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นายเจษฎา ตัณฑนุช

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APPLICATION OF GROUP ANALYSIS TO FUNCTIONAL DIFFERENTIAL EQUATIONS

Mr. Jessada Tanthanuch

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APPLICATION OF GROUP ANALYSIS TO FUNCTIONAL DIFFERENTIAL EQUATIONS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy

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วิทยานิพนธ์นี้ศึกษาการพัฒนาขั้นตอนวิธีสำหรับการประยุกต์กลุ่มวิเคราะห์ เพื่อใช้หาผลเฉลยของสมการเชิงอนุพันธ์ฟังก์ชันนัล โดยเสนอบทนิยามและทฤษฎีบทที่เกี่ยวข้องกับกลุ่มวิเคราะห์สำหรับสมการเชิงอนุพันธ์ประวิงและสมการเชิงอนุพันธ์ฟังก์ชันนัล และขั้นตอนวิธีที่ได้พัฒนาขึ้นในงานวิจัยนี้แสดงการหากลุ่มสมมาตรจากสมการเชิงอนุพันธ์ประวิงและสมการเชิงอนุพันธ์ฟังก์ชันนัล ระเบียบวิธีสำหรับการประยุกต์กลุ่มวิเคราะห์เพื่อใช้แก้สมการเชิงอนุพันธ์ฟังก์ชันนัลประกอบด้วย การสร้างสมการกำหนด, การแบ่งแยกสมการกำหนด และ การแก้สมการกำหนด

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FUNCTIONAL DIFFERENTIAL EQUATION/ DELAY DIFFERENTIAL EQUATION/ GROUP ANALYSIS/ SYMMETRY GROUP

This thesis is devoted to developing an algorithm for applying group analysis to functional differential equations. The definitions and theorems concerning group analysis for delay differential equations (DDEs) and functional differential equations (FDEs) were established in the thesis. The algorithms developed show how one can obtain symmetry groups for DDEs and FDEs. The method of applying group analysis to FDEs consists of constructing determining equations, splitting determining equations, and solving determining equations.

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Jessada Tanthanuch

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Chapter I

Introduction

The purpose of this thesis is to develop an algorithm for applying group analysis to functional differential equations (FDEs).

In applications, many phenomena in Mathematics, Physics, Chemistry and Biology are modelled by FDEs. A present, most solutions of these equations are of numerical type, with only approximate solutions obtained. Among all the methods used for finding exact solutions of differential equations, *group analysis of differential equations* is one of the most powerful methods.

Group analysis was first introduced by Sophus Lie in the 1870s. For more than a hundred years, group analysis has been applied to many types of differential equations, e.g. ordinary differential equations (ODEs), partial differential equations (PDEs), differential-difference equations, integro-differential equations, etc.

Group analysis involves the study of symmetries of differential equations, with the emphasis on using the symmetries for finding solutions. For ODEs and PDEs, it provides a systematic procedure to find all symmetries of the equations.

In the case of an ODE, the existence of a symmetry allows us to reduce the order of the equation. The solution of the original equation can then be obtained by solving the reduced equation. For the single first order ODE, this method provides an explicit formula for the general solution.

For a given system of PDEs, group analysis usually cannot determine the general solution. However, tt may indicate when the system can be transformed into an easier form. One can use symmetry groups to determine special types of solutions, which are invariant under some subgroup of the full symmetry group of the system. Such solutions are found by solving a reduced system of the differential equations depending on fewer independent variables than the original system.

Group analysis has been applied to differential-difference equations[1] and integro-differential equations[2]. There are also a few results related to delay differential equations (DDEs) and functional differential equations (FDEs).

The main obstacle for applying group analysis to solve DDEs and FDEs is the non-locality of the equations. To overcome this complexity, Linchuk (2002) used the theory of formal operators and the principle of factorization. But even with this approach, the full symmetry group of DDEs and FDEs could not be found.

While preparing the text of this thesis, the author found an article presenting an approach similar to the approach developed in this thesis (Zawistowski, 2002). But there are no examples and there is no analysis for splitting the determining equations.

This thesis is devoted to developing an approach which can give all symmetries of DDEs and FDEs. Here, the definition of a symmetry group is a group of transformations converting every solution of an equation into another solution of the same equation. With this definition of symmetry group, the determining equations of delay differential equations and functional differential equations can be obtained. The key for solving the determining equations is the existence of

[1]See, e.g. Yanenko and Shokin (1973, quoted in Ibragimov, 1996), Dorodnitsyn (1987, quoted in Ibragimov), and Levi and Winternitz (1991)

[2]See, e.g. Taranov (1976, quoted in Ibragimov), Bunimovich and Krasnoslobodtsev (1982, quoted in Ibragimov), Grigoriev and Meleshko (1986, quoted in Ibragimov), and Kovalev, Pustovalov and Senashov (1992, quoted in Ibragimov)

solutions of initial value problems. The existence theory provides the arbitrariness of variables and functions that enables us to split the determining equations into systems of several equations. After solving the split equations, one obtains symmetries of DDEs and FDEs.

The thesis begins with definitions and properties of FDEs and DDEs. A review of group analysis is then presented. The following two chapters show how one can apply group analysis to DDEs and FDEs. Examples and conclusion are presented in the last two chapters.

Chapter II

Functional and Delay Differential Equations

Functional differential equations were first encountered in the late eighteenth century by the Bernoullis, Laplace and Condorcet (Hale, 1971). This type of differential equation plays a large role in mathematical, physical and biological modelling : the distribution of prime numbers (Driver, 1977), the two-body problem of electrodynamics (Driver), geometrical problems (Driver), control systems (Driver), prey-predator population models (Driver), the modelling gene expression with differential equations (Chen, He, and Church, 1999), population growth models, financial mathematics, weather forecasting, etc. Because of their use in many branches of science, the theory of FDEs has been and is still being developed.

This chapter presents the definition of an FDE and that of a particular type of FDE, the delay differential equation (DDE).

2.1 Functional Differential Equations

Definition 2.1 (FDE). An equation involving functionals[1] of independent variables, dependent variables and derivatives of dependent variables with respect to one or more independent variables is called *a functional differential equation.*

[1]Some familiarity with the concept of "*functional*" and related concepts is assumed but a review is included in C.1, Appendix C. One may find the definition and its concepts from textbooks, e.g. Kreyszig (1978).

Examples of FDEs (Hale; Kolmanovskii and Myshkis, 1992) follow:

- a linear retarded FDE ,

$$u'(x) = ku(x - r),$$

where k, r are constant, $r > 0$ and $k \neq 0$.

- an advanced FDE,

$$u'(x) = u(x + r),$$

where r is constant, $r > 0$.

- a mixed type FDE,

$$u'(x) + au(x - r) + bu(x + r) = 0,$$

where r, a, b are constant and $r > 0,\ a \neq 0,\ b \neq 0$.

- an FDE with aftereffect,

$$u^{(m)}(x) = f(x, u^{(m_1)}(x - h_1(x)), ..., u^{(m_k)}(x - h_k(x))), \tag{2.1}$$

where $u(x) \in \mathbb{R}^n$, $u^{(m_i)}$ is the m_i-order derivative of u with respect to x and all $m_i \geq 0,\ h_i(x) \geq 0,\ i = 1, ..., k$.

In the literature, equation (2.1) is called

  - a functional differential equation of *retarded type* or *retarded differential equation* (RDE), if $\max\{m_1, ..., m_k\} < m$;
  - a functional differential equation of *neutral type* (NDE), if $\max\{m_1, ..., m_k\} = m$; and
  - a functional differential equation of *advanced type* (ADE), if $\max\{m_1, ..., m_k\} > m$.

- a functional integro-differential equation of *Volterra type*,

$$u'(x) = f\left(x, \int_{x-h(x)}^{x} K(x, \theta, u(\theta))d\theta\right),$$

where $h(x)$ is a real functional of x such that $h(x) \geq 0$.

- a model of two interacting species,

$$\frac{du_1}{dx}(x) = a_1 u_1(x) - a_2 u_1(x) u_2(x) - a_3 u_1^2(x),$$
$$\frac{du_2}{dx}(x) = -a_4 u_2(x) + a_5 u_1(x-h) u_2(x-h),$$

where $a_i > 0$ are constants, $i = 1, ..., 4$, and delay $h > 0$ is the average time between the death of preys and the birth of subsequent predators.

- a model of the human immunodeficiency virus (HIV) epidemic,

$$\frac{dS}{dt}(t) = \Lambda - B(t) - S(t),$$
$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau})i(t, \tau) = -[1 + \alpha(\tau)]i(t, \tau),$$
$$i(t, 0) = B(t) = S(t)C(T(t))\frac{W(t)}{T(t)},$$
$$T = I + S, \quad I(t) = \int_0^\infty i(t, \tau)d\tau,$$
$$W(t) = \int_0^\infty \lambda(\tau)i(t, \tau)d\tau,$$
$$\frac{dA}{dt}(t) = \int_0^\infty \alpha(\tau)i(t, \tau)d\tau - (1 + \nu)A(t),$$

where S, I and A are the populations of uninfected but susceptible, HIV infected and fully developed acquired immunodeficiency syndrome (AIDS) cells respectively, t is time, τ is time elapsed from the moment of infection, Λ is the (constant) rate of growth of the sexually active population, $B(t)$ is the number of new cases of infection per unit time, $i(t, r)$ is the infection-age density, $C(T(t))$ is the average number of sexual contacts an average person has in unit time, ν is the average death rate of a person with AIDS, and $\tau \mapsto \lambda(\tau)$ is a given nonnegative function.

Similarly to the classification of differential equations by order, we classify FDEs according to the order of the highest derivative appearing in the equation.

Definition 2.2. The *order* of an FDE is the order of the highest derivative of the unknown function entering in the equation.

Definition 2.3. A *solution* of an FDE in some region $\mathcal{R}$ of the space of the independent variables is a function that has derivatives and functionals of derivatives appearing in the equation in some domain containing $\mathcal{R}$ and satisfies the equation everywhere in $\mathcal{R}$.

As there is yet no general theory for solving FDEs, it is difficult to deal with an abstract and general equation under the given definition of an FDE. For the sake of simplicity, the study will begin from a particular case of FDEs, namely the delay differential equation (DDE), and its theory.

2.2 Delay Differential Equations

Although a DDE is a particular type of FDE, it also plays a role in many fields: nuclear reactors (Ergen, 1954; Levin and Nohel, 1960; Gorjačenko, 1971, quoted in Driver, 1977), electron energy distribution in a gas discharge (Sherman, 1960, quoted in Driver), transistor circuits (Gumowski, 1962, quoted in Driver), photo emulsions (Silberstein, 1940, quoted in Driver), elasticity (Volterra, 1909; Gurtin and Sternberg, 1962, quoted in Driver), the spread of infectious diseases (Lotka and Sharpe, 1923; Wilson and Burke, 1942; London and Yorke, 1973, quoted in Driver), neurology (Melzak, 1961, quoted in Driver), the respiratory system (Grodins, Buell and Bart, 1967, quoted in Driver), business cycles and economic growth (Kalecki, 1935; Goodwin 1951; Cooke and Yorke, 1972, quoted in Driver) and the production and death of read blood cells (Chow, 1974, quoted in Driver).

Definition 2.4 (DDE). *Delay differential equations with one independent variable*, or *functional differential equations of retarded type*, are of the form

$$u'(x) = f(x, u(g_1(x)), ..., u(g_q(x))), \tag{2.2}$$

where $x \in [x_0, \beta)$, $u : [\gamma, x] \mapsto \mathcal{D}$, $\mathcal{D}$ is an open subset in $\mathbb{R}^n$, u and f are n-vector-valued, sufficiently time differentiable functions, $f : [x_0, \beta) \times \mathcal{D}^q \mapsto \mathbb{R}^n$, and for each $\lambda = 1, ..., q,\ \ \gamma \leq g_\lambda(x) \leq x,$ for $x_0 \leq x < \beta$.

Note that g_1 is usually chosen to be the identity mapping.

Definition 2.5. A *solution* of equations (2.2), with the initial condition $\theta(x)$ defined on $[\gamma, x_0]$, is a continuous function $u : [\gamma, \beta_1) \mapsto D$, for some $\beta_1 \in (x_0, \beta]$ such that

1. $u(x) = \theta(x)$ for $\gamma \leq x \leq x_0$, and
2. $u'(x) = f(x, u(g_1(x)), ..., u(g_q(x)))$ for $x_0 \leq x \leq \beta_1$.

Remark. The derivative of u at the point x_0 is considered only from the right-hand side.

Definitions 2.4 and 2.5 indicate that initial values of DDEs have to be satisfied for the whole interval considered. In other words, they are of *non-local differential equation* type.

2.3 Existence Theory of Solutions of DDEs

One of the main requirements for solving determining equations[2] is the existence of solutions of DDEs. This section presents definitions and theorems concerning

[2]See definition of *determining equation* in 3.5, Chapter III.

the existence theory of solution of DDEs. They are similar, but more general, to the existence theory for ODEs. The following definitions and theorems come from Driver (1977).

Consider a delay differential system

$$u'(x) = f(x, u(g_1(x)), ..., u(g_q(x))). \tag{2.3a}$$

By definition 2.4, we may assume that

$$x - r \leq g_\lambda(x) \leq x \quad \text{for} \quad x \geq x_0, \quad \lambda = 1, ..., q,$$

for some constant $r \geq 0$. The initial condition takes the form

$$u(x) = \theta(x) \quad \text{for} \quad x_0 - r \leq x \leq x_0.$$

Note that system (2.3a) is reduced to a system of ODEs if $r = 0$. It is assumed that f is defined on $[x_0, \beta) \times \mathcal{D}^q \mapsto \mathbb{R}^n$ for some $\beta > x_0$ and some open set $\mathcal{D} \subset \mathbb{R}^n$.

Since the notation of system (2.3a) is cumbersome, it would be better to have a simpler notation.

If u is a function defined at least on $[x - r, x] \mapsto \mathbb{R}^n$, then we define a new function $u_x : [-r, 0] \mapsto \mathbb{R}^n$ by

$$u_x(\sigma) = u(x + \sigma) \quad \text{for} \ -r \leq \sigma \leq 0.$$

From another point of view, u_x is obtained by considering only $u(s)$ for $x - r \leq s \leq x$ and then translating this segment of u to the interval $[-r, 0]$. If u is a continuous function, then u_x is a continuous function on $[-r, 0]$.

Let real numbers $r \geq 0$ and x_0 be given and let $x_0 < \beta \leq \infty$. Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$, and let F be defined on $[x_0, \beta) \times \mathcal{C}_\mathcal{D} \mapsto \mathbb{R}^n$, where $\mathcal{C}_\mathcal{D}$ is the set of all continuous functions mapping $[\gamma - x, 0] \mapsto \mathcal{D}$, or $\mathcal{C}_\mathcal{D} = \mathcal{C}([\gamma - x, 0], \mathcal{D})$. Define

$$F(x, u_x) \equiv f(x, u(g_1(x)), ..., u(g_q(x))).$$

Then system (2.3a) can be written as

$$u'(x) = F(x, u_x). \tag{2.3b}$$

Given any $\phi \in \mathcal{C}_{\mathcal{D}}$, we seek a continuous function $u : [x_0 - r, \beta_1) \mapsto \mathcal{D}$ for some $\beta_1 \in (x_0, \beta]$ such that system (2.3b) is satisfied on $[x_0, \beta_1)$ and

$$u_{x_0} = \phi. \tag{2.4}$$

For the existence of solutions of system (2.3b), it is sufficient to require the following conditions on F.

Definition 2.6. A function $F(x, u_x)$ satisfies the *Continuity Condition (C)* if $F(x, u_x)$ is continuous with respect to x in $[x_0, \beta)$ for any given continuous function $u : [x_0 - r, \beta) \mapsto \mathcal{D}$.

If F satisfies the *Continuity Condition (C)* then a continuous function $u : [x_0, \beta_1) \mapsto \mathcal{D}$ is a solution of equations (2.3b) and (2.4) if and only if

$$u(x) = \begin{cases} \phi(x - x_0) & \text{for} \quad x_0 - r \le x \le x_0, \\ \phi(0) + \int_{x_0}^{x} F(s, u_s) ds & \text{for} \quad x_0 \le x \le \beta_1. \end{cases} \tag{2.5}$$

In order to define a *Lipschitz condition*, a means for measuring the magnitude of elements of $\mathcal{C}_{\mathcal{D}}$ is required.

For a function $\psi \in \mathcal{C}_{\mathcal{D}}$,

$$|\psi|_r = \sup_{-r \le \varrho \le 0} |\psi(\varrho)|.$$

Definition 2.7. Let $F : [x_0, \beta) \times \mathcal{C}_{\mathcal{D}} \mapsto \mathbb{R}^n$ and let $\mathcal{E}$ be a subset of $[x_0, \beta) \times \mathcal{C}_{\mathcal{D}}$. If for some $K \ge 0$

$$|F(x, \psi) - F(x, \bar{\psi})| \le K|\psi - \bar{\psi}|_r, \tag{2.6}$$

whenever (x, ψ) and $(x, \bar{\psi}) \in \mathcal{E}$, we say that F satisfies a *Lipschitz condition* (or F is *Lipschitzian*) on $\mathcal{E}$ with *Lipschitz constant* K.

Definition 2.8. A functional $F : [x_0, \beta) \times \mathcal{C}_{\mathcal{D}} \mapsto \mathbb{R}^n$ is *locally Lipschitzian* if for each given $(\bar{x}, \bar{\psi}) \in [x_0, \beta) \times \mathcal{C}_{\mathcal{D}}$ there exist numbers $a > 0$ and $b > 0$ such that

$$\mathcal{E} \equiv (\ [\bar{x} - a, \bar{x} + a] \cap [x_0, \beta)\ ) \times \{\psi \in \mathcal{C}_{\mathcal{D}} : |\psi - \bar{\psi}|_r \leq b\}$$

is a subset of $[x_0, \beta) \times \mathcal{C}_{\mathcal{D}}$ and F is Lipschitzian on $\mathcal{E}$.

Remark. The Lipschitz constant for F depends on the particular set $\mathcal{E}$.

Theorem 2.1 (Local Existence). *Let $F : [x_0, \beta) \times \mathcal{C}_{\mathcal{D}} \mapsto \mathbb{R}^n$ satisfy Continuity Condition (C) and be locally Lipschitzian. Then, for each $\phi \in \mathcal{C}_{\mathcal{D}}$, equations (2.3b) and (2.4) have a unique solution on $[x_0 - r, x_0 + \Delta)$ for some $\Delta > 0$.*

Chapter III

Group Analysis

In the latter part of the 19th century, Sophus Lie, a Norwegian mathematician (1842-1899), introduced the notion of continuous transformation groups, now known as *Lie groups*, in order to create a theory of integrating ordinary differential equations, which is similar to the Abelian theory of solving algebraic equations. He gave a definition and investigated the fundamental concepts of the group admitted by a given system of differential equations. Later, these groups were applied to many types of differential equations. At present, treatment of Lie group of transformations and the differential equations admitted by these groups is called *group analysis of differential equations.*

This chapter introduces the basic ideas of Lie groups, which are necessary for the later chapters. The material in this chapter is reviewed from Bluman and Kumei (1996), Ibragimov (1996), Ibragimov (1999), Lie (1891), Olver (1993), Ovsiannikov (1978) and Stephani (1989).

3.1 Lie-point Transformations

Let $x = (x_1, \ldots, x_n)$ be n-tuples of the independent variables and $u = (u^1, \ldots, u^m)$ be m-tuples of the dependent variables. Consider invertible transformations of x and u

$$\begin{aligned} \bar{x} &= (\bar{x}_1, \ldots, \bar{x}_n) = (\varphi_1^x(x,u;a), \ldots, \varphi_n^x(x,u;a)) = \varphi^x(x,u;a), \\ \bar{u} &= (\bar{u}^1, \ldots, \bar{u}^m) = (\varphi_1^u(x,u;a), \ldots, \varphi_m^u(x,u;a)) = \varphi^u(x,u;a), \end{aligned} \tag{3.1}$$

depending upon a real continuous parameter a, which lies in an open symmetric interval $\mathcal{S}$, with conditions

$$\begin{aligned} \varphi_i^x(x,u;0) &= x_i, \qquad i=1,\dots,n, \\ \varphi_\alpha^u(x,u;0) &= u^\alpha, \qquad \alpha=1,\dots,m. \end{aligned} \tag{3.2}$$

These transformations are assumed to be sufficiently differentiable with respect to the variables x_i and u^α, and to be analytic functions of the parameter a.

It is said that these transformations form a *one-parameter group* G if the successive action of two transformations is equivalent to the action of another transformation of the form (3.1), i.e.

$$\begin{aligned} \varphi^x(\bar{x},\bar{u};b) &= \varphi^x(\varphi^x(x,u;a),\varphi^u(x,u;a);b) = \varphi^x(x,u;a+b), \\ \varphi^u(\bar{x},\bar{u};b) &= \varphi^u(\varphi^x(x,u;a),\varphi^u(x,u;a);b) = \varphi^u(x,u;a+b). \end{aligned} \tag{3.3}$$

In practice, it often happens that the group property is valid only locally, i.e. only for $|a|$ and $|b|$ sufficiently small. In this case, G is referred to as a *local one-parameter transformation group.* In group analysis, local groups are used, which for brevity will simply be called *groups.*

The transformations (3.1) are called *point transformations*, and the group G is called a *group of point transformations.* It is readily seen from formulas (3.2) and (3.3) that the inverse transformation can be obtained by changing the sign of the parameter:

$$x = \varphi(\bar{x},\bar{u},-a), \quad u = \psi(\bar{x},\bar{u},-a) \tag{3.4}$$

Let T_a denote the transformation (3.1) of a point (x,u) into the point $(\bar{x},\bar{u})$, I denote the identity transformation, T_a^{-1} denote the transformation inverse to T_a, and T_bT_a denote the composition of two transformations. Then one may summarize properties (3.1)-(3.4) as follows:

A set G of transformations T_a is a ***group of point transformations*** *if the following hold:*

1. $T_0 = I \in G$,

2. $T_b T_a = T_{a+b} \in G$,

3. If $a \in \mathcal{S}$ and $T_a\left((x,u)\right) = (x,u)$ for all (x,u), then $a = 0$.

The functions φ^x and φ^u can be represented via their Taylor series expansions with respect to the parameter a in the neighborhood of the expansion point 0 and thus the transformations in (3.1) can be written as follows:

$$\bar{x}_i = \varphi_i^x(x,u;a) = x_i + \xi_i(x,u)a + \cdots,$$
$$\bar{u}^\alpha = \varphi_\alpha^u(x,u;a) = u^\alpha + \eta^\alpha(x,u)a + \cdots,$$

or

$$\bar{x}_i \approx x_i + \xi_i(x,u)a, \qquad \bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x,u)a, \tag{3.5}$$

where

$$\xi_i(x,u) = \left.\frac{\partial \varphi_i^x(x,u;a)}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x,u) = \left.\frac{\partial \varphi_\alpha^u(x,u;a)}{\partial a}\right|_{a=0}.$$

Given an infinitesimal transformation (3.5), the corresponding group can be completely determined by the following system of differential equations, called *Lie equations*, with appropriate initial conditions:

$$\begin{aligned} \frac{d\varphi_i^x}{da} &= \xi_i(\varphi^x, \varphi^u), \quad \left.\varphi_i^x\right|_{a=0} = x_i, \\ \frac{d\varphi_\alpha^u}{da} &= \eta^\alpha(\varphi^x, \varphi^u), \quad \left.\varphi_\alpha^u\right|_{a=0} = u^\alpha. \end{aligned} \tag{3.6}$$

Consider the first-order differential operator

$$X = \xi_1(x,u)\frac{\partial}{\partial x_1} + \cdots + \xi_n(x,u)\frac{\partial}{\partial x_n} + \eta^1(x,u)\frac{\partial}{\partial u^1} + \cdots + \eta^m(x,u)\frac{\partial}{\partial u^m}. \tag{3.7}$$

Sophus Lie called the operator (3.7) a *symbol* of the infinitesimal transformation (3.5). In this thesis, the words *infinitesimal generator, infinitesimal operator, group generator, group operator* and *Lie operator* are used interchangeably.

The first-order differential operator (3.7) is written briefly as

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{3.8}$$

where the repeated index i means summation with respect to i from $i = 1$ to n and the repeated index α means summation with respect to α from $\alpha = 1$ to m.

3.2 Contact Transformations

Unlike point transformations, contact transformations depend on not only the independent and dependent variables but also on the first order derivatives.

Let u' denote the set of first order derivatives $u^\alpha_{,i} = \dfrac{\partial u^\alpha}{\partial x_i}$, $\alpha = 1, \dots, m$, $i = 1, \dots, n$, and consider a one-parameter group of transformations

$$\bar{x}_i = \varphi_i(x,u,u';a), \quad \bar{u}^\alpha = \psi^\alpha(x,u,u';a), \quad \text{and} \quad \bar{u}^\alpha_{,i} = \omega^\alpha_i(x,u,u';a) \tag{3.9}$$

in the space of variables (x, u, u'). The transformations (3.9) are referred to as *contact transformations* when $\bar{u}^\alpha_{,i} = \dfrac{\partial \bar{u}^\alpha}{\partial \bar{x}_i}$ and they leave invariant the first-order tangency conditions

$$d\bar{u}^\alpha - \bar{u}^\alpha_{,i}\, d\bar{x}_i = 0, \quad \alpha = 1, \dots, m.$$

Note that the repeated index i means the summation with respect to i from $i = 1$ to n. From here on, summation with respect to the repeated index will be assumed implicitly.

The corresponding infinitesimal transformations are

$$\bar{x}_i \approx x_i + \xi_i(x,u,u')a, \quad \bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x,u,u')a, \quad \bar{u}^\alpha_{,i} \approx u^\alpha_{,i} + \zeta^\alpha_i(x,u,u')a, \tag{3.10}$$

where

$$\begin{aligned}
\frac{d\varphi_i}{da} &= \xi_i(\varphi,\psi,\omega), \quad \varphi_i\Big|_{a=0} = x_i,\\
\frac{d\psi^\alpha}{da} &= \eta^\alpha(\varphi,\psi,\omega), \quad \psi^\alpha\Big|_{a=0} = u^\alpha,\\
\frac{d\omega^\alpha_i}{da} &= \zeta^\alpha_i(\varphi,\psi,\omega), \quad \omega^\alpha_i\Big|_{a=0} = u^\alpha_{,i},
\end{aligned}$$

and ω denotes the set of transformations of the first derivatives, or $\omega = \{\omega_i^\alpha\}$.

First-order tangency conditions require ζ_i^α to satisfy

$$\zeta_i^\alpha = D_i(\eta^\alpha - \xi_j u_{,j}^\alpha) + \xi_j u_{,ji}^\alpha, \tag{3.11}$$

where $D_i = \dfrac{\partial}{\partial x_i} + u_{,i}^\beta \dfrac{\partial}{\partial u^\beta} + u_{,ij}^\beta \dfrac{\partial}{\partial u_{,j}^\beta} + \ldots + u_{,ii_1\cdots i_n}^\beta \dfrac{\partial}{\partial u_{,i_1\cdots i_n}^\beta} + \ldots$

Finally the corresponding infinitesimal generator is

$$X = \xi_i(x, u, u')\frac{\partial}{\partial x_i} + \eta^\alpha(x, u, u')\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x, u, u')\frac{\partial}{\partial u_{,i}^\alpha}. \tag{3.12}$$

3.3 Lie-Bäcklund Transformations

Hereafter we employ the notations[1] $\underset{1}{u}, \underset{2}{u}, \ldots,$ for the sets of first-order, second-order, and other high order partial derivatives $\{u_{,i}^\alpha\}, \{u_{,i_1i_2}^\alpha\}, \ldots,$ where $u_{,i}^\alpha = \dfrac{\partial u^\alpha}{\partial x_i},\ u_{,i_1i_2}^\alpha = \dfrac{\partial^2 u^\alpha}{\partial x_{i_2}\partial x_{i_1}},\ \ldots$

Let z denote sequence

$$z = (x, u, \underset{1}{u}, \underset{2}{u}, \ldots) \tag{3.13}$$

with elements $z^\nu,\ \nu \geq 1$, where, e.g.,

$$z^i = x_i, \quad i = 1, \ldots, n, \quad z^{n+\alpha} = u^\alpha, \quad \alpha = 1, \ldots, m,$$

with the remaining elements representing the derivatives of u. However, in applications, one invariably utilizes only finite subsequences of z, denoted by $[z]$.

Definition 3.1. A *differential function* $f([z])$ is a locally analytic function : f is locally expandable in a Taylor series with respect to all arguments. The highest order of derivatives appearing in f is called the *order* of the differential function and is denoted by ord(f), e.g. if $f([z]) = f(z, u, \underset{1}{u}, \ldots, \underset{s}{u})$, then ord$(f) = s$.

[1]The notations and material in this section follow Ibragimov (1996)

The space of all differential functions of all finite orders is denoted by $\mathcal{A}$. This space is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is the usual multiplication of functions. Furthermore, it has the important property of being closed under the differentiation given by $D_i = \frac{\partial}{\partial x_i} + u^\alpha_{,i}\frac{\partial}{\partial u^\alpha} + u^\alpha_{,ij}\frac{\partial}{\partial u^\alpha_{,j}} + \ldots$.

Consider an operator of the form

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^\alpha_i \frac{\partial}{\partial u^\alpha_{,i}} + \zeta^\alpha_{i_1 i_2} \frac{\partial}{\partial u^\alpha_{,i_1 i_2}} + \ldots, \tag{3.14}$$

where $\xi_i([z]),\ \eta^\alpha([z]) \in \mathcal{A}$ are the differential functions, and

$$\begin{aligned} \zeta^\alpha_i &= D_i(\eta^\alpha - \xi_j u^\alpha_{,j}) + \xi_j u^\alpha_{,ji}, \\ \zeta^\alpha_{i_1 i_2} &= D_{i_2} D_{i_1}(\eta^\alpha - \xi_j u^\alpha_{,j}) + \xi_j u^\alpha_{,j i_1 i_2}, \\ &\ldots \end{aligned} \tag{3.15}$$

An operator given by equations (3.14) and (3.15) is called a *Lie-Bäcklund operator*. In fact, the operator (3.14) is the infinite-order prolongation[2] of

$$X = \xi_i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi_i,\ \eta^\alpha \in \mathcal{A}. \tag{3.16}$$

Let $[[\mathcal{A}]]$ denote the space of formal power series in one symbol a with coefficients in $\mathcal{A}$. One can prove that, for any Lie-Bäcklund operator (3.14), there exists a unique solution of the Lie-Bäcklund equations

$$\begin{aligned} \frac{d\bar{x}_i}{da} &= \xi_i(\bar{x}, \bar{u}, \ldots, \underset{s}{\bar{u}}), \quad \bar{x}_i\big|_{a=0} = x_i, \\ \frac{d\bar{u}^\alpha}{da} &= \eta^\alpha(\bar{x}, \bar{u}, \ldots, \underset{s}{\bar{u}}), \quad \bar{u}^\alpha\big|_{a=0} = u^\alpha, \\ &\ldots \end{aligned} \tag{3.17}$$

[2]The concept of the prolongation group and prolonged generator will be given in Section 3.4

in the space $[[\mathcal{A}]]$. This solution is given by formal power series, i.e.,

$$\bar{x}_i = x_i + \sum_{\beta=1}^{\infty} A^i_\beta a^\beta,$$
$$\bar{u}^\alpha = u^\alpha + \sum_{\beta=1}^{\infty} B^\alpha_\beta a^\beta, \tag{3.18}$$
$$\ldots$$

with coeffcients $A^i_\beta, B^\alpha_\beta \in \mathcal{A}$, where $A^i_1 = \xi_i(x, u, \ldots, \underset{s}{u})$ and $B^\alpha_1 = \eta^\alpha(x, u, \ldots, \underset{s}{u})$. It is a *formal one-parameter group* that leaves invariant the infinite-order tangency condition

$$d\bar{u}^\alpha - \bar{u}^\alpha_{,j} d\bar{x}_j = 0, d\bar{u}^\alpha_{,i} - \bar{u}^\alpha_{,ij} d\bar{x}_j = 0, \ldots.$$

The formal groups obtained above are called *formal Lie-Bäcklund groups of transformations.* The formal Lie-Bäcklund group of transformations is called a *one-parameter Lie-Bäcklund group of transformations* if the series in equations (3.18) converge.

Definition 3.2. A Lie-Bäcklund operator (3.16) of the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \eta^\alpha \in \mathcal{A},$$

is called a *canonical operator.*

3.4 Prolongations

By definition, groups of point transformations act only on the space of (x, u) of $n + m$ variables. However, to apply these groups to differential equations, one needs the transformations of derivatives. It is thus necessary to extend a group of point transformations acting on the (x, u)-space to groups of point transformations acting on the $(x, u, \underset{1}{u})$-space, $(x, u, \underset{1}{u}, \underset{2}{u})$-space, $\ldots$, $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u})$-space, $s \geq 1$, for a given differential equation with order s. These groups are called *the* *first prolongation group, the second prolongation group, ...,the s-times prolongation group*, respectively, where the transformations are of the form

$$\begin{aligned}
\bar{x} &= \varphi^x(x,u;a) = x + \xi(x,u)a + \cdots\,,\\
\bar{u} &= \varphi^u(x,u;a) = u + \eta(x,u)a + \cdots\,,\\
\underset{1}{\bar{u}} &= \varphi^{\underset{1}{u}}(x,u,\underset{1}{u};a) = \underset{1}{u} + \zeta^{(1)}(x,u,\underset{1}{u})a + \cdots\,,\\
&\vdots\\
\underset{s}{\bar{u}} &= \varphi^{\underset{s}{u}}(x,u,\underset{1}{u},\ldots,\underset{s}{u};a) = \underset{s}{u} + \zeta^{(s)}(x,u,\underset{1}{u},\ldots,\underset{s}{u})a + \cdots\,.
\end{aligned}$$

The prolongation transformation formulas[3] of the components $\{\bar{u}^{\alpha}_{,i}\}$ of $\underset{1}{\bar{u}}$ are determined by

$$\begin{bmatrix} \bar{u}^{\alpha}_{,1} \\ \bar{u}^{\alpha}_{,2} \\ \vdots \\ \bar{u}^{\alpha}_{,n} \end{bmatrix} = \begin{bmatrix} (\varphi^{\underset{1}{u}})^{\alpha}_{1}(x,u,\underset{1}{u};a) \\ (\varphi^{\underset{1}{u}})^{\alpha}_{2}(x,u,\underset{1}{u};a) \\ \vdots \\ (\varphi^{\underset{1}{u}})^{\alpha}_{n}(x,u,\underset{1}{u};a) \end{bmatrix} = A^{-1}\begin{bmatrix} D_1\varphi^u(x,u;a) \\ D_2\varphi^u(x,u;a) \\ \vdots \\ D_n\varphi^u(x,u;a) \end{bmatrix},$$

where A^{-1} is the inverse (assumed to exist) of the matrix

$$A = \begin{bmatrix} D_1\varphi^x_1 & D_1\varphi^x_2 & \cdots & D_1\varphi^x_n \\ D_2\varphi^x_1 & D_2\varphi^x_2 & \cdots & D_2\varphi^x_n \\ \vdots & \vdots & & \vdots \\ D_n\varphi^x_1 & D_n\varphi^x_2 & \cdots & D_n\varphi^x_n \end{bmatrix},$$

and the prolongation transformations formulas of the components $\{\bar{u}^{\alpha}_{,i_1\cdots i_s}\}$ of $\underset{s}{\bar{u}}$

[3]See more detail in Bluman and Kumei (1996)

are determined by

$$\begin{bmatrix} \bar{u}^{\alpha}_{,i_1\cdots i_{s-1}1} \\ \bar{u}^{\alpha}_{,i_1\cdots i_{s-1}2} \\ \vdots \\ \bar{u}^{\alpha}_{,i_1\cdots i_{s-1}n} \end{bmatrix} = \begin{bmatrix} (\varphi^{\underset{s}{u}})^{\alpha}_{i_1\cdots i_{s-1}1}(x,u,\underset{1}{u},\ldots,\underset{s}{u};a) \\ (\varphi^{\underset{s}{u}})^{\alpha}_{i_1\cdots i_{s-1}2}(x,u,\underset{1}{u},\ldots,\underset{s}{u};a) \\ \vdots \\ (\varphi^{\underset{s}{u}})^{\alpha}_{i_1\cdots i_{s-1}n}(x,u,\underset{1}{u},\ldots,\underset{s}{u};a) \end{bmatrix}$$

$$= A^{-1} \begin{bmatrix} D_1[(\varphi^{\underset{s-1}{u}})^{\alpha}_{i_1\cdots i_{s-1}}(x,u,\underset{1}{u},\ldots,\underset{s-1}{u};a)] \\ D_2[(\varphi^{\underset{s-1}{u}})^{\alpha}_{i_1\cdots i_{s-1}}(x,u,\underset{1}{u},\ldots,\underset{s-1}{u};a)] \\ \vdots \\ D_n[(\varphi^{\underset{s-1}{u}})^{\alpha}_{i_1\cdots i_{s-1}}(x,u,\underset{1}{u},\ldots,\underset{s-1}{u};a)] \end{bmatrix}.$$

The formulas of the coefficients, $\zeta^{\alpha}_{i},\ldots,\zeta^{\alpha}_{i_1\cdots i_s}$, of the infinitesimal generator are determined by

$$\begin{aligned} \zeta^{\alpha}_{i} &= D_i(\eta^{\alpha}) - u^{\alpha}_{,j}D_i(\xi_j), \\ \zeta^{\alpha}_{i_1 i_2} &= D_{i_2}(\zeta^{\alpha}_{i_1}) - u^{\alpha}_{,i_1 j}D_{i_2}(\xi_j), \\ &\vdots \\ \zeta^{\alpha}_{i_1\cdots i_s} &= D_{i_s}(\zeta^{\alpha}_{i_1\cdots i_{s-1}}) - u^{\alpha}_{,i_1\cdots i_{s-1}j}D_{i_s}(\xi_j). \end{aligned}$$

Thus the first prolonged generator of (3.8) is

$$X_{(1)} = X + \zeta^{\alpha}_{i}\frac{\partial}{\partial u^{\alpha}_{,i}} = \xi_i\frac{\partial}{\partial x_i} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \zeta^{\alpha}_{i}\frac{\partial}{\partial u^{\alpha}_{,i}},$$

and the s-times prolonged generator is also written similarly:

$$X_{(s)} = X_{(s-1)} + \zeta^{\alpha}_{i_1\cdots i_s}\frac{\partial}{\partial u^{\alpha}_{,i_1\cdots i_s}}.$$

This concept of the prolongation group can also be applied to groups of contact transformations and Lie-Bäcklund groups of transformations.

3.5 Symmetry Groups

Lie related groups and differential equations through the following idea.

Definition 3.3 (Admitted group). A *symmetry group of a system of differential equations* is a group of transformations mapping every solution to another solution of the same system. A symmetry group is also termed the *group admitted by the system*, or an *admitted group,* and that system of differential equations is said to be *invariant* under the symmetry group.

Consider a system of differential equations,

$$F(x, u, \underset{1}{u}, \cdots, \underset{s}{u}) = 0. \tag{3.19}$$

Let $u = \upsilon(x)$ be a solution of system (3.19) and let the transformations depending on a parameter a, $\bar{x} = \varphi^x(x, u; a)$, $\bar{u} = \varphi^u(x, u; a)$, belong to a group admitted by system (3.19). Therefore, by the definition of an admitted group,

$$\bar{x} = \varphi^x(x, \upsilon(x); a),$$
$$\bar{u} = \varphi^u(x, \upsilon(x); a),$$

must be another solution of system (3.19). Hence

$$F(\bar{x}, \bar{u}, \underset{1}{\bar{u}}, \cdots, \underset{s}{\bar{u}}) = 0, \tag{3.20}$$

whenever u satisfies system (3.19). This implies that system (3.20) is invariant with respect to the parameter a:

$$\left.\frac{\partial F(\bar{x}, \bar{u}, \underset{1}{\bar{u}}, \cdots, \underset{s}{\bar{u}})}{\partial a}\right|_{a=0,\ (3.19)} \equiv 0. \tag{3.21}$$

Definition 3.4. Equation (3.21) is called the *determining equation.*

Chapter IV

Application of Group Analysis to DDEs

In this thesis, an algorithm for finding a symmetry group for a given DDE is suggested beginning from constructing the determining equation. The determining equation obtained is an equation of unknowns ξ and η (or $\xi,\ \eta$ and ζ in the case of contact transformations, or $\xi,\ \eta,\ \zeta, \ldots,$ in the case of Lie-Bäcklund transformations). After solving the determining equation, one can find groups of transformations by solving the Lie equations, equations (3.6), as mentioned in the previous chapter.

For the sake of simplicity, only equations of a single dependent variable and a single independent variable are considered in this chapter.

4.1 Constructing Determining Equations

Let G be a symmetry group of transformations φ_a admitted by the DDE (2.2). Here φ_a is defined by

$$\bar{x} = \varphi^x(x,u;a), \quad \bar{u} = \varphi^u(x,u;a),$$

which depend on a real parameter a, $x \equiv \varphi^x(x,u;0)$ and $u \equiv \varphi^u(x,u;0)$. Suppose $u = v(x)$ is a solution of the equation (2.2).

One can relate this solution with another solution, a transformed solution, of the same equation by letting

$$\bar{x} = \varphi^x(x, \upsilon(x); a), \tag{4.1}$$

$$\bar{u} = \varphi^u(x, \upsilon(x); a). \tag{4.2}$$

In order to write $\bar{u}$ as a function of $\bar{x}$, we first have to derive

$$x = \psi(\bar{x}; a) \tag{4.3}$$

from equation (4.1).

For ODEs, the *local inverse function theorem* guarantees equation (4.3). But for DDEs, the domain of the function φ^x is considered not only in a neighborhood of x, but also in an interval $[x - r, x]$. By definition of an admitted group, it is required that the symmetry group possesses the property that allows equation (4.1) to be derivable to equation (4.3).

The transformed solution $\bar{u} = \upsilon_a(\bar{x})$ can be written as a function of $\bar{x}$ by

$$\bar{u} = \upsilon_a(\bar{x}) = \varphi^u(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a). \tag{4.4}$$

So

$$\bar{u}' = \frac{d\upsilon_a(\bar{x})}{d\bar{x}} = (\varphi^u_{,1} + \varphi^u_{,2}\upsilon'(\psi(\bar{x}; a)))\frac{\partial\psi(\bar{x}; a)}{\partial\bar{x}}, \tag{4.5}$$

where $f_{,i}$ denotes the partial derivative of f with respect to i-th argument.

By equation (2.2), $F(\bar{x}, \bar{u}_{\bar{x}})$ can be written as

$$f(\bar{x}, \varphi^u(\psi(g_1(\bar{x}); a), \upsilon(\psi(g_1(\bar{x}); a)); a), ..., \varphi^u(\psi(g_q(\bar{x}); a), \upsilon(\psi(g_q(\bar{x}); a)); a).$$

Let $\Xi(x, u_x, u') = u' - F(x, u_x)$. Then

$$\Xi(x, u_x, u') \equiv 0, \tag{4.6}$$

for $u = \upsilon(x)$ and x in some interval considered. Since the transformation φ_a transforms a solution to another solution of the same system, this means that $\Xi(\bar{x}, \bar{u}_{\bar{x}}, \bar{u}')$

depending on the independent variable $\bar{x}$ and the parameter a is identically equal to zero :

$$\Xi(\bar{x}, \bar{u}_{\bar{x}}, \bar{u}') \equiv 0.$$

With the symmetry transformation φ_a, the function Ξ is invariant with respect to the parameter a. Thus

$$\left. \frac{\partial \Xi(\bar{x}, \bar{u}_{\bar{x}}, \bar{u}')}{\partial a} \right|_{a=0} = 0, \tag{4.7}$$

where equation (4.6) is satisfied, is a *determining equation for the DDE.*

Remark. In differential equations there are two approaches for finding determining equations, using variables $(\bar{x}, a)$ and using the variables (x, a). The determining equations obtained by both approaches are equivalent, that is, they are identical on the manifold of solutions. For FDEs and DDEs, however, finding determining equations via the approach using $(\bar{x}, a)$ is simpler.

4.2 Splitting the Determining Equations

One can simplify the obtained determining equation by splitting it with respect to the variables and functions involved into several equations.

Let

$$\xi(x, u) = \left. \frac{\partial \varphi^x(x, u; a)}{\partial a} \right|_{a=0} \quad \text{and} \quad \eta(x, u) = \left. \frac{\partial \varphi^u(x, u; a)}{\partial a} \right|_{a=0}.$$

By equations (4.4) and (4.5), one obtains[1]

$$\begin{aligned} \left.\frac{\partial \bar{u}'}{\partial a}\right|_{a=0} = \left.\frac{\partial \left(dv_a(\bar{x})/d\bar{x}\right)}{\partial a}\right|_{a=0} = \; & \eta_{,1}(x, v(x)) \\ & + \left[\eta_{,2}(x, v(x)) - \xi_{,1}(x, v(x))\right] v'(x) \\ & - \xi_{,2}\,(x, v(x))[v'(x)]^2 - \xi(x, v(x))v''(x), \quad (4.8) \\ \left.\frac{\partial \bar{u}(g_\lambda(\bar{x}))}{\partial a}\right|_{a=0} = \left.\frac{\partial v_a(g_\lambda(\bar{x}))}{\partial a}\right|_{a=0} = & - \xi(g_\lambda(x), v(g_\lambda(x)))v'(g_\lambda(x)) \\ & + \eta(g_\lambda(x), v(g_\lambda(x))), \end{aligned}$$

[1]See details in Appendix B.

where $g_\lambda(x)$ is a delay term which is $x - r \leq g_\lambda(x) \leq x$, $\lambda = 1, \ldots, q$. Note that the partial derivatives in equation (4.8) hold $\bar{x}$ constant.

Let Δ and Θ_λ denote the right hand side terms of each of the equations (4.8) respectively. Equation (4.7) then becomes

$$\left[\sum_{\lambda=1}^{q} \Xi_{,\lambda+1}\,(x, u_x, u')\Theta_\lambda + \Xi_{,q+2}\,\Delta\right]_{(4.6)} \equiv 0. \tag{4.9}$$

Then the variables and functions involved are:

- $x, \upsilon(x), \upsilon(g_\lambda(x)), \upsilon'(g_\lambda(x)), \upsilon''(g_\lambda(x)), ...$
- the functions η and ξ of $(x, \upsilon(x))$ or of $(g_\lambda(x), \upsilon(g_\lambda(x)))$ and their derivatives $\eta_{,1}\,, \xi_{,1}\,, \eta_{,2}\,, \xi_{,2}\,, ...$

where $\lambda = 1, ..., q$.

Note that $\upsilon'(x)$ is not an involved function because equation (4.9) is considered on the manifold (4.6), i.e. $\upsilon'(x) = F(x, u_x)$.

The determining equation (4.9) is a first order delay partial differential equation for the unknowns ξ and η, which must be satisfied identically for any $x \geq x_0$ in a neighborhood of x_0 and for any solution $\upsilon(x)$ of equation (4.6). For any initial function $\theta(x),\ x \in [\gamma, x_0]$, the *local existence theorem 2.1* guarantees the existence of a solution $\upsilon(x)$. Therefore, by the arbitrariness of $x \in [\gamma, \beta_1]$ and the initial function $\theta(x)$, the variable x and the functions $\upsilon(x), \upsilon(g_\lambda(x)), \upsilon'(g_\lambda(x)), \upsilon''(g_\lambda(x)), ...,$ can be considered as arbitrary elements. If the determining equation is written as a polynomial of these arbitrary elements, the coefficients of monomials in the equation must vanish. This allows us to split the determining equation into a system of several equations, becoming an overdetermined system which can be solved analytically. Examples of splitting the determining equations and solving the resulting overdetermined systems are given in Chapter VI.

4.3 Another Representation of the Determining Equations for DDEs

One found that

$$\left.\frac{\partial \Xi(\bar{x}, \bar{u}_{\bar{x}}, \bar{u}')}{\partial a}\right|_{a=0} = \sum_{\lambda=1}^{q} \Xi_{,\lambda+1}(x, u_x, u')\Theta_\lambda + \Xi_{,q+2}\,\Delta. \tag{4.10}$$

Equation (4.10) also coincides with the equation obtained by applying the prolongation of the *canonical Lie-Bäcklund operator*,

$$\tilde{X} = (\eta(x, u) - \xi(x, u)u')\,\partial_u + \sum_{\lambda=1}^{q}(\eta(g_\lambda(x), u_\lambda) - \xi(g_\lambda(x), u_\lambda)u'_\lambda)\partial_{u_\lambda},$$

where $u_\lambda = \upsilon(g_\lambda(x))$, to equation (4.6), i.e.

$$\left.\frac{\partial \Xi(\bar{x}, \bar{u}_{\bar{x}}, \bar{u}')}{\partial a}\right|_{a=0} = \tilde{X}_{(1)}\Xi, \tag{4.11}$$

where

$$\tilde{X}_{(1)} = \tilde{X} + \big(\eta_{,1}(x, u) + [\eta_{,2}(x, u) - \xi_{,1}(x, u)]\,u' - \xi_{,2}(x, u)[u']^2 - \xi(x, u)u''\big)\,\partial_{u'}.$$

Thus, the determining equation (4.7) can also be written as

$$\left.\tilde{X}_{(1)}\Xi\right|_{(4.6)} \equiv 0. \tag{4.12}$$

Chapter V

Application of Group Analysis to FDEs

Although FDEs are defined in a more general way than DDEs, the algorithm for finding a symmetry group for an FDE is only slightly different from the algorithm discussed in the previous chapter.

By definition of a symmetry group as a group of transformations converting any solution of a differential equation into a solution of the same equation, the possibility where the equation admitting the group has no solution can be excluded. However, there is no general theory which guarantees the existence of a solution of a given FDE. Hereafter, the considered FDEs are assumed to have solutions.

5.1 Constructing Determining Equations

The algorithm for finding a symmetry group for a given FDE also starts from constructing a determining equation *a priori.*

Let

$$\Xi(\chi_1, \ldots, \chi_q) = 0 \tag{5.1}$$

be a system of functional differential equations, where $\chi_1, \ldots, \chi_q$ are functionals of $(x, u, \underset{1}{u}, \ldots, \underset{s}{u})$. Let G be a symmetry group of transformation φ_a admitted by a system of FDEs (5.1):

$$\bar{x} = \varphi^x(x, u; a), \quad \bar{u} = \varphi^u(x, u; a),$$

which depend on a real parameter a, $x \equiv \varphi^x(x, u; 0)$ and $u \equiv \varphi^u(x, u; 0)$. Suppose $u = v(x)$ is a solution of equation (5.1).

The solution can be related with another solution of the same equation by letting

$$\bar{x} = \varphi^x(x, \upsilon(x); a), \tag{5.2}$$

$$\bar{u} = \varphi^u(x, \upsilon(x); a). \tag{5.3}$$

One must assume a priori that the group possesses the property that equation (5.2) can be derived to

$$x = \psi(\bar{x}; a) = (\psi_1(\bar{x}; a), \ldots, \psi_n(\bar{x}; a)). \tag{5.4}$$

Then the transformed solution $\bar{u} = \upsilon_a(\bar{x})$ and its derivatives can be written as functions of $\bar{x}$ by

$$\begin{aligned}
\bar{u}^\alpha &= \upsilon_a^\alpha(\bar{x}) = \varphi_\alpha^u(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a), \\
\bar{u}_{,i}^\alpha &= \frac{\partial \upsilon_a^\alpha(\bar{x})}{\partial \bar{x}_i} \\
&= \sum_{\beta=1}^{n} (\varphi_\alpha^u)_{,\beta} \frac{\partial \psi_\beta(\bar{x}; a)}{\partial \bar{x}_i} + \sum_{\gamma=1}^{m} \left((\varphi_\alpha^u)_{,(n+\gamma)} \sum_{\beta=1}^{n} \upsilon_{,\beta}^\gamma(\psi(\bar{x}; a)) \frac{\partial \psi_\beta(\bar{x}; a)}{\partial \bar{x}_i} \right), \\
&\vdots
\end{aligned}$$

Let $\underset{1}{\bar{u}}, \ldots, \underset{s}{\bar{u}}$ be the sets $\{\bar{u}_{,i}^\alpha\}, \ldots, \{\bar{u}_{,i_1 \cdots i_s}^\alpha\}$ and $\bar{\chi}_1, \ldots, \bar{\chi}_q$ be the functionals $\chi_1, \ldots, \chi_q$ of $(\bar{x}, \bar{u}, \underset{1}{\bar{u}}, \ldots, \underset{s}{\bar{u}})$ respectively. It can be said that $\Xi(\bar{\chi}_1, \ldots, \bar{\chi}_p)$ must be invariant with respect group G if $u = \upsilon(x)$ is an solution of system (5.1).

Therefore the *determining equations for FDEs (5.1)* can be defined by

$$\left. \frac{\partial \Xi(\bar{\chi}_1, \ldots, \bar{\chi}_p)}{\partial a} \right|_{a=0} \equiv 0, \tag{5.5}$$

where system (5.1) satisfies.

5.2 Splitting the Determining Equations

To solve determining equations for FDEs, one can simplify them by splitting into several equations with respect to the variables and functions involved.

Let

$$\xi_i(x,u) = \left.\frac{\partial \varphi_i^x(x,u;a)}{\partial a}\right|_{a=0} \quad \text{and} \quad \eta^\alpha(x,u) = \left.\frac{\partial \varphi_\alpha^u(x,u;a)}{\partial a}\right|_{a=0},$$

$i = 1, \ldots, n,\ \alpha = 1, \ldots, m.$

The determining equations (5.5) are equations for unknown functions ξ_i and η^α. Because of the assumption of existence of a solution of a given FDE, the initial value problem has solutions for arbitrary initial values and functions. Hence, the solution $v(x)$, its derivatives, and the functionals of $v(x)$ and its derivatives can also be considered as arbitrary elements. If the determining equations (5.5) are written as polynomials of these arbitrary elements, the coefficients of monomials in the equations must vanish. This enables us to split the equations with respect to arbitrary elements. After splitting, the process of solving the determining equations for FDEs is similar to the process of obtaining solutions of the determining equations of ODEs and PDEs.

A demonstration will be given in the next chapter.

5.3 Another Representation of the Determining Equations for FDEs

Comparing determining equations for ODEs, PDEs and DDEs, one can also write the determining equations for FDEs in an infinitesimal generator form.

Let

$$\check{X}\Xi = \left.\frac{\partial \Xi(\bar{\chi}_1, \ldots, \bar{\chi}_p)}{\partial a}\right|_{a=0}, \tag{5.6}$$

where $\check{X} = \vartheta^\beta \partial_{\chi_\beta}$. The coefficient ϑ^β is determined by the partial Fréchet derivative[1] of $\bar{\chi}_\beta$ with respect to a parameter a at $a = 0$, $\beta = 1, \ldots, p$.

[1]See the definition of partial Fréchet derivative in Section C.3, appendix C.

The determining equations (5.5) can also be written in the form

$$\check{X}\Xi\Big|_{(5.1)} \equiv 0.$$

Thus we have another representation of the determining equations for FDEs.

Chapter VI

Applications of the Developed Algorithm to DDEs and FDEs

In this chapter, the algorithm developed is applied to some DDEs and FDEs. Determining equations and solutions of determining equations of DDEs and FDEs are constructed here. For the examples in Sections 6.1-6.3, we only construct determining equations. For the examples in Sections 6.4 and 6.5, we more fully explore the implications of our work by solving the determining equations and examining the admitted Lie groups.

6.1 A Linear Retarded Equation

Consider the linear retarded equation,

$$u'(x) = ku(x-r), \tag{6.1}$$

where $k, r =$const, $r > 0$ and $k \neq 0$. This equation is of DDE type.

Let G be an admitted group of transformations

$$\bar{x} = \varphi^x(x, u; a), \quad \bar{u} = \varphi^u(x, u; a),$$

and

$$\xi(x, u) = \left.\frac{\partial \varphi^x(x, u; a)}{\partial a}\right|_{a=0} \quad \text{and} \quad \eta(x, u) = \left.\frac{\partial \varphi^u(x, u; a)}{\partial a}\right|_{a=0}.$$

We find that:

$$\tilde{X}_{(1)}\left[u'(x)-ku(x-r)\right]=\eta_{,1}(x,u(x))+\left[\eta_{,2}(x,u(x))-\xi_{,1}(x,u(x))\right]u'(x)$$
$$-\,\xi_{,2}\,(x,u(x))[u'(x)]^2-\xi(x,u(x))u''(x)$$
$$-\,k\left[\eta(x-r,u(x-r))-\xi(x-r,u(x-r))u'(x-r)\right],$$

where

$$\tilde{X}=(\eta(x,u(x))-\xi(x,u(x))u'(x))\partial_u+(\eta(x-r,u(x-r))-\xi(x-r,u(x-r))u'(x-r))\partial_{u_r}$$

and $u_r=u(x-r)$.

Hence, the determining equation for equation (6.1) is

$$\eta_{,1}(x,u(x))+\left[\eta_{,2}(x,u(x))-\xi_{,1}(x,u(x))\right]ku(x-r)$$
$$-\xi_{,2}\,(x,u(x))[ku(x-r)]^2-\xi(x,u(x))ku'(x-r)$$
$$-k\left[\eta(x-r,u(x-r))-\xi(x-r,u(x-r))u'(x-r)\right]=0.$$

6.2 An Integro-Differential Equation

For an integro-differential equation

$$u'(x)=\int_{-r}^{0}u_x(s)ds, \qquad (6.2)$$

where $r=$const and $r>0$, one obtains

$$\tilde{X}_{(1)}\left(\bar{u}'(\bar{x})-\int_{-r}^{0}\bar{u}_{\bar{x}}(s)ds\right)=$$
$$\eta_{,1}(x,u(x))+\left[\eta_{,2}(x,u(x))-\xi_{,1}(x,u(x))\right]u'(x)$$
$$-\xi_{,2}\,(x,u(x))[u'(x)]^2-\xi(x,u(x))u''(x)$$
$$-\int_{-r}^{0}\left[\eta(x+s,u(x+s))-\xi(x+s,u(x+s))u'(x+s)\right]ds.$$

The determining equation for the integro-differential equation (6.2) follows:

$$\eta_{,1}(x,u(x))+\left[\eta_{,2}(x,u(x))-\xi_{,1}(x,u(x))\right]\int_{-r}^{0}u_x(s)ds$$
$$-\xi_{,2}\,(x,u(x))[\int_{-r}^{0}u_x(s)ds]^2-\xi(x,u(x))\int_{-r}^{0}u'(x+s)ds$$
$$-\int_{-r}^{0}\left[\eta(x+s,u(x+s))-\xi(x+s,u(x+s))u'(x+s)\right]ds=0.$$

6.3 The Equation $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = g(u, \hat{u})$

Let us consider the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = g(u, \hat{u}), \tag{6.3}$$

where $u = u(x, t)$, $\hat{u} = u(x, t - r)$, $r > 0$, and g is a function of u and $\hat{u}$ such that $\dfrac{\partial g}{\partial \hat{u}} \neq 0$ (otherwise it is not a delay differential equation.)

Let G be an admitted group of transformations

$$\bar{x} = \varphi^x(x, t, u; a), \quad \bar{t} = \varphi^t(x, t, u; a), \quad \bar{u} = \varphi^u(x, t, u; a),$$

and

$$\xi(x, t, u) = \left.\frac{\partial \varphi^x(x, t, u; a)}{\partial a}\right|_{a=0}, \quad \eta(x, t, u) = \left.\frac{\partial \varphi^t(x, t, u; a)}{\partial a}\right|_{a=0},$$
$$\zeta(x, t, u) = \left.\frac{\partial \varphi^u(x, t, u; a)}{\partial a}\right|_{a=0}.$$

The determining equation for equation (6.3) is

$$\tilde{X}_{(1)}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - g(u, \hat{u})\right)\Bigg|_{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = g(u, \hat{u})} = 0, \tag{6.4}$$

where

$$\tilde{X}_{(1)} = \zeta^u \partial_u + \zeta^{\hat{u}} \partial_{\hat{u}} + \zeta^{u_x} \partial_{u_x} + \zeta^{u_t} \partial_{u_t},$$
$$\zeta^u = -\xi(x, t, u)u_{,1} - \eta(x, t, u)u_{,2} + \zeta(x, t, u),$$
$$\zeta^{\hat{u}} = -\xi(x, t - r, \hat{u})\hat{u}_{,1} - \eta(x, t - r, \hat{u})\hat{u}_{,2} + \zeta(x, t - r, \hat{u}),$$
$$\zeta^{u_x} = -\xi_{,1}(x, t, u)u_{,1} - \xi_{,3}(x, t, u)[u_{,1}]^2 - \xi(x, t, u)u_{,11}$$
$$- \eta_{,1}(x, t, u)u_{,2} - \eta_{,3}(x, t, u)u_{,1}u_{,2} - \eta(x, t, u)u_{,12}$$
$$+ \zeta_{,1}(x, t, u) + \zeta_{,3}(x, t, u)u_{,1},$$
$$\zeta^{u_t} = -\xi_{,2}(x, t, u)u_{,1} - \xi_{,3}(x, t, u)u_{,1}u_{,2} - \xi(x, t, u)u_{,12}$$
$$- \eta_{,2}(x, t, u)u_{,2} - \eta_{,3}(x, t, u)[u_{,2}]^2 - \eta(x, t, u)u_{,22}$$
$$+ \zeta_{,2}(x, t, u) + \zeta_{,3}(x, t, u)u_{,2},$$

and $u_{,1} = \dfrac{\partial u}{\partial x}(x,t)$, $u_{,2} = \dfrac{\partial u}{\partial t}(x,t)$, $\hat{u}_{,1} = \dfrac{\partial u}{\partial x}(x,t-r)$, $\hat{u}_{,2} = \dfrac{\partial u}{\partial t}(x,t-r)$, $g_{,1}(u,\hat{u}) = \dfrac{\partial g}{\partial u}(u,\hat{u})$, $g_{,2}(u,\hat{u}) = \dfrac{\partial g}{\partial \hat{u}}(u,\hat{u})$.

Hence, the determining equation of equation (6.3) is

$$\begin{aligned}
&\zeta_{,2}(x,t,u) + [\zeta_{,3}(x,t,u) - \eta_{,2}(x,t,u)]\, g(u,\hat{u}) - \eta_{,3}(x,t,u)[g(u,\hat{u})]^2 \\
&-\zeta(x,t,u)g_{,1}(u,\hat{u}) - \zeta(x,t-r,\hat{u})g_{,2}(u,\hat{u}) \\
&+ [\zeta_{,1}(x,t,u) - \eta_{,1}(x,t,u)g(u,\hat{u})]\, u \\
&+ [\zeta(x,t,u) - \xi_{,2}(x,t,u) - \xi_{,3}(x,t,u)g(x,\hat{u})]\, u_{,1} \\
&+ [\eta_{,2}(x,t,u) - \xi_{,1}(x,t,u)) + \eta_{,3}(x,t,u)g(u,\hat{u})]\, [uu_{,1}] \\
&+\eta_{,1}(x,t,u)[u^2u_{,1}] \\
&+ [\xi(x,t-r,\hat{u}) - \xi(x,t,u)]\, g_{,2}(u,\hat{u})\hat{u}_{,1} \\
&+ [\eta(x,t-r,\hat{u}) - \eta(x,t,u)]\, g_{,2}(u,\hat{u})\hat{u}_{,2} = 0.
\end{aligned} \tag{6.5}$$

6.4 The Equation $u''(x) = u(x) - u(x-r)$

6.4.1 Solutions of the Determining Equation

Consider the initial value problem

$$u''(x) = u(x) - u(x-r), \tag{6.6a}$$

$$u'(x_0) = u_1, \quad u_{x_0}(s) = \psi(s), \quad s \in [x_0 - r, x_0], \tag{6.6b}$$

where r is a constant, $r > 0$.

The initial value problem (6.6a),(6.6b) is equivalent to the problem

$$u'(x) = \int_{-r}^{0} u_x(s)ds, \quad u_{x_0}(s) = \psi(s), \quad s \in [x_0 - r, x_0]. \tag{6.7}$$

Let G be an admitted group of transformations

$$\bar{x} = \varphi^x(x,u;a), \quad \bar{u} = \varphi^u(x,u;a),$$

and

$$\xi(x,u) = \left.\frac{\partial \varphi^x(x,u;a)}{\partial a}\right|_{a=0} \quad \text{and} \quad \eta(x,u) = \left.\frac{\partial \varphi^u(x,u;a)}{\partial a}\right|_{a=0}.$$

Then the determining equation for equation (6.6a), which is considered in a neighborhood $N_\epsilon^+(x_0)$[1] and a solution $u(x)$ of equation (6.6a), is

$$\tilde{X}_{(2)}\left(u''(x) - u(x) + u(x-r)\right)\Big|_{u''(x)=u(x)-u(x-r)} = 0. \tag{6.8}$$

Here,

$$\begin{aligned}
\tilde{X}_{(2)} &= \tilde{X}_{(1)} + \zeta^{u''}\partial_{u''},\\
\tilde{X}_{(1)} &= (\eta(x,u(x)) - \xi(x,u(x))u'(x))\partial_u\\
&\quad + (\eta(x-r,u(x-r)) - \xi(x-r,u(x-r))u'(x-r))\partial_{u_r} + \zeta^{u'}\partial_{u'},\\
\zeta^{u'} &= \eta_{,1}(x,u) + [\eta_{,2}(x,u) - \xi_{,1}(x,u)]\,u' - \xi_{,2}\,(x,u)[u']^2 - \xi(x,u)u'',\\
\zeta^{u''} &= \eta_{,11}(x,u) + [2\eta_{,12}(x,u) - \xi(x,u)]\,u' + [\eta_{,22}(x,u) - 2\xi_{,2}(x,u)]\,[u']^2\\
&\quad - \xi_{,22}(x,u)\,[u']^3 + [\eta_{,2}(x,u) - 2\xi_{,1}(x,u)]\,u'' - 3\xi_{,2}(x,u)u'u'' - \xi(x,u)u'''.
\end{aligned}$$

The explicit form of the determining equation (6.8) is

$$\begin{gathered}
\eta_{,11}(x,u(x)) - \eta(x,u(x)) + \eta(x-r,u(x-r))\\
+ [\eta_{,2}(x,u(x)) - 2\xi_{,1}(x,u(x))]\,u(x)\\
+ [2\xi_{,1}(x,u(x)) - \eta_{,2}(x,u(x))]\,u(x-r)\\
+ [2\eta_{,12}(x,u(x)) - \xi_{,11}(x,u(x)) - 3\xi_{,2}(x,u(x))u(x)]\,u'(x)\\
+3\xi_{,2}(x,u(x))[u(x-r)u'(x)]\\
+ [\xi(x,u(x)) - \xi(x-r,u(x-r))]\,u'(x-r)\\
+ [\eta_{,22}(x,u(x)) - 2\xi_{,12}(x,u(x))]\,[u'(x)]^2\\
+\xi_{,22}(x,u(x))[u'(x)]^3 = 0.
\end{gathered} \tag{6.9}$$

Equation (6.9) must be identical for any solution of the problem (6.6a),(6.6b). For the fixed value x_0, the values of a solution $u(x)$ and its first

[1]Neighborhood $N_\epsilon^+(x_0)$ means a neighborhood of x_0, which contains x, $x > x_0$.

derivative at the points x_0 and $x_0 - r$ are:

$$\begin{aligned} u(x_0) &= \psi(x_0), \\ u(x_0 - r) &= \psi(x_0 - r), \\ u'(x_0) &= u_1, \\ u'(x_0 - r) &= \psi'(x_0 - r). \end{aligned}$$

Since the value u_1 and the function ψ are arbitrary, the values $u(x_0)$, $u(x_0 - r)$, $u'(x_0)$ and $u'(x_0 - r)$ must also be arbitrary. Since x_0 is arbitrary, this implies that, in the determining equation (6.9), the elements $u(x)$, $u(x - r)$, $u'(x)$ and $u'(x - r)$ are arbitrary.

In the determining equation (6.9), the unknown functions ξ and η and their derivatives depend only on x, $u(x)$ and $u(x - r)$. Then equation (6.9) can be split with respect to $u'(x)$ and $u'(x - r)$ into 5 equations, i.e.

$$u'(x) \;:\; 2\eta_{,12}(x,u) - \xi_{,11}(x,u) - 3\xi_{,2}(x,u)u + 3\xi_{,2}(x,u)u_r = 0, \tag{6.10}$$

$$u'(x-r) \;:\; \xi(x,u) - \xi(x-r,u_r) = 0, \tag{6.11}$$

$$[u'(x)]^2 \;:\; \eta_{,22}(x,u) - 2\xi_{,12}(x,u) = 0, \tag{6.12}$$

$$[u'(x)]^3 \;:\; \xi_{,22}(x,u) = 0, \tag{6.13}$$

$$\begin{aligned} 1 \;:\; & \eta_{,11}(x,u(x)) - \eta(x,u(x)) + \eta(x-r,u(x-r)) \\ & + [\eta_{,2}(x,u(x)) - 2\xi_{,1}(x,u(x))]\, u(x) \\ & + [2\xi_{,1}(x,u(x)) - \eta_{,2}(x,u(x))]\, u(x-r), \end{aligned} \tag{6.14}$$

where $u_r = u(x - r)$. Because u and u_r are independent, equation (6.11) implies that the function $\xi(x,u)$ does not depend on u. Thus equations (6.10)-(6.12) become

$$2\eta_{,12}(x,u) - \xi''(x) = 0, \tag{6.15}$$

$$\xi(x) - \xi(x-r) = 0, \tag{6.16}$$

$$\eta_{,22}(x,u) = 0. \tag{6.17}$$

Equation (6.17) implies $\eta = \kappa_1(x)u + \kappa_2(x)$. The determining equation is now reduced to

$$\begin{gathered}\kappa_2''(x) - \kappa_2(x) + \kappa_2(x-r) \\ + \left[\kappa_1''(x) - 2\xi'(x)\right] u + \left[2\xi'(x) - \kappa_1(x) + \kappa_1(x-r)\right] u_r \\ + \left[2\kappa_1'(x) - \xi''(x)\right] u' = 0.\end{gathered} \tag{6.18}$$

The equation $2\kappa_1'(x) - \xi''(x) = 0$, which is obtained by splitting equation (6.18) with respect to u', implies

$$\kappa_1(x) = \frac{\xi'(x)}{2} + C_1, \tag{6.19}$$

where C_1 is a constant.

Equation (6.16) shows that ξ is periodic, i.e. $\xi(x) = \xi(x-r)$. By equation (6.19), κ_1 must also be periodic, i.e. $\kappa_1(x) = \kappa_1(x-r)$. The determining equation (6.18) is again reduced to

$$\kappa_2''(x) - \kappa_2(x) + \kappa_2(x-r) + \left[\frac{\xi'''(x)}{2} - 2\xi'(x)\right] u + 2\xi'(x)u_r = 0. \tag{6.20}$$

After splitting equation (6.20) with respect to u_r, it shows that $\xi'(x) = 0$ or $\xi = C_2$, where C_2 is constant.

The above process gives us

$$\xi = C_2,$$

$$\eta = C_1 u + \kappa_2(x),$$

where C_1 and C_2 are arbitrary constants and $\kappa_2(x)$ is an arbitrary solution of the equation

$$\kappa_2''(x) - \kappa_2(x) + \kappa_2(x-r) = 0.$$

The functions ξ and η can be used for finding group of transformations by using Lie equations.

The infinitesimal generator corresponding to the admitted Lie group is

$$X = C_2\partial_x + (C_1 u + \kappa_2(x))\partial_u. \tag{6.21}$$

6.4.2 The Admitted Lie Group

Even though we know that one can find the group corresponding to the infinitesimal generator X (6.21) by using the Lie equation, it is difficult to solve the Lie equation, which includes the undetermined function $\kappa_2(x)$. Here, a particular function $\kappa_2(x)$ will be considered in order to find a particular group admitted by equation (6.6a).

Consider

$$\kappa_2(x) = C_3 e^{\mu x},$$

where C_3 is an arbitrary constant, and μ is the positive solution of the equation

$$e^{-\mu r} + \mu^2 - 1 = 0. \tag{6.22}$$

Let

$$\mathcal{F}(\mu) = e^{-\mu r} + \mu^2 - 1,$$

hence, the derivative of $\mathcal{F}$ with respect to μ is

$$\mathcal{F}'(\mu) = -re^{-\mu r} + 2\mu.$$

Here, the function $\mathcal{F}$ is continuous, $\mathcal{F}(0) = 0$, $\mathcal{F}'(\mu) < 0$ for $\mu \leq 0$. Obviously, there exists only one $\mu^* > 0$ such that

$$\mathcal{F}'(\mu^*) = 0.$$

Since $\mathcal{F}'(\mu) > 0$, where $\mu > \mu^*$, the function $\mathcal{F}$ is strictly increasing where $\mu > \mu^*$ and $\mathcal{F}(\mu) > 0$ when μ is sufficiently large. Thus there exists only one positive μ^{**} such that $\mathcal{F}(\mu^{**}) = 0$. This guarantees the existence of the positive solution of equation (6.22).

The infinitesimal generator X corresponding to the particular $\kappa_2(x)$ is

$$X = C_2\partial_x + (C_1 u + C_3 e^{\mu x})\partial_u.$$

By the Lie equations, the group corresponding to the infinitesimal generator X can be found by solving the problem

$$\frac{d\bar{x}}{da} = C_2, \quad \bar{x}\Big|_{a=0} = x,$$
$$\frac{d\bar{u}}{da} = C_1\bar{u} + C_3 e^{\mu\bar{x}}, \quad \bar{u}\Big|_{a=0} = u.$$

In this case, the group of transformations is

$$\begin{aligned}
C_1 = 0, C_2 = 0, C_3 \neq 0 &: \quad \bar{x} = x, \quad \bar{u} = C_3 e^{\mu x} a + u; \\
C_1 = 0, C_2 \neq 0 &: \quad \bar{x} = x + aC_2, \quad \bar{u} = C_4 e^{\mu x}\left(e^{\mu C_2 a} - 1\right) + u, \quad C_4 = \frac{C_3}{C_2}; \\
C_1 \neq 0, \mu C_2 - C_1 \neq 0 &: \quad \bar{x} = x + aC_2, \quad \bar{u} = e^{C_1 a}\left(u + C_5 e^{\mu x}\left[e^{a(\mu C_2 - C_1)} - 1\right]\right), \\
&\qquad C_5 = \frac{C_3}{\mu C_2 - C_1}; \\
C_1 \neq 0, \mu C_2 = C_1 &: \quad \bar{x} = x + aC_2, \quad \bar{u} = e^{C_1 a}\left(u + aC_3 e^{\mu x}\right).
\end{aligned}$$

6.5 The Equation $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \hat{u} - u$

6.5.1 Solutions of the Determining Equation

Consider the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \hat{u} - u, \tag{6.23}$$

where $u = u(x,t)$, $\hat{u} = u(x, t-r)$, $r > 0$, with initial condition

$$u(x,s) = \varpi(x,s), \quad s \in [t_0 - r, t_0].$$

The initial value problem (6.23) has a solution for any arbitrary value t_0 and any arbitrary given function $\varpi(x,s),\ s \in [t_0 - r, t_0]$ (Brandi, 2002).

Let G be an admitted group of transformations

$$\bar{x} = \varphi^x(x,t,u;a), \quad \bar{t} = \varphi^t(x,t,u;a), \quad \bar{u} = \varphi^u(x,t,u;a),$$

and

$$\xi(x,t,u) = \left.\frac{\partial \varphi^x(x,t,u;a)}{\partial a}\right|_{a=0}, \quad \eta(x,t,u) = \left.\frac{\partial \varphi^t(x,t,u;a)}{\partial a}\right|_{a=0},$$

$$\zeta(x,t,u) = \left.\frac{\partial \varphi^u(x,t,u;a)}{\partial a}\right|_{a=0}.$$

The determining equation of equation (6.23) is

$$\begin{aligned}
&\zeta_{,2}(x,t,u) + \zeta(x,t,u) - \zeta(x,t-r,\hat{u}) \\
&+ [\eta_{,2}(x,t,u) + \zeta_{,1}(x,t,u) - \zeta_{,3}(x,t,u)]\,u \\
&+ [\eta_{,1}(x,t,u) - \eta_{,3}(x,t,u)]\,u^2 + [2\eta_{,3}(x,t,u) - \eta_{,1}(x,t,u)]\,[u\hat{u}] \\
&+ [\zeta_{,3}(x,t,u) - \eta_{,2}(x,t,u)]\,\hat{u} - \eta_{,3}(x,t,u)\hat{u}^2 \\
&+ [\zeta(x,t,u) - \xi_{,2}(x,t,u)]\,u_{,1} + [\eta_{,2}(x,t,u) + \xi_{,3}(x,t,u) - \xi_{,1}(x,t,u)]\,uu_{,1} \\
&+ [\eta_{,1}(x,t,u) - \eta_{,3}(x,t,u)]\,u^2u_{,1} - \xi_{,3}(x,t,u)\hat{u}u_{,1} + \eta_{,3}(x,t,u)u\hat{u}u_{,1} \\
&+ [\xi(x,t-r,\hat{u}) - \xi(x,t,u)]\,\hat{u}_{,1} + [\eta(x,t-r,\hat{u}) - \eta(x,t,u)]\,\hat{u}_{,2} = 0,
\end{aligned} \tag{6.24}$$

where $u_{,1} = \dfrac{\partial u}{\partial x}(x,t),\ u_{,2} = \dfrac{\partial u}{\partial t}(x,t),\ \hat{u}_{,1} = \dfrac{\partial u}{\partial x}(x,t-r)$, and $\hat{u}_{,1} = \dfrac{\partial u}{\partial t}(x,t-r)$.

Because of the arbitrariness of the value t_0 and the function $\varpi(x,s)$, one can consider the variables $u_{,1}, \hat{u}_{,1}, \hat{u}_{,2}, u, \hat{u}, x$ and t as independent and arbitrary in the determining equation (6.24).

Splitting the determining equation (6.24) with respect to $\hat{u}_{,1}$ and $\hat{u}_{,2}$, one obtains

$$\xi(x,t-r,\hat{u}) - \xi(x,t,u) = 0, \tag{6.25}$$

$$\eta(x,t-r,\hat{u}) - \eta(x,t,u) = 0, \tag{6.26}$$

respectively.

Because u and $\hat{u}$ are independent, then equations (6.25) and (6.26) imply that the functions $\xi(x,t,u)$ and $\eta(x,t,u)$ do not depend on the variable u and $\xi(x,t)=\xi(x,t-r),\ \eta(x,t)=\eta(x,t-r)$.

Determining equation (6.24) is now simplified to

$$\begin{gathered}\zeta_{,2}(x,t,u)+\zeta(x,t,u)-\zeta(x,t-r,\hat{u})\\ +\left[\eta_{,2}(x,t)+\zeta_{,1}(x,t,u)-\zeta_{,3}(x,t,u)\right]u\\ +\eta_{,1}(x,t)u^2-\eta_{,1}(x,t)[u\hat{u}]\\ +\left[\zeta_{,3}(x,t,u)-\eta_{,2}(x,t)\right]\hat{u}\\ +\left[\zeta(x,t,u)-\xi_{,2}(x,t)\right]u_{,1}+\left[\eta_{,2}(x,t)-\xi_{,1}(x,t)\right]uu_{,1}\\ +\eta_{,1}(x,t)u^2u_{,1}=0.\end{gathered}\tag{6.27}$$

Splitting the determining equation (6.27) with respect to $u_{,1}$, one obtains $\zeta(x,t,u)=\xi_{,2}(x,t)-\left[\eta_{,2}(x,t)-\xi_{,1}(x,t)\right]u-\eta_{,1}(x,t)u^2$.

After that, the determining equation (6.27) is simplified to

$$\begin{gathered}\xi_{,22}(x,t)+\left[2\xi_{,12}(x,t)+\eta_{,2}(x,t)-\eta_{,22}(x,t)\right]u\\ +\left[\xi_{,11}(x,t)+2\eta_{,1}(x,t)-2\eta_{,12}(x,t)\right]u^2\\ -\eta_{,11}(x,t)u^3-3u\hat{u}\eta_{,1}(x,t)\\ -\eta_{,2}(x,t)\hat{u}+\eta_{,1}(x,t)\hat{u}^2=0.\end{gathered}\tag{6.28}$$

Splitting determining equation (6.28) with respect to $\hat{u}$ and $\hat{u}^2$ implies that η is constant. Let $\eta=C_3$, where C_3 is an arbitrary constant. After splitting the determining equation (6.28) with respect to u and u^2, one obtains

$$\begin{aligned}\xi_{,11}(x,t)&=0,\\ \xi_{,12}(x,t)&=0,\\ \xi_{,22}(x,t)&=0,\end{aligned}$$

or $\xi(x,t)=C_1x+C_2+C_4t$ where C_1,C_2 and C_4 are arbitrary constants. However, equation (6.25) implies that C_4 must vanish.

Therefore $\xi = C_1 x + C_2$, $\eta = C_3$ and $\zeta = C_1 u$.

The infinitesimal generator corresponding to the Lie group is

$$X = (C_1 x + C_2)\partial_x + C_3 \partial_t + C_1 u \partial_u. \tag{6.29}$$

One of these nonzero constants can be assumed equal to 1.

6.5.2 The Admitted Lie Group

The Lie group admitted by the equation (6.23) can be obtained by solving the problem

$$\begin{aligned} \frac{d\bar{x}}{da} &= C_1 \bar{x} + C_2, \quad \bar{x}\Big|_{a=0} = x, \\ \frac{d\bar{t}}{da} &= C_3, \quad \bar{t}\Big|_{a=0} = t, \\ \frac{d\bar{u}}{da} &= C_1 \bar{u}, \quad \bar{u}\Big|_{a=0} = u. \end{aligned}$$

Here, by theorem in C.4, appendix C, there must be two functionally independent invariants. In order to find the group of transformations and invariants, one may calculate according the following cases :

Case 1. $C_1 = 0$, $C_2 = 0$, $C_3 = 1$:

The group of transformations is

$$\bar{x} = x, \quad \bar{t} = t + a, \quad \bar{u} = u.$$

After solving the characteristic system, the invariants are u and x. According to the algorithm for constructing invariant solutions, one has to assume dependence of one invariant on another. Then the representation of an invariant solution is

$$u = H_1(x),$$

where H_1 is defined by the equation

$$H_1(x) H_1'(x) = 0.$$

This equation is obtained by substituting the representation of the invariant solution into equation (6.23).

Case 2. $C_1 = 0$, $C_2 = 1$, $C_3 = 0$:

The group of transformations is

$$\bar{x} = x + a, \quad \bar{t} = t, \quad \bar{u} = u.$$

The invariants are u and t. Then the representation of an invariant solution is

$$u = H_2(t),$$

where H_2 is defined by the equation

$$H_2'(t) = H_2(t - r) - H_2(t).$$

Case 3. $C_1 = 0$, $C_2 \neq 0$, $C_3 = 1$:

The group of transformations is

$$\bar{x} = x + C_2 a, \quad \bar{t} = t + a, \quad \bar{u} = u.$$

The invariants are u and $x - C_2 t$. Then the representation of an invariant solution is

$$u = H_3(x - C_2 t),$$

where H_3 is defined by the equation

$$H_3'(\tilde{x})\,(H_3(\tilde{x}) - C_2) = H_3(\tilde{x} + C_2 r) - H_3(\tilde{x}), \quad \tilde{x} = x - C_2 t.$$

Case 4. $C_1 = 1$, $C_2 = 0$, $C_3 = 0$:

The group of transformations is

$$\bar{x} = xe^a, \quad \bar{t} = t, \quad \bar{u} = ue^a.$$

The invariants are $\dfrac{u}{x}$ and t. Then the representation of an invariant solution is

$$u = xH_4(t),$$

where H_4 is defined by the equation

$$H_4'(t) + H_4(t)^2 + H_4(t) - H_4(t - r) = 0.$$

Case 5. $C_1 \neq 0$, $C_2 = 0$, $C_3 = 1$:

The group of transformations is

$$\bar{x} = xe^{C_1 a}, \quad \bar{t} = t + a, \quad \bar{u} = ue^{C_1 a}.$$

The invariants are $\dfrac{u}{x}$ and $C_1 xe^{-C_1 t}$. Then the representation of an invariant solution is

$$u = xH_5(C_1 xe^{-C_1 t}),$$

where H_5 is defined by the equation

$$\tilde{t}H_5'(\tilde{t})\left(H_5'(\tilde{t}) - C_1\right) + H_5(\tilde{t})^2 + H_5(\tilde{t}) - H_5(e^{C_1 r}\tilde{t}) = 0, \quad \tilde{t} = C_1 xe^{-C_1 t}.$$

Case 6. $C_1 = 1$, $C_2 \neq 0$, $C_3 = 0$:

The group of transformations is

$$\bar{x} = (x + C_2)e^a - C_2, \quad \bar{t} = t, \quad \bar{u} = ue^a.$$

The invariants are $\dfrac{u}{x + C_2}$ and t. Then the representation of an invariant solution is

$$u = (x + C_2)\, H_6(t),$$

where H_6 is defined by the equation

$$H_6'(t) + H_6(t)^2 + H_6(t) - H_6(t - r) = 0.$$

Case 7. $C_1 = 1$, $C_2 \neq 0$, $C_3 \neq 0$:

The group of transformations is

$$\bar{x} = (x + C_2)e^a - C_2, \quad \bar{t} = t + C_3 a, \quad \bar{u} = ue^a.$$

The invariants are $\dfrac{u}{x + C_2}$ and $(x + C_2)e^{-\frac{t}{C_3}}$. Then the representation of an invariant solution is

$$u = (x + C_2)\, H_7\left((x + C_2)e^{-\frac{t}{C_3}}\right),$$

where H_7 is defined by the equation

$$\tau H_7'(\tau)\left(H_7(\tau) - \frac{1}{C_3}\right) + H_7(\tau)^2 H_7(\tau) - H_7(\tau e^{\frac{r}{C_3}}) = 0, \quad \tau = (x + C_2)e^{-\frac{t}{C_3}}.$$

Chapter VII

Conclusion

The goal of this thesis was to develop an algorithm for applying group analysis to functional differential equations.

The definitions and theorems concerning group analysis for DDEs and FDEs were established in the thesis. The given algorithms show how one can obtain a symmetry group for DDEs and FDEs. The method of applying group analysis to FDEs consists of the following steps:

- Constructing the determining equations.

  The construction of determining equations plays a key role in this algorithm. Determining equations are obtained through invariance of the given FDEs with respect to symmetry transformations: transformations which map any solution of the equation to a solution of the same equation. Thus, the existence of solutions of the considered FDE is required *a priori*.

- Splitting the determining equations.

  In order to simplify the determining equations, one can split them into several equations. The main idea that allows us to split the determining equations is the arbitrariness of the initial elements of the problem.

- Solving the determining equations.

  After solving the system of split equations, the result obtained will be used for finding the symmetry transformations via the Lie equations. These symmetry transformations can be used to find invariant solutions as is done for applications of group analysis to ODEs and PDEs.

Through the above discussion, we have obtained a tool for solving functional differential equations.

Even though the definition of an admitted group given in the thesis follows the classical definition of an admitted group given by Lie (1891), the algorithm developed in the thesis allows constructing determining equations without the requirement of the existence of the solution. This leads us to the definition of an admitted group which has weaker condition:

An admitted group is a group of transformations, which is obtained from solving the determining equation.

For ODEs and PDEs, the classical definition of an admitted group and the latter definition are equivalent for some classes of equations. For FDEs, however, the admitted groups obtained by the alternative definition are more general than the admitted groups obtained by the classical definition. The study of the difference between symmetry groups for FDEs obtained by both definitions are subjects of further research.

The concept of admitted group suggested in this thesis applied to DDEs and FDEs is extendable to other types of transformations such as contact transformations and Lie-Bäcklund transformations.

Finally, the author expects that one may extend the concepts of group analysis given in this thesis to other type of equations, e.g. stochastic equations.

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Appendices

Appendix A

Notation

SET-THEORETICAL SYMBOLS

$\emptyset$	Empty set
$\in$	Element of a set
$\subset$	Set inclusion
$\cap$	Intersection of sets
$\cup$	Union of sets
$\times$	Product of sets
$\otimes$	Tensor multiplication
$\oplus$	Direct sum of sets
$\{...\}$	Set, characterized by elements inside braces
$\{x\|S\}$	Set of elements, possessing property S
$\mathcal{B}(X,Y)$	Vector space of all bounded linear operators from a normed space X into a normed space Y
$\mathcal{C}(A,B)$	Set of all continuous mappings $A \to B$
$\mathcal{C}^s(\Omega)$	s-times continuously differentiable real-valued functions on Ω
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex number
$\mathbb{R}^n$	Real Euclidean space of n-dimensional column vectors, with norm $\|\cdot\|$

Appendix B

Derived Equations

Let x be the independent variable and u be the dependent variable. Consider an ordinary differential equation

$$F(x, u, u', ..., u^{(s)}) = 0. \tag{B.1}$$

Let

$$\bar{x} = \varphi^x(x, u; a),$$

$$\bar{u} = \varphi^u(x, u; a),$$

with conditions

$$x \equiv \varphi^x(x, u; 0),$$

$$u \equiv \varphi^u(x, u; 0),$$

be transformations of group admitted by equation (B.1).

Define

$$\xi(x, u) = \left.\frac{\partial \varphi^x(x, u; a)}{\partial a}\right|_{a=0},$$

$$\eta(x, u) = \left.\frac{\partial \varphi^u(x, u; a)}{\partial a}\right|_{a=0}.$$

If $u = \upsilon(x)$ is a solution of equation (B.1), then it implies that

$$\bar{x} = \varphi^x(x, \upsilon(x); a), \tag{B.2}$$

$$\bar{u} = \varphi^u(x, \upsilon(x); a), \tag{B.3}$$

is also a solution of equation (B.1).

Suppose that one can derive

$$x = \psi(\bar{x}; a),$$

from equation (B.2), where $x \equiv \psi(x; 0)$.

Hereafter, all constructions are considered in the space of variables $(\bar{x}, \bar{u}, a)$, i.e.

$$\bar{x} \equiv \varphi^x(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a), \tag{B.4}$$

$$\bar{u} = \upsilon_a(\bar{x}) = \varphi^u(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a), \tag{B.5}$$

where $\bar{x}$ is the independent variable and $\bar{u}$ is the dependent variable.

B.1 The Function $\left.\dfrac{\partial \bar{u}\,(g_\lambda(\bar{x}))}{\partial a}\right|_{a=0}$

Differentiating equation (B.4) with respect the parameter a, one obtains

$$\frac{\partial \bar{x}}{\partial a} = \frac{\partial \varphi^x(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a)}{\partial a}. \tag{B.6}$$

If the variable $\bar{x}$ is fixed, equation (B.6) is

$$\begin{aligned} 0 = {} & \varphi^x_{,1}(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a)\frac{\partial \psi(\bar{x}; a)}{\partial a} \\ & + \varphi^x_{,2}(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a)\upsilon'(\psi(\bar{x}; a))\frac{\partial \psi(\bar{x}; a)}{\partial a} \\ & + \varphi^x_{,3}(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a). \end{aligned}$$

Hence

$$\frac{\partial \psi(\bar{x}; a)}{\partial a} = \frac{-\varphi^x_{,3}(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a)}{\varphi^x_{,1}(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a) + \varphi^x_{,2}(\psi(\bar{x}; a), \upsilon(\psi(\bar{x}; a)); a)\upsilon'(\psi(\bar{x}; a))}. \tag{B.7}$$

Other derivatives are

$$
\begin{aligned}
&\varphi^x_{,1}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\Big|_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^x(\psi(\bar{x};a)+h,\upsilon(\psi(\bar{x};a));a)-\varphi^x(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^x(\psi(\bar{x};a)+h,\upsilon(\psi(\bar{x};0));0)-\varphi^x(\psi(\bar{x};a),\upsilon(\psi(\bar{x};0));0)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\psi(\bar{x};a)+h-\psi(\bar{x};a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{h}{h}\right]_{a=0}\\
&=1,
\end{aligned}
$$

$$
\begin{aligned}
&\varphi^x_{,2}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\Big|_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^x(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a))+h;a)-\varphi^x(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^x(\psi(\bar{x};0),\upsilon(\psi(\bar{x};a))+h;0)-\varphi^x(\psi(\bar{x};0),\upsilon(\psi(\bar{x};a));0)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\psi(\bar{x};0)-\psi(\bar{x};0)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{0}{h}\right]_{a=0}\\
&=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi^x_{,3}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\Big|_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^x(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a+h)-\varphi^x(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^x(\psi(\bar{x};0),\upsilon(\psi(\bar{x};0));a+h)-\varphi^x(\psi(\bar{x};0),\upsilon(\psi(\bar{x};0));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^x(\bar{x},\upsilon(\bar{x});a+h)-\varphi^x(\bar{x},\upsilon(\bar{x});a)}{h}\right]_{a=0}\\
&=\left[\xi(\bar{x},\upsilon(\bar{x}))\right]_{a=0}\\
&=\xi(x,\upsilon(x)).
\end{aligned}
$$

Similarly, one can find derivatives

$$
\begin{aligned}
&\varphi^u_{,1}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\Big|_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^u(\psi(\bar{x};a)+h,\upsilon(\psi(\bar{x};a));a)-\varphi^u(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^u(\psi(\bar{x};a)+h,\upsilon(\psi(\bar{x};0));0)-\varphi^u(\psi(\bar{x};a),\upsilon(\psi(\bar{x};0));0)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\upsilon(\bar{x})-\upsilon(\bar{x})}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{0}{h}\right]_{a=0}\\
&=0,
\end{aligned}
$$

$$
\begin{aligned}
&\varphi^u_{,2}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\Big|_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^u(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a))+h;a)-\varphi^u(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^u(\psi(\bar{x};0),\upsilon(\psi(\bar{x};a))+h;0)-\varphi^u(\psi(\bar{x};0),\upsilon(\psi(\bar{x};a));0)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\upsilon(\psi(\bar{x};a))+h-\upsilon(\psi(\bar{x};a))}{h}\right]_{a=0}\\
&=\left[\lim_{h\to h}\frac{h}{h}\right]_{a=0}\\
&=1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi^u_{,3}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\Big|_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^u(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a+h)-\varphi^u(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^u(\psi(\bar{x};0),\upsilon(\psi(\bar{x};0));a+h)-\varphi^u(\psi(\bar{x};0),\upsilon(\psi(\bar{x};0));a)}{h}\right]_{a=0}\\
&=\left[\lim_{h\to 0}\frac{\varphi^u(\bar{x},\upsilon(\bar{x});a+h)-\varphi^u(\bar{x},\upsilon(\bar{x});a)}{h}\right]_{a=0}\\
&=[\eta(\bar{x},\upsilon(\bar{x}))]_{a=0}\\
&=\eta(x,\upsilon(x)).
\end{aligned}
$$

Hence

$$\left.\frac{\partial\psi(\bar{x},a)}{\partial a}\right|_{a=0} = \frac{-\xi(x,\upsilon(x))}{1+0\upsilon'(x)} = -\xi(x,\upsilon(x)). \tag{B.8}$$

Differentiating (B.5) with respect to the parameter a (the variable $\bar{x}$ is fixed), one obtains

$$\begin{aligned}\frac{\partial\bar{u}}{\partial a} = \frac{\partial\upsilon_a(\bar{x})}{\partial a} &= \frac{\partial\varphi^u(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)}{\partial a}\\ &= \varphi^u_{,1}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\frac{\partial\psi(\bar{x};a)}{\partial a}\\ &\quad+\varphi^u_{,2}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a)\upsilon'(\psi(\bar{x};a))\frac{\partial\psi(\bar{x};a)}{\partial a}\\ &\quad+\varphi^u_{,3}(\psi(\bar{x};a),\upsilon(\psi(\bar{x};a));a).\end{aligned}$$

Thus

$$\begin{aligned}\left.\frac{\partial\bar{u}}{\partial a}\right|_{a=0} &= (0+1\upsilon'(x))\,(-\xi(x,\upsilon(x)))+\eta(x,\upsilon(x))\\ &= \eta(x,\upsilon(x))-\upsilon'(x)\xi(x,\upsilon(x)).\end{aligned}$$

For the function $\bar{u}$ with delay term, one obtains

$$\begin{aligned}\frac{\partial\bar{u}\,(g_\lambda(\bar{x}))}{\partial a} &= \frac{\partial\upsilon_a\,(g_\lambda(\bar{x}))}{\partial a}\\ &= \varphi^u_{,1}\left(\partial\psi(g_\lambda(\bar{x});a),\upsilon(\psi(g_\lambda(\bar{x});a));a\right)\frac{\partial\psi(g_\lambda(\bar{x});a)}{\partial a}\\ &\quad+\varphi^u_{,2}\left(\psi(g_\lambda(\bar{x});a),\upsilon(\psi(g_\lambda(\bar{x});a));a\right)\upsilon'(\psi(g_\lambda(\bar{x});a))\frac{\psi(g_\lambda(\bar{x});a)}{\partial a}\\ &\quad+\varphi^u_{,3}\left(\psi(g_\lambda(\bar{x});a),\upsilon(\psi(g_\lambda(\bar{x});a));a\right).\end{aligned}$$

This gives

$$\begin{aligned}\left.\frac{\partial\bar{u}\,(g_\lambda(\bar{x}))}{\partial a}\right|_{a=0} &= (-\xi(g_\lambda(x),\upsilon(g_\lambda(x))))+\eta(g_\lambda(x),\upsilon(g_\lambda(x)))\\ &= \eta(g_\lambda(x),\upsilon(g_\lambda(x)))-\upsilon'(g_\lambda(x))\xi(g_\lambda(x),\upsilon(g_\lambda(x))).\end{aligned}$$

B.2 The Function $\left.\dfrac{\partial \bar{u}'}{\partial a}\right|_{a=0}$

The derivative of the transformed function $\bar{u}(\bar{x})$ is defined as the following:

$$\begin{aligned}\bar{u}' = \frac{d\bar{u}}{d\bar{x}} &= \frac{\partial \varphi^u(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)}{\partial \bar{x}} \\ &= \varphi^u_{,1}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)\frac{\partial \psi(\bar{x};a)}{\partial \bar{x}} \\ &\quad + \varphi^u_{,2}\upsilon'(\psi(\bar{x};a))(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)\frac{\partial \psi(\bar{x};a)}{\partial \bar{x}}. \end{aligned} \tag{B.9}$$

Since

$$\begin{aligned}\frac{\partial \bar{x}}{\partial \bar{x}} &= \frac{\partial \varphi^x(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)}{\partial \bar{x}}, \\ 1 &= \varphi^x_{,1}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)\frac{\partial \psi(\bar{x};a)}{\partial \bar{x}} \\ &\quad + \varphi^x_{,2}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)\upsilon'(\psi(\bar{x};a))\frac{\partial \psi(\bar{x};a)}{\partial \bar{x}}. \end{aligned}$$

Hence

$$\frac{\partial \psi(\bar{x};a)}{\partial \bar{x}} = \frac{1}{\varphi^x_{,1}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a) + \varphi^x_{,2}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)\upsilon'(\psi(\bar{x};a))}. \tag{B.10}$$

Equations (B.9) and (B.10) imply

$$\bar{u}' = \frac{\Phi_1}{\Phi_2}, \tag{B.11}$$

where

$$\Phi_1 = \varphi^u_{,1}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a) + \varphi^u_{,2}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)\upsilon'(\psi(\bar{x};a))$$

and

$$\Phi_2 = \varphi^x_{,1}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a) + \varphi^x_{,2}(\psi(\bar{x};a), \upsilon(\psi(\bar{x};a));a)\upsilon'(\psi(\bar{x};a)).$$

Differentiating equation (B.11) with respect to the parameter a, one has

$$\frac{\partial \bar{u}'}{\partial a} = \frac{\Phi_2 \frac{\partial \Phi_1}{\partial a} - \Phi_1 \frac{\partial \Phi_2}{\partial a}}{[\Phi_2]^2},$$

where

$$\begin{aligned}\frac{\partial \Phi_1}{\partial a} &= \left[\varphi^u_{,11} + \varphi^u_{,12}\upsilon'\right]\frac{\partial \psi}{\partial a} + \varphi^u_{,13} \\ &+ \left[\left[\varphi^u_{,21} + \varphi^u_{,22}\upsilon'\right]\frac{\partial \psi}{\partial a} + \varphi^u_{,23}\right]\upsilon' \\ &+ \varphi^u_{,2}\upsilon''\frac{\partial \psi}{\partial a},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \Phi_2}{\partial a} &= \left[\varphi^x_{,11} + \varphi^x_{,12}\upsilon'\right]\frac{\partial \psi}{\partial a} + \varphi^x_{,13} \\ &+ \left[\left[\varphi^x_{,21} + \varphi^x_{,22}\upsilon'\right]\frac{\partial \psi}{\partial a} + \varphi^x_{,23}\right]\upsilon' \\ &+ \varphi^x_{,2}\upsilon''\frac{\partial \psi}{\partial a}.\end{aligned}$$

The second order derivatives in the last expressions are

$$\begin{aligned}\varphi^x_{,11}\Big|_{a=0} &= 0, \\ \varphi^x_{,12}\Big|_{a=0} = \varphi^x_{,21}\Big|_{a=0} &= 0, \\ \varphi^x_{,13}\Big|_{a=0} &= \xi_{,1}(x, \upsilon(x)), \\ \varphi^x_{,23}\Big|_{a=0} &= \xi_{,2}(x, \upsilon(x)), \\ \varphi^u_{,11}\Big|_{a=0} &= 0, \\ \varphi^u_{,12}\Big|_{a=0} = \varphi^u_{,21}\Big|_{a=0} &= 0, \\ \varphi^u_{,13}\Big|_{a=0} &= \eta_{,1}(x, \upsilon(x)), \\ \varphi^u_{,23}\Big|_{a=0} &= \eta_{,2}(x, \upsilon(x)).\end{aligned}$$

Hence

$$\begin{aligned}\Phi_1\Big|_{a=0} &= 0 + 1\cdot\upsilon'(x) = \upsilon'(x), \\ \Phi_2\Big|_{a=0} &= 1 + 0\cdot\upsilon'(x) = 1,\end{aligned}$$

$$\left.\frac{\partial \Phi_1}{\partial a}\right|_{a=0} = [0 + 0\upsilon'(x)]\,[-\xi(x,\upsilon(x))] + \eta_{,1}\,(x,\upsilon(x))$$

$$+ [[0 + 0\upsilon'(x)]\,[-\xi(x,\upsilon(x))] + \eta_{,2}\,(x,\upsilon(x))]\,\upsilon'(x)$$

$$+ 1 \cdot \upsilon''(x)\,[-\xi(x,\upsilon(x))]$$

$$= \eta_{,1}\,(x,\upsilon(x)) + \eta_{,2}\,(x,\upsilon(x))\,\upsilon'(x) - \xi(x,\upsilon(x))\upsilon''(x),$$

and

$$\left.\frac{\partial \Phi_2}{\partial a}\right|_{a=0} = [0 + 0\upsilon'(x)]\,[-\xi(x,\upsilon(x))] + \xi_{,1}\,(x,\upsilon(x))$$

$$+ [[0 + 0\upsilon'(x)]\,[-\xi(x,\upsilon(x))] + \xi_{,2}\,(x,\upsilon(x))]\,\upsilon'(x)$$

$$+ 0 \cdot \upsilon''(x)\,[-\xi(x,\upsilon(x))]$$

$$= \xi_{,1}\,(x,\upsilon(x)) + \xi_{,2}\,(x,\upsilon(x))\,\upsilon'(x).$$

Thus

$$\left.\frac{\partial \bar{u}'}{\partial a}\right|_{a=0} = \frac{1 \cdot [\eta_{,1} + \eta_{,2}\upsilon'(x) - \xi\upsilon''(x)] - \upsilon'(x)\,[\xi_{,1} + \xi_{,2}\upsilon'(x)]}{1^2}$$

$$= \eta_{,1} + [\eta_{,2} - \xi_{,1}]\,\upsilon'(x) - \xi_{,2}\,[\upsilon'(x)]^2 + \xi\upsilon''(x).$$

Appendix C

Some Material for Review and Reference

C.1 Definition of a Functional

Mapping. Let X and Y be sets and $A \subset X$ be any nonempty subset. A *mapping* (or *transformation*) T from A into Y is obtained by associating with each $x \in A$ a single $y \in Y$, written $y = Tx$ and called the *image of x with respect to T*.

Operator. In Calculus, the real line $\mathbb{R}$ and real-valued functions on $\mathbb{R}$ (or on a subset of $\mathbb{R}$) are ususally considered. Obviously, any such function is a *mapping* of its domain into $\mathbb{R}$. Generally we consider more general spaces, such as *metric spaces*, or *normed spaces*, and mappings of these spaces.

In the case of vector spaces and in particular, normed spaces, a mapping is called an *operator*.

Functional. A *functional* is an *operator* whose range lies on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$.

C.2 Inverse Function Theorem

Inverse function theorem. *Let E and F be Euclidean spaces and U be open in E. Let $x_0 \in U$, and $f : U \mapsto F$ be a $\mathcal{C}^s$ map. Assume that the derivative $f'(x_0) : E \mapsto F$ is invertible. Then f is locally $\mathcal{C}^s$-invertible at x_0. If φ is its local inverse, and $y = f(x)$, then $\varphi'(x) = f'(x)^{-1}$.*

See proof in Lang (1997).

C.3 Fréchet Derivatives

Fréchet derivatives. Let X, Y be normed linear spaces, $U \subset X$ open, $f : X \mapsto Y$ and $x \in U$. Then f is *Fréchet differentiable* at x if there is an element $A \in \mathcal{B}(X, Y)$ such that if

$$R(x, h) \stackrel{\text{def}}{=} f(x + h) - f(x) - Ah,$$

then

$$\lim_{h \to 0} \frac{\|R(x, h)\|_Y}{\|h\|_X} = 0.$$

We call A the *Fréchet derivative* of f at x and denote it by

$$f'(x) = Df(x).$$

Higher order Fréchet derivatives.

$$D^n f(x) = DD^{n-1} f(x), \ n = 2, 3, \ldots$$

Note: Let X, Y be normed linear spaces, $U \subset X$ open, $f : X \mapsto Y$ and $x \in U$. Then

$$\begin{aligned}
f(x) &\in Y, \\
Df(x) &\in \mathcal{B}(X, Y), \\
D^2 f(x) &\in \mathcal{B}(X, \mathcal{B}(X, Y)) = \mathcal{B}^2(X, Y), \\
&\vdots \\
D^n f(x) &\in \mathcal{B}(X, \mathcal{B}^{n-1}(X, Y)) = \mathcal{B}^n(X, Y),
\end{aligned}$$

where $\mathcal{B}^n(X, Y)$ is the space of bounded n-linear maps (i.e. linear in each variable separately)

$$f : \underbrace{X \times \cdots \times X}_{n-\text{components}} \mapsto Y.$$

Partial Fréchet derivatives. Let X_i and Y be normed linear spaces and $X = X_1 \oplus \cdots \oplus X_m$. Let $U \subset X$ be open and $F : U \mapsto Y$. Let $x = (x_1, \ldots, x_m) \in U$. Fix $k \in \{1, \ldots, m\}$. For z near x_k in X_k, $(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_m)$ lies in U. Define $f_k(z) = F(x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_m)$. (Thus $f_k(z)$ maps an open subset of X_k into Y.)

If f_k has a Fréchet derivative at $z = x_k$, we say F has a *k-th partial derivative* at x and define

$$D_k F(x) = D f_k(x_k).$$

(Note $D_k F(x) \in \mathcal{B}(X_k, Y)$.)

C.4 Invariants

Invariant. A function $F(x)$ is called an *invariant* of a group G of transformations (3.1) if F remains unaltered where one moves along any path curve of the group G. In other words, F is an invariant if $F\left(T_a(x)\right) = F(x)$ identically in x and a in a neighborhood of $a = 0$.

A Basis of Invariants. *A one-parameter group G of transformations in $\mathbb{R}^n$ has precisely $n-1$ functionally independent invariants. Any set of independent invariants, $\psi_1(x), \ldots, \psi_{n-1}(x)$, is termed a **basis of invariants of** G. The basis is not unique. One can obtain basic invariants, the left-hand sides of $n-1$ first integrals*

$$\psi_1(x) = C_1, \ldots, \psi_{n-1}(x) = C_{n-1},$$

*from the **characteristic system** of equations*

$$X(F) \equiv \xi^i(x) \frac{\partial F(x)}{\partial x_i} = 0,$$

i.e.

$$\frac{dx^1}{\xi^1(x)} = \cdots = \frac{dx^n}{\xi^n(x)}.$$

An arbitrary invariant $F(x)$ of G is given by the formula

$$F = \Phi(\psi_1(x), \ldots, \psi_{n-1}(x)).$$

See more details and proofs in Ibragimov (1999).

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