-1}-t_i} B_{i,n-1}(t) + \frac{t_{i+n}-t}{t_{i+n}-t_{i+1}} B_{i+1,n-1}(t) \\ &= \frac{\hat{t}-\hat{t}_i}{\hat{t}_{i+n-1}-\hat{t}_i} \hat{B}_{i,n-1}(\hat{t}) + \frac{\hat{t}_{i+n}-\hat{t}}{\hat{t}_{i+n}-\hat{t}_{i+1}} \hat{B}_{i+1,n-1}(\hat{t}) = \hat{B}_{i,n}(\hat{t}). \end{aligned}$$

If now $S$ and $\hat{S}$ are interpolating splines on the meshes $\Delta$ and $\hat{\Delta}$ respectively connected by the affine transformation $\hat{t} = pt + q$ $(p \neq 0)$, then by virtue of the uniqueness of the spline representation as a linear combination of B-splines, the following representation holds:

$$S(t) = \sum_{i=1}^{\rho} b_i B_{i,n}(t) = \sum_{i=1}^{\rho} b_i \hat{B}_{i,n}(\hat{t}) = \hat{S}(\hat{t}). \tag{2}$$

Therefore, the interpolating spline $S$ is invariant with respect to affine transformations of its variable. This proves the lemma. □

Let us obtain the relation between $S'$ and $\hat{S}'$. By differentiation of the equality proven above, $B_{i,n}(t) = \hat{B}_{i,n}(\hat{t})$, $i = 1, \dots, \rho$, one obtains

$$B'_{i,n}(t) = \frac{d}{dt}\left[\hat{B}_{i,n}(\hat{t})\right] = \frac{d}{d\hat{t}}\left[\hat{B}_{i,n}(\hat{t})\right]\frac{d\hat{t}}{dt} = p\hat{B}'_{i,n}(\hat{t}).$$

Differentiating now equality (2) one can write down

$$S'(t) = \sum_i b_i B'_{i,n}(t) = \sum_i b_i p \hat{B}'_{i,n}(\hat{t}) = p\hat{S}'(\hat{t}). \tag{3}$$

By repeated differentiation of the last and next to last equalities we arrive at

$$S^{(r)}(t) = \sum_i b_i B^{(r)}_{i,n}(t) = \sum_i b_i p^r \hat{B}^{(r)}_{i,n}(\hat{t}) = p^r \hat{S}^{(r)}(\hat{t}),$$
$$r = 1, \dots, n-1.$$

Note that the invariance of cubic weighted $\nu$-splines, and in particular of interpolating $C^2$ cubic splines with respect to affine transformations of the real line, was shown in [10]. The invariance with respect to affine transformations of generalized tension splines was proven in [22, chapter 7].

## 3. Shape-Preserving Parametrization

Let us consider the behaviour of the parabola $\mathbb{L}_{i,2}$ passing through the points $(t_j, f_j)$, $j = i, i+1, i+2$, depending on the mesh choice. We will seek the knot $t_{i+1}$ such that the parabola turns out to be monotonicity preserving for the initial data, that is, such that the following relations are met

$$\mathbb{L}'_{i,2}(t)(f_{j+1} - f_j) > 0, \quad j = i, i+1.$$

We introduce the following notation,

$$\alpha_i = (t_{i+1} - t_i)/(t_{i+2} - t_i), \quad T_i = t_{i+2} - t_i.$$

As a corollary of the Lemma 1 the knots $t_i = 0$ and $t_{i+2} = 1$ can be fixed. Then

$$\alpha_i = t_{i+1}, \quad T_i = 1,$$

$$\frac{d}{dt}\mathbb{L}_{i,2}(t) = \frac{(1-\alpha_i)(f_{i+1} - f_i) + (2t - \alpha_i)[\alpha_i(f_{i+2} - f_i) - f_{i+1} + f_i]}{\alpha_i(1-\alpha_i)}. \tag{4}$$

Let $F_i \equiv (f_{i+1} - f_i)(f_{i+2} - f_{i+1})$. We consider three possible configurations of the initial data:

A. $F_i > 0$;

B. $F_i < 0$;
C. $F_i = 0$.

A. By assumption, the sequence $f_i < f_{i+1} < f_{i+2}$ or $f_i > f_{i+1} > f_{i+2}$ is monotone. Since $\mathbb{L}'_{i,2}$ is a linear function, the monotonicity of $\mathbb{L}_{i,2}$ is equivalent to the fulfillment of the two inequalities

$$\mathbb{L}'_{i,2}(t_i)(f_{i+1} - f_i) > 0, \quad \mathbb{L}'_{i,2}(t_{i+2})(f_{i+1} - f_i) > 0.$$

Simple manipulations, based on the fact that $0 < \alpha_i < 1$, readily yield the inequalities

$$-\alpha_i^2 + \frac{f_{i+1} - f_i}{f_{i+2} - f_i} > 0, \quad -(1 - \alpha_i)^2 + \frac{f_{i+2} - f_{i+1}}{f_{i+2} - f_i} > 0.$$

From these inequalities it follows that

$$\alpha^f_{\min} = 1 - \sqrt{\frac{f_{i+2} - f_{i+1}}{f_{i+2} - f_i}} < \alpha_i < \sqrt{\frac{f_{i+1} - f_i}{f_{i+2} - f_i}} = \alpha^f_{\max}. \tag{5}$$

The same argument implies the following result:

**Lemma 3.** *If the restriction imposed on the initial data that $F_i > 0$ holds, then monotonicity of the parabola is equivalent to the knot $t_{i+1}$ being in the interval $T_i^f = (t_i + \alpha^f_{\min} T_i,\ t_i + \alpha^f_{\max} T_i)$.*

One value of the knot $t_{i+1}$ in the range $T_i^f$ can be obtained by minimizing the parabola length. Since $f_{i+2} \neq f_i$, the equality (4) can be transformed into the form

$$(f_{i+2} - f_i)^{-1} \mathbb{L}'_{i,2}(t) = 1 + \frac{1 - 2t}{\alpha_i(1 - \alpha_i)} \left( \frac{f_{i+1} - f_i}{f_{i+2} - f_i} - \alpha_i \right).$$

Hence setting

$$\alpha_i = (f_{i+1} - f_i)/(f_{i+2} - f_i), \tag{6}$$

we have the equality $(f_{i+2} - f_i)^{-1} \mathbb{L}'_{i,2}(t) = 1$, that is, the parabola degenerates into a straight line, and it is easily verified that $t_{i+1} = t_i + \alpha_i T_i \in T_i^f$.

B. In this case the knot $t_{i+1}$ is chosen so that

$$\mathbb{L}'_{i,2}(t_{i+1}) = 0.$$

Since by virtue of (4) it follows that

$$\begin{aligned} \mathbb{L}'_{i,2}(t_{i+1}) &= \alpha_i^{-1}(1 - \alpha_i)^{-1}[\alpha_i^2(f_{i+2} - f_i) - 2\alpha_i(f_{i+1} - f_i) + f_{i+1} - f_i] \\ &= \alpha_i^{-1}(1 - \alpha_i)^{-1}[\alpha_i^2(f_{i+2} - f_{i+1}) + (1 - \alpha_i)^2(f_{i+1} - f_i)], \end{aligned}$$

we have

$$\alpha_i = \frac{\sqrt{|f_{i+1} - f_i|}}{\sqrt{|f_{i+1} - f_i| + |f_{i+2} - f_{i+1}|}}. \tag{7}$$

C. There are three possible configurations of the initial data:

(i) $f_i = f_{i+1} = f_{i+2}$;
(ii) $f_i = f_{i+1},\ f_{i+1} \neq f_{i+2}$;
(iii) $f_i \neq f_{i+1},\ f_{i+1} = f_{i+2}$.

In the first case, the position of the knot can be chosen arbitrarily because $\mathbb{L}_{i,2}(t) = constant$. The selection range for $t_{i+1}$ will be the interval $T_i^f = (0, 1)$. In the second and third cases the parabola is non-monotone for any position of the knot $t_{i+1}$ (for case (ii) in the interval $[t_i, t_{i+1}]$ and for case (iii) in the interval $[t_{i+1}, t_{i+2}]$, respectively). For continuity of formulae (6) and (7) we set

$$\alpha_i = \begin{cases} \varepsilon & \text{if } f_i = f_{i+1}, \quad f_{i+1} \neq f_{i+2}\ , \\ 1-\varepsilon & \text{if } f_i \neq f_{i+1}, \quad f_{i+1} = f_{i+2}, \end{cases} \tag{8}$$

where $\varepsilon$ is a small number (e.g. $\varepsilon =$(computer precision)$\times 100$). In cases B and C the interval $T_i^f$ degenerates into the point $t_{i+1}$.

Therefore, formulae (6)–(8) determine a location of the knot $t_{i+1}$ for any possible configuration of the initial data.

Let us formally denote the set of parameters obtained from formulae (6)–(8) for the data $f_0, f_1, \dots, f_N$ as $\alpha_i^f = \alpha_i$, $i = 0, \dots, N-2$.

The mesh $\Delta$ for the curve $C(t) = (C_x(t), C_y(t))$ is constructed by using the two above found meshes for each of the component functions $C_x$ and $C_y$. It follows from the above that

**Lemma 4.** *The curve $C$ will be rendered monotonicity preserving for the initial data if $T_i^x \cap T_i^y \neq \emptyset$ and $t_{i+1} \in T_i^x \cap T_i^y$ for all $i$, $i = 0\dots, N-2$. Otherwise the conditions of monotonicity preservation are violated.*

Let us specify the algorithm for parametrizing the curve $C$ in the general case. We define the parameters

$$\hat{\alpha}_i = \frac{d_i^e}{d_i^e + d_{i+1}^e}, \quad 0 \le e \le 1; \quad d_j = |P_{j+1} - P_j|,$$
$$j = i, i+1; \quad i = 0, \dots, N-2,$$

which fix a mesh normed by cumulative chord length. Then we choose $t_{i+1} = t_i + \hat{\alpha}_i$ if $t_{i+1} \in T_i^x \cap T_i^y$. If $t_i + \hat{\alpha}_i \notin T_i^x \cap T_i^y$ or $T_i^x \cap T_i^y = \emptyset$ we set

$$t_{i+1} = t_i + \alpha_i,$$

where

$$\alpha_i = \begin{cases} \alpha_i^x & \text{if } \quad T_i^x \subset T_i^y, \\ \alpha_i^y & \text{if } \quad T_i^y \subset T_i^x, \\ (\alpha_i^x + \alpha_i^y)/2, & \text{otherwise}\ . \end{cases} \tag{9}$$

The value $t_{i+1} \notin T_i^x \cap T_i^y \neq \emptyset$ is additionally corrected to the nearest point of the interval $T_i^x \cap T_i^y$. Thus, the mesh has been constructed.

Denote the step size of the mesh $\Delta$ by $h_i = t_{i+1} - t_i$, $i = 0, \dots, N-1$. For a curve $C(t)$ passing through the points $P_i$, $i = 0, \dots, N$, the mesh $\Delta : t_0 < t_1 < \dots < t_N$ is uniquely determined by a set of parameters $\alpha_0, \alpha_1, \dots, \alpha_{N-2}$ assigning the relations of the mesh steps, $h_i/h_{i-1}$, $i = 1, \ldots, N-1$, and the values $h_0$ or $h_{N-1}$.

For a vector-function $C_3(t) = (C_x(t), C_y(t), C_z(t))$ which defines the curve in the $xyz$-space passing through the points $Q_i = (x_i, y_i, z_i)$, $i = 0, \ldots, N$, the general parametrization algorithm is similar to the 2-D case with formal substitution of the interval $T_i^x \cap T_i^y \cap T_i^z$ for the interval $T_i^x \cap T_i^y$. In formula (9), instead of $(\alpha_i^x + \alpha_i^y)/2$ we choose $\alpha_i = (\alpha_i^x + \alpha_i^y + \alpha_i^z)/3$.

Furthermore, the parametrization will be called shape-preserving if the parameters $\alpha_0, \ldots, \alpha_{N-2}$ are chosen by the algorithm described above.

## 4. Parametrization for Cubic Splines

Let us consider a conventional cubic spline on the mesh $\Delta : t_0 < t_1 < \cdots < t_N$ satisfying the interpolation conditions

$$S(t_i) = f_i, \quad i = 0, \ldots, N,$$

as well as the endpoint constraints which are preferable in practical calculations [3]:

$$S'(t_i) = f_i', \quad i = 0, N. \tag{10}$$

Note that in order to preserve the invariance of the spline under affine transformations of the parameter space, by virtue of (3), the values $f_i'$ must be modified by $\hat{f}_i' = q^{-1} f_i'$, $i = 0, N$.

The given data will be called strictly monotone if $f_0 < f_1 < \cdots < f_N$ or $f_0 > f_1 > \cdots > f_N$ and if in addition the following inequalities are met,

$$f_0'(f_1 - f_0) > 0, \quad f_N'(f_N - f_{N-1}) > 0.$$

We will say that the spline $S$ preserves the strict monotonicity of the initial data if $f_0 < f_1 < \cdots < f_N$ and $S'(t) > 0$, $t \in [t_0, t_N]$, or $f_0 > f_1 > \cdots > f_N$ and $S'(t) < 0$, $t \in [t_0, t_N]$. In [22, chapter 4], sufficient conditions for the initial data are established which ensure that if the data is nondecreasing $f_0 \le f_1 \le \cdots \le f_N$ then the derivative of the cubic spline is nonnegative $S'(t) \ge 0$, $t \in [t_0, t_N]$. By virtue of Theorem 4.4 in [22], the following assertion is valid.

**Lemma 5.** *Let a cubic spline $S \in C^2[t_0, t_N]$ with endpoint conditions (10) interpolate the strictly monotonically increasing (decreasing) data $\{f_i\}$, $i = 0, \ldots, N$. If the following inequalities are valid*

$$\frac{f_0'}{f[t_0, t_1]} < 3, \quad \frac{f_N'}{f[t_{N-1}, t_N]} < 3,$$
$$\frac{f[t_{i-1}, t_i]}{f[t_i, t_{i+1}]} < 2 + \frac{h_{i-1}}{h_i}, \quad \frac{f[t_i, t_{i+1}]}{f[t_{i-1}, t_i]} < 2 + \frac{h_i}{h_{i-1}}, \quad i = 1, \ldots, N-1,$$

*then $S'(x) > 0$ $(< 0)$ for all $x \in [t_0, t_N]$, that is, $S$ is strictly monotone on $[t_0, t_N]$.*

With the above notation $\alpha_i = (t_{i+1} - t_i)/(t_{i+2} - t_i)$, $i = 0, \ldots, N-2$, one has

$$\frac{h_i}{h_{i-1}} = \frac{1 - \alpha_{i-1}}{\alpha_{i-1}}, \quad i = 0, \ldots, N-2.$$

Then from Lemma 5 one readily obtains the following restrictions on $\alpha_i$:

$$1 - \sqrt{\frac{f_{i+2} - f_{i+1}}{f_{i+2} - f_i}} < \alpha_i < \sqrt{\frac{f_{i+1} - f_i}{f_{i+2} - f_i}}, \qquad i = 0, \ldots, N-2.$$

These inequalities coincide with the conditions (5) of monotonicity preserving parametrization. Thus we have proved

**Theorem 1.** *Let the initial data $\{f_i\}$, $i = 0, \ldots, N$, be strictly monotone and given on the mesh constructed by the algorithm of shape-preserving parametrization. Then a cubic spline $S$, that interpolates this data and satisfies the endpoint conditions (10), preserves the strict monotonicity of the initial data provided that the following inequalities are met:*

$$\frac{f'_0}{f[t_0, t_1]} < 3, \qquad \frac{f'_N}{f[t_{N-1}, t_N]} < 3. \tag{11}$$

The following two corollaries follow immediately from Theorem 1.

**Corollary 1.** *Let the initial data $P_i = (x_i, y_i)$, $i = 0, \ldots, N$, be strictly monotone. The interpolating parametric cubic spline $C(t) = (C_x(t), C_y(t))$ with knots on the mesh $\Delta$ constructed by the algorithm for shape-preserving parametrization is monotonicity preserving for the initial data if*

(a) $T_i^x \cap T_i^y \neq \emptyset, \quad i = 0, \ldots, N-2;$

(b) $\max\left(\frac{x'_0}{x[t_0, t_1]}, \frac{y'_0}{y[t_0, t_1]}\right) < 3, \qquad \max\left(\frac{x'_N}{x[t_{N-1}, t_N]}, \frac{y'_N}{y[t_{N-1}, t_N]}\right) < 3.$

**Corollary 2.** *Let $x_0 < x_1 < \cdots < x_N$ and $x'_0 > 0$, $x'_N > 0$. The interpolating cubic spline with knots on the mesh $\Delta$ such that*

(a) $t_{i+1} \in T_i^x$, $i = 0, \ldots, N-2$, *and*

(b) $\frac{x'_0}{x[t_0, t_1]} < 3, \qquad \frac{x'_N}{x[t_{N-1}, t_N]} < 3,$

*increases monotonically on the interval $[t_0, t_N]$.*

If the restrictions of Corollary 2 are fulfilled, then for single-valued functional data there is a one-to-one correspondence between the points of the $x$-axis and the curve points, that is, there exists a single-valued function $y = y(x)$ with a graph $(C_x, C_y)$.

Conditions (11) can be rewritten in the form

$$h_0 < 3(f_1 - f_0)/f'_0, \quad h_{N-1} < 3(f_N - f_{N-1})/f'_N. \tag{12}$$

To fulfill these conditions one can apply two different algorithms.

(i) Starting from specified values $f'_0$ and $f'_N$ one finds $h_0$ and $h_{N-1}$;

(ii) Fixing values of $h_0$ and $h_{N-1}$ one corrects $f'_0$ and $f'_N$.

Since $h_{i+1} = h_i(1 - \alpha_i)/\alpha_i$, $i = 0, \ldots, N-2$, then setting

$$R = \prod_{i=0}^{N-2} \frac{h_{i+1}}{h_i} = \prod_{i=0}^{N-2} \frac{1 - \alpha_i}{\alpha_i},$$

one has $h_{N-1} = h_0 R$. Now choosing

$$h_0 = 3 \min\left( \frac{f_1 - f_0}{f_0'}, \frac{f_N - f_{N-1}}{R f_N'} \right)$$

one fulfills the conditions (12).

If the nodes $t_0$ and $t_N$ of the mesh $\Delta$ are fixed, e.g. if we use the normed parametrization $\Delta : 0 = t_0 < t_1 < \cdots < t_N = 1$, then we set

$$f_0' = \begin{cases} \mathbb{L}_{0,2}'(t_0) & , \text{ if } \mathbb{L}_{0,2}'(t_0)(f_1 - f_0) > 0; \\ \varepsilon \, sign(f_1 - f_0), & \text{otherwise,} \end{cases}$$

$$f_N' = \begin{cases} \mathbb{L}_{N-2,2}'(t_N) & \text{if } \mathbb{L}_{N-2,2}'(t_N)(f_N - f_{N-1}) > 0; \\ \varepsilon \, sign(f_N - f_{N-1}), & \text{otherwise.} \end{cases}$$

One can easily check that in this case conditions (12) are again satisfied. In order to set $f_0'$ and $f_N'$, the cubic Lagrange polynomials may also be employed.

**Remark 1.** By virtue of formula (6), the step size of the mesh $\Delta$ for shape-preserving parametrization is substantially smaller in the regions of sharply increasing "gradient" of the initial data. Hence the points of the spline whose values are obtained on a uniform partition of the interval $[t_0, t_N]$ will be concentrated in such domains. This property of the suggested parametrization is useful in applications.

## 5. Parametrization under Surface Construction

Let us consider the application of shape-preserving parametrization in the construction of surfaces by a discrete set of points given in the manner described in [22, chapter 8].

Let the domain $G : [c, d] \times [0, 1]$ in the $WU$ plane be divided into $N$ rectangular subdomains by the straight lines $w = w_i$, $i = 0, \ldots, N$, of the grid $\Delta_w : c = w_0 < w_1 < \cdots < w_N = d$. Suppose that on each of the lines $w = w_i$, the grid

$$\Delta_u^i : 0 = u_0^i < u_1^i < \cdots < u_{M_i}^i = 1, \quad i = 0, \ldots, N,$$

is given. The number of nodes and their position on the grids $\Delta_u^i$, $i = 0, \ldots, N$, are independent of one another. Cartesian surface coordinates $P_{ij} = (x_{ij}, y_{ij}, z_{ij})$ are given in the nodes $u_j^i$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$. The surface is constructed as a triple of 2-D splines

$$x = x(u, w), \quad y = y(u, w), \quad z = z(u, w)$$

on the corresponding coordinates using the methods of shape-preserving approximation of [22, chapter 8] based on generalized tension splines. Tension is introduced to improve the correspondence between the geometry of the surface and the initial data. It is now natural to choose the partitions $\Delta_w$ and $\Delta_u^i$, $i = 0, \ldots, N$, by applying shape-preserving parametrization.

**Algorithm 1.** *Construction of meshes* $\Delta_u^i$, $i = 0, \ldots, N$.

(i) *For each* $i = 0, \ldots, N$, *start with the points* $P_{ij}$, $j = 0, \ldots, M_i$, *and use the algorithm of shape-preserving parametrization to find a set of parameters* $\hat{\alpha}_j^i$, $j = 0, \ldots, M_i - 2$, *which gives the subsidiary mesh* $\hat{\Delta}_u^i : 0 = \hat{u}_0^i < \hat{u}_1^i < \cdots < \hat{u}_{M_i}^i = 1$.

(ii) *Consider the set* $\hat{u}_j^i$, $j = 0, \ldots, M_i$, *for fixed* $i$, $0 \le i \le N$, *as the values of a linear interpolating spline* $l_i$ *with knots on the uniform partition of the interval* $[0, 1]$ *with the step size* $h = 1/M_i$, *that is,*

$$l_i(j/M_i) = \hat{u}_j^i, \quad j = 0, \ldots, M_i.$$

*Then find the mesh* $\Delta_u$ *for each* $i$, $0 \le i \le N$, *from the following formula*

$$u_j^i = \frac{1}{N+1} \sum_{k=0}^{N} l_k(j/M_i), \quad j = 0, \ldots, M_i,$$

*as an arithmetic mean of all the subsidiary parametrizations.*

Clearly, $0 = u_0^i < u_1^i < \cdots < u_{M_i}^i = 1$ for all $i$. If $M_i = M_j$ then the meshes $\Delta_u^i$ and $\Delta_u^j$ coincide. Therefore, if $M_i = M$, $i = 0, \ldots, N$, then all the meshes $\Delta_u^i$ are the same.

The second step of this algorithm takes the "averaged" geometry of the initial points along all the cross-sections into account through the parametrization.

**Algorithm 2.** *Construction of the mesh* $\Delta_w$.

*Let* $P_i(u) = (x_i(u), y_i(u), z_i(u))$, $i = 0, \ldots, N$, *be a curve passing through the points* $P_{ij}$, $j = 0, \ldots, M_i$, *where the functions* $x_i(u), y_i(u), z_i(u)$ *are choosen to be linear interpolating splines on the mesh* $\Delta_u^i$. *Let us assume a certain fixed set of parameters* $\hat{u}_l$, $l = 0, \ldots, L$ *(*$L$ *is a sufficiently large number) is given. For example we can set* $\hat{u}_l = l/L$, $l = 0, \ldots, L$.

(i) *For each fixed* $l$, $0 \le l \le L$, *start with the points* $P_i(\hat{u}_i)$, $i = 0, \ldots, N$, *and*

(a) *obtain the cumulative chord length* $H_l$ *joining those points;*

(b) *use the algorithm of shape-preserving parametrization to find the set of parameters* $\alpha_{0,l}, \ldots, \alpha_{N-2,l}$ *assigning the subsidiary mesh*

$$0 = w_{0,l} < w_{1,l} < \cdots < w_{N,l} = H_l.$$

(ii) *Set*

$$w_i = \frac{1}{L+1} \sum_{l=0}^{L} w_{i,l}, \quad i = 0, \ldots, N.$$

*The mesh* $\Delta_w$ *is thus constructed. Evidently, its length is* $H = w_N - w_0 = \frac{1}{L+1} \sum_{l=0}^{L} H_l$.

**Remark 2.** Let us assume that the number of initial points is the same in all the cross-sections, that is, $M_i = M$, $i = 0, \ldots, N$. When the mesh points $\Delta_u = \Delta_u^i$ ($\hat{u}_l = u_l^i$, $l = 0, \ldots, M$) are used in Algorithm 2 as $\hat{u}_l$, the mesh $\Delta_w$ constructed by Algorithm 2 coincides with the mesh of Algorithm 1.

## 6. Graphical Examples

The figures below illustrate the employment of shape-preserving parametrization (sp-parametrization for short) in interpolations by the parametric cubic and generalized tension splines of [22, chapter 4] with the defining functions

$$\psi_i(t) = t^3/[1 + q_i t(1 - t)]Q_i, \quad Q_i^{-1} = 2(1 + q_i)(3 + q_i), \quad \varphi_i(t) = \psi_i(1 - t).$$

For the purpose of comparison, spline curves with centripetal, cumulative chord length, and uniform parametrizations, which are the most common, are given. These three methods are similar with the method suggested in this report in terms of implementation complexity and computer resources consumed. The

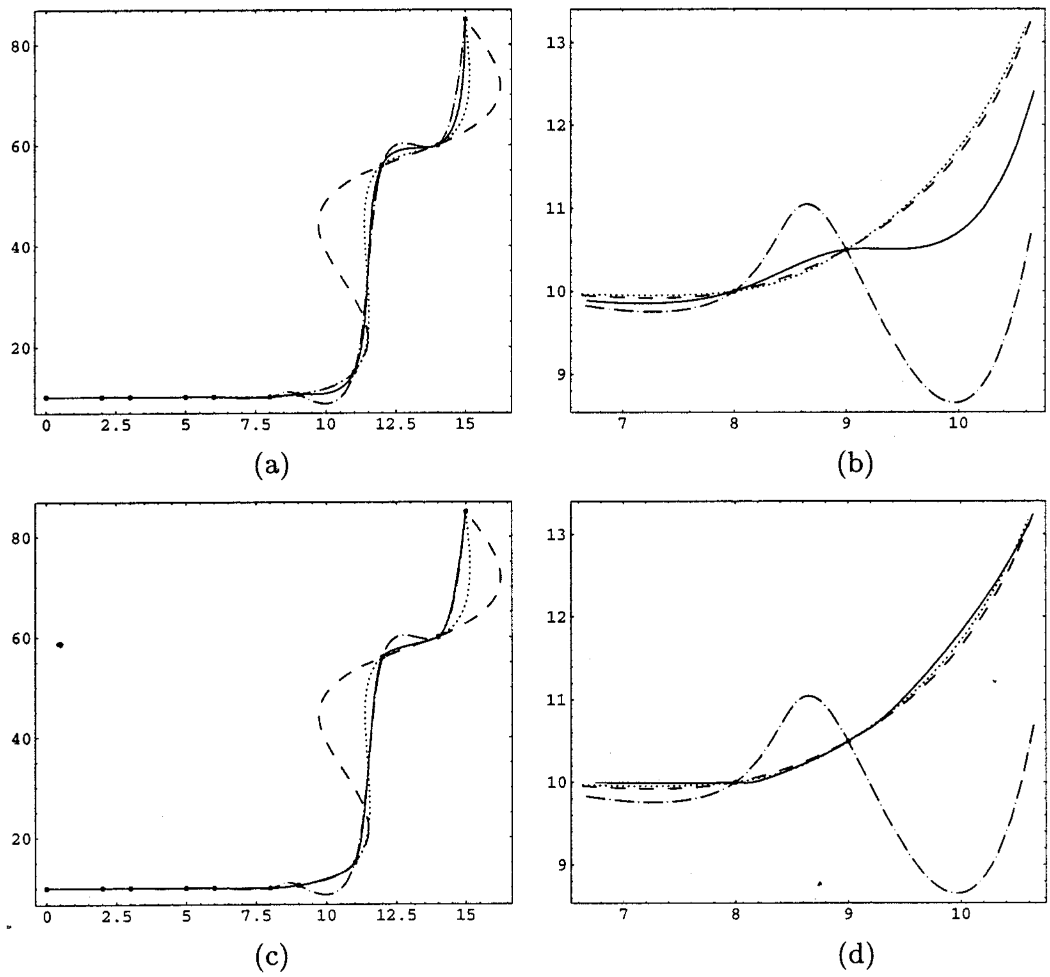


**Figure 1.** Akima's data with sharp gradient increase. (a) Interpolation using sp-, centripetal, cumulative chord length and uniform parametrizations ($q_i = 0$). (b) Magnification of the interval [7, 10]. (c), (d) The same as (a), (b) but with $q_0 = 1$, $q_1 = q_2 = 10$, $q_3 = 16$, $q_4 = 35$, $q_5 = 5.5$, $q_6 = 3.8$, $q_7 = 4$, $q_8 = 5$, $q_9 = 2$.

solid, dotted, dashed and dash-dotted lines show, respectively, the curves with sp, centripetal, cumulative chord length, and uniform parametrizations. The bullet signs denote the data points. In the construction of the cubic and generalized tension splines, endpoint conditions of type (10) were used where the derivatives were computed by means of the second degree Lagrange interpolating polynomials: $S'(t_0) = \mathbb{L}'_{0,2}(t_0)$ and $S'(t_N) = \mathbb{L}'_{N-2,2}(t_N)$.

As our first example we have interpolated Akima's [1] data of Table 1. The effects of using four different parametrizations are depicted in Figure 1. Figures 1(a) and 1(b) are obtained setting $q_i = 0$ for all $i$, that is considering the parametric cubic splines interpolating the data. Uniform and cumulative chord length parametrizations are utterly unsatisfactory. The graph of the spline with centripetal parametrization fails to show a one-to-one correspondence between the points of the $x$-axis and the curve. The spline with sp-parametrization has small oscillations along the data because the conditions 1 of Corollary 1 are violated. The magnification in Figure 1(b) shows this effect clearly. In Figures 1(c) and 1(d) new interpolants with tension parameters $q_0 = 1$, $q_1 = 11$, $q_2 = 10$, $q_3 = 16$, $q_4 = 35$, $q_5 = 5.5$, $q_6 = 3.8$, $q_7 = 4$, $q_8 = 5$, $q_9 = 2$ are displayed for the same data, and the stretching effect of the increase in tension parameters is evident.

**Table 1.** Data for Figure 1:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 0 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 | 15 |
| $y_i$ | 10 | 10 | 10 | 10 | 10 | 10 | 10.5 | 15 | 56 | 60 | 85 |

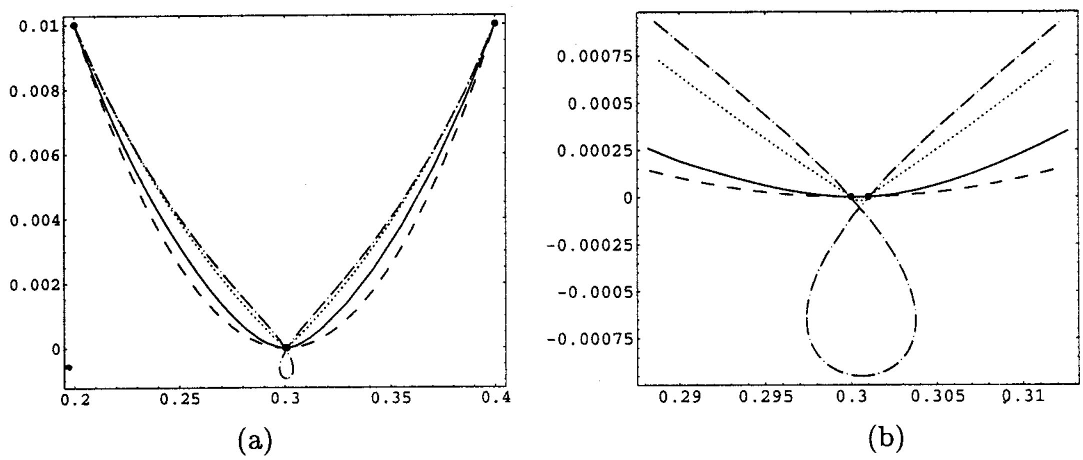


**Figure 2.** C. de Boor example. Parametrizations for the parabola $f(x) = (x - 0.3)^2$ with $x = 0, 0.1, 0.2, 0.3, 0.301, 0.4, 0.5, 0.6$ and tension parameters $q_0 = 1.5$, $q_1 = 2.5$, $q_2 = 0.75$, $q_3 = 1.12$, $q_4 = 7.5$, $q_5 = q_6 = 0$. (a) Magnification of the interval $[0.2, 0.4]$. (b) Magnification of the interval $[0.29, 0.31]$.

Figure 2 illustrates an example from C. de Boor's book [6]. The data points have been obtained from the function $f(x) = (x-0.3)^2$ with $x = 0, 0.1, 0.2, 0.3, 0.301, 0.4, 0.5, 0.6$. Figures 2(a) and 2(b) present magnifications of the intervals $[0.2, 0.4]$ and $[0.29, 0.31]$. We have used tension parameters $q_0 = 1.5$, $q_1 = 2.5$, $q_2 = 0.75$, $q_3 = 1.12$, $q_4 = 7.5$, $q_5 = q_6 = 0$. Here uniform and centripetal parametrizations give a loop and a cusp correspondently. The solution with sp-parametrization permits perfectly reproduce the data shape.

The data for Figure 3 (Table 2) has been taken from the book by Späth [33]. Figures 3(a) and 3(b) show the plots of the interpolating cubic splines produced by a uniform choice of tension parameters, namely, $q_i = 0$. In Figures 3(c) and 3(d), in

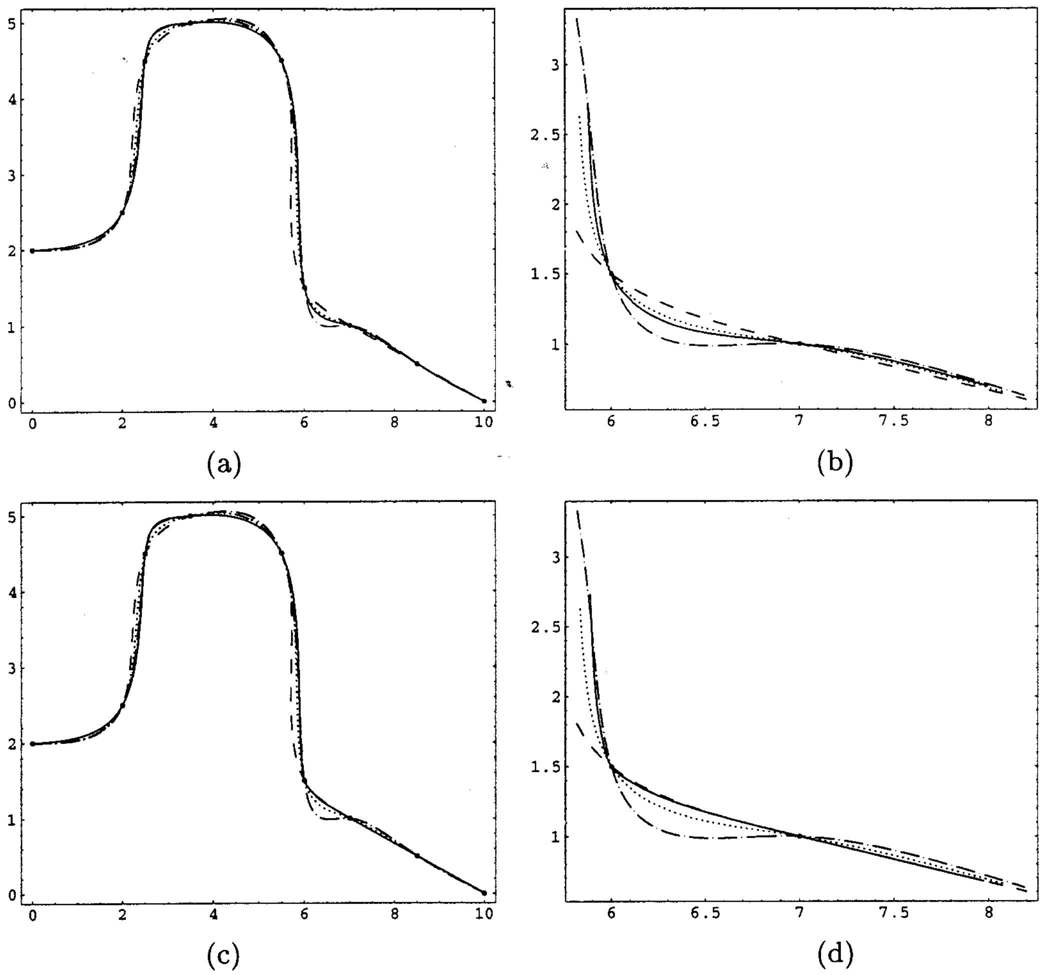

**Figure 3.** Späth's data. (a) Interpolation by parametric cubic splines. (b) Magnification of the interval [6, 8]. (c),(d) The same as (a),(b) but with $q_i = 10$, $i = 5, 6, 7$ while the remaining parameters $q_i$ are unchanged.

order to approximate the segment of a straight line passing through the last three data points, we have set $q_5 = q_6 = q_7 = 10$ while the remaining $q_i$ are unchanged.

The data for Figure 4 (Table 3) has been obtained from the function $f(x) = (x-5)^4+2$ with $x = 2.5+i$, $i = 0, \ldots, 5$, considered by Goodman and Unsworth [15]. We have used tension parameters $q_0 = 1.5$, $q_1 = 2.5$, $q_2 = 0.75$, $q_3 = 1.12$, $q_4 = 7.5$. Here cumulative chord length parametrization is unsatisfactory. Magnification of the interval [4, 6] shows that sp-parametrization preserves monotonicity of the data.

**Table 2.** Data for Figure 3:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 0 | 2 | 2.5 | 3.5 | 5.5 | 6 | 7 | 8.5 | 10 |
| $y_i$ | 2 | 2.5 | 4.5 | 5 | 4.5 | 1.5 | 1 | 0.5 | 0 |

**Table 3.** Data for Figure 4:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_i$ | 2.5 | 3.5 | 4.5 | 5.5 | 6.5 | 7.5 |
| $y_i$ | 41.0625 | 7.0625 | 2.0625 | 2.0625 | 7.0625 | 41.0625 |

Figure 5 illustrates the behaviour of the splines for the data selected by us on the

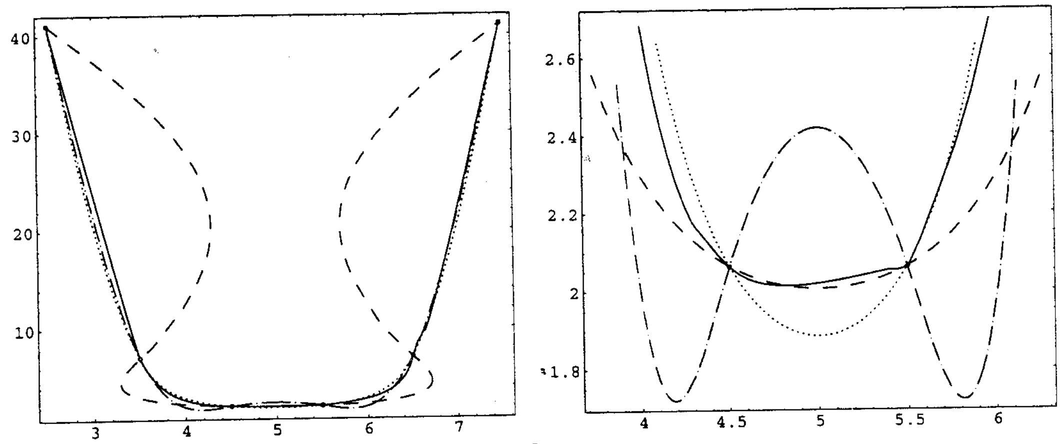


**Figure 4.** (a) Parametrizations for the data obtained from the function $f(x) = (x-5)^4 + 2$ with $x = 2.5 + i$, $i = 0, \ldots, 5$, and $q_0 = 1.5$, $q_1 = 2.5$, $q_2 = 0.75$, $q_3 = 1.12$, $q_4 = 7.5$. (b) Magnification of the interval $[4, 6]$.

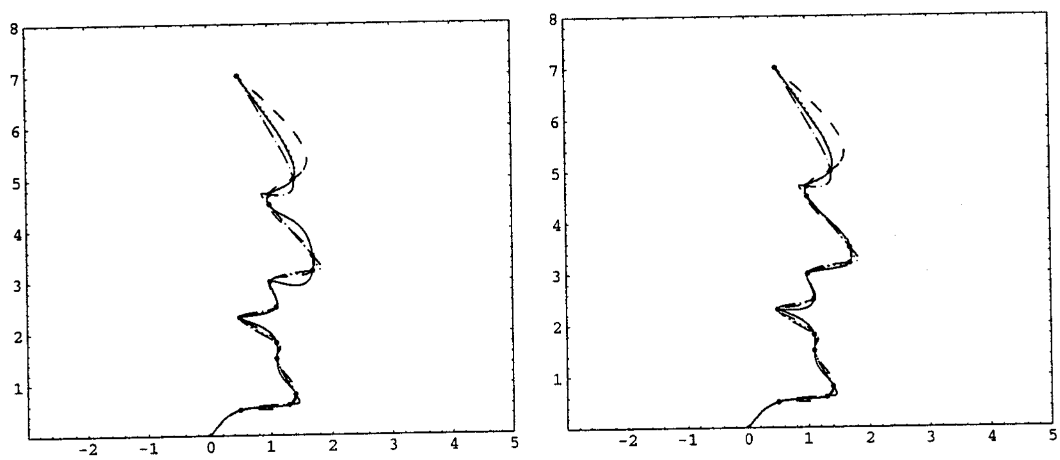


**Figure 5.** Example with the data of a "human face". (a) No tension. (b) The same as (a) but with tension parameters $q_7 = 2$, $q_8 = 18.5$, $q_9 = 1$, $q_{10} = 3.8$, $q_{11} = 0.8$ while the remaining parameters $q_i$ are unchanged.

plane using a drawing: $\{x\} = \{0, 0.5, 1.3, 1.4, 1.1, 1.1, 0.5, 1.1, 1, 1.7, 1.7, 1, 1, 1.4, 0.5\}$, $\{y\} = \{0, 0.5, 0.6, 0.8, 1.5, 1.8, 2.3, 2.5, 3, 3.2, 3.5, 4.5, 4.7, 5, 7\}$.. To produce Figure 5(a) we have used parametric cubic spline interpolants. Figure 5(b) is obtained setting $q_i = 0$, $i = 0, \ldots, 6$, $q_7 = 2$, $q_8 = 18.5$, $q_9 = 1$, $q_{10} = 3.8$, $q_{11} = 0.8$.

An example of a surface constructed by the algorithm of shape-preserving approximation of [22, chapter 8] with the uniform and shape-preserving (Algorithm 1) parametrizations is shown in Figures 6 and 7 respectively. The mesh in Figure 7 is characterized by a concentration of lines in the regions of "sharp gradient variation" in both directions which illustrates Remark 1 above. The improved correspondence between the surface geometry and the initial data under shape-preserving parametrization enables us to reduce the number of additional knots and the values of tension parameters introduced when splines are constructed by the algorithms of [22, chapter 8]. In this example, the number of additional knots

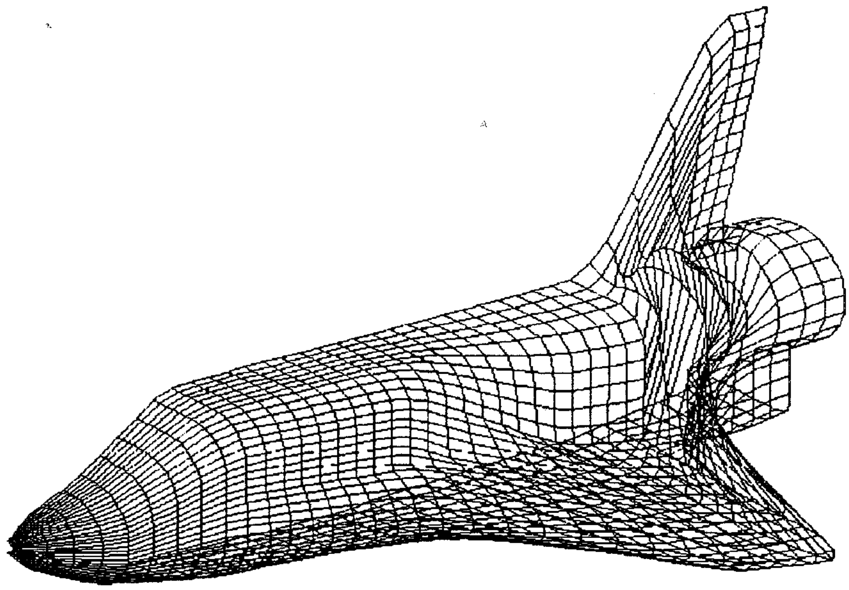

**Figure 6.** Jet's data. Uniform parametrization.

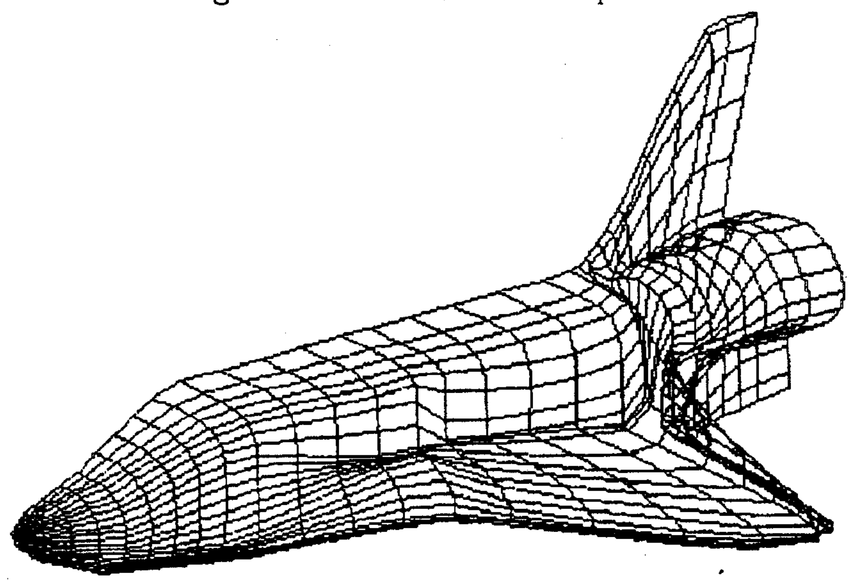

**Figure 7.** Jet's data. Shape-preserving parametrization gives a mesh concentration in the domains of rapid gradient growth.

of the spline in the $u$ direction is by approximately 20 percent (92 knots) lower than when uniform parametrization is used. This allowed us to reduce computer memory requirements.

## 7. Computer Programs for Shape-Preserving Surface Approximation

The methods of shape-preserving spline parametrization considered in this report were used in a package of computer programs in FORTRAN for the description of multivalued surfaces in three-dimensional space.

Let a topologically rectangular surface $\Pi$ be given, on which one has, in cartesian $xyz$-coordinate, a set of points $T_{ij} = (x_{ij}, y_{ij}, z_{ij}) \in \Pi$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$. A system of nonintersecting lines $L_i$, $i = 0, \ldots, N$, on $\Pi$ contains all points $T_{ij}$ and has the property that by means of some homeomorphic transformation of the surface $\Pi$ onto a rectangle $G : [c, d] \times [0, 1]$ in the $wu$-plane, the curves $L_i$ are transformed into a set of parallel straight lines $w = w_i$, $i = 0, \ldots, N$, connecting two opposite sides of the rectangle.

In the computer program, meshes $\Delta_u^i : 0 = u_0^i < u_1^i < \cdots < u_{M_i}^i = 1$, $i = 0, \ldots, N$ are first choosen with step $l_i = 1/M_i$ and then are optimized according to the algorithm of shape-preserving parametrization of this report. The $w$-axis coincides with the $x$-axis. For the mesh $\Delta_w : c = w_0 < w_1 < \cdots < w_N = d$, under the assumption $x_{00} < x_{10} < \cdots < x_{N0}$, one takes $w_i = x_{i0}$, $i = 0, \ldots, N$.

At each of the given points $T_{ij}$ on the surface $\Pi$, a quantity $\varepsilon_{ij} = (\varepsilon_{ij}^x, \varepsilon_{ij}^y, \varepsilon_{ij}^z)$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$ is fixed which specifies an admissible tolerance of the approximation spline $S(w, u)$ at the point which in turn is a triple $(S^x(w, u), S^y(w, u), S^z(w, u))$.

The two-dimensional spline $S(w, u)$ is in the class $C^2$ for fixed $u$, and is a twice continuously differentiable of $u$ for fixed $w$, preserving the angles and the non-smoothness in the data according to the algorithm of [22, chapter 5].

As a final result, the computer package generates the values of the spline $S(w, u)$ in the nodes of a regular mesh $\Delta = \tilde{\Delta}_w \times \tilde{\Delta}_u$, where $\tilde{\Delta}_w : c \le \tilde{w}_0 < \tilde{w}_1 < \cdots < \tilde{w}_{\tilde{N}} \le d$ and $\tilde{\Delta}_u : 0 \le \tilde{u}_0 < \tilde{u}_1 < \cdots < \tilde{u}_{\tilde{M}} \le 1$.

Let us formulate the main steps in the calculation of the values of the spline $S(u, u)$.

**Step 1.** Construct the one-dimensional interpolating splines $S_i(u)$ along the sections $w = w_i$, $i = 0, \ldots, N$. For this, the computer program package RSPIZG for shape-preserving interpolation by generalized tension splines of [22, chapter 5] is used. Then we shall turn to a basis of GB-splines.

**Step 2.** Calculate the parameters $p_i$, $q_i$ on the mesh $\Delta_w$.

Calculation of the parameters $p_i$, $q_i$ for the curve $L_i$ is performed in a loop over all points of the initial data $T_{ij}$, $j = 0, \ldots, M_i$, $i = 0, \ldots, N$.

(a) First we take $p_i = q_i = 0$, $i = 0, \ldots, N$.

(b) For the mesh $T_{ij}$ with fixed $i$ $(2 \le i \le N - 2)$ we calculate the values of the generalized splines $S_k(u_j)$, $k = i - 2, \ldots, i + 2$, $j = 0, \ldots, M_i$. This permits, considering the local approximation spline in the variable $w$ and using the shape-preserving conditions in [22, chapter 8], to find the values of the parameters $p_{ij}$, $q_{ij}$.

If $i = 0, 1$ $(i = N - 1, N)$ we calculate the values $S_k(u_j)$, $k = 0, \ldots, 3$ $(k = N - 3, \ldots, N)$, $j = 0, \ldots, M_i$. Using quadratic and cubic Lagrange polynomials in the variable $w$ we find the values of the first derivative of the spline on the boundary (see (8.28) in [22]). Then using the formulae for the coefficients (see (8.27) in [22]), we construct a local approximating spline in $w$ and analogously find the parameters $p_{ij}$, $q_{ij}$ for $i = 0, 1, N-1, N$.

Finally set $p_i = \max p_{ij}$, $q_i = \max q_{ij}$, $j = 0, \ldots, M_i$.

(c) Set $h_j = h_0$, $p_j = q_j = q_0$, $j = -2, -1$; $h_j = h_{N-1}$, $p_j = q_j = p_{N-1}$, $j = N, N+1$.

**Step 3.** For each $\tilde{u}_j$, $j = 0, \ldots, \tilde{M}$ find the coefficients of generalized spline

$$S_j(w) = \sum_{k=-1}^{N+1} \beta_k B_k(w), \quad \beta_k = b_k(\tilde{u}_j).$$

**Step 4.** Find the values of splines $S_j(w)$, $j = 0, \ldots, \tilde{M}$ in the nodes of the mesh $\tilde{\Delta}_w$.

A call to the main program is:

CALL GEOM1(N, M, N0, T, EPS, NN, WN, MN, XN, YN, ZN).

Input data:

N is the number of sections in $w$.

M is an array of dimension N, where M($i$) is the number of points in the $i$th section.

N0 $= \sum_{i=1}^{N}$M($i$) is the total number of the initial data.

T is a two-dimensional array of size N0×3, which is a linear list of coordinates of the initial data ordered sequentially by $x$, $y$, $z$. In particular, T(1, 1),..., T(M(1),1) are the $x$-coordinates of points for the first section. T(M(1) + 1, 1),...,T(M(1)+M(2),1) are the $x$-coordinates of points for the second section, T(N0−M(N)+1,1),...,T(N0,1) are the $x$-coordinates of the points for the $N$th section, etc.

EPS is an array of length N0 of given tolerances of the approximating spline at the initial points. We assume that the tolerances in all three coordinates are the same.

NN is the number of nodes of the mesh $\tilde{\Delta}_w$.

WN is an array of NN coordinates of the mesh $\tilde{\Delta}_w$.

MN is the number of nodes of the uniform mesh $\tilde{\Delta}_u$.

Output data:

XN, YN, ZN are arrays of MN×NN elements of the cartesian coordinates of the points of the surface in the knots of the mesh $\Delta = \tilde{\Delta}_w \times \tilde{\Delta}_u$.

This package of computer programs has a modular structure and is based on calls to the one-dimensional programs of shape-preserving interpolation [22, chapter 5], shape-preserving local approximation [22, chapter 8] and shape-preserving parametrization described in this report.

# References


1. H. Akima, A new method of interpolation and smooth curve fitting based on local procedures, *J. Assoc. Comput. Mach.* **17** (1970) 589–602.
2. L. Alt, Parametrization for data approximation, in: *Curves and Surfaces*, eds. P. J. Laurent, A. Le Méhauté, and L. L. Schumaker (Academic Press, Boston, 1991), pp. 1–4.
3. R. K. Beatson and E. Chacko, A quantative comparison of end conditions for cubic spline interpolation, in: *Approximation Theory VI: Proceedings of the Sixth International Symposium on Approximation Theory.* Vol. I, eds. C. K. Chui, L. L. Schumaker and J. D. Ward (Academic Press, Boston, 1989), pp. 77–79.
4. W. Böhm, Parameterdarstellung, kubischer und bikubischer splines, *Computing* **17** (1976) 87–92.
5. E. Cohen, T. Lyche, and R. Riesenfeld, Discrete B-splines and subdivision techniques in computer aided geometric design and computer graphics, *Comp. Graphics and Image Proc.* **14** (1980) 87–111.
6. C. De Boor, *A Practical Guide to Splines* (Springer Verlag, New York, 1978).
7. W. L. F. Degen, Best approximation of parametric curves by splines, in: *Mathematical Methods in Computer Aided Geometric Design II*, eds. T. Lyche and L. L. Schumaker (Academic Press, New York, 1992) pp. 171–184.
8. M. P. Epstein, On the influence of parametrization in parametric interpolation, *SIAM J. Numer. Anal.* **13** (1976) 261–268.
9. G. Farin, *Curves and Surfaces for Computer Aided Geometric Design. A Practical Guide* (Academic Press, San Diego, 1993).
10. T. A. Foley, Interpolation with interval and point tension control using cubic weighted $\nu$-splines, *ACM Trans. Math. Soft.* **13** (1987) 68–96.
11. T. A. Foley, T. N. T. Goodman, and K. Unsworth, An algorithm for shape preserving parametric interpolation curves with $GC^2$ continuity, in: *Mathematical Methods in Computer Aided Geometric Design*, eds. T. Lyche and L. L. Schumaker (Academic Press, New York, 1989), pp. 249–259.
12. T. A. Foley and G. M. Nielson, Knot selection for parametric spline interpolation, in: *Mathematical Methods in Computer Aided Geometric Design*, eds. T. Lyche and L. L. Schumaker (Academic Press, New York, 1989), pp. 261–271.
13. T. N. T. Goodman, Shape preserving interpolation by parametric rational cubic splines, in: *Numerical Mathematics Singapore'1988*, eds. R. P. Agarwal, Y. M. Chow, and S. J. Wilson (International Series of Numerical Mathematics, Vol. 86, Birkhäuser, Basel, 1988), pp. 149–158.
14. T. N. T. Goodman and K. Unsworth, Shape preserving interpolation by curvature continuous parametric curves, *Comput. Aided Geom. Design* **5** (1988) 323–340.

15. T. N. T. Goodman and K. Unsworth, Shape-preserving interpolation by parametrically defined curves, *SIAM J. Numer. Anal.* **25** (1988) 1453–1465.
16. G. Greiner, Variational design and fairing of spline surfaces, *Computer Graphics Forum* **13** (1994) 144–154.
17. P. J. Hartley and C. J. Judd, Parametrization and shape of B-spline curves for CAD, *Computer-Aided Design* **12** (1980) 235–238.
18. J. Hoschek, Intrinsic parametrization for approximation, *Comput. Aided Geom. Design* **5** (1988) 27–31.
19. P. D. Kaklis and N. S. Sapidis, Preserving interpolatory parametric splines of non-uniform polynomial degree, *Comput. Aided Geom. Design* **12** (1995) 1–26.
20. E. Yu. Kurchatov and V. F. Snigirev, Best choice of spline knots in automation of contour design, in: *Mathematical and Experimental Methods of Technical Systems Synthesis*, Kazan, 1989, pp. 38–43 (in Russian).
21. B. I. Kvasov, GB-splines and their properties, *Annals of Numerical Mathematics* **3** (1996) 139–149.
22. B. I. Kvasov, *Methods of Shape-Preserving Spline Approximation* (World Scientific Publ. Co. Pte. Ltd., Singapore, 2000).
23. E. T. Y. Lee, Choosing nodes in parametric curve interpolation, *Computer Aided Design* **21** (1989) 363–370.
24. E. T. Y. Lee, Energy, fairness and a counterexample, *Computer-Aided Design*, **22** (1990) 37–40.
25. E. T. Y. Lee, Corners, cusps, and parametrization: Variations on a theorem of Epstein, *SIAM J. Numer. Anal.* **29** (1992) 553–565.
26. E. T. Y. Lee, On a class of data parametrizations: Variations on a theme of Epstein, II, in: *Mathematical Methods in Computer Aided Geometric Design II*, eds. T. Lyche and L. L. Schumaker (Academic Press, New York, 1992), pp. 381–390.
27. S. P. Marin, An approach to data parametrization in parametric cubic spline interpolation problems, *J. Approx. Theory* **41** (1984) 64–86.
28. V. L. Miroshnichenko, Convex and monotone spline interpolation, in: *Constructive Theory of Functions'84*, Sofia, 1984, pp. 610–620.
29. G. M. Nielson and T. A. Foley, A survey of applications of an affine invariant norm, in: *Mathematical Methods in Computer Aided Geometric Design*, eds. T. Lyche and L. L. Schumaker (Academic Press, New York, 1989), pp. 445–467,
30. D. F. Rogers and J. A. Adams, *Mathematical Elements for Computer Graphics* (McGraw-Hill Publ. Comp., New York, 1990).
31. N. Sapidis and G. Farin, Automatic fairing algorithm for B-spline curves, *Computer-Aided Design* **22** (1990) 121–129.
32. C. Seymour and K. Unsworth, Interactive shape preserving interpolation by curvature continuous rational cubic splines, *J. Comput. Appl. Math.* **102** (1999) 87–117.

33. H. Späth, *Spline Algorithms for Curves and Surfaces* (Utilitas Mathematica Publishing, Inc., Winnipeg, 1974).

34. C. Y. Wang, Shape classification of the parametric cubic curve and parametric B-spline cubic curve, *Computer-Aided Design* **13** (1981) 199–206.

35. C. D. Woodward, B2-splines: a local representation for cubic spline interpolation, *The Visual Computer* **3** (1987) 152–161.

36. Yu. S. Zavyalov, B. I. Kvasov, and V. L. Miroshnichenko, *Methods of Spline Functions* (Nauka, Moscow, 1980, in Russian).

# CURRICULUM VITAE

## Boris I. KVASOV

## PRESENT ADDRESS:

Suranaree University of Technology
School of Mathematics
111, University Avenue, Muang District
Nakhon Ratchasima 30000, Thailand
Tel.: (66-44) 224283, Fax: (66-44) 224185
Email: boris@math.sut.ac.th

## DEGREES:

D.Sc., Habilitation Degree in Computational Mathematics (CM for short),
Russian Academy of Sciences (RAS for short), Moscow, 1997
Senior Researcher Degree in CM, RAS, Moscow, 1986
Associate Professor Degree in CM, Ministry of Higher Education, Moscow, 1986
Ph.D. in CM, Institut of Mathematics RAS, Novosibirsk, 1973
M.Sc. in CM, Novosibirsk State University (NSU for short), Novosibirsk, 1969

## POSITIONS:

| | |
|---|---|
| 1995/to date | Suranaree University of Technology, Associate Professor of Mathematics |
| 1991–1995 | Institute of Computational Technology RAS, Senior Researcher, Leading Researcher |
| 1978–1991 | Institute of Theoretical and Applied Mechanics RAS, Senior Researcher |
| 1976–1978 | Institute of Mathematics Belorus Acad. of Sci., Gomel, Head of Laboratary |
| 1974–1976 | Institute of Mathematics RAS, Novosibirsk, Assistant Researcher |
| 1975 (2 mo.) | Stefan Banach Mathematical Center, Warsaw, Poland, Visiting Research Fellow |

## ACADEMIC EXPERIENCE:

| | |
|---|---|
| 1996 (1 mo.) | University degli Studi di Firenze, Italy, Visiting Professor of Mathematics |
| 1985–1995 | NSU, Associate Professor of CM |
| 1978–1985 | NSU, Assistant Professor of CM |
| 1983 (2 mo.) | Grodno State University, Visiting Assistant Professor of CM |
| 1976–1978 | Gomel State University, Senior Lecturer |
| 1974–1976 | NSU, Lecturer |
| 1972–1974 | Krasnoyarsk State University, Senior Lecturer |

Graduate/Postgraduate students: I supervised 15 M.Sc. and 2 Ph.D. in CM.

## PRINCIPAL RESEARCH INTERESTS:

Numerical Analysis, Mathematical Methods in CAGD, Approximation Theory,
Spline Based Curve and Surface Approximation, Scientific Visualization

## CURRENT RESEARCH INTERESTS:

Shape Preserving Approximation, Subdivisions, Difference Methods for Constructing Splines,
Tension and Discrete Splines and GB-splines, Curve and Surface Parametrization

## RESEARCH GRANTS:

*Difference Method for Constructing Shape Preserving Splines* / Principal Investigator, The Thailand Research Fund, Thailand, November 1, 1999 to October 31, 2001 (code BRG/08/2543)

*Shape-Preserving Parametrization for Spline Interpolation* / Principal Investigator, Suranaree University of Technology, Thailand, October 1, 1999 to April 30, 2001

*Discrete B-Spline Approximation through Lagrange-Newton Polynomials* / Principal Investigator, Suranaree University of Technology, Thailand, October 1, 1998 to December 31, 1999

*Algorithms of Shape-Preserving Spline Approximation* / Coinvestigator, The Thailand Research Fund, Thailand, July 1, 1997 to June 30, 1999 (code BRG/16/2540)

*Difference Method for Construction Tension Splines* / Principal investigator, MURST, Università Degli Studi di Firenze, Italy, 20 November to 20 December 1996

*Geometric Splines for Curves and Surfaces Design* / Principal Investigator, State Committee on Higher Education, Saint-Petersburg Technical University, January 1994 to December 1995 (code PG-13)

*Shape-Preserving Approximation for Curves and Surfaces* / Principal Investigator, The Russian Foundation for Basic Research, RAS, Moscow, January 1993 to December 1995 (code 93–012–495)

## PROFESSIONAL ACTIVITIES:

Thailand Research Fund's Royal Golden Jubilee Ph.D. Grantee, 1998, 2000

Organizer, Third All-union Conference on Approximation Theory and Problems of CM, Novosibirsk, January 28 - February 1, 1991

Guest Editor, *Aproximation Theory and Problems of CM*, Modelirovanie v Mekh., **5(22)** (1991) No. 5

The USSR Government State Prize in Science and Technology Semifinalist, 1986

Referee for Zh. Vychisl. Mat. i Mat. Fiz., Moscow; Numerical Algorithms, France

Reviewer for Mathematical Reviews (in the field of Numerical Analysis, 1975-1985)

## CONFERENCE PRESENTATIONS: (recent years)

Approximation by discrete GB-splines, Tenth International Conference on Approximation Theory, Saint Louis, USA, 26–29 March, 2001

Approximation by Lagrange splines, Fourth Annual National Symposium on Computational Science and Engineering, Kasetsart University, Bangkok, March 27–29, 2000

On generalized discrete tension splines, Computational Techniques and Applications Conference and Workshops, Canberra, Australian National University, September 20–24, 1999

A general approach to discrete splines, Third Annual National Symposium on Computational Science and Engineering, Chulalongkorn University, Bangkok, March 24–26, 1999

A general approach to discrete splines, Conference on Surface Approximation and Visualization, University of Canterbury, New Zealand, February 15–18, 1999

## INVITED LECTURES:

- National University of Singapore, Singapore, 1998
- Suranaree University of Technology, Nakhon Ratchasima, Thailand, 1995, 1997
- Università degli Studi di Firenze, Italy, 1995, 1996
- Università di Milano, Italy, 1996
- Università di Siena, Italy, 1996
- Institutt for Informatikk, Universitetet i Oslo, Norway, 1991

## LANGUAGE PROFICIENCY:

Fluent in English and Russian. Read, understand and can translate from German and French into English and Russian. I translated from English and French into Russian four books on splines and the finite element method: (Publ. House "Mir", Moscow)

1. Laurent, P. J., *Approximation et Optimisation*, 1975, 496 pp.
2. Descloux, J., *Methode des Elements Finis*, 1976, 96 pp.
3. Ciarlet, Ph., *The Finite Element Method for Elliptic Problems*, 1980, 512 pp.
4. Zienkiewicz, O. C. and K. Morgan, *Finite Elements and Approximation*, 1986, 320 pp.

## SCIENTIFIC PUBLICATIONS: Two books and more than 70 research publications.