Lecture Notes in

Real Analysis I

(Math 103 621)

Eckart Schulz

School of Mathematics
Suranaree University of Technology

August, 2547

Contents

Chapter 1

Metric Spaces

1.1 Normed Spaces

In the following, let X denote a vector space over the field $\mathbb{R}$ (respectively, $\mathbb{C}$). Then X is called a *real vector space* (respectively, *complex vector space*), and the elements of the underlying field $\mathbb{R}$ (respectively $\mathbb{C}$) are called *scalars.*

If X is a vector space, we will usually denote its elements by simple letters, such as x, y, or f, g. Scalars will be denoted by Greek letters, $\alpha, \beta, \ldots$. This notation is ambiguous at times, for example, 0 can denote both the zero vector and the number zero. It usually is clear from the context, however, what is meant. We choose not to introduce special symbols for vectors, because this is inconvenient when writing by hand. In some of the examples we may use different notation though.

1.1.1 Review of Concepts from Linear Algebra

First let us review of concepts from linear algebra:

1. Given finitely many vectors $x_1, \ldots, x_n \in X$ and scalars $\alpha_1, \ldots, \alpha_n$, the vector

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

   is called a *linear combination* of $x_1, \ldots, x_n$.

2. A finite subset $S = \{x_1, x_2, \ldots, x_n\}$ of X is called *linearly independent,* if whenever

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$$

   then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. That is, the only linear combination of $x_1, x_2, \ldots, x_n$ which is zero is the trivial linear combination.

   The concept of linear independence can be generalized to infinite sets:

3. An arbitrary subset S of X is called *linearly independent,* if every finite subset M of S is linearly independent. A subset S of X which is not linearly independent is called *linearly dependent.* Thus S is linearly dependent iff there exist

finitely many vectors $x_1, \ldots, x_n$ in X and scalars $\alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0.$$

4. Let $X \neq \{0\}$. Then X is called *finite dimensional* if there exists $n \in \mathbb{N}$ such that

   (a) there exists a linearly independent subset

   $$S_o = \{x_1, \ldots, x_n\}$$

   of X of cardinality n,

   (b) any subset

   $$\{y_1, \ldots, y_{n+1}\}$$

   of X of cardinality $n+1$ is linearly dependent.

   The number n is unique and called the *dimension* of X. The set S_o is called a *basis* of X. Note that a basis is not unique. If such an n does not exist, then X is called *infinite dimensional*.

1.1.2 Definition of a Normed Linear Space

Throughout, X will be a finite dimensional or infinite dimensional, real or complex vector space.

Definition 1.1.1. Let X be a real or complex vector space. A *norm* on X is a function

$$\|\cdot\| : X \to \mathbb{R}$$

satisfying

(N1)	$\|x\| \geq 0$	(positive)
(N2)	$\|x\| = 0 \quad \Leftrightarrow \quad x = 0$	(definite)
(N3)	$\|\alpha x\| = \|\alpha\| \, \|x\|$	(positive homogeneous)
(N4)	$\|x + y\| \leq \|x\| + \|y\|$	(triangle inequality)

for all $x, y \in X$ and scalars α. The pair $(X, \|\cdot\|)$ is called a *normed linear space*, abbreviated n.l.s.

Remark 1.1.1. 1. If $(X, \|\cdot\|)$ is a normed linear space, we often simply say that "X is a normed linear space".

2. If condition $(N2)$ does not hold, then $\|\cdot\|$ is called a *semi norm* on X. Note that by condition (N3), $x = 0$ implies $\|x\| = 0$.

3. If $X = \mathbb{R}^n$ with the usual vector norm, then the triangle inequality can be visualized geometrically as indicated in figure 1.1.

Figure 1.1: The triangle equality for norms: $\|x+y\| \leq \|x\| + \|y\|$.

4. The triangle inequality generalizes to any finite number of vectors. If $x_1, \ldots, x_n \in X$, then
$$\|x_1 + x_2 + \cdots + x_n\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_n\|.$$
This can be proved by induction on n.

5. For all $x, y \in X$, we have by (N3),
$$\|-x\| = \|(-1)x\| = |-1|\, \|x\| \underset{\text{(N3)}}{=} \|x\|$$
and hence, replacing x by $x - y$,
$$\|y - x\| = \|-(x-y)\| = \|x - y\|. \tag{1.1}$$

6. From (N3) and (N4) we can derive a *second triangle inequality.* In fact, for all $x, y \in X$,
$$\|x\| = \|y + (x - y)\| \underset{\text{(N4)}}{\leq} \|y\| + \|x - y\|$$
so that
$$\|x\| - \|y\| \leq \|x - y\|. \tag{1.2}$$
Since 1.2 holds for all x and y, we may exchange x and y, and obtain
$$\|y\| - \|x\| \leq \|y - x\|$$
or extracting a minus sign on both sides,
$$-\big(\,\|x\| - \|y\|\,\big) \leq \|-(y-x)\| = \|x - y\|. \tag{1.3}$$
Combining (1.2) and (1.3) we obtain
$$\Big|\,\|x\| - \|y\|\,\Big| = \max\big\{\,\|x\| - \|y\|,\ -\big(\,\|x\| - \|y\|\,\big)\,\big\} \leq \|x - y\|.$$
That is
$$\text{(N4')} \qquad \Big|\,\|x\| - \|y\|\,\Big| \leq \|x - y\|.$$

1.1.3 Examples of Normed Linear Spaces

Example 1.1.1. It is easy to see that if $X = \mathbb{R}$, then the absolute value $|x|$ has the properties of a norm. That is, $(\mathbb{R}, |\cdot|)$ is a normed linear space.

Example 1.1.2. (Euclidean/unitary spaces). Let $X = \mathbb{R}^n$, or $X = \mathbb{C}^n$. Given $\vec{x} = (x_1, \ldots, x_n) \in X$, set

$$\|\vec{x}\|_1 := \sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + \cdots + |x_n| \qquad (L^1\text{-norm})$$

$$\|\vec{x}\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} \qquad (L^2\text{-norm, or Euclidean norm if } X = \mathbb{R}, \text{ unitary norm if } X = \mathbb{C})$$

$$\|\vec{x}\|_\infty := \max_{i=1\ldots n} |x_i| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}. \qquad (L^\infty\text{-norm, or maximum norm})$$

The spaces $(\mathbb{R}^n, \|\cdot\|_2)$ are called *Euclidean spaces*, and the spaces $(\mathbb{C}^n, \|\cdot\|_2)$ *unitary spaces.*

Exercise 1.1.1. Show that $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are indeed norms on $\mathbb{R}^n$ and $\mathbb{C}^n$. (Hint: To prove (N4) for $\|\cdot\|_2$, proceed as follows:

1. Show that $2ab \leq a^2 + b^2 \ \forall a, b \in \mathbb{R}$.
2. Show that $\sum_{i=1}^n |x_i|\,|y_i| \leq 1$ if $\sum_{i=1}^n |x_i|^2 = \sum_{i=1}^n |y_i|^2 = 1$.
3. Show that $\sum_{i=1}^n |x_i|\,|y_i| \leq \|\vec{x}\|_2 \|\vec{y}\|_2 \ \forall \vec{x} = (x_1, \ldots, x_n).\ \vec{y} = (y_1, \ldots, y_n) \in X$.
4. Prove (N4).

Also note that the proofs for $\mathbb{R}^n$ and $\mathbb{C}^n$ are essentially the same.)

Definition 1.1.2. Two norms $\|\cdot\|$ and $\|\cdot\|_o$ on a linear space X are said to be equivalent, if there exists constants $a, b > 0$ such that

$$a\|x\| \leq \|x\|' \leq b\|x\|$$

for all $x \in X$.

Exercise 1.1.2. Let $X = \mathbb{R}^n$ or $X = \mathbb{C}^n$. Show:

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$$

for all $x \in X$. Then show that any two of the three norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.

Recall from linear algebra: If X is any set, then set set of real valued (respectively complex valued) functions defined on X,

$$V_{(X,\mathbb{R})} := \{f : X \to \mathbb{R}\}, \quad \text{(respectively } V_{(X,\mathbb{C})} := \{f : X \to \mathbb{C}\})$$

is a real (respectively complex) vector space, with vector space operations $f+g$ and αf defined pointwise,

$$(f+g)(x) = f(x) + g(x)$$
$$(\alpha f)(x) = \alpha f(x)$$

for $f, g \in V_{(X,\mathbb{R})}$ (respectively $V_{(X,\mathbb{C})}$) and α scalar. To simplify notation, and since most proofs for the real and complex spaces are identical, we will use the notation V_X to denote either of $V_{(X,\mathbb{R})}$ and $V_{(X,\mathbb{C})}$. (Careful: X here denotes the domain of the functions f, and is not a vector space !)

Many interesting normed linear spaces arise as subspaces of the spaces V_X for various choices of X. In the next two examples, we will consider subspaces of $V_{[a,b]}$ and $V_{\mathbb{N}}$, respectively.

Recall that in order to show that a subset $W \subset V_X$ is a vector space, we only need to verify that it is closed under vector space operations, that is, we need to show that whenever $f, g \in W$ and α is scalar, then $f+g \in W$ and $\alpha f \in W$.

Example 1.1.3. (The space $C[a,b]$). Given a closed interval $[a,b]$, let us set

$$C([a,b],\mathbb{R}) := \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$$
$$C([a,b],\mathbb{C}) := \{ f : [a,b] \to \mathbb{C} \mid f \text{ is continuous} \}$$

with

$$\|f\|_u := \max_{x\in[a,b]} |f(x)|.$$

Note that $\|f\|_u$ exists by the Extreme Value Theorem.

For simplicity, let us denote both of the above spaces by $C[a,b]$. We need to show that $C[a,b]$ is a vector space, and that $\|\cdot\|_u$ is a norm.

In fact, from basic analysis we know that if f, g are continuous on $[a,b]$, and α is any constant, then $f+g$ and αf are also continuous. Hence, $C[a,b]$ is a subspace of $V_{[a,b]}$.

To check that $\|\cdot\|_u$ is a norm, let $f, g \in C[a,b]$ be arbitrary, and α a scalar.

(N1): Since $|f(x)| \geq 0$ for all $x \in [a,b]$, it follows that $\|f\|_u \geq 0$. Thus, (N1) holds.

(N2): We have

$$\begin{aligned} \|f\|_u = 0 \quad &\Leftrightarrow \quad \max_{x\in[a,b]} |(f(x)| = 0 \\ &\Leftrightarrow \quad |f(x)| = 0 \quad \forall x \in [a,b] \\ &\Leftrightarrow \quad f(x) = 0 \quad \forall x \in [a,b] \\ &\Leftrightarrow \quad f = 0 \quad \text{in } C[a,b]. \end{aligned}$$

Thus, (N2) holds.

(N3): Note that

$$\begin{aligned}\|\alpha f\|_u &= \max_{x\in[a,b]} |(\alpha f)(x)| = \max_{x\in[a,b]} |\alpha f(x)| \\ &= \max_{x\in[a,b]} |\alpha||f(x)| = |\alpha| \max_{x\in[a,b]} |f(x)| \\ &= |\alpha| \, \|f\|_u,\end{aligned}$$

and hence (N3) holds.

(N4): Finally,

$$\begin{aligned}\|f+g\|_u &= \max_{x\in[a,b]} |(f+g)(x)| = \max_{x\in[a,b]} |f(x)+g(x)| \\ &\le \max_{x\in[a,b]} \big(|f(x)| + |g(x)|\big) \\ &\le \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)| \\ &= \|f\|_u + \|g\|_u\end{aligned}$$

so that (N4) holds as well.

We have thus shown that $\big(C[a,b], \|\cdot\|_u\big)$ is a normed linear space.

Exercise 1.1.3. Let

$$\begin{aligned}C_b(\mathbb{R}) &:= \big\{f:\mathbb{R}\to\mathbb{C} \;\big|\; f \text{ is continuous and bounded}\big\} \\ C_o(\mathbb{R}) &:= \big\{f:\mathbb{R}\to\mathbb{C} \;\big|\; f \text{ is continuous and } \lim_{|x|\to\infty} f(x)=0\big\}.\end{aligned}$$

$C_o(\mathbb{R})$ is called the *set of continuous functions vanishing at infinity.* Also, define the *uniform norm* on $C_b(\mathbb{R})$ by

$$\|f\|_u := \sup_{x\in\mathbb{R}} |f(x)| \qquad \forall f\in C_b(\mathbb{R}).$$

1. Show that $C_b(\mathbb{R})$ and $C_o(\mathbb{R})$ are vector spaces.
2. Show that $\|f\|_u$ is indeed a norm on $C_b(\mathbb{R})$.
3. Show that $C_o(\mathbb{R}) \subset C_b(\mathbb{R})$.
4. Show that when $f\in C_b(\mathbb{R})$, then $\max_{x\in\mathbb{R}} |f(x)|$ need not exist.
5. Show that when $f\in C_o(\mathbb{R})$, then $\|f\|_u = \max_{x\in\mathbb{R}} |f(x)|$.

Example 1.1.4. (The space ℓ^∞). Let us set

$$\begin{aligned}\ell^\infty_{\mathbb{R}} &:= \big\{\, f:\mathbb{N}\to\mathbb{R} \;\big|\; f \text{ is bounded}\big\} \\ \ell^\infty_{\mathbb{C}} &:= \big\{\, f:\mathbb{N}\to\mathbb{C} \;\big|\; f \text{ is bounded}\big\}.\end{aligned}$$

For simplicity we will denote both of the above spaces by ℓ^∞. Also, set

$$\|f\|_\infty := \sup_n |f(n)|$$

for $f \in \ell^\infty$.

One usually uses a different representation for these spaces. Recall that a function f defined on $\mathbb{N}$ is also called sequence, by setting

$$x_n = f(n)$$

for all $n \in \mathbb{N}$. Then

$$\ell^\infty = \{ \, x = \{x_n\}_{n=1}^\infty : \{x_n\} \text{ is bounded } \},$$

the space of all bounded sequences in $\mathbb{R}$ (respectively $\mathbb{C}$), and for $x \in \ell^\infty$,

$$\|x\|_\infty = \sup_n |x_n|.$$

For convenience, we may also write a sequence by listing its terms,

$$x = (x_1, x_2, x_3, x_4, \dots)$$

Note that if $x = (x_1, x_2, x_3, x_4, \dots) = \{x_n\}_{n=1}^\infty$ and $y = (y_1, y_2, y_3, y_4, \dots) = \{y_i\}_{i=1}^\infty$ are two sequences and α is scalar, then

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, \dots) = \{x_n + y_n\}_{n=1}^\infty$$

and

$$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \dots) = \{\alpha x_n\}_{n=1}^\infty.$$

Thus, if x and y are bounded, say $|x_n| \le M$ and $|y_n| \le N$ for some $M, N > 0$ and all n, then

$$|x_n + y_n| \le |x_n| + |y_n| \le M + N \qquad \forall n$$

and

$$|\alpha x_n| \le |\alpha| M \qquad \forall n$$

which shows that $x + y$ and αx are again bounded. Hence, ℓ^∞ is a vector space. It is left to show that $\|\cdot\|_\infty$ is a norm:

(N1): Since $|x_n| \ge 0$ for all n, it follows that $\|x\|_\infty \ge 0$, that is, (N1) holds.

(N2): We have

$$\begin{aligned} \|x\|_\infty = 0 \quad &\Leftrightarrow \quad \sup_n |x_n| = 0 \\ &\Leftrightarrow \quad |x_n| = 0 \quad \forall n \\ &\Leftrightarrow \quad x_n = 0 \quad \forall n \\ &\Leftrightarrow \quad x = 0 \quad \text{in } \ell^\infty. \end{aligned}$$

Thus, (N2) holds.

(N3): Note that

$$\begin{aligned}\|\alpha x\|_\infty &= \sup_n |\alpha x_n| \\ &= \sup_n |\alpha||x_n| \\ &= |\alpha| \sup_n |x_n| \\ &= |\alpha| \, \|x\|_\infty,\end{aligned}$$

and hence (N3) holds.

(N4): Finally,

$$\begin{aligned}\|x+y\|_\infty &= \sup_n |x_n + y_n| \\ &\leq \sup_n \left(|x_n| + |y_n| \right) \\ &\leq \sup_n |x_n| + \sup_n |y_n| \\ &= \|x\|_\infty + \|y\|_\infty\end{aligned}$$

so that (N4) holds as well.

This shows that $(\ell^\infty, \|\cdot\|_\infty)$ is a normed linear space.

Exercise 1.1.4. Show that ℓ^∞ and $C[a,b]$ are infinite dimensional.

Exercise 1.1.5. Let

$$\ell^1 := \left\{ f : \mathbb{N} \to \mathbb{C} \;\Big|\; \sum_{n=1}^{\infty} |f(n)| < \infty \right\} = \left\{ x = \{x_n\}_{n=1}^{\infty} \subseteq \mathbb{C} \;\Big|\; \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

with

$$\|x\|_1 := \sum_{n=1}^{\infty} |x_n|.$$

Show that ℓ^1 is a vector space, that $\ell^1 \subset \ell^\infty$, and that $\|\cdot\|_1$ is a norm on ℓ^1.

1.1.4 Construction of Normed Spaces from Normed Spaces

Subspaces

Let $(X, \|\cdot\|)$ be a normed linear space, and V a subspace of X. Then obviously, $(V, \|\cdot\|)$ is also a normed linear space, called a *subspace* of $(X, \|\cdot\|)$.

Remark 1.1.2. The meaning of the word "subspace" is ambiguous. In order to avoid confusion, we will use the following notation: If X is a vector space, and V a subspace, then we will call V a *subvectorspace* of X. If $(X, \|\cdot\|)$ is a normed linear space, and V a subvectorspace of X, then we will call $(V, \|\cdot\|)$ a *subspace* of $(X, \|\cdot\|)$.

Example 1.1.5. Let $X = C[a,b]$ with the uniform norm. Given $K \subseteq [a,b]$, set

$$V := \{ f \in C[a,b] : f(x) = 0 \qquad \forall x \in K \}.$$

Then $(V, \|\cdot\|_u\|)$ is a subspace of $(X, \|\cdot\|_u\|)$. (Check !)

Product Spaces

Recall from linear algebra: If $X_1, X_2, \ldots, X_n$ are sets, then the *Cartesian Product* $X_1 \times X_2 \times \cdots \times X_n$ is the set of n-tuples,

$$\prod_{i=1}^{n} X_i = X_1 \times X_2 \times \cdots \times X_n := \{ (x_1, x_2, \ldots, x_n) : x_i \in X_i, \ i = 1 \ldots n \}.$$

So if $E_1, E_2, \ldots, E_n$ are sets with $E_i \subseteq X_i$ for $i = 1 \ldots n$, then

$$E_1 \times E_2 \times \cdots \times E_n = \{ (x_1, x_2, \ldots, x_n) : x_i \in E_i, \ i = 1 \ldots n \}.$$

is a subset of $X_1 \times X_2 \times \cdots \times X_n$.

If each X_i is also a real (respectively, complex) vector space, then $X_1 \times X_2 \times \cdots \times X_n$ becomes a vector space when we define the vector space operations componentwise,

$$\begin{aligned} (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &:= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ \alpha(x_1, x_2, \ldots, x_n) &:= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \end{aligned}$$

for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X_1 \times X_2 \times \cdots \times X_n$, and α scalar. This vector space is called the *product space of* $X_1, X_2, \ldots, X_n$. (It is a straightforward exercise to show that the vector space axioms hold in $X_1 \times X_2 \times \cdots \times X_n$.)

Now suppose, each X_i is also a normed linear space. For simplicity of notation, let's denote the norm on each X_i by $\|\cdot\|$. (Be aware, however, that the spaces X_i are usually different, and thus their norms are defined differently.) There are many ways to introduce norms in the product space $X_1 \times X_2 \times \cdots \times X_n$, some of which are:

$$\begin{aligned} \|x\|_1 &:= \sum_{i=1}^{n} \|x_i\| = \|x_1\| + \|x_2\| + \cdots + \|x_n\| \\ \|x\|_2 &:= \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{1/2} = \sqrt{\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2} \\ \|x\|_\infty &:= \max_{i=1\ldots n} \|x_i\| = \max\{ \|x_1\|, \|x_2\|, \ldots, \|x_n\| \}. \end{aligned} \tag{1.4}$$

This looks just like example 1.1.2 ! In fact, starting with n copies of $(\mathbb{R}, |\cdot|)$ we can consider $\mathbb{R}^n$ as a product space

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$$

with the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$ defined as in (1.4). Thus, the following exercise can be solved similarly to exercises 1.1.1 and 1.1.2.

Exercise 1.1.6. Show:

1. $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are indeed norms on $X_1 \times X_2 \times \cdots \times X_n$.

2. We have

$$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$$
$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$$

for all $x \in X_1 \times X_2 \times \cdots \times X_n$.

3. Any two of the three norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.

Example 1.1.6. Let $M_{m,n}$ denote the set of all $m \times n$ matrices with real entries. Then $M_{m,n}$ is a vector space. There are many ways in which one can define norms on this space, one of which is the following.

Start with $\left(\mathbb{R}^m, \|\cdot\|_1\right)$, and write the elements of $\mathbb{R}^m$ as column vectors,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

Since matrices are added columnwise, we can consider $M_{m,n}$ as the product of n copies of $\mathbb{R}^m$,

$$M_{m,n} = \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m.$$

That is, if

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \in M_{m,n}$$

then

$$A = (\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n) \qquad \text{where} \qquad \vec{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \in \mathbb{R}^m, \quad j = 1 \dots n.$$

Then by exercise 1.1.6,

$$\|A\| := \|(\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n)\|_\infty = \max_{j=1\dots n} \|a_j\|_1 = \max_{j=1\dots n} \sum_{i=1}^{m} |a_{ij}|$$

is a norm on $M_{m.n}$.

1.2 Introduction to Metric Spaces

1.2.1 Definition of a Metric Space

The idea is to generalize the notion of "distance" to arbitrary sets.

Definition 1.2.1. Let X be a set. A *metric* on X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying

(M1)	$d(x,y) \geq 0$	(positive)
(M2)	$d(x,y) = 0 \quad \Leftrightarrow \quad x = y$	(definite)
(M3)	$d(y,x) = d(x,y)$	(symmetric)
(M4)	$d(x,y) \leq d(x,z) + d(z,y)$	(triangle inequality)

for all $x, y, z \in X$. The pair (X, d) is called a *metric space.*

Remark 1.2.1.

1. If (X, d) is a metric space, we often simply say that "X is a metric space". Elements $x \in X$ are called *points* and $d(x, y)$ is called the *distance* between the points x and y.
2. If instead of (M2) only the weaker condition

   (M2') $\qquad d(x,x) = 0 \quad \forall x \in X$

   holds, then d is called a *pseudometric*
3. Property (M4) is called the *triangle inequality*

Figure 1.2: The triangle equality for metrics.

4. By induction, one easily generalizes the triangle inequality to

   $$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$$

   for points $x_1, \ldots, x_n \in X$.

Figure 1.3: The usual metric on $\mathbb{R}$

Example 1.2.1. 1. (The real line). Let $X = \mathbb{R}$. Then

$$d(x, y) := |x - y| \qquad (x, y \in \mathbb{R})$$

defines a metric on $\mathbb{R}$, called the *usual metric* or the *Euclidean metric.*

2. Let $X = \mathbb{R}$ again, but set

$$\sigma(x, y) := \left|\tan^{-1} x - \tan^{-1} y\right|, \qquad (x, y \in \mathbb{R}). \tag{1.5}$$

Then σ is a metric on $\mathbb{R}$. Note that for all $x, y \in \mathbb{R}$,

$$\sigma(x, y) \leq |\tan^{-1} x| + |\tan^{-1} y| < \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Exercise 1.2.1. Verify that the above are metrics.

Example 1.2.2. Let $X \neq \emptyset$ be any set, and define

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then d is a metric on X called the *discrete metric.* In fact, (M1) – (M3) are obvious. As for (M4), note that if $x = y$ and z is arbitrary, then

$$d(x, y) = 0 \leq d(x, z) + d(z, y).$$

On the other hand, if $x \neq y$ and $z \in X$ is arbitrary, we separate into three cases: $z = x$, or $z = y$, or $x \neq z \neq y$. Then $d(x, y) = 1$ while

$$d(x, z) + d(z, y) \begin{cases} 0 + 1 = 1 & \text{if } z = x \\ 1 + 0 = 1 & \text{if } z = y \\ 1 + 1 = 2 & \text{if } x \neq z \neq y, \end{cases}$$

that is $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Hence, (M4) holds.

Figure 1.4: The discrete metric.

1.2.2 Metrics from Norms

The concepts of norms and metrics look similar. In fact, there is a direct connection between the two:

Given a normed linear space $(X, \|\cdot\|)$, let us set

$$d(x, y) := \|x - y\| \tag{1.6}$$

for $x, y \in X$. We claim that d is a metric on X. In fact, for all $x, y, z \in X$ we have

(M1) $d(x, y) = \|x - y\| \underset{(N1)}{\geq} 0.$

(M2) $d(x, y) = 0 \quad \Leftrightarrow \quad \|x - y\| = 0 \underset{(\mathrm{N2})}{\Leftrightarrow} x - y = 0 \quad \Leftrightarrow \quad x = y.$

(M3) $d(y, x) = \|y - x\| \underset{(1.1)}{=} \|x - y\| = d(x, y)$

(M4) $$\begin{aligned} d(x, y) = \|x - y\| &= \|(x - z) + (z - y)\| \\ &\underset{(\mathrm{N3})}{\leq} \|x - z\| + \|z - y\| = d(x, z) + d(z, y) \end{aligned}$$

which shows that d is indeed a metric. It is called the *metric on X determined by the norm* $\|\cdot\|$.

Example 1.2.3. Let $X = \mathbb{R}^n$ or $X = \mathbb{C}^n$, and consider the norms discussed in example 1.1.2. The corresponding metrics on X are

$$d_1(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_1 = \|(x_1 - y_1, \ldots, x_n - y_n)\|_1 = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_2(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 = \|(x_1 - y_1, \ldots, x_n - y_n)\|_2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$$

(Euclidean metric if $X = \mathbb{R}^n$, unitary metric if $X = \mathbb{C}^n$)

$$d_\infty(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_\infty = \|(x_1 - y_1, \ldots, x_n - y_n)\|_\infty = \max_{i=1\ldots n} |x_i - y_i|$$

for $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_n) \in X$. Note that if $n = 1$, then all three metrics coincide.

Figure 1.5: Metrics on $\mathbb{R}^n$

Example 1.2.4. Consider $X = C[a,b]$ with the uniform norm $\|\cdot\|_u$. The corresponding metric is

$$d(f,g) = \|f-g\|_u = \max_{x\in[a,b]} |(f-g)(x)| = \max_{x\in[a,b]} |f(x)-g(x)|.$$

Figure 1.6: The uniform metric on $C[a,b]$.

Exercise 1.2.2. Let X be a vector space.

1. Suppose, X carries a norm $\|\cdot\|$. Show that the metric d determined by this norm has the following special properties:

   (a) (translation invariance)

   $$d(x+z, y+z) = d(x,y) \tag{1.7}$$

   (b) (scaling property)

   $$d(\alpha x, \alpha y) = |\alpha| d(x,y) \tag{1.8}$$

   for all $x, y, z \in X$ and scalars α.

2. Conversely, suppose that X carries a metric d which satisfies properties (1.7) and (1.8). Show that

   $$\|x\| := d(x,0)$$

   defines a norm on X, and that d is the metric on X determined by this norm.

Thus, there is a one-to-one correspondence between norms on X and metrics satisfying properties (1.7) and (1.8).

Exercise 1.2.3. Show that the discrete metric on $\mathbb{R}^n$ is not determined by any norm.

1.2.3 Construction of Metric Spaces from Metric Spaces

Subspaces

Definition 1.2.2. Let (X, d) be a metric space, and $Y \subset X$. Then the restriction of d to Y is a metric on Y, called the metric on Y *induced* by d. (Y, d) is called a (metric) *subspace* of (X, d).

Example 1.2.5. $\mathbb{Q}$ is a metric space in the metric induced from $\mathbb{R}$,

$$d(x, y) = |x - y|, \qquad x, y \in \mathbb{Q}.$$

A Bounded Metric

Example 1.2.6. Let d be a metric on a space X. Set

$$\sigma(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \qquad x, y \in X. \tag{1.9}$$

We claim that σ is also a metric on X. In fact, (M1), (M2) and (M3) are obvious, since d possesses these properties. To prove (M4), note that for all $0 \le a \le b$,

$$\frac{b}{1+b} - \frac{a}{1+a} = \frac{b(1+a) - a(1+b)}{(1+b)(1+a)} = \frac{b-a}{(1+b)(1+a)} \ge 0$$

that is,

$$\frac{a}{1+a} \le \frac{b}{1+b}. \tag{1.10}$$

Setting $a = d(x, y)$ and $b = d(x, z) + d(z, y)$, the triangle inequality for d shows that $a \le b$, and thus by (1.9),

$$\begin{aligned}
\sigma(x, y) = \frac{a}{1+a} \le \frac{b}{1+b} &= \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} \\
&= \frac{d(x, z)}{1 + d(x, z) + d(z, y)} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\
&\le \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} = \sigma(x, z) + \sigma(z, y).
\end{aligned}$$

This proves the claim.

Note that by definition of σ,

$$\sigma(x, y) < 1 \qquad \forall\, x, y \in X.$$

Exercise 1.2.4. Let d denote the Euclidean metric on $\mathbb{R}^n$, and σ the metric on $\mathbb{R}^n$ derived from d as in (1.9). Show that σ is not determined by any norm.

Exercise 1.2.5. (Construction of a metric from a separating family of pseudometrics).

1. Give an example of a pseudometric on $\mathbb{R}^2$ which is not a metric.

2. Let X be a set, and $\{d_i\}_{i=1}^{\infty}$ be a family of pseudometrics on X. Set
$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x,y)}{1+d_i(x,y)}.$$
Show that d is a pseudometric on X, and that $d(x,y) \leq 1$ for all $x, y \in X$.

3. We say that the family $\{d_i\}_{i=1}^{\infty}$ *separates to points of* X if for all pairs $x, y \in X$ with $x \neq y$, there exists an i such that $d_i(x,y) \neq 0$. Show: If $\{d_i\}_{i=1}^{\infty}$ separates the points of X, then d is a metric.

Product Metrics

Let $(X_1, \sigma_1), (X_2, \sigma_2) \ldots, (X_n, \sigma_n)$ be metric spaces. There are many ways in which one can turn the Cartesian product $X = X_1 \times X_2 \times \cdots \times X_n$ into a metric space. The three most important ones are:

$$\begin{aligned} d_1(x,y) &:= \sigma_1(x_1,y_1) + \sigma_2(x_2,y_2) + \cdots + \sigma_n(x_n,y_n) = \sum_{i=1}^{n} \sigma_i(x_i,y_i) \\ d_2(x,y) &:= \sqrt{\sigma_1(x_1,y_1)^2 + \sigma_2(x_2,y_2)^2 + \cdots + \sigma_n(x_n,y_n)^2} = \sqrt{\sum_{i=1}^{n} \sigma(x_i,y_i)^2} \\ d_\infty(x,y) &:= \max\{\sigma_1(x_1,y_1), \sigma_2(x_2,y_2), \ldots, \sigma_n(x_n,y_n)\} = \max_{i=1\ldots n} \sigma(x_i,y_i) \end{aligned} \tag{1.11}$$

for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots . y_n) \in X$.

Exercise 1.2.6. Show that d_1, d_2 and d_∞ are indeed metrics on $X = X_1 \times X_2 \times \cdots \times X_n$. Furthermore,
$$d_\infty(x,y) \leq d_1(x,y) \leq n\, d_\infty(x,y)$$
and
$$d_\infty(x,y) \leq d_2(x,y) \leq \sqrt{n}\, d_\infty(x,y).$$
for all $x, y \in X$. (Hint: Proceed similar to exercises 1.1.1 and 1.1.2.)

Example 1.2.7. Let $X_1 = X_2 = \cdots = X_n = \mathbb{R}$ with the usual metric, $\sigma_i(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, $i = 1 \ldots n$. Then $\mathbb{R}^n$ is an n-fold Cartesian product,
$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ factors}}.$$
The metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$, as defined in example 1.2.3, are simply the product metrics (1.11).

1.3 Topology of Metric Spaces

1.3.1 Open and Closed Balls

Definition 1.3.1. Let (X, d) be a metric space. Given a point $x_o \in X$ and $r > 0$, let us set

$$B_r(x_o) = \{\, x \in X \ : \ d(x, x_o) < r \,\} \quad \text{(the } \textit{open ball} \text{ with center } x_o \text{ and radius } r\text{)}$$
$$\overline{B}_r(x_o) = \{\, x \in X \ : \ d(x, x_o) \le r \,\} \quad \text{(the } \textit{closed ball} \text{ with center } x_o \text{ and radius } r\text{)}$$
$$S_r(x_o) = \{\, x \in X \ : \ d(x, x_o) = r \,\} \quad \text{(the } \textit{sphere} \text{ with center } x_o \text{ and radius } r\text{)}$$

Note that $\overline{B}_r(x_o) = B_r(x_o) \cup S_r(x_o)$, a disjoint union.

Remark 1.3.1. If X is a normed linear space, then since $d(x, x_o) = \|x - x_o\|$, these sets can also be described by their norms:

$$B_r(x_o) = \{\, x \in X \ : \ \|x - x_o\| < r \,\}$$
$$\overline{B}_r(x_o) = \{\, x \in X \ : \ \|x - x_o\| \le r \,\}$$
$$S_r(x_o) = \{\, x \in X \ : \ \|x - x_o\| = r \,\}.$$

Example 1.3.1. 1. Let $X = \mathbb{R}$. Given $x_o \in \mathbb{R}$ and $r > 0$, then

$$B_r(x_o) = \{\, x \in \mathbb{R} : |x - x_o| < r \,\} = (x_o - r, x_o + r) \quad \text{(an open interval)}$$
$$\overline{B}_r(x_o) = \{\, x \in \mathbb{R} : |x - x_o| \le r \,\} = [x_o - r, x_o + r] \quad \text{(a closed interval)}$$
$$S_r(x_o) = \{\, x \in \mathbb{R} : |x - x_o| = r \,\} = \{x_o - r,\ x_o + r\} \quad \text{(a two-point set).}$$

2. Let $X = \mathbb{R}^2$.

   (a) Give $\mathbb{R}^2$ the Euclidean metric d_2. Given $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, the open ball

   $$B_r(\vec{x}) = \{\, \vec{y} = (y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < r \,\}$$

   is called an *open disc* with center $\vec{x}$ and radius r, and similarly,

   $$\overline{B}_r(\vec{x}) = \{\, \vec{y} = (y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \le r \,\}$$

   is called a *closed disk* with center $\vec{x}$ and radius r.

   $$S_r(\vec{x}) = \{\, \vec{y} = (y_1, y_2) \in \mathbb{R}^2 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = r \,\}$$

   is a *circle* with center $\vec{x}$ and radius r.

   (b) Now give $\mathbb{R}^2$ the metric d_∞. Given $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, we have

   $$S_r(\vec{x}) = \{\, \vec{y} = (y_1, y_2) \in \mathbb{R}^2 : |y_1 - x_1| = |y_2 - x_2| = r \,\}.$$

   This is the square whose vertices are the points $P(x_1 - r, x_2 - r)$, $Q(x_1 + r, x_2 - r)$, $R(x_1 + r, x_2 + r)$ and $S(x_1 - r, x_2 + r)$.

   $$B_r(\vec{x}) = \{\, \vec{y} = (y_1, y_2) \in \mathbb{R}^2 : |y_1 - x_1| < r,\ \ |y_2 - x_2| < r \,\}$$

is the inside of this square, also called the *open interval* $(x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r)$, and

$$\overline{B}_r(\vec{x}) = \{ \vec{y} = (y_1, y_2) \in \mathbb{R}^2 : |y_1 - x_1| \leq r, \ |y_2 - x_2| \leq r \}$$

is called the *closed interval* $[x_1 - r, x_1 + r] \times [x_2 - r, x_2 + r]$.

(c) Finally, give $\mathbb{R}^2$ the metric d_1. Given $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, we have

$$B_r(\vec{x}) = \{ \vec{y} = (y_1, y_2) \in \mathbb{R}^2 : |y_1 - x_1| + |y_2 - x_2| < r \}.$$

This is the inside of a square whose vertices are the points $P(x_1 - r, x_2)$, $Q(x_1, x_2 - r)$, $R(x_1 + r, x_2)$ and $S(x_1, x_2 + r)$.

Figure 1.7: Balls $B_r(\vec{x})$ in $\mathbb{R}^2$ in various metrics.

3. Let $X = \mathbb{R}^3$ with the Euclidean metric d_2. Let $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then

$$B_r(\vec{x}) = \{ \vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2} < r \}$$

and

$$\overline{B}_r(\vec{x}) = \{ \vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2} \leq r \}$$

are indeed open and closed balls, respectively, while

$$S_r(\vec{x}) = \{ \vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2} = r \}$$

is a sphere. This is why the sets $B_r(x_o)$, $\overline{B}_r(x_o)$ and $S_r(x_o)$ carry these names.

4. Let X be an arbitrary set with the discrete metric. Then

$$B_r(x_o) = \begin{cases} \{x_o\} & \text{if } 0 < r \leq 1 \\ X & \text{if } r > 1 \end{cases}$$

$$\overline{B}_r(x_o) = \begin{cases} \{x_o\} & \text{if } 0 < r < 1 \\ X & \text{if } r \geq 1 \end{cases}$$

$$S_r(x_o) = \begin{cases} \emptyset & \text{if } 0 < r, r \neq 1 \\ X \backslash \{x_o\} & \text{if } r = 1. \end{cases}$$

5. Let $X = C[a,b]$ with the *uniform* norm $\|\cdot\|_u$. Given $f \in C[a,b]$ we have

$$\begin{aligned} B_\epsilon(f) &= \{\, g \in C[a,b] : \|f-g\|_u < \epsilon \,\} \\ &= \{\, g \in C[a,b] : \max_{x\in[a,b]} |f(x)-g(x)| < \epsilon \,\} \end{aligned}$$

Figure 1.8: The ball B_ϵ in $C[a,b]$ consists of all functions g inside the ϵ-strip.

Exercise 1.3.1. Describe the open balls, closed balls and spheres for the space l^∞ with norm $\|\cdot\|_\infty$.

Definition 1.3.2. (Bounded Set). Let (X,d) be a metric space. A non-empty set $M \subseteq X$ is said to be *bounded*, if

$$\delta(M) := \sup_{x,y\in M} d(x,y) < \infty.$$

$\delta(M)$ is called the *diameter* of the set M.

Remark 1.3.2. Let $M \subseteq X$ be bounded with diameter $\delta = \delta(M) \geq 0$. Fix any $x_o \in M$. Then for each $x \in M$ we have $d(x,x_o) \leq \delta < \delta + 1$, and thus $M \subseteq \overline{B}_\delta(x_o) \subseteq B_{\delta+1}(x_o)$.

Conversely, if $M \subseteq X$ is contained in some ball $\overline{B}_r(x_o)$ or $B_r(x_o)$, then obviously, for all $x,y \in M$,

$$d(x,y) \leq d(x,x_o) + d(x_o,y) \leq r + r = 2r$$

which shows that M is bounded of diameter less or equal to $2r$.

Figure 1.9: A bounded set.

We have shown: $M \subseteq X$ is bounded $\Leftrightarrow$ M is contained in some open ball $B_r(x_o)$ (respectively closed ball $\overline{B}_r(x_o)$).

Example 1.3.2. Let (X, d) be a metric space.

1. Let σ the metric derived from d as in example 1.2.6,

$$\sigma(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

   Then X is bounded, and hence every set $M \subseteq X$ is bounded. (If X is a bounded set, then we call d a *bounded metric.*)

2. Let d denote the discrete metric on X. Then d is bounded.

3. By example 1.2.1, the metric $\sigma(x, y) = \left|\tan^{-1} x - \tan^{-1} y\right|$ on $\mathbb{R}$ is bounded.

Exercise 1.3.2. Let $(X, \|\cdot\|)$ be a normed linear space. Show: $M \subseteq X$ is bounded $\Leftrightarrow$ there exists $m > 0$ such that $\|x\| \leq m$ for all $x \in M$.

1.3.2 Open and Closed Sets, Neighborhoods

Definition 1.3.3. Let (X, d) be a metric space.

1. A subset U of X is called *open* if for each point $x \in U$ there exists an open ball $B_\epsilon(x)$ such that

$$B_\epsilon(x) \subseteq U.$$

   (Note that ϵ depends on x.)

2. A subset F of X is called *closed* if F^c is open.

Figure 1.10: An open set and a closed set.

Example 1.3.3. Let (X, d) be a metric space.

1. Every open ball $B_r(x_o)$ is an open set. In fact, let $x \in B_r(x_o)$ be given. Since $d(x, x_o) < r$, we can pick $\epsilon > 0$ such that $d(x, x_o) < d(x, x_o) + \epsilon < r$. Now for every $y \in B_\epsilon(x)$ we have $d(y, x) < \epsilon$, and thus by the triangle inequality,

$$d(y, x_o) \leq d(y, x) + d(x, x_o) < \epsilon + d(x, x_o) < r,$$

   which shows that $B_\epsilon(x) \subseteq B_r(x_o)$. Since $x \in B_r(x_o)$ was arbitrary, we conclude that $B_r(x_o)$ is an open set.

Figure 1.11: An open ball is an open set — a closed ball is a closed set.

2. Every closed ball $\overline{B}_r(x_o)$ is a closed set. In fact, we need to show that $\left[\overline{B}_r(x_o)\right]^c$ is open. So let $x \in \left[\overline{B}_r(x_o)\right]^c$ be given. Since $d(x, x_o) > r$, we can pick $\epsilon > 0$ such that $r < d(x, x_o) - \epsilon < d(x, x_o)$. Now for every $y \in B_\epsilon(x)$ we have $d(y, x) < \epsilon$, and thus by the triangle inequality,

$$d(x, x_o) \leq d(x, y) + d(y, x_o) < \epsilon + d(y, x_o).$$

Subtract ϵ,

$$r < d(x, x_o) - \epsilon < d(y, x_o),$$

which shows that $B_\epsilon(x) \subseteq \left[\overline{B}_r(x_o)\right]^c$. Since $x \in \left[\overline{B}_r(x_o)\right]^c$ was arbitrary, we conclude that $\left[\overline{B}_r(x_o)\right]^c$ is indeed open, an hence $\overline{B}_r(x_o)$ closed.

3. A set $F = \{x_o\}$ consisting of one point only is is closed. (Such a set is called a *singleton.*) In fact, let $x \neq x_o$ be given. Then $\epsilon := d(x, x_o) > 0$. For every $y \in B_\epsilon(x)$ we have

$$d(y, x) < \epsilon = d(x_o, x)$$

and hence $y \neq x_o$, that is, $y \in F^c$. This shows that $B_\epsilon(x) \subseteq F^c$. Since $x \in F^c$ was arbitrary, it follows F^c is open, so that F is closed.

Definition 1.3.4. Let (X, d) be a metric space, and $x \in X$. A set $M \subseteq X$ is called a *neighborhood of* X, if there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq M$.

If M itself is an open set, then it is called an *open neighborhood* of X.

Figure 1.12: Neighborhood of x.

Remark 1.3.3. The following two remarks are obvious:

1. For each $\epsilon > 0$, the ball $B_\epsilon(x)$ is an open neighborhood of x. For this reason, $B_\epsilon(x)$ is also called an (open) ϵ*-neighborhood of* x.

2. If M is a neighborhood of x, and $M \subseteq N$, then N is also a neighborhood of x.

Exercise 1.3.3. Let (X, d) be a metric space, $Y \subseteq X$, and $\tilde{d}$ the metric on Y induced by the metric d on X. Show:

1. Given $x_o \in Y$, if $M = B_\epsilon(x_o)$ denotes the open ball in Y with respect to $\tilde{d}$, and $N = B_\epsilon(x_o)$ the open ball in X with respect to d, then $M = N \cap Y$.
2. $U \subseteq Y$ is open in Y $\Leftrightarrow$ $U = V \cap Y$ for some open subset V of X.
3. $F \subseteq Y$ is closed in Y $\Leftrightarrow$ $F = G \cap Y$ for some closed subset G of X.

Exercise 1.3.4. Let (X, d) be a discrete metric space. (That is, d is the discrete metric). Show that every set $M \subseteq X$ is both, open and closed..

1.3.3 The Topology Determined by a Metric

Theorem 1.3.1. *Let (X, d) be a metric space, and let*

$$\tau := \{ U \subseteq X : U \textit{ is open} \}$$

be the collection of all open sets. Then

(T1) $\emptyset \in \tau$ *and* $X \in \tau$.
("$\emptyset$ and X are open sets.")

(T2) *If* $\{U_\alpha\}_{\alpha \in A} \subseteq \tau$, *then* $\bigcup_{\alpha \in A} U_\alpha \in \tau$.
("The arbitrary union of open sets is open.")

(T3) *If* $U_1, U_2, \ldots, U_n \in \tau$, *then* $\bigcap_{i=1}^n U_i \in \tau$.
("The finite intersection of open sets is open.")

Proof.

(T1): Since $\emptyset$ contains no points, the statement "For every $x \in \emptyset$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \emptyset$" is true. Hence, $\emptyset$ is an open set.

On the other hand, for every $x \in X$ and $\epsilon > 0$, $B_\epsilon(x) \subseteq X$. Hence, X is an open set.

(T2): Let $x \in \bigcup_{\alpha \in A} U_\alpha$ be arbitrary. Then $x \in U_{\alpha_o}$ for at least one $\alpha_o \in A$. As U_{α_o} is open, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U_{\alpha_o}$, and hence

$$B_\epsilon(x) \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

Thus, $\bigcup_{\alpha \in A} U_\alpha$ is open.

(T3): Let $x \in \bigcap_{i=1}^n U_i$ be arbitrary. Then $x \in U_i$ for all $i = 1 \ldots n$. Since each U_i is open, there exist ϵ_i, $i = 1 \ldots n$, such that

$$B_{\epsilon_i}(x) \subseteq U_i, \qquad i = 1 \ldots n.$$

Set $\epsilon = \min_{i=1\ldots n} \epsilon_i > 0$. Since $\epsilon \leq \epsilon_i$ for all i we have

$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq U_i \qquad \text{for all } i = 1 \ldots n,$$

so that

$$B_\epsilon(x) \subseteq \bigcap_{i=1}^{n} U_i$$

This shows that $\bigcap_{i=1}^{n} U_i$ is open.

□

Remark 1.3.4. (General definition of topology) The above properties can be used to define the notion of open or closed sets without making use of a metric or open balls. In general, given a set X and a collection τ of subsets of X satisfying (T1)–(T3), we call (X, τ) a *topological space* and τ a *topology* on X. Each set $U \in \tau$ is called an *open set* and each $F = U^c$, $U \in \tau$, a *closed set.* Many of the properties in this and the next section carry over to this abstract setting. However, since most of the topologies we are interested in in this course come from a metric, we will restrict our attention to metric spaces.

Corollary 1.3.2. *Let (X, d) be a metric space. Then*

(T1') $\emptyset$ and X are closed sets.

(T2') If $\{F_\alpha\}_{\alpha \in A}$ is an arbitrary collection of closed sets in X, then $\bigcap_{\alpha \in A} F_\alpha$ is also closed. ("The arbitrary intersection of closed sets is closed.")

(T3') If $F_1, F_2, \ldots, F_n$ are closed sets in X, then $\bigcup_{i=1}^{n} F_i$ is closed. ("The finite union of closed sets is closed.")

Proof.

(T1'): Since $\emptyset = X^c$ and X is open, it follows that $\emptyset$ is closed.

Similarly, since $X = \emptyset^c$ and $\emptyset$ is open, it follows that X is closed.

(T2'): Note that

$$\bigcap_{\alpha \in A} F_\alpha = \left[\bigcup_{\alpha \in A} F_\alpha^c\right]^c$$

Since each F_α^c is open, (T2) shows that $\bigcup_{\alpha \in A} F_\alpha^c$ is open; hence $\bigcap_{\alpha \in A} F_\alpha$ is closed.

(T3'): Similarly,

$$\bigcup_{i=1}^{n} F_i = \left[\bigcap_{i=1}^{n} F_i^c\right]^c.$$

Since each F_i^c is open,(T3) shows that $\bigcap_{i=1}^{n} F_i^c$ is open; hence $\bigcup_{i=1}^{n} F_i$ is closed.

□

Remark 1.3.5. Let (X, d) be a metric space.

1. The infinite intersection of open sets need not be open. For example, let $X = \mathbb{R}$ with the usual metric. For each $i \in \mathbb{N}$, set $U_i = (-\frac{1}{i}, \frac{1}{i})$. Then U_i is open, and
$$\bigcap_{i=1}^{\infty} U_i = \bigcap_{i=1}^{\infty} (-\tfrac{1}{i}, \tfrac{1}{i}) = \{0\}.$$
But the singleton $\{0\}$ is not open in $\mathbb{R}$. (Why ?)

2. Similarly, the countable union of closed sets need not be closed. For example, consider $X = \mathbb{R}$ again. For each $i \in \mathbb{N}$, set $F_i = [\frac{1}{n}, 1 - \frac{1}{n}]$. Then F_i is closed, and
$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} [\tfrac{1}{n}, 1 - \tfrac{1}{n}] = (0, 1).$$
But the interval $(0, 1)$ is not closed in $\mathbb{R}$. (Why ?)

3. Every finite subset $F = \{x_1, \ldots, x_n\}$ of X is closed, since it is a finite union of closed sets, $F = \bigcup_{i=1}^{n} \{x_i\}$,

Exercise 1.3.5. Let (X, d) be a metric space.

1. Show: If $U \subseteq X$ is open, and $F \subseteq X$ is closed, then
$$F \backslash U \quad \text{is closed, and} \quad U \backslash F \quad \text{is open.}$$

2. Use 1. to show that every sphere $S_r(x_o)$ is closed.

3. Show: $U \subseteq X$ is open $\quad \Leftrightarrow \quad U$ is a union of open balls.

1.3.4 Interior, Closure and Boundary

Interior of a Set

Definition 1.3.5. Let (X, d) be a metric space, and $M \subseteq X$. A point $x \in M$ is called an *interior point of* M, if M is a neighborhood of x. (That is, if there exists $\epsilon > 0$ so that $B_\epsilon(x) \subseteq M$.)

Figure 1.13: x is an interior point of $M \subset \mathbb{R}^2$, y is not.

We set $M^\circ = \text{Int}(M) := \{ x \in M : x \text{ is an interior point of M} \}$. M° is called the *interior* of M.

Example 1.3.4. 1. If $X = \mathbb{R}$ and $M = [0,1] \cup \{2\}$, then $M^\circ = (0,1)$. (Why ?)

2. If $X = \mathbb{R}$ and $M = \mathbb{Q}$, then $\mathbb{Q}^\circ = \emptyset$ by the density of the irrationals in $\mathbb{R}$.

3. If $X = \mathbb{R}^2$ and $M = [a,b) \times [c,d)$, then $M^\circ = (a,b) \times (c,d)$. (Why ?)

Theorem 1.3.3. *Let X be a metric space, and $M \subseteq X$. Then*

1. *M° is open.*

2. *If $U \subseteq M$ and U is open in X, then $U \subseteq M^\circ$. (That is, M° is the* largest *open subset of M.)*

3. *M is open $\quad \Leftrightarrow \quad M^\circ = M$.*

Proof. 1. Let $x_o \in M^\circ$ be arbitrary. Since x_o is an interior point of M, there exists $\epsilon > 0$ so that $B_\epsilon(x_o) \subseteq M$. We claim that $B_\epsilon(x_o) \subseteq M^\circ$. In fact, since $B_\epsilon(x_o)$ is an open set (by example 1.3.3), for each $x \in B_\epsilon(x_o)$, there exists an open ball $B_\delta(x)$ such that $x \in B_\delta(x) \subseteq B_\epsilon(x_o) \subseteq M$, that is, x is an interior point of M, i.e. $x \in M^\circ$. We conclude that $B_\epsilon(x_o) \subseteq M^\circ$, and the claim holds. Since $x_o \in M^\circ$ was arbitrary, it follows from the definition of open sets that M° is open.

2. Let $U \subseteq M$ be open. Then for each $x_o \in U$, there exists $\epsilon > 0$ so that $B_\epsilon(x_o) \subseteq U$, and hence $B_\epsilon(x_o) \subseteq M$. That is, x_o is an interior point of M, i.e. $x_o \in M^\circ$. Since $x_o \in U$ was arbitrary, it follows that $U \subseteq M^\circ$.

3. $\Rightarrow$: Suppose, M is open. Then M itself is an open set contained in M, so that by part 2., $M \subseteq M^\circ$. The reverse inclusion, $M^\circ \subseteq M$, always holds by definition of M°. Hence, $M^\circ = M$.

$\Leftarrow$: Suppose, $M^\circ = M$. Since M° is open by part 1., it follows that M is open. □

Remark 1.3.6. It is tempting to believe that the interior of the closed ball $\overline{B}_\epsilon(x_o)$ is the open ball $B_\epsilon(x_o)$. However, this is not true in general: Let (X,d) be a discrete metric space, let $\epsilon = 1$ and fix $x_o \in X$. Then $\overline{B}_\epsilon(x_o) = X$ is open, so that $\left[\overline{B}_\epsilon(x_o)\right]^\circ = X$, while $B_\epsilon(x_o) = \{x_o\}$. This shows that $\operatorname{Int}\left[\overline{B}_\epsilon(x_o)\right] \neq B_\epsilon(x_o)$.

However, in case of a normed linear space, this is true:

Exercise 1.3.6. Let $(X, \|\cdot\|)$ be a normed linear space. Show: For each closed ball $\overline{B}_\epsilon(x_o)$ we have $\operatorname{Int}\left[\overline{B}_\epsilon(x_o)\right] = B_\epsilon(x_o)$.

Cluster Points

Definition 1.3.6. Let (X,d) be a metric space and $M \subseteq X$. A point $x_o \in X$ is called a *cluster point* of M (also an *accumulation point* or a *limit point* of M), if every open neighborhood U of x_o contains at least one point $x \in M$ with $x \neq x_o$. (That is, $M \cap U \backslash \{x_o\} \neq \emptyset$.)

Let use set

$$\operatorname{Acc}(M) := \{x \in X : x \text{ is a cluster point of M}\}.$$

Remark 1.3.7. The notion of accumulation point, cluster point and limit point is not standard in the literature. Some authors may give slightly varying definitions for these notions.

Example 1.3.5.

1. Let $X = \mathbb{R}$ and $M = \{\frac{1}{n} : n \in \mathbb{N}\}$. We claim that 0 is a cluster point of M. In fact, given an open neighborhood U of 0, pick $\epsilon > 0$ so that $(-\epsilon, \epsilon) \subseteq U$. Now if $n > \frac{1}{\epsilon}$ then $0 < \frac{1}{n} < \epsilon$ so that $\frac{1}{n} \in U$. This proves the claim.

   Note that 0 is the only cluster point of M. (Why ?)

2. Let $X = \mathbb{R}$. Then $\mathrm{Acc}(\mathbb{Q}) = \mathbb{R}$ by the density of the rationals in $\mathbb{R}$.

3. Let $X = \mathbb{R}$ and $M = (0, 1] \cup \{2\}$. Then $\mathrm{Acc}(M) = [0, 1]$. (Why ?) $x = 2$ is not a cluster point of M, since $B_{1/2}(2) = (3/2, 5/2)$ contains no point of M different from 2. ($x = 2$ is called an *isolated point* of M, since it is a point of M but not a cluster point.)

4. Let $X = \mathbb{R}^2$ with the usual metric. Set

$$M = \{\, (x, y) \in \mathbb{R}^2 : x > 0,\ y = \sin\frac{1}{x} \,\}.$$

   Then $\mathrm{Acc}(M) = M \cup \{(0, y) : -1 \leq y \leq 1)$. (Why ?)

Exercise 1.3.7. Let (X, d) be a metric space and $M \subseteq X$. Show that the following are equivalent:

1. x_o is a cluster point of M.
2. Every open ball $B_\epsilon(x_o)$ contains a point x of M with $x \neq x_o$.
3. Every open ball $B_\epsilon(x_o)$ contains infinitely many points of M.
4. Every open neighborhood U of x_o contains infinitely many points of M.

Closure of a Set

Definition 1.3.7. Let (X, d) be a metric space and $M \subseteq X$. The set

$$\overline{M} := M \cup \text{Acc(M)}.$$

is called the *closure of* M.

Theorem 1.3.4. *Let (X, d) be a metric space and $M \subseteq X$. Then*

1. *$\overline{M}$ is closed.*
2. *If $M \subseteq F$ and F is closed in X, then $\overline{M} \subseteq F$. (That is, $\overline{M}$ is the* smallest *closed subset of X containing M.)*
3. *M is closed $\Leftrightarrow$ $\overline{M} = M$.*

Proof. 1. Let us show that $[\overline{M}]^c$ is open. For this, let $x_o \in [\overline{M}]^c$. Then $x_o \notin M$ and $x_o \notin \text{Acc}(M)$. Since x_o is not a cluster point, there exists an open neighborhood U of x_o such that

$$U \cap (M \backslash \{x_o\}) = \emptyset.$$

Since as $x_o \notin M$ either, then

$$U \cap M = \emptyset.$$

Next we need to "reduce" U in order that $U \cap \overline{M} = \emptyset$. Since U is an open neighborhood of x_o, there exists $\epsilon > 0$ so that

$$B_\epsilon(x_o) \subseteq U \subseteq M^c \tag{1.12}$$

We claim: $B_{\epsilon/2}(x_o) \subseteq [\overline{M}]^c$. For suppose to the contrary, that there exists $x_1 \in B_{\epsilon/2}(x_o) \cap \overline{M}$. Then by (1.12), $x_1 \notin M$, and hence $x_1 \in \text{Acc}(M)$. Thus, the open neighborhood $B_{\epsilon/2}(x_1)$ of x_1 contains at least one point x of M. But by the triangle inequality,

$$d(x, x_o) \leq d(x, x_1) + d(x_1, x_o) < \epsilon/2 + \epsilon/2 = \epsilon$$

so that $x \in B_\epsilon(x_o)$ which contradicts (1.12). This proves the claim, and since $x_o \in [\overline{M}]^c$ was arbitrary, that $[\overline{M}]^c$ is open. Thus, $\overline{M}$ is closed.

2. Let $F \subseteq X$ be closed with $M \subseteq F$. We claim that $\text{Acc}(M) \subseteq F$. For if $x_o \in F^c$, then since F^c is open, there exists $\epsilon > 0$ such that $B_\epsilon(x_o) \subseteq F^c \subseteq M^c$. That is, $B_\epsilon(x_o)$ is an open neighborhood of x_o containing no point of M, hence x_o can not be an accumulation point of M. This proves the claim.

   It follows now that

$$\overline{M} = M \cup \text{Acc}(M) \subseteq F \cup F = F.$$

3. $\Rightarrow$: Suppose, M is closed. Then $F = M$ is a closed subset of X containing M, so by part 2., $\overline{M} \subseteq F = M$. The reverse inclusion, $M \subseteq \overline{M}$, always holds by definition 1.3.7. It follows that $\overline{M} = M$.

   $\Leftarrow$: Suppose, $\overline{M} = M$. Since by part 1., $\overline{M}$ is closed, it follows that M is closed.

□

Remark 1.3.8. Since $\overline{M} = M \cup \mathrm{Acc}(M)$, part 3. of the theorem can be rephrased as: *M is closed $\Leftrightarrow$ $Acc(M) \subseteq M$.*

Example 1.3.6. 1. Let $X = \mathbb{R}$ and $M = (0,1] \cup \{2\}$. Then by example 1.3.5,

$$\overline{M} = M \cup \mathrm{Acc}(M) = \big((0,1] \cup \{2\}\big) \cup [0,1] = [0,1] \cup \{2\}.$$

2. Let $X = \mathbb{R}$. If (a,b) is a bounded, open interval, then $\overline{(a,b)} = [a,b]$.

3. Let $X = \mathbb{R}$. If $M = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $\overline{M} = M \cup \{0\}$ by example 1.3.5.

4. Let $X = \mathbb{R}^2$. If $M = \{\, (x,y) \in \mathbb{R}^2 : x > 0,\ y = \sin\frac{1}{x} \,\}$, then

$$\overline{M} = M \cup \{(0,y) \in \mathbb{R}^2 : -1 \leq y \leq 1).$$

Exercise 1.3.8. 1. Let (X,d) be a metric space. Show: $\overline{B_r(x_o)} \subseteq \overline{B}_r(x_o)$, but equality does not hold in general.

2. Show that if $(X, \|\cdot\|)$ is a normed linear space, then $\overline{B_r(x_o)} = \overline{B}_r(x_o)$.

Exercise 1.3.9. Let (X,d) be a metric space, and $A, B \subseteq X$. Show:

1. If $A \subseteq B$, then $A^\circ \subseteq B^\circ$ and $\overline{A} \subseteq \overline{B}$.
2. $(A \cap B)^\circ = A^\circ \cap B^\circ$ and $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.
3. $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$ and $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$. Equality does not hold in general.

Boundary of a Set

Definition 1.3.8. Let (X,d) be a metric space and $M \subseteq X$. Then the set

$$\partial(M) := \overline{M} \backslash M^\circ$$

is called the *boundary* of M.

Figure 1.14: Boundary of M.

Note that $\overline{M} = M^\circ \cup \partial(M)$, a disjoint union.

Example 1.3.7. 1. Let $X = \mathbb{R}$ and $M = (0,1] \cup \{2\}$. Then

$$\partial(M) = \overline{M} \backslash M^\circ = \big((0,1] \cup \{2\}\big) \backslash (0,1) = \{0,1,2\}.$$

2. Let $X = \mathbb{R}$ and $M = \mathbb{Q}$. Then $\overline{Q} = \mathbb{R}$ and $\mathbb{Q}^\circ = \emptyset$, hence

$$\partial(\mathbb{Q}) = \overline{\mathbb{Q}} \backslash \mathbb{Q}^\circ = \mathbb{R} \backslash \emptyset = \mathbb{R}.$$

Exercise 1.3.10. Let (X,d) be a metric space, $M \subsetneq X$ be nonempty, and $x \in X$.

1. Show that the following are equivalent:
   (a) $x \in \overline{M}$.
   (b) Every open neighborhood U of x contains a point of M.
   (c) Every open ball $B_\epsilon(x)$ contains a point of M.
   (d) For each $\epsilon > 0$ there exists $y \in M$ with $d(x,y) < \epsilon$.

2. Show that the following are equivalent:
   (a) $x \in \partial(M)$.
   (b) Every open neighborhood U of x contains points of both M *and* M^c.
   (c) Every open ball $B_\epsilon(x)$ contains points of both M *and* M^c.
   (d) For each $\epsilon > 0$ there exist $y \in M$ and $z \in M^c$ with $d(x,y) < \epsilon$ and $d(x,z) < \epsilon$.

1.4 Continuous Mappings

1.4.1 Continuous Mappings Between Metric Spaces

Definition 1.4.1. Let (X, d) and (Y, σ) be two metric spaces, and

$$T : X \to Y$$

a mapping (i.e. a function). For simplicity, we write Tx instead of $T(x)$. Then T is said to be

1. *continuous at a point* $x_o \in X$, if for every $\epsilon > 0$ there exists $\delta = \delta(x_o, \epsilon) > 0$ such that

$$\sigma(Tx, Tx_o) < \epsilon \qquad \text{whenever } x \in X \text{ and } d(x, x_o) < \delta. \tag{1.13}$$

Figure 1.15: T is continuous at x_o

2. *continuous*, if it is continuous at every $x_o \in X$.

3. *uniformly continuous*, if δ can be chosen independent of x_o, that is, if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\sigma(Tx, Ty) < \epsilon \qquad \text{whenever } x, y \in X \text{ and } d(x, y) < \delta. \tag{1.14}$$

Remark 1.4.1. 1. Condition 1.13 can be restated as follows:

$$T\big(B_\delta(x_o)\big) \subseteq B_\epsilon(Tx_o).$$

2. Obviously, uniform continuity implies continuity.

Example 1.4.1. 1. Let $X = Y = \mathbb{R}$ with the usual metric. Then $f(x) = x^2$ is continuous, but not uniformly continuous.

2. Let $X = Y = \mathbb{R}$ and $f(x) = \frac{1}{1+x^2}$. Then $f(x)$ is uniformly continuous.

3. Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. The function

$$f(x) = \begin{cases} \frac{x}{x^2+y^2} & \text{if } (x, y) \neq (0,0) \\ 1 & \text{if } (x, y) = (0,0) \end{cases}$$

is continuous at every $x_o \in X$, except at $x_o = (0, 0)$. The function

$$f(x) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0,0) \\ 1 & \text{if } (x, y) = (0,0) \end{cases}$$

is uniformly continuous. (Use polar coordinates !)

Example 1.4.2. If (X, d) is a discrete metric space, then every mapping $T : X \to Y$ is uniformly continuous.

Example 1.4.3. Let (X, d) and (Y, σ) be metric spaces, and $T : X \to Y$ a mapping. Then T is called an *isometry* if $\sigma(Tx, Ty) = d(x, y)$ for all $x, y \in X$. (That is, an isometry preserves distances). Obviously, every isometry is uniformly continuous (Just choose $\delta = \epsilon$.)

Example 1.4.4. Let $X = Y = \mathbb{R}^2$ with the usual metric, and write elements of $\mathbb{R}^2$ as column vectors. Fix a matrix $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. (Note that A is orthogonal.) Then the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{x}) = A\vec{x}$ is an isometry. (Check !)

Remark 1.4.2. Suppose, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, and $T : X \to Y$. Then the above definitions become:

1. T is *continuous* at $x_o \in X$, if for every $\epsilon > 0$ there exists $\delta = \delta(x_o, \epsilon) > 0$ such that
$$\|Tx - Tx_o\|_Y < \epsilon \qquad \text{whenever } x \in X \text{ and } \|x - x_o\|_X < \delta.$$
2. T is *uniformly continuous* if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that
$$\|Tx - Ty\|_Y < \epsilon \qquad \text{whenever } x, y \in X \text{ and } \|x - y\|_X < \delta.$$

1.4.2 Continuity and Open Sets

Let X, Y be two sets, and $T : X \to Y$ a mapping. Given $V \subseteq Y$, the set
$$T^{-1}(V) := \{x \in X : T(x) \in V\}$$
is called the *preimage* of V in X. It is easy to verify that for an arbitrary collection $\{V_\alpha\}_{\alpha \in A}$ of subsets of Y,
$$T^{-1}\Big(\bigcap_{\alpha\in A} V_\alpha\Big) = \bigcap_{\alpha\in A} T^{-1}(V_\alpha) \qquad \text{and} \qquad T^{-1}\Big(\bigcup_{\alpha\in A} V_\alpha\Big) = \bigcup_{\alpha\in A} T^{-1}(V_\alpha).$$
Also, if $V \subseteq Y$ then $V = T\big(T^{-1}(V)\big)$, while if $U \subseteq X$ then $U \subseteq T^{-1}\big(T(U)\big)$.

Theorem 1.4.1. *(Characterization of continuity through open sets) Let (X, d) and (Y, σ) be metric spaces, and $T : X \to Y$ a mapping. Then T is continuous $\Leftrightarrow$ $T^{-1}(V)$ is open in X for each open set $V \subseteq Y$.*

Figure 1.16: Preimage.

Proof. $\Rightarrow$: Suppose T is continuous. Let $V \subseteq Y$ be open. We need to show that $U := T^{-1}(V)$ is open in X.

If $U = \emptyset$ we are done. Otherwise, let $x_o \in U$ be arbitrary. Since $T(x_o) \in V$ and V is open, there exists $\epsilon > 0$ such that

$$B_\epsilon(Tx_o) \subseteq V.$$

Now as T is continuous, there exists $\delta > 0$ such that

$$T\big(B_\delta(x_o)\big) \subseteq B_\epsilon(Tx_o) \subseteq V,$$

that is,

$$B_\delta(x_o) \subseteq T^{-1}(V) = U.$$

Since $x_o \in U$ was arbitrary, it follows that U is open.

Figure 1.17: $T\big(B_\delta(x_o)\big) \subseteq B_\epsilon(Tx_o)$.

$\Leftarrow$: Now suppose that $T^{-1}(V)$ is open in X for each open set $V \subseteq Y$. Let $x_o \in X$ be arbitrary, and $\epsilon > 0$ be given. Since $B_\epsilon(Tx_o)$ is open in Y, it follows from the assumption that $T^{-1}\big(B_\epsilon(Tx_o)\big)$ is open in X. Now $x_o \in T^{-1}\big(B_\epsilon(Tx_o)\big)$, and thus there exists $\delta > 0$ such that

$$B_\delta(x_o) \subseteq T^{-1}\big(B_\epsilon(Tx_o)\big)$$

and hence

$$T\big(B_\delta(x_o)\big) \subseteq B_\epsilon(Tx_o).$$

This shows that T is continuous at x_o. As $x_o \in X$ was arbitrary, we conclude that T is continuous on X. □

Exercise 1.4.1. Let (X, d) and (Y, σ) be metric spaces, and $T : X \to Y$. Show that T is continuous $\Leftrightarrow$ $T^{-1}(F)$ is closed in X for each closed set $F \subseteq Y$.

1.4.3 Continuity on Subsets

Let (X, d) and (Y, σ) be metric spaces. Recall that if $E \subseteq X$, then (E, d) is itself a metric space. Now let $T : D(T) \subseteq X \to Y$ be a mapping. If $E \subseteq D(T)$, then we say that T is *continuous on* E (respectively *uniformly continuous on* E), if the mapping $T : E \to Y$ is continuous (respectively uniformly continuous). That is,

1. T is *continuous on* E, iff for every $x_o \in E$ and every $\epsilon > 0$ there exists $\delta = \delta(x_o, \epsilon) > 0$ such that

$$\sigma(Tx, Tx_o) < \epsilon \qquad \text{whenever } x \in E \text{ and } d(x, x_o) < \delta.$$

2. T is *uniformly continuous on* E, iff for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\sigma(Tx, Ty) < \epsilon \qquad \text{whenever } x, y \in E \text{ and } d(x, y) < \delta.$$

Figure 1.18: T is continuous on E.

Example 1.4.5. Let $X = Y = \mathbb{R}$ with the usual metric.

1. $f(x) = \frac{1}{x}$ is continuous on $E = (0, \infty)$, but not uniformly continuous.
2. $g(x) = \sqrt{x}$ is uniformly continuous on $E = [0, \infty)$.

1.4.4 Continuous Linear Maps

In the case of linear maps between normed linear spaces we obtain a simpler characterization for continuity. This comes from the fact that the norm is translation invariant.

Recall: If X and Y are vector spaces, then a map $T : X \to Y$ is called a *linear mapping* (or a *linear transformation*) if

1. $T(\alpha x) = \alpha T(x)$, and
2. $T(x + y) = T(x) + T(y)$

for all $x, y \in X$ and scalars α. We often simply write Tx instead of $T(x)$. If 0 denotes the zero vector in X, then by 2., $T(0) = T(0+0) = T(0) + T(0)$, and hence $T(0) = 0$.

Theorem 1.4.2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces, and $T : X \to Y$ a linear mapping. Then T.F.A.E. (The following are equivalent):*

1. *T is continuous at some $x_o \in X$.*
2. *T is continuous.*
3. *T is uniformly continuous.*
4. *There exists a constant $K > 0$ so that*

$$\|Tx\|_Y \leq K \|x\|_X \tag{1.15}$$

for all $x \in X$.

Proof. 1. $\Rightarrow$ 4.: Suppose, T is continuous at x_o. Then for $\epsilon = 1$ there exists $\delta > 0$ such that

$$\|Tz - Tx_o\|_Y < 1 \qquad \text{whenever } z \in X \text{ and } \|z - x_o\|_X < \delta. \tag{1.16}$$

First let $x \in X$ be such that $\|x\|_X = \frac{\delta}{2}$. Set $z = x + x_o$. Then

$$\|z - x_o\|_X = \|x\|_X = \frac{\delta}{2} < \delta,$$

and hence by linearity of T and (1.16),

$$\|Tx\|_Y = \|T(x + x_o) - Tx_o\|_Y = \|Tz - Tx_o\|_Y < 1. \tag{1.17}$$

Now if $x \in X$ is arbitrary, $x \neq 0$, set $u = \frac{\delta}{2\|x\|_X}x$. Then $\|u\|_X = \frac{\delta}{2}$, so by (1.17),

$$\|Tu\|_Y < 1.$$

That is,

$$\left\|T\left(\frac{\delta}{2\|x\|_X}x\right)\right\|_Y = \left\|\frac{\delta}{2\|x\|_X}(Tx)\right\|_Y = \frac{\delta}{2\|x\|_X}\|Tx\|_Y < 1$$

where we have used (N3) and linearity of T. Setting $K = \frac{2}{\delta}$ we thus have that

$$\|Tx\|_Y < K\,\|x\|_X.$$

for all $x \neq 0$. Finally, if $x = 0$ then (1.15) holds trivially.

4. $\Rightarrow$ 3.: Suppose, $\|Tx\|_Y \leq K\,\|x\|_X$ for all $x \in X$. Given $\epsilon > 0$, set $\delta = \frac{\epsilon}{K}$. Then whenever $x, y \in X$ with $\|x - y\|_X < \delta$ we have

$$\|Tx - Ty\|_Y = \|T(x - y)\|_Y \leq K\,\|x - y\|_X < K\delta = \epsilon.$$

This shows that T is uniformly continuous.

3. $\Rightarrow$ 2. and 2. $\Rightarrow$ 1. are obvious. □

Example 1.4.6. Consider the real normed linear space $X = \big(C[a,b], \|\cdot\|_u\big)$.

1. Fix a point $x_o \in X$, and define $T : C[a,b] \to \mathbb{R}$ by

$$T(f) = f(x_o).$$

   Obviously, T is linear, and $|T(f)| = |f(x_o)| \leq \|f\|_u$. Thus, by the theorem, T is continuous.

2. The mapping $T : C]a,b] \to \mathbb{R}$ given by

$$T(f) = \int_a^b f(t)\,dt$$

   is linear, and since $|T(f)| \leq (b-a)\|f\|_u$ for all f, also continuous.

3. The mapping $S : C]a, b] \to C[a, b]$ given by
$$(Sf)(x) = \int_a^x f(t)\, dt$$
is linear. Now for each $x \in [a, b]$,
$$|(Sf)(x)| \leq (x - a)\|f\|_u \leq (b - a)\|f\|_u.$$
Thus, $\|Sf\|_u \leq (b - a)\|f\|_u$ for all f, which shows that S is continuous.

Exercise 1.4.2. Let $T : C[a, b] \to C[a, b]$ be given by $T(f) = f^2$. Is T linear ? Show that T is continuous, but not uniformly continuous.

Exercise 1.4.3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, and let $\|\cdot\|$ be any norm on $\mathbb{R}^m$. Show:

1. If $\mathbb{R}^n$ carries the norm $\|\cdot\|_1$, then T is continuous.
2. If $\mathbb{R}^n$ carries the norm $\|\cdot\|_\infty$, then T is continuous.
3. If $\mathbb{R}^n$ carries the norm $\|\cdot\|_2$, then T is continuous.

Definition 1.4.2. Let d and σ be two metrics on a set X.

1. We say that d and σ are *equivalent*, if there exist constants $a, b > 0$ such that
$$ad(x, y) \leq \sigma(x, y) \leq bd(x, y)$$
for all $x, y \in X$.
2. Set
$$\tau_d = \{U \subseteq X : U \text{ is open with respect to } d\}$$
and
$$\tau_\sigma = \{U \subseteq X : U \text{ is open with respect to } \sigma\}.$$
We say that d and σ *generate the same topology*, if $\tau_d = \tau_\sigma$.

Exercise 1.4.4. Let d and σ be two metrics on a set X. Show:

1. d and σ generate the same topology iff the identity maps $I : (X, d) \to (X, \sigma)$ and $I : (X, \sigma) \to (X, d)$, given by $I(x) = x$, are continuous.
2. If d and σ on X are equivalent, then they generate the same topology.
3. Let (X, d) be an unbounded metric space, (by unbounded we mean that $\delta(X) = \infty$), and let σ be the metric on X as defined in example 1.2.6. Then d and σ generate the same topology, but are not equivalent. (Thus, inequivalent metrics may generate the same topology.)
4. Let X be a vector space, and $\|\cdot\|$ and $\|\cdot\|'$ two norms on X. Denote the metrics determined by these norms by d and σ, respectively. Then $\|\cdot\|$ and $\|\cdot\|'$ are equivalent $\Leftrightarrow$ d and σ generate the same topology.

1.5 Sequences

1.5.1 Convergence of Sequences

One particular feature of metric spaces is that many topological properties can be described by the behavior of sequences.

Definition 1.5.1. Let (X, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X is said to *be convergent*, if there exists an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{1.18}$$

Then x is called the *limit* of the sequence $\{x_n\}_{n=1}^{\infty}$, and we write

$$\lim_{n\to\infty} x_n = x \qquad \text{or} \qquad x_n \to x \quad (\text{as } n \to \infty)$$

and say that $\{x_n\}_{n=1}^{\infty}$ *converges to* x. If the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge, then it is said to *diverge*, or *be divergent.*

Remark 1.5.1. Using the definition of convergence of the sequence $\{d(x_n, x)\}_{n=1}^{\infty}$ in $\mathbb{R}$, it follows from (1.18) that:

$$\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad \text{given } \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad d(x_n, x) < \epsilon \quad \forall n \geq N, \tag{1.19}$$

or also

$$\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad \text{given } \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad x_n \in B_\epsilon(x) \quad \forall n \geq N. \tag{1.20}$$

That is, every open neighborhood of x contains the tail of the sequence $\{x_n\}_{n=1}^{\infty}$.

Figure 1.19: Convergent sequence

Remark 1.5.2. Let $(X, \|\cdot\|)$ be a normed linear space, and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Then (1.18) becomes

$$\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad \lim_{n\to\infty} \|x_n - x\| = 0.$$
$$\Leftrightarrow \quad \text{given } \epsilon > 0 \quad \exists N \in \mathbb{N} \text{ such that } \|x_n - x\| < \epsilon \text{ for all } n \geq N.$$

Example 1.5.1. Let $X = \big(C[a,b], \|\cdot\|_u\big)$. If $\{f_n\}_{n=1}^{\infty} \subset C[a,b]$, $f \in C[a,b]$ then

$$\begin{aligned} f_n \to f \quad &\Leftrightarrow \quad \text{given } \epsilon > 0 \quad \exists N \in \mathbb{N} \text{ such that } \max_{x\in[a,b]} |f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq N \\ &\Leftrightarrow \quad \text{given } \epsilon > 0 \quad \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \epsilon \quad \forall n \geq N,\ \forall x \in [a,b]. \end{aligned}$$

That is, the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f in $C[a,b]$ if and only if $f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$.

Example 1.5.2. Let (X,d) be a discrete metric space. Suppose, $x_n \to x$. Then given $\epsilon = 1$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ for all $n > N$. But then $d(x_n, x) = 0$, that is, $x_n = x$, for all $n > N$. So the only convergent sequences are those whose tails are constant.

Theorem 1.5.1. *Let (X,d) be a metric space, and $\{x_n\}_{n=1}^{\infty}$ a convergent sequence in X. Then*

1. *the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded, and*

2. *its limit is unique.*

Proof. 1. To prove boundedness, suppose that $x_n \to x$. Then, for $\epsilon = 1$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ for all $n \geq N$. Set

$$M := \max\{\, d(x_1, x),\ d(x_2, x),\ \ldots,\ d(x_{N-1}, x),\ 1\,\}.$$

Then $d(x_n, x) \leq M$ for all n. Then by the triangle inequality, for all $n, m \in \mathbb{N}$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \leq M + M = 2M.$$

This shows that the set $\{x_n\}_{n=1}^{\infty}$ is bounded in X.

2. To prove uniqueness, suppose that $x_n \to x$ and $x_n \to y$ in X. Then by the triangle inequality, we have for all $n \in \mathbb{N}$,

$$0 \leq d(x,y) \leq d(x, x_n) + d(x_n, y).$$

Let $n \to \infty$. Then

$$0 \leq d(x,y) \leq \lim_{n\to\infty} d(x, x_n) + \lim_{n\to\infty} d(x_n, y) = 0 + 0 = 0$$

by assumption of convergence. It follows that $d(x,y) = 0$, and hence by definiteness of the metric, that $x = y$. □

1.5.2 Topological Properties and Sequences

Theorem 1.5.2. *(Sequential Characterization of Continuity)*
Let (X,d) and (Y,σ) be metric spaces, $T : X \to Y$ a mapping and $x_o \in X$. Then T is continuous at x_o $\Leftrightarrow$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X with $x_n \to x_o$ in X, then $Tx_n \to Tx_o$ in Y.

Proof. $\Rightarrow$: Suppose, T is continuous at x_o, and $x_n \to x_o$ in X. Let $\epsilon > 0$ be given. As T is continuous at x_o, there exists $\delta > 0$ such that

$$\sigma(Tx, Tx_o) < \epsilon \quad \text{whenever} \quad d(x, x_o) < \delta. \tag{1.21}$$

On the other hand, as $x_n \to x_o$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_o) < \delta$ for all $n \geq N$. Hence by (1.21),

$$\sigma(Tx_n, Tx_o) < \epsilon \quad \text{whenever} \quad n \geq N.$$

Since ϵ was arbitrary, it follows that $Tx_n \to Tx_o$.

$\Leftarrow$: Suppose to the contrary, that $x_n \to x_o$ implies $Tx_n \to Tx_o$, but T is not continuous at x_o. Then there exists $\epsilon > 0$ such that for *every* $\delta > 0$ we can find $x \in X$ with $d(x, x_o) < \delta$ but $\sigma(Tx, Tx_o) \geq \epsilon$. Choosing $\delta = \frac{1}{n}$ for $n = 1, 2, \ldots$, we thus find elements $x_n \in X$ with $d(x_n, x_o) < \frac{1}{n}$, but $\sigma(Tx_n, Tx_o) \geq \epsilon$. That is, $x_n \to x_o$ while $Tx_n \not\to Tx_o$, a contradiction to our assumption. Thus, T must be continuous at x_o. □

Figure 1.20: T is continuous at x_o

Theorem 1.5.3. *(Sequential Characterization of Closed Sets)*
Let (X, d) be a metric space and $M \subseteq X$ be non-empty. Then

1. $x \in Acc(M)$ $\Leftrightarrow$ *there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ with $x_n \neq x$ for all n, such that $x_n \to x$.*
2. $x \in \overline{M}$ $\Leftrightarrow$ *there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ such that $x_n \to x$.*
3. *M is closed* $\Leftrightarrow$ *whenever $\{x_n\}_{n=1}^{\infty} \subseteq M$ converges in X, say $x_n \to x$, then $x \in M$.*

Proof. 1. $\Rightarrow$: Let $x \in \text{Acc}(M)$. Then for each $\epsilon = \frac{1}{n}$ the ball $B_{\frac{1}{n}}(x)$ contains an element $x_n \in M$ with $x_n \neq x$. Since

$$0 \leq d(x_n, x) < \frac{1}{n} \to 0 \qquad \text{as } n \to \infty$$

it follows from the squeeze theorem that $d(x_n, x) \to 0$, that is, $x_n \to x$. Hence, $\{x_n\}_{n=1}^{\infty}$ is the desired sequence.

$\Leftarrow$: Suppose there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$, $x_n \neq x$, such that $x_n \to x$. Let U be any open neighborhood of x. Pick $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Now since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$. In particular, x_N is an element of M with $x_N \neq x$ and $x_N \in B_\epsilon(x) \subseteq U$. Since U was arbitrary, it follows that x is a cluster point of M.

2. $\Rightarrow$: Let $x \in \overline{M} = M \cup \text{Acc}(M)$. Then $x \in M$ or $x \in \text{Acc}(M)$. If $x \in M$, let $\{x_n\}_{n=1}^{\infty}$ be the constant sequence, $x_n = x$ for all n. Then obviously, $x_n \to x$ as $n \to \infty$. On the other hand, if $x \in \text{Acc}(M)$, then by 1., there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in M such that $x_n \to x$.

   $\Leftarrow$: Let $x \in X$ and suppose, there exists sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ such that $x_n \to x$. If $x \in M$ then obviously, $x \in \overline{M}$ and we are done. If $x \notin M$, then $x_n \neq x$ for all n, and hence by part 1., $x \in \text{Acc}(M) \subseteq \overline{M}$.

3. $\Rightarrow$: Suppose, M is closed. Let $\{x_n\}_{n=1}^{\infty} \subseteq M$ with $x_n \to x \in X$. Then by part 2., $x \in \overline{M}$. But M is closed, that is, $\overline{M} = M$, and hence $x \in M$.

   $\Leftarrow$: Assume that whenever $\{x_n\}_{n=1}^{\infty} \subseteq M$ and $x_n \to x \in X$, then $x \in M$. Let $x \in \text{Acc}(M)$. Then by part 1., there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq M$ such that $x_n \to x$. Then by assumption, $x \in M$. We have shown that $\text{Acc}(M) \subseteq M$. Hence, $\overline{M} = M \cup \text{Acc}(M) \subseteq M \cup M = M$ which shows that M is closed.

□

Example 1.5.3. Let $X = \mathbb{R}^2$ and $M = \{\vec{x} = (x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$.

We claim that M is closed. In fact, let $\{\vec{x}_n\}_{n=1}^{\infty} \subset M$ be convergent, say $\vec{x}_n = (x_n, y_n) \to \vec{x} = (x, y)$. Then $x_n \to x$ and $y_n \to y$. Since $|x_n y_n| \leq 1$ for all n, by the comparison test,

$$|xy| = |(\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)| = \lim_{n\to\infty} |x_n y_n| \leq 1$$

as well, which shows that $\vec{x} = (x, y) \in M$. Hence, M is closed by the theorem.

Example 1.5.4. Let $X = C[a, b]$ with the uniform norm. Fix $E \subset [a, b]$, and let

$$M = \{f \in C[a, b] : f(x) = 0 \quad \forall\, x \in E\}.$$

We claim that M is closed in $C[a, b]$. In fact, suppose $\{f_n\}_{n=1}^{\infty} \subset M$, $f \in C[a, b]$ and $f_n \to f$ in $C[a, b]$. Then by example 1.5.1, $f_n(x) \to f(x)$ uniformly on $[a, b]$. In particular, for all $x \in E$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$$

which shows that $f \in M$ also. Thus by the theorem, M is closed.

Theorem 1.5.4. *(Continuity of the Metric) Let (X, d) be a metric space.*

1. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in X, say $x_n \to x$, then for all $y \in X$,*

$$d(x_n, y) \to d(x, y).$$

2. *If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent sequences in X, say $x_n \to x$ and $y_n \to y$, then*

$$d(x_n, y_n) \to d(x, y).$$

Proof. Let us prove 2. first. Suppose $x_n \to x$ and $y_n \to y$. Then by the triangle inequality, for all $n \in \mathbb{N}$ we have

$$d(x_n, y_n) - d(x, y) \leq \big(d(x_n, x) + d(x, y) + d(y, y_n)\big) - d(x, y) = d(x_n, x) + d(y, y_n) \tag{1.22}$$

Interchanging x_n and x, and y_n and y in (1.22), we obtain

$$d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y_n, y)$$

Both inequalities can be combined to

$$0 \leq |d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y)$$

for all n. Now the right-hand side goes to zero as $n \to \infty$. It follows from the squeeze theorem that

$$\lim_{n\to\infty} |d(x_n, y_n) - d(x, y)| = 0$$

and hence

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

Part 1. now follows immediately by choosing $y_n = y$ for all n. □

For normed linear spaces we have:

Theorem 1.5.5. *Let $(X, \|\cdot\|)$ be a normed linear space.*

1. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in X, say $x_n \to x$, then*

$$\lim_{n\to\infty} \|x_n\| = \|x\|.$$

(That is, the norm is a continuous function.)

2. *If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent sequences in X, say $x_n \to x$ and $y_n \to y$, and if $\{\alpha_n\}_{n=1}^{\infty}$ is a convergent sequence of scalars, say $\alpha_n \to \alpha$,then*

   (a) $x_n + y_n \to x + y$.

   (b) $\alpha_n x_n \to \alpha x$.

(That is, the vector space operations are continuous.)

Proof. 1. Suppose, $x_n \to x$. Then by theorem 1.5.4, part 1,

$$\|x_n\| = \|x_n - 0\| = d(x_n, 0) \to d(x, 0) = \|x - 0\| = \|x\|.$$

2. Suppose, $x_n \to x$ and $y_n \to y$. Then

$$\begin{aligned}\|(x_n + y_n) - (x + y)\| &= \|(x_n - x) - (y - y_n)\| \\ &\leq \|x_n - x\| + \|y - y_n\| \to 0 + 0 = 0\end{aligned}$$

which shows that $x_n + y_n \to x + y$. Now if $\alpha_n \to \alpha$, then there exists $M > 0$ such that $|\alpha_n| \leq M$ for all n, so that

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| = \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\| \\ &\leq \|\alpha_n(x_n - x)\| + \|(\alpha_n - \alpha)x\| = |\alpha_n| \, \|x_n - x\| + |\alpha_n - \alpha| \, \|x\| \\ &\leq M \, \|x_n - x\| + |\alpha_n - \alpha| \, \|x\| \to 0 + 0 = 0\end{aligned}$$

which shows that $\alpha_n x_n \to \alpha x$.

□

1.5.3 Cauchy Sequences

Definition 1.5.2. Let (X, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is called *Cauchy* if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon \quad \text{for all } m, n \geq N. \tag{1.23}$$

Remark 1.5.3. It is is easy to see that (1.23) can be replaced by

$$d(x_m, x_n) < \epsilon \quad \text{for all } m > n \geq N.$$

Theorem 1.5.6. *Let (X, d) be a metric space. Then*

1. *Every Cauchy sequence is bounded.*
2. *Every convergent sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is Cauchy.*

Proof. 1. Suppose, $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Then given $\epsilon = 1$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. In particular, $d(x_n, x_N) < 1$ for all $n \geq N$. Set

$$M := \max\{\, d(x_1, x_N),\, d(x_2, x_N), \ldots, d(x_{N-1}, x_N),\, 1\,\}.$$

Then for all $m, n \in \mathbb{N}$,we have

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) \leq M + M = 2M$$

which shows that the set $\{x_n\}_{n=1}^{\infty}$ is bounded.

2. Suppose, $x_n \to x$ in X. Let $\epsilon > 0$ be arbitrary, but given. By convergence, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon/2 \qquad \text{for all } n \geq N.$$

Thus if $m, n \geq N$ we have by the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. □

Example 1.5.5. The converse statement of 2. is false: If $\{x_n\}_{n=1}^\infty$ is Cauchy in X, it need not converge in X.

For example, let $X = (0, 1]$ with the usual metric, and $x_n = \frac{1}{n}$. Since the sequence $\{x_n\}_{n=1}^\infty$ converges in $\mathbb{R}$, it is Cauchy in $\mathbb{R}$, and hence in X. However, as $x_n \to 0$, and $0 \notin X$, and limits are unique, this sequence has no limit in X.

Loosely speaking, a Cauchy sequence 'wants' to have a limit, but the space X may not be large enough to contain a limit.

Definition 1.5.3. A metric space (X, d) is said to be *complete*, if every Cauchy sequence $\{x_n\}_{n=1}^\infty \subseteq X$ converges in X. A normed linear space which is complete (in the metric induced by its norm) is called a *Banach space.*

Example 1.5.6. (Example 1.5.5 continued) $X = (0, 1]$, with the usual metric, is not complete.

Example 1.5.7. In a basic course on mathematical analysis one learns that $\mathbb{R}$ is complete in the usual topology. (This is a consequence of the completeness axiom for $\mathbb{R}$.) It follows from exercise 1.5.5 below that $\mathbb{C}$, $\mathbb{R}^n$ and $\mathbb{C}^n$ are all complete in the usual metric.

Example 1.5.8. Suppose (X, d) is a discrete metric space. Let $\{x_n\}_{m=1}^\infty$ be a Cauchy sequence in X. Then for $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < 1 \qquad \text{for all } m, n \geq N,$$

that is, $x_n = x_m$ for all $m, n \geq N$. Thus, the tail of every Cauchy sequence is constant, and hence X is complete.

Exercise 1.5.1. Let (X, d) be a metric space and $\{x_n\}_{n=1}^\infty$ a Cauchy sequence in X. Show: If $\{x_n\}_{n=1}^\infty$ has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which converges, say $x_{n_k} \to x$ as $k \to \infty$, then $x_n \to x$ as $n \to \infty$.

Exercise 1.5.2. Let X be a set, and d and σ be two equivalent metrics on X. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X, and $x \in X$. Show:

1. $x_n \to x$ in $(X, d) \quad \Leftrightarrow \quad x_n \to x$ in (X, σ).
2. $\{x_n\}$ is Cauchy in $(X, d) \quad \Leftrightarrow \quad \{x_n\}$ is Cauchy in (X, σ).
3. (X, d) is complete $\quad \Leftrightarrow \quad (X, \sigma)$ is complete.

Exercise 1.5.3. Let $X = \mathbb{R}$, with the usual metric d, and let

$$\sigma(x, y) = |\arctan x - \arctan y| \qquad (x, y \in \mathbb{R})$$

be the metric of exercise 1.5. Show that

1. d and σ generate the same topology.
2. d and σ are not equivalent.

3. $(\mathbb{R}, \sigma)$ is not complete.

(So even though both metrics have the same open sets, one metric is complete why the other is not.)

Example 1.5.9. $(C[a,b], \|\cdot\|_u)$ is complete, i.e. a Banach space. In fact, let $\{f_n\}_{n=1}^{\infty} \subseteq C[a,b]$ be Cauchy. Then given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_u = \max_{x \in [a,b]} |f_n(x) - f_m(x)| < \epsilon$$

for all $n, m \geq N$, which is equivalent to

$$|f_n(x) - f_m(x)| < \epsilon \qquad \forall x \in [a,b],\ \forall\, n, m \geq N.$$

That is, the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ is *uniformly Cauchy* on $[a,b]$. In particular, for fixed $x \in [a,b]$, the sequence of real numbers $\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ (respectively $\mathbb{C}$), and thus converges by completeness of $\mathbb{R}$ (or $\mathbb{C}$). We thus can define a function f by setting

$$f(x) = \lim_{n \to \infty} f_n(x)$$

for $x \in [a,b]$.

Claim: $f_n(x) \to f(x)$ uniformly on $[a,b]$. For let $\epsilon > 0$ be given. Choose $n \in \mathbb{N}$ so that

$$|f_n(x) - f_m(x)| < \epsilon/2 \qquad \forall x \in [a,b],\ \forall\, n, m \geq N.$$

Now let $m \to \infty$. By continuity of the absolute value,

$$|f_n(x) - f(x)| = |f_n(x) - \lim_{m \to \infty} f_m(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \epsilon/2 < \epsilon \qquad (1.24)$$

for all $n > N$ and all $x \in [a,b]$. This proves the claim.

Now from basic analysis we know that the uniform limit of a sequence of continuous functions is continuous. Thus, $f \in C[a,b]$ as well. Finally, (1.24) shows that

$$\|f_n - f\|_u < \epsilon$$

for all $n > N$. As ϵ was arbitrary, it follows that $f_n \to f$ in $C[a,b]$. We have shown that $C[a,b]$ is complete.

Example 1.5.10. (A normed linear space which is not complete)
Let $X = C[0,1]$, and define

$$\|f\|_1 = \int_0^1 |f(x)|\, dx$$

for $f \in C[0,1]$. Then $\|\cdot\|_1$ is a norm on $C[0,1]$. (Check !)

Figure 1.21: A Cauchy sequence which does not converge

We will show that $(X, \|\cdot\|_1)$ is not complete. In fact, consider the sequence $\{f_n\}_{n=2}^{\infty}$ in $C[0,1]$ given by

$$f_n(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ n(x - \frac{1}{2}) & \text{if } \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} < x \leq 1. \end{cases}$$

We claim that $\{f_n\}_{n=2}^{\infty}$ is Cauchy. In fact, for $m > n$ we have

$$\begin{aligned} d(f_n, f_m) = \|f_n - f_m\|_1 &= \int_0^1 |f_n(x) - f_m(x)|\, dx \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - f_m(x)|\, dx \leq \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} 1\, dx = \frac{1}{n} \end{aligned}$$

where we have used the fact that $f_n(x) \neq f_m(x)$ on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]$ only, and $0 \leq f_n(x) \leq f_m(x) \leq 1$. Hence,

$$d(f_n, f_m) < \epsilon$$

provided that $m > n \geq \frac{1}{\epsilon}$, which shows that $\{f_n\}_{n=2}^{\infty}$ is Cauchy.

Next we claim that $\{f_n\}$ does not converge in $(X, \|\cdot\|_1)$. For suppose to the contrary that there exists $f \in C[a,b]$ such that $\|f_n - f\|_1 \to 0$. Then,

$$\begin{aligned} 0 = \lim_{n\to\infty} \|f_n - f\|_1 &= \lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|\, dx \\ &= \lim_{n\to\infty} \left(\int_0^{\frac{1}{2}} |f(x)|\, dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - f(x)|\, dx + \int_{\frac{1}{2}+\frac{1}{n}}^1 |1 - f(x)|\, dx \right) \\ &= \int_0^{\frac{1}{2}} |f(x)|\, dx + 0 + \int_{\frac{1}{2}}^1 |1 - f(x)|\, dx \end{aligned}$$

which only can happen if

$$\int_0^{\frac{1}{2}} |f(x)|\, dx = 0 \qquad \text{and} \qquad \int_{\frac{1}{2}}^1 |1 - f(x)|\, dx = 0.$$

By continuity of f, the left equality implies that $f(x) = 0$ on $[0, \frac{1}{2}]$. and the right equality that $1 - f(x) = 0$ on $[\frac{1}{2}, 1]$. That is, $f(\frac{1}{2}) = 0$ while also, $f(\frac{1}{2}) = 1$, which is impossible. This proves the claim, and hence that $(X, \|\cdot\|_1)$ is not complete.

Note that by exercise 1.5.2, the two norms $\|\cdot\|_u$ and $\|\cdot\|_1$ on $C[0,1]$ are not equivalent.

Exercise 1.5.4. Show that l^∞ and c_o are complete. (Hint: proceed as in example 1.5.9.)

The next theorem can sometimes be used to test for completeness.

Theorem 1.5.7. *Let (X, d) be a complete metric space, and $M \subseteq X$. (Thus, M itself is a metric space in the metric d). Then M is complete $\Leftrightarrow$ M is closed in X.*

Proof. $\Rightarrow$: Suppose, M is complete. We need to show that $\overline{M} \subseteq M$. We may assume that M is non-empty, for the assertion is obvious if $M = \emptyset$.

To this end, let $x \in \overline{M}$ be arbitrary. Then by theorem 1.5.3, there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq M$ such that $x_n \to x$. Since this sequence converges in X, it is Cauchy in X, and thus also in M. But by completeness of M, there exists $y \in M$ such that $x_n \to y$ in M, and hence in X. Now as limits are unique, it follows that $y = x$, that is, $x \in M$. This shows that $\overline{M} \subseteq M$, i.e. that M is closed.

$\Leftarrow$: Suppose that M is closed in X. Let $\{x_n\}_{n=1}^\infty \subseteq M$ be any Cauchy sequence. Then $\{x_n\}_{n=1}^\infty$ is also Cauchy in X, so that by completeness of X, there exists $x \in X$ such that $x_n \to x$. Now as $x_n \in M$ for all n and M is closed, the limit x must be in M as well by theorem 1.5.3. This shows that M is complete. □

Example 1.5.11.

1. $M = [0,1]$ is a complete metric space (in the usual metric), since M is a closed subset of $\mathbb{R}$, and $\mathbb{R}$ is complete.

2. $M = (0,1]$ is not a complete metric space since M is not closed in $\mathbb{R}$.

3. $M = \{(x,y) \in \mathbb{R}^2 : y = \sin\frac{1}{x},\ x > 0\}$ is not a complete metric space as M is not closed in $\mathbb{R}^2$.

Exercise 1.5.5. Let $(X_1, \sigma_2), (X_1, \sigma_2), \ldots, (X_n, \sigma_n)$ be metric spaces, and let d_1, d_2 and d_∞ denote the metrics on $X = X_1 \times X_2 \times \cdots \times X_n$ as defined in example 1.11.

1. Denote sequences in X by $\{x^{(k)}\}_{k=1}^\infty$ where $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$, $x_i^{(k)} \in X_i$, $i = 1, \ldots, n$. Show:

   (a) $\{x^{(k)}\}_{k=1}^\infty$ converges in X $\Leftrightarrow$ each component sequence $\{x_i^{(k)}\}_{k=1}^\infty$ converge in X_i $(i = 1, \ldots, n)$. Furthermore

   $$\lim_{k\to\infty} x^{(k)} = \big(\lim_{k\to\infty} x_1^{(k)}, \lim_{k\to\infty} x_2^{(k)}, \ldots, \lim_{k\to\infty} x_n^{(k)}\big).$$

   (b) $\{x^{(k)}\}_{k=1}^\infty$ is Cauchy in X $\Leftrightarrow$ each component sequence $\{x_i^{(k)}\}_{k=1}^\infty$ is Cauchy in X_i, $i = 1, \ldots, n$.

2. Show that X is complete $\Leftrightarrow$ $(X_1, \sigma_2), (X_1, \sigma_2), \ldots, (X_n, \sigma_n)$ are all complete.

1.5.4 Infinite Series in a Normed Linear Space

Let $(X\|\cdot\|)$ be a normed linear space, and

$$\sum_{k=1}^{\infty} x_k \qquad (x_k \in X) \tag{1.25}$$

be an infinite series. The notion of convergence of this series is defined just as in the case of a series of real numbers, by considering its partial sums

$$S_n = \sum_{k=1}^{n} x_k.$$

Definition 1.5.4. We say that the series (1.25) *converges* in X, if its sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges. In this case, we define its sum by

$$\sum_{k=1}^{\infty} x_k := \lim_{n\to\infty} S_n \left(= \lim_{n\to\infty} \sum_{k=1}^{n} x_k\right).$$

Note that by definition of convergence of the sequence $\{S_n\}_{n=1}^{\infty}$,

$$\sum_{k=1}^{\infty} x_k = x \quad \Leftrightarrow \quad \text{given } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that}$$

$$\left\|\sum_{k=1}^{N} x_k - x\right\| < \epsilon \quad \text{for all } n \geq N.$$

Definition 1.5.5. We say that the series (1.25) *converges absolutely*, if

$$\sum_{k=1}^{\infty} \|x_k\|$$

converges in $\mathbb{R}$.

Example 1.5.12. From basic analysis we know that every absolutely convergent series of real numbers converges. This is no longer true in a general normed linear space.

For example, let $X = C[0,1]$ with the norm $\|\cdot\|_1$, and $\{f_n\}_{n=1}^{\infty}$ as in example 1.5.10,

$$f_n(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ n(x-\frac{1}{2}) & \text{if } \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } \frac{1}{2} + \frac{1}{n} < x \leq 1. \end{cases}$$

Since $\{f_n\}_{n=1}^{\infty}$ diverges and is Cauchy, by exercise 1.5.1, the subsequence $\{f_{2^k}\}_{k=1}^{\infty}$ must also diverge. Now set

$$g_k(x) = f_{2^{k+1}} - f_{2^k} \qquad (k = 1, 2, \dots)$$

Figure 1.22: The integral is evaluated easily geometrically

We claim that $\sum_{k=1}^{\infty} g_k$ converges absolutely, but does not converge in $\big(C[0,1], \|\cdot\|_1\big)$.

To see this, note that $\|g_k\|_1 = \int_0^1 g(x)\,dx$ is the area of the indicated triangle, that is,

$$\|g_k\|_1 = \frac{1}{2}\frac{1}{2^{k+1}} = \frac{1}{2^{k+2}}.$$

Thus,

$$\sum_{k=1}^{\infty} \|g_k\|_1 = \sum_{k=1}^{\infty} \frac{1}{2^{k+2}} = \frac{1}{4} < \infty$$

so that $\sum_{k=1}^{\infty} g_k$ converges absolutely. On the other hand, for each N,

$$\sum_{k=1}^{N} g_k = \sum_{k=1}^{N} (f_{2^{k+1}} - f_{2^k}) = f_{2^{N+1}} - f_2.$$

Since $\{f_{2^N}\}_{N=1}^{\infty}$ diverges, it follows that $\sum_{k=1}^{\infty} g_k$ must diverge also. This proves the claim.

In fact, we have the following:

Theorem 1.5.8. *Let $(X, \|\cdot\|)$ be a normed linear space. Then X is a Banach space $\Leftrightarrow$ every absolutely convergent series converges.*

Proof. $\Rightarrow$: Suppose that X is a Banach space. Let $\sum_{k=1}^{\infty} x_k$ be an absolutely convergent series, that is

$$\sum_{k=1}^{\infty} \|x_k\| < \infty. \tag{1.26}$$

Then the sequence of partial sums of 1.26 is Cauchy in $\mathbb{R}$, that is, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} \|x_k\| < \epsilon$$

for all $n > m \geq N$. So if $S_n = \sum_{k=1}^{n} x_k$, then for all $n > m \geq N$,

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| < \epsilon,$$

that is, the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is Cauchy in X. By completeness, this sequence converges in X, that is,

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} S_n$$

exists in X.

$\Leftarrow$: Now suppose that every absolutely convergent series is convergent in X. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. We must show that $\{x_n\}_{n=1}^{\infty}$ converges. The idea is to construct a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ first. In fact, since $\{x_n\}_{n=1}^{\infty}$ is Cauchy we can pick $n_1 \in \mathbb{N}$ such that

$$\|x_n - x_{n_1}\| < \frac{1}{2} \qquad \text{for all } n \geq n_1.$$

Next we pick $n_2 > n_1$ such that

$$\|x_n - x_{n_2}\| < \frac{1}{2^2} \qquad \text{for all } n \geq n_2.$$

Continuing inductively, we obtain a sequence $n_1 < n_2 < n_3 < \ldots$ of positive integers such that

$$\|x_n - x_{n_k}\| < \frac{1}{2^k} \qquad \text{for all } n \geq n_k.$$

Set $y_k = x_{n_{k+1}} - x_{n_k}$ $(k = 1, 2 \ldots)$. Then as $n_{k+1} > n_k$, we have

$$\sum_{k=1}^{\infty} \|y_k\| = \sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

that is, $\sum_{k=1}^{\infty} y_k$ converges absolutely. Then by assumption,

$$\sum_{k=1}^{\infty} y_k$$

converges to some $x \in X$. But this is a telescoping series, so that

$$x = \lim_{N\to\infty} \sum_{k=1}^{N} \left(x_{n_{k+1}} - x_{n_k}\right) = \lim_{N\to\infty} \left(x_{n_{N+1}} - x_{n_1}\right)$$

which shows that

$$\lim_{N\to\infty} x_{n_{N+1}} = x + x_{n_1},$$

that is, $\{x_{n_k}\}_{k=1}^{\infty}$ converges in X. Now that we have found this convergent subsequence of $\{x_n\}_{n=1}^{\infty}$, it follows from exercise 1.5.1 that the sequence $\{x_n\}_{n=1}^{\infty}$ itself converges in X. $\square$

1.6 Completion of Metric Spaces

We want to show now that every metric space can be "enlarged" to a complete metric space.

1.6.1 Density

Definition 1.6.1. Let (X, d) be a metric space. A set $M \subseteq X$ is said to be *dense in* X, if $\overline{M} = X$.

Remark 1.6.1. Let (X, d) be a metric space and $M \subseteq X$ non-empty. Since $\overline{M} = M \cup \mathrm{Acc}(M)$ it follows directly:

$$
\begin{aligned}
M \text{ is dense in } X \quad &\Leftrightarrow \quad \overline{M} = X \\
&\underset{\text{exer. 1.3.10}}{\Leftrightarrow} \text{ for each } x_o \in X \text{ and each open ball } B_\epsilon(x_o) \text{ there exists } x \in M \cap B_\epsilon(x_o) \\
&\Leftrightarrow \quad \text{for each } x_o \in X \text{ and } \epsilon > 0 \text{ there exists } x \in M \text{ such that } d(x, x_o) < \epsilon \\
&\underset{\text{thm 1.5.3}}{\Leftrightarrow} \text{ for each } x_o \in X \text{ there exists a sequence } \{x_n\}_{n=1}^\infty \text{ in } M \text{ such that } x_n \to x_o.
\end{aligned}
$$

Definition 1.6.2. Let (X, d) be a metric space. X is called *separable*, if there exists a *countable* dense subset M in X.

Example 1.6.1. In a first analysis course, one usually proves the following *density theorem*: Given $x_1 < x_2 \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $x_1 < q < x_2$.

Now let $x_o \in \mathbb{R}$ be arbitrary. Then for each n, there exists $x_n \in \mathbb{Q}$ with $x_o < x_n < x_o + \frac{1}{n}$. By the squeeze theorem, $\lim_{n\to\infty} x_n = x_o$. This shows that $\mathbb{Q}$ is dense in $\mathbb{R}$, that is, $\mathbb{R}$ is separable.

Exercise 1.6.1. Let $(X_1, \sigma_1), \ldots, (X_n, \sigma_n)$ be metric spaces, and for each $i = 1 \ldots n$, let M_i be a dense subset of X_i. Give $X_1 \times \cdots \times X_n$ the metric d_∞. Show:

1. $M_1 \times \cdots \times M_n$ is dense in $X_1 \times \cdots \times X_n$.
2. If the spaces $X_1, \ldots, X_n$ are all separable, then so is $X := X_1 \times \cdots \times X_n$.

(Since the metrics d_1, d_2 and d_∞ are all equivalent on $X_1 \times \cdots \times X_n$, they have the same open sets, and the same convergent sequences by exercise 1.5.5. It follows that the above statements also hold for the metrics d_1 and d_2. Furthermore, as $\mathbb{Q}$ is dense in $\mathbb{R}$ it follows that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, thus $\mathbb{R}^n$ is separable as well, in any of the metrics d_1, d_2 and d_∞.)

Exercise 1.6.2. Let $V = \{f : \mathbb{N} \to \mathbb{C} \,\Big|\, \exists N = N(f) \text{ s.t. } f(n) = 0 \;\forall n > N\}$. Show:

1. V is dense in $(\ell^1, \|\cdot\|_1)$ and in $(c_o, \|\cdot\|_\infty)$.
2. $(\ell^1, \|\cdot\|_1)$ and in $(c_o, \|\cdot\|_\infty)$ are separable. (Hint: Recall that the Cartesian product of *finitely many* countable sets is countable, and the countable union of countable sets is countable.)

Remark 1.6.2. One can show that $C[a, b]$ is separable. This is a consequence of the Weierstrass Approximation Theorem (See [10], theorem 14.18). On the other hand, ℓ^∞ is not separable. (See [4], Example 1.3-9).

1.6.2 Isometries

Definition 1.6.3. Let (X, d) and (Y, σ) be two metric spaces.

1. A map $T : X \to Y$ is called an *isometry* if
$$\sigma(Tx_1, Tx_2) = d(x_1, x_2) \tag{1.27}$$
for all $x_1, x_2 \in X$. (That is, T preserves distances.)
2. Two metric spaces (X, d) and (Y, σ) are said to be *isometric*, if there exists an isometry T of X onto Y.

Remark 1.6.3. 1. Every isometry $T : X \to Y$ is one-to-one. In fact, by definiteness [property (M2)] of a metric,
$$Tx_1 = Tx_2 \quad \Leftrightarrow \quad \sigma(Tx_1, Tx_2) = 0 \quad \Leftrightarrow \quad d(x_1, x_2) = 0 \quad \Leftrightarrow \quad x_1 = x_2.$$
Thus, X and Y are isometric iff there exists a bijection T of X onto Y preserving distances. Isometric spaces are indistinguishable as metric spaces.

2. Every isometry $T : X \to Y$ is uniformly continuous. In fact, given $\epsilon > 0$, choose $\delta = \epsilon$. Then whenever $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, we have
$$\sigma(Tx_1, Tx_2) = d(x_1, x_2) < \delta = \epsilon.$$

Example 1.6.2. Let $X = \mathbb{C}$ and $Y = \mathbb{R}^2$, with the usual metrics. It is easy to see that the map $T : \mathbb{C} \to \mathbb{R}^2$ given by $T(x + iy) = (x, y)$ is an isometry of $\mathbb{C}$ onto $\mathbb{R}^2$. Hence, $\mathbb{C}$ and $\mathbb{R}^2$ are isometric spaces.

Remark 1.6.4. Suppose, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are both normed linear spaces. Denote the metrics determined by these two norms by d and σ, respectively. If $T : X \to Y$ is a linear map, then
$$\begin{aligned} T \text{ is an isometry} \quad &\Leftrightarrow \quad \sigma(Tx_1, Tx_2) = d(x_1, x_2) \quad \forall\, x_1, x_2 \in X \\ &\Leftrightarrow \quad \|Tx_1 - Tx_2\|_Y = \|x_1 - x_2\|_X \quad \forall\, x_1, x_2 \in X \\ &\Leftrightarrow \quad \|T(x_1 - x_2)\|_Y = \|x_1 - x_2\|_X \quad \forall\, x_1, x_2 \in X \\ &\Leftrightarrow \quad \|Tz\|_Y = \|z\|_X \quad \forall\, z \in X \qquad (z = x_1 - x_2) \end{aligned}$$
A linear mapping which is an isometry is called a *linear isometry*.

Example 1.6.3. Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space (for example, $X = \mathbb{R}^n$ with the Euclidean norm), and $T : X \to X$ a linear isometry. Then T is one-to-one, that is, $\ker(T) = \{0\}$, so that $\dim(\mathrm{Ran}(T)) = n$. Thus, T maps X onto X.

Example 1.6.4. Let $X = \ell^\infty$ with the supremum norm $\|\cdot\|_\infty$. Define $T : \ell^\infty \to \ell^\infty$ by

$$(Tf)(n) = \begin{cases} 0 & \text{if } n = 1 \\ f(n-1) & \text{if } n > 1. \end{cases}$$

In sequence notation, $T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$. One easily checks that T is a linear isometry, but is not surjective.

Lemma 1.6.1. *Let (X, d) and (Y, σ) be complete metric spaces, $V \subseteq X$ and $T : V \to Y$ an isometry. Then there exists a unique isometry $\widetilde{T} : \overline{V} \xrightarrow{onto} \overline{T(Y)}$ such that $\widetilde{T}x = Tx$ for all $x \in V$. (We say that $\widetilde{T}$* extends *T* from *V* to *$\overline{V}$.)*

Proof. 1. Define $\widetilde{T}$. Let $x \in \overline{V}$ be given. Then by theorem 1.5.3, there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq V$ such that $x_n \to x$. In particular, $\{x_n\}_{n=1}^\infty$ is Cauchy in V. Since T is an isometry,

$$\sigma(Tx_n, Tx_m) = d(x_n, x_m) \qquad \text{for all } m, n$$

and thus $\{Tx_n\}_{n=1}^\infty$ is Cauchy in $T(V)$. Since Y is complete, this sequence converges, and we can thus define

$$\widetilde{T}x = \lim_{n\to\infty} Tx_n.$$

Obviously, $\widetilde{T}x \in \overline{T(V)}$.

2. Show that $\widetilde{T}$ is well defined. Suppose, $\{x'_n\}_{n=1}^\infty$ is another sequence in V with $x'_n \to x$. We need to show that $\lim_{n\to\infty} Tx'_n = \lim_{n\to\infty} Tx_n$.

In fact, since metrics are continuous (theorem 1.5.4) and T is an isometry,

$$\begin{aligned} \sigma\big(\lim_{n\to\infty} Tx_n, \lim_{n\to\infty} Tx'_n\big) &= \lim_{n\to\infty} \sigma\big(Tx_n, Tx'_n\big) \\ &= \lim_{n\to\infty} d(x_n, x'_n) = d\big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x'_n\big) = d(x, x) = 0. \end{aligned}$$

By definiteness of σ, it follows that

$$\lim_{n\to\infty} Tx'_n = \lim_{n\to\infty} Tx_n$$

in Y. Note that if $x \in V$, then we can choose $x_n = x$ for all n, and obtain $\widetilde{T}x = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tx = Tx$. Thus, $\widetilde{T}$ extends T to $\overline{V}$.

3. Show that $\widetilde{T}$ is an isometry. Let $x, x' \in X$, and pick sequences $\{x_n\}_{n=1}^{\infty}$ and $\{x'_n\}_{n=1}^{\infty}$ in V such that $x_n \to x$ and $x'_n \to x'$. Then by theorem 1.5.4 and the fact that T is an isometry, we have

$$\begin{aligned}\sigma\big(\widetilde{T}x, \widetilde{T}x'_n\big) &= \sigma\big(\lim_{n\to\infty} Tx_n, \lim_{n\to\infty} Tx'_n\big) = \lim_{n\to\infty} \sigma\big(Tx_n, Tx'_n\big) \\ &= \lim_{n\to\infty} d\big(x_n, x'_n\big) = d\big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x'_n\big) = d(x, x'),\end{aligned}$$

which shows that $\widetilde{T}$ is an isometry.

4. Show that $\widetilde{T}$ is unique. Suppose that $\widehat{T} : \overline{V} \to Y$ is another isometry satisfying $\widehat{T}x = Tx$ for all $x \in V$. Let $x \in \overline{V}$ be given. Pick a sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ such that $x_n \to x$. Then by continuity of isometries,

$$\widehat{T}x = \widehat{T}(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \widehat{T}x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \widetilde{T}x_n = \widetilde{T}(\lim_{n\to\infty} x_n) = \widetilde{T}x,$$

that is, $\widehat{T}x = \widetilde{T}x$ for all $x \in \overline{V}$.

5. Show that $\widetilde{T}$ maps $\overline{V}$ onto $\overline{T(V)}$. Let $y \in \overline{T(V)}$ be given. Then by theorem 1.5.3, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subseteq T(V)$ such that $y_n \to y$. Since T is one-to-one, we can set $x_n = T^{-1}(y_n) \in V$. Since $\{y_n\}_{n=1}^{\infty}$ is Cauchy, and T is an isometry, it follows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy in V, and thus converges to some $x \in \overline{V}$ by completeness of X. Then

$$\widetilde{T}x = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$$

Thus, $\widetilde{T}$ is onto.

Figure 1.23: $T : V \to Y$ extends to $\widetilde{T} : \overline{V} \xrightarrow{\text{onto}} \overline{T(V)}$. □

Remark 1.6.5. If V is dense in X and $T(V)$ is dense in Y, then by the theorem, T can be extended uniquely to an isometry $\widetilde{T} : X \xrightarrow{\text{onto}} Y$.

In the above proof, completeness of X was only required to show that the range of $\widetilde{T}$ is all of $\overline{T(V)}$. So even if X is not complete, we can still extend T to an isometry $\widetilde{T} : \overline{V} \xrightarrow{\text{into}} \overline{T(V)}$.

Because $\widetilde{T}x = Tx$ for all $x \in V$, we often simply use the symbol T for this extension.

Exercise 1.6.3. Let $(X, \|\cdot\|)$ be a normed linear space, and V a subspace (i.e. a sub-vectorspace) of X. Show: $\overline{V}$ is also a subspace of X.

Corollary 1.6.2. *Let X and Y be Banach spaces, V a subspace of X, and $T : X \to Y$ a linear isometry. Then T extends uniquely to a linear isometry $\widetilde{T} : \overline{V} \xrightarrow{\text{onto}} \overline{T(Y)}$.*

Proof. Let $\widetilde{T}$ be the unique extension of T as in the lemma. We need to show that $\widetilde{T}$ is linear.

Let $x, z \in \overline{V}$ and α, β be scalars. Pick sequences $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ in V such that $x_n \to x$ and $z_n \to z$. Since the vector space operations are continuous, by theorem 1.5.5,

$$\alpha x_n + \beta z_n \to \alpha x + \beta z.$$

Thus by definition of $\widetilde{T}$,

$$\begin{aligned}
\widetilde{T}(\alpha x + \beta z) &= \lim_{n\to\infty} T(\alpha x_n + \beta z_n) \\
&= \lim_{n\to\infty} (\alpha T x_n + \beta T z_n) \qquad (T \text{ is linear}) \\
&= \alpha \lim_{n\to\infty} T x_n + \beta \lim_{n\to\infty} T z_n \\
&= \alpha \widetilde{T} x + \beta \widetilde{T} z.
\end{aligned}$$

Thus, $\widetilde{T}$ is linear. □

Exercise 1.6.4. Show that the lemma and the corollary can be generalized as follows:

1. Let (X, d) and (Y, σ) be metric spaces, Y complete, $V \subseteq X$ and $T : V \to Y$ uniformly continuous. Then there exists a unique uniformly continuous map $\widetilde{T} : \overline{V} \to Y$ such that $\widetilde{T}x = Tx$ for all $x \in V$.

2. Show by example that ordinary continuity of T is not sufficient above.

3. Let X, Y be normed linear spaces, Y complete, V a subspace of X and $T : V \to Y$ a continuous linear map, say

$$\|Tx\| \le k\|x\| \quad \forall\, x \in V.$$

Show that $\widetilde{T}$ is also linear, and

$$\|Tx\| \le k\|x\| \quad \forall\, x \in \overline{V}.$$

1.6.3 Existence of Completion

Theorem 1.6.3. *(Completion of Metric Space)*
Let (X, d) be a metric space. Then there exist a complete metric space $(\widehat{X}, \hat{d})$ and an isometry $T : X \to \widehat{X}$ such that $T(X)$ is dense in $\widehat{X}$.

Furthermore, $\widehat{X}$ is unique in the sense that if $(\widetilde{X}, \tilde{d})$ is another complete metric space, and $\widetilde{T}$ an isometry of X into $\widetilde{X}$ such that $\widetilde{T}(X)$ is dense in $\widetilde{X}$, then there exists an isometry U of $\widehat{X}$ onto $\widetilde{X}$ such that $U(Tx) = \widetilde{T}x$ for all $x \in X$.

Figure 1.24: Essential uniqueness of the completion of X.

Remark 1.6.6. We call $(\widehat{X}, \hat{d})$ the *completion of* (X, d).

Proof. The proof is very lengthy, and we thus proceed in steps:

1. Construct $(\widehat{X}, \hat{d})$.
2. Construct the isometry T.
3. Show that $(\widehat{X}, \hat{d})$ is complete.
4. Show that $\widehat{X}$ is unique up to an isometry.

1. Construct $(\widehat{X}, \hat{d})$.

   Let us call two Cauchy sequences $\{x_n\}$ and $\{x'_n\}$ in X *equivalent*, and write $\{x_n\} \sim \{x'_n\}$, if
   $$\lim_{n\to\infty} d(x_n, x'_n) = 0. \tag{1.28}$$
   It is easy to verify (do it !) that $\sim$ is an equivalence relation on the collection of Cauchy sequences in X. Denote the equivalence class of the Cauchy sequence $\{x_n\}$ by $\widehat{\{x_n\}}$. We now set
   $$\widehat{X} = \{\hat{x} = \widehat{\{x_n\}} : \{x_n\} \text{ is a Cauchy sequence in } X\}.$$
   and define $\hat{d} : \widehat{X} \times \widehat{X} \to \mathbb{R}$ by
   $$\hat{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n) = 0 \tag{1.29}$$
   where $\hat{x} = \widehat{\{x_n\}}$ and $\hat{y} = \widehat{\{y_n\}}$.

   First we must show that $\hat{d}$ is well defined, that is, we must show that

   (a) the limit in (1.29) exists, and

   (b) the limit in (1.29) is independent of the representatives $\{x_n\}$ and $\{y_n\}$ of $\hat{x}$ and $\hat{y}$.

To prove existence, let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences with $\hat{x} = \widehat{\{x_n\}}$ and $\hat{y} = \widehat{\{y_n\}}$. Then by the triangle inequality,

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

so that

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_n, y_m). \tag{1.30}$$

Exchanging x_n with y_n and x_m with y_m, we also obtain

$$-\big[d(x_n, y_n) - d(x_m, y_m)\big] = d(x_m, y_m) - d(x_n, y_n) \leq d(y_n, y_m) + d(x_n, x_m). \tag{1.31}$$

The last two equations can be combined to

$$\big|d(x_n, y_n) - d(x_m, y_m)\big| \leq d(x_n, x_m) + d(y_n, y_m). \tag{1.32}$$

for all n and m. Let $\epsilon > 0$ be given. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy, we can pick $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon/2 \qquad \text{and} \qquad d(y_n, y_m) < \epsilon/2$$

for all $n, m \geq N$. Thus,

$$\big|d(x_n, y_n) - d(x_m, y_m)\big| < \epsilon$$

for all $n, m \geq N$, that is the sequence $\{d(x_n, y_n)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ and as $\mathbb{R}$ is complete, converges in $\mathbb{R}$. Thus, the limit (1.29) exists.

Now let $\{x'_n\}$ and $\{y'_n\}$ be arbitrary Cauchy sequences in X with $\{x'_n\} \sim \{x_n\}$ and $\{y'_n\} \sim \{y_n\}$. By (1.32) (with x'_n and y'_n instead of x_m and y_m) we obtain

$$\big|d(x_n, y_n) - d(x'_n, y'_n)\big| \leq d(x_n, x'_n) + d(y_n, y'_n). \tag{1.33}$$

for all n. Now as $d(x_n, x'_n) \to 0$ and $d(y_n, y'_n) \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \big|d(x_n, y_n) - d(x'_n, y'_n)\big| = 0,$$

that is,

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

This shows that $\hat{d}$ in (1.29) is well defined.

Next we show that $\hat{d}$ is a metric on $\widehat{X}$. For this, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be Cauchy sequences in X. Then

(M1): Clearly, $\hat{d}(\hat{x}, \hat{y}) \geq 0$ since $d(x_n, y_n) \geq 0$ for all n.

(M2): $\hat{d}(\hat{x}, \hat{y}) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} d(x_n, \hat{y}_n) = 0 \quad \Leftrightarrow \quad \{x_n\} \sim \{y_n\} \quad \Leftrightarrow \quad \hat{x} = \hat{y}$.

(M3): Since (M3) holds for d, we have

$$\hat{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = \hat{d}(\hat{y}, \hat{x}).$$

(M4): Since (M4) holds for d, we have

$$\begin{aligned}\hat{d}(\hat{x},\hat{y}) &= \lim_{n\to\infty} d(x_n,\hat{y}_n) \le \lim_{n\to\infty}\big[d(x_n,z_n)+d(z_n,y_n)\big] \\ &= \lim_{n\to\infty} d(x_n,z_n) + \lim_{n\to\infty} d(z_n,y_n) = \hat{d}(\hat{x},\hat{z}) + \hat{d}(\hat{z},\hat{y}).\end{aligned}$$

This shows that $\hat{d}$ is indeed a metric.

2. Construct the isometry T.

   It is now easy to see how T should be defined. Given $x \in X$, we identify x with the constant sequence $\{x_n\}$, where $x_n = x$ for all n, and set

   $$T(x) = \widehat{\{x_n\}} \qquad \big(\{x_n\} = (x,x,x,\dots)\big).$$

   Let us check that T is an isometry. Given $x, y \in X$, set $x_n = x$ and $y_n = y$ for all n. Then by definition of $\hat{d}$,

   $$\hat{d}(Tx,Ty) = \hat{d}\big(\widehat{\{x_n\}},\widehat{\{y_n\}}\big) = \lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x,y) = d(x,y)$$

   which shows that T is an isometry.

   We now can show that $T(X)$ is dense in $\widehat{X}$, that is, that $\overline{T(X)} = \widehat{X}$. For this, let $\hat{x} \in \widehat{X}$ be given, say $\hat{x} = \widehat{\{x_n\}}$ for some Cauchy sequence in X, and let $\epsilon > 0$ be arbitrary. Pick $N \in \mathbb{N}$ such that

   $$d(x_n,x_m) < \frac{\epsilon}{2}$$

   for all $n, m \ge N$. Then in particular,

   $$d(x_n,x_N) < \frac{\epsilon}{2} \tag{1.34}$$

   for all $n \ge N$. Denote by $\{y_n\}$ the constant sequence with $y_n = x_N$ for all n. Then by (1.34)

   $$\hat{d}(\hat{x},Tx_N) = \hat{d}\big(\widehat{\{x_n\}},\widehat{\{y_n\}}\big) = \lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x_n,x_N) \le \frac{\epsilon}{2} < \epsilon$$

   It follows from exercise 1.3.10 that $\hat{x} \in \overline{T(X)}$. Since $\hat{x} \in \widehat{X}$ was arbitrary it follows that $\widehat{X} = \overline{T(X)}$.

3. Now we are ready to show that $(\widehat{X},\hat{d})$ is complete.

   Let $\{\hat{x}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\widehat{X}$. Construct a Cauchy sequence $\{z_n\}$ in X as follows. Since $T(X)$ is dense in $\widehat{X}$, for each $\epsilon = \frac{1}{n}$ we can find $z_n \in X$ such that

   $$\hat{d}(\hat{x}_n,Tz_n) < \frac{1}{n}. \tag{1.35}$$

Claim: the sequence $\{z_n\}$ is Cauchy in X. In fact, note that

$$d(z_n, z_m) = \hat{d}(Tz_n, Tz_m) \leq \hat{d}(Tz_n, \hat{x}_n) + \hat{d}(\hat{x}_n, \hat{x}_m) + \hat{d}(\hat{x}_m, z_m).$$

Now given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

1. $\frac{1}{N} < \frac{\epsilon}{3}$, and
2. $\hat{d}(\hat{x}_n, \hat{x}_m) < \frac{\epsilon}{3}$

for all $m, n > N$. By (1.35) it follows that if $n, m > N$ then

$$d(z_n, z_m) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

which proves the claim.

Next we claim that $\hat{x}_n \to \hat{z} := \widehat{\{z_n\}}$. In fact, for each n we have

$$\begin{aligned}\hat{d}(\hat{x}_n, \hat{z}) &\leq \hat{d}(\hat{x}_n, Tz_n) + \hat{d}(Tz_n, \hat{z}) \\ &< \frac{1}{n} + \hat{d}(\widehat{\{z_n\}_{m=1}^{\infty}}, \widehat{\{z_m\}_{m=1}^{\infty}}) = \frac{1}{n} + \lim_{m\to\infty} d(z_n, z_m).\end{aligned}$$

where we have used the fact that $Tz_n = \widehat{\{z_n\}_{m=1}^{\infty}}$ and $\hat{z} = \widehat{\{z_m\}_{m=1}^{\infty}}$. Since $\{z_n\}$ is Cauchy, the last term on the right can be made as small as we wish by choosing n sufficiently large. Thus, $\hat{d}(\hat{x}_n, \hat{z}) \to 0$ as $n \to \infty$, that is, $\hat{x}_n \to \hat{z}$. This proves the claim.

4. It is left to show uniqueness of $(\widehat{X}, \hat{d})$.

   Suppose that $(\widetilde{X}, \tilde{d})$ is another complete metric space, and $\widetilde{T} : X \to \widetilde{X}$ is an isometry such that $\widetilde{T}(X)$ is dense in $\widetilde{X}$.

   Given $\hat{x} = Tx \in T(X) \subseteq \widehat{X}$, set

   $$U(\hat{x}) = U(Tx) := \widetilde{T}(T^{-1}(\hat{x})) = \widetilde{T}x$$

   Note that U is well defined since T is one-to-one, and U maps $T(X)$ onto $\widetilde{T}(X)$. Since

   $$\tilde{d}(U(Tx), U(Ty)) = \tilde{d}(\widetilde{T}x, \widetilde{T}y) = d(x, y) = \hat{d}(Tx, Ty)$$

   it follows that U is an isometry. Then by lemma 1.6.1, U extends to an isometry, also called U, from $\overline{T(X)} = \widehat{X}$ onto $\overline{\widetilde{T}(X)} = \widetilde{X}$, and essential uniqueness of $\widehat{X}$ is proved.

□

Remark 1.6.7. For ease of notation, one usually identifies X with its isometric image $T(X)$ in $\widehat{X}$, and writes $X \subseteq \widehat{X}$.

1.6.4 Completions of Normed Linear Spaces

If X is a normed linear space, then its completion will also be a normed linear space:

Theorem 1.6.4. *(Completion of Normed Linear Space)*
Let $(X, \|\cdot\|)$ be a normed linear space. Then there exist a Banach space $(\widehat{X}, \|\cdot\|_\wedge)$ and a linear isometry $T : X \to \widehat{X}$ such that $T(X)$ is dense in $\widehat{X}$.

Furthermore, $\widehat{X}$ is unique in the sense that if $(\widetilde{X}, \|\cdot\|_\sim)$ is another Banach space, and $\widetilde{T}$ a linear isometry of X into $\widetilde{X}$ such that $\widetilde{T}(X)$ is dense in $\widetilde{X}$, then there exists a linear isometry U of $\widehat{X}$ onto $\widetilde{X}$ such that $U(Tx) = \widetilde{T}x$ for all $x \in X$.

Proof. Let d be the metric on X determined by its norm $\|\cdot\|$, and let $(\widehat{X}, \hat{d})$ be the completion of (X, d) as in theorem 1.6.3. Keeping the notation of theorem 1.6.3, we need to show that

1. $\widehat{X}$ can be made into a linear space, so that T is linear.

2. $\|\hat{x}\|_{\|\wedge} := \hat{d}(\hat{x}, 0)$ defines a norm on $\widehat{X}$, and $\|Tx\|_\wedge = \|x\|$ for all $x \in X$.

3. U is a linear map.

1. First consider $T(X) \subseteq \widehat{X}$. Since T maps X onto $T(X)$ in a one-to-one fashion, we can identify X with $T(X)$ as a set, and thus transfer the vector space operations and norm from X onto $T(X)$. That is, if $\hat{x}, \hat{y} \in T(X)$ and α is scalar, then we define

$$\hat{x} + \hat{y} = T(x) + T(y) := T(x+y)$$
$$\alpha\hat{x} = \alpha T(x) := T(\alpha x)$$

and

$$\|\hat{x}\|_{\|\wedge} = \|T(x)\|_\wedge := \|x\|.$$

where $\hat{x} = Tx$ and $\hat{y} = Ty$. In this way, $T(X)$ becomes a normed linear space (over the same field as X), and T is a linear and preserves norms. Note that

$$\hat{d}(Tx, Ty) = d(x, y) = \|x - y\| = \|T(x - y)\|_\wedge = \|Tx - Ty\|_\wedge,$$

that is,

$$\hat{d}(\hat{x}, \hat{y}) = \|\hat{x} - \hat{y}\|_\wedge \tag{1.36}$$

for all $\hat{x}, \hat{y} \in T(X)$. So $\hat{d}$ is in fact the metric on $T(X)$ determined by $\|\cdot\|_\wedge$. Setting $y = 0$ in the above, we have $\hat{y} = \hat{0} = 0$, and hence

$$\|\hat{x}\|_\wedge = \|\hat{x} - 0\|_\wedge = \hat{d}(\hat{x}, 0).$$

for all $\hat{x} = Tx \in T(X)$.

Next we make all of $\widehat{X}$ into a linear space. For this, let $\hat{x}, \hat{y} \in \widehat{X}$ be arbitrary. Pick sequences $\{\hat{x}_n\}$ and $\{\hat{y}_n\}$ in $T(X)$ such that $\hat{x}_n \to \hat{x}$ and $\hat{y}_n \to \hat{y}$. We claim that the sequence $\{\hat{x}_n + \hat{y}_n\}$ is Cauchy in $T(X)$. In fact, by (1.36) and since $\|\cdot\|_\wedge$ is a norm on $T(X)$, we have for all m and n,

$$\begin{aligned}\hat{d}(\hat{x}_n + \hat{y}_n, \hat{x}_m + \hat{y}_m) &= \|(\hat{x}_n + \hat{y}_n) - (\hat{x}_m + \hat{y}_m)\|_\wedge \\ &\leq \|\hat{x}_n - \hat{x}_m\|_\wedge + \|\hat{y}_n - \hat{y}_m\|_\wedge \\ &= \hat{d}(\hat{x}_n, \hat{x}_m) + \hat{d}(\hat{y}_n, \hat{y}_m).\end{aligned}$$

Since the sequences $\{\hat{x}_n\}$ and $\{\hat{y}_n\}$ converge, they are Cauchy, and this inequality shows that $\{\hat{x}_n + \hat{y}_n\}$ is also Cauchy, and hence by completeness of $\widehat{X}$, converges. We thus can define

$$\hat{x} + \hat{y} := \lim_{n\to\infty} (\hat{x}_n + \hat{y}_n). \tag{1.37}$$

Similarly, for each scalar α,

$$\hat{d}(\alpha\hat{x}_n, \alpha\hat{x}_m) = \|\alpha\hat{x}_n - \alpha\hat{x}_m\|_\wedge = |\alpha|\,\|\hat{x}_n - \hat{x}_m\|_\wedge = \hat{d}(\hat{x}_n, \hat{x}_m)$$

which shows that $\{\alpha\hat{x}_n\}$ is also Cauchy, so that by completeness of $\widehat{X}$ we can define

$$\alpha\hat{x} := \lim_{n\to\infty} \alpha\hat{x}_n. \tag{1.38}$$

We must still verify that these operations are well defined, that is, that they are independent of the choices of $\{\hat{x}_n\}$ and $\{\hat{y}_n\}$. For this, suppose that $\{\hat{x}'_n\}$ and $\{\hat{y}'_n\}$ are also sequences in $T(X)$ such that $\hat{x}'_n \to \hat{x}$ and $\hat{y}'_n \to \hat{y}$. Then by theorem 1.5.4

$$\begin{aligned}&\hat{d}\big(\lim_{n\to\infty}(\hat{x}'_n + \hat{y}'_n), \lim_{n\to\infty}(\hat{x}_n + \hat{y}_n)\big) = \lim_{n\to\infty} \hat{d}\big((\hat{x}'_n + \hat{y}'_n), (\hat{x}_n + \hat{y}_n)\big) \\ &= \lim_{n\to\infty} \|(\hat{x}'_n + \hat{y}'_n) - (\hat{x}_n + \hat{y}_n)\|_\wedge \\ &\leq \lim_{n\to\infty} \|\hat{x}'_n - \hat{x}_n\|_\wedge + \|\hat{y}'_n - \hat{y}_n\|_\wedge \\ &= \lim_{n\to\infty} \big(\hat{d}(\hat{x}'_n, \hat{x}_n) + \hat{d}(\hat{y}'_n, \hat{y}_n)\big) \\ &\leq \lim_{n\to\infty} \big(\hat{d}(\hat{x}'_n, \hat{x}) + \hat{d}(\hat{x}, \hat{x}_n) + \hat{d}(\hat{y}'_n, \hat{y})) + \hat{d}(\hat{y}, \hat{y}_n)\big) = 0\end{aligned}$$

since $\hat{x}'_n \to \hat{x}$, $\hat{x}_n \to \hat{x}$ and $\hat{y}'_n \to \hat{y}$ and $\hat{y}_n \to \hat{y}$. In a similar way, one easily checks that

$$\hat{d}(\lim_{n\to\infty} \alpha\hat{x}'_n, \lim_{n\to\infty} \alpha\hat{x}_n) = 0.$$

It follows from definiteness of the metric that

$$\lim_{n\to\infty} (\hat{x}'_n + \hat{y}'_n) = \lim_{n\to\infty} (\hat{x}_n + \hat{y}_n)$$

and

$$\lim_{n\to\infty} \alpha\hat{x}'_n = \lim_{n\to\infty} \alpha\hat{x}_n,$$

so that the vector space operations are well defined.

It is straightforward, but tedious to show that $\widehat{X}$ with the above defined operations is a vector space whose zero element is $T(0)$, so we leave this as an exercise.

In each case, one uses the fact that $T(X)$ is a vector space, and uses continuity of $\hat{d}$. For example, commutativity is shown as follows:

$$\begin{aligned}\hat{d}(\hat{x}+\hat{y},\hat{y}+\hat{x}) &= \hat{d}\big(\lim_{n\to\infty}(\hat{x}_n+\hat{y}_n),\lim_{n\to\infty}(\hat{y}_n+\hat{x}_n)\big)\\ &= \lim_{n\to\infty}\hat{d}\big(\hat{x}_n+\hat{y}_n,\, \hat{y}_n+\hat{x}_n\big) = \lim_{n\to\infty 0} = 0\end{aligned}$$

since $\hat{x}_n+\hat{y}_n = \hat{y}_n+\hat{x}_n$ for all $\hat{x}_n, \hat{y}_n \in T(X)$. It follows from definiteness of the metric that

$$\hat{x}+\hat{y} = \hat{y}+\hat{x}$$

for all $\hat{x}, \hat{y} \in \widehat{X}$.

2. Next we need to check that $\|\hat{x}\|_\wedge := \hat{d}(\hat{x}, 0)$ defines a norm on $\widehat{X}$. (In part 1. above we have already seen that it is a norm on $T(X)$.) So given $\hat{x}, \hat{y} \in X$, let again $\{\hat{x}_n\}$ and $\{\hat{y}_n\}$ be sequences in $T(X)$ with $\hat{x}_n \to \hat{x}$ and $\hat{y}_n \to \hat{y}$. Note that if $\|\cdot\|_\wedge$ is defined on $\widehat{X}$ as above, then by theorem 1.5.4 and (1.36),

$$\|\hat{x}\|_\wedge = \hat{d}(\hat{x},0) = \hat{d}\big(\lim_{n\to\infty}\hat{x}_n, 0\big) = \lim_{n\to\infty}\hat{d}(\hat{x}_n,0) = \lim_{n\to\infty}\|\hat{x}_n\|_\wedge. \tag{1.39}$$

Since $\hat{d}$ is a metric on $\widehat{X}$, positivity and definiteness of $\|\cdot\|_\wedge$ follow directly from the corresponding properties of $\hat{d}$ and the definition of $\|\cdot\|_\wedge$. Then by (1.38) and (1.39),

$$\|\alpha\hat{x}\|_\wedge = \|\lim_{n\to\infty}(\alpha\hat{x}_n)\|_\wedge = \lim_{n\to\infty}\|\alpha\hat{x}_n\|_\wedge = |\alpha|\lim_{n\to\infty}\|\hat{x}_n\|_\wedge = |\alpha|\,\|\hat{x}\|_\wedge.$$

while also by (1.37) and (1.39),

$$\begin{aligned}\|\hat{x}+\hat{y}\|_\wedge &= \|\lim_{n\to\infty}(\hat{x}_n+\hat{y}_n)\|_\wedge = \lim_{n\to\infty}\|\hat{x}_n+\hat{y}_n\|_\wedge\\ &\le \lim_{n\to\infty}\big(\|\hat{x}_n\|_\wedge+\|\hat{y}_n\|_\wedge\big) = \lim_{n\to\infty}\|\hat{x}_n\|_\wedge + \lim_{n\to\infty}\|\hat{y}_n\|_\wedge\\ &= \|\hat{x}\|_\wedge + \|\hat{y}\|_\wedge\end{aligned}$$

Hence, $\|\cdot\|_\wedge$ is a norm in $\widehat{X}$.

3. Finally, we must check that the map U is linear. Note that if $\hat{x} = Tx, \hat{y} = Ty \in T(X)$ with $x, y \in X$, then by linearity of T and $\widetilde{T}$ and definition of U,

$$\begin{aligned}U(\alpha\hat{x}+\beta\hat{y}) &= U\big(\alpha T(x)+\beta T(y)\big) = U\big(T(\alpha x+\beta y)\big)\\ &= \widetilde{T}(\alpha x+\beta y) = \alpha\widetilde{T}(x)+\beta\widetilde{T}(y)\\ &= \alpha U\big(Tx\big)+\beta\big(Ty\big) = \alpha U(\hat{x})+\beta U(\hat{y}).\end{aligned} \tag{1.40}$$

which shows that U is linear on $T(X)$. Since $\overline{T(X)} = X$, and $U : \widehat{X} \xrightarrow{\text{onto}} \widetilde{X}$ is the only isometry extending the linear isometry $U : T(X) \to \widetilde{X}$, U must coincide with the linear isometry of corollary 1.6.2. □

Example 1.6.5. Let

$$X = V = \{f : \mathbb{N} \to \mathbb{C} \;:\; \exists N \text{ such that } f(n) = 0 \;\; \forall\, n \ge N\}$$

endowed with the supremum norm $\|\cdot\|_\infty$. Since $\overline{V} = c_o$ and c_o is complete (see exercises 1.6.2 and 1.5.4), it follows by (essential) uniqueness of completions, that the completion of the normed linear space $(V, \|\cdot\|_\infty)$ is $(c_o, \|\cdot\|_\infty)$.

1.7 Compactness

Throughout this section, (X, d) will denote a metric space.

1.7.1 Compact sets

Definition 1.7.1. Let X be a metric space, and $M \subseteq X$. A family $\mathcal{O} = \{O_\alpha\}_{\alpha\in A}$ of subsets of X is called a *cover* (or a *covering*) of M, if

$$M \subseteq \bigcup_{\alpha\in A} O_\alpha.$$

(Note that the index set A need not be countable !) If in addition, each O_α is an open set, then we call $\{O_\alpha\}_{\alpha\in A}$ an *open covering* of M.

Let $\{O_\alpha\}_{\alpha\in A}$ be a covering of M. A subcollection $\{O_\alpha\}_{\alpha\in A_o}$ $(A_o \subseteq A)$ satisfying

$$M \subseteq \bigcup_{\alpha\in A_o} O_\alpha$$

is called a *subcovering* of $\{O_\alpha\}_{\alpha\in A}$.

If A_o is a countable (respectively finite) set, then $\{O_\alpha\}_{\alpha\in A_o}$ is called a *countable* (respectively *finite*) subcover.

Example 1.7.1. Let $X = \mathbb{R}$ and $M = [-1, 1]$. Fix $\epsilon > 0$.

$$\mathcal{O} = \{\,(x-\epsilon, x+\epsilon)\,\}_{x\in[-1,1]}$$

is an open cover of $[-1, 1]$. The collection

$$\{\,(x-\epsilon, x+\epsilon)\,\}_{x\in[-1,1]\cap\mathbb{Q}}$$

is a *countable subcover* of $\mathcal{O}$. Fix a positive integer n with $\frac{1}{n} < \epsilon$. Then the collection

$$\{\,(\tfrac{k}{n}-\epsilon, \tfrac{k}{n}+\epsilon)\,\}_{-n\le k\le n}$$

is a *finite subcover* of $\mathcal{O}$.

Definition 1.7.2. Let (X, d) be a metric space. A set $K \subseteq X$ is called *compact*, if every open cover $\mathcal{O} = \{O_\alpha\}_{\alpha\in A}$ of K possesses a finite subcover $\{O_{\alpha_j}\}_{j=1}^n$.

If X itself is a compact set, then we call (X, d) a *compact metric space.*

Example 1.7.2. $\mathbb{R}$ is not compact. For example, the open cover

$$\mathcal{O} = \{(n-\frac{3}{4}, n+\frac{3}{4})\}_{n\in\mathbb{N}}$$

has no finite subcover, since every integer is contained in exactly one of these intervals.

Example 1.7.3. Every finite subset M of a metric space X is compact. (Why ?)

Continuous maps take compact sets to compact sets:

Theorem 1.7.1. *Let (X,d) and (Y,σ) be metric spaces, and $T : X \to Y$ be continuous. If $K \subseteq X$ is compact, then its image $T(K)$ is compact in Y.*

Proof. Let $\{O_\alpha\}_{\alpha\in A}$ be an open covering of $T(K)$ in Y. For each α, set

$$U_\alpha := T^{-1}(O_\alpha).$$

Since T is continuous, each U_α is open in X. Furtheremore, since $T(K) \subseteq \bigcup_{\alpha\in A} O_\alpha$ it follows that

$$K \subseteq T^{-1}(T(K)) \subseteq T^{-1}\Big(\bigcup_{\alpha\in A} O_\alpha\Big) = \bigcup_{\alpha\in A} T^{-1}(O_\alpha) = \bigcup_{\alpha\in A} U_\alpha.$$

Thus $\{U_\alpha\}_{\alpha\in A}$ is an open covering of K. Now since K is compact, there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ for K, that is,

$$K \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}.$$

Then

$$T(K) \subseteq T\Big(\bigcup_{j=1}^{n} U_{\alpha_j}\Big) = \bigcup_{j=1}^{n} T(U_{\alpha_j}) = \bigcup_{j=1}^{n} O_{\alpha_j},$$

that is, $\{O_{\alpha_j}\}_{j=1}^{n}$ is a a finite subcover of $\{O_\alpha\}_{\alpha\in A}$ for $T(K)$. Hence, $T(K)$ is compact. □

Exercise 1.7.1. Show by example that if T is continuous and $T(K)$ is compact, then K need not be compact.

Since the definition of compact sets is not easy to use, we want a characterization of compact sets.

Theorem 1.7.2. *Let (X,d) be a metric space, and $K \subseteq X$ be compact.*

1. *For each $x \in K^c$ there exist open sets U and V such that*

$$x \in U, \quad K \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$

2. *K is closed.*

Proof. 1. Let $x \in K^c$ be given. For each $y \in K$, pick open neighborhoods U_y of x and V_y of y such that

$$U_y \cap V_y = \emptyset. \tag{1.41}$$

(For example, let $U_y = B_{\epsilon/2}(x)$ and $V_y = B_{\epsilon/2}(y)$, where $\epsilon < d(x,y)$.) Then $\{V_y\}_{y\in K}$ is an open covering of K. By compactness of K, there exists a finite subcovering $\{V_{y_1}, \ldots, V_{y_n}\}$ for K, that is, $K \subseteq \bigcup_{j=1}^{n} V_{y_j}$. Set

$$U = \bigcap_{j=1}^{n} U_{y_j} \quad \text{and} \quad V = \bigcup_{j=1}^{n} V_{y_j}.$$

Figure 1.25: Point x and set K can be separated by disjoint open sets.

Both sets are open, and $x \in U$ (as $x \in U_{y_j}$, $j = 1 \ldots n$). Note that if $y \in V$, then $y \in V_{y_j}$ for some j, and hence $y \notin U_{y_j}$ by (1.41). It follows that $y \notin U$, which shows that $U \cap V = \emptyset$.

2. Let $x \in K^c$ be arbitrary. By part 1), there exists an open neighborhood U_x of x such that $U_x \cap K = \emptyset$, that is, $U_x \subseteq K^c$. It follows that

$$K^c \subseteq \bigcup_{x \in K^c} U_x \subseteq \bigcup_{x \in K^c} K^c = K^c$$

that is,

$$K^c = \bigcup_{x \in K^c} U_x.$$

Since the right-hand set is open, if follows that K^c is open and hence, K is closed.

□

Remark 1.7.1. The important ingredient in this proof was property (1.41). Topological spaces in which distinct points x and y can be separated by open disjoint open sets in this way are called *Hausdorff.* Thus, every metric space is Hausdorff.

The first property can be generalized as follows:

Exercise 1.7.2. Let (X, d) be a metric space, and $K, M \subseteq X$ be compact with $K \cap M = \emptyset$. Then there exists open sets U and V such that

$$K \subseteq U, \quad M \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$

Closed subsets of compact sets are always compact:

Theorem 1.7.3. *Let (X, d) be a metric space, and $K \subseteq X$ be compact. If $F \subseteq K$ is closed in X, then F is also compact.*

Proof. Let $\{O_\alpha\}_{\alpha \in A}$ be any open covering of F in X. As

$$K \subseteq X = F \cup F^c \subseteq \Big(\bigcup_{\alpha \in A} O_\alpha\Big) \cup F^c$$

and F^c is open, we see that

$$\{O_\alpha\}_{\alpha \in A} \cup \{F^c\}$$

is an open cover of K. Then by compactness of K, there exist finitely many sets $O_{\alpha_1}, \ldots, O_{\alpha_n}$ such that

$$K \subseteq \Big(\bigcup_{j=1}^{n} O_{\alpha_j}\Big) \cup F^c.$$

Then

$$F \subseteq \bigcup_{j=1}^{n} O_{\alpha_j},$$

that is, $\{O_{\alpha_1}, \ldots, O_{\alpha_n}\}$ is a finite subcover of $\{O_\alpha\}$ for F. Hence, F is compact. □

1.7.2 Sequential Compactness

Definition 1.7.3. Let (X, d) be a metric space. A set $K \subseteq X$ is called *sequentially compact* if every sequence $\{x_n\}_{n=1}^{\infty} \subseteq K$ possesses a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converges to some $x \in K$.

Example 1.7.4. Let $X = \mathbb{R}^n$, and let $K \subseteq \mathbb{R}^n$ be closed and bounded. Thus, every sequence $\{x_m\}_{m=1}^{\infty} \subseteq K$ is bounded, and by the Bolzano-Weierstrass theorem, has a convergent subsequence $\{x_{m_k}\}_{k=1}^{\infty}$, say $x_{m_k} \to x \in \mathbb{R}^n$. Note that $x \in K$ also, since K is closed. This shows that every closed and bounded subsset K of $\mathbb{R}^n$ is sequentially compact. (See also corollary 1.7.7 below).

Exercise 1.7.3. Consider

1. $X = C[0,1]$ with the uniform norm $\|\cdot\|_u$,

2. $X = \ell^\infty$ with the supremum norm $\|\cdot\|_\infty$.

In both cases, let $K = \overline{B}_1(0)$ denote the closed unit ball. Thus, K is a closed and bounded set. Show that K is not sequentially compact. (Hint: Find a sequence $\{f_n\}$ in K which has no convergent subsequence.)

1.7.3 Totally Bounded Sets

Definition 1.7.4. Let (X, d) be a metric space, and $M \subseteq X$. Fix $\epsilon > 0$. A finite set

$$M_\epsilon := \{y_1, y_2, \ldots, y_n\} \subseteq X$$

is called an *ϵ-net for M in X*, if

$$M \subseteq \bigcup_{i=1}^{n} B_\epsilon(y_i).$$

(That is, if every $x \in M$ lies within ϵ-distance of at least one point y_i).

A set $M \subseteq X$ is called *totally bounded* if for each $\epsilon > 0$, there exists an ϵ-net M_ϵ for M in X.

Exercise 1.7.4. Show directly, *using the above definition*: Every bounded subset $M \subseteq \mathbb{R}^n$ is totally bounded. (Hint: First prove this for the metric d_∞. Then use the equivalence of d_2 and d_∞.)

Exercise 1.7.5. Consider $X = (\ell^\infty, \|\cdot\|_\infty)$, and let $K = \overline{B}_1(0)$ denote the closed unit ball. Thus, K is a closed and bounded set. Show that K is not totally bounded.

In fact, the notion of totally boundedness is stronger than boundedness:

Theorem 1.7.4. *Let (X, d) be a metric space. Then every totally bounded set $M \subseteq X$ is bounded.*

Proof. Let $M \subseteq X$ be totally bounded. If $M = \emptyset$ the assertion is obvious. Thus, we may assume that $M \neq \emptyset$. Fix any $\epsilon > 0$, and let

$$\{y_1, y_2, \ldots, y_n\}$$

be a corresponding ϵ-net. Thus, if x, y are any two points in M, then there exist i, j $(1 \leq i, j \leq n)$ such that

$$d(x, y_i) < \epsilon \qquad \text{and} \qquad d(y, y_j) < \epsilon$$

Set

$$C := \max_{1 \leq i,j \leq n} d(y_i . y_j).$$

Then

$$d(x, y) \leq d(x, y_i) + d(y_i, y_j) + d(y_j, y) < \epsilon + C + \epsilon = C + 2\epsilon.$$

This shows that M is bounded. □

The next theorem says that the three notions of compactness, sequential compactness, and total boundedness are essentially equivalent:

Theorem 1.7.5. *let (X, d) be a metric space, and $K \subseteq X$. Then T.F.A.E:*

1. *K is compact.*

2. *K is sequentially compact.*

3. *K is complete and totally bounded.*

Proof. We proceed by first showing that 2. ⇔ 3. Then we will show that 1. ⇔ 2. If $K = \emptyset$ then the assertion is trivial; thus we may assume throughout that $K \neq \emptyset$.
2) ⇒ 3): Suppose, K is sequentially compact. We first show that K is complete. To this end, let $\{x_n\}_{n=1}^\infty \subseteq K$ be Cauchy. Since K is sequentially compact, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ converging to some $x \in K$. Then by exercise 1.5.1, $x_n \to x$ also. Hence, K is complete.

To show that K is totally bounded, let $\epsilon > 0$ be given. We construct an ϵ-net $\{y_1, \ldots, y_m\}$ for K inductively as follows.

(i) Pick $y_1 \in K$ arbitrary.

(ii) If $K \subseteq B_\epsilon(y_1)$, then stop. Otherwise, pick $y_2 \in K \backslash B_\epsilon(y_1)$.

(iii) Suppose, we have picked $y_1, \ldots, y_{m-1} \in K$. If $K \subseteq \bigcup_{i=1}^{m-1} B_\epsilon(y_i)$, then stop here. Otherwise, pick $y_m \in K \backslash \bigcup_{i=1}^{m-1} B_\epsilon(y_i)$.

We claim that this process will stop after finitely many steps. For suppose to the contrary, that it does not stop. By construction, we then obtain a sequence $\{y_m\}_{m=1}^\infty \subseteq K$ such that

$$y_m \notin \bigcup_{i=1}^{m-1} B_\epsilon(y_i)$$

for all $m \in \mathbb{N}$. In particular,

$$d(y_m, y_n) > \epsilon$$

for all $m > n$, $m, n \in \mathbb{N}$. This means that no subsequence of $\{y_m\}_{m=1}^\infty$ can be Cauchy, which contradicts the assumption that K be sequentially compact. Thus the claim holds, which means that there exists $m \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{i=1}^{m} B_\epsilon(y_i),$$

that is, $\{y_1, \ldots, y_m\}$ is an ϵ-net for K. Hence, K is totally bounded.

3) $\Rightarrow$ 2): Next, suppose K be complete and totally bounded. We need to show that K is sequentially compact.

To this end, let $\{x_n\}_{n=1}^\infty$ be an arbitrary sequence in K. We extract a Cauchy subsequence as follows:

(i) Since K is totally bounded, given $\epsilon = 1/2$ there exist $y_1, \ldots, y_m$ in X such that

$$K \subseteq \bigcup_{i=1}^{m} B_{1/2}(y_i).$$

Now as $\{x_n\}$ has infinitely many terms, at least one of the balls $B_{1/2}(y_i)$, call it B_1, must contain infinitely many x_n. Set

$$N_1 = \{n \in \mathbb{N} : x_n \in B_1\}.$$

Then N_1 is an infinite set.

(ii) Similarly, given $\epsilon = 1/2^2$ theree exist $y_1, \ldots, y_m$ in X (m and y_i are of course different from above) such that

$$K \subseteq \bigcup_{i=1}^{m} B_{1/2^2}(y_i).$$

Now as N_1 is an infinite set, at least one of the balls $B_{1/2^2}(y_i)$, call it B_2, must contain infinitely many of the terms x_n in $\{x_n\}_{n \in N_1}$. Set

$$N_2 = \{n \in N_1 : x_n \in B_2\}.$$

Then N_2 is an infinite set.

(iii) Continue inductively. Suppose we have chosen an infinite set $N_k \subseteq \mathbb{N}$ and a ball B_k of radius $1/2^k$ such that $x_n \in B_k$ for all $n \in N_k$. Then given $\epsilon = 1/2^{k+1}$ there exist $y_1, \ldots, y_m$ in X (m and y_i are of course different from above) such that

$$K \subseteq \bigcup_{i=1}^{m} B_{1/2^{k+1}}(y_i).$$

Now as N_k is an infinite set, at least one of the balls $B_{1/2^k}(y_i)$, call it B_{k+1}, must contain infinitely many of the terms x_n in $\{x_n\}_{n \in N_k}$. Set

$$N_{k+1} = \{n \in N_k : x_n \in B_{k+1}\}.$$

Then N_{k+1} is an infinite set.

We thus arrive at the decreasing sequence

$$\mathbb{N} \supseteq N_1 \supseteq N_2 \cdots \supseteq N_k \supseteq N_{k+1} \supseteq \cdots \tag{1.42}$$

of infinite index sets, and balls B_k of radii $1/2^k$ such that $x_n \in B_k$ for all $n \in N_k$.

Now pick $n_1 \in N_1$, $n_2 \in N_2$ such that $n_2 > n_1$, and continue inductively. In general, having picked $n_k \in N_k$, pick $n_{k+1} \in N_{k+1}$ such that $n_{k+1} > n_k$. This is possible as the sets N_k are all infinite. We thus arrive at a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ with $x_{n_k} \in B_k$ for all k.

Note that if $j \geq k$ is arbitrary, then by (1.42), $x_{n_j} \in B_k$ as well, and hence

$$d(x_{n_k}, x_{n_j}) < \text{diameter}(B_k) = 2 \cdot \frac{1}{2^k} = \frac{1}{2^{k-1}}$$

which can be made arbitrarily small ($< \epsilon/2$) by choosing k sufficiently large. This shows that$\{x_{n_k}\}_{k=1}^{\infty}$ is Cauchy.

Now as K is complete, $\{x_{n_k}\}_{k=1}^{\infty}$ converges to some $x \in K$. This shows that K is sequentially compact.

1) $\Rightarrow$ 2): Suppose, K is compact. We want to show that K is sequentially compact. To this end, let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence in K.

Claim: There exists $z \in K$ so that every ball $B_\epsilon(z)$ contains infinitely many terms x_n.

For suppose to the contrary, that the claim is false. Then for every $z \in K$ there exists $\epsilon_z > 0$ such that $B_{\epsilon_z}(z)$ contains only finitely many x_n. Now as $\{B_{\epsilon_z}(z)\}_{z \in K}$ is an open cover of K and K is compact, there exists a finite subcover $\{B_{\epsilon_{z_i}}(z_i)\}_{i=1}^{m}$ for K, that is,

$$K \subseteq \bigcup_{i=1}^{m} B_{\epsilon_{z_i}}(z_i).$$

But then K contains only finitely many terms x_n, which contradicts the fact that all x_n lie in K. This proves the claim.

Let z be as in the claim 1. We now construct a subsequence of $\{x_n\}_{n=1}^{\infty}$ which converges to z. In fact, pick $n_1 \in \mathbb{N}$ such that

$$x_{n_1} \in B_{1/2}(z)$$

Then pick $n_2 \in \mathbb{N}$ such that

$$x_{n_2} \in B_{1/2^2}(z) \qquad \text{and} \qquad n_2 > n_1$$

Suppose we have chosen n_k so that $x_{n_k} \in B_{1/2^k}(z)$. Then since $B_{1/2^{k+1}}(z)$ contains infinitely many terms of x_n, we can pick $n = n_{k+1} \in \mathbb{N}$ such that

$$x_{n_{k+1}} \in B_{1/2^{k+1}}(z) \qquad \text{and} \qquad n_{k+1} > n_k.$$

Continuing inductively, we thus obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, z) < \frac{1}{2^k},$$

for all k, that is, $x_{n_k} \to z$ as $k \to \infty$. This proves the claim, and that K is sequentially compact.

2) $\Rightarrow$ 1): Suppose that K is sequentially compact. Then as shown above, K is totally bounded as well. Let $\{O_\alpha\}_{\alpha \in A}$ be an open cover of K. We need to extract a finite subcover.

We claim that there exists $\epsilon > 0$ so that every ball $B_\epsilon(x)$ satisfying $B_\epsilon(x) \cap K \neq \emptyset$ is contained in some $\{O_\alpha\}$. For suppose, the claim does not hold. Then for each $\epsilon = \frac{1}{n}$ $(n \in \mathbb{N})$ there exists a ball $B_n := B_{\frac{1}{n}}(y_n)$ such that $B_n \cap K \neq \emptyset$, and

$$B_n \not\subseteq O_\alpha \tag{1.43}$$

for all α. For each n, pick $x_n \in B_n \cap K$. Since K is sequentially compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $x \in K$. Now $\{O_\alpha\}$ is a cover of K, so there exists $\alpha_o \in A$ such that $x \in O_{\alpha_o}$. Since O_{α_o} is open, there exists $\delta > 0$ such that $B_\delta(x) \subseteq O_{\alpha_o}$. Now as $x_{n_k} \to x$, we can pick k so that

$$\frac{1}{n_k} < \frac{\delta}{3} \qquad \text{and} \qquad d(x_{n_k}, x) < \frac{\delta}{3}.$$

Then for all $y \in B_{n_k}$ we have

$$d(y, x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \text{diameter}(B_{n_k}) + \frac{\delta}{3} = \frac{2\delta}{3} + \frac{\delta}{3} = \delta,$$

that is,

$$B_{n_k} \subseteq B_\delta(x) \subseteq O_{\alpha_o}$$

contradicting (1.43). Thus, the claim must hold.

Now let ϵ be as in the claim. Since K is totally bounded, there exist $y_1, \ldots, y_m \in X$ such that $K \subseteq \bigcup_{i=1}^m B_\epsilon(y_i)$. Removing some of the y_i if necessary, we may assume that $K \cap B_\epsilon(y_i) \neq \emptyset$ for all $i = 1 \ldots m$. Then by the claim, $B_\epsilon(y_i) \subseteq O_{\alpha_i}$ for some $\alpha_i, \in A$, and all $i = 1 \ldots m$. (Note that the indices α_i need not be distinct), and hence,

$$K \subseteq \bigcup_{i=1}^m B_\epsilon(y_i) \subseteq \bigcup_{i=1}^m O_{\alpha_i}.$$

Thus, $\{O_{\alpha_i}\}_{i=1}^m$ is a finite subcover of $\{O_\alpha\}$ for K. This shows that K is compact. □

Theorem 1.7.6. *Let (X, d) be a complete metric space, and $K \subseteq X$. Then K is complete $\Leftrightarrow$ K is closed.*

Proof. $\Rightarrow$: Suppose that K is complete. Let $\{x_n\}_{n=1}^{\infty}$ be any convergent sequence in K, say $x_n \to x \in X$. Then $\{x_n\}_{n=1}^{\infty}$ is Cauchy in X, and hence in K as well, and by completeness of K, $x_n \to y$ for some $y \in K$. It follows from uniqueness of limits that $y = x$; in particular, $x \in K$. Hence, K is closed.

$\Leftarrow$: Now suppose, K is closed. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in K. Since X is complete, there exists $x \in X$ such $x_n \to x$. But $x_n \in K$ for all n, and K is closed, hence $x \in K$ as well. This proves that K is complete. □

Remark 1.7.2. Note that in the "$\Rightarrow$" part of the above proof, completeness of X is not required.

Combining theorems 1.7.5 and 1.7.6 we obtain directly:

Corollary 1.7.7. *let (X, d) be a complete metric space, and $K \subseteq X$. Then T.F.A.E:*

1. *K is compact.*
2. *K is sequentially compact.*
3. *K is closed and totally bounded.*

For $X = \mathbb{R}^n$ with the Euclidean metric we have the well known characterization of compact sets:

Corollary 1.7.8. *(Heine-Borel Theorem)*
Let $K \subseteq \mathbb{R}^n$. Then K is compact $\Leftrightarrow$ K is closed and bounded.

Proof. $\Rightarrow$: Follows from corollary 1.7.7 and theorem 1.7.4.

$\Leftarrow$: Suppose, K is closed an bounded. By exercise 1.7.4, K is totally bounded. Now apply corollary 1.7.7. □

Corollary 1.7.9. *Every compact metric space (X, d) is separable.*

Proof. Let X be a compact metric space. Then by theorem 1.7.5, X is totally bounded. Thus, for each $\epsilon = \frac{1}{n}$ there exists $y_1^{(n)}, \ldots, y_{m_n}^{(n)}$ in X such that

$$X = \bigcup_{i=1}^{m_n} B_{\frac{1}{n}}(y_i^{(n)}). \tag{1.44}$$

Set

$$Y = \bigcup_{n=1}^{\infty} \{y_1^{(n)}, \ldots, y_{m_n}^{(n)}\},$$

a countable set.

Now let $x \in X$ and $\epsilon > 0$ be given. Pick $n > \frac{1}{\epsilon}$. Then by (1.44) there exists $y_i^{(n)} \in Y$ such that

$$d(x, y_i^{(n)}) < \frac{1}{n} < \epsilon.$$

This shows that Y is dense in X. □

Corollary 1.7.10. *(Extreme Value Theorem) Let (X, d) be a metric space, and $T : X \to \mathbb{R}$ be continuous. If $K \subseteq X$ is compact, then there exist $x_m, x_M \in K$ such that*

$$T(x_m) \leq T(x) \leq T(x_M)$$

for all $x \in K$.

Proof. Since K is compact and T is continuous, the image $T(K)$ is also compact by theorem 1.7.1. By the Heine-Borel theorem, $T(K)$ is closed and bounded. Thus,

$$s := \sup T(K) = \sup\{T(x) : x \in K\} < \infty.$$

For each $n \in \mathbb{N}$, $s - \frac{1}{n}$ is not an upper bound of $T(K)$, hence there exists a sequence $T(x_n)$ in $T(K)$ such that $T(x_n) \to s$. But as $T(K)$ is closed, then $s \in T(K)$ as well, that is, $s = T(x_M)$ for some $x_M \in K$. Hence, $T(x) \leq T(x_M)$ for all $x \in K$.

The existence of a minimizing element x_m is proved similarly, using $\inf T(K)$. □

1.8 The Contraction Principle

Definition 1.8.1. Let X be a set, and $T : X \to X$ a mapping. A point $x \in X$ is called a *fixed point* of T, if

$$T(x) = x.$$

Example 1.8.1. let $X = \mathbb{R}$ and $f(x) = x^2$.

put0,0

Figure 1.26: Fixed points for $f(x) = x^2$.

Then $x = 0$ and $x = 1$ are the only fixed points of f.

Exercise 1.8.1. Let $F : [0, 1] \to [0, 1]$ be continuous. Show that f has at least one fixed point. (Hint: use the Intermediate Value Theorem.)

Definition 1.8.2. Let (X, d) be a metric space.A mapping $T : X \to X$ satisfying

$$d\big(T(x), T(y)\big) \leq k\, d(x, y)$$

for some constants c, $0 \leq k < 1$, and all $x, y \in X$ is called a *contraction.*

Note that every contraction is uniformly continuous. (Give $\epsilon > 0$, simply choose $\delta = \epsilon$.)

Theorem 1.8.1. *(Banach Fixed Point Theorem, Contraction principle) Let (X, d) be a complete metric space, and $TX \to X$ a contraction. Then T has a unique fixed point.*

Proof. Let us first show that T has at least one fixed point. For this, we construct a Cauchy sequence as follows: Pick $x_o \in X$ arbitrary. Then set $x_1 = T(x_o)$, $x_2 = T(x_1)$, $x_3 = T(x_2)$, ..., and in general, $x_{n=1} = T(x_n)$. Since T is a contraction, we have

$$\begin{aligned} d(x_2, x_1) &= d\big(T(x_1), T(x_o)\big) \le kd(x_1, x_o) \\ d(x_3, x_2) &= d\big(T(x_2), T(x_1)\big) \le kd(x_2, x_1) \le k^2 d(x_1, x_o) \\ &\vdots \qquad \vdots \\ d(x_{n+1}, x_n) &= d\big(T(x_), T(x_{n-1})\big) \le kd(x_n, x_{n-1}) \le \cdots \le k^n d(x_1, x_o). \end{aligned}$$

We claim that $\{x_n\}$ is Cauchy in X. In fact, if $m > n$, then by the triangle inequality,

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\le k^{m-1} d(x_1, x_o) + k^{m-2} d(x_1, x_o) + \cdots + k^n d(x_1, x_o) \\ &= d(x_1, x_o) k^n \sum_{i=0}^{m-n-1} k^i \le d(x_1, x_0) \frac{k^n}{1-k} \end{aligned} \tag{1.45}$$

Since $k^n \to 0$ as $n \to \infty$, we can conclude that $\{x_n\}$ is Cauchy (why ?). This proves the claim.

Now as X is complete, the sequence $\{x_n\}$ converges to some $x \in X$. Since $T(x_n) = x_{n+1}$ and T is continuous, it now follows that

$$T(x) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Hence, x is a fixed point for T.

Next we show that x is the unique fixed point. For suppose, y is another fixed point for T, that is, $T(y) = y$. Then

$$d(x, y) = d\big(T(x), T(y)\big) \le k\, d(x, y)$$

which can only hold if $d(x, y) = 0$, since $0 \le k < 1$. Thus, $x = y$. □

Remark 1.8.1. Note that the starting point x_o in the above proof is completely arbitrary. In practical computations, one can only compute finitely many of the terms x_n, and thus obtain an approximation for the fixed point x. Thus, one needs to know how well the n-th term x_n approximates the fixed point x.

Let us begin with (1.45),

$$d(x_m, x_n) \le d(x_1, x_o) \frac{k^n}{1-k}.$$

Now let $m \to \infty$. Then by continuity of the metric,

$$d(x, x_n) = \lim_{m\to\infty} d(x_m, x_n) \le d(x_1, x_o) \frac{k^n}{1-k}. \tag{1.46}$$

This is called an *apriori* estimate, because we can obtain this error estimate before computing x_n. However, another estimate is often more precise: From (1.46) we obtain

$$d(x, x_1) \le d(x_1, x_o)\frac{k}{1-k}.$$

for *any* starting point x_o. Thus we may consider x_{n-1} as the staring point, and obtain

$$d(x, x_n) \le d(x, x_{n-1})\frac{k}{1-k}.$$

This is called a *posterior* estimate, since we can only obtain the estimate once we have computed x_n.

Example 1.8.2. let $f : [a, b] \to [a, b]$ be differentiable, and suppose that $|f'(x)| \le k < 1$ for all $x \in [a, b]$. Then by the Mean Value Theorem, for all $x, y \in [a, b]$ we have

$$|f(x) - f(y)| \le |f'(\xi)|\,|x - y| \le k\,|x - y|$$

where ξ is between x and y. It follows from the contraction principle that f has a unique fixed point.

This argument shows, for example, that $f(x) = x^2$ has a unique fixed point on $[-c, c]$, for any $c < 1/2$. (This fixed point is of course $x = 0$.)

An important application of Banach's Fixed Point Theorem is the proof of Picard's theorem in the course on ordinary differential equations.

Exercise 1.8.2. Show by example, that completeness of X is necessary for the contraction principle to hold in general.

Exercise 1.8.3. Show by example that the weaker condition

$$d\big(T(x), T(y)\big) \le d(x, y)$$

is not sufficient for the contraction principle to hold in general. (Hint: Let $X = [1, \infty)$ and consider $T(x) = x + \frac{1}{x}$.)

Exercise 1.8.4. Use the proof of the contraction principle to find a sequence $\{x_n\}$ converging to $\sqrt{2}$, and give apriori and posterior estimates for $|x_n - \sqrt{2}|$. (Hint: Consider the function $f(x) = x^2 + x - \frac{1}{8}$.)

Chapter 2

Measure Theory

2.1 Sigma-Algebras and Measurable Spaces

2.1.1 Algebras

Definition 2.1.1. Let X be a set. A non-empty collection $\mathcal{M}$ of subsets of X is called an *algebra of subsets of* X if the following hold:

(A1) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$,

(A2) If $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.

Remark 2.1.1. Let X be a set, and $\mathcal{M}$ be an algebra of subsets of X.

1. Pick any $A \in \mathcal{M}$. Then by (A1) and (A2) above we have $X = A \cup A^c \in \mathcal{M}$, and then also $\emptyset = X^c \in \mathcal{M}$. So every algebra contains both, the empty set and the space X itself.

2. Let $A, B \in \mathcal{M}$. Then
$$A \cap B = \left[A^c \cup B^c\right]^c \in \mathcal{M}$$
by properties (A1) and (A2) above, and then also
$$A \backslash B = A \cap B^c \in \mathcal{M}.$$

3. By induction on n, one easily shows that if $A_1, \ldots, A_n \in \mathcal{M}$, then
$$\bigcup_{i=1}^{n} A_i \in \mathcal{M} \qquad \text{and} \qquad \bigcap_{i=1}^{n} A_i \in \mathcal{M}.$$

Thus, an algebra is closed under formation of differences, finite unions and finite intersections.

Example 2.1.1. Let X be any set, and denote by $\mathcal{P}(X)$, or 2^X, the collection of all subsets of X. ($\mathcal{P}(X)$ is called the *power set* of X).

1. $\mathcal{M} = \{\emptyset, X\}$ is an algebra. In fact, it is the smallest algebra of subsets of X.

2. $\mathcal{P}(X)$ is an algebra. In fact, it is the largest algebra of subsets of X.

3. Fix any $E \subseteq X$. Then $\mathcal{M} = \{\emptyset, E, E^c, X\}$ is an algebra of subsets of X.

4. Suppose that X is infinite. Then
$$\mathcal{M}_1 := \{E \subseteq X : E \text{ is finite}\}$$
is *not* an algebra as (A1) does not hold. However,
$$\mathcal{M}_2 := \{E \subseteq X : E \text{ is finite, or } E^c \text{ is finite}\}$$
is an algebra. (check !)

Exercise 2.1.1. Let X be a set, and let $\mathcal{M}$ be a non-empty collection of subsets of X satisfying

1. If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$,
2. If $A, B \in \mathcal{M}$, then $A \cap B \in \mathcal{M}$.

Show that $\mathcal{M}$ is an algebra of subsets of X.

We will often make use of the next theorem which allows us to replace any finite or countable collection of sets in $\mathcal{M}$ with a collection of disjoint sets.

Theorem 2.1.1. *Let $\mathcal{M}$ be an algebra of subsets of X. If $\{A_i\}_{i=1}^{\infty}$ is a countable family (=sequence) of sets in $\mathcal{M}$, then there exists a family $\{B_i\}_{i=1}^{\infty}$ of pairwise disjoint sets of $\mathcal{M}$ satisfying $B_n \subseteq A_n$ for all i, and*
$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i \tag{2.1}$$
for all $n = 1, 2, \ldots$. Furthermore,
$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$

Proof. We construct the sets B_i inductively. First, set $B_1 = A_1$. Then the assertion is true for $n = 1$. In general, suppose we have constructed pairwise disjoint sets $B_1, \ldots, B_n \in \mathcal{M}$ satisfying $B_i \subseteq A_i$ for all $i = 1, \ldots, n$ and
$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i. \tag{2.2}$$
Set $B_{n+1} = A_{n+1} \setminus \bigcup_{i=1}^{n} B_i \in \mathcal{M}$. Then by construction, the sets $B_1, \ldots, B_{n+1}$ are pairwise disjoint, and
$$\begin{aligned}\bigcup_{i=1}^{n+1} B_i &= B_{n+1} \cup \left(\bigcup_{i=1}^{n} B_i\right) = \left(A_{n+1} \setminus \bigcup_{i=1}^{n} B_i\right) \cup \left(\bigcup_{i=1}^{n} B_i\right) \\ &= A_{n+1} \cup \left(\bigcup_{i=1}^{n} B_i\right) \overset{(2.2)}{=} A_{n+1} \cup \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n+1} A_i.\end{aligned}$$
Thus, (2.1) holds. Finally, by (2.1) we have
$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{\infty} A_i.$$
□

2.1.2 Sigma-Algebras

Since we often will deal with countable unions of sets, we need to refine the concept of an algebra.

Definition 2.1.2. Let X be a set. A non-empty collection $\mathcal{M}$ of subsets of X is called a *σ-algebra of subsets of X* if the following hold:

(A1) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$,

(A2') If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ is a countable collection of subsets of $\mathcal{M}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Remark 2.1.2. Let X be a set and $\mathcal{M}$ a σ-algebra of subsets of X.

1. Let $A, B \in \mathcal{M}$. Set
$$A_n = \begin{cases} A & \text{if } n \text{ is odd} \\ B & \text{if } n \text{ is even.} \end{cases}$$
Then
$$A \cup B = (A \cup B) \cup (A \cup B) \cup (A \cup B) \cup \cdots = \bigcup_{n=1}^{\infty} A_n \in \mathcal{M},$$
that is, (A2) holds. Thus, every σ-algebra is also an algebra.

2. If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$, we have
$$\bigcap_{n=1}^{\infty} A_n = \left[\bigcup_{n=1}^{\infty} A_n^c\right]^c \in \mathcal{M}$$
by (A1) and (A2').

Thus, a σ-algebra is closed under formation of differences, countable unions and countable intersections.

Exercise 2.1.2. Let X be a set, and let $\mathcal{M}$ be a non-empty collection of subsets of X. Suppose, $\mathcal{M}$ satisfies:

1. Whenever $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$,

2. Whenever $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

Show that $\mathcal{M}$ is a σ-algebra.

Example 2.1.2. Let X be a set. Then

1. $\mathcal{M} = \{\emptyset, X\}$ is a σ-algebra. In fact, it is the smallest σ-algebra of subsets of X.

2. $\mathcal{P}(X)$ is a σ-algebra. In fact, it is the largest σ-algebra of subsets of X.

3. Fix any $E \subseteq X$. Then $\mathcal{M} = \{\emptyset, E, E^c, X\}$ is a σ-algebra.

4. Suppose that X is infinite. Then
$$\mathcal{M}_2 := \{E \subseteq X : E \text{ is finite, or } E^c \text{ is finite}\}$$
is *not* a σ-algebra. In fact, let $\{x_1, x_2, x_3, \dots\}$ be a countable subset of X. Set $E = \bigcup_{k=1}^{\infty} \{x_{2k}\} = \{x_2, x_4, x_6, \dots\}$. Now each singleton $\{x_{2k}\}$ is in $\mathcal{M}_2$, while E and E^c are both infinite sets, an hence, $E \notin \mathcal{M}_2$. Thus, (A2') does not hold. However,
$$\mathcal{M}_3 := \{F \subseteq X : F \text{ is countable, or } F^c \text{ is countable}\}$$
is a σ-algebra. (Check !) Note that by "countable" we mean either finite, or countably infinite.

5. Let X be infinite, and $\{E_n\}_{n=1}^{\infty}$ be a countable family of pairwise disjoint subsets of X. Set
$$\mathcal{M} := \{E \subseteq X : E \text{ is the countable union of some sets } E_n\}.$$
Then $\mathcal{M}$ is a σ-algebra.

Theorem 2.1.2. *Let Λ be an index set, and for each $\alpha \in \Lambda$, let $\mathcal{M}_\alpha$ be a σ-algebra of subsets of X. Then*
$$\bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha$$
is again a σ-algebra of subsets of X.

Proof. We need to show that properties (A1) and (A2') hold.

1. Let $A \in \bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha$. Then $A \in \mathcal{M}_\alpha$, for each $\alpha \in \Lambda$. Since each $\mathcal{M}_\alpha$ is a σ-algebra, then $A^c \in \mathcal{M}_\alpha$ for each α, and hence
$$A^c \in \bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha.$$

2. Let $\{A_n\}_{n=1}^{\infty} \subseteq \bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha$. Then $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}_\alpha$ for each $\alpha \in \Lambda$. Since $\mathcal{M}_\alpha$ is a σ-algebra, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}_\alpha$, for all α, and hence
$$\bigcup_{n=1}^{\infty} A_n \in \bigcap_{\alpha \in \Lambda} \mathcal{M}_\alpha.$$

□

Remark 2.1.3. Let K be a collection of subsets of X. Let us set
$$\mathcal{M}_o := \bigcap \{\mathcal{M}_\alpha : \mathcal{M}_\alpha \text{ is a } \sigma\text{-algebra containing } K\}.$$
Since $\mathcal{P}(X)$ is a σ-algebra containing K, then obviously, $\mathcal{M}_o \neq \emptyset$. Also, as $K \subseteq \mathcal{M}_\alpha$ for all α, we have $K \subseteq \mathcal{M}_o$. Thus by the theorem, $\mathcal{M}_o$ is a σ-algebra containing K. Furthermore, if $\mathcal{M}_1$ is any σ-algebra containing K, then by the above definition of $\mathcal{M}_o$, we have $\mathcal{M}_o \subseteq \mathcal{M}_1$. Thus $\mathcal{M}_o$ is the *smallest* σ-algebra containing K, called the *σ-algebra generated by K*, and denoted by $\sigma(K)$.

Example 2.1.3. 1. Let X be any set. If $E \subseteq X$, then

$$\sigma(E) = \{\emptyset, E, E^c, X\}.$$

2. Let X be an uncountable set, and $K = \{E \subseteq X : E \text{ is finite}\}$. Then

$$\sigma(K) = \{F \subseteq X : F \text{ is countable, or } F^c \text{ is countable}\}.$$

(Check !)

2.2 The Extended Real Numbers

2.2.1 Arithmetic of Extended Real Numbers

The set of *extended real numbers* is defined to be the set

$$\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$$

which we also may write as $\mathbb{R}^* = [-\infty, \infty]$.

Algebraic operations are extended to $\mathbb{R}^*$ by setting

1. $$\infty + x = \infty \quad \text{and} \quad (-\infty) + x = -\infty$$
$$\infty + \infty = \infty \quad \text{and} \quad (-\infty) + (-\infty) = -\infty$$
(Note that $\infty + (-\infty)$ and $-\infty + \infty$ are undefined.)

2. $$(\pm\infty) \cdot x = \begin{cases} \pm\infty & \text{if } x > 0 \\ \mp\infty & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

3. $$\infty \cdot \infty = \infty \quad \text{and} \quad (-\infty) \cdot (-\infty) = \infty$$
$$(-\infty) \cdot (\infty) = -\infty \quad \text{and} \quad (-\infty) \cdot (\infty) = -\infty$$

for all $x \in \mathbb{R}$.

Note that the distributive law, and the usual cancellation laws for addition and multiplication no longer hold:

$$\begin{array}{lcl} x + y = x + z & \not\Rightarrow & y = z \\ x + y = z & \not\Rightarrow & x = z - y \\ xy = xz \quad (x \neq 0) & \not\Rightarrow & y = z \end{array}$$

The order "$<$" on $\mathbb{R}$ is extended to $\mathbb{R}^*$ by setting

$$-\infty < x < \infty$$

for all $x \in \mathbb{R}$. Then ∞ is an upper bound for every set $E \subseteq \mathbb{R}^*$, and $-\infty$ is a lower bound. Note that $\sup E$ always exists. In fact,

$$\sup E = \begin{cases} M \text{ is finite, if } E \subseteq \mathbb{R},\ E \neq \emptyset \text{ and } E \text{ is bounded above in } \mathbb{R}, \\ \infty, \text{ if } E \subseteq \mathbb{R} \text{ is not bounded above in } \mathbb{R}, \text{ or if } \infty \in E, \\ -\infty, \text{ if } E = \emptyset. \end{cases}$$

Similarly, $\inf E$ always exist.

In order to avoid introducing a topology into $\mathbb{R}^*$ (which is not difficult to do) we extend the concept of convergent sequences as follows: Let $\{x_n\} \subset \mathbb{R}^*$. Then for $L \in \mathbb{R}$ we have

$$\lim_{n\to\infty} x_n = L \quad \Leftrightarrow \quad \forall \epsilon > 0 \quad \exists N \in \mathbb{N} \text{ such that } L - \epsilon < x_n < L + \epsilon \quad \forall n \geq N,$$
$$\lim_{n\to\infty} x_n = \infty \quad \Leftrightarrow \quad \forall M \in \mathbb{R} \quad \exists N \in \mathbb{N} \text{ such that } x_n > M \quad \forall n \geq N,$$
$$\lim_{n\to\infty} x_n = -\infty \quad \Leftrightarrow \quad \forall m \in \mathbb{R} \quad \exists N \in \mathbb{N} \text{ such that } x_n < m \quad \forall n \geq N.$$

Note that every increasing sequence $\{x_n\} \uparrow \subset \mathbb{R}^*$ converges in $\mathbb{R}^*$, either to a real number (if $x_n \in \mathbb{R}\ \forall n$, and the sequence is bounded above in $\mathbb{R}$), or to ∞.

Note that in general,

$$\lim_{n\to\infty} (x_n y_n) \neq \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right). \tag{2.3}$$

For example, if $x_n = n^2$, $y_n = \frac{1}{n}$, then $\lim_{n\to\infty} (x_n y_n) = \infty$ while $(\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n) = 0$. However, if $\{x_n\} \uparrow$, $\{y_n\} \uparrow$, and $x_n > 0$, $y_n > 0$ for all n, then (2.3) holds.

2.2.2 Unordered Sums

Recall that every absolutely convergent series $\sum_{n=1}^{\infty} a_n$ in $\mathbb{R}$ can be rearranged arbitrarily. In particular,

$$\sum_{n=1}^{\infty} a_n := \sum_{\substack{n=1\\ a_n \geq 0}}^{\infty} a_n + \sum_{\substack{n=1\\ a_n < 0}}^{\infty} a_n.$$

In order to extend this concept to $\mathbb{R}^*$ we make the following definition:

A sum of the form

$$\sum_{n\in\mathbb{N}} a_n \qquad (a_n \in \mathbb{R}^*)$$

(or more general, of the form $\sum_{n\in P} a_n$ with P a countable set), will be called an *unordered sum*. Its sum is defined as follows:

1. If $0 \leq a_n \leq \infty$ for all n, we define

$$\sum_{n\in\mathbb{N}} a_n := \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$
$$= \begin{cases} \sum\limits_{n=1}^{\infty} a_n & \text{if } a_n \in \mathbb{R} \text{ for all } n, \text{ and this series converges in } \mathbb{R} \\ \infty & \text{if } a_n \in \mathbb{R} \text{ for all } n, \text{ and this series diverges in } \mathbb{R} \\ \infty & \text{if } a_n = \infty \text{ for some } n. \end{cases}$$

2. Similarly, if $a_n < 0$ for all n, we define

$$\sum_{n\in\mathbb{N}} a_n := \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$
$$= \begin{cases} \sum_{n=1}^{\infty} a_n & \text{if } a_n \in \mathbb{R} \text{ for all } n, \text{ and this series converges in } \mathbb{R} \\ -\infty & \text{if } a_n \in \mathbb{R} \text{ for all } n, \text{ and this series diverges in } \mathbb{R} \\ -\infty & \text{if } a_n = -\infty \text{ for some } n. \end{cases}$$

3. In general, set

$$P^+ := \{n \in \mathbb{N} : a_n \geq 0\} \qquad \text{and} \qquad P^- := \{n \in \mathbb{N} : a_n < 0\}$$

and define

$$\sum_{n\in\mathbb{N}} a_n := \sum_{n\in P^+} a_n + \sum_{n\in P^-} a_n$$

provided that the right-hand sum is not of the form $\infty - \infty$.

Note that by the above remarks,

$$\sum_{n\in\mathbb{N}} a_n = \begin{cases} \sum_{n=1}^{\infty} a_n & \text{if } \sum_{n=1}^{\infty} |a_n| \text{ converges in } \mathbb{R} \\ \infty & \text{if } \sum_{n\in P^+} a_n = \infty \quad \text{and} \quad \sum_{n\in P^-} a_n \text{ converges in } \mathbb{R} \\ -\infty & \text{if } \sum_{n\in P^-} a_n = -\infty \quad \text{and} \quad \sum_{n\in P^+} a_n \text{ converges in } \mathbb{R}. \end{cases}$$

For example,

$$\sum_{n\in\mathbb{N}} \frac{(-1)^{n+1}}{n}$$

is undefined as an unordered sum in the sense above, while

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

2.3 Measures

2.3.1 Definition of Measure

Definition 2.3.1. Let X be a set, and $\mathcal{M}$ a σ-algebra of subsets of X. The pair $(X, \mathcal{M})$ is called a *measurable space.*

A *measure* on $(X, \mathcal{M})$ is a function

$$\mu : \mathcal{M} \to [0, \infty]$$

satisfying

(Meas1) $\mu(\emptyset) = 0$,

(Meas2) For any countable collection $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ of pairwise disjoint sets we have

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n) \qquad (\text{"}\sigma\text{-additivity"}).$$

The triple $(X\mathcal{M}, \mu)$ is called a *measure space.*

If $\mu(X) < \infty$ then μ is called a *finite measure*, and $(X, \mathcal{M}, \mu)$ a *finite measure space.*

If there exists a countable collection $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ such that $\mu(E_n) < \infty$ for each n, and $X = \bigcup_{n=1}^{\infty} E_n$, then μ is called *σ-finite*, and $(X, \mathcal{M}, \mu)$ is called a *σ-finite measure space.*

Example 2.3.1. Let X be any set. One easily verifies that all of the following are measure on $\mathcal{M} = \mathcal{P}(X)$. (Exercise !)

1. Two *trivial measures* on $\mathcal{P}(X)$ are given by
   (a) $\mu(E) = 0$ for all $E \subseteq X$.
   (b) $\mu(\emptyset) = 0$, $\mu(E) = \infty$ for all $E \subseteq X$, $E \neq \emptyset$.

2. The *counting measure* on $\mathcal{P}(X)$ is defined by

$$\mu(E) = \begin{cases} \text{card}(E) & \text{if } E \text{ is a finite set} \\ \infty & \text{if } E \text{ is an infinite set} \end{cases}$$

   for $E \subseteq X$.

   μ is a finite measure $\Leftrightarrow$ X is a finite set.
   μ is a σ-finite measure $\Leftrightarrow$ X is a countable set.

3. Fix $x \in X$. The corresponding *Dirac-point measure* on $\mathcal{P}(X)$ is defined by

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

   for $E \subseteq X$. This is obviously a finite measure.

4. *Discrete measures* on $\mathcal{P}(X)$: Suppose, X is infinite. Fix a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in X, and a sequence $\{a_n\}_{n=1}^{\infty} \subset [0, \infty]$. For $E \subseteq X$, define

$$\mu(E) = \sum_{\{n: x_n \in E\}} a_n \quad \Big(= \sum_{n=1}^{\infty} a_n \delta_{x_n}(E)\Big).$$

   μ is a finite measure $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n < \infty$.
   μ is a σ-finite measure $\Leftrightarrow$ $a_n < \infty$ for all n.

5. Let us modify the discrete measure. Suppose that $X = \{x_1, x_2, x_3, \ldots\}$ is countably infinite. Given $E \subseteq X$, let

$$\mu(E) = \begin{cases} \sum\limits_{\{n:x_n \in E\}} \frac{1}{2^n} & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite} \end{cases}$$

It is not difficult to verify that μ is finitely additive, i.e. $\mu\Big(\bigcup\limits_{n=1}^{N} E_n\Big) = \sum\limits_{n=1}^{N} \mu(E_n)$ for every finite collection $\{E_n\}_{n=1}^{N} \subseteq \mathcal{P}(X)$ of pairwise disjoint sets and every N. However, μ is not σ-additive. (Why ?) Hence, μ is not a measure.

Example 2.3.2. In probability theory, one considers measure spaces $(X, \mathcal{M}, \mu)$ with $\mu(X) = 1$. Then μ is called a *probability measure.* Elements of $\mathcal{M}$ are called *events*, $\mathcal{M}$ is called *the event space*, and $(X, \mathcal{M}, \mu)$ a *probability space.* Given an event E, $\mu(E)$ is the *probability* that event E occurs. $\emptyset$ is the *impossible event*, and X the *sure event.*

For example, throwing a dice once, we could have

$$X = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad \mathcal{M} = \mathcal{P}(X).$$

Then $E = \{1, 2, 3\}$ is the event that the number on the dice is ≤ 3, and $\mu(\{1, 2, 3\}) = \frac{1}{2}$ if the dice is not loaded.

Exercise 2.3.1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $A \subseteq X$ be given. Set

$$\mathcal{M}_A := \{E \cap A : E \in \mathcal{M}\}.$$

1. Show that $\mathcal{M}_A$ is a σ-algebra of subsets of A.

2. Show that if $A \in \mathcal{M}$, then $\mathcal{M}_A \subseteq \mathcal{M}$, and $\mu_{|A}(E) := \mu(E)$ for all $E \in \mathcal{M}_A$ defines a measure on the measurable space $(A, \mathcal{M}_A)$. (We call $\mu_{|A}$ the *restriction of* μ *to* $(A, \mathcal{M}_A)$ and sometimes denote it simply by the same symbol μ.)

2.3.2 Properties of Measures

Theorem 2.3.1. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Then*

1. *For any finite collection* $E_1, \ldots, E_n$ *of pairwise disjoint sets in* $\mathcal{M}$*, we have*

$$\mu\Big(\bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} \mu(E_i) \qquad \text{("finite additivity").}$$

2. *Whenever* $E, F \in \mathcal{M}$ *and* $E \subseteq F$*, then*

$$\mu(E) \leq \mu(F) \qquad \text{("monotonicity").}$$

3. Whenever $E, F \in \mathcal{M}$ with $E \subseteq F$ and $\mu(E) < \infty$, then

$$\mu(F \backslash E) = \mu(F) - \mu(E) \tag{2.4}$$

4. For any countable collection $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ we have

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \sum_{i=1}^{\infty} \mu(E_i) \qquad \text{("}\sigma\text{-subadditivity").}$$

5. For any finite collection $E_1, \dots, E_n$ of sets in $\mathcal{M}$ we have

$$\mu\Big(\bigcup_{i=1}^{n} E_i\Big) \leq \sum_{i=1}^{n} \mu(E_i) \qquad \text{("subadditivity").}$$

Proof. 1. Set $E_i = \emptyset \in \mathcal{M}$ for $i = n+1, n+2, \dots$. Then the sets $\{E_i\}_{i=1}^{\infty}$ are pairwise disjoint, hence by σ-additivity,

$$\mu\Big(\bigcup_{i=1}^{n} E_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{n} \mu(E_i) + \sum_{i=n+1}^{\infty} \mu(E_i)$$
$$= \sum_{i=1}^{n} \mu(E_i) + \sum_{i=n+1}^{\infty} 0 = \sum_{i=1}^{n} \mu(E_i).$$

2. Since E and $F \backslash E$ are disjoint and $\mu \geq 0$, we have by 1. (additivity),

$$\mu(E) \leq \mu(F \backslash E) + \mu(E) = \mu\big((F \backslash E) \cup E\big) = \mu(F).$$

3. Since E and $F \backslash E$ are disjoint, we have by 1. (additivity),

$$\mu(F) = \mu\big(E \cup (F \backslash E)\big) = \mu(E) + \mu(F \backslash E).$$

Since $\mu(E) < \infty$ we can subtract,

$$\mu(F) - \mu(E) = \mu(F \backslash E).$$

4. Let B_i be the disjoint sets in $\mathcal{M}$ obtained from E_i as in the proof of theorem 2.1.1. Then $B_i \subseteq E_i$ for each i, and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} E_i.$$

Now by 2. (monotonicity), $\mu(B_i) \leq \mu(E_i)$ for each i, so that by σ-additivity,

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

5. This is derived from 4. by arguing as in to 1.

□

Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets with $E_1 \subseteq E_2 \subseteq E_3 \cdots \subseteq E_n \subseteq E_{n+1} \subseteq \ldots$. We call this sequence *increasing*, and write $\{E_n\}\uparrow$.

Similarly, if $E_1 \supseteq E_2 \supseteq E_3 \cdots \supseteq E_n \supseteq E_{n+1} \supseteq \ldots$, then we call the sequence *decreasing*, and write $\{E_n\}\downarrow$.

Theorem 2.3.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\{E_n\}_{n=1}^{\infty}$ a sequence of sets in $\mathcal{M}$.*

1. *If $\{E_n\}\uparrow$, then* $\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)$.

2. *If $\{E_n\}\downarrow$, and $\mu(E_{n_o}) < \infty$ for some n_o, then* $\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)$.

Proof. 1. Let B_n be the pairwise disjoint sets obtained from E_n as in the proof of theorem 2.1.1. That is, $B_1 = E_1$, $B_n = E_n \backslash \bigcup_{i=1}^{n-1} E_i = E_n \backslash E_{n-1}$ for $n \geq 2$ (in the last equality, we have used the fact that $\{E_n\}\uparrow$), and

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} E_i \quad \text{for each } n \text{ and } \quad \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} E_i.$$

Then

$$\begin{aligned}\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) &= \mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) \\ &= \lim_{n\to\infty} \mu\Big(\bigcup_{i=1}^{n} B_i\Big) = \lim_{n\to\infty} \mu\Big(\bigcup_{i=1}^{n} E_i\Big) = \lim_{n\to\infty} \mu(E_n)\end{aligned}$$

since the sets B_i are pairwise disjoint and $E_n = \cup_{i=1}^{n} E_i$.

2. For each $n \geq n_o$, set $F_n = E_{n_o} \backslash E_n$. Then $\{F_n\}_{n=n_o}^{\infty}\uparrow$, so that by part 1. above,

$$\begin{aligned}\mu\Big(\bigcup_{n=n_o}^{\infty} F_n\Big) &= \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(E_{n_o} \backslash E_n) \\ &\underset{(2.4)}{=} \lim_{n\to\infty} \mu(E_{n_o}) - \mu(E_n) = \mu(E_{n_o}) - \lim_{n\to\infty} \mu(E_n)\end{aligned}$$

where we have also used the fact that $\mu(E_n) \leq \mu(E_{n_o}) < \infty$. But

$$\bigcup_{n=n_o}^{\infty} F_n = \bigcup_{n=n_o}^{\infty} (E_{n_o} \backslash E_n) = E_{n_o} \backslash \bigcap_{n=n_o}^{\infty} E_n$$

and hence

$$\mu(E_{n_o})-\mu\Big(\bigcap_{n=n_o}^{\infty} E_n\Big) \underset{(2.4)}{=} \mu\Big(E_{n_o}\setminus \bigcap_{n=n_o}^{\infty} E_n\Big) = \mu\Big(\bigcup_{n=n_o}^{\infty} F_n\Big) = \mu(E_{n_o})-\lim_{n\to\infty}\mu(E_n)$$

from which the assertion follows, again since $\mu(E_{n_o})$ is finite.

□

Remark 2.3.1. In the second case, the condition that $\mu(E_{n_o}) < \infty$ can not be dropped. For example, if $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ and μ is the counting measure, set

$$E_n = \{n, n+1, n+2, \dots\}.$$

Then $\mu(E_n) = \infty$ for all n, $\{E_n\} \downarrow$, and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, and thus

$$\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0 \neq \infty = \lim_{n\to\infty}\mu(E_n).$$

2.4 Construction of the Lebesgue Measure

Now that we have defined the concept of a measure, we would like to construct a measure on $\mathbb{R}^n$ which generalizes the notion of volume of an interval. It turns out that this construction can be applied to a large class of sets, and we therefore introduce it in a general way.

2.4.1 Semirings

Consider the set $X = \mathbb{R}^n$. An *n-interval* is a set of the form

$$E = I_1 \times I_2 \times \cdots \times I_n$$

where each I_j $(j = 1\dots n)$ is an interval in $\mathbb{R}$. If each I_j is an open interval,

$$E = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$$

then E is called an *open n-interval.* Similarly, if each I_j is a closed interval,

$$E = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

then E is called a *closed n-interval.*

Exercise 2.4.1. Show that every open n-interval is an open set, and every closed n-interval is a closed set, in the usual metric of $\mathbb{R}^n$.

In the following, we will consider the collection

$$\mathcal{S}_o := \{\, E \subseteq \mathbb{R}^n \ :\ E = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n] \,\}$$

with $-\infty < a_j \leq b_j < \infty$.

Note that the collection $\mathcal{S} = \mathcal{S}_o$ has the following properties:

Figure 2.1: n-intervals.

(SR1) $\emptyset \in \mathcal{S}$.

(SR2) If $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$.

(SR3) If $A, B \in \mathcal{S}$, then $A \backslash B$ is a finite disjoint union of elements of $\mathcal{S}$,

$$A \backslash B = \bigcup_{i=1}^{m} C_i, \qquad C_i \in \mathcal{S},\ C_i \cap C_j = \emptyset \ \text{ if } \ i \neq j.$$

These properties are intuitively clear, and they are easy to prove if $n = 1$. For arbitrary n they can be proved by induction on n. (Exercise).

Definition 2.4.1. Let X be a set, and $\mathcal{S}$ a non-empty collection of subsets of X. Then $\mathcal{S}$ is called a *semiring*, provided that (SR1) – (SR3) above hold.

Example 2.4.1. 1. Every algebra $\mathcal{M}$, and hence every σ-algebra $\mathcal{M}$, is a also semiring, by property (A1) and remark 2.1.1.

2. Let X be an infinite set, and $\mathcal{S} = \{E \subseteq X : E \text{ is finite}\}$. Then $\mathcal{S}$ is a semiring (but not an algebra).

If $\mathcal{S}$ is a semiring, then the finite or countable union of elements of $\mathcal{S}$ need not be in $\mathcal{S}$. Since we will need to deal which such unions, we make the following definition:

Definition 2.4.2. Let $\mathcal{S}$ be a semiring of subsets of X. Then a set $E \subseteq X$ is called a *σ-set*, if E is the countable disjoint union of elements of $\mathcal{S}$,

$$E = \bigcup_{i=1}^{\infty} A_i, \qquad A_i \in \mathcal{S},\ A_i \cap A_j = \emptyset \text{ if } i \neq j.$$

Remark 2.4.1. Obviously, if E is a finite disjoint union of elements of $\mathcal{S}$,

$$E = \bigcup_{i=1}^{m} A_i, \qquad A_i \in \mathcal{S},\ A_i \cap A_j = \emptyset \text{ if } i \neq j.$$

then E is also a σ-set. Simply choose $A_{m+1} = A_{m+2} = \cdots = \emptyset$.

We need the following generalization of (SR3) and of theorem 2.1.1:

Theorem 2.4.1. *Let $\mathcal{S}$ be a semiring of subsets of X. Then*

1. *If $A \in \mathcal{S}$, and $A_1, A_2, \ldots, A_n \in \mathcal{S}$, then then $A \backslash \cup_{i=1}^{n} A_i$ is a finite disjoint union of elements of $\mathcal{S}$,*

$$A \backslash \overset{n}{\underset{i=1}{\cup}} A_i = \overset{m}{\underset{k=1}{\cup}} C_k, \qquad C_k \in \mathcal{S},\ C_j \cap C_k = \emptyset \ \text{ if } \ j \neq k. \tag{2.5}$$

(In particular, $A \backslash \cup_{k=1}^{n} A_k$ is a σ-set.)

2. *If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$, then $\overset{\infty}{\underset{n=1}{\cup}} A_n$ is a σ-set. In fact,*

$$\overset{\infty}{\underset{n=1}{\cup}} A_n = \overset{\infty}{\underset{n=1}{\cup}} \overset{m_n}{\underset{k=1}{\cup}} C_{nk}, \qquad C_{nk} \in \mathcal{S},\ \{C_{nk}\}_{n=1}^{\infty},{}_{k=1}^{m_n} \text{ disjoint},\ C_{nk} \subseteq A_n.$$

3. *The countable union of σ-sets is a σ-set.*

4. *The finite intersection of σ-sets is a σ-set.*

Proof. 1. We proceed by induction on n.
a) If $n = 1$, then (2.5) is (SR3), so the statement is true.
b) Suppose, the statement is true for some n. Let $A \in \mathcal{S}$, and $A_1, A_2, \ldots, A_{n+1} \in \mathcal{S}$ be given. Then by induction assumption, (2.5) holds, so that

$$A \backslash \overset{n+1}{\underset{i=1}{\cup}} A_i = \left(A \backslash \overset{n}{\underset{i=1}{\cup}} A_i\right) \backslash A_{n+1} = \left(\overset{m}{\underset{k=1}{\cup}} C_k\right) \backslash A_{n+1} = \overset{m}{\underset{k=1}{\cup}} C_k \backslash A_{n+1},$$

and the collection $\{C_k \backslash A_{n+1}\}_{k=1}^{m}$ is disjoint, since $C_k \backslash A_{n+1} \subseteq C_k$, and $\{C_k\}_{k=1}^{m}$ is disjoint. Now by (SR3), each set $C_k \backslash A_{n+1}$ is a finite, disjoint union of elements of $\mathcal{S}$,

$$C_k \backslash A_{n+1} = \overset{m_k}{\underset{j=1}{\cup}} B_{kj}, \qquad B_{kj} \in \mathcal{S},\ \{B_{kj}\}_{j=1}^{m_k} \text{ disjoint}.$$

Now as $\{C_k \backslash A_{n+1}\}_{k=1}^{m}$ is disjoint, it follows that $\{B_{kj}\}_{k=1}^{m},{}_{j=1}^{m_k}$ is a disjoint collection, and

$$A \backslash \overset{n+1}{\underset{i=1}{\cup}} A_i = \overset{m}{\underset{k=1}{\cup}} C_k \backslash A_{n+1} = \overset{m}{\underset{k=1}{\cup}} \overset{m_k}{\underset{j=1}{\cup}} B_{kj}.$$

Thus, the assertion hold for $n+1$.
c) It follows by induction that (2.5) holds for all n.

2. Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ be given. Let

$$\begin{aligned} B_1 &:= A_1 \\ B_2 &:= A_2 \backslash A_1 \\ &\vdots \\ B_n &:= A_n \backslash \bigcup_{i=1}^{n-1} A_i \end{aligned}$$

be as in the proof of theorem 2.1.1. Then $B_i \cap B_j = \emptyset$ for $i < j$, as $B_i \subseteq A_i$, and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Note that the sets B_n need not be in $\mathcal{S}$ if $\mathcal{S}$ is not an algebra. However, by part 1., for each n,

$$B_n = \bigcup_{k=1}^{m_n} C_{nk}, \qquad C_{nk} \in \mathcal{S},\ \{C_{nk}\}_{k=1}^{m_n} \text{ disjoint},$$

and since $\{B_n\}_{n=1}^{\infty}$ is disjoint, the collection $\{C_{nk}\}_{n=1}^{\infty}{}_{,}{}_{k=1}^{m_k}$ is also disjoint. Since

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} C_{nk},$$

and $C_{nk} \subseteq B_n \subseteq A_n$ for each n, the second assertion follows.

3. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ be a countable collection of σ-sets, and set $A = \cup_{n=1}^{\infty} A_n$. Since each A_n is of the form

$$A_n = \bigcup_{k=1}^{\infty} B_{nk}, \quad B_{nk} \in \mathcal{S},$$

we have

$$A = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{nk}, \quad B_{nk} \in \mathcal{S},$$

a countable union. It now follows from part 2. that A is a σ-set.

4. Let $A_1, \ldots, A_m$ be σ-sets. Then each A_n is of the form

$$A_n = \bigcup_{k=1}^{\infty} B_{nk}, \quad B_{nk} \in \mathcal{S},\ \{B_{nk}\}_{k=1}^{\infty} \text{ disjoint}.$$

Then

$$A := \bigcap_{n=1}^{m} A_n = \bigcap_{n=1}^{m} \left(\bigcup_{k=1}^{\infty} B_{nk} \right) = \bigcup_{k=1}^{\infty} \Big(\underbrace{\bigcap_{n=1}^{m} B_{nk}}_{\in \mathcal{S} \text{ by (SR2)}} \Big)$$

Since for each k, $\bigcap_{n=1}^{m} B_{nk} \subseteq B_{1k}$ and the family $\{B_{1k}\}_{k=1}^{\infty}$ is disjoint, it follows that A is a σ-set. $\square$

2.4.2 Premeasures on Semirings

Definition 2.4.3. Let $\mathcal{S}$ be a semiring. A function $\mu : \mathcal{S} \to [0, \infty]$ satisfying

(PM1) $\mu(\emptyset) = 0$,

(PM2) Whenever $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ are pairwise disjoint with $\cup_{n=1}^{\infty} E_n \in \mathcal{S}$, then

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i), \qquad \text{("}\sigma\text{-additive")}$$

is called a *premeasure (or sometimes simply a measure) on* $\mathcal{S}$.

Examples of Premeasures

Example 2.4.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then obviously, μ is a premeasure on $\mathcal{M}$.. So the above definition is a generalization of the concept of measure from the class of σ-algebras to the class of semirings.

Example 2.4.3. Let X be a set, and $\mathcal{S} = \{E \subseteq X : E \text{ is finite}\}$. Set

$$\mu(E) := \operatorname{card}(E)$$

for all $E \in \mathcal{S}$. Then μ is a premeasure on $\mathcal{S}$.

Example 2.4.4. Let $X = \mathbb{R}$, and $\mathcal{S} = \mathcal{S}_o$ the semiring of half-open intervals,

$$\mathcal{S} = \{\, (a, b] \subset \mathbb{R} : -\infty < a \leq b < \infty\}.$$

Furthermore, fix a function $F : \mathbb{R} \to \mathbb{R}$ with the following properties:

1. F is increasing (i.e. nondecreasing),
2. F is right continuous. That is, for each $a \in \mathbb{R}$ we have

$$\lim_{x \to a^+} F(x) = F(a).$$

Now define $\mu : \mathcal{S} \to [0, \infty)$ by

$$\mu\big(\,(a, b]\,\big) := F(b) - F(a).$$

Let us verify that μ is a premeasure. Property (PM1) is obvious: If $a = b$ then

$$\mu(\emptyset) = \mu\big(\,(a, a]\,\big) = F(a) - F(a) = 0.$$

In order to prove (PM2), let $\{E_k\}_{k=1}^{\infty}$ be a sequence of pairwise disjoint intervals in $\mathcal{S}$, say $E_k = (a_k, b_k]$ for each k, satisfying

$$E := \bigcup_{k=1}^{\infty} E_k \in \mathcal{S}.$$

Then E is also a half-open interval, say say $E = (a, b]$. If $E = \emptyset$ the assertion of the theorem is trivial, so we may assume that $E \neq \emptyset$.

a) For given $N \in \mathbb{N}$, let's consider the first N intervals $E_1, \ldots, E_N$. Relabeling if necessary, we may assume that the intervals are ordered form left to right,

$$a \le a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_k \le b_k \le a_{k+1} \le b_{k+1} \le \cdots \le b_N \le b.$$

Then

$$\begin{aligned}
\sum_{k=1}^{N} \mu(E_k) &= \sum_{k=1}^{N} [F(b_k) - F(a_k)] \\
&\le \sum_{k=1}^{N-1} [F(a_{k+1}) - F(a_k)] + [F(b_N) - F(a_N)] \qquad (b_k \le a_{k+1},\ F\uparrow) \\
&= F(b_N) - F(a_1) \qquad \text{(telescoping sum)} \\
&\le F(b) - F(a) = \mu(E) \qquad (a \le a_1,\ b_N \le b,\ F\uparrow).
\end{aligned}$$

Now let $N \to \infty$, and obtain

$$\sum_{k=1}^{\infty} \mu(E_k) \le \mu(E). \tag{2.6}$$

b) Now we need to prove the reverse inequality. For this, let $\epsilon > 0$, $\delta > 0$ be arbitrary, but fixed, with $0 < \delta < b - a$. We enlarge the sets E_k slightly to open intervals, so that we can use a compactness argument. In fact, since F is continuous from the right and increasing, for each k we can pick a number c_k satisfying $b_k < c_k$ and

$$F(c_k) < F(b_k) + \frac{\epsilon}{2^k}. \tag{2.7}$$

Set $A_k = (a_k, c_k)$. Then $E_k = (a_k, b_k] \subset (a_k, c_k) = A_k$ for all k.

Consider the compact interval $[a + \delta, b]$. Since

$$[a + \delta, b] \subset (a, b] = \bigcup_{k=1}^{\infty} E_k \subset \bigcup_{k=1}^{\infty} A_k$$

and each A_k is open, there exists a finite collection $A_{k_1}, \ldots, A_{k_N}$ covering $[a + \delta, b]$. Note that if one of these intervals is contained in another, say $A_{k_i} \subseteq A_{k_j}$, then we may discard the set A_{k_i} and still have an open cover of $[a+\delta, b]$. Thus we may assume that $A_{k_i} \not\subset A_{k_j}$ for all $i \neq j$, $j = 1 \dots N$. Similarly, after discarding some of the sets A_{k_i} if necessary, we may assume that $A_{k_i} \cap [a+\delta, b] \neq \emptyset$ for all i. Now we reindex so that the left endpoints of the intervals A_{k_i} are ordered from left to right,

$$a \leq a_{k_1} < a_{k_2} < a_{k_3} \cdots < a_{k_N}.$$

Note that as $A_{k_i} \not\subset A_{k_j}$ for $i \neq j$, the right endpoints must also be ordered from left to right,

$$c_{k_1} < c_{k_2} < c_{k_3} \cdots < c_{k_N}.$$

Now since $A_{k_1}, \dots, A_{k_N}$ is a cover of of $[a+\delta, b]$, the intervals (a_k, c_k) and (a_{k+1}, c_{k+1}) must overlap, that is,

$$a_{k_{i+1}} < c_{k_i}$$

for each $i = 1 \dots N$, and also

$$a_{k_1} < a + \delta \qquad \text{and} \qquad b < c_{k_N}.$$

Because of these inequalities, and as $F\uparrow$, we have

$$\begin{aligned}
\mu\big((a+\delta, b]\big) &= F(b) - F(a+\delta) \leq F(c_{k_N}) - F(a_{k_1}) \\
&= F(c_{k_N}) - F(a_{k_N}) + \sum_{i=1}^{N-1} \big[F(a_{k_{i+1}}) - F(a_{k_i})\big] \qquad \text{(telescoping sum)} \\
&\leq \sum_{i=1}^{N} \big[F(c_{k_i}) - F(a_{k_i})\big] \leq \sum_{k=1}^{\infty} \big[F(c_k) - F(a_k)\big] \\
&\leq \sum_{k=1}^{\infty} \big[F(b_k) + \frac{\epsilon}{2^k} - F(a_k)\big] \qquad \text{(by (2.7))} \\
&= \sum_{k=1}^{\infty} \big[F(b_k) - F(a_k)\big] + \epsilon = \sum_{k=1}^{\infty} \mu(E_k) + \epsilon.
\end{aligned}$$

Now as ϵ was arbitrary, we conclude that

$$\mu\big((a+\delta, b]\big) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

But $\delta > 0$ was also arbitrary, hence by right continuity of F,

$$\begin{aligned}
\mu(E) &= \mu\big((a, b]\big) = F(b) - F(a) \\
&= \lim_{\delta \to 0+} F(b) - F(a+\delta) = \lim_{\delta \to 0+} \mu\big((a+\delta, b]\big) \leq \sum_{k=1}^{\infty} \mu(E_k).
\end{aligned}$$

This proves the reverse inequality to (2.6), and hence (PM2) holds.

Remark 2.4.2. The most important case is when $F(x) = x$, in which case simply

$$\mu\big(\,(a,b]\,\big) = b - a.$$

In this case, we use the symbol λ,

$$\lambda\big(\,(a,b]\,\big) = b - a$$

and call λ the *Lebesgue premeasure* on $\mathcal{S}$.

Example 2.4.5. Consider the semialgebra $\mathcal{S} = \mathcal{S}_o$ of "half-open" intervals in $\mathbb{R}^n$,

$$\mathcal{S} = \{\, E \subseteq \mathbb{R}^n \ : \ E = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]\,\}$$

with $-\infty < a_j \leq b_j < \infty$. Set

$$\lambda(E) \ = \ \mathrm{vol}(E) \ := \ (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n).$$

Then λ is a premeasure on $\mathcal{S}$, again called the *Lebesgue premeasure.* (The fact that λ is a premeasure is proved using example 2.4.4 and induction on n. The induction step is easiest proved by using properties of measures which we will study later; thus we postpone the proof until later in theorem 2.8.6.)

Properties of Premeasures

Remark 2.4.3. Let $\mathcal{S}$ be a semiring, and $\mu : \mathcal{S} \to [0, \infty]$ a premeasure on $\mathcal{S}$. Then

a) If $E_1, \ldots, E_m \in \mathcal{S}$ are pairwise disjoint, and $\bigcup\limits_{i=1}^{m} E_i \in \mathcal{S}$, then

$$\mu\Big(\bigcup_{i=1}^{m} E_i\Big) = \sum_{i=1}^{m} \mu(E_i) \qquad \text{("finite additivity").}$$

In fact, we simply set $E_{m+1} = E_{m+2} = \cdots = \emptyset$ and apply (PM1) and (PM2).

b) If $E, F \in \mathcal{S}$ and $E \subseteq F$, then

$$\mu(E) \leq \mu(F) \qquad \text{("monotonicity").}$$

In fact, by theorem 2.4.1, we can write

$$F \backslash E = \bigcup_{i=1}^{m} C_i, \qquad \{C_i\}_{i=1}^{m} \subseteq \mathcal{S} \text{ disjoint.}$$

Thus,

$$F = E \cup \Big(\bigcup_{i=1}^{m} C_i\Big),$$

a finite, disjoint union. It follows from the above remark that

$$\mu(F) = \mu(E) + \sum_{i=1}^{m} \mu(C_i) \geq \mu(E).$$

Theorem 2.4.2. *Let $\mathcal{S}$ be a semiring, and $\mu : \mathcal{S} \to [0, \infty]$ a function satisfying $\mu(\emptyset) = 0$. Then μ is a premeasure $\Leftrightarrow$ μ has the following two properties:*

1. *Whenever $E_1, \ldots, E_n \in \mathcal{S}$ are pairwise disjoint, $E \in \mathcal{S}$ and $\bigcup\limits_{i=1}^{n} E_i \subseteq E$, then*

$$\sum_{i=1}^{n} \mu(E_i) \leq \mu(E).$$

2. *Whenever $E \in \mathcal{S}$, $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ and $E \subseteq \bigcup\limits_{n=1}^{\infty} E_n$, then*

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n) \qquad \text{("}\sigma\text{-subadditivity").}$$

Proof. $\Rightarrow$: Suppose, μ is a premeasure on $\mathcal{S}$.
a) Let $E_1, \ldots, E_n \in \mathcal{S}$ be pairwise disjoint, $E \in \mathcal{S}$ and $\bigcup\limits_{i=1}^{n} E_i \subseteq E$. Then by theorem 2.4.1, $E \setminus \bigcup\limits_{i=1}^{n} E_i$ is a disjoint union of sets in $\mathcal{S}$,

$$E \setminus \bigcup_{i=1}^{n} E_i = \bigcup_{k=1}^{m} C_k, \quad \{C_k\}_{k=1}^{m} \subseteq \mathcal{S} \text{ disjoint.}$$

Thus,

$$E = \left(\bigcup_{i=1}^{n} E_i\right) \cup \left(\bigcup_{k=1}^{m} C_k\right),$$

a disjoint union. Then by finite additivity of μ (see remark 2.4.3),

$$\mu(E) = \sum_{i=1}^{n} \mu(E_i) + \sum_{k=1}^{m} \mu(C_k) \geq \sum_{i=1}^{n} \mu(E_i).$$

Thus, a) holds.
b) Let $E \in \mathcal{S}$, $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ with $E \subseteq \bigcup\limits_{n=1}^{\infty} E_n$. Then by theorem 2.4.1,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} C_{nk}, \qquad C_{nk} \in \mathcal{S},\ C_{nk} \subseteq E_n,\ \{C_{nk}\}_{n=1}^{\infty}{}_{k=1}^{m_n} \text{ disjoint.}$$

Now by a) we have, since $\cup_{k=1}^{m_n} C_{nk} \subseteq E_n$, that

$$\sum_{k=1}^{m_n} \mu(C_{nk}) \leq \mu(E_n) \tag{2.8}$$

for all n. Now

$$E = E \cap \Big(\bigcup_{n=1}^{\infty} E_n \Big) = E \cap \Big(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} C_{nk} \Big) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} (E \cap C_{nk}),$$

a disjoint union. Since $E \cap C_{nk} \in \mathcal{S}$ by (SR2), we have

$$\mu(E) = \mu\Big(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} (E \cap C_{nk}) \Big) \underset{\text{(PM2)}}{=} \sum_{n=1}^{\infty} \sum_{n=1}^{m_n} \mu(E \cap C_{nk}) \underset{(2.8)}{\leq} \sum_{n=1}^{\infty} \mu(E_n)$$

which shows that b) holds.

$\Leftarrow$: Now suppose that $\mu : \mathcal{S} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$ together with a) and b). Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ be disjoint, with $E := \bigcup_{n=1}^{\infty} E_n \in \mathcal{S}$. then for each $N \in \mathbb{N}$,

$$\bigcup_{n=1}^{N} E_n \subseteq E$$

and hence

$$\sum_{n=1}^{N} \mu(E_n) \underset{\text{a)}}{\leq} \mu(E) \underset{\text{b)}}{\leq} \sum_{n=1}^{\infty} \mu(E_n).$$

Let $N \to \infty$. We obtain

$$\sum_{n=1}^{\infty} \mu(E_n) \leq \mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n),$$

that is,

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

Thus, (PM2) holds, so that μ is a premeasure on $\mathcal{S}$. □

2.4.3 Outer Measures

Suppose, $\mathcal{S}$ is a semiring of subsets of X, and $\mu : \mathcal{S} \to [0, \infty]$ a premeasure on $\mathcal{S}$. (For example, $\mathcal{S} = \mathcal{S}_o$ is the semialgebra of "half-open" intervals defined earlier, and μ the volume of elements of $\mathcal{S}_o$.) We now want to extend μ to a measure on the σ-algebra generated by $\mathcal{S}$. To do so, we first we extend μ to a function on all of $\mathcal{P}(X)$ which, while not a measure, is still monotone and σ-subadditive.

Definition 2.4.4. Let X be a set, $\mathcal{S}$ a collection of subsets of X containing the empty set, and $\mu : \mathcal{S} \to [0, \infty]$ a function satisfying $\mu(\emptyset) = 0$. (For example, $\mathcal{S}$ is a semiring of subsets of X and μ a premeasure.) For $E \subseteq X$, set

$$\mu_*(E) := \inf\Big\{ \sum_{i=1}^{\infty} \mu(A_i) \, : \, \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{S}, \;\; E \subseteq \bigcup_{i=1}^{\infty} A_i \Big\}. \tag{2.9}$$

Then $0 \leq \mu_*(E) \leq \infty$. (Note that $\inf(\emptyset) = \infty$ is a possibility if E can not be covered at all by some countable family $\{A_i\}_{i=1}^{\infty}$ of sets in $\mathcal{S}$.) We call μ_* the *outer measure induced on X by $(\mathcal{S}, \mu)$*.

Theorem 2.4.3. *μ_* has the following properties.*

(OM1) $\mu_*(\emptyset) = 0$,

(OM2) If $E \subseteq F$ then

$$\mu_*(E) \leq \mu_*(F), \qquad \text{("monotonicity")}$$

(OM3) For any countable collection $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ we have

$$\mu_*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} \mu_*(E_n), \qquad \text{("}\sigma\text{-subadditivity")} \tag{2.10}$$

(4) For all $E \in \mathcal{S}$, we have

$$\mu_*(E) \leq \mu(E),$$

(5) If $\mathcal{S}$ is a semiring and μ a premeasure on $\mathcal{S}$, then

(a) for all $E \in \mathcal{S}$, we have

$$\mu_*(E) = \mu(E),$$

(b) the sets A_i in definition 2.4.4 may be assumed to be pairwise disjoint, that is,

$$\mu_*(E) = \inf\Big\{ \sum_{j=1}^{\infty} \mu(C_j) \, : \, \{C_j\}_{j=1}^{\infty} \subseteq \mathcal{S}, \; \{C_j\}_{j=1}^{\infty} \text{ disjoint}, \;\; E \subseteq \bigcup_{j=1}^{\infty} C_j \Big\}. \tag{2.11}$$

Proof. (4): Let $E \in \mathcal{S}$ be given. Set $B_1 = E$ and for $i \geq 2$, set $B_i = \emptyset$. Then $E \subseteq \bigcup_{i=1}^{\infty} B_i$, so that by definition of μ_*,

$$\mu_*(E) = \inf\Big\{ \sum_{i=1}^{\infty} \mu(A_i) \, : \, \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{S}, \;\; E \subseteq \bigcup_{i=1}^{\infty} A_i \Big\} \leq \sum_{i=1}^{\infty} \mu(B_i) = \mu(B_1) = \mu(E),$$

that is, $\mu_*(E) \leq \mu(E)$.

(OM1): If $E = \emptyset$, then by (4) and assumption on μ we have $0 \leq \mu_*(\emptyset) \leq \mu(\emptyset) = 0$, so that $\mu_*(\emptyset) = 0$.

(OM2): Let $E \subseteq F$ be subsets of X. If $\{A_i\}_{i=1}^\infty \subseteq \mathcal{S}$ is such that $F \subseteq \bigcup_{i=1}^\infty A_i$, then obviously, $E \subseteq \bigcup_{i=1}^\infty A_i$ as well, so that

$$\mu_*(E) = \inf\Big\{ \sum_{i=1}^\infty \mu(B_i) : \{B_i\}_{i=1}^\infty \subseteq \mathcal{S}, \ \ E \subseteq \bigcup_{i=1}^\infty B_i \Big\} \leq \sum_{i=1}^\infty \mu(A_i).$$

Taking the infimum over all possible collections $\{A_i\}_{i=1}^\infty \subseteq \mathcal{S}$ covering F, we have

$$\mu_*(E) \leq \inf\Big\{ \sum_{i=1}^\infty \mu(A_i) : \{A_i\}_{i=1}^\infty \subseteq \mathcal{S}, \ \ F \subseteq \bigcup_{i=1}^\infty A_i \Big\} = \mu_*(F).$$

(OM3): Let $\{E_n\}_{n=1}^\infty \subseteq \mathcal{P}(X)$. If $\mu_*(E_n) = \infty$ for some n, then 2.10 is obvious. Thus, we may assume that $\mu_*(E_n) < \infty$ for all n.

Let $\epsilon > 0$ be given. By definition of μ_*, for each n there exists a collection of sets $\{A_i^{(n)}\}_{i=1}^\infty \subseteq \mathcal{S}$ such that

$$E_n \subseteq \bigcup_{i=1}^\infty A_i^{(n)} \qquad \text{and} \qquad \sum_{i=1}^\infty \mu(A_i^{(n)}) < \mu_*(E_n) + \frac{\epsilon}{2^n}.$$

Then

$$\bigcup_{n=1}^\infty E_n \subseteq \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty A_i^{(n)}$$

and hence by definition of μ_*,

$$\mu_*\Big(\bigcup_{n=1}^\infty E_n\Big) \underset{\text{def}}{\leq} \sum_{n=1}^\infty \sum_{i=1}^\infty \mu(A_i^{(n)}) < \sum_{n=1}^\infty \Big(\mu_*(E_n) + \frac{\epsilon}{2^n}\Big) = \sum_{n=1}^\infty \mu_*(E_n) + \epsilon.$$

Since ϵ was arbitrary, it follows that

$$\mu_*\Big(\bigcup_{n=1}^\infty E_n\Big) \leq \sum_{n=1}^\infty \mu_*(E_n).$$

(5): Suppose that $\mathcal{S}$ is a semiring and μ a premeasure on $\mathcal{S}$.

(a) Let $E \in \mathcal{S}$ be given. Note that if $\{A_i\}_{i=1}^\infty$ is any collection in $\mathcal{S}$ such that $E \subseteq \bigcup_{i=1}^\infty A_i$, then by theorem 2.4.2,

$$\mu(E) \leq \sum_{i=1}^\infty \mu(A_i)$$

and hence, taking the infimum over all coverings $\{A_i\}_{i=1}^\infty \subseteq \mathcal{S}$ of E,

$$\mu(E) \leq \inf\Big\{ \sum_{i=1}^\infty \mu(A_i) : \{A_i\}_{i=1}^\infty \subseteq \mathcal{S}, \ \ E \subseteq \bigcup_{i=1}^\infty A_i \Big\} = \mu_*(E).$$

It now follows from (4) that $\mu_*(E) = \mu(E)$.

(b) Let $E \in \mathcal{S}$ be given. If $\{A_i\}_{i=1}^\infty$ is any collection of sets in $\mathcal{S}$ with $E \subseteq \bigcup_{i=1}^\infty A_i$, then by theorem 5, there exists a disjoint collection $\{C_{ik}\}_{i=1}^\infty{}_{,}{}_{k=1}^{m_i}$ of sets in $\mathcal{S}$ with

$$\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty \bigcup_{k=1}^{m_i} C_{ik}$$

and $C_{ik} \subseteq A_i$. Thus by theorem 6, for each i,

$$\sum_{k=1}^{m_i} \mu(C_{ik}) \le \mu(A_i)$$

so that

$$\sum_{i=1}^\infty \sum_{k=1}^{m_i} \mu(C_{ik}) \le \sum_{i=1}^\infty \mu(A_i).$$

Reindexing the countable collection $\{C_{ik}\}_{i=1}^\infty{}_{,}{}_{k=1}^{m_i}$ to $\{C_j\}_{j=1}^\infty$ we have

$$\sum_{j=1}^\infty \mu(C_j) \le \sum_{i=1}^\infty \mu(A_i).$$

Taking the infimum over all disjoint collections $\{C_j\}_{j=1}^\infty$ of subsets of $\mathcal{S}$ covering E we have

$$\mu_*(E) \underset{\text{def}}{=} \inf\Big\{\sum_{j=1}^\infty \mu(A_j) \,:\, \{A_j\}_{j=1}^\infty \subseteq \mathcal{S},\ \ E \subseteq \bigcup_{j=1}^\infty A_j\Big\}$$

$$\le\ \inf\Big\{\sum_{j=1}^\infty \mu(C_j) \,:\, \{C_j\}_{j=1}^\infty \subseteq \mathcal{S},\ \{C_j\}_{j=1}^\infty \text{ disjoint},\ \ E \subseteq \bigcup_{j=1}^\infty C_j\Big\}\ \le\ \sum_{i=1}^\infty \mu(A_i).$$

Now this inequality holds for every countable collection $\{A_i\}_{i=1}^\infty$ of subsets of $\mathcal{S}$ covering E. Thus we have by definition of μ_*,

$$\mu_*(E)\ \le\ \inf\Big\{\sum_{j=1}^\infty \mu(C_j) \,:\, \{C_j\}_{j=1}^\infty \subseteq \mathcal{S},\ \{C_j\}_{j=1}^\infty \text{ disjoint},\ \ E \subseteq \bigcup_{j=1}^\infty C_j\Big\}$$

$$\le\ \inf\Big\{\sum_{i=1}^\infty \mu(A_i) \,:\, \{A_i\}_{i=1}^\infty \subseteq \mathcal{S},\ \ E \subseteq \bigcup_{i=1}^\infty A_i\Big\} \underset{\text{def}}{=} \mu_*(E).$$

It follows that (2.11) holds.

□

Remark 2.4.4. For the proof of (5), we had to make use of σ-subadditivity of μ which, by the proof of theorem 2.4.2, is a consequence of the semiring structure of $\mathcal{S}$ and of (PM2). If $\mathcal{S}$ is not a semiring, it may happen that $\mu_*(E) \ne \mu(E)$ for some $E \in \mathcal{S}$.

For example, let $X = \{1,2,3\}$, $\mathcal{S} = \{\emptyset, \{1\}, \{1,2\}\}$ and $\mu(\emptyset) = 0$, $\mu(\{1\} = 2$ and $\mu(\{1,2\}) = 1$. Then obviously, $\mu_*(\{1\}) = 1 \neq \mu(\{1\})$. The reason for this is easy to see: While $\{1\} \subset \{1,2\}$ we don't have the monotonicity property; $\mu(\{1\}) \not\leq \mu(\{1,2\})$.

In general, we define:

Definition 2.4.5. Let X be a set. A function $\mu_* : \mathcal{P}(X) \to [0,\infty]$ satisfying (OM1), (OM2) and (OM3) above is called an *outer measure on* X.

Remark 2.4.5. Let μ_* be an outer measure on X. Then for any finite collection $\{E_i\}_{i=1}^n \subseteq \mathcal{P}(X)$ we have

$$\mu_*\Big(\bigcup_{i=1}^n E_i\Big) \leq \sum_{i=1}^n \mu_*(E_i). \qquad \text{("finite subadditivity")}$$

This is proved by setting $E_i = \emptyset$ for $i > n$ and using σ-subadditivity, just as in the proof of theorem 2.3.1.

Example 2.4.6. Consider the semialgebra $\mathcal{S} = \{E \subseteq X : E \text{ is finite}\}$ and premeasure $\mu(E) := \text{card}(E)$ of example 2.4.3. The corresponding outer measure on X is

$$\mu_*(E) = \begin{cases} \text{card}(E) & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite.} \end{cases}$$

Note that μ_* is even a measure on $\mathcal{P}(X)$, namely the counting measure.

In the following, in order to simplify notation, we will simply use the symbol μ to denote an outer measure.

Example 2.4.7. (An outer measure which is not a measure.) Let $X = [0,1]$ and for each $E \subseteq X$, define

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E \text{ is uncountable.} \end{cases}$$

Let us verify that μ is an outer measure.

Obviously, $\mu(\emptyset) = 0$ and $\mu(E) \leq \mu(F)$ if $E \subseteq F$. Now let $\{E_n\}$ be a countable family of subsets of X. We distinguish two possibilities:

1. All E_n are countable. Then $\bigcup_{n=1}^\infty E_n$ is also countable, so that

$$\mu\Big(\bigcup_{n=1}^\infty E_n\Big) = 0 = \sum_{n=1}^\infty 0 = \sum_{n=1}^\infty \mu(E_n).$$

2. At least one E_n, say E_{n_o} is uncountable. Then $\bigcup_{n=1}^\infty E_n$ is also uncountable, so that

$$\mu\Big(\bigcup_{n=1}^\infty E_n\Big) = 1 = \mu(E_{n_o}) \leq \sum_{n=1}^\infty \mu(E_n).$$

Thus, μ is an outer measure.

Note that μ is not a measure in $\mathcal{P}(X)$. To see this, let $E = [0, \frac{1}{2}]$ and $F = (\frac{1}{2}, 1]$. Then E and F are disjoint, while

$$\mu(E \cup F) = \mu([0,1]) = 1 \neq 2 = \mu(E) + \mu(F).$$

Thus, μ is not even finitely additive.

This example shows that in general, an outer measure is not a measure. The reason is that the σ-algebra $\mathcal{P}(X)$ is way to large. However, if we restrict our attention to a σ-subalgebra of $\mathcal{P}(X)$, then we can obtain a measure:

2.4.4 Measurable Sets

Definition 2.4.6. Let μ be an outer measure on a set X. A set $E \in \mathcal{P}(X)$ is said to be *μ-measurable*, if for all sets $A \subseteq X$ we have

$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c). \tag{2.12}$$

Set

$$\mathcal{M}_\mu := \{E \subseteq X : E \text{ is } \mu\text{-measurable}\}.$$

To check whether a set E is measurable, it is enough to verify a weaker condition than (2.12):

Lemma 2.4.4. *Let μ be an outer measure on X, and $E \subseteq X$. Then*

$$E \in \mathcal{M}_\mu \quad \Leftrightarrow \quad \mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c)$$

for all $A \subseteq X$ with $\mu(A) < \infty$.

Proof. $\Rightarrow$ This implication is obvious from (2.12).
$\Leftarrow$. Suppose, the inequality $\mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c)$ holds for all sets A with $\mu(A) < \infty$. Then trivially, it also holds for all $A \subseteq X$ with $\mu(A) = \infty$.

On the other hand, since μ is an outer measure, the reverse inequality,

$$\mu(A) = \mu\big((A \cap E) \cup (A \cap E^c)\big) \underset{\text{(OM3)}}{\leq} \mu(A \cap E) + \mu(A \cap E^c)$$

holds for every $A \subseteq X$ by subadditivity. Thus, (2.12) holds, that is, $E \in \mathcal{M}_\mu$. □

Definition 2.4.7. Let μ be an outer measure on X. A set $E \subseteq X$ with $\mu(E) = 0$ is called a *null set.*

Lemma 2.4.5. *Let μ be an outer measure on X. Then every null set is measurable.*

Proof. Suppose, $\mu(E) = 0$. Then for all $A \subseteq X$,

$$\begin{aligned} \mu(A) &= \mu\big((A \cap E) \cup (A \cap E^c)\big) \underset{\text{(OM3)}}{\leq} \mu(A \cap E) + \mu(A \cap E^c) \\ &\underset{\text{(OM2)}}{\leq} \mu(E) + \mu(A) = \mu(A). \end{aligned}$$

That is,

$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c),$$

which shows that $E \in \mathcal{M}_\mu$. □

Example 2.4.8. Recall example 2.4.7 of an outer measure: $X = [0,1]$ and

$$\mu(E) := \begin{cases} 0 & \text{if } E \subseteq X \text{ is countable} \\ 1 & \text{if } E \subseteq X \text{ is uncountable.} \end{cases}$$

Let us find all the μ-measurable sets.

Suppose that $E \subseteq X$ is μ-measurable. Then in particular, choosing $A = X$,

$$1 = \mu(X) = \mu(X \cap E) + \mu(X \cap E^c) = \mu(E) + \mu(E^c).$$

Thus, exactly *one* of E or E^c must be uncountable.

Conversely, suppose that $E \subseteq X$ has the property that exactly one of E or E^c is uncountable. Let $A \subseteq X$ be given.

Case 1: A is countable. Then $A \cap E$ and $A \cap E^c$ are both countable, so that

$$\mu(A \cap E) + \mu(A \cap E^c) = 0 + 0 = 0 = \mu(A).$$

Case 2: A is uncountable. Then *at least* one of $A \cap E$ and $A \cap E^c$ must be uncountable, since $A = (A \cap E) \cup (A \cap E^c)$. On the other hand, by assumption on E, *at most* one of $A \cap E$ and $A \cap E^c$ can be uncountable. Thus, exactly one of the two sets is uncountable.

$$\mu(A \cap E) + \mu(A \cap E^c) = \left\{ \begin{array}{ll} 1 + 0 & (\text{if } A \cap E \text{ is uncountable}) \\ 0 + 1 & (\text{if } A \cap E^c \text{ is uncountable}) \end{array} \right\} = 1 = \mu(A).$$

We thus have shown that

$$\mathcal{M}_\mu = \{E \subseteq [0,1] : \text{exactly one of } E \text{ or } E^c \text{ is uncountable } \}. \tag{2.13}$$

Note that the countable sets are the null sets.

The next theorem says that the collection $\mathcal{M}_\mu$ in (2.13) is a σ-algebra.

Theorem 2.4.6. *Let μ be an outer measure on X. Then $\mathcal{M}_\mu$ is a σ-algebra, and μ is a measure on $\mathcal{M}_\mu$.*

Proof. We already know that $\emptyset \in \mathcal{M}_\mu$, and that $\mu(\emptyset) = 0$. For the remainder of the theorem, we proceed as follows:

1. Show that $\mathcal{M}_\mu$ is an algebra.

2. Show that μ is finitely additive on $\mathcal{M}_\mu$.

3. Show that μ is σ-additive on $\mathcal{M}_\mu$.

4. Show that $\mathcal{M}_\mu$ is a σ-algebra.

1. We need to show that (A1) and (A2) hold. If $E \in \mathcal{M}_\mu$, then it follows directly from (2.12) that $E^c \in \mathcal{M}_\mu$, Thus, (A1) holds.

To show (A2), suppose that $E, F \in \mathcal{M}_\mu$. We need to show that $E \cup F \in \mathcal{M}_\mu$. By lemma 2.4.4, it is enough to show that for all $A \subseteq X$,

$$\mu(A) \geq \mu\big(A \cap [E \cup F]\big) + \mu\big(A \cap [E \cup F]^c\big).$$

Note that

$$A \cap [E \cup F] = [A \cap E] \cup [A \cap E^c \cap F]$$

and hence,

$$\begin{aligned}
&\mu\big(A \cap [E \cup F]\big) + \mu\big(A \cap [E \cup F]^c\big) \\
&\quad = \mu\big([A \cap E] \cup [A \cap E^c \cap F]\big) + \mu\big(A \cap [E \cup F]^c\big) \\
&\quad \underset{\text{subadditive}}{\leq} \mu\big(A \cap E\big) + \mu\big(A \cap E^c \cap F\big) + \mu\big(A \cap E^c \cap F^c\big) \\
&\quad \underset{F \in \mathcal{M}_\mu}{=} \mu(A \cap E) + \mu(A \cap E^c) \underset{E \in \mathcal{M}_\mu}{=} \mu(A).
\end{aligned}$$

Thus, $E \cup F$ is measurable, so that (A2) holds.

2. We want to show that μ is a finitely additive measure on $\mathcal{M}_\mu$. Since $\mathcal{M}_\mu$ is an algebra, $\emptyset \in \mathcal{M}_\mu$. Furthermore, as μ is an outer measure, $\mu(\emptyset) = 0$. Thus, (Meas1) holds. It is left to show finite additivity.

Claim: If $E_1, \ldots, E_n$ are pairwise disjoint sets in $\mathcal{M}_\mu$, then for each $A \subseteq X$,

$$\mu\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} \mu(A \cap E_i). \tag{2.14}$$

Fix A and proceed by induction on n.

(i) n=1: Obvious.

(ii) Assume we have shown that for some $k < n$,

$$\mu\left(A \cap \left(\bigcup_{i=1}^{k} E_i\right)\right) = \sum_{i=1}^{k} \mu(A \cap E_i). \tag{2.15}$$

Then

$$\begin{aligned}
\mu\Big(A \cap \Big(\bigcup_{i=1}^{k+1} E_i\Big)\Big) &\underset{E_{k+1} \in \mathcal{M}_\mu}{=} \mu\Big(A \cap \Big(\bigcup_{i=1}^{k+1} E_i\Big) \cap E_{k+1}\Big) + \mu\Big(A \cap \Big(\bigcup_{i=1}^{k+1} E_i\Big) \cap E_{k+1}^c\Big) \\
&= \mu\Big(A \cap E_{k+1}\Big) + \mu\Big(A \cap \Big(\bigcup_{i=1}^{k} E_i\Big)\Big) \quad \text{(since the } \{E_i\} \text{ are disjoint)} \\
&\underset{(2.15)}{=} \mu\Big(A \cap E_{k+1}\Big) + \sum_{i=1}^{k} \mu(A \cap E_i) = \sum_{i=1}^{k+1} \mu(A \cap E_i).
\end{aligned}$$

This proves the claim. Now choosing $A = X$ in (2.14) we obtain that

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i). \tag{2.16}$$

Thus, μ is finitely additive.

3. Next let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}_\mu$ be pairwise disjoint. Set $E = \cup_{i=1}^{\infty} E_i$. We want to show that $E \in \mathcal{M}_\mu$ and $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$.

To show that E is measurable, let $A \subseteq X$ with $\mu(A) < \infty$ be arbitrary. Then

$$\mu(A \cap E) = \mu\Big(A \cap \Big(\bigcup_{i=1}^{\infty} E_i\Big)\Big) = \mu\Big(\bigcup_{i=1}^{\infty}(A \cap E_i)\Big) \underset{\text{(OM3)}}{\leq} \sum_{i=1}^{\infty} \mu(A \cap E_i). \tag{2.17}$$

Now as $\mathcal{M}_\mu$ is an algebra, $\cup_{i=1}^{n} E_i \in \mathcal{M}_\mu$ for each n, and hence

$$\begin{aligned} \infty > \mu(A) &= \mu\Big(A \cap \Big(\bigcup_{i=1}^{n} E_i\Big)\Big) + \mu\Big(A \cap \Big(\bigcup_{i=1}^{n} E_i\Big)^c\Big) \\ &\underset{\text{(OM2)}}{\geq} \mu\Big(\bigcup_{i=1}^{n}(A \cap E_i)\Big) + \mu(A \cap E^c) \qquad \text{as } E^c \subseteq \Big(\bigcup_{i=1}^{n} E_i\Big)^c \\ &\underset{(2.14)}{=} \sum_{i=1}^{n} \mu(A \cap E_i) + \mu(A \cap E^c). \end{aligned}$$

Now $\mu(A \cap E^c) \leq \mu(A) < \infty$, and hence

$$\sum_{i=1}^{n} \mu(A \cap E_i) \leq \mu(A) - \mu(A \cap E^c)$$

for each n. Let $n \to \infty$. We obtain that

$$\sum_{i=1}^{\infty} \mu(A \cap E_i) \leq \mu(A) - \mu(A \cap E^c).$$

Hence by (2.17),

$$\mu(A \cap E) \leq \mu(A) - \mu(A \cap E^c),$$

that is,

$$\mu(A) \geq \mu(A \cap E) + \mu(A \cap E^c).$$

It follows from lemma 2.4.4 that $E \in \mathcal{M}_\mu$.

Now by monotonicity, subadditivity and (2.16) we have for each n,

$$\sum_{i=1}^{n} \mu(E_i) = \mu\Big(\bigcup_{i=1}^{n} E_i\Big) \leq \mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Let $n \to \infty$. We obtain that

$$\sum_{i=1}^{\infty} \mu(E_i) = \mu(E).$$

Thus, μ is σ-additive on $\mathcal{M}_\mu$.

4. Next let $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}_\mu$ be arbitrary. We need to show that $\cup_{i=1}^{\infty} A_i \in \mathcal{M}_\mu$.

Since $\mathcal{M}_\mu$ is an algebra, by theorem 2.1.1, there exists a sequence $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}_\mu$ of pairwise disjoint sets such that $E_i \subseteq A_i$ for all i, and $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} A_i$. Now in 3. above we have shown that, since the sets E_i are pairwise disjoint, $E := \cup_{i=1}^{\infty} E_i \in \mathcal{M}_\mu$. Hence, $\cup_{i=1}^{\infty} A_i \in \mathcal{M}_\mu$. This proves the theorem. □

Definition 2.4.8. A measure space $(X, \mathcal{M}, \mu)$ is said to be *complete*, if every subset of a null set is measurable. (That is, $E \in \mathcal{M}$ and $\mu(E) = 0$ imply that $A \in \mathcal{M}$ for all $A \subseteq E$.)

Remark 2.4.6. Let $(X, \mathcal{M}_\mu, \mu_*)$ be the measure space constructed from an outer measure μ_* as above. Then $(X, \mathcal{M}_\mu, \mu_*)$ is complete.

In fact, let $A \subseteq E$ where $E \in \mathcal{M}_\mu$ and $\mu_*(E) = 0$. Then by monotonicity of the outer measure,

$$0 \le \mu_*(A) \le \mu_*(E) = 0$$

which shows that $\mu_*(A) = 0$. By lemma 2.4.5, A is measurable.

Remark 2.4.7. In definition 2.4.6 above, we have introduced the notion of measurable set through an outer measure μ_*. We have found that the collection $\mathcal{M}_\mu$ of measurable sets forms a σ-algebra, and that μ_* is a measure on $\mathcal{M}_\mu$. That is, $(X, \mathcal{M}_\mu, \mu_*)$ is a measure space. In general, if $(X, \mathcal{M}, \mu)$ is a measure space, then we call the elements of $\mathcal{M}$ *μ-measurable* sets.

2.4.5 Extensions of Measures

Let us summarize what he have done so far.

1. We started with a semiring $\mathcal{S}$ of subsets of X, and a premeasure μ on $\mathcal{S}$.

2. Then we defined an outer measure μ_* on $\mathcal{P}(X)$ by setting

$$\mu_*(E) = \inf\Big\{ \sum_{i=1}^{\infty} \mu(A_i),\ A_i \in \mathcal{S},\ E \subseteq \bigcup_{i=1}^{\infty} A_i \Big\}.$$

   If $E \in \mathcal{S}$, then $\mu_*(E) = \mu(E)$.

3. If we restrict μ_* to the σ-algebra $\mathcal{M}_\mu$ of all μ_* measurable sets, then μ_* is a measure on $\mathcal{M}_\mu$.

The next question is: Is $\mathcal{S} \subseteq \mathcal{M}_\mu$, that is, is every set E in $\mathcal{S}$ measurable ? The answer is yes:

Theorem 2.4.7. *Let $\mathcal{S}$ be a semiring of subsets of X, and $\mu : \mathcal{S} \to [0, \infty]$ a premeasure on $\mathcal{S}$. Let μ_* denote the outer measure on X determined by μ as in (2.9), and let $\mathcal{M}_\mu$ denote the σ-algebra of μ_*-measurable subsets of X. Then $\mathcal{S} \subseteq \mathcal{M}_\mu$.*

Proof. Let $E \in \mathcal{S}$. We need to show that

$$\mu_*(A) \geq \mu_*(A \cap E) + \mu_*(A \cap E^c) \tag{2.18}$$

for all $A \subseteq X$ with $\mu_*(A) < \infty$. If $E = \emptyset$, this is obvious, so we may assume that $E \neq \emptyset$.

Let $\epsilon > 0$ be given. By definition of μ_*, there exist $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ such that

$$A \subseteq \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mu(A_n) \leq \mu_*(A) + \epsilon.$$

First consider $A \cap E$.

$$A \cap E \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right) \cap E = \bigcup_{n=1}^{\infty} \underbrace{(A_n \cap E)}_{\in \mathcal{S}}$$

so that

$$\mu_*(A \cap E) \underset{\text{(OM2)}}{\leq} \mu_*\left(\bigcup_{n=1}^{\infty} (A_n \cap E)\right) \underset{\text{(OM3)}}{\leq} \sum_{n=1}^{\infty} \mu_*(A_n \cap E) \underset{\text{thm 2.4.3, (5)}}{=} \sum_{n=1}^{\infty} \mu(A_n \cap E). \tag{2.19}$$

Next consider $A \cap E^c$. Since $\mathcal{S}$ is a semiring, by (SR2) for each n,

$$A_n \cap E^c = A_n \backslash E = \bigcup_{j=1}^{m_n} C_{nj}, \qquad C_{nj} \in \mathcal{S}, \ \{C_{nj}\}_{j=1}^{\infty} \text{ disjoint.}$$

Hence

$$A \cap E^c \subseteq \left(\bigcup_{n=1}^{\infty} A_n\right) \cap E^c = \bigcup_{n=1}^{\infty} (A_n \cap E^c) = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n} C_{nj}$$

so that

$$\mu_*(A \cap E^c) \underset{\substack{\text{(OM2)} \\ \text{(OM3)}}}{\leq} \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} \mu_*(C_{nj}) \underset{\text{thm 2.4.3, (5)}}{=} \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} \mu(C_{nj}). \tag{2.20}$$

Combining (2.19) and (2.20) we have

$$\begin{aligned} \mu_*(A \cap E) + \mu_*(A \cap E^c) \quad &\leq \sum_{n=1}^{\infty} \left(\mu(A_n \cap E) + \sum_{j=1}^{m_n} \mu(C_{nj}) \right) \\ &\underset{\text{(PM2)}}{=} \sum_{n=1}^{\infty} \mu(A_n) < \mu_*(A) + \epsilon. \end{aligned}$$

where we have used the fact that $(A_n \cap E) \cup \left(\bigcup_{j=1}^{m_n} C_{nj}\right) = (A_n \cap E) \cup (A_n \cap E^c) = A_n$ and this union is disjoint, because $C_{nj} \subseteq A_n \cap E^c$. As ϵ was arbitrary, it follows that (2.18) holds. Then by lemma 2.4.4, $E \in \mathcal{M}_\mu$ which proves the theorem. □

Remark 2.4.8. It follows from the theorem that $\sigma(\mathcal{S}) \subseteq \mathcal{M}_\mu$. The two σ-algebras, however, are usually different. (See example 2.4.6 where

$$\sigma(\mathcal{S}) = \{E \subseteq X : \ E \text{ is countable or } E^c \text{ is countable }\}$$

while $\mathcal{M}_\mu = \mathcal{P}(X)$.)

Remark 2.4.9. Because $\mathcal{S} \subseteq \mathcal{M}_\mu$, μ_* is a measure on $\mathcal{M}_\mu$ and $\mu_*(E) = \mu(E)$ for all $E \in \mathcal{S}$, we call μ_* an *extension of μ (from $\mathcal{S}$) to $\mathcal{M}_\mu$*. For ease of notation, we often simply denote this extension by μ.

Now that we have extended the measure μ from the semiring $\mathcal{S}$ to the σ-algebra $\mathcal{M}_\mu$, the question is, is this extension unique ? First some notation.

Definition 2.4.9. Let $\mathcal{S}$ be a semiring of subsets of X, and $\mu : \mathcal{S} \to [0,\infty]$ a premeasure on $\mathcal{S}$. Then μ is called *σ-finite*, if there exists a countable family $\{E_n\}_{n=1}^\infty \subseteq \mathcal{S}$ such that $\mu(E_n) < \infty$ for all n and $X = \bigcup\limits_{n=1}^{\infty} E_n$. (Compare with the definition of a σ-finite measure !)

Remark 2.4.10. In the above definition, we may assume that the sets $\{E_n\}_{n=1}^\infty$ are pairwise disjoint. In fact, by theorem 5, part 2., we can write

$$X = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} C_{nk}, \qquad C_{nk} \in \mathcal{S},\ C_{nk} \subseteq E_n,\ \{C_{nk}\}_{n=1}^{\infty}{}_{k=1}^{m_n} \text{ disjoint.}$$

Then by monotonicity of μ, $\mu(C_{nk}) \leq \mu(E_n) < \infty$ for each n and each k. That is, X is the countable disjoint union of sets in $\mathcal{S}$ of finite measure.

Example 2.4.9. Let $X = \mathbb{R}^n$, $\mathcal{S}$ the semiring of "half-open" intervals,

$$\mathcal{S} = \mathcal{S}_o = \{E = (a_1, b_1] \times \cdots \times (a_n, b_n] \ : \ -\infty < a_i \leq b_i < \infty, \ i = 1 \ldots n\}$$

and λ the Lebesgue premeasure on $\mathcal{S}$,

$$\lambda(E) = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n).$$

Then $\mathbb{R}^n$ is σ-finite. In fact, we can write $\mathbb{R}^n = \bigcup\limits_{k=1}^{\infty} E_k$ where

$$E_k = (-k, k] \times (-k, k] \times \cdots \times (-k, k].$$

and $\lambda(E_k) = 2^n k^n < \infty$ for all k.

We can also write $\mathbb{R}^n$ as a disjoint union: For each $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, set

$$E_{\boldsymbol{k}} := (k_1, k_1 + 1] \times (k_2, k_2 + 1] \times \cdots \times (k_n, k_n + 1].$$

Then the sets $E_{\boldsymbol{k}}$ are unit cubes whose vertices are the points of $\mathbb{Z}^n$. The sets $\{E_{\boldsymbol{k}}\}_{\boldsymbol{k} \in Z^n}$ are pairwise disjoint, $\lambda(E_{\boldsymbol{k}}) = 1$ for all $\boldsymbol{k}$, and $\mathbb{R}^n = \bigcup\limits_{\boldsymbol{k} \in \mathbb{Z}^n} E_{\boldsymbol{k}}$.

Theorem 2.4.8. *(Uniqueness of extension). Let $\mathcal{S}$ be a semiring of subsets of X, and $\mu : \mathcal{S} \to [0, \infty]$ a premeasure on $\mathcal{S}$. Let μ_* denote the outer measure on X determined by μ as in (2.9), and let $\mathcal{M}_\mu$ denote the σ-algebra of μ_*-measurable subsets of X. Suppose, $\mathcal{M}$ is another σ-algebra with*

$$\mathcal{S} \subseteq \mathcal{M} \subseteq \mathcal{M}_\mu,$$

and ν is a measure on $\mathcal{M}$ satisfying $\nu(E) = \mu(E)$ for all $E \in \mathcal{S}$. (That is, μ_* and ν are both extensions of μ from $\mathcal{S}$ to $\mathcal{M}$). If X is σ-finite, then $\nu(E) = \mu_*(E)$ for all $E \in \mathcal{M}$.

Proof. 1) First we show that $\nu(E) \leq \mu_*(E)$ for all $E \in \mathcal{M}$.
Let $E \in \mathcal{M}$ be arbitrary. If $\{A_n\}_{n=1}^\infty \subseteq \mathcal{S}$ is such that $E \subseteq \bigcup_{n=1}^\infty A_n$, then by monotonicity and σ-subadditivity of ν,

$$\nu(E) \;\leq\; \nu\Big(\bigcup_{n=1}^\infty A_n\Big) \leq \sum_{n=1}^\infty \nu(A_n) \underset{\substack{\nu=\mu \\ \text{on } \mathcal{S}}}{=} \sum_{n=1}^\infty \mu(A_n).$$

Taking the infimum over all covers of E by countably many sets in $\mathcal{S}$ we get

$$\nu(E) \;\leq\; \inf\Big\{\sum_{n=1}^\infty \mu(A_n) \;:\; \{A_n\}_{n=1}^\infty \subseteq \mathcal{S},\;\; E \subseteq \bigcup_{n=1}^\infty A_n\Big\} \underset{\text{def}}{=} \mu_*(E).$$

2) Now we need to show the reverse inequality.

a) First let $E \in \mathcal{M}$ be such that $\mu_*(E) < \infty$. Let $\epsilon > 0$ be arbitrary. Then by (2.11), there exist $\{A_n\}_{n=1}^\infty \subseteq \mathcal{S}$, pairwise disjoint, so that

$$E \subseteq \bigcup_{n=1}^\infty A_n \qquad \text{and} \qquad \sum_{n=1}^\infty \mu(A_n) < \mu_*(E) + \epsilon.$$

Set $B = \bigcup_{n=1}^\infty A_n \in \mathcal{M}$. Then $E \subseteq B$ and by σ-additivity,

$$\mu_*(B) = \sum_{n=1}^\infty \mu_*(A_n) = \sum_{n=1}^\infty \mu(A_n) < \mu_*(E) + \epsilon.$$

By part 1), and since $\mu_*(E)$ is finite,

$$\nu(B \backslash E) \leq \mu_*(B \backslash E) = \mu_*(B) - \mu_*(E) < \epsilon.$$

Since μ_* and ν are measures and coincide on $\mathcal{S}$, we have

$$\mu_*(E) \leq \mu_*(B) = \sum_{n=1}^\infty \mu(A_n) \underset{\substack{\nu=\mu \\ \text{on } \mathcal{S}}}{=} \sum_{n=1}^\infty \nu(A_n) = \nu(B) = \nu(E) + \nu(B \backslash E) < \nu(E) + \epsilon.$$

As ϵ was arbitrary, it follows that $\mu_*(E) \leq \nu(E)$, and thus by part 1),

$$\nu(E) = \mu_*(E).$$

b) Now let $E \in \mathcal{M}$ be arbitrary. As X is σ-finite, there exist $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$, pairwise disjoint, so that

$$\mu(E_n) < \infty \ \ \forall n \qquad \text{and} \qquad X = \bigcup_{n=1}^{\infty} E_n.$$

Now as $\mu_*(E \cap E_n) \leq \mu_*(E_n) < \infty$, we have by part a) that

$$\nu(E \cap E_n) = \mu_*(E \cap E_n)$$

for all n. Also,

$$E = E \cap X = E \cap \left(\bigcup_{n=1}^{\infty} E_n \right) = \bigcup_{n=1}^{\infty} (E \cap E_n),$$

a disjoint union, so that by σ-additivity,

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap E_n) = \sum_{n=1}^{\infty} \mu_*(E \cap E_n) = \mu_*(E).$$

This proves the theorem.

□

Remark 2.4.11. The assumption that X be σ-finite can not be dropped. For example, let $X = \mathbb{R}$,

$$\mathcal{S} = \{(a, b] \, : -\infty < a \leq b < \infty\}$$

and define $\mu : \mathcal{S} \to [0, \infty]$ by

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ \infty & \text{else} \end{cases}$$

for all $E \in \mathcal{S}$. It is straightforward to check that μ is a premeasure on $\mathcal{S}$, that $\mathcal{M}_\mu = \mathcal{P}(X)$ and that

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ \infty & \text{else} \end{cases}$$

for all $E \subset X$. However, the *counting measure*,

$$\nu(E) = \begin{cases} \text{card}(E) & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite} \end{cases}$$

is another extension from $\mathcal{S}$ to $\mathcal{P}(X)$ different from μ_*.

Exercise 2.4.2. Let $\mathcal{S}$ be a semiring of subsets of X, $\mu : \mathcal{S} \to [0, \infty]$ a premeasure on $\mathcal{S}$ and μ_* the outer measure induced by $(\mathcal{S}, \mu)$. Show: For each $E \subseteq X$ there exists $A \in \sigma(\mathcal{S})$ such that

$$E \subseteq A \qquad \text{and} \qquad \mu_*(A) = \mu_*(E).$$

2.5 Properties of the Lebesgue Measure

2.5.1 The Lebesgue Measure

Using the procedure outlined in the previous section, we now construct a measure on $\mathbb{R}^n$.

Start with the semiring $\mathcal{S}_o$ of "half-open" intervals,

$$\mathcal{S}_o = \Big\{ E = \prod_{i=1}^{n}(a_i, b_i] = (a, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n] : -\infty < a_i \leq b_i < \infty, \quad i = 1 \ldots n \Big\} \tag{2.21}$$

and the Lebesgue premeasure λ on $\mathcal{S}_o$,

$$\lambda(E) = \prod_{i=1}^{n}(b_i - a_i) = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n)$$

for $E = \prod_{i=1}^{n}(a_i, b_i] \in \mathcal{S}_o$. The outer measure λ_* on $\mathbb{R}^n$ determined by λ is called the *Lebesgue outer measure*,

$$\begin{aligned}\lambda_*(E) &= \inf\Big\{ \sum_{k=1}^{\infty}\lambda(A_k) \ : \ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}_o, \ E \subseteq \bigcup_{k=1}^{\infty} A_k \Big\} \\ &\underset{\text{thm 2.4.3, (5)}}{=} \inf\Big\{ \sum_{k=1}^{\infty}\lambda(A_k) \ : \ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}_o, \ \{A_k\}_{k=1}^{\infty} \text{ disjoint}, \ E \subseteq \bigcup_{k=1}^{\infty} A_k \Big\}\end{aligned}$$

for $E \subseteq \mathbb{R}^n$ arbitrary. The σ-algebra of λ_*-measurable subsets of $\mathbb{R}^n$ will be denoted by $\mathcal{M}_\lambda$, and its elements are called *Lebesgue measurable sets.* By theorems 2.4.6 and 2.4.7, λ_* is the (unique by theorem 2.4.8) extension of λ from $\mathcal{S}_o$ to $\mathcal{M}_\lambda$, called the *Lebesgue measure* on $\mathcal{M}_\lambda$, and is simply denoted by λ. The measure space $(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$ is complete, by remark 2.4.7.

The following question now arises naturally: Can we characterize the elements of $\mathcal{M}_\lambda$?

2.5.2 Borel Sets

Definition 2.5.1. Given a metric space X (or more generally, a topological space X), let

$$\tau := \{ \ U \subseteq X \ : \ U \text{ is open } \}$$

be the collection of open sets. Then $\sigma(\tau)$, the σ-algebra generated by the open sets, is called the *Borel σ-algebra on X*, and is denoted by $\mathcal{B}(X)$. The elements of $\mathcal{B}(X)$ are called *Borel sets.*

Remark 2.5.1. A set of the form $M = \bigcap_{i=1}^{\infty} G_i$, with G_i open for all i, is called a G_δ set. Note that a G_δ set need not be open. Similarly, a set of the form $M = \bigcup_{i=1}^{\infty} F_i$, with F_i closed for all i, is called an F_σ set. An F_σ set need not be closed.

Since σ-algebras are closed under formation of complements, countable unions and countable intersections, it follows that $\mathcal{B}(X)$ contains all closed sets, all G_δ sets, and all F_σ sets.

Lemma 2.5.1. *(Lindelöf's Theorem for n-intervals)*
Every open set $U \subseteq \mathbb{R}^n$ is the countable union of bounded, open n-intervals whose vertices have rational coordinates.

Proof. Let U be an open subset of $\mathbb{R}^n$. Since the metrics d_2 and d_∞ are equivalent, U is also open in the metric d_∞. Thus, for each $x = (x_1, \dots, x_n) \in U$, there exists $\epsilon > 0$ such that

$$B_\epsilon(x) = \prod_{i=1}^{n}(x_i - \epsilon, x_i + \epsilon) \subseteq U,$$

$B_\epsilon(x)$ denoting an open ball in the d_∞ metric. For each $i = 1 \dots n$, pick $a_i, b_i \in \mathbb{Q}$ such that

$$x_i - \epsilon < a_i < x_i < b_i < x_i + \epsilon.$$

Then $(a_i, b_i) \subset (x_i - \epsilon, x_i + \epsilon)$ for all i, and thus

$$J_x := \prod_{i=1}^{n}(a_i, b_i) \subset \prod_{i=1}^{n}(x_i - \epsilon, x_i + \epsilon) \subseteq U.$$

That is, J_x is a bounded, open n-interval whose vertices have rational coordinates, and $x \in J_x \subseteq U$. Then

$$U = \bigcup_{x \in U}\{x\} \subseteq \bigcup_{x \in U} J_x \subseteq U$$

which shows that

$$U = \bigcup_{x \in U} J_x. \tag{2.22}$$

But there exist only countably many bounded, open non-empty n-intervals whose vertices have rational coordinates, in fact, the map

$$\prod_{i=1}^{n}(a_i, b_i) \to (a_1, \dots, a_n, b_1, \dots b_n)$$

is an isomorphism of the collection of such n-intervals into $\mathbb{R}^{2n}$. Hence, the union in (2.22) is really a union of countably many distinct intervals only. This proves the lemma. □

Theorem 2.5.2. *Let $\mathcal{S}_o$ denote the semiring of "half-open" intervals in $\mathbb{R}^n$, as in (2.21). Then $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{S}_o)$.*

Proof. 1) First we show that $\sigma(\mathcal{S}_o) \subseteq \mathcal{B}(\mathbb{R}^n)$. Since $\sigma(\mathcal{S}_o)$ is the smallest σ-algebra containing $\mathcal{S}_o$, it is enough to show that $S_o \subseteq \mathcal{B}(\mathbb{R}^n)$. To this end, let $E \in \mathcal{S}_o$ be arbitrary, say

$$E = \prod_{i=1}^{n} (a_i, b_i].$$

Then E is a countable intersection of open intervals,

$$E = \bigcap_{k=1}^{\infty} E_k \qquad \text{where} \qquad E_k = \prod_{i=1}^{n} (a_i, b_i + \frac{1}{k}).$$

Since each E_k is open (E_k is an open ball in the d_∞ metric, and as the metrics d_2 and d_∞ are equivalent, also an open set in the d_2 metric), $E_k \in \mathcal{B}(\mathbb{R}^n)$. Now every σ-algebra is closed under countable intersections; hence $E \in \mathcal{B}(\mathbb{R}^n)$. We have shown that $\mathcal{S}_o \subseteq \mathcal{B}(\mathbb{R}^n)$, and hence $\sigma(\mathcal{S}_o) \subseteq \mathcal{B}(\mathbb{R}^n)$.

2) Now we show that $\mathcal{B}(\mathbb{R}^n) \subseteq \sigma(\mathcal{S}_o)$. First, if $I = \prod_{i=1}^{n} (a_i, b_i)$ is a bounded, open n-interval, then

$$I = \bigcup_{k=1}^{\infty} \Big[\underbrace{\prod_{i=1}^{n} (a_i, b_i - \frac{1}{k}]}_{\in \mathcal{S}_o} \Big].$$

Hence, $I \in \sigma(\mathcal{S}_o)$.

In general, if $U \subseteq \mathbb{R}^n$ is open, then by lemma 2.5.1,

$$U = \bigcup_{k=1}^{\infty} I_k$$

with each I_k a bounded, open n-interval. By the above, $I_k \in \sigma(\mathcal{S}_o)$ for each k, and hence $U \in \sigma(\mathcal{S}_o)$. We have shown that $\sigma(\mathcal{S}_o)$ is a σ-algebra containing all open sets. But $\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra containing all open sets, and hence $\mathcal{B}(\mathbb{R}^n) \subseteq \sigma(\mathcal{S}_o)$. □

Exercise 2.5.1. 1. Show that $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{F})$, where $\mathcal{F}$ denotes the collection of all closed subsets of $\mathbb{R}^n$.

2. Show that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{H})$, where $\mathcal{H}$ denotes the collection of all infinite half-open intervals in $\mathbb{R}$,

$$\mathcal{H} = \{ \ (-\infty, b] : b \in \mathbb{R} \ \}.$$

Remark 2.5.2. Since $\mathcal{M}_\lambda$ is a σ-algebra containing $\mathcal{S}_o$, it follows that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}_\lambda$. One can show (see [1], [3]) that this is a proper inclusion, that is, there exist Lebesgue measurable sets which are not Borel sets.

Example 2.5.1. Let us present some Borel subsets of $\mathbb{R}^n$.

1. let $E = \{\vec{x}\}$ be a singleton, where $\vec{x} = (x_1, \ldots, x_n)$. Then E is closed, hence $E \in \mathcal{B}(\mathbb{R}^n)$. Furthermore,

$$\lambda(\{\vec{x}\}) = \lambda\Big(\bigcap_{k=1}^{\infty} \prod_{i=1}^{n} (x_i - \frac{1}{k}, x_i]\Big) \underset{\text{thm } 2.3.2}{=} \lim_{k\to\infty} \lambda\Big(\prod_{i=1}^{n}(x_i - \frac{1}{k}, x_i]\Big) = \lim_{k\to\infty} \frac{1}{k^n} = 0.$$

2. Next let $E = \{\vec{x}^{(k)}\}_{k=1}^{M}$ be a countable subset of $\mathbb{R}^n$, with $M \in \mathbb{N} \cup \{\infty\}$. Then by finite additivity, respectively σ-additivity,

$$\lambda(E) = \lambda\Big(\bigcup_{k=1}^{M}\{\vec{x}^{(k)}\}\Big) = \sum_{k=1}^{M} \lambda(\{\vec{x}^{(k)}\}) = \sum_{k=1}^{M} 0 = 0.$$

3. Let E be a *bounded*, open n-interval, say $E = \prod_{i=1}^{n}(a_i, b_i)$. Then

$$\lambda(E) = \prod_{i=1}^{n}(b_i - a_i).$$

4. Let E be a *bounded*, closed n-interval, say $E = \prod_{i=1}^{n}[a_i, b_i]$. Then

$$\lambda(E) = \prod_{i=1}^{n}(b_i - a_i).$$

5. Let E be a *bounded* n-interval, say $E = \prod_{i=1}^{n} I_i$, with each I_i an interval in $\mathbb{R}$ having endpoints $a_i \le b_i$. Then $E \in \mathcal{B}(\mathbb{R}^n)$, and

$$\lambda(E) = \prod_{i=1}^{n}(b_i - a_i).$$

6. Let E be an *unbounded*, proper n-interval, say $E = \prod_{i=1}^{n} I_i$, with each I_i an interval in $\mathbb{R}$ having endpoints a_i, b_i satisfying $-\infty \le a_i < b_i \le \infty$. Then $E \in \mathcal{B}(\mathbb{R}^n)$, and

$$\lambda(E) = \infty.$$

7. Let E be a coordinate plane in $\mathbb{R}^n$, that is,

$$E = \{\, (x_1, \ldots, x_n) \in \mathbb{R}^n : x_{j_o} = 0 \text{ for some fixed } j_o,\ 1 \le j_o \le n \,\}.$$

Then $E \in \mathcal{B}(\mathbb{R}^n)$, and $\lambda(E) = 0$.

Exercise 2.5.2. Prove 3. – 7. above. (In the case of unbounded intervals, assume for the sake of simplicity that $E = (-\infty, b_1) \times (a_2, b_2] \times (a_3, b_3] \times \cdots \times (a_n, b_n]$ with all a_i, b_i finite numbers. In the case of a coordinate plane, assume that $j_o = 1$.)

Exercise 2.5.3. Show that for all $E \subseteq \mathbb{R}^n$,

1) $\lambda_*(E) = \inf\Big\{ \sum_{k=1}^{\infty} \lambda(A_k) \ : \ A_k$ is a bounded open n-interval, $E \subseteq \bigcup_{k=1}^{\infty} A_k \Big\}$,

2) $\lambda_*(E) = \inf\Big\{ \sum_{k=1}^{\infty} \lambda(A_k) \ : \ A_k$ is a bounded n-interval, $E \subseteq \bigcup_{k=1}^{\infty} A_k \Big\}$.

Example 2.5.2. Recall the *Cantor set*

$$C \ = \ [0,1] \setminus \bigcup_{n=1}^{\infty} G_n$$

where

$$G_1 = (\tfrac{1}{3}, \tfrac{2}{3}), \quad G_2 = (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}), \quad G_3 = (\tfrac{1}{27}, \tfrac{2}{27}) \cup (\tfrac{7}{27}, \tfrac{8}{27}) \cup (\tfrac{21}{27}, \tfrac{22}{27}) \cup (\tfrac{25}{27}, \tfrac{26}{27}),$$

and in general, G_n is the disjoint union of 2^{n-1} open intervals, each of length $\frac{1}{3^n}$, and $G_{n+1} \subseteq G_n^c$. Thus, $\lambda(G_n) = \frac{2^{n-1}}{3^n}$, and the sets $\{G_n\}_{n=1}^{\infty}$ are also disjoint, so that

$$\lambda\Big(\bigcup_{n=1}^{\infty} G_n\Big) \ = \ \sum_{n=1}^{\infty} \lambda(G_n) \ = \ \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} \ = \ \frac{1}{3} \sum_{n=0}^{\infty} \Big(\frac{2}{3}\Big)^n \ = \ \frac{1}{3} \frac{1}{1-\frac{2}{3}} \ = \ 1.$$

Hence,

$$\lambda(C) \ = \ \lambda\big([0,1]\big) - \lambda\Big(\bigcup_{n=1}^{\infty} G_n\Big) \ = \ 1 - 1 \ = 0.$$

The Cantor set is an *uncountable set* of measure zero !

Theorem 2.5.3. *Let λ_* denote the Lebesgue outer measure on $\mathbb{R}^n$, and λ the Lebesgue measure. Then*

1. *For every compact subset K of $\mathbb{R}^n$, $\lambda(K) < \infty$.*

2. *For every subset E of $\mathbb{R}^n$,*

$$\lambda_*(E) = \inf\{\, \lambda(U) : E \subseteq U, \ U \ \textit{open} \,\} \qquad (\textit{"outer regularity"}).$$

3. *For every Lebesgue measurable subset E of $\mathbb{R}^n$,*

$$\lambda(E) = \sup\{\, \lambda(K) : K \subseteq E, \ K \ \textit{compact} \,\} \qquad (\textit{"inner regularity"}).$$

Proof. 1. Let $K \subseteq \mathbb{R}^n$ be compact. Then K is bounded, so there exist $M > 0$ such that $K \subseteq I := \prod_{j=1}^{n} (-M, M]$. By monotonicity of the measure,

$$\lambda(K) \leq \lambda(I) = 2^n M^n < \infty.$$

2. Let $E \subseteq \mathbb{R}^n$ be arbitrary. If U is an open set with $E \subseteq U$, then by monotonicity,

$$\lambda_*(E) \leq \lambda_*(U) = \lambda(U)$$

and hence

$$\lambda_*(E) \leq \inf\{\, \lambda(U) : E \subseteq U,\ U \text{ open } \}.$$

We must show the reverse inequality. If $\lambda_*(E) = \infty$, then obviously,

$$\lambda_*(E) = \infty = \inf\{\, \lambda(U) : E \subseteq U,\ U \text{ open } \}.$$

Thus we may assume that $\lambda_*(E) < \infty$. Let $\epsilon > 0$ be given. Then by exercise 2.5.3, there exists a collection $\{E_k\}_{k=1}^{\infty}$ of bounded, open n-intervals such that

$$\text{i)} \quad E \subseteq \bigcup_{k=1}^{\infty} E_k$$

$$\text{ii)} \quad \sum_{k=1}^{\infty} \lambda(E_k) < \lambda_*(E) + \epsilon.$$

Set $U = \bigcup_{k=1}^{\infty} E_k$. Then U is open, $E \subseteq U$, and by σ-subadditivity,

$$\lambda(U) \;=\; \lambda\Big(\bigcup_{k=1}^{\infty} E_k\Big) \;\leq\; \sum_{k=1}^{\infty} \lambda(E_k) < \lambda_*(E) + \epsilon.$$

Thus,

$$\inf\{\, \lambda(U) : E \subseteq U,\ U \text{ open } \} \;<\; \lambda_*(E) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, it follows that

$$\inf\{\, \lambda(U) : E \subseteq U,\ U \text{ open } \} \;\leq\; \lambda_*(E).$$

This proves 2.

3. Let $E \in \mathcal{M}_\lambda$. If $K \subseteq E$ is compact, then by monotonicity, $\lambda(K) \leq \lambda(E)$, and hence,

$$\sup\{\, \lambda(K) : K \subseteq E, K \text{ is compact } \} \leq \lambda(E).$$

We must show the reverse inequality.

Case 1: E is bounded. Then $\overline{E}$ is also bounded, and hence is compact. In particular, $\lambda(E) \leq \lambda(\overline{E})) < \infty$.

Let $\epsilon > 0$ be given. Applying part 2. to the set $\overline{E} \backslash E$, there exists $U \subset \mathbb{R}^n$ open such that

$$\overline{E} \backslash E \subseteq U \qquad \text{and} \qquad \lambda(U) < \lambda(\overline{E} \backslash E) + \epsilon.$$

Set $K := \overline{E} \backslash U$. Then K is also compact, and

$$K = \overline{E} \backslash U \subseteq \overline{E} \backslash (\overline{E} \backslash E) = E.$$

Now

$$\overline{E} \subseteq (\overline{E} \backslash U) \cup U = K \cup U,$$

a disjoint union, and hence

$$\lambda(\overline{E}) = \lambda(K) + \lambda(U) < \lambda(K) + \lambda(\overline{E} \backslash E) + \epsilon = \lambda(K) + \lambda(\overline{E}) - \lambda(E) + \epsilon$$

as $\lambda(E)$ is finite. Thus,

$$\lambda(E) < \lambda(K) + \epsilon$$

so that

$$\lambda(E) < \sup\{ \lambda(K) : K \subseteq E, K \text{ is compact } \} + \epsilon.$$

Since ϵ was arbitrary, it follows that

$$\lambda(E) \leq \sup\{ \lambda(K) : K \subseteq E, K \text{ is compact } \}.$$

Case 2: E is unbounded. Set

$$E_k := E \cap B_k(0), \qquad k = 1, 2, \ldots.$$

Then $\{E_k\}\uparrow$ and $E = \bigcup_{k=1}^{\infty} E_k$. By case 1, for each k there exists $C_k \subseteq E_k$ compact, such that

$$\lambda(E_k) - \frac{1}{k} \leq \lambda(C_k).$$

Set $K_k := \bigcup_{j=1}^{k} C_j$. Then K_k is compact, $\{K_k\}\uparrow$, $K_k \subseteq \bigcup_{j=1}^{k} E_j = E_k$, and thus

$$\lambda(E_k) - \frac{1}{k} \leq \lambda(C_k) \leq \lambda(K_k) \leq \lambda(E_k)$$

for all k. Let $k \to \infty$. We obtain

$$\lambda(E) = \lim_{k\to\infty} \left(\lambda(E_k) - \frac{1}{k}\right) \leq \lim_{k\to\infty} \lambda(K_k) \leq \lim_{k\to\infty} \lambda(E_k) = \lambda(E).$$

Since $\{\lambda(K_k)\}\uparrow$, and $K_k \subseteq E_k$ for all k, we have

$$\lambda(E) = \lim_{k\to\infty} \lambda(K_k) = \sup_k \lambda(K_k) \leq \sup\{ \lambda(K) : K \subseteq E, K \text{ is compact } \}.$$

This proves the theorem. □

Definition 2.5.2. Let X be a metric space (or more generally, a topological space), and μ a measure on $\mathcal{B}(X)$. (i.e., μ is a Borel measure.) Then μ is called a *regular Borel measure* if

1. $\mu(K) < \infty$ for all $K \subseteq X$ compact,
2. $\mu(E) = \inf\{\ \mu(U),\ E \subseteq U,\ U \text{ open}\ \}$ for all $E \in \mathcal{B}(X)$,
3. $\mu(E) = \sup\{\ \mu(K) : K \subseteq E,\ K \text{ is compact}\ \}$ for all $E \subseteq X$ open.

Thus by the theorem, the Lebesgue measure is a regular Borel measure.

Exercise 2.5.4. Let $E \subseteq \mathbb{R}^n$. Show: $E \in \mathcal{M}_\lambda \quad \Leftrightarrow \quad$ for every $\epsilon > 0$ there exist an open set $U \subseteq X$ and a closed set $F \subseteq X$ such that

1. $F \subseteq E \subseteq U$,
2. $\lambda(U \backslash F) < \epsilon$.

2.6 Measurable Functions

2.6.1 Characterization of Measurable Functions

Definition 2.6.1. Let $(X, \mathcal{M})$ be a measurable space. An extended real valued function $f : X \to \mathbb{R}^*$ is called $\mathcal{M}$-measurable, if

1. $f^{-1}(U) \in \mathcal{M}$ for all $U \subseteq \mathbb{R}$, U open,
2. $f^{-1}(-\infty) \in \mathcal{M}$ and $f^{-1}(\infty) \in \mathcal{M}$.

Remark 2.6.1. In case that f is real valued, $f : X \to \mathbb{R}$, we have $f^{-1}(-\infty) = f^{-1}(\infty) = \emptyset \in \mathcal{M}$, so that the above definition reduces to

$$f \text{ is } \mathcal{M}\text{-measurable} \quad \Leftrightarrow \quad f^{-1}(U) \in \mathcal{M} \text{ for all } U \subseteq \mathbb{R},\ U \text{ open}.$$

Remark 2.6.2. Suppose that X is also a metric space (or more general, a topological space), and $\mathcal{B}(X) \subseteq \mathcal{M}$. (For example, $X = \mathbb{R}^n$ and $\mathcal{M} = \mathcal{B}(\mathbb{R}^n)$, or $\mathcal{M} = \mathcal{M}_\lambda$.) Let $f : X \to \mathbb{R}$ be continuous. Then for all $U \subseteq \mathbb{R}$ is open, $f^{-1}(U)$ is open in X, and thus $f^{-1}(U) \in \mathcal{B}(X) \subseteq \mathcal{M}$. Thus, every continuous function is $\mathcal{M}$-measurable.

Theorem 2.6.1. *Let $(X, \mathcal{M})$ be a measurable space, and $f : X \to \mathbb{R}^*$. Then T.F.A.E.:*

1. *f is $\mathcal{M}$-measurable.*
2. *$\{x \in X : f(x) > a\} \in \mathcal{M}$ for all $a \in \mathbb{R}$ ($a \in \mathbb{Q}$).*
3. *$\{x \in X : f(x) \leq a\} \in \mathcal{M}$ for all $a \in \mathbb{R}$ ($a \in \mathbb{Q}$).*
4. *$\{x \in X : f(x) < a\} \in \mathcal{M}$ for all $a \in \mathbb{R}$ ($a \in \mathbb{Q}$).*

5. $\{x \in X : f(x) \geq a\} \in \mathcal{M}$ *for all* $a \in \mathbb{R}$ $(a \in \mathbb{Q})$.

Proof. We first prove that 2.–5. all all equivalent. Then we will prove that 1. and 2.–5. are equivalent.

2. $\Rightarrow$ 3.: Suppose, that 2. holds. Then for each fixed $a \in \mathbb{R}$ $(a \in \mathbb{Q})$, we have

$$\{x \in X : f(x) \leq a\} = X \setminus \underbrace{\{x \in X : f(x) > a\}}_{\in \mathcal{M} \text{ by 2.}} \in \mathcal{M}.$$

3. $\Rightarrow$ 4.: Suppose, that 3. holds. Then for each fixed $a \in \mathbb{R}$ $(a \in \mathbb{Q})$, we have

$$\{x \in X : f(x) < a\} = \bigcup_{n=1}^{\infty} \underbrace{\{x \in X : f(x) \leq a + \frac{1}{n}\}}_{\in \mathcal{M} \text{ by 3.}} \in \mathcal{M}.$$

(Note that if $a \in \mathbb{Q}$ then $a + \frac{1}{n} \in \mathbb{Q}$ also.

4. $\Rightarrow$ 5.: Suppose, that 4. holds. Then for each fixed $a \in \mathbb{R}$ $(a \in \mathbb{Q})$, we have

$$\{x \in X : f(x) \geq a\} = X \setminus \underbrace{\{x \in X : f(x) < a\}}_{\in \mathcal{M} \text{ by 4.}} \in \mathcal{M}.$$

5. $\Rightarrow$ 2.: Suppose, that 5. holds. Then for each fixed $a \in \mathbb{R}$ $(a \in \mathbb{Q})$, we have

$$\{x \in X : f(x) > a\} = \bigcup_{n=1}^{\infty} \underbrace{\{x \in X : f(x) \geq a + \frac{1}{n}\}}_{\in \mathcal{M} \text{ by 5.}} \in \mathcal{M}.$$

(Note that if $a \in \mathbb{Q}$ then $a + \frac{1}{n} \in \mathbb{Q}$ also.

1. $\Rightarrow$ 2.: Suppose that f is $\mathcal{M}$-measurable. Then for each $a \in \mathbb{R}$ we have by definition 2.6.1,

$$\{x \in X : f(x) > a\} = f^{-1}\big(\underbrace{(a, \infty)}_{\text{open}}\big) \cup f^{-1}(\infty) \in \mathcal{M}.$$

2.–5. $\Rightarrow$ 1.: Suppose that 2–5. hold for all $a \in \mathbb{Q}$.

a)
$$f^{-1}(\infty) = \bigcap_{n=1}^{\infty} \{\underbrace{x \in X : f(x) \geq n}_{\in \mathcal{M} \text{ by 5.}}\} \in \mathcal{M}.$$

b)
$$f^{-1}(-\infty) = \bigcap_{n=1}^{\infty} \{\underbrace{x \in X : f(x) \leq -n}_{\in \mathcal{M} \text{ by 3.}}\} \in \mathcal{M}.$$

c) Let $U \subseteq \mathbb{R}$ be open. Then by Lindelöfs theorem (lemma 2.5.1) we can write

$$U = \bigcup_{k=1}^{\infty} I_k \tag{2.23}$$

where each I_k is an open interval with rational endpoints, $I_k = (a_k, b_k)$, and $a_k < b_k$, $a_k, b_k \in \mathbb{Q}$. Now for each k,

$$f^{-1}(I_k) = \underbrace{\{x \in X : f(x) > a_k\}}_{\in \mathcal{M} \text{ by } 2.} \cap \underbrace{\{x \in X : f(x) < b_k\}}_{\in \mathcal{M} \text{ by } 4.} \in \mathcal{M}.$$

Thus by (2.23),

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{k=1}^{\infty} I_k\Big) = \bigcup_{k=1}^{\infty} f^{-1}(I_k) \in \mathcal{M}$$

since $\mathcal{M}$ is a σ-algebra.

Hence, f is $\mathcal{M}$-measurable. □

Exercise 2.6.1. Let $(X, \mathcal{M})$ be a measurable space, and $f : X \to \mathbb{R}$. Show: f is $\mathcal{M}$-measurable $\Leftrightarrow$ $f^{-1}(B) \in \mathcal{M}$ for every Borel set $B \subseteq \mathbb{R}$.

If $f, g : X \to \mathbb{R}$ are functions and $g(x) = 0$ for some x, then $\frac{f(x)}{g(x)}$ is not defined. For this reason, let us make the convention that $\frac{f}{g}$ is the function defined by

$$\left(\frac{f}{g}\right)(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{if } g(x) = 0. \end{cases}$$

This convention is not problematic when discussing measurable functions:

Theorem 2.6.2. *Let $(X, \mathcal{M})$ be a measurable space, and $f, g : X \to \mathbb{R}$ be $\mathcal{M}$-measurable. Then*

1. *the constant functions $h(x) = c$,*
2. *$f + g$,*
3. *$\alpha f \quad (\alpha \in \mathbb{R})$,*
4. *fg,*
5. *$\frac{f}{g}$*

are all $\mathcal{M}$-measurable.

Proof. We will make use of theorem 2.6.1.

1. Let $h(x) = c$. Then for each $a \in \mathbb{R}$,

$$\{x \in X : f(x) < a\} = \begin{cases} X & \text{if } c < a \\ \emptyset & \text{if } c \geq a \end{cases} \in \mathcal{M}.$$

2. Note that for all $a \in \mathbb{Q}$,

$$\{x \in X : (f+g)(x) < a\} = \bigcup_{r \in \mathbb{Q}} \{x \in X : f(x) < r \text{ and } g(x) < a - r\}$$

$$= \bigcup_{r \in \mathbb{Q}} \Big[\underbrace{\{x \in X : f(x) < r\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in X : g(x) < a - r\}}_{\in \mathcal{M}} \Big] \in \mathcal{M}$$

3. We separate into 3 cases: $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. If $\alpha = 0$ then $\alpha f = 0$; hence αf is $\mathcal{M}$-measurable by part 1. If $\alpha > 0$ then

$$\{x \in X : (\alpha f)(x) < a\} = \{x \in X : f(x) < \frac{a}{\alpha}\} \in \mathcal{M}$$

while if $\alpha < 0$ then

$$\{x \in X : (\alpha f)(x) < a\} = \{x \in X : f(x) > \frac{a}{\alpha}\} \in \mathcal{M}.$$

4. Note that

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}.$$

By 2. and 3., it is enough to show that whenever h is $\mathcal{M}$-measurable, then h^2 is $\mathcal{M}$-measurable. So let h be $\mathcal{M}$-measurable. Then for all $a \in \mathbb{R}$,

$$\{x \in X : h^2(x) < a\}$$
$$= \begin{cases} \underbrace{\{x \in X : h(x) > -\sqrt{a}\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in X : h(x) < \sqrt{a}\}}_{\in \mathcal{M}} & (\text{if } a > 0) \\ \emptyset & (\text{if } a \leq 0) \end{cases} \in \mathcal{M}.$$

5. Exercise. (By 4., it is enough to show that $\frac{1}{g}$ is $\mathcal{M}$-measurable.) □

If f, g are extended real valued, then $(f+g)(x)$ does not exist if $f(x) + g(x)$ is of the form $\infty - \infty$ or $-\infty + \infty$. We thus only consider non-negative functions.

Theorem 2.6.3. *Let $(X, \mathcal{M})$ be a measurable space, and $f, g : X \to [0, \infty]$ be $\mathcal{M}$-measurable. Then*

1. $f + g$,
2. $\alpha f \quad (\alpha \in \mathbb{R})$,

are all $\mathcal{M}$-measurable.

Proof. The proof is exactly the same as that of theorem 2.6.2, parts 2. and 3. □

Definition 2.6.2. Given a function $f : X \to \mathbb{R}^*$, let us set

$$f^+ := \max(f, 0) \qquad \text{and} \qquad f^- := -\min(f, 0).$$

Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Theorem 2.6.4. *Let $(X, \mathcal{M})$ be a measurable space, and $f, g : X \to \mathbb{R}^*$ be $\mathcal{M}$-measurable. Then*

1. $\max(f, g)$,
2. $\min(f, g)$,
3. f^+, f^- *and* $|f|$

are all $\mathcal{M}$-measurable.

Proof. 1. For all $a \in \mathbb{R}$,

$$\{x \in X : \max(f, g)(x) < a\} = \underbrace{\{x \in X : f(x) < a\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in X : g(x) < a\}}_{\in \mathcal{M}} \in \mathcal{M}.$$

2. For all $a \in \mathbb{R}$,

$$\{x \in X : \min(f, g)(x) < a\} = \underbrace{\{x \in X : f(x) < a\}}_{\in \mathcal{M}} \cup \underbrace{\{x \in X : g(x) < a\}}_{\in \mathcal{M}} \in \mathcal{M}.$$

3. Since $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$ and $|f| = f^+ + f^-$, this follows from parts 1. and 2., and theorem 2.6.3. □

Lemma 2.6.5. *Let $(X, \mathcal{M})$ be a measurable space, and $f, g : X \to \mathbb{R}^*$ be $\mathcal{M}$-measurable. Then*

1. $\{x \in X : f(x) > g(x)\} \in \mathcal{M}$, *and*
2. $\{x \in X : f(x) = g(x)\} \in \mathcal{M}$.

Proof. 1. Note that

$$\begin{aligned} \{x \in X : f(x) > g(x)\} &= \bigcup_{r \in \mathbb{Q}} \{x \in X : f(x) > r > g(x)\} \\ &= \bigcup_{r \in \mathbb{Q}} \Big[\underbrace{\{x \in X : f(x) > r\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in X : g(x) < r\}}_{\in \mathcal{M}}\Big] \in \mathcal{M}. \end{aligned}$$

2. It follows that

$$\begin{aligned} \{x \in X : f(x) = g(x)\} &= \{x \in X : f(x) \le g(x)\} \cap \{x \in X : f(x) \ge g(x)\} \\ &= \underbrace{\{x \in X : f(x) > g(x)\}}_{\in \mathcal{M} \text{ by 1.}}{}^c \cap \underbrace{\{x \in X : f(x) < g(x)\}}_{\in \mathcal{M} \text{ by 1.}}{}^c \in \mathcal{M} \end{aligned}$$

□

Recall that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of extended real numbers, then

$$\limsup_n a_n = \inf_n (\sup_{k\geq n} a_k) = \lim_{n\to\infty} (\sup_{k\geq n} a_k);$$

the right equality holds since the sequence $\{\sup_{k\geq n} a_k\}_{n=1}^{\infty}$ is decreasing. Similarly,

$$\liminf_n a_n = \sup_n (\inf_{k\geq n} a_k) = \lim_{n\to\infty} (\inf_{k\geq n} a_k).$$

Then $\lim_{n\to\infty} a_n$ exists $\Leftrightarrow$ $\limsup_n a_n = \liminf_n a_n$, in which case

$$\lim_{n\to\infty} a_n = \limsup_n a_n = \liminf_n a_n.$$

We now apply these concepts to sequences of functions.

Definition 2.6.3. Let X by a set and $\{f_n\}_{n=1}^{\infty}$ a sequence of functions, $f_n : X \to \mathbb{R}^*$. Define the following functions from X to $\mathbb{R}^*$,

$\sup f_n$:	by	$(\sup f_n)(x) := \sup_n f_n(x),$
$\inf f_n$:	by	$(\inf f_n)(x) := \inf_n f_n(x),$
$\limsup f_n$:	by	$(\limsup f_n)(x) := \limsup_n f_n(x),$
$\liminf f_n$:	by	$(\liminf f_n)(x) := \liminf_n f_n(x).$

Note that these functions always exist in the system of extended real numbers ! Furthermore, if the sequence $\{f_n(x)\}$ converges for each $x \in X$, then we say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise*, define

$\lim f_n$:	by	$(\lim f_n)(x) := \lim_{n\to\infty} f_n(x),$

and write $f_n \to f$ where $f = \lim f_n$.

We also denote these functions by $\sup_n f_n$, $\liminf_n f_n$, $\lim_n f_n$, $\lim_{n\to\infty} f_n$. etc.

Remark 2.6.3. 1. By the above definition, we have for all x,

$$\begin{aligned}(\limsup f_n)(x) &= \limsup_n f_n(x) = \inf_n \sup_{k\geq n} f_n(x)\\ &= \inf_n\big((\sup_{k\geq n} f_k)(x)\big) = \big(\inf_n(\sup_{k\geq n} f_k)\big)(x).\end{aligned}$$

That is,

$$\limsup f_n = \inf_n(\sup_{k\geq n} f_k)$$

Similarly,

$$\liminf f_n = \sup_n(\inf_{k\geq n} f_k)$$

2. By the above definition,

$$\begin{aligned}\lim f_n \text{ exists} \quad &\Leftrightarrow \quad \lim_{n\to\infty} f_n(x) \text{ exists for all } x \in X \\ &\Leftrightarrow \quad \limsup_n f_n(x) = \liminf_n f_n(x) \text{ for all } x \in X \\ &\Leftrightarrow \quad \limsup f_n = \liminf f_n,\end{aligned}$$

and in this case, since

$$\lim_{n\to\infty} f_n(x) = \limsup_n f_n(x) \qquad \text{and} \qquad \lim_{n\to\infty} f_n(x) = \liminf_n f_n(x)$$

for all $x \in X$, then

$$\lim f_n = \limsup f_n = \liminf f_n. \tag{2.24}$$

3. Note that if $f_n : X \to \mathbb{R}$ and $f : X \to \mathbb{R}$, then

$$f_n \to f \Leftrightarrow \text{ Given } x \in X \text{ and } \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t } |f_n(x) - f(x)| < \epsilon \quad \forall n \geq N.$$

If ϵ can be chosen independent of x, then we say that $\{f_n\}$ *converges uniformly to f on X*, and write $f_n \underset{X}{\rightrightarrows} f$. That is

$$f_n \underset{X}{\rightrightarrows} f \Leftrightarrow \text{ Given } \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t } |f_n(x) - f(x)| < \epsilon \quad \forall n \geq N,\ x \in X.$$

It is easy to show that if $f_n \underset{X}{\rightrightarrows} f$ and $g_n \underset{X}{\rightrightarrows} g$, then $\alpha f_n + \beta g_n \underset{X}{\rightrightarrows} \alpha f + \beta g$, for any choice of constants α and β.

Recall from analysis that the pointwise limit of continuous functions need not be continuous. However, the pointwise limit of measurable functions is always measurable:

Theorem 2.6.6. *Let $(X, \mathcal{M})$ be a measurable space, and $f_n : X \to \mathbb{R}^*$. If each f_n is $\mathcal{M}$-measurable, then so are*

$$\sup f_n, \quad \inf f_n, \quad \limsup f_n, \quad \liminf f_n \quad \textit{and} \quad \lim f_n \ (\textit{if it exists}).$$

Proof. Let $a \in \mathbb{R}$ be arbitrary. Then

$$\{x \in X : \inf_n f_n(x) < a\} = \bigcup_{n=1}^{\infty} \underbrace{\{x \in X : f_n(x) < a\}}_{\in \mathcal{M}} \in \mathcal{M}.$$

Hence $\inf f_n$ is $\mathcal{M}$-measurable. Similarly,

$$\{x \in X : \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty} \underbrace{\{x \in X : f_n(x) > a\}}_{\in \mathcal{M}} \in \mathcal{M}.$$

Hence $\sup f_n$ is $\mathcal{M}$-measurable.

It now follows that for each n,

$$g_n := \sup_{k \geq n} f_k \qquad \text{and} \qquad h_n := \inf_{k \geq n} f_k$$

are $\mathcal{M}$-measurable, and hence

$$\limsup f_n = \inf g_n \qquad \text{and} \qquad \liminf f_n = \sup h_n$$

are $\mathcal{M}$-measurable. In particular, by (2.24) $\lim_n f_n$ is $\mathcal{M}$-measurable if it exists. □

2.6.2 Simple Functions

Definition 2.6.4. Let X be a set, and $A \subseteq X$. The *characteristic function of* A, denoted χ_A, is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Remark 2.6.4. 1. Let $A, B \subseteq X$. It is easy to verify that

$$\chi_{A \cap B} = \chi_A \chi_B$$

and

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}.$$

(Exercise !) In particular,

$$\chi_{A \cup B} = \chi_A + \chi_B \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

2. Now let $(X, \mathcal{M})$ be a measurable space. Then

$$\chi_A \text{ is } \mathcal{M}\text{-measurable} \quad \Leftrightarrow \quad A \in \mathcal{M}.$$

Proof. Given $a \in \mathbb{R}$ set

$$S_a := \{x \in X : \chi_A(x) > a\} = \begin{cases} \emptyset \in \mathcal{M} & \text{if } a \geq 1 \\ A & \text{if } 0 \leq a < 1 \\ X \in \mathcal{M} & \text{if } a < 0. \end{cases}$$

Then χ_A is $\mathcal{M}$-measurable $\Leftrightarrow$ $S_a \in \mathcal{M}$ for all a $\Leftrightarrow$ $A \in \mathcal{M}$. □

Definition 2.6.5. A function $\varphi : X \to \mathbb{R}$ (or $\varphi : X \to \mathbb{C}$) is called *simple* if its range is a finite set.

Remark 2.6.5. 1. Obviously, every characteristic function χ_A is simple.

2. Let φ be a simple function with range $\{c_1, c_2, \ldots . c_n\}$. (all c_k distinct.) Set

$$A_k := \varphi^{-1}(\{c_k\}) = \{x \in X : \varphi(x) = c_k\},$$

for $k = 1, \ldots, n$. Then

(a) The sets $A_1, A_2, \dots, A_n$ are pairwise disjoint,

(b) $X = \bigcup_{k=1}^{n} A_k$, and

(c) $\varphi = \sum_{k=1}^{n} c_k A_k$.

This is called the *canonical representation of* φ, and obviously, it is unique up to rearrangement of the sum. If $c_k = 0$ for some k, then we may drop the corresponding term in (c) to obtain an even simpler representation of φ.

3. Every linear combination of simple functions is again a simple function. For suppose, $\varphi_1, \varphi_2, \dots, \varphi_m$ are simple functions, with finite ranges $M_1, M_2, \dots, M_m$ respectively, and let $\alpha_1, \alpha_2, \dots, \alpha_m$ be scalars numbers. Set

$$\varphi = \sum_{j=1}^{m} \alpha_j \varphi_j$$

Then

$$\text{range}(\varphi) \subseteq M := \Big\{ \sum_{j=1}^{m} \alpha_j y_j : y_j \in M_j \Big\}.$$

Note that M is a finite set, as each M_j contains only finitely many numbers y_j.

4. In particular, every function of the form

$$\varphi = \sum_{k=1}^{n} c_k A_k$$

($A_k \subseteq X$, $\{A_k\}_{k=1}^{n}$ not necessarily disjoint, c_k scalar) is simple, as it is a linear combination of characteristic functions.

5. Finite products of simple functions are simple. In fact, let

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k} \qquad \text{and} \qquad \psi = \sum_{j=1}^{m} b_j \chi_{B_j}.$$

Then

$$\varphi\psi = \sum_{k=1}^{n} \sum_{j=1}^{m} a_k b_j \chi_{A_k} \chi_{B_j} = \sum_{k=1}^{n} \sum_{j=1}^{n} \underbrace{a_k b_j}_{c_{k,j}} \underbrace{\chi_{A_k \cap B_j}}_{C_{k,j}}$$

which is a simple function. The assertion now follows by induction on the number of factors.

6. Let $(X, \mathcal{M})$ be a measurable space, and $\varphi = \sum_{k=1}^{n} c_k A_k : X \to \mathbb{R}$. If each $A_k \in \mathcal{M}$, then by remark 2.6.3, each χ_{A_k} is an $\mathcal{M}$-measurable function, so that by theorem 2.6.2, φ is also $\mathcal{M}$-measurable.

Conversely, let $\varphi : X \to \mathbb{R}$ be an $\mathcal{M}$-measurable simple function, and

$$\varphi = \sum_{k=1}^{n} c_k A_k \tag{2.25}$$

its canonical representation. Then

$$A_k = \{x \in X : \varphi(x) \geq a\} \cap \{x \in X : \varphi(x) \leq a\} \in \mathcal{M},$$

for each $k = 1, \ldots, n$. That is, each set A_k is measurable.

(Note that if (2.25) is not the canonical representation of φ, then the sets A_k need not be measurable in general !)

7. In the special case where $X = \mathbb{R}^n$ and each I_k is an n-interval, a simple function

$$\varphi = \sum_{k=1}^{n} c_k I_k$$

is called a *step function.* Note that a step function is $\mathcal{B}(\mathbb{R}^n)$-measurable (and hence $\mathcal{M}_\lambda$-measurable) since each I_k is a Borel set.

2.6.3 The Structure Theorem for Measurable Functions

Theorem 2.6.7. *Let $(X, \mathcal{M})$ be a measurable space, and $f : X \to \mathbb{R}^*$ an $\mathcal{M}$-measurable function. Then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of real valued, $\mathcal{M}$-measurable simple functions such that*

1. $|\varphi_n(x)| \leq |f(x)|$ *for all* $x \in X$,
2. $\lim_{n\to\infty} \varphi_n(x) = f(x)$ *for all* $x \in X$.

If $f \geq 0$, then we may choose $\varphi_n \geq 0$ and $\{\varphi_n\}\uparrow$.
If f is bounded, then we may choose $\{\varphi_n\}$ so that $\varphi_n \underset{X}{\rightrightarrows} f$ (i.e. φ_n converges uniformly to f).

Proof. 1. Assume first that $f \geq 0$. Set

$$\varphi_1(x) = \begin{cases} 0 & \text{if } 0 \leq f(x) < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq f(x) < 1 \\ 1 & \text{if } f(x) \geq 1. \end{cases}$$

Then set

$$\varphi_2(x) = \begin{cases} 0 & \text{if } 0 \leq f(x) < \frac{1}{4} \\ \frac{1}{4} & \text{if } \frac{1}{4} \leq f(x) < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq f(x) < \frac{3}{4} \\ \vdots & \qquad \vdots \\ \frac{7}{4} & \text{if } \frac{7}{4} \leq f(x) < 2 \\ 2 & \text{if } f(x) \geq 2. \end{cases}$$

In general, given $x \in X$ we set

$$\varphi_n(x) = \begin{cases} \frac{i-1}{2^n} & \text{if } f(x) < n \text{ and } \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n} \text{ for some } 1 \le i \le n\,2^n \\ n & \text{if } f(x) \ge n. \end{cases} \tag{2.26}$$

Note that

$$\varphi_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{A_i} + n\chi_{A_0}$$

where we have set

$$A_i = A_i^{(n)} := \{x \in X : \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}\} \qquad (1 \le i \le n2^n), \text{ and}$$
$$A_0 = A_0^{(n)} := \{x \in X : f(x) \ge n\}.$$

Then by theorem 2.6.1, $A_i \in \mathcal{M}$ for all i, $0 \le i \le n2^n$. Hence, each φ_n is an $\mathcal{M}$-measurable simple function, and $\varphi_n \ge 0$. Furthermore by (2.26), $\varphi_n \le f$.

We now show that $\{\varphi_n\}\uparrow$. In fact let n be fixed, and $x \in X$.

a) Suppose, $0 \le f(x) < n$. Then there exists i, $1 \le i \le n2^n$, so that

$$\frac{i-1}{2^n} \le f(x) < \frac{i}{2^n},$$

and $\varphi_n(x) = \frac{i-1}{2^n}$. Now compute $\varphi_{n+1}(x)$. The above inequality gives

$$\frac{2i-2}{2^{n+1}} \le f(x) < \frac{2i}{2^{n+1}}.$$

If $\frac{2i-2}{2^{n+1}} \le f(x) < \frac{2i-1}{2^{n+1}}$, then by (2.26),

$$\varphi_{n+1}(x) = \frac{2i-2}{2^{n+1}} = \varphi_n(x),$$

while if $\frac{2i-1}{2^{n+1}} \le f(x) < \frac{2i}{2^{n+1}}$, then

$$\varphi_{n+1}(x) = \frac{2i-1}{2^{n+1}} \ge \frac{2i-2}{2^{n+1}} = \varphi_n(x).$$

b) Suppose, $n \le f(x) < n+1$. Then $\varphi_n(x) = n$, and

$$\varphi_{n+1}(x) = \frac{i-1}{2^{n+1}}$$

where i is the unique positive integer satisfying $\frac{i-1}{2^{n+1}} \le f(x) < \frac{i}{2^{n+1}}$ Note that $\frac{i-1}{2^{n+1}} \ge n$ since $f(x) \ge n$. Hence,

$$\varphi_{n+1}(x) = \frac{i-1}{2^{n+1}} \ge n\varphi_n(x).$$

c) Finally, suppose that $f(x) \ge n+1$. then

$$\varphi_{n+1}(x) = n+1 > n = \varphi_n(x).$$

This shows that $\varphi_{n+1}(x) \geq \varphi_x$ for all $x \in X$.

Next we show that $\varphi_n \to f$. In fact, let $x \in X$ be given.
a) If $f(x) = \infty$, then $\varphi_n = n$ for all n, and hence

$$\varphi_n(x) = n \to \infty = f(x)$$

as $n \to \infty$.
b) If $0 \leq f(x) < \infty$, then there exists $N = N(x) \in \mathbb{N}$ such that $f(x) < N$. By construction (2.26) of φ_n, we have for all $n \geq N$ that

$$0 \leq f(x) - \varphi_n(x) \leq \frac{1}{2^n} \to 0$$

as $n \to \infty$. Hence, $\varphi_n(x) \to f(x)$.

Note that if f is bounded, then there exists $N \in \mathbb{N}$ such that $f(x) < N$ for all $x \in X$. By construction (2.26) of φ_n, we have

$$0 \leq f(x) - \varphi_n(x) \leq \frac{1}{2^n}$$

for all $x \in X$ and all $n \geq N$. Hence, $\varphi_n(x) \underset{X}{\rightrightarrows} f(x)$. This proves the theorem in case $f \geq 0$.

2. Next let f be arbitrary, $\mathcal{M}$-measurable. Then f^+ and f^- are $\mathcal{M}$-measurable. Let $\{\varphi_n\}\uparrow$, $\{\psi_n\}\uparrow$ be sequences of simple, $\mathcal{M}$-measurable functions constructed in 1., with

$$0 \leq \varphi_n \leq f^+, \qquad 0 \leq \psi_n \leq f^-$$

for all n, and

$$\varphi_n \to f^+, \quad \psi_n \to f^-.$$

Then $\{\varphi_n - \psi_n\}$ is a sequence of $\mathcal{M}$-measurable simple functions satisfying

1. $|\varphi_n - \psi_n| \leq |\varphi_n| + |\psi_n| = \varphi_n + \psi_n \leq f^+ + f^- = |f|$, and
2. $\lim_{n\to\infty} (\varphi_n - \psi_n)(x) = \lim_{n\to\infty} \varphi_n(x) - \lim_{n\to\infty} \psi_n(x) = f^+(x) - f^-(x) = f(x)$, for all $x \in X$.

Finally, if f is bounded, then f^+ and f^- are also bounded, as $0 \leq f^+, f^- \leq |f|$, so that by part 1., $\varphi_n \underset{X}{\rightrightarrows} f^+$ and $\psi_n \underset{X}{\rightrightarrows} f^-$. Then $\varphi_n - \psi_n \underset{X}{\rightrightarrows} f^+ - f^- = f$. □

2.6.4 Almost Everywhere

Definition 2.6.6. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let P be a statement about the elements of X. Set

$$B := \{x \in X : P \text{ is not valid}\}.$$

We say that P *holds μ-almost everywhere* (written "a.e."), if there exists $E \in \mathcal{M}$, with $\mu(E) = 0$ and $B \subseteq E$.

Note that the set B itself need not be measurable. However, if $(X, \mathcal{M}, \mu)$ is complete, then the above definition is equivalent to

$$P \text{ holds } \mu\text{-a.e.} \quad \Leftrightarrow \quad B \in \mathcal{M} \text{ and } \mu(B) = 0.$$

Example 2.6.1. 1. Let $f, g : X \to \mathbb{R}^*$. Then "$f(x) = g(x)$ a.e." means that there exists $E \in \mathcal{M}$, $\mu(E) = 0$, such that

$$f(x) = g(x) \quad \forall\, x \in X,\ x \notin E.$$

2. Let $f : X \to \mathbb{R}^*$. Then "$f(x)$ is finite a.e." means that there exists $E \in \mathcal{M}$, $\mu(E) = 0$, such that

$$-\infty < f(x) < \infty \quad \forall\, x \in X,\ x \notin E.$$

3. Let $f_n, f : X \to \mathbb{R}^*$. Then "$f_n(x) \to f(x)$ a.e." means that there exists $E \in \mathcal{M}$, $\mu(E) = 0$, such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X,\ x \notin E.$$

4. Let λ denote the Lebesgue measure on $\mathbb{R}$.

   Set

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

   Then $f(x) = 0$ λ-a.e.

   Next let $f_n(x) = e^{-nx^2}$. Then

$$f_n(x) \to \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

   that is, $f_n(x) \to 0$ λ-a.e.

   Finally, let $g_n(x) = \cos^n(x)$. Then

$$\begin{cases} g_n(x) \to 0 & \text{if } x \neq k\pi \\ g_n(x) \to 1 & \text{if } x = 2k\pi \\ g_n(x) \text{ diverges} & \text{if } x = (2k+1)\pi. \end{cases}$$

   That is, $g_n(x) \to 0$ λ-a.e.

Theorem 2.6.8. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f, g : X \to \mathbb{R}^*$. Suppose that*

1. *f is $\mathcal{M}$-measurable*

2. *$f(x) = g(x)$ μ-a.e.*

3. $(X, \mathcal{M}, \mu)$ *is complete.*

Then g *is also* $\mathcal{M}$*-measurable.*

Proof. Let
$$B := \{x \in X : f(x) \neq g(x)\}.$$
By assumption, $B \in \mathcal{M}$ and $\mu(B) = 0$. Now let $a \in R$ be arbitrary. Then

$$\begin{aligned}\{x \in X : g(x) > a\} &= \{x \in B^c : g(x) > a\} \cup \{x \in B : g(x) > a\} \\ &= \{x \in B^c : f(x) > a\} \cup \{x \in B : g(x) > a\} \\ &= \big(\underbrace{B^c}_{\in\mathcal{M}} \cap \underbrace{\{x \in X : f(x) > a\}}_{\in\mathcal{M}}\big) \cup \underbrace{\{x \in B : g(x) > a\}}_{\in\mathcal{M}\text{by completeness}} \in \mathcal{M}.\end{aligned}$$

Hence, g is $\mathcal{M}$-measurable. □

Exercise 2.6.2. Show that if $(X, \mathcal{M}, \mu)$ is not complete, then the assertion still holds, provided that there exists a set $E \in \mathcal{M}$ of measure zero such that $f(x) = g(x)$ for all $x \notin E$, and that g is constant on E.

2.6.5 Complex Valued Measurable Functions

Recall: If $z = x + iy$ is a complex number, then its *real part* and *imaginary part* are given by
$$\Re(z) = x = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad \Im(z) = y = \frac{z - \overline{z}}{2i}.$$
Furthermore, the *absolute value*, or *modulus*, of z is

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

Then obviously,

$$\Re(z) \leq |\Re(z)| \leq |z| \qquad \text{and} \qquad \Im(x) \leq |\Im(z)| \leq |z|.$$

Now let X be a set, and $f : X \to \mathbb{C}$ a function. We define its *conjugate* $\overline{f}$, its *real part* $\Re(f)$ and its *imaginary part* $\Im(f)$ pointwise, that is, by

$$\overline{f}(x) := \overline{f(x)}, \qquad \big[\Re(f)\big](x) := \Re\big(f(x)\big), \qquad \big[\Im(f)\big](x) := \Im\big(f(x)\big)$$

It is easy to verify that

$$\Re(f) = \frac{f + \overline{f}}{2}, \qquad \Im(f) = \frac{f - \overline{f}}{2}, \qquad \text{and} \qquad F = \Re(f) + i\Im(f)$$

where as usual, the vector space operations on functions are also defined pointwise.

If $f_n : X \to \mathbb{C}$ is a sequence of functions, then by (???), $f_n(x) \to f(x) \quad \Leftrightarrow \quad \Re(f_n)(x) \to \Re(f)$ and $\Im(f_n)(x) \to \Im(f)(x)$.

Definition 2.6.7. Let $(X, \mathcal{M})$ be a measurable space, and $f : X \to \mathbb{C}$. We call f *$\mathcal{M}$-measurable* if $\Re(f) : X \to \mathbb{R}$ and $\Im(f) : X \to \mathbb{R}$ are both $\mathcal{M}$-measurable.

Theorem 2.6.9. *Let $(X, \mathcal{M})$ be a measurable space, and $f, f_n, g : X \to \mathbb{C}$ be $\mathcal{M}$-measurable. Then $f+g$, fg, $\overline{f}$, $|f|$, $\frac{1}{f}$, $\lim_{n\to\infty} f_n$ (if it exists) are all $\mathcal{M}$-measurable.*

Proof. Since f, g, f_n are all $\mathcal{M}$-measurable, so are their real and imaginary parts, by definition.

1. By theorem 2.6.2,

$$\Re(f+g) = \Re(f) + \Re(g) \qquad \text{and} \qquad \Im(f+g) = \Im(f) + \Im(g)$$

   are both $\mathcal{M}$-measurable. Hence, $f+g$ is $\mathcal{M}$-measurable.

2. Note that

$$\Re(fg) = \Re(f)\Re(g) - \Im(f)\Im(g) \qquad \text{and} \qquad \Im(fg) = \Re(f)\Im(g) + \Re(g)\Im(f).$$

   By theorem 2.6.2, both these functions are $\mathcal{M}$-measurable. Hence, fg is $\mathcal{M}$-measurable.

3. Note that by theorem 2.6.2,

$$|f|^2 = \Re(f)^2 + \Im(f)^2$$

   is $\mathcal{M}$-measurable. Now if $a \in \mathbb{R}$ is arbitrary, we have

$$\{x \in X : |f(x)| > a\} = \begin{cases} \{x \in X : |f(x)|^2 > a^2\} \in \mathcal{M} & \text{if } a \geq 0 \\ X \in \mathcal{M} & \text{if } a < 0. \end{cases}$$

   Hence, $|f|$ is $\mathcal{M}$-measurable.

4. Since

$$\overline{f} = \Re(f) - i\Im(f) = \Re(f) + i(-\Im(f))$$

   it follows from theorem 2.6.2 and the definition that $\overline{f}$ is $\mathcal{M}$-measurable.

5. Recall that as in theorem 2.6.2, we define $\frac{f}{g}$ by

$$\left(\frac{f}{g}\right)(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if} g(x) \neq 0 \\ 0 & \text{if } g(x) = 0. \end{cases}$$

   With this definition applying to each fraction below, we have

$$\frac{f}{g} = \frac{f\overline{g}}{g\overline{g}} = f\overline{g}\,\frac{1}{|g|^2}.$$

   It now follows from parts 2.,3., and 4. above, and theorem 2.6.2 that $\frac{f}{g}$ is $\mathcal{M}$-measurable.

6. Now suppose that $\lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. Then

$$\lim_{n\to\infty} f_n = \underbrace{\lim_{n\to\infty} \Re(f_n)}_{\mathcal{M}\text{-meas. by thm 2.6.2}} + i \underbrace{\lim_{n\to\infty} \Im(f_n)}_{\mathcal{M}\text{-meas. by thm 2.6.2}}.$$

Hence, $\lim_{n\to\infty} f_n$ is $\mathcal{M}$-measurable.

□

Theorems 2.6.7 and 2.6.8 can easily be extended to complex valued $\mathcal{M}$-measurable functions (Exercise).

2.7 Definition of the Lebesgue Integral

Throughout, $(X, \mathcal{M}, \mu)$ will denote a fixed measure space.

2.7.1 The Integral of a Non-negative Simple Function

Definition 2.7.1. Let us set

$$\mathcal{S}^+ = \mathcal{S}^+(X, \mathcal{M}) := \{f : X \to [0, \infty) \mid f \text{ is simple and } \mathcal{M}\text{-measurable}\}.$$

Let $\varphi \in \mathcal{S}^+$ have canonical representation

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k},$$

where $a_1, \ldots, a_n \geq 0$ and $A_1, \ldots, A_n \in \mathcal{M}$ are disjoint. We define its integral by

$$\int \varphi \, d\mu := \sum_{k=1}^{n} a_k \mu(A_k). \tag{2.27}$$

Note that $\int \varphi \, d\mu \in [0, \infty]$.

Remark 2.7.1. The integral of $\varphi \in \mathcal{S}^+$ is independent of its representation, in the following sense: If

$$\varphi = \sum_{j=1}^{m} b_j \chi_{B_j}$$

is any representation of φ, where $b_1, \ldots, b_m \geq 0$ and $B_1, \ldots, B_m \in \mathcal{M}$ are disjoint, then

$$\int \varphi \, d\mu := \sum_{j=1}^{m} b_j \mu(B_j). \tag{2.28}$$

Proof. Since the union of the sets B_j need not be the whole of X, we set $B_0 := X \setminus \bigcup_{j=1}^{m} B_j \in \mathcal{M}$ and $b_0 = 0$, so that

$$\varphi = \sum_{j=0}^{m} b_j \chi_{B_j},$$

with $b_0, \dots, b_m \geq 0$, $B_0, \dots, B_m \in \mathcal{M}$ disjoint, and $X = \bigcup_{j=0}^{m} B_j$. Then

$$\begin{aligned}
\int \varphi \, d\mu &= \sum_{k=1}^{n} a_k \mu(A_k) = \sum_{k=1}^{n} a_k \mu(A_k \cap X) = \sum_{k=1}^{n} a_k \mu\Big(A_k \cap \bigcup_{j=0}^{m} B_j\Big) \\
&= \sum_{k=1}^{n} a_k \mu\Big(\bigcup_{j=0}^{m} (A_k \cap B_j)\Big) \qquad \text{(a disjoint union)} \\
&= \sum_{k=1}^{n} \sum_{j=0}^{m} a_k \mu(A_k \cap B_j)
\end{aligned}$$

Now note that if $A_k \cap B_j = \emptyset$, then

$$a_k \mu(A_k \cap B_j) = 0 = b_j \mu(A_k \cap B_j)$$

while if $A_k \cap B_j \neq \emptyset$, then for all $x \in A_k \cap B_j$ we have $a_k = \varphi(x) = b_j$, and hence also

$$a_k \mu(A_k \cap B_j) = b_j \mu(A_k \cap B_j).$$

Thus the above becomes

$$\begin{aligned}
\int \varphi \, d\mu &= \sum_{k=1}^{n} \sum_{j=0}^{m} b_j \mu(A_k \cap B_j) = \sum_{j=1}^{m} \sum_{k=0}^{n} b_j \mu(A_k \cap B_j) \\
&= \sum_{j=1}^{m} b_j \mu\Big(\bigcup_{k=1}^{n} (A_k \cap B_j)\Big) \qquad \text{(a disjoint union)} \\
&= \sum_{j=1}^{m} b_j \mu\Big(\Big(\bigcup_{k=1}^{n} A_k\Big) \cap B_j\Big) \\
&= \sum_{j=1}^{m} b_j \mu(X \cap B_j) = \sum_{j=1}^{m} b_j \mu(B_j).
\end{aligned}$$

□

Theorem 2.7.1. *Let $\varphi, \psi \in \mathcal{S}^+$ and $\alpha \geq 0$. Then*

1. $\displaystyle\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu$ *("additive")*
2. $\displaystyle\int \alpha\varphi \, d\mu = \alpha \int \varphi \, d\mu$ *("positive homogeneous")*
3. *If* $\varphi \leq \psi$ *then* $\displaystyle\int \varphi \, d\mu \leq \int \psi \, d\mu$ *("monotone")*

Proof. Let

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k} \qquad \text{and} \qquad \psi = \sum_{j=1}^{n} b_j \chi_{B_j} \tag{2.29}$$

be the canonical representations of φ and ψ, respectively. Then $a_k, b_j \geq 0$, $A_k, B_j \in \mathcal{M}$, $\{A_k\}$ are disjoint, $\{B_j\}$ are disjoint, and $X = \bigcup_{k=1}^{n} A_k = \bigcup_{j=1}^{m} B_j$.

1. Obviously

$$\varphi + \psi = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \chi_{A_k \cap B_j} + \sum_{j=1}^{m}\sum_{k=1}^{n} b_j \chi_{A_k \cap B_j} = \sum_{k=1}^{n}\sum_{j=1}^{m} (a_k + b_j)\chi_{A_k \cap B_j}.$$

Thus by remark 2.7.1 and definition of the integral,

$$\begin{aligned}
\int (\varphi + \psi)\, d\mu \underset{\text{rem. 2.7.1}}{=} & \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (a_k + b_j)\mu(A_k \cap B_j) \\
&= \Big[\sum_{k=1}^{n}\sum_{j=1}^{m} a_k \mu(A_k \cap B_j)\Big] + \Big[\sum_{k=1}^{n}\sum_{j=1}^{m} b_j \mu(A_k \cap B_j)\Big] \\
&= \Big[\sum_{k=1}^{n} a_k \sum_{j=1}^{\infty} \mu(A_k \cap B_j)\Big] + \Big[\sum_{j=1}^{m} b_j \sum_{k=1}^{n} \mu(A_k \cap B_j)\Big] \\
&= \Big[\sum_{k=1}^{n} a_k \mu\Big(A_k \cap \bigcup_{j=1}^{m} B_j\Big)\Big] + \Big[\sum_{j=1}^{m} b_j \mu\Big(\Big(\bigcup_{k=1}^{n} A_k\Big) \cap B_j\Big)\Big] \\
&= \Big[\sum_{k=1}^{n} a_k \mu(A_k)\Big] + \Big[\sum_{j=1}^{m} b_j \mu(B_j)\Big] = \int \varphi\, d\mu + \int \psi\, d\mu.
\end{aligned}$$

2. Since

$$\alpha\varphi = \alpha \sum_{k=1}^{n} a_k \chi_{A_k} = \sum_{k=1}^{n} \alpha a_k \chi_{A_k},$$

we have by definition of the integral,

$$\int \alpha\varphi\, d\mu = \sum_{k=1}^{n} (\alpha a_k)\mu(A_k) = \alpha \sum_{k=1}^{n} a_k \mu(A_k) = \alpha \int \varphi\, d\mu.$$

3. Suppose, $\varphi \leq \psi$. Then as in part 1.,

$$\varphi = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \chi_{A_k \cap B_j} \qquad \text{and} \qquad \psi = \sum_{k=1}^{n}\sum_{j=1}^{m} b_j \chi_{A_k \cap B_j},$$

and the sets $\{A_k \cap B_j\}_{k=1}^{n}{}_{j=1}^{m}$ are pairwise disjoint. Now if $x \in A_k \cap B_j$, then by assumption,

$$a_k = \varphi(x) \leq \psi(x) = b_j,$$

and hence

$$a_k\mu(A_k \cap B_j) \le b_j\mu(A_k \cap B_j),$$

and summing over all k and j,

$$\int \varphi\, d\mu = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k\mu(A_k \cap B_j) \le \sum_{k=1}^{n}\sum_{j=1}^{m} b_j\mu(A_k \cap B_j) = \int \psi\, d\mu.$$

□

2.7.2 The Integral of a Non-negative Function

Definition 2.7.2. Let us set

$$\mathcal{L}^+ = \mathcal{L}^+(X, \mathcal{M}) := \{f : X \to [0, \infty] \mid f \text{ is } \mathcal{M}\text{-measurable}\}.$$

By the Structure Theorem for measurable functions, given $f \in \mathcal{L}^+$, there exists a sequence $\{\varphi_n\}\uparrow$ in $\mathcal{S}^+$ such that $\varphi_n(x) \to f(x)$ for each $x \in X$. (For simplicity, we will only write $\varphi_n \to f$). By theorem 2.7.1, the sequence of integrals $\{\int \varphi_n\, d\mu\}$ is also increasing in $\mathbb{R}^*$, and we can thus define

$$\int f\, d\mu := \lim_{n\to\infty} \int \varphi_n\, d\mu. \tag{2.30}$$

Note that $\int f\, d\mu \in [0, \infty]$.

Theorem 2.7.2. *$\int f\, d\mu$ is well defined for $f \in \mathcal{L}^+$.*

Proof. We must show: If $\{\varphi_n\}\uparrow$, $\{\psi_n\}\uparrow$ are two sequences in $\mathcal{S}^+$ with $\varphi_n \to f$ and $\psi_n \to f$, then

$$\lim_{n\to\infty} \int \varphi_n\, d\mu = \lim_{n\to\infty} \int \psi_n\, d\mu. \tag{2.31}$$

The main part of the proof consists of the following lemma:

Lemma 2.7.3. *Suppose, $h \in \mathcal{S}^+$ satisfies*

$$h \le \lim \varphi_n.$$

Then

$$\int h\, d\mu \le \lim_{n\to\infty} \int \varphi_n\, d\mu.$$

Proof of the Lemma: Let

$$h = \sum_{j=1}^{m} c_j \chi_{E_j}$$

be the canonical representation of h. Then

$$\int h\, d\mu = \sum_{j=1}^{m} c_j \mu(E_j).$$

In the following, given $\epsilon > 0$ we will consider the sets

$$A_n = A_{n,\epsilon} := \{x \in X \,|\, \varphi_n(x) > h(x) - \epsilon\} \in \mathcal{M}.$$

Then $\{A_n\}\uparrow$. Furthermore, given $x \in X$, we have by assumption that

$$\lim_{n\to\infty} \varphi x \geq h(x) > h(x) - \epsilon,$$

so that $x \in A_n$ for n sufficiently large. That is,

$$X = \bigcup_{n=1}^{\infty} A_n.$$

Case 1: Assume first that $\int h\, d\mu < \infty$. Set

$$E := \bigcup_{\{j:c_j>0\}} E_j \quad \Big(= \{x \in X \,|\, h(x) > 0\}\Big)$$

and set

$$c := \max_{1\leq j\leq m} c_j \qquad \Big(= \max_{x\in X} h(x)\Big).$$

Then by assumption, $\mu(E) < \infty$. Let $\epsilon > 0$ be arbitrary, and $A_n = A_{n,\epsilon}$ as above. Then for each n,

$$\begin{aligned} h = h\chi_E &= h(\chi_{E\cap A_n} + \chi_{E\setminus A_n}) \\ &\leq (\varphi_n + \epsilon)\chi_{E\cap A_n} + c\chi_{E\setminus A_n} \qquad (\text{as } h \leq \varphi_n + \epsilon \text{ on } A_n) \\ &\leq \varphi_n + \epsilon\chi_{E\cap A_n} + c\chi_{E\setminus A_n}. \end{aligned}$$

By theorem 2.7.1,

$$\begin{aligned} \int h\, d\mu &\leq \int \varphi_n\, d\mu + \epsilon \int \chi_{E\cap A_n} d\mu + c\int \chi_{E\setminus A_n}\, d\mu \\ &= \int \varphi_n\, d\mu + \epsilon\mu(E\cap A_n) + c\mu(E\setminus A_n). \end{aligned}$$

Let $n \to \infty$. Since $\mu(E) < \infty$ we obtain

$$\begin{aligned} \int h\, d\mu &\leq \lim_{n\to\infty} \int \varphi_n\, d\mu + \epsilon \lim_{n\to\infty} \mu(E\cap A_n) + c \lim_{n\to\infty} \mu(E\cap A_n^c) \\ &= \lim_{n\to\infty} \int \varphi_n d\mu + \epsilon\mu(E) + c\mu\Big(\underbrace{\bigcap_{n=1}^{\infty}(E\cap A_n^c)}_{=\emptyset}\Big) \\ &= \lim_{n\to\infty} \int \varphi_n d\mu + \epsilon\mu(E). \end{aligned}$$

As $\epsilon > 0$ was arbitrary, we conclude that

$$\int h\, d\mu \leq \lim_{n\to\infty} \int \varphi_n d\mu.$$

This proves the assertion.

Case 2: Now assume that $\int h\, d\mu = \infty$. Then there exists j, $1 \leq j \leq m$, so that $\mu(E_j) = \infty$. Pick any ϵ with $0 < \epsilon < c_j$. We have

$$\infty = \mu(E_j) = \mu\Big(E_j \cap \bigcup_{n=1}^{\infty} A_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} \mu(E_j \cap A_n)\Big) = \lim_{n\to\infty} \mu(E_j \cap A_n)\Big);$$

the last equality holds since $\{A_n\}\uparrow$. Now since $\varphi_n(x) > h(x) - \epsilon$ for all $x \in A_n$, we have

$$\varphi_n \geq \varphi_n \chi_{E_j \cap A_n} > [h(x) - \epsilon]\chi_{E_j \cap A_n} = [c_j - \epsilon]\, \chi_{E_j \cap A_n}$$

for all n, so that by monotonicity of the integral in $\mathcal{S}^+$,

$$\int \varphi_n\, d\mu \geq \int [c_j - \epsilon]\chi_{E_j \cap A_n} d\mu = [c_j - \epsilon]\mu(E_j \cap A_n).$$

Letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \int \varphi_n\, d\mu \geq [c_j - \epsilon] \lim_{n\to\infty} \mu(E_j \cap A_n) = [c_j - \epsilon] \cdot \infty = \infty = \int h\, du.$$

Thus, the lemma is proved.

Return to the proof of the theorem. Since

$$\psi_k \leq f = \lim_{n\to\infty} \varphi_n$$

for each k, then by the lemma,

$$\int \psi_k\, d\mu \leq \lim_{n\to\infty} \int \varphi_n\, d\mu$$

for each k. Let $k \to \infty$. We obtain

$$\lim_{k\to\infty} \int \psi_k\, d\mu \leq \lim_{n\to\infty} \int \varphi_n\, d\mu.$$

Exchanging φ_n and ψ_k in the above argument, we obtain similarly that

$$\lim_{n\to\infty} \int \varphi_n\, d\mu \leq \lim_{k\to\infty} \int \psi_k\, d\mu.$$

Thus, (2.31) holds. □

Theorem 2.7.4. *For all $f, g \in \mathcal{L}^+$ and $\alpha \geq 0$ we have*

1. $\displaystyle\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$ *("additive")*
2. $\displaystyle\int \alpha f\,d\mu = \alpha \int f\,d\mu$ *("positive homogeneous")*
3. *If* $f \leq g$ *then* $\displaystyle\int f\,d\mu \leq \int g\,d\mu$ *("monotone")*

Proof. Let $f, g \in \mathcal{L}^+$ be given. Pick sequences $\{\varphi_n\}\uparrow$ and $\{\psi_n\}\uparrow$ in $\mathcal{S}^+$ such that $\varphi_n \to f$ and $\psi_n \to g$.

1. Obviously, $\varphi_n + \psi_n \in \mathcal{S}^+$ for each n, $\{\varphi_n + \psi_n\}\uparrow$ and $\varphi_n + \psi_n \to f + g$. Hence by definition of the integral in $\mathcal{L}^+$,

$$\int (f+g)\,d\mu \underset{\text{def.}}{=} \lim_{n\to\infty} \int (\varphi_n + \psi_n)\,d\mu \underset{\text{thm 2.7.2}}{=} \lim_{n\to\infty} \Big(\int \varphi_n\,d\mu + \int \psi_n\,d\mu \Big)$$
$$= \lim_{n\to\infty} \int \varphi_n\,d\mu + \lim_{n\to\infty} \int \psi_n\,d\mu \underset{\text{def.}}{=} \int f\,d\mu + \int g\,d\mu.$$

2. Similarly, $\alpha\varphi_n \in \mathcal{S}^+$ for each n, $\{\alpha\varphi_n\}\uparrow$ and $\alpha\varphi_n \to \alpha f$, so that by definition of the integral in $\mathcal{L}^+$,

$$\int \alpha f\,d\mu \underset{\text{def.}}{=} \lim_{n\to\infty} \int \alpha\varphi_n\,d\mu \underset{\text{thm 2.7.2}}{=} \lim_{n\to\infty} \Big(\alpha \int \varphi_n\,d\mu\Big)$$
$$= \alpha \lim_{n\to\infty} \int \varphi_n\,d\mu \underset{\text{def.}}{=} \alpha \int f\,d\mu.$$

3. Now suppose that $f \leq g$. Since it is not assured that $\phi_n \leq \psi_n$, we modify each ψ_n and set

$$\psi'_n = \max\{\varphi_n, \psi_n\}.$$

Then ψ'_n is $\mathcal{M}$-measurable, simple, and $\psi'_n \geq 0$, that is, $\psi'_n \in \mathcal{S}^+$. Furthermore, as $\{\varphi_n\}$ and $\{\psi_n\}$ are increasing, we have

$$\psi'_{n+1} = \max\{\varphi_{n+1}, \psi_{n+1}\} \geq \max\{\varphi_n, \psi_n\} = \psi'_n,$$

that is, $\{\psi'_n\}\uparrow$. Now as $\psi_n \leq \psi'_n \leq g$ and $\psi_n \to g$, we conclude that $\psi'_n \to g$ as well. Since by definition, $\varphi_n \leq \psi'_n$ for each n, then by monotonicity of the integral in $\mathcal{S}^+$ (theorem 2.7.2),

$$\int \varphi_n\,d\mu \leq \int \psi'_n\,d\mu$$

for all n. Hence,

$$\int f\,d\mu \underset{\text{def.}}{=} \lim_{n\to\infty} \int \varphi_n\,d\mu \leq \lim_{n\to\infty} \int \psi'_n\,d\mu \underset{\text{def.}}{=} \int g\,d\mu.$$

□

Exercise 2.7.1. Consider the measure space $(\mathbb{R}, \mathcal{M}_\lambda, \lambda)$. Use the definition of the integral to compute

$$\int f\,d\lambda \qquad \text{where} \quad f(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

2.7.3 The Integral of an Extended Real Valued Function

Definition 2.7.3. Let $f : X \to \mathbb{R}^*$ be $\mathcal{M}$-measurable. Then $f^+, f^- : X \to [0, \infty]$ are also $\mathcal{M}$-measurable, that is, $f^+, f^- \in \mathcal{L}^+$, and $f = f^+ - f^-$.

1. If at least one of
$$\int f^+\,d\mu \qquad \text{and} \qquad \int f^-\,du \tag{2.32}$$
is finite, we define $\int f\,d\mu$ by
$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu. \tag{2.33}$$
If *both* integrals in (2.32) are infinite, then $\int f\,d\mu$ is undefined.

2. We say that f *is integrable*, if both integrals in (2.32) are finite.

Thus, f is integrable iff $\int f\,d\mu$ is defined, and is finite.

Remark 2.7.2. 1. f is integrable $\Leftrightarrow$ $|f|$ is integrable. In this case,
$$\left|\int f\,d\mu\right| \leq \int |f|\,d\mu.$$

Proof. Suppose that f is integrable. Then $\int f^+\,d\mu < \infty$ and $\int f^-\,d\mu < \infty$ so that
$$\int |f| d\mu = \int (f^+ + f^-) d\mu \underset{\text{thm 2.7.4}}{=} \int f^+ d\mu + \int f^- d\mu < \infty.$$
Hence, $|f|$ is integrable.

Conversely, suppose that $|f|$ is integrable. Since $0 \leq f^+, f^- \leq |f|$ it follows monotonicity of the integral in $\mathcal{L}^+$ (theorem 2.7.4, part 3.) that
$$0 \leq \int f^+\,d\mu \leq \int |f|\,d\mu < \infty \qquad \text{and} \qquad 0 \leq \int f^-\,d\mu \leq \int |f|\,d\mu < \infty.$$
Hence, f is integrable.

In this case, by the triangle inequality,
$$\begin{aligned} \left|\int f\,d\mu\right| &= \left|\int f^+ d\mu - \int f^- d\mu\right| \\ &\leq \left|\int f^+ d\mu + \int f^- d\mu\right| = \left|\int |f|\,d\mu\right| = \int |f|\,d\mu. \end{aligned}$$

□

2. If $\mu(X) < \infty$, then every *bounded*, $\mathcal{M}$-measurable function f is integrable. For suppose, $|f(x)| \leq M$ for all $x \in X$. Then by monotonicity of the integral in Theorem $\mathcal{L}^+$ (theorem 2.7.2, part 3.),

$$0 \leq \int |f| \, d\mu \leq \int M\chi_X \, d\mu = M\,\mu(X) < \infty.$$

3. Let φ be a simple, $\mathcal{M}$-measurable function, say

$$\varphi = \sum_{k=1}^{n} c_k \chi_{A_k}, \qquad A_k \in \mathcal{M},\ \{A_k\} \text{ disjoint.}$$

Then

$$\varphi^+ = \sum_{\{k:c_k>0\}}^{n} c_k\chi_{A_k} \qquad \text{and} \qquad \varphi^- = \sum_{\{k:c_k<0\}}^{n} (-c_k)\chi_{A_k}$$

so that

$$\begin{aligned}\int \varphi \, d\mu \underset{\text{def.}}{=} \int \varphi^+ \, d\mu - \int \varphi^- \, d\mu &= \sum_{\{k:c_k>0\}} c_k\mu(A_k) - \sum_{\{k:c_k<0\}} (-c_k)\mu(A_k) \\ &= \sum_{\{k:c_k>0\}} c_k\mu(A_k) + \sum_{\{k:c_k<0\}} c_k\mu(A_k) + \sum_{\{k:c_k=0\}} \underbrace{c_k\mu(A_k)}_{=0}.\end{aligned}$$

That is,

$$\int \varphi \, d\mu = \sum_{k=1}^{n} c_k\mu(A_k).$$

4. If f is integrable, then f is finite a.e.

Proof. Let

$$\begin{aligned} A &:= \{x \in X : f(x) = \infty\} \in \mathcal{M} \\ B &:= \{x \in X : f(x) = -\infty\} \in \mathcal{M}. \end{aligned}$$

We need to show that both sets have measure zero.

For each $n \in \mathbb{N}$,

$$f^+ \geq n\chi_A \qquad \text{and} \qquad f^- \geq n\chi_B,$$

so that by monotonicity of the integral in $\mathcal{L}^+$,

$$\begin{aligned} \int f^+ \, d\mu &\geq \int n\chi_A \, d\mu = n\mu(A) \\ \int f^- \, d\mu &\geq \int n\chi_B \, d\mu = n\mu(B). \end{aligned}$$

That is,

$$0 \leq \mu(A) \leq \frac{1}{n}\int f^+ \, d\mu \qquad \text{and} \qquad 0 \leq \mu(B) \leq \frac{1}{n}\int f^- \, d\mu$$

for each n. Let $n \to \infty$. Since $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$ we obtain that $\mu(A) = 0$ and $\mu(B) = 0$. □

Let us set

$$\mathcal{L} = \mathcal{L}_{\mathbb{R}} = \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R}) := \{f : X \to \mathbb{R} \,|\, f \text{ is } \mathcal{M}\text{-measurable and integrable}\}.$$

So if f is real-valued and $\mathcal{M}$-measurable, then

$$f \in \mathcal{L} \quad \Leftrightarrow \quad \int |f| \, d\mu < \infty \quad \Leftrightarrow \quad \int f^+ \, d\mu < \infty \text{ and } \int f^- \, d\mu < \infty.$$

Theorem 2.7.5. *$\mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$ is a real vector space, and the integral is linear and monotone. That is, for all $f, g \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$ and $\alpha \in \mathbb{R}$ we have*

1. $\displaystyle\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$ *("additive")*
2. $\displaystyle\int \alpha f \, d\mu = \alpha \int f \, d\mu$ *("homogeneous")*
3. *If* $f \leq g$ *then* $\displaystyle\int f \, d\mu \leq \int g \, d\mu$ *("monotone")*

Proof. Let $f, g \in \mathcal{L}$ and α scalar. Then

$$\int |f| \, d\mu < \infty \qquad \text{and} \qquad \int |g| \, d\mu < \infty.$$

Since $|f + g| \leq |f| + |g|$ and $|\alpha f| = |\alpha| \, |f|$, it follows from theorem 2.7.5 that

$$\int |f + g| \, d\mu \leq \int (|f| + |g|) \, d\mu = \int |f| \, d\mu + \int |g| \, d\mu < \infty$$

and

$$\int |\alpha f| \, d\mu \leq \int |\alpha||f| \, d\mu = |\alpha| \int |f| \, d\mu < \infty.$$

Thus, $f + g \in \mathcal{L}$ and $\alpha f \in \mathcal{L}$, which shows that $\mathcal{L}$ is a real vector space.

1. Note that

$$f + g = (f + g)^+ - (f + g)^- \tag{2.34}$$

while also

$$f + g = (f^+ - f^-) + (g^+ - g^-). \tag{2.35}$$

Equating (2.34) and (2.35) we obtain

$$(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+$$

so that

$$\int \Big((f + g)^+ + f^- + g^-\Big) \, d\mu = \int \Big((f + g)^- + f^+ + g^+\Big) \, d\mu.$$

Since all functions involved are in $\mathcal{L}^+$, we obtain by additivity of the integral in $\mathcal{L}^+$ (theorem 2.7.5) that,

$$\int (f+g)^+\,d\mu + \int f^-\,d\mu + \int g^-\,d\mu = \int (f+g)^-\,d\mu + \int f^+\,d\mu + \int g^+\,d\mu.$$

As all integrals are finite, we may subtract,

$$\int (f+g)^+\,d\mu - \int (f+g)^-\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu + \int g^+\,d\mu - \int g^-\,d\mu,$$

that is,

$$\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu.$$

2. Suppose first that $\alpha \geq 0$. Then $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$. Hence,

$$\begin{aligned}\int \alpha f\,d\mu &\underset{\text{def.}}{=} \int (\alpha f)^+\,d\mu - \int (\alpha f)^-\,d\mu = \int \alpha f^+\,d\mu - \int \alpha f^-\,d\mu \\ &\underset{\text{thm 2.7.5}}{=} \alpha \int f^+\,d\mu - \alpha \int f^-\,d\mu = \alpha\Big(\int f^+\,d\mu - \int f^-\,d\mu\Big) \underset{\text{def.}}{=} \alpha \int f\,d\mu.\end{aligned}$$

Similarly, if $\alpha < 0$, then $(\alpha f)^+ = |\alpha| f^-$ and $(\alpha f)^- = |\alpha| f^+$, so that

$$\begin{aligned}\int \alpha f\,d\mu &\underset{\text{def.}}{=} \int (\alpha f)^+\,d\mu - \int (\alpha f)^-\,d\mu = \int |\alpha| f^-\,d\mu - \int |\alpha| f^+\,d\mu \\ &\underset{\text{thm 2.7.5}}{=} (|\alpha| \int f^-\,d\mu - |\alpha| \int f^+\,d\mu = |\alpha|\Big(\int f^-\,d\mu - \int f^+\,d\mu\Big) \\ &= -|\alpha|\Big(\int f^+\,d\mu - \int f^-\,d\mu\Big) \underset{\text{def.}}{=} \alpha \int f\,d\mu.\end{aligned}$$

3. Now suppose, $f \leq g$. Then $g - f \geq 0$, so that

$$\int (g-f)\,d\mu \geq 0.$$

By parts 1. and 2. above,

$$\int g\,d\mu - \int f\,d\mu \geq 0$$

so that

$$\int f\,d\mu \leq \int g\,d\mu.$$

□

2.7.4 The Integral of a Complex Valued Function

Definition 2.7.4. Let $f : X \to \mathbb{C}$ be $\mathcal{M}$-measurable. We say that f *is integrable*, if both, $\Re(f), \Im(f) : X \to \mathbb{R}$ are integrable. In this case we define

$$\int f \, d\mu := \int \Re(f) \, d\mu + i \int \Im(f) \, d\mu.$$

We set

$$\mathcal{L}_{\mathbb{C}} = \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C}) := \{f : X \to \mathbb{C} \,|\, f \text{ is } \mathcal{M}\text{-measurable and integrable}\}.$$

Theorem 2.7.6. *$\mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C})$ is a complex vector space, and the integral is linear. That is, for all $f, g \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C})$ and $\alpha \in \mathbb{C}$ we have*

1. $\displaystyle\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$ *("additive")*
2. $\displaystyle\int \alpha f \, d\mu = \alpha \int f \, d\mu$ *("homogeneous")*

Furthermore

3. $f \in \mathcal{L}_{\mathbb{C}} \quad \Leftrightarrow \quad \overline{f} \in \mathcal{L}_{\mathbb{C}},$ *in which case* $\displaystyle\overline{\int f \, d\mu} = \int \overline{f} \, d\mu,$
4. $f \in \mathcal{L}_{\mathbb{C}} \quad \Leftrightarrow \quad |f| \in \mathcal{L}_{\mathbb{R}},$ *in which case* $\displaystyle\left|\int f \, d\mu\right| \leq |f| \, d\mu.$

Proof. 1–2. Let $f, g \in \mathcal{L}_{\mathbb{C}}$ and $\alpha \in \mathbb{C}$.

Since f and g are integrable, then $\Re(f), \Im(f), \Re(g), \Im(g) : X \to \mathbb{R}$ are integrable. It follows from theorem 2.7.5 that $\Re(f+g) = \Re(f)+\Re(g)$ and $\Im(f+g) = \Im(f)+\Im(g)$ are also integrable; hence $f + g$ is integrable.

Note that if $\alpha = a + ib$, then

$$\alpha f = (a + ib)\big(\Re(f) + i\Im(f)\big) = \big(a\Re(f) - b\Im(f)\big) + i\big(a\Im(f) + b\Re(f)\big).$$

Applying theorem 2.7.5 again, it follows that $\Re(\alpha f) = a\Re(f) - b\Im(f)$ and $\Im(\alpha f) = a\Im(f) + b\Re(f)$ are integrable. Hence, af is integrable.

This shows that $\mathcal{L}_{\mathbb{C}}$ is a vector space. Using the above notation, we now obtain

$$\begin{aligned}
\int (f + g) \, d\mu &\underset{\text{def.}}{=} \int \Re(f + g) \, d\mu + i \int \Im(f + g) \, d\mu \\
&= \int \big(\Re(f) + \Re(g)\big) \, d\mu + i \int \big(\Im(f) + \Im(g)\big) \, d\mu \\
&\underset{\text{thm. 2.7.6}}{=} \Big(\int \Re(f) \, d\mu + \int \Re(g) \, d\mu\Big) + i\Big(\int \Im(f) \, d\mu + \int \Im(g) \, d\mu\Big) \\
&= \int f \, d\mu + \int g \, d\mu.
\end{aligned}$$

and

$$\begin{aligned}\int (\alpha f)\,d\mu &= \int \Big(\big(a\Re(f) - b\Im(f)\big) + i\big(a\Im(f) + b\Re(f)\big)\Big) d\mu \\ &\underset{\text{def.}}{=} \int \big(a\Re(f) - b\Im(f)\big)\,d\mu + i\int \big(a\Im(f) + b\Re(f)\big)\,d\mu \\ &\underset{\text{thm. 2.7.6}}{=} \Big(a\int \Re(f)\,d\mu - b\int \Im(f)\,d\mu\Big) + i\Big(a\int \Im(f)\,d\mu + b\int \Re(f)\,d\mu\Big) \\ &= (a+ib)\Big(\int \Re(f)\,d\mu + i\int \Im(f)\,d\mu\Big) \;=\; \alpha\int f\,d\mu.\end{aligned}$$

3. Since $\overline{f} = \Re(f) - i\Im(f) = \Re(f) + i\big(-\Im(f)\big)$ we have

$$f \in \mathcal{L}_{\mathbb{C}} \underset{\text{def.}}{\Leftrightarrow} \Re(f), \Im(f) \in \mathcal{L}_{\mathbb{R}} \underset{\text{thm 2.7.5}}{\Leftrightarrow} \Re(f), -\Im(f) \in \mathcal{L}_{\mathbb{R}} \quad \Leftrightarrow \quad \overline{f} \in \mathcal{L}_{\mathbb{C}}$$

and

$$\begin{aligned}\int \overline{f}\,d\mu &= \int \Big(\Re(f) + i\big(-\Im(f)\big)\Big)\,d\mu \underset{\text{def.}}{=} \int \Re(f)\,d\mu + i\int \big(-\Im(f)\big)\,d\mu \\ &\underset{2.}{=} \int \Re(f)\,d\mu - i\int \Im(f)\,d\mu \;=\; \overline{\int \Re(f)\,d\mu + i\int \Im(f)\,d\mu} \underset{\text{def.}}{=} \overline{\int f\,d\mu}.\end{aligned}$$

4. Since

$$|\Re(f)|,\ |\Im(f)| \underset{(A)}{\leq} \sqrt{\Re(f)^2 + \Im(f)^2} = |f| \underset{(B)}{\leq} |\Re(f)| + |\Im(f)|$$

we see by (A) that

$$\int |f|\,d\mu < \infty \;\Rightarrow\; \int |\Re(f)|\,d\mu < \infty \text{ and } \int |\Im(f)|\,d\mu < \infty \;\Rightarrow\; f \text{ is integrable}$$

while by (B),

$$f \text{ is integrable} \;\Rightarrow\; \int |\Re(f)|\,d\mu < \infty \text{ and } \int |\Im(f)|\,d\mu < \infty \;\Rightarrow\; \int |f|\,d\mu < \infty.$$

That is $f \in \mathcal{L}_{\mathbb{C}} \quad \Leftrightarrow \quad |f| \in \mathcal{L}_{\mathbb{R}}$.

Now if $f \in \mathcal{L}_{\mathbb{C}}$, let us use the polar notation for its integral and write

$$\int f\,d\mu = re^{i\theta}, \qquad r \geq 0,\ -\pi < \theta \leq \pi. \tag{2.36}$$

Multiplying by $e^{-i\theta}$ we obtain

$$r = e^{-i\theta}\int f\,d\mu \underset{2.}{=} \int e^{-i\theta} f\,d\mu \underset{\text{def.}}{=} \int \Re\big(e^{-i\theta}f\big)\,d\mu + i\int \Im\big(e^{-i\theta}f\big)\,d\mu. \tag{2.37}$$

Since $r \geq 0$, all terms in this equation are real, and in particular, $\int \Im\big(e^{-i\theta}f\big)\,d\mu = 0$. Since $\Re(z) \leq |z|$ for all $z \in \mathbb{C}$, it follows from (2.37) together with monotonicity of the integral in $\mathcal{L}^+$ that

$$\Big|\int f\,d\mu\Big| = |re^{i\theta}| = r = \int \Re\big(e^{-i\theta}f\big)\,d\mu \leq \int |e^{-i\theta}f|\,d\mu = \int |f|\,d\mu.$$

This proves the theorem. □

Exercise 2.7.2. Consider the measure space $(\mathbb{R}, \mathcal{M}_\lambda, \lambda)$. Show that each of the following functions f is $\mathcal{B}(\mathbb{R})$-measurable, and using the definition of the Lebesgue integral, find $\int f\,d\lambda$ if defined.

1. $f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ -2 & 1 \leq x < 2 \\ 0 & \text{else} \end{cases}$

2. $f(x) = \begin{cases} \frac{1}{n} & n-1 \leq x < n, \ n \in \mathbb{N} \\ 0 & \text{else} \end{cases}$

3. $f(x) = \begin{cases} \frac{1}{n} & n-1 \leq x < n, \ n \in \mathbb{N} \text{ even} \\ -\frac{1}{n} & n-1 \leq x < n, \ n \in \mathbb{N} \text{ odd} \\ 0 & \text{else.} \end{cases}$

2.8 Properties of the Lebesgue Integral

Throughout this section, $(X, \mathcal{M}, \mu)$ will denote a measure space. By "f is $\mathcal{M}$-measurable" or "f is measurable" we will mean that $f : X \to \mathbb{R}^*$, or $f : X \to \mathbb{C}$, and f is $\mathcal{M}$-measurable. Recall from the previous section that if f is $\mathcal{M}$-measurable, then

$$f \text{ is integrable} \quad \Leftrightarrow \quad \int |f|\,d\mu < \infty.$$

For ease of notation, we will often drop the symbol "$d\mu$" in the integral, and simply write $\int f$ instead of $\int f\,d\mu$.

2.8.1 Integral Over a Set

Definition 2.8.1. Let f be $\mathcal{M}$-measurable and $A \in \mathcal{M}$. Then $f\chi_A$ is also $\mathcal{M}$-measurable, and we can define

$$\int_A f\,d\mu := \int f\chi_A\,d\mu$$

provided that the integral on the right-hand side is defined.

Remark 2.8.1. 1. If $\int f\,d\mu$ is defined (for example, if $f \in \mathcal{L}^+$, or f is integrable) then obviously $\int_A f\,d\mu$ is also defined for all $A \in \mathcal{M}$.

2. If $\mu(A) = 0$, then $\int_A f\,d\mu$ is always defined, and $\int_A f\,d\mu = 0$.

Proof. (a) First let $f \in \mathcal{S}^+$. Then f is bounded, so there exists $M > 0$ such that $0 \le f(x) \le M$ for all x. Thus by monotonicity,

$$0 \le \int\limits_A f\,d\mu = \int f\chi_A\,d\mu \le \int M\chi_A\,d\mu = M\mu(A) = 0,$$

which shows that $\int\limits_A f\,d\mu = 0$.

(b) Next let $f \in \mathcal{L}^+$. Pick $\{\varphi_n\} \subseteq \mathcal{S}^+$, $\{\varphi_n\}\uparrow$, such that $\varphi_n \to f$. Then $\{\varphi_n\chi_A\} \subseteq \mathcal{S}^+$, $\{\varphi_n\chi_A\}\uparrow$, and $\varphi_n\chi_A \to f\chi_A$. Hence

$$\int\limits_A f \underset{\text{def.}}{=} \int f\chi_A \underset{\text{def.}}{=} \lim_{n\to\infty} \int \varphi_n\chi_A \underset{\text{(a)}}{=} \lim_{n\to\infty} 0 = 0.$$

(c) Now let $f : X \to \mathbb{R}^*$. Note that $(f\chi_A)^+ = f^+\chi_A$ and $(f\chi_A)^- = f^-\chi_A$. Then by (b),

$$\int (f\chi_A)^+ = \int f^+\chi_A = 0 \qquad \text{and} \qquad \int (f\chi_A)^- = \int f^-\chi_A = 0.$$

Hence, $\int f\chi_A$ is defined, and

$$\int\limits_A f \underset{\text{def.}}{=} \int f\chi_A \underset{\text{def.}}{=} \int (f\chi_A)^+ - \int (f\chi_A)^- = 0 - 0 = 0.$$

(d) Finally, let $f : X \to \mathbb{C}$. Note that $\Re(f\chi_A) = \Re(f)\chi_A$ and $\Im(f\chi_A) = \Im(f)\chi_A$. By part (c), these functions are integrable, and

$$\int \Re(f\chi_A) = \int \Re(f)\chi_A = 0, \qquad \int \Im(f\chi_A) = \int \Im(f)\chi_A = 0. \tag{2.38}$$

Hence, $\int f\chi_A$ is defined, and

$$\int\limits_A f \underset{\text{def.}}{=} \int f\chi_A \underset{\text{def.}}{=} \int \Re(f\chi_A) + i\int \Im(f\chi_A) = 0 + i0 = 0.$$

For example, $\int_{\mathbb{Q}} x^2\,d\lambda = 0$. □

3. Let $A, B \in \mathcal{M}$ be disjoint. Then

$$\int\limits_{A\cup B} f\,d\mu \quad \text{is defined} \quad \Leftrightarrow \quad \int\limits_A f\,d\mu + \int\limits_B f\,d\mu \quad \text{is defined.} \tag{2.39}$$

In this case,

$$\int\limits_{A\cup B} f\,d\mu = \int\limits_A f\,d\mu + \int\limits_B f\,d\mu.$$

Proof. The main ingredient in the proof is the fact that $\chi_{A\cup B} = \chi_A + \chi_B$, because $A \cap B = \emptyset$.

(a) First let $f \in \mathcal{L}^+$. Then both sides in (2.39) are defined, and

$$\int\limits_{A\cup B} f = \int f\chi_{A\cup B} = \int f(\chi_A + \chi_B)$$
$$= \int f\chi_A + \int f\chi_B = \int\limits_A f + \int\limits_B f. \tag{2.40}$$

(b) Next let $f : X \to \mathbb{R}^*$. Note that

$$(f\chi_{A\cup B})^+ = f^+\chi_{A\cup B} = f^+\chi_A + f^+\chi_B$$
$$(f\chi_{A\cup B})^- = f^-\chi_{A\cup B} = f^-\chi_A + f^-\chi_B,$$

so that

$$\begin{aligned} \int (f\chi_{A\cup B})^+ &= \int f^+\chi_A + \int f^+\chi_B \quad \text{and} \\ \int (f\chi_{A\cup B})^- &= \int f^-\chi_A + \int f^-\chi_B \end{aligned} \tag{2.41}$$

It follows that $\int f\chi_{A\cup B}\, d\mu$ is defined

$\Leftrightarrow$ at least one of the two sums

$$\int f^+\chi_A + \int f^+\chi_B \qquad \text{and} \qquad \int f^-\chi_A + \int f^-\chi_B$$

is finite

$\Leftrightarrow$ $\int f\chi_A + \int f\chi_B$ is not of the form $\infty + (-\infty)$ or $-\infty + \infty$.

Thus, (2.39) holds. In this case

$$\begin{aligned} \int\limits_{A\cup B} f &= \int f\chi_{A\cup B} = \int (f\chi_{A\cup B})^+ - \int (f\chi_{A\cup B})^- \\ &= \int (f^+\chi_A + f^+\chi_B) - \int (f^-\chi_A + f^-\chi_B) \\ &= \left(\int f^+\chi_A - \int f^-\chi_A\right) + \left(\int f^+\chi_B - \int f^-\chi_B\right) \\ &= \left(\int (f\chi_A)^+ - \int (f\chi_A)^-\right) + \left(\int (f\chi_B)^+ - \int (f\chi_B)^-\right) \\ &= \int f\chi_A + \int f\chi_B = \int\limits_A f + \int\limits_B f. \end{aligned}$$

(c) Now let $f : X \to \mathbb{C}$. Since $|\Re(f)|, |\Im(f)| \leq |f| \leq |\Re(f)| + |\Im(f)|$ we have by monotonicity of the integral and (a) above,

$$\begin{aligned} &\int |f|\chi_{A\cup B} < \infty \\ \Leftrightarrow \quad &\int |\Re(f)|\chi_{A\cup B} + \int |\Im(f)|\chi_{A\cup B} < \infty \\ \Leftrightarrow \quad &\int |\Re(f)|\chi_A + \int |\Re(f)|\chi_B + \int |\Im(f)|\chi_A + \int |\Im(f)|\chi_B < \infty \\ \Leftrightarrow \quad &\int |f|\chi_A + \int |f|\chi_B < \infty. \end{aligned}$$

This shows that

$$\int_{A\cup B} f\,d\mu \quad \text{is defined} \quad \Leftrightarrow \quad \int_A f\,d\mu + \int_B f\,d\mu \quad \text{is defined.}$$

In this case, as $f\chi_{A\cup B}, f\chi_A, f\chi_B \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C})$, the computations of equation (2.40) remain valid by additivity of the integral.

□

4. Recall that by exercise 2.3.1, $\mathcal{M}_A = \{E \subseteq A : E \in \mathcal{M}\}$ is a σ-algebra of subsets of A, and the restriction $\mu_{|A}$ of μ to $\mathcal{M}_A$ is a measure, where $\mu_{|A}(E) := \mu(E)$ for all $E \in \mathcal{M}_A$. That is, $(A, \mathcal{M}_A, \mu_{|A})$ is a measure space. Now let $f : X \to \mathbb{R}^*$ (resp. $\mathbb{C}$) be $\mathcal{M}$-measurable. It is obvious that the restriction $f_{|A} : A \to \mathbb{R}^*$ (resp. $\mathbb{C}$) given by $f_{|A}(x) := f(x)$ for $x \in A$, is $\mathcal{M}_A$-measurable, and it is an easy exercise to show, using the definition of the integrals, that

$$\int_A f\,d\mu \quad \text{is defined} \quad \Leftrightarrow \quad \int f_{|A}\,d\mu_{|A} \quad \text{is defined,}$$

and that both integrals coincide whenever they are defined.

Changing the values of a function on a set of measure zero does not affect its integral:

Theorem 2.8.1. *Let f, g be $\mathcal{M}$-measurable, with $f(x) = g(x)$ a.e. Then*

$$\int f\,d\mu \quad \textit{is defined} \quad \Leftrightarrow \quad \int g\,d\mu \quad \textit{is defined.}$$

Furthermore, both integrals coincide whenever they are defined.

Proof. Set $E := \{\, x \in X : f(x) \neq g(x) \,\}$. Then $E \in \mathcal{M}$ as f, g are measurable (why ?), and by assumption, $\mu(E) = 0$.

$\Rightarrow$: Suppose, $\int f\,d\mu$ is defined. Then by part 3. of remark 2.8.1,

$$\int f\,d\mu = \int_{(X\backslash E)\cup E} f = \int_{X\backslash E} f + \int_E f = \int_{X\backslash E} f + 0 = \int_{X\backslash E} g + 0 = \int_{X\backslash E} g + \int_E g$$

as $f = g$ on $X\backslash E$ and $\mu(E) = 0$. It follows by the same remark that $\int g\,d\mu$ exists, and

$$\int g = \int_{X\backslash E} g + \int_E g = \int f.$$

$\Leftarrow$: Follows by symmetry. □

Theorem 2.8.2. *Let $f \in \mathcal{L}^+$. Then*

$$\int f\,d\mu = 0 \quad \Leftrightarrow \quad f = 0 \quad a.e.$$

Proof. $\Leftarrow$: Follows directly from theorem 2.8.1.
$\Rightarrow$: Suppose, that $\int f\,d\mu = 0$. Set

$$A := \{\, x \in X : f(x) > 0 \,\} \in \mathcal{M} \qquad \text{and} \qquad A_n := \{\, x \in X : f(x) > \tfrac{1}{n} \,\} \in \mathcal{M}$$

for $n = 1, 2, \ldots$. Then $\chi_{A_n} \leq nf$ for all n, so that by monotonicity of the integral,

$$0 \leq \mu(A_n) = \int \chi_{A_n}\,d\mu \leq \int nf\,d\mu = n\int f\,d\mu = 0,$$

that is $\mu(A_n) = 0$. Now since $A = \bigcup_{n=1}^{\infty} A_n$, it follows from subadditivity of the measure that

$$0 \leq \mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0.$$

Hence $\mu(A) = 0$, that is, $f(x) = 0$ a.e. □

2.8.2 Functions Defined Almost Everywhere

Suppose, f is defined a.e. on X and measurable. By this we mean that there exists $Y \in \mathcal{M}$ with $\mu(X\backslash Y) = 0$ and $f : Y \to \mathbb{R}^*$ (resp. $f : Y \to \mathbb{C}$) is $\mathcal{M}_Y$-measurable, where $\mathcal{M}_Y := \{E \in \mathcal{M} : E \subseteq Y\}$ is defined as in exercise 2.3.1.

There are many ways to extend f to an $\mathcal{M}$-measurable function $\tilde{f} : X \to \mathbb{R}^*$ (resp. $f : Y \to \mathbb{C}$). Since for any measurable extension $\tilde{f}$,

$$\int_X |\tilde{f}|\,d\mu = \int_Y |\tilde{f}|\,d\mu + \int_{X\backslash Y} |\tilde{f}|\,d\mu = \int_Y |f|\,d\mu + 0, \tag{2.42}$$

it does not matter for the integral how we define $\tilde{f}$, so for convenience we set

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{if } x \in X\backslash Y. \end{cases}$$

It is easy to check that $\tilde{f}$ is $\mathcal{M}$-measurable. Furthermore, computing as in (2.42) we see that $\tilde{f} \in \mathcal{L}^+(X)$ (resp. $\int_X \tilde{f}\, d\mu$ is defined, resp. $\tilde{f} \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$, resp. $\tilde{f} \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C})$) $\Leftrightarrow$ $f \in \mathcal{L}^+(Y)$ (resp. $\int_Y \tilde{f}\, d\mu$ is defined, resp. $f \in \mathcal{L}(Y, \mathcal{M}, \mu; \mathbb{R})$, resp. $f \in \mathcal{L}(Y, \mathcal{M}, \mu; \mathbb{C})$), in which case

$$\int_X \tilde{f}\, d\mu = \int_Y f d\mu.$$

We usually identify f with $\tilde{f}$, and thus consider f an $\mathcal{M}$-measurable function defined on X.

Example 2.8.1. 1. $f(x) = \frac{1}{\sqrt{x}}$ can be considered an element of $\mathcal{L}^+([0,1], \mathcal{M}_{[0,1]})$.

2. $g(x) = \begin{cases} \frac{1}{n^2} & n-1 < x < n,\ n \text{ even} \\ -\frac{1}{n^2} & n-1 < x < n,\ n \text{ odd} \end{cases}$

   is not defined if x is integer. However, it can be considered an element of $\mathcal{L}(\mathbb{R}, \mathcal{M}_\lambda, \lambda; \mathbb{R})$.

Example 2.8.2. Let $\{f_n\}_{n=1}^\infty$ be a sequence of $\mathcal{M}$-measurable functions converging a.e. That is, there exists $Y \in \mathcal{M}$, $\mu(X \backslash Y) = 0$ such that $\{f_n(x)\}_{n=1}^\infty$ converges for all $x \in Y$. We can thus define $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in Y$. By theorem 2.6.6, f is $\mathcal{M}_Y$-measurable. The measurable extension $\tilde{f}$ of f above will simply be denoted by $\lim f_n$, that is

$$(\lim_{n\to\infty} f_n)(x) = \begin{cases} \lim\limits_{n\to\infty} f_n(x) & \text{if } x \in Y \\ 0 & \text{if } x \in X \backslash Y. \end{cases}$$

2.8.3 Convergence Theorems

In this section we investigate conditions under which we may exchange limits with integrals. That is, are we allowed to write

$$\lim_{n\to\infty} \int f_n\, d\mu = \int (\lim_{n\to\infty} f_n)\, d\mu \quad ? \tag{2.43}$$

In general, this is not true. For example, consider the measure space $(\mathbb{R}, \mathcal{M}_\lambda, \lambda)$ and

$$f_n(x) = \begin{cases} n & 0 < x \le \frac{1}{n} \\ 0 & \text{else.} \end{cases}$$

Then $\lim\limits_{n\to\infty} f_n(x) = 0$, but $\lim\limits_{n\to\infty} \int f_n = 1 \neq 0 = \int (\lim f_n)$. Recall that in the case of the Riemann integral, we don't even know whether $\lim f_n$ is integrable, unless convergence of $\{f_n\}$ is uniform, in which case (2.43) holds. On the other hand, in the case of the Lebesgue integral, since each f_n is measurable, the pointwise limit $f(x) := \lim f_n(x)$ is always measurable, and is integrable provided that $\int |f|\, d\mu < \infty$.

Theorem 2.8.3. (Monotone Convergence Theorem).
Let $f_n \geq 0$ be $\mathcal{M}$-measurable, and $\{f_n\}\uparrow$. Then

$$\int (\lim_{n\to\infty} f_n)\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu. \tag{2.44}$$

Proof. Since the sequence $\{f_n\}_{n=1}^{\infty}$ is increasing, by monotonicity of the integral, the sequence of integrals $\{\int f_n\, d\mu\}_{n=1}^{\infty}$ is also increasing. Thus, both sides of (2.44) are defined in $\mathbb{R}^*$. Set $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

By the Structure Theorem for measurable functions, for each n there exists a sequence $\{\varphi_{n,k}\}_{k=1}^{\infty} \subseteq \mathcal{S}^+$, $\{\varphi_{n,k}\}\uparrow$, such that $\lim\limits_{k\to\infty} \varphi_{n,k}(x) = f_n(x)$. Consider the following diagram:

$$\begin{array}{ccccccccccccc}
\varphi_{11} & \leq & \varphi_{1,2} & \leq & \varphi_{1,3} & \leq & \cdots & \leq & \varphi_{1,n} & \leq & \cdots & \to & f_1 \\
 & & & & & & & & & & & & \leq \\
\varphi_{21} & \leq & \varphi_{2,2} & \leq & \varphi_{2,3} & \leq & \cdots & \leq & \varphi_{2,n} & \leq & \cdots & \to & f_2 \\
 & & & & & & & & & & & & \leq \\
\varphi_{31} & \leq & \varphi_{3,2} & \leq & \varphi_{3,3} & \leq & \cdots & \leq & \varphi_{3,n} & \leq & \cdots & \to & f_3 \\
 & & & & & & & & & & & & \leq \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
 & & & & & & & & & & & & \leq \\
\varphi_{n1} & \leq & \varphi_{n,2} & \leq & \varphi_{n,3} & \leq & \cdots & \leq & \varphi_{n,n} & \leq & \cdots & \to & f_n \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \downarrow \\
 & & & & & & & & & & & & f
\end{array}$$

For each k set

$$\psi_k := \max\{\, \varphi_{1,k}, \varphi_{2,k}, \varphi_{3,k}, \ldots, \varphi_{k,k}\}.$$

Then ψ_k is simple, $\mathcal{M}$-measurable, and $\psi_k \geq 0$, that is $\psi_k \in \mathcal{S}^+$ for all k. Also, as $\varphi_{n,k+1} \geq \varphi_{n.k}$ for all k, n, we have

$$\begin{aligned}\psi_{k+1} &= \max\{\, \varphi_{1,k+1}, \varphi_{2,k+1}, \ldots, \varphi_{k,k+1}, \varphi_{k+1,k+1}\} \\ &\geq \max\{\, \varphi_{1,k},\ \varphi_{2,k},\ \ldots,\ \varphi_{k,k}\} = \psi_k.\end{aligned}$$

That is, $\{\psi_k\}\uparrow$; in particular, $\lim\limits_{k\to\infty} \psi_k$ exists. By construction,

$$\varphi_{n,k} \leq \psi_k \leq f_k \qquad \forall n \leq k, \tag{2.45}$$

so letting $k \to \infty$,

$$f_n = \lim_{k\to\infty} \varphi_{n,k} \leq \lim_{k\to\infty} \psi_k \leq \lim_{k\to\infty} f_k = f \qquad \forall\, n.$$

Letting $n \to \infty$,

$$f = \lim_{n\to\infty} f_n \leq \lim_{k\to\infty} \psi_k \leq f,$$

that is, $f = \lim\limits_{k\to\infty} \psi_k$. Then by the definition and monotonicity of the integral in $\mathcal{L}^+$,

$$\int f \underset{\text{def.}}{=} \lim_{k\to\infty} \int \psi_k \underset{(2.45)}{\leq} \lim_{k\to\infty} \int f_k \leq \lim_{k\to\infty} \int f = \int f.$$

Hence,

$$\int f = \lim_{k\to\infty} \int f_k.$$

This proves the theorem. □

Exercise 2.8.1. Show that the theorem remains valid if its assumptions are changed from

1. "$f_n(x) \geq 0$ *for all* x" to "$f_n(x) \geq 0$ *a.e.*"
2. "$f_n(x) \geq 0$ *a.e.*" to "*there exists* $g \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$ *with* $f_n(x) \geq g(x)$ *a.e.*"

Exercise 2.8.2. Use exercise 2.8.1 to show:
Let $f_n \geq 0$ a.e. be $\mathcal{M}$-measurable for all n. Then $\sum_{n=1}^{\infty} f_n$ is defined a.e., and

$$\int \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

If the sequence $\{f_n\}$ is not increasing, then it need not converge. However, we still have:

Theorem 2.8.4. (Fatou's Lemma). *Let $f_n \geq 0$ be $\mathcal{M}$-measurable for all n. Then*

$$\int (\liminf_{n\to\infty} f_n) \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu. \tag{2.46}$$

Proof. For each n, set

$$g_n = \inf_{k\geq n} f_k$$

Then g_n is $\mathcal{M}$-measurable, $0 \leq g_n \leq f_n$ and $\{g_n\}\uparrow$. Furthermore,

$$\liminf_{n\to\infty} f_n = \lim_{n\to\infty} \inf_{k\geq n} f_k = \lim_{n\to\infty} g_n. \tag{2.47}$$

Applying the Monotone Convergence Theorem to $\{g_n\}$, we obtain

$$\int \left(\lim_{n\to\infty} g_n\right) = \lim_{n\to\infty} \int g_n.$$

That is,

$$\int \left(\liminf_{n\to\infty} f_n\right) = \lim_{n\to\infty} \int g_n = \liminf_{n\to\infty} \int g_n \leq \liminf_{n\to\infty} \int f_n$$

where on the right we have used monotonicity of the integral. □

Exercise 2.8.3. Show that the theorem remains valid if its assumptions are changed from

1. "$f_n(x) \geq 0$ *for all* x" to "$f_n(x) \geq 0$ *a.e.*"
2. "$f_n(x) \geq 0$ *a.e.*" to "*there exists* $g \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$ *with* $f_n(x) \geq g(x)$ *a.e.*"

Theorem 2.8.5. (Lebesgue Dominated Convergence Theorem).
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{M}$-measurable functions. Suppose $\{f_n(x)\}$ converges (pointwise), and there exists $g \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Then f_n and $\lim\limits_{n\to\infty} f_n$ are integrable, and

$$\int (\lim_{n\to\infty} f_n)\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu. \tag{2.48}$$

Proof. Since

$$|f_n(x)| \leq g(x)$$

for all x, then also

$$|\lim_{n\to\infty} f_n(x)| \leq g(x)$$

for all x. By monotonicity of the integral,

$$\int |f_n| \leq \int g < \infty \qquad \text{and} \qquad \int \big|\lim_{n\to\infty} f_n\big| \leq \int g < \infty$$

which shows that f_n and $\lim_{n\to\infty} f_n$ are integrable.
1. Assume first that $f_n : X \to \mathbb{R}^*$. Then as $|f_n(x)| \leq g(x)$ for all x, we have that $f_n : X \to \mathbb{R}$, so by the above, $f_n \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$. Since

$$-g(x) \leq f_n(x) \leq g(x)$$

for all n and x, we obtain that

$$g + f_n \geq 0 \qquad \text{and} \qquad g - f_n \geq 0 \tag{2.49}$$

for all n. The left-hand inequality gives

$$\int g + \int \lim_{n\to\infty} f_n \underset{\text{thm 2.7.5}}{=} \int \lim_{n\to\infty} (g + f_n) \underset{\text{Fatou}}{\leq} \liminf_{n\to\infty} \int (g + f_n) = \int g + \liminf_{n\to\infty} \int f_n,$$

that is,

$$\int \lim_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Similarly, the right-hand inequality in (2.49) gives

$$\begin{aligned}\int g - \int \lim_{n\to\infty} f_n &\underset{\text{thm 2.7.5}}{=} \int \lim_{n\to\infty} (g - f_n) \underset{\text{Fatou}}{\leq} \liminf_{n\to\infty} \int (g - f_n) \\ &= \int g + \liminf_{n\to\infty} \Big(-\int f_n\Big) = \int g - \limsup_{n\to\infty} \int f_n,\end{aligned}$$

that is,

$$\limsup_{n\to\infty} \int f_n \leq \int \lim_{n\to\infty} f_n.$$

Combining the two inequalities we obtain that

$$\int \lim_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int \lim_{n\to\infty} f_n$$

which shows that $\lim_{n\to\infty} \int f_n$ exists, and

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n.$$

2. Now let $f_n : X \to \mathbb{C}$. Set $f(x) = \lim_{n\to\infty} f_n(x)$. By exercise 1.5.5,

$$\Re(f_n)(x) \to \Re(f)(x) \qquad \text{and} \qquad \Im(f_n)(x) \to \Im(f)(x).$$

Since $|\Re(f_n)(x)|, |\Im(f_n)(x)| \le |f_n(x)| \le g(x)$, we can apply part 1. to $\int \lim_{n\to\infty} \Re(f_n)$ and to $\int \lim_{n\to\infty} \Im(f_n)$ and obtain that

$$\int \lim_{n\to\infty} f_n = \int f \underset{\text{def.}}{=} \int \Re(f) + i \int \Im(f) \underset{\text{by 1.}}{=} \lim_{n\to\infty} \int \Re(f_n) + i \lim_{n\to\infty} \int \Im(f_n)$$
$$= \lim_{n\to\infty} \Big(\int \Re(f_n) + i \int \Im(f_n) \Big) = \lim_{n\to\infty} \int f_n.$$

This proves the theorem. □

Exercise 2.8.4. Show that the theorem remains valid if its assumptions are changed from "$\{f_n(x)\}$ *converges (pointwise) and* $|f_n(x)| \le g(x)$ *for all* x" to "$\{f_n(x)\}$ *converges a.e. and* $|f_n(x)| \le g(x)$ *a.e.*".

Exercise 2.8.5. Use exercise 2.8.4 to show:
Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C})$ such that $\sum_{n=1}^\infty f_n(x)$ exists *a.e.* on X, and such that $\sum_{n=1}^\infty \int |f_n| \, d\mu < \infty$. Then

$$\sum_{n=1}^\infty f_n \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C}) \qquad \text{and} \qquad \int \Big(\sum_{n=1}^\infty f_n \Big) \, du = \sum_{n=1}^\infty \int f_n \, d\mu.$$

Exercise 2.8.6. Let $f \in \mathcal{L}^+$, or $f \in \mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C})$, and let $\{A_k\}_{k=1}^\infty \subseteq \mathcal{M}$. Show:

1. If $\{A_k\}$ are disjoint, then

$$\int_{\bigcup_{k=1}^\infty A_k} f \, d\mu = \sum_{k=1}^\infty \int_{A_k} d\mu.$$

2. If $\{A_k\}\uparrow$, then

$$\int_{\bigcup_{k=1}^\infty A_k} f \, d\mu = \lim_{k\to\infty} \int_{A_k} d\mu.$$

We are now ready to complete example 2.4.5 of section 2.4.

Theorem 2.8.6. *Consider the semialgebra $\mathcal{S} = \mathcal{S}^n$ of "half-open" intervals in $\mathbb{R}^n$,*

$$\mathcal{S}^n := \{ E \subseteq \mathbb{R}^n \ : \ E = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n] \}$$

with $-\infty < a_j \leq b_j < \infty$. *Set*

$$\lambda(E) = \lambda^n(E) := (b_1 - a_1)(b_2 - a_2) \dots (b_n - a_n),$$

for $E \in \mathcal{S}$. Then λ^n is a premeasure on $\mathcal{S}^n$.

Proof. It is obvious that $\lambda^n(\emptyset) = 0$. Thus, we must only prove that λ^n is σ-additive. We proceed by induction on n.
1. n=1. This was already proved in example 2.4.4.
2. Suppose, we have shown that λ^1 and λ^n are premeasures on $\mathcal{S}^1$ and $\mathcal{S}^n$, respectively. Then as shown in section 2.4, λ^1 and λ^n extend to the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^n$, respectively, and we thus have the integration machinery in $\mathbb{R}$ and in $\mathbb{R}^n$ available.

Consider the semiring S^{n+1}. Let $E \in S^{n+1}$ be a countable disjoint union of elements of S^{n+1},

$$E = \bigcup_{j=1}^{\infty} E_j, \qquad E_j \in S^{n+1}, \ \{E_j\} \text{ disjoint.}$$

Then E and E_j are of the form

$$E = I \times (a, b] \qquad \text{and} \qquad E_j = I_j \times (a_i, b_j]$$

for some n-intervals $I, I_j \in S^n$. Since the sets $\{E_j\}$ are disjoint, we have

$$\chi_{I \times (a,b]} = \chi_E = \chi_{\cup_{j=1}^{\infty} E_j} = \sum_{j=1}^{\infty} \chi_{E_j} = \sum_{j=1}^{\infty} \chi_{I_j \times (a_j . b_j]}.$$

Write elements of $\mathbb{R}^{n+1}$ as pairs $(\vec{x}, t)$, with $\vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$. Then

$$\chi_I(\vec{x}) \chi_{(a,b]}(t) = \chi_{I \times (a,b]}(\vec{x}, t) = \sum_{j=1}^{\infty} \chi_{I_j \times (a_j . b_j]}(\vec{x}, t) = \sum_{j=1}^{\infty} \chi_{I_j}(\vec{x}) \chi_{(a_j . b_j]}(t).$$

For fixed $\vec{x}$, consider the functions $\varphi_k : \mathbb{R} \to [0, 1]$ consisting of the partial sums

$$\varphi_k(t) = \sum_{j=1}^{k} \chi_{I_j}(\vec{x}) \chi_{(a_j . b_j]}(t).$$

Then $\varphi_k \in \mathcal{S}^+(\mathbb{R})$, $\{\varphi_k\}\uparrow$ and

$$\varphi_k(t) \to \varphi(t) := \chi_I(\vec{x}) \chi_{(a,b]}(t)$$

for each t. Thus by the Monotone Convergence Theorem,

$$\int_{\mathbb{R}} \varphi_k \, d\lambda^1 \to \int_{\mathbb{R}} \varphi \, d\lambda^1 = \int_{\mathbb{R}} \chi_I(\vec{x})\chi_{(a,b]}(t) \, d\lambda^1,$$

and by monotonicity of the integral, $\{\int_{\mathbb{R}} \varphi_k \, d\lambda^1\}\uparrow$. That is,

$$\psi_k(\vec{x}) := \sum_{j=1}^{k} \chi_{I_j}(\vec{x})(b_j - a_j) \to \psi(\vec{x}) := \chi_I(\vec{x})(b - a)$$

and $\{\psi_k(\vec{x})\}\uparrow$. Applying the Monotone Convergence Theorem again, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \psi_k \, d\lambda^n = \int_{\mathbb{R}^n} \psi \, d\lambda^n,$$

that is

$$\sum_{j=1}^{\infty} \lambda^n(I_j)(b_j - a_j) = \lambda^n(I)(b - a)$$

or equivalently,

$$\sum_{j=1}^{\infty} \lambda^{n+1}(E_j) = \lambda^{n+1}(E).$$

This shows that λ^{n+1} is σ-additive, and proves the theorem. □

Chapter 3

Function Spaces

3.1 Spaces of Integrable Functions

Throughout, $(X, \mathcal{M}, \mu)$ will denote a measure space. All functions (real or complex valued) defined on X will be assumed to be $\mathcal{M}$-measurable; for simplicity we call such functions simply *measurable.*

Definition 3.1.1. For a fixed real number p, $1 \le p < \infty$, let

$$\mathcal{L}_p(X, \mathcal{M}, \mu) := \{f : X \to \mathbb{C} \,\big|\, f \text{ is } \mathcal{M}\text{-measurable, and } \int |f|^p \, d\mu < \infty\},$$

and for each $f \in \mathcal{L}_p$, set

$$\|f\|_p := \left[\int |f|^p \, d\mu\right]^{1/p}. \tag{3.1}$$

Definition 3.1.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. An $\mathcal{M}$- measurable function f is called *essentially bounded,* if there exists $M \ge 0$ so that

$$\mu\big(\{\, x \in X \;:\; |f(x)| > M \,\}\big) = 0.$$

Such a number M is called an *essential bound* of f. Set

$$\mathcal{L}_\infty(X, \mathcal{M}, \mu) := \{f : X \to \mathbb{C} \,\big|\, f \text{ is } \mu\text{-measurable, and essentially bounded}\},$$

and for each $f \in \mathcal{L}_\infty$, set

$$\|f\|_\infty = \text{ess-sup}\, f := \inf\{\, M : M \text{ is an essential bound of } f \,\}. \tag{3.2}$$

Example 3.1.1. Consider the measure space $([0,1], \mathcal{M}_{[0,1]}, \lambda)$.

1. If
$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$
then $f \in \mathcal{L}_p[0,1] \ \big(= \mathcal{L}_p([0,1], \mathcal{M}_{[0,1]}, \lambda)\big) \quad \Leftrightarrow \quad 1 \le p < 2.$

2. If
$$f(x) = \begin{cases} n & \text{if } x = \frac{1}{n} \\ 1 & \text{else} \end{cases}$$
then $f \in \mathcal{L}_p[0,1]$ for all $1 \le p \le \infty$, and in fact, $\|f\|_p = 1$, since $f(x) = 1$ a.e. Note that f is essentially bounded, but not bounded.

Remark 3.1.1. 1. For ease of notation, the spaces $\mathcal{L}_p(X, \mathcal{M}, \mu)$ are often denoted simply by $\mathcal{L}_p(X)$ or $\mathcal{L}_p$.

2. Sometimes, $\mathcal{L}_p(X, \mathcal{M}, \mu)$ is considered as a space of *real-valued* functions $f : X \to \mathbb{R}$. If we want to make this distinction explicit, we may use the notation $\mathcal{L}_p(X, \mathcal{M}, \mu; \mathbb{R})$ and $\mathcal{L}_p(X, \mathcal{M}, \mu; \mathbb{C})$, respectively.

   Since $\mathcal{L}_p(X, \mathcal{M}, \mu; \mathbb{R}) \subset \mathcal{L}_p(X, \mathcal{M}, \mu; \mathbb{C})$ we will always assume in this section that $\mathcal{L}_p(X, \mathcal{M}, \mu)$ consist of complex valued functions; all results apply to the case of real valued function spaces as well without modification of the proofs.

3. If $f \in \mathcal{L}_\infty$, then $\|f\|_\infty$ is an essential bound of f.
   In fact, note that every $M > \|f\|_\infty$ is an essential bound of f. (Why ?) So if we set
$$X_n := \left\{x \in X : |f(x)| > \|f\|_\infty + \frac{1}{n}\right\}, \qquad n = 1, 2, \ldots$$
   then $\mu(X_n) = 0$. Let
$$X_o := \{x \in X : |f(x)| > \|f\|_\infty\}.$$
   Then $X_o = \bigcup_{n=1}^\infty X_n$, and hence by sub-additivity, $\mu(X_o) = 0$ also. This shows that $\|f\|_\infty$ is an essential bound of f, in fact, by (3.2) it is the smallest essential bound of f.

4. Let f be an $\mathcal{M}$-measurable function defined on X, and $1 \le p < \infty$. Then
$$\|f\|_p = \left[\int |f|^p \, d\mu\right]^{1/p}$$
   is always defined in $\mathbb{R}^*$, possibly ∞. Thus,
$$f \in \mathcal{L}_p \quad \Leftrightarrow \quad \|f\|_p < \infty.$$
   Similarly,
$$f \in \mathcal{L}_\infty \quad \Leftrightarrow \quad \|f\|_\infty < \infty.$$

5. We obviously have
$$\|f\|_p^p = \int |f|^p \, d\mu = \|\, |f|^p \,\|_1 \qquad (1 \le p < \infty). \tag{3.3}$$

6. We have for $1 \le p < \infty$,
$$\|f\|_p = 0 \;\Leftrightarrow\; \int |f|^p \, d\mu = 0 \underset{\text{thm. 2.8.2}}{\Leftrightarrow} |f(x)|^p = 0 \text{ a.e.} \;\Leftrightarrow\; f(x) = 0 \text{ a.e.}$$
   Also
$$\|f\|_\infty = 0 \;\Leftrightarrow\; \mu(\{x \in X : |f(x)| > 0\}) = 0 \;\Leftrightarrow\; f(x) = 0 \text{ a.e.}$$

3.1.1 Hölder's Inequality

Definition 3.1.3. For $1 < p < \infty$, set

$$q = \frac{p}{p-1}.$$

q is called the *conjugate* of p.

Remark 3.1.2. 1. Obviously, $1 < q < \infty$.

2. It is easy to see that

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{3.4}$$

3. If $p = 1$, we set $q = \infty$. If $p = \infty$, we set $q = 1$. Then (3.4) still holds if we set $\frac{1}{\infty} = 0$.

Remark 3.1.3. 1. If $p = 2$ then $q = 2$. We say that $p = 2$ is *self-conjugate.*

2. Since $(p-1)q = p$ we have

$$\int |f|^p \, d\mu = \int |f|^{(p-1)q} \, d\mu = \int \left(|f|^{p-1}\right)^q \, d\mu.$$

Thus,

$$f \in \mathcal{L}_p(X) \quad \Leftrightarrow \quad |f|^{p-1} \in \mathcal{L}_q(X)$$

and also

$$\|f\|_p^p = \left\| \, |f|^{p-1} \, \right\|_q^q$$

for $1 < p < \infty$.

Lemma 3.1.1. *Let $1 < p < \infty$, and q its conjugate. Then for all $a, b \geq 0$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{3.5}$$

Proof. If $a = 0$ or $b = 0$, then (3.5) holds trivially, so we need to show that (3.5) holds for all $a > 0$ and $b > 0$.

Note that setting $u = a^p$ and $v = b^q$, inequality (3.5) is equivalent to

$$u^{1/p} v^{1/q} \leq \frac{u}{p} + \frac{v}{q},$$

or dividing by $v \neq 0$, to

$$\left(\frac{u}{v}\right)^{1/p} \leq \frac{1}{p}\frac{u}{v} + \frac{1}{q}$$

for all $u, v > 0$. Setting $t = \frac{u}{v}$, this in turn is equivalent to

$$t^{1/p} \leq \frac{t}{p} + \frac{1}{q} \tag{3.6}$$

for all $t > 0$.

Thus, we must show that (3.6) holds for all $t > 0$. For this, consider the function

$$f(t) = \frac{t}{p} + \frac{1}{q} - t^{1/p} \quad (t > 0).$$

Using elementary calculus (check!) one easily shows that $f(t)$ has an absolute minimum at $t = 1$, and $f(1) = 0$. Hence,

$$0 = f(1) \leq f(t) = \frac{t}{p} + \frac{1}{q} - t^{1/p}$$

for all $t > 0$, that is, (3.6) holds. This proves the lemma. □

Theorem 3.1.2. *(Hölder's Inequality; $p = 2$: Cauchy-Schwarz Inequality) Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathcal{L}_p(X, \mathcal{M}, \mu)$ and $g \in \mathcal{L}_q(X, \mathcal{M}, \mu)$, then $fg \in \mathcal{L}_1(X, \mathcal{M}, \mu)$ and*

$$\|fg\|_1 \;\leq\; \|f\|_p \, \|g\|_q. \tag{3.7}$$

Proof. We consider several cases.
Case 1: $1 < p < \infty$. Then $1 < q < \infty$ also. Let $f \in \mathcal{L}_p$ and $g \in \mathcal{L}_q$ be given.

1. If $\|f\|_p = 0 \underset{\text{rem. } 3.1.1}{\Rightarrow} f = 0$ a.e. $\Rightarrow fg = 0$ a.e. $\underset{\text{rem. } 3.1.1}{\Rightarrow} \|fg\|_1 = 0$, so that $fg \in \mathcal{L}_1$ and (3.7) holds.

2. Similarly, if $\|g\|_q = 0$, then $fg \in \mathcal{L}_1$ and (3.7) holds.

3. If $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$ we can apply lemma 3.1.1 to $a = \frac{|f(x)|}{\|f\|_p}$ and $b = \frac{|g(x)|}{\|g\|_q}$ and obtain
$$\frac{|f(x)|}{\|f\|_p} \, \frac{|g(x)|}{\|g\|_q} \;\leq\; \frac{1}{p} \, \frac{|f(x)|^p}{\|f\|_p^p} \;+\; \frac{1}{q} \, \frac{|g(x)|^q}{\|g\|_q^q}$$
for all $x \in X$. We integrate both sides and obtain by linearity and monotonicity of the integral,
$$\begin{aligned} \frac{1}{\|f\|_p \, \|g\|_q} \int |f(x)g(x)| \, d\mu \;&\leq \frac{1}{p} \, \frac{1}{\|f\|_p^p} \int |f(x)|^p \, d\mu \;+\; \frac{1}{q} \, \frac{1}{\|g\|_q^q} \int |g(x)|^q \, d\mu \\ &= \frac{1}{p} \, \frac{1}{\|f\|_p^p} \, \|f\|_p^p \;+\; \frac{1}{q} \, \frac{1}{\|g\|_q^q} \, \|g\|_q^q \;=\; 1. \end{aligned}$$
Multiply by $\|f\|_p \, \|g\|_q$,
$$\int |f(x)g(x)| \, d\mu \;\leq\; \|f\|_p \, \|g\|_q.$$
This shows that $fg \in \mathcal{L}_1$ and (3.7) holds.

Case 2: $p = 1$ and $q = \infty$. Let $f \in \mathcal{L}_1$ and $g \in \mathcal{L}_\infty$ be given. Since $\|g\|_\infty$ is an essential bound for g, then

$$|g(x)| \leq \|g\|_\infty \text{ a.e.}$$

and hence

$$|f(x)g(x)| \;=\; |f(x)|\,|g(x)| \;\leq\; |f(x)|\,\|g\|_\infty \text{ a.e.}$$

By linearity and monotonicity of the integral in $\mathcal{L}^+$,,

$$\int |f(x)g(x)|\,d\mu \leq \int |f(x)|\,\|g\|_\infty \underset{\text{thm 2.7.5}}{=} \|g\|_\infty \int |f(x)|\,d\mu = \|f\|_1\,\|g\|_\infty.$$

Thus, $fg \in \mathcal{L}_1$ and (3.7) holds.

Case 3: $p = \infty$ and $q = 1$. This follows from case 2 by symmetry. □

3.1.2 Minkowski's Inequality

Theorem 3.1.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \leq p \leq \infty$. then*

1. *$\mathcal{L}_p(X, \mathcal{M}, \mu)$ is a vector space,*
2. *$\|\cdot\|_p$ is a semi-norm on $\mathcal{L}_p(X, \mathcal{M}, \mu)$. In particular,*

$$\|f+g\|_p \;\leq\; \|f\|_P + \|g\|_p \qquad \text{("Minkowski's Inequality")}$$

for all $f, g \in \mathcal{L}_p(X, \mathcal{M}, \mu)$.

Proof. Since $\mathcal{L}_p(X, \mathcal{M}, \mu)$ is a subset of the vector space V_X discussed on page 5, in order to prove 1 we only need to show that $\mathcal{L}_p(X, \mathcal{M}, \mu)$ is closed under vector space operations. Furthermore, it is obvious that $\|f\|_p \geq 0$ for all $f \in \mathcal{L}_p$ and all p, so in order to prove 2 we only need to show that $\|\cdot\|_p$ is positive homogeneous and that the triangle inequality holds. We separate again into three cases.

Case 1: $p = 1$.

Note that $\mathcal{L}_1(X, \mathcal{M}, \mu)$ coincides with the space $\mathcal{L}(X, \mathcal{M}, \mu; \mathbb{R})$, respectively $\mathcal{L}(X, \mathcal{M}, \mu; \mathbb{C})$, hence by theorems 2.7.5 and 2.7.6 is a real, respectively complex, vector space. Thus we only need to show that $\|\cdot\|_1$ is a seminorm.

Let $f, g \in \mathcal{L}_1$ and α be scalar. Then by linearity and monotonicity of the integral,

$$\text{i)} \qquad \|\alpha f\|_1 = \int |\alpha f|\,d\mu = \int |\alpha|\,|f|\,d\mu = |\alpha| \int |f|\,d\mu = |\alpha|\,\|f\|_1$$

$$\begin{aligned} \text{ii)} \qquad \|f+g\|_1 &= \int |f+g|\,d\mu \leq \int (|f| + |g|)\,d\mu \\ &= \int |f|\,d\mu + \int |g|\,d\mu = \|f\|_1 + \|g\|_1. \end{aligned}$$

This proves the theorem in case $p = 1$.

Case 2: $1 < p < \infty$. Let $f, g \in \mathcal{L}_p$ and α be scalar. Then

i) By homogeneity of the integral,

$$\int |\alpha f|^p \, d\mu = |\alpha|^p \int |f|^p \, d\mu = |\alpha|^p \, \|f\|_p^p < \infty.$$

This shows that $\alpha f \in \mathcal{L}_p$ and taking p-th roots, that

$$\|\alpha f\|_p = |\alpha| \, \|f\|_p. \tag{3.8}$$

ii) Since the function $t \longrightarrow t^p$ is increasing on $[0, \infty)$, we have

$$\begin{aligned} |f(x) + g(x)|^p &\le \left[2 \max\{ |f(x)|, |g(x)| \} \right]^p \\ &= 2^p \max\{ |f(x)|^p, |g(x)|^p \} \le 2^p \left[|f(x)|^p + |g(x)|^p \right] \end{aligned}$$

for all $x \in X$. Thus, by monotonicity of the integral,

$$\begin{aligned} \int |f + g|^p \, d\mu &\le \int 2^p \left[|f|^p + |g|^p \right] d\mu \\ &= 2^p \left[\int |f|^p \, d\mu + \int |g|^p \, d\mu \right] < \infty. \end{aligned}$$

Hence, $f + g \in \mathcal{L}_p$.

This shows that $\mathcal{L}_p$ is a vector space. Next we show that $\| \cdot \|_p$ is a seminorm. By (3.8) we only need to prove that the triangle inequality holds. Using monotonicity and linearity of the integral together with Hölder's inequality, we have

$$\begin{aligned} \|f + g\|_p^p &= \int |f + g|^p \, d\mu = \int |f + g| \, |f + g|^{p-1} \, d\mu \\ &\le \int \left[|f| \, |f + g|^{p-1} + |g| \, |f + g|^{p-1} \right] d\mu \\ &= \int \underbrace{|f|}_{\in \mathcal{L}_p} \; \underbrace{|f + g|^{p-1}}_{\in \mathcal{L}_q \text{ (rem. 3.1.3)}} \, d\mu + \int \underbrace{|g|}_{\in \mathcal{L}_p} \; \underbrace{|f + g|^{p-1}}_{\in \mathcal{L}_q \text{ (rem. 3.1.3)}} \, d\mu \\ &\underset{\text{Hölder}}{\le} \|f\|_p \, \| \, |f + g|^{p-1} \|_q + \|g\|_p \, \| \, |f + g|^{p-1} \|_q \\ &= \left(\|f\|_q + \|g\|_q \right) \|f + g\|_p^{p/q} \end{aligned}$$

where in the last line we have used the fact that $\| \, |f + g|^{p-1} \|_q^q = \|f + g\|_p^p$; see remark 3.1.3. Thus,

$$\|f + g\|_p^{p - p/q} \le \|f\|_p + \|g\|_p$$

(Even if $\|f + g\|_p = 0$!) But $p - \frac{p}{q} = p(1 - \frac{1}{q}) = p\frac{1}{p} = 1$, so we obtain

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

This shows that $\| \cdot \|_p$ is a seminorm.

Case 3: $p = \infty$. Let $f, g \in \mathcal{L}_\infty$ and α scalar.We set

$$N_1 := \{ x \in X : |f(x)| > \|f\|_\infty \} \quad \text{and} \quad N_2 := \{ x \in X : |g(x)| > \|g\|_\infty \}$$

so that $\mu(N_1) = \mu(N_2) = 0$.

i) If $x \notin N_1$ then

$$|(\alpha f)(x)| = |\alpha|\,|f(x)| \le |\alpha|\,\|f\|_\infty$$

which shows that $|\alpha|\,\|f\|_\infty$ is an essential bound for αf. It follows that $\alpha f \in \mathcal{L}_\infty$, and

$$\|\alpha f\|_\infty \le |\alpha|\,\|f\|_\infty.$$

Arguing as in case 2, from here we obtain easily that

$$\|\alpha f\|_\infty = |\alpha|\,\|f\|_\infty.$$

ii) If $x \notin N_1 \cup N_2$, then

$$|(f+g)(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty$$

which shows that $\|f\|_\infty + \|g\|_\infty$ is an essential bound for $f+g$. It follows that $f+g \in \mathcal{L}_\infty$ and

$$\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$$

This proves the theorem. □

In general, $\|\cdot\|_p$ is not a norm on $\mathcal{L}_p$. In fact, we have already seen in remark 3.1.1 that

$$\|f\|_p = 0 \quad \Leftrightarrow \quad f = 0 \text{ a.e.}$$

We thus need to "reduce" our space as follows:

Exercise 3.1.1. For fixed p, define a relation $\sim$ on $\mathcal{L}_p$ by

$$f \sim g \quad \Leftrightarrow \quad f = g \text{ a.e.}$$

1. Show that $\sim$ is an equivalence relation on $\mathcal{L}_p$.

2. Given $f \in \mathcal{L}_p$, let $[f]$ denote its equivalence class, and set set

$$L^p(X, \mathcal{M}, \mu) = L_p(X, \mathcal{M}, \mu) := \{\ [f] : f \in \mathcal{L}_p(X, \mathcal{M}, \mu)\ \}.$$

   Show that the following operations on $L^p(X, \mathcal{M}, \mu)$ are well defined:

   (a) $[f] + [g] := [f+g]$
   (b) $\alpha[f] := [\alpha f]$
   (c) $\|\,[f]\,\|_p := \|f\|_p$

   where $f, g \in \mathcal{L}_p$ and α is scalar.

3. Show that $L^p(X, \mathcal{M}, \mu)$ is a vector space with the above vector space operations.

4. Show that $\|\,[f]\,\|_p$ is a norm on $L^p(X, \mathcal{M}, \mu)$.

Remark 3.1.4. One often confuses the spaces $L^p(X, \mathcal{M}, \mu)$ and $\mathcal{L}_p(X, \mathcal{M}, \mu)$, and considers the elements in $\mathcal{L}_p(X, \mathcal{M}, \mu)$ as functions.

For example, when on writes "let f be a function in L^p" one really means "let f be a representative in $\mathcal{L}_p$ of the equivalence class $[f] \in L^p$.

3.1.3 Completeness of L^p

Theorem 3.1.4. *The spaces $L^p(X, \mathcal{M}, \mu)$ are Banach spaces for $1 \leq p \leq \infty$.*

Proof. The idea is to show that very absolutely convergent series in L^p converges. The case $p = \infty$ needs to be treated separately.

Case 1: $1 \leq p < \infty$. For this, let $\{f_k\} \subseteq L^p$ be such that $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$. We proceed as follows:

1. Show that $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise a.e. on X.
2. Show that $f(x) := \sum_{k=1}^{\infty} f_k(x) \in L^p$.
3. Show that $\sum_{k=1}^{n} f_k \overset{\|\cdot\|_p}{\to} f$ as $n \to \infty$.

1. For each $n \in \mathbb{N}$, set

$$g_n(x) := \left(\sum_{k=1}^{n} |f_k(x)|\right)^p \qquad \text{and set} \qquad g(x) := \left(\sum_{k=1}^{\infty} |f_k(x)|\right)^p$$

Then $g_n, g : X \to [0, \infty]$ are measurable (as sums, limits and powers of measurable functions are measurable), $\{g_n\}\uparrow$ and $g_n(x) \to g(x)$ for all $x \in X$. Note that

$$\begin{aligned}\int g_n \, d\mu &= \int \left(\sum_{k=1}^{n} |f_k|\right)^p d\mu = \left[\left\|\sum_{k=1}^{n} |f_k|\right\|_p\right]^p \\ &\underset{\text{Minkowski}}{\leq} \left[\sum_{k=1}^{n} \big\| |f_k| \big\|_p\right]^p \leq \left[\sum_{k=1}^{\infty} \|f_k\|_p\right]^p < \infty\end{aligned}$$

by assumption. Applying the Monotone Convergence Theorem, then

$$\int g \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu \underset{\text{MCT}}{=} \lim_{n\to\infty} \int g_n \, d\mu \leq \left[\sum_{k=1}^{\infty} \|f_k\|_p\right]^p < \infty.$$

Thus, g is integrable, and in particular, $g(x)$ is finite valued a.e. Then by definition of g,

$$\sum_{k=1}^{\infty} |f_k(x)|$$

is finite a.e., that is, there exists a set $N \in \mathcal{M}$ such that

1. $\mu(N) = 0$, and
2. $\sum\limits_{k=1}^{\infty} |f_k(x)|$ converges for each $x \in X \backslash N$.

However, every absolutely convergent series in $\mathbb{R}$ (respectively $\mathbb{C}$) is convergent, thus $\sum_{k=1}^{\infty} f_k(x)$ converges for each $x \in X \backslash N$.

2. We can thus set

$$f(x) \; := \; \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if } x \notin N \\ 0 & \text{if } x \in N. \end{cases}$$

Then f is measurable, and

$$|f(x)|^p \; \leq \; \begin{cases} \left(\sum_{k=1}^{\infty} |f_k(x)| \right)^p & = g(x) \quad \text{if } x \notin N \\ 0 & \leq g(x) \quad \text{if } x \in N. \end{cases}$$

Since g is integrable, it follows that $|f|^p$ is integrable as well, that is, $f \in L^p(X, \mathcal{M}, \mu)$.

3. Now for every $x \notin N$ we have by definition of f that

$$\lim_{n\to\infty} \Big| f(x) - \sum_{k=1}^{n} f_k(x) \Big|^p \; = \; \left[\lim_{n\to\infty} \Big| f(x) - \sum_{k=1}^{n} f_k(x) \Big| \right]^p \; = \; 0^p \; = \; 0 \qquad (3.9)$$

while also

$$\Big| f(x) - \sum_{k=1}^{n} f_k(x) \Big|^p \; = \; \Big| \sum_{k=n+1}^{\infty} f_k(x) \Big|^p \; \leq \; \Big[\sum_{k=1}^{\infty} |f_k(x)| \Big]^p \; = g(x).$$

As g is integrable, we can apply the Lebesgue Dominated Convergence Theorem to obtain that

$$\begin{aligned} \lim_{n\to\infty} \Big\| f - \sum_{k=1}^{n} f_k \Big\|_p^p \; &= \; \lim_{n\to\infty} \int \Big| f(x) - \sum_{k=1}^{n} f_k(x) \Big|^p d\mu \\ &\underset{\text{LDCT}}{=} \int \lim_{n\to\infty} \Big| f(x) - \sum_{k=1}^{n} f_k(x) \Big|^p d\mu \underset{\text{by (3.9)}}{=} \int 0 \, d\mu \; = \; 0 \end{aligned}$$

and thus taking p-th roots,

$$\lim_{n\to\infty} \Big\| f - \sum_{k=1}^{n} f_k \Big\|_p \; = \; 0.$$

This shows that

$$\sum_{k=1}^{n} f_k \overset{\|\cdot\|_p}{\longrightarrow} f \quad \text{as } n \to \infty.$$

We have shown that every absolutely convergent series converges in L^p; hence L^p is a Banach space by theorem 1.5.8.

Case 2: $p = \infty$. Again, let $\{f_k\} \subseteq L^\infty$ be such that $\sum_{k=1}^\infty \|f_k\|_\infty < \infty$.
1. For each k, set

$$N_k := \{ \, x \in X : |f_k(x)| > \|f_k\|_\infty \, \}$$

and set

$$N := \bigcup_{k=1}^\infty N_k.$$

Then $\mu(N) = 0$. Furthermore, if $x \in X \backslash N$, then $|f_k(x)| \leq \|f_k\|_\infty$ for all k, and hence

$$\sum_{k=1}^\infty |f_k(x)| \; \leq \; \sum_{k=1}^\infty \|f_k\|_\infty < \infty \tag{3.10}$$

which shows that $\sum\limits_{k=1}^\infty f_k(x)$ converges in $\mathbb{R}$ (resp. $\mathbb{C}$.)
2. We thus can set

$$f(x) \; := \; \begin{cases} \sum\limits_{k=1}^\infty f_k(x) & \text{if } x \in X\backslash N \\ 0 & \text{if } x \in N. \end{cases}$$

Then f is measurable, and

$$|f(x)| \; \leq \; \sum_{k=1}^\infty \|f_k\|_\infty$$

for all $x \in X\backslash N$. This shows that $f \in L^\infty$.
3. Finally, if $x \in X\backslash N$ then

$$\Big| \, f(x) - \sum_{k=1}^n f_k(x) \Big| \; = \; \Big| \sum_{k=n+1}^\infty f_k(x) \Big| \; \leq \; \sum_{k=n+1}^\infty |f_k(x)| \; \leq \; \sum_{k=n+1}^\infty \|f_k\|_\infty$$

so that $\sum_{k=n+1}^\infty \|f_k\|_\infty$ is an essential bound for $\big| \, f(x) - \sum_{k=1}^n f_k(x) \, \big|$. Thus,

$$\Big\| f - \sum_{k=1}^n f_k \Big\|_\infty \; = \; \text{ess-sup} \Big| \, f(x) - \sum_{k=1}^n f_n(x) \Big| \; \leq \; \sum_{k=n+1}^\infty \|f_k\|_\infty \; \to \; 0$$

as $n \to \infty$. This shows that

$$\sum_{k=1}^n f_k \overset{\|\cdot\|_\infty}{\longrightarrow} f \quad \text{as } n \to \infty.$$

Applying theorem 1.5.8 again, it follows that L^∞ is a Banach space. $\square$

3.2 Approximation Theorems

3.2.1 Convergence in the Mean and Pointwise Convergence

Let $\{h_n\}_{n=1}^{\infty}$ be as sequence in $L^p(X, \mathcal{M}, \mu)$. We have seen two ways in which this sequence may converge to a measurable function h:

1. *Convergence almost everywhere.*

$$h_n \to h \text{ a.e.} \quad \Leftrightarrow \quad \exists N \in \mathcal{M},\ \mu(N) = 0 \text{ s.t. } h_n(x) \to h(x)\ \ \forall x \in X \backslash N.$$

(Note that because elements of $L^p(X, \mathcal{M}, \mu)$ are equivalence classes of functions which are equal almost everywhere, it does not make sense to talk about *everywhere convergence* here.)

2. *Convergence in the p-mean.* If $h \in L^p(X, \mathcal{M}, \mu)$, then

$$h_n \xrightarrow{\|\cdot\|_p} h \quad \Leftrightarrow \quad \lim_{n\to\infty} \|h_n - h\|_p = 0.$$

In general, both types of convergence are not equivalent:

Example 3.2.1. Consider the space $L^p[0,1] = L^p([0,1], \mathcal{M}_{[0,1]}, \lambda)$, with $1 \le p < \infty$.

a) Let

$$h_n(x) = n\chi_{[0,\frac{1}{n}]}(x) = \begin{cases} n & \text{if } 0 \le x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x \le 1 \end{cases} \in L^p[0,1].$$

Then $h_n(x) \to h(x) = 0$ a.e., while

$$\|h_n - h\|_p = \|n\chi_{[0,\frac{1}{n}]}\|_p = \left[\int n^p \chi_{[0,\frac{1}{n}]}\, d\lambda\right]^{1/p} = [n^{p-1}]^{1/p} = n^{\frac{p-1}{p}} \not\to 0.$$

b) Let

$$h_n(x) = \frac{1}{x}\, \chi_{[\frac{1}{n},1]} = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{n} \\ \frac{1}{x} & \text{if } \frac{1}{n} \le x \le 1 \end{cases} \in L^p[0,1].$$

Then $h_n(x) \to h(x) = \frac{1}{x}$ a.e.; however, $h \notin L^p[0,1]$.

c) Observe that every $n \in N$ can be written uniquely as

$$n = 2^k + m \quad \text{with} \qquad k, m \in \mathbb{N} \cup \{0\}, \quad 0 \le k < \infty, \quad 0 \le m < 2^k.$$

Given $n \in \mathbb{N}$, set

$$h_n(x) = \begin{cases} 1 & \text{if } x \in (\frac{m}{2^k}, \frac{m+1}{2^k}] \\ 0 & \text{else,} \end{cases}$$

and let $h(x) = 0$. Then

$$\|h_n - h\|_p^p = \|\chi_{(\frac{m}{2^k},\frac{m+1}{2^k}]}\|_p^p = \int_{[0,1]} \left[\chi_{(\frac{m}{2^k},\frac{m+1}{2^k}]}\right]^p d\lambda = \frac{1}{2^k} \to 0$$

Figure 3.1: h_n converges in the p-mean, but not pointwise a.e.

as $n \to \infty$. That is, $h_n \xrightarrow{\|\cdot\|_p} h$. However, if $x \in (0,1]$ then $h_n(x) = 1$ for infinitely many values of n, and hence

$$h_n(x) \not\to h(x) \text{ a.e.}$$

Note: While $\{h_n\}$ does not converge to $h(x) = 0$ a.e., there are however many subsequences of $\{h_n\}$ converging to 0. For example, consider the subsequence $\{h_{2^k}\}_{k=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$. Then

$$h_{2^k}(x) = \begin{cases} 1 & \text{if } x \in (0, \frac{1}{2^k}] \\ 0 & \text{else.} \end{cases}$$

Given $x \in (0,1]$ is arbitrary, pick K such that $\frac{1}{K} < x$. Then for all $k \geq K$ we have $h_{2^k}(x) = 0$, and hence $h_{2^k}(x) \to 0$. Thus,

$$h_{2^k}(x) \to 0 \text{ a.e.}$$

The last remark in c) is true in general, and is a consequence of the proofs of theorems 1.5.8 and 3.1.4:

Theorem 3.2.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \leq p < \infty$. If $\{h_n\}_{n=1}^{\infty} \subseteq L^p(X, \mathcal{M}, \mu)$, $h \in L^p(X, \mathcal{M}, \mu)$ are such that $h_n \xrightarrow{\|\cdot\|_p} h$, then there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that*

$$h_{n_k}(x) \to h(x) \qquad a.e.$$

Proof. Since $\{h_n\}$ converges, it is Cauchy in $L^p(X, \mathcal{M}, \mu)$. As shown in the proof of the $\Rightarrow$ part of theorem 1.5.8, we can pick a subsequence $\{h_{n_k}\}$ such that

$$\|h_{n_{k+1}} - h_{n_k}\|_p < \frac{1}{2^k}$$

for $k = 1, 2, \ldots$. Next consider the telescoping series

$$\sum_{k=1}^{\infty} f_k \qquad \text{where} \qquad f_k = h_{n_{k+1}} - h_{n_k}.$$

Proof. Since

$$|h_n(x)|^p \le g(x)^p \quad a.e.$$

then also

$$|h(x)|^p \le g(x)^p \quad a.e.$$

By monotonicity of the integral,

$$\int |h_n|^p \, d\mu \le \int g^p \, d\mu < \infty \qquad \text{and} \qquad \int |h|^p \, d\mu \le \int g^p \, d\mu < \infty.$$

That is, $h_n \in L^p(X, \mathcal{M}, \mu)$ and $h \in L^p(X, \mathcal{M}, \mu)$.

Now for almost all $x \in X$,

$$|h_n(x) - h(x)|^p \le \left[\, |h_n(x)| + |h(x)| \,\right]^p \le \left[2g(x) \right]^p = 2^p g(x)^p \in L^1(X, \mathcal{M}, \mu).$$

It follows from the Lebesgue Dominated Convergence Theorem (see exercise 2.8.4) that

$$\lim_{n\to\infty} \|h_n - h\|_p^p = \lim_{n\to\infty} \int |h_n - h|^p \, d\mu \underset{\text{LDCT}}{=} \int \lim_{n\to\infty} |h_n - h|^p \, d\mu = \int 0 \, d\mu = 0.$$

That is, $h_n \xrightarrow{\|\cdot\|_p} h$. □

The statement of the above theorem is not true if $p = \infty$, even if X is a finite measure space:

Exercise 3.2.2. Find a sequence $\{h_n\}_{n=1}^\infty$ in $L^\infty[0,1]$ such that $h_n(x) \to 0$ a.e., and

a) $\left\{\|h_n\|_\infty\right\}_{n=1}^\infty$ is bounded, but $h_n \not\to 0$ in the norm $\|\cdot\|_\infty$, or

b) $\|h_n\|_\infty \to \infty$.

Corollary 3.2.3. *(Uniqueness of limit under different types of convergence)*
Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p \le \infty$. Suppose, $\{h_n\}_{n=1}^\infty \subseteq L^p(X, \mathcal{M}, \mu)$, $h \in L^p(X, \mathcal{M}, \mu)$ and $g : X \to \mathbb{C}$ are such that

$$h_n \xrightarrow{\|\cdot\|_p} h$$

and

$$h_n(x) \longrightarrow g(x) \quad a.e.$$

Then $g(x) = h(x)$ a.e.

Proof. Since $h_n \xrightarrow{\|\cdot\|_p} h$, by theorem 3.2.1 (or exercise 3.2.1 in case $p = \infty$) there exist a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ and a a set N_1 of measure zero such that

$$h_{n_k}(x) \to h(x)$$

for all $x \in X \backslash N_1$. On the other hand, since $h_n(x) \longrightarrow g(x)$ *a.e.*, there exists a set N_2 of measure zero such that

$$h_{n_k}(x) \to g(x)$$

for all $x \in X \backslash N_2$. By uniqueness of limits (in $\mathbb{C}$, resp. $\mathbb{R}$),

$$h(x) = g(x) \quad \text{for all } x \in X \backslash (N_1 \cup N_2).$$

That is, $g(x) = h(x)$ a.e. □

Exercise 3.2.3. Show: If $\mu(X) < \infty$, then

$$L^q(X, \mathcal{M}, \mu) \subseteq L^p(X, \mathcal{M}, \mu)$$

for all $1 \le p \le q \le \infty$. Furthermore, there exists a constant k (depending on p and q) such that

$$\|\cdot\|_p \le k\,\|\cdot\|_q.$$

Remark 3.2.1. If I is an interval, we usually use the notation

$$L^p(I) := L^p(I, \mathcal{M}_I, \lambda).$$

The above exercise shows that

$$L^q[a,b] \subseteq L^p[a,b]$$

whenever $1 \le p \le q \le \infty$.

Exercise 3.2.4. Let $1 \le p < q \le \infty$. Show that there exist $f \in L^p(\mathbb{R})$ such that $f \notin L^q(\mathbb{R})$, and conversely, that there exists $g \in L^q(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$. That is, neither the inclusion $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$, nor the reverse inclusion $L^q(\mathbb{R}) \subseteq L^p(\mathbb{R})$ hold.

Exercise 3.2.5. Let $X = \{1, 2, \ldots, n\}$ and μ the counting measure on $\mathcal{P}(X)$.

1. Find all $\mathcal{M}$-measurable functions $f : X \to \mathbb{R}$.
2. Find a simple formula for $\int f \, d\mu$.
3. Show: There exists an isometric isomorphism of $L^p(X, \mathcal{P}(X), \mu; \mathbb{R})$ onto
   (a) $(\mathbb{R}^n, \|\cdot\|_2)$ if $p = 2$,
   (b) $(\mathbb{R}^n, \|\cdot\|_\infty)$ if $p = \infty$,
   (c) $(\mathbb{R}^n, \|\cdot\|_1)$ if $p = 1$.

Exercise 3.2.6. Let $X = \mathbb{N}$ and μ the counting measure on $\mathcal{P}(\mathbb{N})$. Set

$$\ell^p := L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu; \mathbb{C}).$$

1. Find all $\mathcal{M}$-measurable functions $f : \mathbb{N} \to \mathbb{C}$.
2. Find a simple formula for $\int f \, d\mu$.
3. Show that
   (a) $\ell^p = \left\{ f : \mathbb{N} \to \mathbb{C} : \sum_{n=1}^{\infty} |f(n)|^p < \infty \right\}$ and $\|f\|_p = \left[\sum_{n=1}^{\infty} |f(n)|^p \right]^{1/p}$ for $1 \le p < \infty$.
   (b) $\ell^\infty = \left\{ f : \mathbb{N} \to \mathbb{C} : f \text{ is bounded} \right\}$ and $\|f\|_\infty = \sup_{n \in \mathbb{N}} |f(n)|$.

(Note that setting $x_n = f(x)$ we have

$$\ell^p = \left\{ \{x_n\}_{n=1}^{\infty} \subseteq \mathbb{C} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}, \quad \|\{x_n\}\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p}$$
$$\ell^\infty = \left\{ \{x_n\}_{n=1}^{\infty} \subseteq \mathbb{C} : \{x_n\} \text{ is bounded} \right\}, \quad \|\{x_n\}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$$

Thus, ℓ^∞ and ℓ^1 coincide with the spaces discussed in example 1.1.4 and exercise 1.1.5, respectively.)

4. Show that $\ell^p \subset \ell^q$ for all $1 \le p < q \le \infty$, and this inclusion is proper.

Exercise 3.2.7. Consider the measure space $\big((0,\infty), \mathcal{M}_{(0,\infty)}, \lambda\big)$.

1. Set
$$g(x) = \frac{1}{\sqrt{x}\,(1 + |\log x|)} \qquad (0 < x < \infty).$$
Show that $g \in L^2(0,\infty)$, but $g \notin L^p(0,\infty)$ for $p \neq 2$.

2. Use part 1. to show that for every p, $1 \le p \le \infty$, there exists a function $f \in L^p(0,\infty)$ such that $f \notin L^r(0,\infty)$ whenever $r \neq p$, $1 \le r \le \infty$.

Exercise 3.2.8. Show: If $\mu(X) < \infty$, then

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty \qquad \forall f \in L^\infty(X, \mathcal{M}, \mu).$$

3.2.2 Approximation by Simple Functions

Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \le p \le \infty$. Denote

$$\mathcal{S} := \{\varphi : X \to \mathbb{C} \text{ (resp.}\mathbb{R}) \mid f \text{ is simple and } \mathcal{M}\text{-measurable.}\}$$

Given $\varphi \in \mathcal{S}$, let

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$$

be any representation with $a_k \neq 0$ and $A_k \in \mathcal{M}$ for all k, and $A_i \cap A_k = \emptyset$ for $i \neq k$. It is now easy to see that

1. If $1 \le p < \infty$, then $\varphi \in L^p(X, \mathcal{M}, \mu) \quad \Leftrightarrow \quad \mu(A_k) < \infty \; \forall k$.

2. $\mathcal{S} \subseteq L^\infty(X, \mathcal{M}, \mu)$.

Note that $\mathcal{S} \cap L^p(X, \mathcal{M}, \mu)$ is a vector space.

Theorem 3.2.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 \le p \le \infty$. Then $\mathcal{S} \cap L^p(X, \mathcal{M}, \mu)$ is dense in $L^p(X, \mathcal{M}, \mu)$.*

Proof. Let $f \in L^p(X, \mathcal{M}, \mu)$. We must find a sequence $\{\varphi_n\} \subseteq \mathcal{S} \cap L^P(X, \mathcal{M}, \mu)$ such that $\|\varphi_n - f\|_p \to 0$.

Case 1: $1 \leq p < \infty$.

1. Suppose first that f is real valued. Then by the Structure Theorem for Measurable Functions, there exists a sequence $\{\varphi_n\} \subseteq \mathcal{S}$ of real valued simple, measurable functions, $0 \leq |\varphi_n| \leq |f|$, such that $\varphi_n(x) \to f(x)$ for all X. As $|f| \in L^p(X, \mathcal{M}, \mu)$, it follows from theorem 3.2.2 that

$$\lim_{n\to\infty} \|\varphi_n - f\|_p^p = 0.$$

2. If f is complex valued, write $f = \Re(f) + \Im(f)$. Note that

$$|\Re(f)|^p \leq |f|^p \qquad \text{and} \qquad |\Im(f)|^p \leq |f|^p$$

and thus $\Re(f), \Im(f) \in L^p(X, \mathcal{M}, \mu; \mathbb{R})$. Now by the first part, there exist sequences $\{\varphi_n\}, \{\psi_n\} \subseteq \mathcal{S} \cap L^p(X, \mathcal{M}, \mu)$ such that

$$\lim_{n\to\infty} \|\varphi_n^i - f\|_p = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|\psi_n^i - f\|_p = 0.$$

Set

$$\phi_n := \varphi_n + i\psi_n \in \mathcal{S} \cap L^p(X, \mathcal{M}, \mu).$$

Then by the triangle inequality,

$$\begin{aligned} \lim_{n\to\infty} \|\phi_n - f\|_p &= \lim_{n\to\infty} \| [\varphi_n + i\psi_n] - [\Re(f) + i\Im(f)] \|_p \\ &\leq \lim_{n\to\infty} \big[\|\varphi_n - \Re(f)\|_p + \|\psi_n - \Im(f)\|_p \big] = 0. \end{aligned}$$

Case 2: $p = \infty$. Then $f \in L^\infty(X, \mathcal{M}, \mu)$.

1. Suppose first that f is real valued. Set

$$N := \{x \in X : f(x) > \|f\|_\infty\}.$$

Then $\mu(N) = 0$.

Now given $\epsilon > 0$, pick real numbers $a_0 < a_1 < \cdots < a_n$ such that

(a) $\big[-\|f\|_\infty, \|f\|_\infty\big] \subset (a_0, a_n]$, and

(b) $a_i - a_{i-1} < \epsilon, \quad (i = 1 \ldots n)$,

and set

$$A_i := f^{-1}(a_{i-1}, a_i] \in \mathcal{M} \qquad (i = 1 \ldots n).$$

Then $\{A_i\}_{i=1}^n$ is a disjoint collection. Set

$$\varphi_\epsilon := \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{S}.$$

If $x \in X \backslash N$, then there exists a unique i such that $a_{i-1} < f(x) \le a_i$, and thus $x \in A_i$ and $\varphi_\epsilon(x) = a_i$. Hence,

$$|\varphi_\epsilon(x) - f(x)| = |a_i - f(x)| = a_i - f(x) < a_i - a_{i-1} < \epsilon$$

for all $x \in X \backslash N$, so that

$$\|\varphi_\epsilon - f\|_\infty < \epsilon.$$

Choosing $\epsilon = \frac{1}{n}$, $(n = 1, 2, \dots)$ we thus obtain a sequence $\{\varphi_n\} \subseteq \mathcal{S}$ such that $\|\varphi_n - f\|_\infty \to 0$.

2. The situation of complex valued f is now treated exactly as in case 1.

□

3.2.3 Approximation by Continuous Functions

Remark 3.2.2. We say that a function $f : \mathbb{R}^n \to \mathbb{C}$ has *compact support*, if there exists a closed n-interval I such that $f(x) = 0$ for all $x \notin I$. Let us set

$$C_c^\infty(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is infinitely differentiable and has compact support}\}.$$

Also, set

$$C_c(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is continuous and has compact support}\}.$$

Then $C_c^\infty(\mathbb{R}^n) \subset C_c(\mathbb{R}^n) \subset L^p(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$ for all p. One can show that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \mathcal{M}_\lambda, \lambda)$ for each $1 \le p < \infty$, but *not* for $p = \infty$.

Similar statements hold for the corresponding spaces of real-valued functions, and for spaces defined over intervals. For example, $C\big([0,1];\mathbb{R}\big)$ is dense in $L^p\big([0,1];\mathbb{R}\big)$ for all $1 \le p < \infty$. It folows that $L^p\big([0,1];\mathbb{R}\big)$ is the completion of $C\big([0,1];\mathbb{R}\big)$ with respect to the norm $\|\cdot\|_p$. (See example 1.5.10).

Bibliography

[1] Charalambos D. Aliprantis and Owen Burkinshaw, *Principles of Real Analysis*, 3rd Edition, Academic Press, 1998.

[2] H. S. Bear, *A Primer of Lebesgue Integration*, Academic Press, 1995.

[3] Donald L. Cohn, *Measure Theory*, Birkhäuser, 1980.

[4] Erwin Kreyszig, *Introductionary Functional Analysis With Applications*, John Wiley & Sons, 1978.

[5] John N. McDonald and Neil A. Weiss, *A Course in Real Analysis*, Academic Press, 1999.

[6] J. Tinsley Oden and Leszek F. Demkowicz, *Applied Functional Analysis*, CRC Press, 1996.

[7] H. L. Royden, *Real Analysis*, 3rd Edition, Macmillan, 1988.

[8] Walter Rudin, *Real and Complex Analysis*, 3rd Edition, McGraw-Hill, 1987.

[9] Daniel W. Stroock, *A Concise Introduction to the Theory of Integration*, 2nd Edition, Birkhäuser, 1994.

[10] William R. Wade, *An Introduction to Analysis*, 2nd Edition, Prentice Hall, 1999.