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รายงานการวิจัย

การศึกษาเรื่องบีสปลายเต็มหน่วย

(Generalrized Discrete Tension Splines)

ได้รับทุนอุดหนุนการวิจัยจาก

มหาวิทยาลัยเทคโนโลยีสุรนารี

ผลงานวิจัยเป็นความรับผิดชอบของหัวหน้าโครงการวิจัยแต่เพียงผู้เดียว

อภินันทนาการ รหัสโครงการ SUT 1-103-41-12-06

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หัวหน้าโครงการ

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ได้รับทุนอุดหนุนการวิจัยจากมหาวิทยาลัยเทคโนโลยีสุรนารี ปีงบประมาณ 2541

ผลงานวิจัยเป็นความรับผิดชอบของหัวหน้าโครงการวิจัยแต่เพียงผู้เดียว

กันยายน 2541

กิตติกรรมประกาศ

งานวิจัยฉบับนี้ได้รับทุนอุดหนุนการวิจัยจากมหาวิทยาลัยเทคโนโลยีสุรนารี ปีงบประมาณ 2541 ข้าพเจ้าขอขอบคุณหัวหน้าสถานวิจัยสำนักวิชาวิทยาศาสตร์ คณบดีสำนักวิชาวิทยาศาสตร์ ผู้อำนวยการและเจ้าหน้าที่สถาบันวิจัยและพัฒนา ผู้ซึ่งให้ความช่วยเหลืออย่างดียิ่งในการสนับสนุนงานวิจัย และในท้ายที่สุดขอขอบคุณ คุณอนุสรณ์ รุจิราภา ที่ช่วยจัดพิมพ์งานวิจัยนี้จนสำเร็จ

บทคัดย่อ

ในรายงานการวิจัยฉบับนี้ ได้มีการสร้างขั้นตอนวิธีสำหรับดิสกรีตเทนชันสปลายในอันดับทั่ว ๆ ไป ได้มีการพิสูจน์คุณสมบัติที่สำคัญของดิสกรีตเทนชันสปลาย นั่นคือคุณสมบัติเกี่ยวกับผลแบ่งกั้นของหนึ่ง ต่อจากนั้นยังได้มีการพิสูจน์ต่อไปว่าดิสกรีตเทนชันสปลายนี้จัดรูปกันเป็นระบบเชบิเชฟอย่างอ่อน และอนุกรมของสปลายนี้มีคุณสมบัติการแปรผันลดลง ได้มีการเสนอตัวอย่างของดิสกรีตเทนชันสปลายด้วย

Abstract

Dircet Algorithms for constructing generalized discrete tension splines of arbitrary degree are given. We derive main property of generalized discrete tension splines, i.e., partition of unity. Moreover, it is shown the generalized discrete tension splines form weak Chebyshev systems and their series have a variation diminishing property. Examples of discrete generalized discrete tension splines are included.

สารบัญ

Generalized Discrete Tension Splines*

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1. Introduction

Most of the theory of polynomial splines deals with the ease where the pieces are tie together by continuity of certain derivatives at the knots. But, in the theory of discrete splines the ties will involve differences instead of derivatives. We will talk about the continuous case when derivatives are involved, and the discrete case when differences are involved.

Discrete splines were introduced by Mangasarian and Schumaker (1971) as solutions to certain minimization problems involving differences instead of derivatives and after that they have been studied extensively. Until recently, Kvasov and Sattayatham (1998) investigated discrete tension splines of degree 3 for the sake of developing algorithm for automatic choice of parameter.

In this paper, we study discrete tension spline of arbitrary degree $n \geq 3$ which is the generalization of the proceeding one.

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2. Discrete tension spline of Arbitrary Degree

Let $\tau > 0$ be a small positive number and f(x) be a continuous function on [a, b]. Suppose that the discrete set $\{x, x+\tau, \ldots, x+k\tau\}$ is a subset of [a, b]. *The forward difference* of f(x) on [a, b] is defined as follows:

$$D_\tau^k\ f(x) \;=\; \sum_{\nu=0}^{k} \binom{k}{0} \frac{(-1)^{k-\nu}\ f(x+\nu\tau)}{\tau^k}. \tag{2.1}$$

We note that if f is k times differentiable at x then

$$f^{(k)}(x) \;=\; \lim_{\tau\to 0} D_\tau^k\ f(x). \tag{2.2}$$

Moreover, the operator D_τ^k has the property

$$D_\tau^j\ D_\tau^k\ f(x) \;=\; D_\tau^{j+k}\ f(x).$$

Now, let a partition $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ be given on the segment [a, b] to which we associate a space of discrete tension spline S_n^D whose restriction to a subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$ is spanned by the system of linearly independent functions

$$\{1, x, \ldots, x^{n-2}, \phi_{i,n}(x), \psi_{i,n}(x)\}$$

where $\phi_{i,n}(x)$ and $\psi_{i,n}(x)$ are continuous function on **R**.

Definition 2.1 *The generalized discrete tension spline of degree n* is a function $S(x) \in S_n^D$ such that for any $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$

(1) $S(x) \;=\; P_{i,n-2}(x) + D_\tau^{(n-1)}\ S(x_i)\ \phi_{i,n}(x) + D_\tau^{(n-1)}\ S(x_{i+1})\ \psi_{i,n}(x)$

where $P_{i,n-2}(x)$ is a polynomial of degree n–2 and $\phi_{i,n}(x)$, $\psi_{i,n}(x)$ are continuous functions on **R** satisfying the following properties:

$$\phi_{i,n}(x_{i+1} + k\tau) \;=\; 0 \quad ; \quad k = 0, 2, \ldots, n-1 \tag{2.3}$$

$$\psi_{i,n}(x_i + k\tau) = 0 \quad ; \quad k = 0, 2, \ldots, n-1 \tag{2.4}$$

$$D_\tau^{(n-1)} \phi_{i,n}(x_i) = D_\tau^{(n-1)} \psi_{i,n}(x_{i+1}) = 1. \tag{2.5}$$

(2) S(x) must satisfy the continuity condition

$$D_\tau^r S_{i-1}(x_i) = D_\tau^r S_i(x_i) \quad ; \quad \begin{array}{l} r = 0, 1, \ldots, n-1 \\ i = 1, 2, \ldots, N-1 \end{array} \tag{2.6}$$

(3) $S(x) \in C[a, b]$.

We note that (2.1), (2.3) and (2.4) imply that

$$D_\tau^{(r)} \phi_{i,n}(x_{i+1}) = D_\tau^{(r)} \psi_{i,n}(x_i) = 0 \quad ; \quad r = 0, 1, \ldots, n-1. \tag{2.7}$$

Moreover the continuity condition (2.6) implies that

$$S_{i-1}(x_i + k\tau) = S_i(x_i + k\tau) \quad ; \quad k = 0, 1, \ldots, n-1. \tag{2.8}$$

We denote S_n^D the set of splines satisfying Definition 2.1 The function $\phi_{i,n}(x)$ and $\psi_{i,n}(x)$ depend on the tension parameters which influence essentially the spline behaviour. We call then the *defining relations*. In practice, one takes

$$\phi_{i,n}(x) = \varphi_i(q_i, t_i) h_i^2, \quad \psi_{i,n}(x) = \psi_{i,n}(p_i, t_i) h_i^2, \quad 0 \le p_i, q_i < \infty$$

Here $t_i = (x - x_i)/h_i$ and $h_i = (x_{i+1} - x_i)$; $i = 1, 2, \ldots, N-1$. In the limiting case when $p_i, q_i \to \infty$, we require that $\lim_{p_i \to \infty} \psi_i(p_i, q_i) \equiv 0$, $\lim_{q_i \to \infty} \varphi_i(p_i, t_i) \equiv 0$ so that the formula (1) in Definition (2.1) turns into a polynomial function of degree (n). If $p_i = q_i = 0$, we get a discrete polynomial spline with

$$\phi_i(x) = (x_{i+1} - x)(x_{i+1} - x + \tau) \ldots (x_{i+1} - x + (n-1)\tau) / n!\, h_i$$

$$\psi_i(x) = (x_i - x)(x_i - x + \tau) \ldots (x_i - x + (n-1)\tau) / n!\, h_i.$$

Consider the problem of construction of a basis in the space S_n^D consisitng of the functions with a local support of minimal length. For this, it is convenience for us to extend the mesh Δ by adding points

$$x_{-n} < \ldots < x_{-1} < a < b < x_{N+1} < \ldots < x_{N+n}.$$

Since $\dim(S_n^D) = (n+1)N - n(N-1) = n+N$, it is suffice to construct the system of linearly independent splines $B_{j,n}(x)$; $j = -n, \ldots, N-1$ in S_n^D such that

$$B_{j,n}(x) > 0 \quad ; \quad x \in (x_j + (n-1)\tau,\ x_{j+n+1}) \tag{2.9}$$

$$B_{j,n}(x) \equiv 0 \quad ; \quad x \notin \left(x_j, x_{j+n+1}\right) \tag{2.10}$$

$$\sum_{j=-n}^{N-1} B_{j,n}(x) \equiv 1 \ , \quad \text{for} \quad x \in [a, b] \tag{2.11}$$

Since the splines $B_{j,n}(x)$ must satisfy definition (2.1), then by refering to the conditions (2.3), (2.4) and (2.9), (2.10) the graph of $B_{j,n}(x)$ has a small ripple in the inteval $[x_j, x_j + k\tau]$ $(k = 1, \ldots, n-1)$ where it can go to negative. See Figure 1.

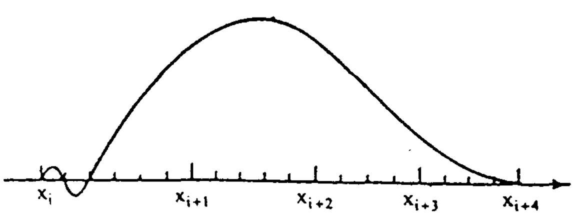

Figure 1. A generalized discrete tension spline $B_{j,3}(x)$

It follows from (2.8) and 2.10) that

$$B_{j,n}(x_j) = B_{j,n}(x_j+\tau) = \dots = B_{j,n}(x_j+(n-1)\tau) = 0. \tag{2.12}$$

The equation (2.12) implies that

$$D_\tau^{(r)} B_{j,n}(x_j) = 0 \quad ; \quad r = 0, 1, \dots, (n-1). \tag{2.13}$$

Moreover, since $B_{j,n}(x) \equiv 0$ outside the interval (x_j, x_{j+n+1}), then

$$D_\tau^{(r)} B_{j,n}(x_{j+n+1}) = 0 \quad ; \quad r = 0, 1, \dots, (n-1). \tag{2.14}$$

According to Definition 2.1, the basis spline $B_{j,n}(x)$ which is different from zero only in the interval (x_j, x_{j+n+1}) should have form

$$B_{j,n}(x) = \begin{cases} D_\tau^{(n-1)} B_{j,n}(x_{j+1})\,\psi_{j,n}(x) \quad ; \quad x_j \le x \le x_{j+1} \\ P_{j,\ell,n-2}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell})\,\phi_{j+\ell,n}(x) \\ \qquad + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell+1})\,\psi_{j+\ell,n}(x) \\ \qquad x_{j+\ell} \le x \le x_{j+\ell+1}, \quad \ell = 1, \dots, n-1 \\ D_\tau^{(n-1)} B_{j,n}(x_{j+n})\,\phi_{j+n,n}(x) \quad , \quad x_{j+n} \le x \le x_{j+n+1} \\ 0 \quad ; \quad x \notin (x_j, x_{j+n+1}). \end{cases} \tag{2.15}$$

The form of $B_{j,n}(x)$ in (2.15) for $x \in [x_{j+k}, x_{j+k+1}]$, $k = 0, n$ has been simplified in virtue of the relation (2.12) and (2.13). Taking into account the continuity condition (2.6), we have the relation

$$P_{j,\ell,n-2}(x) = P_{j,\ell-1,n-2}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell}) \sum_{r=0}^{n-2} D_\tau^{(r)} z_{j+\ell,n} \frac{(x-x_{j+\ell})^r}{r!}$$

$$\ell = 1, \dots, n \tag{2.16}$$

where $D_\tau^{(r)} z_{j+\ell,n} = b_r[D_\tau^{(r)} \psi_{j+\ell-1,n}(x_{j+\ell}) - D_\tau^{(r)} \phi_{j+\ell,n}(x_{j+\ell})]$ for some constants b_r, $r = 0, 1, \ldots, n-2$.

To see this, let us fixed $\ell \in \{1, 2, \ldots, n-1\}$.

It follows from (2.15) that

$$B_{j,n}(x) = P_{j,\ell-1,n-2}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell-1}) \phi_{j+\ell-1,n}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell}) \psi_{j+\ell-1,n}(x) \quad ; \quad x_{j+\ell-1} \le x \le x_{j+\ell} \tag{2.17}$$

$$B_{j,n}(x) = P_{j,\ell,n-2}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell}) \phi_{j+\ell-1,n}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell+1}) \psi_{j+\ell,n}(x) \quad ; \quad x_{j+\ell} \le x \le x_{j+\ell+1} \tag{2.18}$$

we obtain from the equations (2.17) and (2.18) that

$$D_\tau^{(r)} B_{j,n}(x) = D_\tau^{(r)} P_{j,\ell-1,n-2}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell-1}) D_\tau^{(r)} \phi_{j+\ell-1,n}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell}) D_\tau^{(r)} \psi_{j+\ell-1,n}(x) \quad ; \quad x_{j+\ell-1} \le x \le x_{j+\ell}$$

and

$$D_\tau^{(r)} B_{j,n}(x) = D_\tau^{(r)} P_{j,\ell,n-2}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell}) \phi_{j+\ell,n}(x) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell+1}) D_\tau^{(r)} \psi_{j+\ell,n}(x) \quad ; \quad x_{j+\ell} \le x \le x_{j+\ell+1}.$$

By continuity condition (2.6) at $x = x_{j+\ell}$, one gets

$$D_\tau^{(r)} P_{j,\ell,n-2}(x_{j+\ell}) = D_\tau^{(r)} P_{j,\ell-1,n-2}(x_{j+\ell}) + D_\tau^{(n-1)} B_{j,n}(x_{j+\ell}) [D_\tau^{(r)} \psi_{j+\ell-1,n}(x_{j+\ell}) - D_\tau^{(r)} \phi_{j+\ell,n}(x_{j+\ell})]$$

$$r = 0, 1, \ldots, n-2. \tag{2.19}$$

We note that $D_\tau^{(r)}\,\phi_{j+\ell-1,n}(x_{j+\ell}) = D_\tau^{(r)}\,\psi_{j+\ell,n}(x_{j+\ell}) = 0$ by equation (2.3) and (2.4) respectively.

Now, let us suppore that

$$P_{j,\ell,n-2}(x) = \sum_{r=0}^{n-2} a_r \frac{(x - x_{j+\ell})^r}{r!} \tag{2.20}$$

where

$$a_r = P_{j,\ell,n-2}^{(r)}(x_{j+\ell}) = b_r\, D_\tau^{(r)}\, P_{j,\ell,n-2}(x_{j+\ell})$$

for some constatnts b_r , $r = 0, 1, \ldots, n-2$ and by virtue of (2.2), we can easily see that $b_r \to 1$ as $\tau \to 0$.

Substitute (2.19) into (2.20), we get

$$P_{j,\ell,n-2}(x) = \sum_{r=0}^{n-2} b_r\, D_\tau^{(r)}\, P_{j,\ell-1,n-2}(x_{j+\ell}) \cdot \frac{(x - x_{j+\ell})^r}{r!}$$

$$+ D_\tau^{n-1}\, B_{j,n}(x_{j+\ell}) \sum_{r=0}^{n-2} D_\tau^{(r)}\, z_{j+\ell,n} \frac{(x - x_{j+\ell})^r}{r!} \tag{2.21}$$

where $D_r^\tau\, z_{j+\ell,n} = b_r\, [D_\tau^{(r)}\,\psi_{j+\ell-1,n}(x_{j+\ell}) - D_\tau^{(r)}\,\phi_{j+\ell,n}(x_{j+\ell})]$.

Since $b_r\, [D_\tau^{(r)}\, P_{j,\ell-1,n-2}(x_{j+\ell})] \to P_{j,\ell-1,n-2}^{(r)}(x_{j+\ell})$ as $\tau \to 0$, then $P_{j,\ell,n-2}(x)$ in (2.21) can be approximated by

$$P_{j,\ell,n-2}(x) = P_{j,\ell-1,n-2}(x) + D_\tau^{(n-1)}\, B_{j,n}(x_{j+\ell}) \sum_{r=0}^{n-2} D_\tau^{(r)}\, z_{j+\ell,n} \frac{(x - x_{j+\ell})^r}{r!}, \quad \ell = 1, \ldots, n. \tag{2.22}$$

As in (2.15), $P_{j,\ell,n-2}(x) \equiv 0$, $\ell = 0, n$. Then by repeated application of (2.22), we are

Using the normalization condition (2.11), for $x \in [x_j, x_{j+1}]$ and using (2.15), (2.16), we get

$$\sum_{i=j-3}^{j} B_{j,3}(x) = \phi_{j,3}(x)\left[\sum_{i=j-3}^{j-1} D_\tau^{(2)} B_{i,3}(x_j)\right] + \psi_{j,3}(x)\left[\sum_{i=j-2}^{j} D_\tau^{(2)} B_{i,3}(x_{j+1})\right]$$

$$- D_\tau^{(2)} B_{j-2,3}(x_{j+1}) D_\tau^{(1)} z_{j+1,3}(x - y_{j+1,3}) +$$

$$D_\tau^{(2)} B_{j-1,3}(x_j) D_\tau^{(1)} z_{j,3}(x - y_{j,3}) \equiv 1 .$$

Hence, by virtue of the linear independence of functions 1, x, $\phi_{j,3}(x)$, and $\psi_{j,3}(x)$, we obtain the equations

$$\sum_{\ell=j-3}^{j-1} D_\tau^{(2)} B_{i,3}(x_j) = \sum_{\ell=j-2}^{j} D_\tau^{(2)} B_{i,3}(x_{j+1}) = 0$$

$$D_\tau^{(2)} B_{j-2,3}(x_{j+1}) D_\tau^{(1)} z_{j+1,3} - D_\tau^{(2)} B_{j-1,3}(x_j) D_\tau^{(1)} z_{j,3} = 0 \qquad (2.24)$$

$$D_\tau^{(2)} B_{j-2,3}(x_{j+1}) D_\tau^{(1)} z_{j+1,3}\, y_{j+1,3} - D_\tau^{(2)} B_{j-1,3}(x_j) D_\tau^{(1)} z_{j,3}\, y_{j+1,3} = 1.$$

It follows immediately from the system (2.24) and by using the shift of index, we obtain the following formulas

$$D_\tau^{(2)} B_{j,3}(x_{j+1}) = \frac{1}{D_\tau^{(1)} z_{j+1,3}[y_{j+2,3} - y_{j+1,3}]}$$

$$D_\tau^{(2)} B_{j,3}(x_{j+2}) = \frac{-1}{D_\tau^{(1)} z_{j+2,3}}\left[\frac{1}{y_{j+2,3} - y_{j+1,3}} + \frac{1}{y_{j+3,3} - y_{j+2,3}}\right]$$

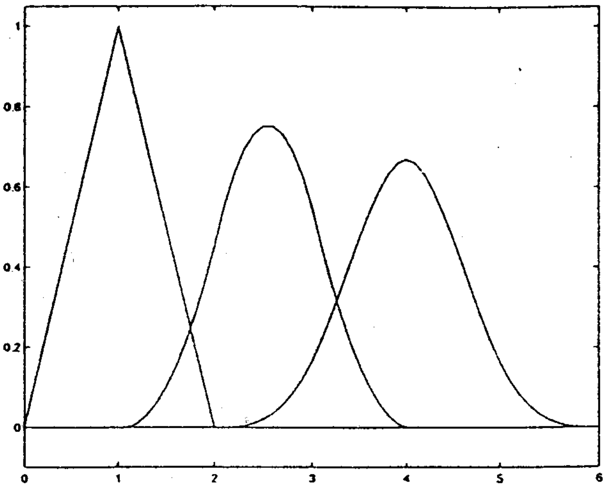

Figure 1. The discrete B-splines of order $k = 1, 2, 3$ (from left to right) on a uniform mesh with step size $h = 1$, no tension and discretization parameter $\tau = 0.1$.

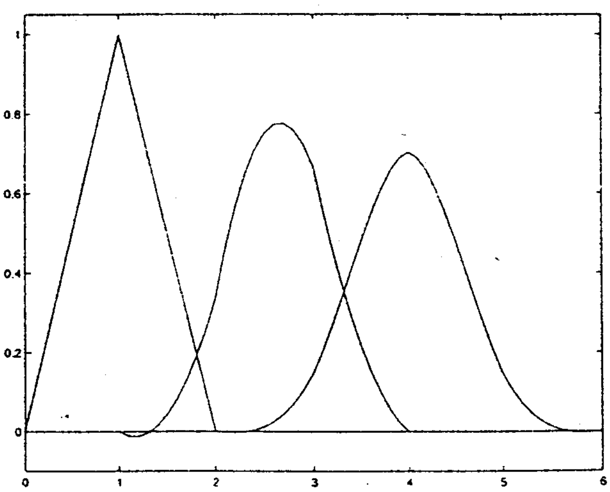

Figure 2. Same as Fig. 1, but with discretization parameter $\tau = 0.33$.

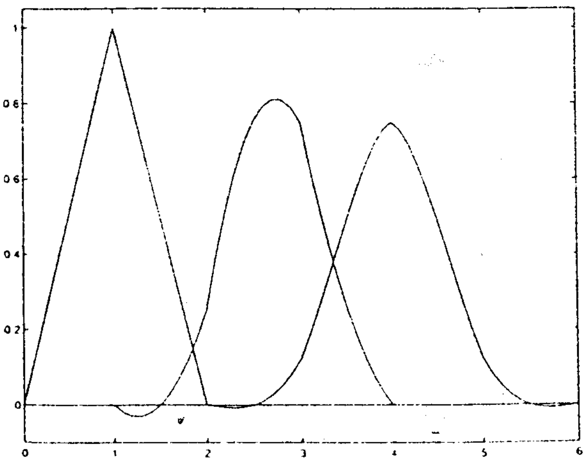

Figure 3. Same as Fig. 1, but with discretization parameter $\tau = 0.5$.

3. Series of Discrete GB-splines

According to Definition 2.1, we have denoted S_n^D to be the set of splines S(x) with n-1 continuous divided differences such that in any subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$, they are spanned by the functions $\left\{1, x, \ldots, x^{n-2}, \phi_i(x), \psi_i(x)\right\}$.

Using the method of Schumaker (1981), it is easy to show that the splines $B_{j,n}(x)$, $j = -n, \ldots, N-1$, have minimum length supports, are linearly independent, and form a basis in the space S_n^D can be uniquely represented in the form

$$S(x) = \sum_{j=-3}^{N-1} b_{j,n} B_{j,n}(x)$$

with some constant coefficient $b_{j,n}$.

Let us suppose that each step size $h_i = x_{i+1} - x_i$ of the mesh $\Delta : a = x_0 < x_1 < \ldots < x_N = b$ is an integer multiple of the same tabulation step τ of some detailed uniform refinement on [a, b].

For $\theta \in \mathbf{R}$, $t > 0$ define

$$\mathbf{R}_{\theta\tau} = \left\{ \theta + i\tau \mid i \text{ is an integer} \right\}$$

and let $\mathbf{R}_{\theta 0} = \mathbf{R}$. For any $a, b \in \mathbf{R}$ and $\tau > 0$ let

$$[a, b]_\tau = [a, b] \cap \mathbf{R}_{a\tau}.$$

By equation (2.9), the splines $B_{j,n}(x)$, $j = -n, \ldots, N-1$ are nonnegative functions on $[a, b]_\tau$ and as a consequence, we can reprove the main results of Sattayatham (1995) for series of generalized discrete tension splines. Even more, one can obtain those results from corresponding assertions for generalized discrcte tension splines as a limiting particular case when $\tau \to 0$.

Let $f(x)$ be a function defined on the discrete set $[a, b]_\tau$. We say that $f(x)$ has a zero at the point $t \in [a, b]_\tau$ provided

$$f(t) \quad \text{or} \quad f(t).f(t+\tau) < 0.$$

When $f(x)$ vanishes at a consecutive set of points of $[a, b]_\tau$, $f(x)$ is 0 at $t, \ldots, t+(r-1)\tau$, but $f(t-\tau).f(t+r\tau) \neq 0$, then we call the set $T = \{t, t+\tau, \ldots, t+(r-1)\tau\}$ a multiple zero of $f(x)$, and we define its multiplicity by

$$Z_T(f) = \begin{cases} r, & \text{if } f(t-\tau).f(t+r\tau) < 0 \text{ and } r \text{ is odd} \\ r, & \text{if } f(t-\tau).f(t+r\tau) > 0 \text{ and } r \text{ is even} \\ r+1, & \text{otherwise}. \end{cases}$$

This definition assures that f changes sign at a zero if and only if the zero is of odd multiplicity (see Figure 4 for example)

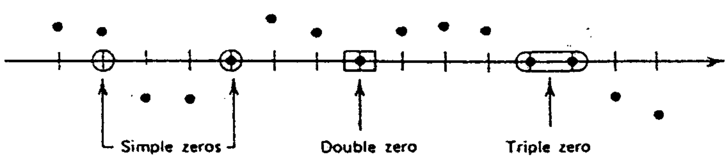

Figure 4. Zeros of a generalized discrete tension spline.

Let $Z_{[a,b]_\tau}(f(x))$ be the number of zero of a function $f(x)$ on the discrete set $[a, b]_\tau$, counted according to their multiplicity.

Theorem 3.1 (Rolle's Theorem for Generalized Discrete Splines). For any $S(x) \in S_n^D$,

$$Z_{[a,b]_\tau}\left(D_\tau^1 S(x)\right) \geq Z_{[a,b]_\tau}\left(S(x)\right) - 1 \tag{3.1}$$

Proof : First, if $S(x)$ has a z-tuple zero on the set $T = \{t,\ldots,t+(r-1)\tau\}$, it follows that $D_\tau^1 S(x)$ has a z-1-tuple zero on the set $T^1 = \{t,\ldots,t+(r-2)\tau\}$. Similarly, if $S(x)$ has a z-tuple zero on an interval, then $D_\tau^1 S(x)$ has a z-1-tuple zero on the same interval. Now if T_1 and T_2 are two consecutive zero sets of S, then it is trivially true that $D_\tau^1 S(x)$ must have sign change at some point between T_1 and T_2. Counting all of these zeros as in the case of ordinary polynomial splines, we arrive at the assertion (3.1). This proves the Theorem #

Theorem 3.2 For every $S(x) \in S_n^D$ that is not identically zero in any subsegment $[x_i, x_{i+1}]$, $i = 0, 1,\ldots, N-1$. We have

$$Z_{[a,b]_\tau}(S(x)) \le N+n-1.$$

Proof : Using the same method of B.I. Kvasov and P. Sattayatham [1998], one can show that

$$D_\tau^{(n-1)} S(x) = \sum_{j=-1}^{N-1} b_{j,1} B_{j,1}(x)$$

where

$$B_{j,1}(x) = \begin{cases} D_\tau^{(n-1)} \psi_j(x), & x_j \le x < x_{j+1} \\ D_\tau^{(n-1)} \phi_{j+1}(x), & x_{j+1} \le x < x_{j+2} \\ 0, & \text{otherwise.} \end{cases}$$

Here the functions $D_\tau^{(n-1)} \phi_i(x)$ and $D_\tau^{(n-1)} \psi_i(x)$ are assume to be monotonous and nonnegative on these subsegments. Hence $Z_{[a,b]_\tau}(D_\tau^{n-1} S(x)) \le N$. Then according to the Rolle's Theorem 3.1, we find that $Z_{[a,b]_\tau}(S(x)) \le N+n-1$. This complete the proof.

Denote by $\text{supp}_\tau B_{j,n}(x) = \{x \in \mathbf{R}_{a\tau} \mid B_{j,n}(x) > 0\}$ the discrete support of the spline $B_{j,n}(x)$, i.e. the discrete set $(x_j + (n-1)\tau\,,\, x_{j+n+1})$.

Theorem 3.3 Assume that $\zeta_{-n} < \zeta_{-n+1} < \dots < \zeta_{N-1}$ are prescribed points in the discrete line $\mathbf{R}_{a,\tau}$. Then

$$D = \det(B_{j,n}(\zeta_k)) \geq 0, \qquad j, k = -n, \dots, N-1$$

and strict positivity holds if and only if

$$\zeta_j \in \mathrm{supp}_\tau B_{j,n}(x), \qquad j = -n, \dots, N-1. \tag{3.1}$$

Proof : let us prove the theorem by induction. It is clear that the theorem holds for one basis function. Assume that it also holds for $(\ell - 1)$ basis functions. Let us show that if (3.1) is satisfied, then $D \neq 0$ for ℓ basis functions.

Let $\zeta_\ell \notin \mathrm{supp}_\tau B_{\ell,n}(x)$, then ζ_ℓ lies to the left with respect to the discrete support of $B_{\ell,n}(x)$. This implies that the last column (line) of the determinant D consists of zeros, i.e., $D = 0$. If $\zeta_\ell \in \mathrm{supp}_\tau B_{\ell,n}(x)$ and $D = 0$, then there exists a nonzero vector $\bar{c} = (c_{-n}, \dots, c_{\ell-n-1})$ such that

$S(\zeta_k) = \sum_{j=-n}^{\ell-n-1} c_j B_{j,n}(\zeta_k)$, $k = -n, \dots, \ell-n-1$, i.e., the spline $S(x)$ has ℓ zeros. But this contradicts to Theorem 3.2, which states that $S(x)$ can have no more than $(\ell - 1)$ zeros. Hence $\bar{c} = \bar{0}$ and $D \neq 0$.

Now it only remains to prove that $D > 0$ if (3.1) is satisfied. Let us choose $x_k + \tau < \zeta_k < x_{k+1} - \tau$ for all k. Then the diagonal elements of D are positive and all the elements above the main diagonal are zero, is $D > 0$. It is clear that D depends continuously on ζ_k, $k = -n, \dots, \ell-n-1$, and $D \neq 0$ for $\zeta_k \in \mathrm{supp}_\tau B_{k,n}(x)$. Hence the determinant D is positive, if condition (3.1) is satisfied. This completes the proof.

The following three corollaries follow immediately from the theorem.

Corollary 3.1 The system of generalized discrete B-splines $\{B_{j,n}(x)\}$, $j = -n, \dots, N-1$, associated with knots on $\mathbf{R}_{a,\tau}$ is a weak Chebyshev system according to the definition given in Schumaker (1981) i.e., for any $\zeta_{-n} < \zeta_{-n+1} < \dots < \zeta_{N-1}$ in $\mathbf{R}_{a,\tau}$, we have $D \geq 0$ and $D > 0$ if and only if condition (3.1) is satisfied. If the latter is satisfied, the generalized spline $S(x) = \sum_{j=-3}^{N-1} b_{j,3} B_{j,3}(x)$ has no more than $N+2$ zeros.

Corollary 3.2 If the condition of Theorem 3.3 are satisfied, the solution of the interpolation problem

$$S(\zeta_i) = f_i, \quad i = -n, \ldots, N-1, \quad f_i \in \mathbf{R} \tag{3.2}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, be a rectangular $m \times n$ matrix with $m \leq n$. The matrix A is said to be totally nonnegative (totally positive) (e.g., see Karlin (1968)) if all the minors of any order of the matrix are nonnegative (positive), i.e. for all $1 \leq p \leq m$, we have

$$\det(a_{i_k j_\ell}) \geq 0 \ (>0) \quad \text{for all} \quad i \leq i_1 < \ldots < i_p \leq m$$
$$i \leq j_1 < \ldots < i_p \leq n.$$

Corollary 3.3 For arbitrary integer $-n \leq \nu_{-n} < \ldots < \nu_{p-n-1} \leq N-1$ and $\zeta_{-n} < \zeta_{-n+1} < \ldots < \zeta_{p-n-1}$, in $\mathbf{R}_{a,\tau}$ we have

$$D_p = \det\left\{B_{\nu_i,n}(\zeta_j)\right\} \geq 0, \quad i,j = -n, \ldots, p-n-1$$

and $D_p > 0$ if and only if $\zeta_i \in \text{supp}_\tau B_{\nu_i,n}(x)$, $i = -n, \ldots, p-n-1$, i.e., the matrix $\left\{B_{j,n}(\zeta_i)\right\}$, $i,j = -n, \ldots, N-1$ is totally negative.

The last statement is proved by induction on the basis of Theorem 3.3 and the recurrence relations for the minors of the matrix $\left\{B_{j,n}(\zeta_i)\right\}$. The proof does not differ from that cited by Schumaker (1981).

De Boor and Pinkus [1977] proved that linear systems with totally nonnegative matrices can be solved by the Gauss method without choosing a point element. Thus the system (3.2) can be solved efficiently by conventional sweeping method.

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1. Sattayatham, P. ***A Convergence to Infinity in Banach Lattices,*** J. Natl. Res. Council Thailand 23 (2), 32-39 (1991).

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B. BOOKS (Written in Thai)

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5. Sattayatham, P. ***Calculus.*** Suranaree Univ. Press, 1993
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C. INVITED ARTICLE

1. Sattayatham P. ***Introduction to the Subject of Wavelets and P.D.E.'s,*** given at the Annual Meeting in Mathematics Khon Kaen Univ., May 1995.