การประยุกต์กลุ่มวิเคราะห์กับสมการสามคลื่นในออพติกส์ไม่เชิงเส้น

นายผลาดร สุวรรณโพธิ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์
มหาวิทยาลัยเทคโนโลยีสุรนารี
ปีการศึกษา 2544
ISBN 974-533-017-5

APPLICATION OF GROUP ANALYSIS TO THREE-WAVE EQUATIONS IN NONLINEAR OPTICS

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in Applied Mathematics
Suranaree University of Technology
Academic Year 2001
ISBN 974-533-017-5

Application of Group Analysis to Three-wave Equations in Nonlinear Optics

Suranaree University of Technology Council has approved this thesis submitted in partial fulfillment of the requirements for a Doctoral Degree.

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อ. ที่ปรึกษา: Prof. Dr. Sergey Meleshko, 104 หน้า. ISBN 974-533-017-5

วิทยานิพนธ์ฉบับนี้ เป็นการนำเอาประโยชน์ของกลุ่มวิเคราะห์ไปใช้กับสมการสามคลื่นในออพติกส์ไม่เชิงเส้น ซึ่งสมการเหล่านี้ได้มาจากปรากฏการณ์จริงของ 2 ลำแสงหลักที่มีความถี่เหมือนกัน แต่มีโพลาไรเซชันต่างกัน ส่องผ่านตัวกลางนอนลิเนียร์ ผลที่ได้นี้จะเกิดเป็นลำแสงที่ 3 ซึ่งมีความถี่เป็นเซ็คกันฮาร์โมนิก (Second harmonic) จากความสัมพันธ์ของสนามไฟฟ้า และสนามแม่เหล็ก ซึ่งแสดงในรูปของสมการแมกซเวลล์ (Maxwell's equations) นำมาสร้างเป็นสมการ ที่เรียกว่าสมการสามคลื่น

การประยุกต์กลุ่มวิเคราะห์กับสมการอนุพันธ์ย่อย เริ่มจากการหากลุ่มแอดมิตเตด (Admitted group) ซึ่งในสมการสามคลื่นมี 11 กลุ่มพารามิเตอร์ สำหรับการสร้างผลเฉลยยืนยงนั้น จำเป็นที่จะต้องทำการจำแนกกลุ่มแอดมิตเตด การจำแนกของกลุ่มแอดมิตเตดจะสมนัยกับการจำแนกพีชคณิตย่อยของพีชคณิต L_{11} จุดประสงค์ของ วิทยานิพนธ์ฉบับนี้ คือการสร้างพีชคณิตย่อยทั้งหมด ของกลุ่มหลักของกลุ่มแอดมิตเตด L_{11} ที่สามารถหาผลเฉลยยืนยงได้ ในการสร้างระบบออพติมอล (Optimal system) สามารถทำได้ง่าย สำหรับพีชคณิตที่มีขนาดน้อย เนื่องจากพีชคณิตแอดมิตเตดของสมการสามคลื่นมีขนาดเท่ากับ 11 ดังนั้นการสร้างระบบออพติมอลจึงเริ่มจากพีชคณิตที่มีขนาดน้อย ผลที่ได้คือพีชคณิตย่อยทั้งหมดใน 3 มิติของพีชคณิต L_{11} (Subalgebra), ผลเฉลยยืนยงที่สอดคล้องกับ บางพีชคณิตย่อยของพีชคณิต L_{11} และใช้วิธีการของรุงแงง คุตตา (Runge-Kutta method) กับระบบสมการที่ลดรูป

สาขาวิชาคณิตศาสตร์ นักศึกษา__________________________

ปีการศึกษา 2544 อาจารย์ที่ปรึกษา___________________

PALADORN SUWANNAPHO: APPLICATION OF GROUP ANALYSIS TO THREE-WAVE EQUATIONS IN NONLINEAR OPTICS
THESIS ADVISOR: PROF. DR. SERGEY MELESHKO, Ph.D. 104 PP. ISBN 974-533-017-5

GROUP ANALYSIS / THREE-WAVE EQUATIONS IN NONLINEAR OPTICS / SECOND HARMONIC / EQUIVALENCE GROUP / ADMITTED GROUP / OPTIMAL SYSTEM / INVARIANT SOLUTIONS / REDUCE SYSTEM

This thesis deals with an application of group analysis to three-wave equations in nonlinear optics. These equations are obtained from the phenomenon that two fundamental beams of light with the same frequency and different polarizations propagate through quadratic nonlinear media. As the result of interaction, a third beam of second harmonic frequency is generated. Maxwell's equations and the slow envelope approximation is used to construct the three-wave equations.

The application of group analysis to partial differential equations starts from finding an admitted group. The admitted group of the three-wave equations is an eleven-parameter group. To construct invariant solutions one needs to classify this group. A classification of the group is equivalent to a classification of subalgebras of the admitted algebra L_{11}. The main point of the thesis was to construct all subalgebras of the admitted algebra L_{11}, which can be source of invariant solutions. In this case a reduced system is a system of ordinary differential equations. The constructing of an optimal system of subalgebras could be done easily only for small dimensions. Because the admitted algebra of the three-wave equations is eleven-dimensional, the problem of its classification was reduced to the study of algebras with small dimensions. The first result is that all essentially different three-dimensional subalgebras of the algebra L_{11} were obtained. Then, invariant solutions with respect to some of subalgebras of the algebra L_{11} were studied. And finally the Runge-Kutta method was employed to study the reduced system.

School of Mathematics Student ________________

Academic Year 2001 Advisor ________________

Acknowledgements

I would like to acknowledge gratefully the personal and profressional support received from my advisor, Prof. Dr. Sergey V.Meleshko and Prof. Dr. Alexander V. Gorchakov.

I am also very grateful to the Chair of the School of Mathematics of Suranaree University of Technology, Assoc. Prof. Dr. Prapasri Asawakun and all the lecturers who taught and helped me during my study at Suranaree University of Technology. They are Assoc. Prof. Dr. Suwan Tangmanee, Assoc. Prof. Dr. Pairote Sattayatham, Assoc. Prof. Dr. Nikolay P. Moshkin, Assoc. Prof. Dr. Boris I. Kvasov, and Asst. Prof. Dr. Arjuna Chaiyasena.

I am most grateful to Asst. Prof. Dr. Eckart Schulz, Issra Pramoolsook and Keota Ketsuwan for their admirable job of correcting the language and the style of my manuscript.

I want to express my appreciation to Anirut Laauddsong, Jessada Tantanuch, Peiangpob Mounnunprang, Apichaai Hematulin, Mana Kaomek, Wei Wei and Anuson Ruchirabha for their invaluable professional guidance and friendly encouragement.

In addition, I wish to express my special thanks to National Science and Technology Development Agency Scholarship and Piboonbumpen Demonstration School for financial support.

Last but not least, I would like also to thank my family, my aunt and my friends for their understanding, supporting and encouraging over the years of my study.

Paladorn Suwannapho

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Chapter I

Introduction

Physical and mathematical modelling are very closely related. Physical phenomena are used to construct mathematical models, and on the other hand, the solutions of mathematical models are used to explain the physical phenomena. The most important mathematical models applied in engineering and physics today are in the form of nonlinear partial differential equations. General solutions of these equations are usually very difficult to find. By using numerical methods, only approximate solutions can be obtained, which do not allow for analyzing the properties of the equations. However, exact solutions are of more interest, as each exact solution has value, firstly, as an exact description of the real process in the framework of a given model; secondly, as an analytical solution to compare with the numerical solutions obtained by various numerical schemes; thirdly, as a basis to improve the models used. Among all the methods used for finding exact solutions, group analysis (Ovsiannikov (1978), Olver (1986), Bluman and Kumei (1996)) can be used with a great variety of equations, and its aim is to reduce the number of the independent variables. A historical review of group analysis can be found in Ibragimov (1999) and Lawrence (1999). Review of modern results in group analysis are collected in the Handbook of Lie group analysis of differential equations (Ibragimov, 1994, 1995, 1996).

In this research, an application of group analysis is used to find exact solutions of the three-wave equations in nonlinear optics. These equations are obtained from the fact that two fundamental beams of light with the same frequency ω and different polarizations propagate through quadratic nonlinear medium such as Quartz, Potassium di hydrogen phosphate (KH_2PO_4) and Ammonium di hydrogen phosphate ($NH_4H_2PO_4$). As the result of interaction, a third beam of second harmonic frequency is generated. Maxwell's equations and the slow envelope approximation derivation of the three-wave equations can be found, for example, in Yariv (1989,1991), Butcher and Colter (1990), Dmitriev, Gurzadyan and Nikogosyan (1991) Robert (1992), Nail and Adrian (1997), Maimistov and Basharov (1999), Orszag (1999). These equations can be written as follows:

$$\begin{aligned} M_1A_1 &= i\sigma_1 A_3 A_2^* e^{i\Delta kz}, \\ M_2A_2 &= i\sigma_2 A_3 A_1^* e^{i\Delta kz}, \\ M_3A_3 &= i\sigma_3 A_1 A_2 e^{-i\Delta kz} \end{aligned} \tag{1.1}$$

Here

$$A_1 = u_1 + iu_2,\ A_2 = u_3 + iu_4,\ A_3 = u_5 + iu_6,$$

A_1, A_2 are the complex valued amplitudes of two fundamental harmonic fields with different polarizations and A_3 is the complex valued amplitude of the second-harmonic field, M_j are the linear differential operators

$$M_j = \frac{\partial}{\partial z} + \beta_j \frac{\partial}{\partial x} + \frac{i}{2k_j}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{k_j}{\omega}\frac{\partial}{\partial t};\ (j = 1, 2, 3),$$

z is the coordinate along the propagation direction, (x, y) are the transverse coordinates, t is time, $k_{1,2}$ are the linear wave numbers of the fundamental frequencies, k_3 is the linear wave number of the second–harmonic frequency, $\Delta k = k_3 - (k_1 + k_2)$ is the wave vector–mismatch, the symbol $*$ denotes the complex conjugation, β_j are the walk-off angles of the fundamental and second harmonic, ω is the frequency of the light, σ_j are nonlinear coupling coefficients. We will consider the case of exact phase-matched condition:only $\Delta k = k_3 - (k_1 + k_2) = 0$. System (1.1) is a system of nonlinear partial differential equations.

There are several reasons for the high level of current interest in studying the three-wave equations. One of them is that the intense study of nonlinear effects in optics offers new facilities for optical signal processing as well as long-distance communication systems. Mostly one-dimensional temporal or spatial solutions were considered by Yariv (1991), Torner et all (1995). Group analysis allows constructing more complex representations of solutions.

The application of group analysis to system (1.1) consists of several steps. The first step is to obtain an equivalence group and admitted group. An equivalence group is a group of transformations that transfers the system of equations (1.1) into a system with the same differential structure, but with different arbitrary elements. A group of transformations is called an admitted group if any solution of system (1.1) by means of any transformation from this group is transferred into a solution of the same system of equations. For the three-wave equations, the equivalence group and admitted group were constructed in Gorchakov and Meleshko (1997). The equivalence group is nine-dimensional and the admitted group is eleven-dimensional G_{11}.

The admitted group G_{11} allows dividing all exact solutions of equations (1.1) into classes of essentially different solutions with respect to G_{11}. Two solutions u_1, u_2 are nonessentially different if one is transformable into the other by a transformation belonging to the group G_{11}. Therefore, essentially different solutions are obtained with respect to different classes of similar subgroups. The set of all representatives (one from each class) is called an optimal system of subgroups.

One of the main goals of application of group analysis to differential equations is construction of representations of solutions. Solutions whose representations are obtained with the help of the admitted group are called invariant or partially invariant. These solutions can be constructed with respect to any subgroup of the group G_{11}. The concept of invariant solutions permits an organization of the search process for particular solutions of system (1.1) admitted by group G_{11}, with the aid of different subgroups $H \subset G_{11}$. It is expedient in this research to begin with the lowest possible number of independent variables and move to successively higher number of independent variables of the studied $H-$solutions. It is easier to find solutions with lower rank because the rank is equal to the number of independent variables in the factor system (S/H).

All invariant solutions (as all solutions) can be divided into essentially different classes. The problem of enumeration of all invariant and partially invariant solutions of a given finite–dimensional Lie group G_{11} can be done in two steps. In the first step, the classification of all subgroups of the admitted group has to be done. The classification of subgroups is equivalent to the classification of all subalgebras. All subalgebras are divided into equivalent classes with respect to the automorphisms of the admitted Lie group G_{11}. A list of representatives from each class is called an optimal system. Note that the determination of an optimal system can be done relatively easy for small dimensionality. For large dimensionality, we need to use the two–step algorithm developed by Ovsiannikov (1994). This algorithm allows reducing the problem of the construction of the optimal system of subalgebras of L_{11} to the classification of subalgebras of lesser dimension. The construction of the optimal system of three-dimensional subgroups of the group G_{11} is one of the main result of the thesis. For the invariant solutions, which are related with three-dimensional subgroups of the group G_{11}, the original system of the three-wave equations reduces to a system of ordinary differential equations.

Therefore, to find invariant or partially invariant solutions, one needs to find the optimal system of subalgebras and then for each subalgebra, one has to find a universal invariant, a representation of a solution, substitute it into the given system and study the compatibility of the resulting system.

Many parts of this method require the carrying out of a lot of complicated symbolic manipulations. Because this is a very laborious part, we used a computer for this task. All calculations were done with the REDUCE program (Hearn, 1999).

The framework of the thesis is as follows. Chapter II introduces notations of group analysis and provides references to well-known facts on application of group analysis for constructing exact solutions of partial differential equations. Chapter III deals with obtaining the three-wave equations from Maxwell's equations. Chapter IV is devoted to constructing a three-dimensional part of the optimal system of subalgebras. At first the factor algebra L_9 is studied and then by using the optimal system of subalgebras of the algebra L_9 we obtain the optimal system of subalgebras of the algebra L_{11}. In Chapter V invariant solutions for which the original system of the three-wave equations reduces to a system of ordinary differential equations are studied. To reduced system the Runge-Kutta method is used. Results of representative calculations are presented.

Chapter II

Symmetries and Invariant Solutions

2.1 Introduction

In this chapter, we discuss the classical method for constructing invariant solutions of partial differential equations. We use this result to determine the point transformations and to generate invariant solutions of the three wave equations. Discussions of the standard group method for constructing exact solutions can be found in the textbooks (Ibragimov, 1994, 1995, 1996, and Ovsiannikov, 1994).

2.2 Admitted groups

In this section, we review the general theory related to group of continuous transformations.

Let $x = (x_1, ..., x_n) \in \Re^n$ be a set of independent variables, $u = (u^1, u^2, ..., u^m) \in \Re^m$ be a set of dependent variables and Δ be a symmetrical interval of $\Re^1$.

We consider the set of invertible point transformations of the space $\Re^{n+m}$:

$$\begin{aligned} \overline{x}_i = &\ \Phi^i(x,u;a), \qquad \overline{u}^j &= \Psi^i(x,u;a) \\ (i = &\ 1,2,...,n); \qquad (j &= 1,2,...,m). \end{aligned} \tag{2.1}$$

Here $a \in \Delta$ is a parameter of transformations.

Group analysis uses special kind of invertible transformations. These transformations compose a local Lie group.

Definition 1 *A set of transformations (2.1) is a local one-parameter group* $\boldsymbol{G}^1$ *if it has the following properties:*

P.1. $\Phi(x,u;0) = x, \quad \Psi(x,u;0) = u$ *for any* $(x,u) \in \Re^{n+m}$

P.2. $\Phi\left(\Phi(x,u;a),\ \Psi(x,u;a);b\right) = \Phi(x,u;a+b)$ *and*
$\Psi\left(\Phi(x,u;a),\ \Psi(x,u;a);b\right) = \Psi(x,u;a+b)$
for any $a,\ b,\ a+b \in \Delta,\ (x,u) \in \Re^{n+m}$

P.3. If $\Phi(x,u;a)=x$ *and* $\Psi(x,u;a)=u$ *then* $a=0$

P.4. $\Phi,\ \Psi\ \in C_\infty(\Re^{n+m}\times\Delta)$

If we expand the function $\Phi^i,\ \Psi^j$ into the Taylor series with respect to parameter a in a neighborhood of $a=0$, then we obtain

$$\begin{aligned}\overline{x}_i &\approx x_i + a\left(\frac{\partial\Phi^i}{\partial a}\right)_{a=0} + \dots \\ \overline{u}^j &\approx u^j + a\left(\frac{\partial\Psi^j}{\partial a}\right)_{a=0} + \dots\end{aligned} \tag{2.2}$$

where $\left(\frac{\partial\Phi^i}{\partial a}\right)_{a=0},\ \left(\frac{\partial\Psi^i}{\partial a}\right)_{a=0}$ are functions of x and u that are denoted by $\xi^i(x,u)$ and $\zeta^j(x,u)$. If a is sufficiently small, then we can write the coordinates of the image point $(\overline{x},\overline{u})$ as

$$\begin{aligned}\overline{x}_i &\approx x_i + a\xi^i,\\ \overline{u}^j &\approx u^j + a\zeta^j\end{aligned} \tag{2.3}$$

Transformation (2.3) is called *an infinitesimal transformation* and the vector (ξ,η) is called *a tangent vector field of the group* $\mathbf{G}$. Here $\xi=(\xi^1,\xi^2,\dots,\xi^n)$, $\zeta=(\zeta^1,\zeta^2,\dots,\zeta^m)$. It can be written in the term of the first-order differential operator [1]

$$\mathbf{X}=\xi^i(x,u)\frac{\partial}{\partial x_i}+\zeta^j(x,u)\frac{\partial}{\partial u^j} \tag{2.4}$$

that is also called *an infinitesimal generator.*

Let us consider the property (P.2) of group transformation (2.1). By differentiating the left and right sides of it, we obtain the relations

$$\begin{aligned}\frac{\partial}{\partial a}\left(\Phi^i(x,u;a)\right) &= \left(\xi^i\circ(\Phi,\Psi)\right)(x,u;a),\\ \frac{\partial}{\partial a}\left(\Psi^j(x,u;a)\right) &= \left(\zeta^j\circ(\Phi,\Psi)\right)(x,u;a)\end{aligned} \tag{2.5}$$

The last equations and property (P.1) of any one parameter group of transformations (2.1) yields the functions $\overline{x}=\Phi(x,u;a)$ and $\overline{u}=\Psi(x,u;a)$ which are solutions of the Cauchy problem

$$\begin{aligned}\frac{d\overline{x}_i}{da} &= \xi^i(\overline{x},\overline{u}),\\ \frac{d\overline{u}^j}{da} &= \zeta^j(\overline{x},\overline{u}),\end{aligned} \tag{2.6}$$

$$\overline{x}_i\big|_{a=0} = x_i,\qquad \overline{u}^j\big|_{a=0}=u^j \tag{2.7}$$

Equations (2.6) are called *Lie equations.*

[1] We will use the summation convention that a repeated index implies summation over the values of the index.

Applications of groups of transformations to differential equations require to know the transformations of derivatives. For the sake of simplicity, we explain this idea by the case when $n = 1$ and $m = 1$.

Let $u_0(x)$ be a known function. The transformed function $u_a(x)$ can be obtained by the following way. Firstly, one has to solve the equation

$$\overline{x} = \Phi\left(x, u_0(x); a\right)$$

with respect to x. Since the Jacobian $\frac{\partial \overline{x}}{\partial x}\big|_{a=0} = \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u} u_0'\right)\big|_{a=0} = 1$, then by the inverse function theorem, one can find $x = g(\overline{x}, a)$ in some neighborhood of $a = 0$. Note that there is the identity

$$\overline{x} = \Phi\left(g(\overline{x}, a), u_0\left(g(\overline{x}, a)\right), a\right). \tag{2.8}$$

The transformed function $u_a(x)$ is

$$u_a(\overline{x}) = \Psi\left(g(\overline{x}, a), u_0\left(g(\overline{x}, a)\right), a\right). \tag{2.9}$$

Differentiating the last expression, one gets

$$\begin{aligned} \frac{du_a(\overline{x})}{d\overline{x}} &= \frac{\partial \Psi}{\partial x} \cdot \frac{\partial g}{\partial \overline{x}} + \frac{\partial \Psi}{\partial u} \cdot \frac{\partial u_0}{\partial x} \cdot \frac{\partial g}{\partial \overline{x}} \\ &= \left(\frac{\partial \Psi}{\partial x} + u_0' \frac{\partial \Psi}{\partial u}\right) \frac{\partial g}{\partial \overline{x}}. \end{aligned} \tag{2.10}$$

The derivative $\frac{\partial g}{\partial \overline{x}}$ is obtained by differentiating (2.8) with respect to $\overline{x}$:

$$\begin{aligned} 1 &= \frac{\partial \Phi}{\partial x} \cdot \frac{\partial g}{\partial \overline{x}} + \frac{\partial \Phi}{\partial u} \cdot \frac{\partial u_0}{\partial x} \cdot \frac{\partial g}{\partial \overline{x}} \\ &= \left(\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u} \cdot u_0'\right) \frac{\partial g}{\partial \overline{x}}. \end{aligned}$$

Hence,

$$\frac{\partial g}{\partial \overline{x}} = \frac{1}{\frac{\partial \Phi}{\partial x} + u_0' \frac{\partial \Phi}{\partial u}}. \tag{2.11}$$

Substituting (2.11) into (2.10), we obtain derivative of the transformed function

$$\frac{du_a}{d\overline{x}} = \frac{\Psi_x + u_0' \Psi_u}{\Phi_x + u_0' \Phi_u} = h(x, u_0, u_0', a). \tag{2.12}$$

Note that $\frac{du_0}{d\overline{x}}\big|_{a=0} = u_0'(x)$. Therefore the infinitesimal transformation of the derivative is

$$u_a' = u_0' + a\zeta_1 + \ldots$$

where $\zeta_1 = D(\eta) - u' D(\xi)$. Here $D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \ldots$. Similarly, we obtain the infinitesimal transformation of the second derivative

$$u_a'' = u_0'' + a\zeta_2, \text{where } \zeta_2 = D(\zeta_1) - u' D(\xi).$$

where $\zeta_2 = D(\zeta_1) - u' D(\xi)$.

Let $x = \{x_i\}$ be the set of independent variables and $u = \{u^j\}$ the set of dependent variables. Derivatives of the dependent variables are given by the sets $u_{(1)} = \{u_i^j\}$, $u_{(2)} = \{u_{i\alpha}^j\}, \ldots$, where $j = 1, \ldots, m$ and $i, \alpha = 1, \ldots, n$. The derivatives of the differentiable functions u^j can be written in terms of the operator of the total differentiation D_i given below

$$\begin{aligned} u_i^j &= D_i(u^j), \\ u_{i\alpha}^j &= D_\alpha(u_i^j). \end{aligned}$$

where

$$D_i = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + u_{i\alpha}^j \frac{\partial}{\partial u_\alpha^j} + \ldots, \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \quad (2.13)$$

In (2.13) there is a summation over the values of the repeated index α from 1 to n. The set of transformations in (x, u) space is

$$\begin{aligned} \overline{x}_i &= \Phi^i(x, u; a), \quad \Phi^i|_{a=0} &= x_i, \quad (i = 1, 2, \ldots, n), \\ \overline{u}^j &= \Psi^j(x, u; a), \quad \Psi^j|_{a=0} &= u^j, \quad (j = 1, 2, \ldots, m), \end{aligned}$$

where a is a real parameter of a one-parameter group of point transformations in the space of the dependent and independent variables if the group properties apply. The generator of the group is

$$\mathbf{X} = \xi^i(x, u) \frac{\partial}{\partial x_i} + \zeta^j(x, u) \frac{\partial}{\partial u^j}, \quad (2.14)$$

where $\xi^i = \frac{\partial \Phi^i}{\partial a}\Big|_{a=0}$, $\zeta^j = \frac{\partial \Psi^j}{\partial a}\Big|_{a=0}$. The first prolongation of the generator (2.14) is given by

$$\mathbf{X}^{[1]} = \xi^i \frac{\partial}{\partial x_i} + \zeta^j \frac{\partial}{\partial u^j,} + \zeta_i^j \frac{\partial}{\partial u_i^j},$$

where

$$\zeta_i^j = D_i\left(\zeta^j\right) - u_\alpha^j D_i\left(\xi^\alpha\right).$$

The second prolonged generator is

$$\mathbf{X}^{[2]} = \mathbf{X}^{[1]} + \zeta_{i_1 i_2}^j \frac{\partial}{\partial u_{i_1 i_2}^j},$$

where

$$\zeta_{i_1 i_2}^j = D_{i_2}\left(\zeta_{i_1}^j\right) - u_{\alpha i_1}^j D_{i_2}\left(\xi^\alpha\right); \quad (i_1, i_2 = 1, 2, \ldots, n).$$

The higher order prolonged generator is

$$\mathbf{X}^{[s]} = \mathbf{X}^{[s-1]} + \zeta_{i_1 \cdots i_s}^j \frac{\partial}{\partial u_{i_1 \cdots i_s}^j},$$

where

$$\zeta^j_{i_1\cdots i_s} = D_{i_s}\left(\zeta_{i_1\cdots i_{s-1}}\right) - u^j_{\alpha i_1\cdots i_{s-1}} D_{i_s}\left(\xi^\alpha\right); \ \ (i_1, i_2, \ldots, i_s = 1, 2, \ldots, n).$$

Now we relate a local Lie group and differential equations. Let

$$F_l(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \quad l = 1, 2, \ldots \tag{2.15}$$

be a system of s-th order differential equations and **G** be a local Lie group of transformations.

Definition 2 *A system of differential equations (2.15) is said to be invariant under a group* $\boldsymbol{G}$ *of point transformations if the solutions of the system are merely permuted among themselves by every transformation of the group* $\boldsymbol{G}$*. The group* $\boldsymbol{G}$ *is also termed a group admitted by the system. Consequently any solutions of system of equations (2.15) are converted into solutions of the same system.*

Finding symmetries of F_l leads to a system of linear and homogeneous differential equations with respect to the components of an admitted generator. These equations are called determining equations.

Definition 3 *(determining equations)*
The equations

$$\left[\boldsymbol{X}^{[s]} F_l\right]\Big|_{F_l=0} = 0 \tag{2.16}$$

are called determining equations.

The determining equations can be split with respect to parametric derivatives. After splitting, one gets overdetermined system of equations for the coefficients of the generator. Because this system of equations is overdetermined, one can solve it.

Theorem 1 *If a differential equation admits the operators* $\zeta \cdot \partial$ *and* $\zeta' \cdot \partial$*, then it also admits their commutator* $[\zeta, \zeta'] \cdot \partial$*.*

2.3 Equivalence group of transformations

A system of PDE can be classified by the symbol $E(m, n, s, l)$ where m is the number of the dependent variables, n is the number of the independent variables, s is the order of the highest derivative and l is the number of differential equations. Normally the differential equations include arbitrary elements (θ). For searching Lie group which are admitted by the original system, one needs to determine group of transformations that changes arbitrary elements but does not change the differential structure. An infinitesimal approach (Meleshko, 1996) was applied for finding this group.

Definition 4 *A nondegenerate change of dependent, independent variables and arbitrary elements which transfers any system of the differential equations of the given class*

$$F_l(x, u, p, \theta) = 0. \tag{2.17}$$

to the system of equations of the same class but with different arbitrary elements is called an equivalence transformation. Here p are the partial derivative from $(u_{(1)}, u_{(2)}, \dots, u_{(s)})$

Group of these transformations with a parameter a can be written as the following

$$\overline{x}_i = \Phi^i(x, u, \theta; a), \;\; \overline{u}^j = \Psi^j(x, u, \theta; a), \;\; \overline{\theta}^k = \Pi^k(x, u, \theta; a), \tag{2.18}$$

where $k = (k^1, k^2 \cdots, k^\gamma)$ be a set of arbitrary elements. Generators of this group have the form

$$\mathbf{X}^e = \xi^i \partial_x + \zeta^j \partial_u + \zeta^{\theta^k} \partial_\theta$$

where

$$\xi^i = \xi^i(x, u, \theta) = \left.\frac{\partial \Phi^i}{\partial a}\right|_{a=0}, \; \zeta^j = \zeta^j(x, u, \theta) = \left.\frac{\partial \Psi^j}{\partial a}\right|_{a=0}, \; \zeta^{\theta^k} = \zeta^{\theta^k}(x, u, \theta) = \left.\frac{\partial \Pi^k}{\partial a}\right|_{a=0}.$$

Arbitrary elements are obtained by the following way. Let $\theta_0(x, u)$ be given. By the inverse function theorem with equation (2.18), we can find $x = f(\overline{x}, \overline{u}; a)$ and $u = g(\overline{x}, \overline{u}; a)$. The transformed arbitrary elements are

$$\theta_0(\overline{x}, \overline{u}) = \Pi\left(f(\overline{x}, \overline{u}; a), g(\overline{x}, \overline{u}; a), \theta_0(f(\overline{x}, \overline{u}; a), g(\overline{x}, \overline{u}; a); a)\right).$$

If $u_0(x)$ is a solution of system (2.17) and $\theta_0(x, u)$ is a concrete value of the arbitrary element, then we have

$$\overline{x} = \Phi(x, u_0(x), \theta_0(x, u_0(x)); a).$$

By the inverse function theorem, we can find

$$x = f(\overline{x}; a)$$

and we also obtain the transformed function

$$u_0(\overline{x}) = \Psi(f(\overline{x}, a), u_0(f(\overline{x}, a)), \;\; \theta_0(f(\overline{x}, a), u_0(f(\overline{x}, a)); a)). \tag{2.19}$$

Differentiating (2.19) with respect to $\overline{x}$, we get the transformation $\overline{p}$. Since $u_a(\overline{x})$ is a solution of the same system with transformed arbitrary elements $\theta_a(\overline{x}, \overline{u})$ then

$$F_l\left(\overline{x}, u_a(\overline{x}), \overline{p_a}(\overline{x}), \theta_a\left(\overline{x}, u_a(\overline{x})\right)\right) = 0, \quad l = 1, 2, \dots.$$

The s-th prolongation of infinitesimal generator $\mathbf{X}^e$ is

$$\overline{\mathbf{X}}_e^{[s]} = \mathbf{X}^e + \zeta_i^j \partial_{u_i^j} + \zeta_{x_i}^{\theta^k} \partial_{\theta_{x_i}^k} + \zeta_{u^j}^{\theta^k} \partial_{\theta_{u^j}^k} + \dots \tag{2.20}$$

where

$$
\begin{aligned}
\zeta_i^j &= D_{x_i}\zeta^j - u_\alpha^j D_{x_i}\xi^\alpha, \\
\zeta_{x_i}^{\theta^k} &= D_{x_i}^e \zeta^{\theta^k} - \theta_{x_\alpha}^k D_{x_i}^e \xi^\alpha - \theta_{u^\beta}^k D_{x_i}^e \zeta^\beta, \\
\zeta_{u^j}^{\theta^k} &= D_{u^j}^e \zeta^{\theta^k} - \theta_{x_\alpha}^k D_{u^j}^e \xi^\alpha - \theta_{u^\beta}^k D_{u^j}^e \zeta^\beta.
\end{aligned}
$$

Here

$$
\begin{aligned}
D_{x_i} &= \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + \left(\theta_{x_i}^k + \theta_{u^j}^k u_i^j\right) \frac{\partial}{\partial \theta^k} + \ldots, \\
D_{x_i}^e &= \frac{\partial}{\partial x_i} + \theta_{x_i}^k \frac{\partial}{\partial \theta^k} + \ldots, \quad D_{u^j}^e = \frac{\partial}{\partial u^j} + \theta_{u^j}^k \frac{\partial}{\partial \theta^k} + \ldots \quad .
\end{aligned}
$$

By the same way as for the admitted group, one can obtain the determining equations for the equivalence group. Let $\mathbf{G}(\theta)$ be admitted group with the arbitrary element θ. A set of groups $\mathbf{G}(\theta)$ that is admitted by equations for all arbitrary elements is called *a kernel of groups*, corresponding to Lie-algebra, which is called *a kernel of algebra*.

2.4 Lie algebra

Before giving a definition of Lie algebra, we need to introduce the commutator. Let $\mathbf{X}_1 = \xi_1\partial_x + \zeta_1\partial_u$, $\mathbf{X}_2 = \xi_2\partial_x + \zeta_2\partial_u$ be two generators. We define a new generator $\mathbf{X}$ denoted by $[\mathbf{X}_1, \mathbf{X}_2]$ with the following formula

$$
\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2] = (\mathbf{X}_1\xi_2 - \mathbf{X}_2\xi_1)\,\partial_x + (\mathbf{X}_1\zeta_2 - \mathbf{X}_2\zeta_1)\,\partial_u.
$$

This generator is called *a commutator of the generators* $\mathbf{X}_1, \mathbf{X}_2$.

Definition 5 *(Lie algebra) The vector space L over the field of real numbers is called a Lie algebra if the operation of the commutator $[\,\cdot\,,\,\cdot\,]$ satisfies the axioms:*

a.1 (bilinearity) : for any $\boldsymbol{X}_1,\ \boldsymbol{X}_2,\ \boldsymbol{X}_3 \in L$ and $a,\ b \in \Re$

$$
\begin{aligned}
[a\boldsymbol{X}_1 + b\boldsymbol{X}_2, \boldsymbol{X}_3] &= a\,[\boldsymbol{X}_1, \boldsymbol{X}_3] + b\,[\boldsymbol{X}_2, \boldsymbol{X}_3] \\
[\boldsymbol{X}_1, a\boldsymbol{X}_2 + b\boldsymbol{X}_3] &= a\,[\boldsymbol{X}_1, \boldsymbol{X}_2] + b\,[\boldsymbol{X}_1, \boldsymbol{X}_3]
\end{aligned}
$$

a.2 (antisymmetry) : for any $\boldsymbol{X}_1,\ \boldsymbol{X}_2 \in L$

$$
[\boldsymbol{X}_1, \boldsymbol{X}_2] = -\,[\boldsymbol{X}_2, \boldsymbol{X}_1]
$$

a.3 (the Jacobi identity) : for any $\boldsymbol{X}_1,\ \boldsymbol{X}_2,\ \boldsymbol{X}_3 \in L$

$$
[[\boldsymbol{X}_1, \boldsymbol{X}_2]\,\boldsymbol{X}_3] + [[\boldsymbol{X}_2, \boldsymbol{X}_3]\,\boldsymbol{X}_1] + [[\boldsymbol{X}_3, \boldsymbol{X}_1]\,\boldsymbol{X}_2] = 0
$$

Remark 1 *Let L_r be an r-dimensional vector space with a basis $\boldsymbol{X}_1,\ \boldsymbol{X}_2,\dots,\boldsymbol{X}_r$: i.e., any vector $\boldsymbol{X}\in L_r$ can be decomposed as the following*

$$\boldsymbol{X}=\sum_{k=1}^{r} x_k \boldsymbol{X}_k$$

where x_k are coordinates of the vector $\boldsymbol{X}$ in the basis $\{\boldsymbol{X}_1,\dots,\boldsymbol{X}_r\}$, then L_r is a Lie algebra if

$$[\boldsymbol{X}_i,\boldsymbol{X}_j]=\sum_{k=1}^{r} c_{ij}^k \boldsymbol{X}_k;\quad i,\ j=1,2,\dots,r$$

with real constants c_{ij}^k.

Definition 6 *(Subalgebra) The vector space $N\subset L$ is called a subalgebra of the Lie algebra L if $[u,v]\in N$ for any $u,\ v\ \in N$.*

Definition 7 *(Ideal) A subalgebra $I\subset L$ is called an ideal of the Lie algebra L if for any $u\in L,\ v\in I$ it is also true that $[u,v]\in I$.*

One of the main aims of group analysis is a construction of exact solutions. All solutions can be divided into equivalent classes of solutions.

Definition 8 *(Equivalent solutions) Two solutions u_1 and u_2 are equivalent with respect to the group $\boldsymbol{G}$ if one is transformable into the another by a transformation belonging to the group $\boldsymbol{G}$.*

The problem of classification of exact solutions is equivalent to classification of subgroups (or subalgebras) of the group $\mathbf{G}$ (or the subalgebra L). Because there is a one-to-one correspondence between groups and algebras we explain here about classification of subalgebras. In order to give the classification of subalgebras, we need to give some definitions.

Definition 9 *(Automorphism) Let $L,\ \overline{L}$ be Lie algebras and $\boldsymbol{X}_1,\ \boldsymbol{X}_2\ \in L$. A linear one-to-one map f of L onto $\overline{L}$ is called an isomorphism if it satisfies the equation*

$$f\left([\boldsymbol{X}_1,\boldsymbol{X}_2]_L\right)=[f(\boldsymbol{X}_1),f(\boldsymbol{X}_2)]_{\overline{L}}$$

where the indices L and $\overline{L}$ denotes the commutators in the corresponding algebras. An isomorphism of L onto itself is called an automorphism of the Lie algebra L.

In the finite-dimensional case, isomorphic Lie algebras have the same dimension. The criterion for the Lie algebras to be equivalent can be stated in terms of their structural constants. If the Lie algebra L and $\overline{L}$ are isomorphic, then there exist bases in them for which their structural constants are correspondingly equal.

Let L be the Lie algebra. If a set $\{\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_n,\}$ is a basis of L, then we have

$$[\mathbf{X}_i,\mathbf{X}_j]=\sum_{k=1}^{n} c_{ij}^{\alpha}\mathbf{X}_\alpha;\quad (i,\ j=1,2,\dots,n),$$

where c_{ij}^{α} are the structural constants. From the table of commutators of the algebra L, we can write a linear system of scalar equations with constant coefficients for the automorphism $A_i, (i = 1, 2, \ldots, n)$:

$$\frac{d\overline{x}_j}{da} = \sum_{\beta=1}^{n} c_{\beta i}^{j} \overline{x}_\beta, \quad (j = 1, 2, \ldots, n)\,. \tag{2.21}$$

Initial values for this system are $\overline{x}_j = x_j$ at $a = 0$. The solutions of these equations consist of a set of the automorphisms.

The set of all subalgebras is divided into equivalent classes with respect to these automorphisms of the admitted Lie algebra L. A list of representatives, where each element of this list is one representative from every class, is called *an optimal system of subalgebras.*

Next, we consider a method for constructing the optimal system. Because of the difficulties in constructing the optimal system of subalgebras for the large dimensional Lie algebras, there is a two-step algorithm (Ovsiannikov, 1978), which reduces this problem to the problem for constructing an optimal system of algebras with less dimensions.

Let us consider an algebra L_k with a basis $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k\}$. According to the algorithm, the algebra L_k is decomposed on $I_1 \oplus N_1$, where I_1 is an ideal of the algebra L_k and N_1 is a subalgebra of the algebra L_k. In the same way, the subalgebra N_1 can also be decomposed as $N_1 = I_2 \oplus N_2$. Following the same process $(\alpha - 1)$ times up to the algebra N_α, the optimal system of subalgebras can be easily constructed.

By gluing ideals I_l and subalgebras N_l starting from $l = \alpha$ to $l = 1$, one constructs the optimal system of subalgebras for the algebra L_k. Note that for every subalgebra N_l one needs to check subalgebra conditions and to use the automorphisms to simplify coefficients of these systems. Therefore, the problem for constructing an optimal system of subalgebras of the algebra L_k by this method is reduced to the problem of classification of algebras with fewer dimensions.

After constructing the optimal system, one can start seeking invariant and partially invariant solutions of subalgebras from the optimal system.

The concept of invariant solution is based on the fact that the problem of discovery of an invariant solution is reduced to the integration of a new system of differential equations, in which the unknown functions depend on a fewer number of independent variables in comparison with the original system. In this sense, invariant solutions are found more easily than arbitrary solutions.

Let us consider some subgroup H of the group G. A function $F(x, u)$ $(x \in \Re^n,\ u \in \Re^m,\ N = n + m)$ is said to be an invariant of the subgroup H if for each point (x, u), it is constant along the trajectory determined by the totality of transformed point $(\overline{x}, \overline{u})$: $F(\overline{x}, \overline{u}) = F(x, u)$. The function $F(x, u)$ is an invariant of the group H with generators

$$X_{(k)} = \xi_{(k)}^{i}(x, u)\partial_{x_i} + \zeta_{(k)}^{j}(x, u)\partial_{u^j}, (k = 1, 2, ..., r)$$

if and only if

$$X_{(k)}(F) = \xi^i_{(k)}(x,u)\frac{\partial F}{\partial x_i} + \zeta^j_{(k)}(x,u)\frac{\partial F}{\partial u^j} = 0, \ (k = 1, 2, ..., r). \tag{2.22}$$

Any r-parameter group H has exactly $N - r$ functionally independent invariants $J = (J^1(x,u), ..., J^{N-r}(x,u))$. To find an invariant solution with respect to the subalgebra H, one has to find a universal invariant and to divide the universal invariant into two parts. The first part of the universal invariant is a function depending on the second part. For the invariant solutions, it is required that

$$rank\left(\frac{\partial(J^1, ..., J^{N-r})}{\partial(u^1, ..., u^m)}\right) = m.$$

Invariant solutions are not the only types of particular solutions that can be obtained by group analysis. One can consider a class of solutions which generalizes the concept of an invariant solution. The property which these new solutions have in common is that they all are found from the knowledge of a certain subgroup H of the main group G, therefore, such solutions are called partially invariant solutions.

The concept of invariant and partially invariant solutions permits an organization of the search process for particular solutions of system (1.1) admitted by group G_{11}, with the aid of different subgroups $H \subset G_{11}$. It is expedient in this search to begin with the lowest possible number of independent variables and move to successively higher number of independent variables of the studied $H-$solutions.

It is easier to find solutions with lower rank because the rank is equal to the number of independent variables in the factor system (S/H).

The algorithm for finding invariant and partially invariant solutions contains an arbitrariness related to the choice of the defect δ, which satisfies the following inequalities

$$\max\{r_* - n, 0\} \le \delta \le \min\{r_* - 1, m - 1\}. \tag{2.23}$$

Here

$$\delta = m - rank\frac{\partial\left(J^1, ..., J^{N-r}\right)}{\partial\left(u^1, ..., u^m\right)}.$$

Where r_* denoted general rank of its tangential mapping ξ of the group G. If $\delta = 0$ then the solution is invariant, otherwise it is partially invariant.

In the general case, the number of different types of solutions which are obtained for given n and m, associated with the different pairs (r_*, δ), depends only on m and n. A calculation shows that the maximum possible number of types of invariant and partially invariant solutions is equal to mn. Of course, for a specific equation $E(n, m, k, s)$, the actual number of types of partially invariant solutions admitted by the equation, which can be sought by using the subgroup H of the group G may be less than mn. This can occur if the dimensionality of the group G is not sufficiently large or because of the necessary condition $m - \delta \le r_*(\partial_u J)$.

Generally, the classification of the invariant and partially invariant solutions of the given differential equations is done on the same basis. If the rank of the Jacobi matrix of the first part with respect to the dependent function u is equal to the number of the functions u, then it is an invariant solution, otherwise it is a partially invariant solution. However, for partially invariant solutions, it is necessary to consider an additional characteristic of the defect δ. As it was noted, a knowledge of the admitted group helps to reduce the number of independent variables by constructing invariant and partially invariant solutions.

Chapter III

Three-wave Equations

3.1 Introduction

This chapter is devoted to deriving the three–wave equations. We consider the propagation of two monochromatic waves with the same frequencies ω but different polarizations in quadratic nonlinear medium. The interaction of this phenomenon generates the second harmonic and parametric oscillation.

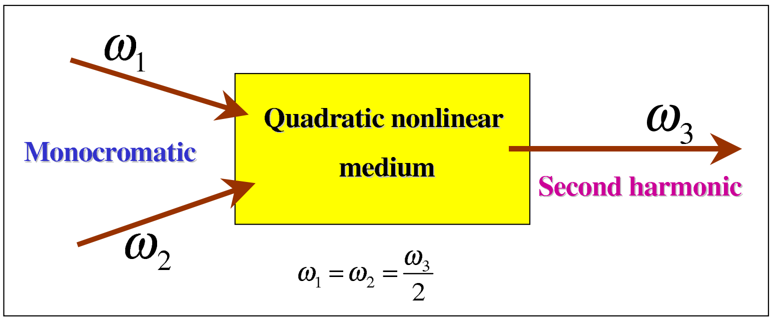

Figure 3.1: Second harmonic generation.

To construct the three–wave equations, one needs to study relations between electric and magnetic field. The interaction of these fields is described by the Maxwell's equations.

3.2 The three–wave equations in nonlinear optics

In this section, we explain the derivation of the three-wave equations. The starting point is the Maxwell equations:

$$\begin{aligned} \nabla \times \mathrm{H} &= \mathrm{I} + \frac{\partial \mathrm{D}}{\partial t}, \\ \nabla \times \mathrm{E} &= -\frac{\partial \mathrm{B}}{\partial t} \end{aligned} \qquad (3.1)$$

and the constitutive equations relating the polarization of the medium to the displacement vectors

$$\begin{aligned} \mathrm{D} &= \varepsilon_0 \mathrm{E} + \mathrm{P}, \quad \mathrm{I} = \sigma \mathrm{E}, \\ \mathrm{B} &= \mu_0(\mathrm{H} + \mathrm{M}), \end{aligned}$$

where I is the current density, E and H are the electric and magnetic field vectors, respectively; D and B are the electric and magnetic polarizations of the medium, ε_0 and μ_0 are the electric and magnetic permeabilities of vacuum, respectively, σ is the conductivity. The total polarization P is separated into its linear and nonlinear portions

$$\mathrm{P} = \varepsilon_0 \chi_l \mathrm{E} + \mathrm{P}_{nl}.$$

In non-linear optics for the magnetic polarizations, it is required that

$$\mathrm{M} = \chi_m \mathrm{H}.$$

Taking the curl of both sides of equation (3.1) and using $divE = 0$ and the vector identity

$$\nabla \times (\nabla \times \mathrm{E}) = \nabla(\mathrm{divE}) - \Delta \mathrm{E} = -\Delta \mathrm{E}, \tag{3.2}$$

we get

$$\Delta \mathrm{E} = \mu \frac{\partial}{\partial t} \Delta \mathrm{H},$$

where

$$\mu = \mu_0(1 + \chi_m).$$

After differentiating the first equation in (3.1) with respect to t and substituting it into the last equation, we obtain

$$\Delta \mathrm{E} = \mu\sigma \frac{\partial \mathrm{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \mathrm{E}}{\partial t^2} + \mu \frac{\partial^2 \mathrm{P}_{nl}}{\partial t^2}, \tag{3.3}$$

where $\varepsilon = \varepsilon_0(1 + \chi_l)$. System (3.3) is not closed. In strong fields for the nonlinear part of the electric polarization P_{nl} of medium, there are phenomenological expressions

$$\mathrm{P}_{nl} = \mathrm{P}^{(2)} + \mathrm{P}^{(3)} + \dots,$$

where $\mathrm{P}^{(j)}$ are the nonlinear part of j-th order. For $\mathrm{P}^{(j)}$ the following formula is suggested

$$\mathrm{P}^{(j)} = \int \int_0^\infty d\tau_1 d\tau_2 \cdots d\tau_j \chi^{(2)}(\tau_1, \tau_2) \, \mathrm{E}(t - \tau_1) \, \mathrm{E}(t - \tau_1 - \tau_2) \cdots \\ \cdots \mathrm{E}(t - \tau_1 - \tau_2 - \dots - \tau_j),$$

where $\hat{\chi}^{(j)}$ are tensors of nonlinear susceptibility. We consider medium with quadratic susceptibility.

In the general case, equations (3.3) are very complex. To study the nonlinear effects, a perturbation method is applied. According to this method, the electric field vector is represented in the form

$$\mathrm{E}\left(x,t\right)=\frac{1}{2}\sum_{j=1}^{N}\mathrm{E}_{j}\left(x,t\right)exp\left(i\omega_{j}t\right)+c.c., \tag{3.4}$$

where c.c. means complex conjugate terms. These waves generate waves with the polarization

$$\mathrm{P}=\frac{1}{2}\sum\mathrm{P}_{q}exp\left(i\omega_{q}t\right)+c.c. \tag{3.5}$$

on the frequencies

$$\omega_{q}=\sum_{j=1}^{N}\mathrm{m}_{j}\omega_{j},$$

where m_j are integer number. As the result in nonlinear medium, the new waves with these frequencies are generated. At the same time, new waves take part in interaction of (3.4) into formula for the polarization $\mathrm{P}^{(2)}$. Each of the electromagnetic field E_j generates two quadratic polarization of medium on doubled and zero frequencies:

$$\begin{aligned}\mathrm{P}_{q}^{(2)}&=\frac{1}{2}\chi^{(2)}\left(\omega_{j}+\omega_{j}\right)\mathrm{E}_{j}\mathrm{E}_{j},\\ \mathrm{P}_{q}^{(2)}&=\frac{1}{2}\chi^{(2)}\left(\omega_{j}-\omega_{j}\right)\mathrm{E}_{j}\mathrm{E}_{j}^{*}.\end{aligned} \tag{3.6}$$

Two electromagnetic waves with different frequencies ω_j and ω_k generate two more polarizations:

$$\begin{aligned}\mathrm{P}_{q}^{(2)}&=\chi^{(2)}\left(\omega_{j}+\omega_{k}\right)\mathrm{E}_{j}\mathrm{E}_{k},\\ \mathrm{P}_{q}^{(2)}&=\chi^{(2)}\left(\omega_{j}-\omega_{k}\right)\mathrm{E}_{j}\mathrm{E}_{k}^{*}.\end{aligned}$$

Substituting (3.4) and (3.5) into (3.3), we obtain the chain of the Helmholt equations :

$$\begin{aligned}\Delta\mathrm{E}_{j}&=\mu\sigma\left(\frac{\partial\mathrm{E}_{j}}{\partial t}+i\omega_{j}\mathrm{E}_{j}\right)+\mu\varepsilon\left(\frac{\partial^{2}\mathrm{E}_{j}}{\partial t^{2}}+2i\omega_{j}\frac{\partial\mathrm{E}_{j}}{\partial t}-\omega_{j}^{2}\mathrm{E}_{j}\right)+\\ &+\mu\left(\frac{\partial^{2}\mathrm{P}_{j}^{(2)}}{\partial t^{2}}+2i\omega_{j}\frac{\partial\mathrm{P}_{j}^{(2)}}{\partial t}-\omega_{j}^{2}\mathrm{P}_{j}^{(2)}\right).\end{aligned} \tag{3.7}$$

If in the result of one act of the interaction of two waves in medium, the third wave on combined frequency appears, then this is called a three-frequency interaction. In order to analyze this process, a method of slowly changing amplitudes is applied. According to this method, the electromagnetic field and polarization are represented in the forms

$$\begin{aligned}\mathrm{E}_{j}\left(x,y,z,t\right)&=\mathrm{e}_{j}\mathrm{A}_{j}\left(x\sqrt{\mu},y\sqrt{\mu},\mu z,\mu\left(z\frac{k_{j}}{\omega_{j}}-t\right)\right)\exp\left(-ik_{j}z\right),\\ \mathrm{P}_{j}^{(2)}\left(x,y,z,t\right)&=\mathrm{p}_{q}\mathrm{P}_{q}\left(x\sqrt{\mu},y\sqrt{\mu},\mu z,\mu t\right),\exp\left(-ik_{j}z\right),\end{aligned}$$

where e_j and p_q are unit vectors of polarization of the waves, A_j and P_q are slowly changing amplitudes. Substituting these representations into Helmholt equations (3.6) and leaving terms of the first degree with respect to the small parameter μ we obtain

$$\begin{aligned} -2i\frac{k_j^2}{\omega_j}\frac{\partial \mathrm{A}_j}{\partial t} + \Delta_\perp \mathrm{A}_j - 2ik_j\frac{\partial \mathrm{A}_j}{\partial \overline{z}} - \frac{1}{\mu}k_j^2\mathrm{A}_j &= \sigma i\omega_j \mathrm{A}_j - \varepsilon\omega_j^2\mathrm{A}_j - \\ &\quad -\alpha_q\omega_j^2\mathrm{P}_j\exp\left(-i\left(k_{p,j}-k_j\right)z\right) \end{aligned}$$

$$\frac{k_j}{\omega_j}\frac{\partial \mathrm{A}_j}{\partial t} + \frac{\partial \mathrm{A}_j}{\partial \overline{z}} + \frac{i}{2k_j}\Delta_\perp\mathrm{A}_j + i\alpha_j\frac{\omega_j^2}{2k_j}\mathrm{P}_j\exp\left(-i\left(k_{p,j}-k_j\right)z\right) = \mathrm{A}_j\frac{\omega_j^2}{2k_j}\left(\varepsilon_j - \varepsilon + i\frac{\sigma}{\omega_j}\right), \tag{3.8}$$

where $\Delta_\perp$ is the Laplacian in (x,y), $\quad \alpha_j = p_j l_j, \quad k_j^2 = \mu\omega_j^2\varepsilon_j$

Let us limit our consideration to the field made up of three plane waves, two of these fields are of the same frequency ω, for example $\omega_1 = \omega_2 = \omega$. As the result of their interaction, the third harmonic with the frequency $\omega_3 = \omega_1 + \omega_2 = 2\omega$ is generated. In this case the polarizations are the following equations

$$\begin{aligned} \mathrm{P}_1^{(2)} &= \frac{1}{2}\chi^{(2)}\left(\omega_3 - \omega_2\right)\mathrm{E}_3\mathrm{E}_2^*, \\ \mathrm{P}_2^{(2)} &= \frac{1}{2}\chi^{(2)}\left(\omega_3 - \omega_1\right)\mathrm{E}_3\mathrm{E}_1^*, \\ \mathrm{P}_3^{(2)} &= \frac{1}{2}\chi^{(2)}\left(\omega_1 + \omega_2\right)\mathrm{E}_1\mathrm{E}_2. \end{aligned}$$

Therefore, equations (3.8) become

$$\begin{aligned} \frac{\partial \mathrm{A}_1}{\partial \overline{z}} + \frac{i}{2k_1}\Delta_\perp\mathrm{A}_1 + \frac{k_1}{\omega}\frac{\partial \mathrm{A}_1}{\partial t} &= i\sigma_1\mathrm{A}_3\mathrm{A}_2^*\exp\left(-i\Delta kz\right), \\ \frac{\partial \mathrm{A}_2}{\partial \overline{z}} + \frac{i}{2k_2}\Delta_\perp\mathrm{A}_2 + \frac{k_2}{\omega}\frac{\partial \mathrm{A}_2}{\partial t} &= i\sigma_2\mathrm{A}_3\mathrm{A}_1^*\exp\left(-i\Delta kz\right), \\ \frac{\partial \mathrm{A}_3}{\partial \overline{z}} + \frac{i}{2\mathrm{k}_3}\Delta_\perp\mathrm{A}_3 + \frac{k_3}{2\omega}\frac{\partial \mathrm{A}_3}{\partial t} &= i\sigma_3\mathrm{A}_1\mathrm{A}_2\exp\left(i\Delta kz\right) \end{aligned}$$

where $\Delta k = k_3 - k_1 - k_2$. In short, these equations can be written as the following:

$$M_1A_1 = i\sigma_1 A_3 A_2^* e^{i\Delta kz}, \quad M_2A_2 = i\sigma_2 A_3 A_1^* e^{i\Delta kz}, \quad M_3A_3 = i\sigma_3 A_1 A_2 e^{-i\Delta kz}. \tag{3.9}$$

Here M_j are linear differential operators

$$M_j = \partial_{\overline{z}} + \frac{i}{2k_j}(\partial_{x^2}^2 + \partial_{y^2}^2) + k_j\omega^{-1}\partial_t, \quad (j = 1,2,3). \tag{3.10}$$

Derivation of equations (3.9) was done without taking account of anisotropy of a crystal. For anisotropic crystals, the equation $divE = 0$ is not correct. Therefore, in the process of deriving, one needs to use the full version of equation (3.2)

$$\nabla \times (\nabla \times \mathrm{E}) = \nabla(\mathrm{divE}) - \Delta\mathrm{E}.$$

This will lead to equations (3.9) with the corrected differential operators

$$M_j = \partial_{\overline{z}} + \frac{i}{2k_j}(\partial^2_{x^2} + \partial^2_{y^2}) + k_j\omega^{-1}\partial_t + \beta_j\partial_x, \quad (j = 1, 2, 3). \tag{3.11}$$

Note that by the equivalence transformations (Gorchakov, Meleshko, 1997) the coefficients $\beta_j, (j = 1, 2, 3)$ can be transformed to zero. Therefore, in this thesis we study the equations with $\beta_j = 0, (j = 1, 2, 3)$.

Chapter IV

Optimal System

4.1 Admitted group and equivalence group

In this section, we describe the equivalence group and group admitted by system (1.1). This system is written in the individual variables. We set the variables as follows :

$$x_1 = t,\ x_2 = x,\ x_3 = y,\ x_4 = z,\ A_1 = u^1 + iu^2,\ A_2 = u^3 + iu^4,\ A_3 = u^5 + iu^6,$$

$$p_i^j = u_{x_i}^j,\ p_{i\alpha}^j = u_{x_i x_\alpha}^j;\ \ (j = 1, 2, \ldots, 6;\ \ i, \alpha = 1, \ldots, 4)$$

$$\theta^k = \beta_k,\ \theta^{3+k} = \kappa_k,\ \theta^{6+k} = \sigma_k,\ \theta^{10} = \omega,\ \ (k = 1, 2, 3).$$

With these notations the functions $F_l = F_l(x, u, p, \theta), (l = 1, 2, .., 6)$ in (2.15) are defined by (1.1). We assume that the generator has representation in the form

$$X^e = \xi^i(x, u, \theta)\partial_{x_i} + \zeta^j(x, u, \theta)\partial_{u^j} + \zeta^{\theta^k}(x, u, \theta)\partial_{\theta^k}$$

and prolonged operator is

$$\overline{X}^e = X^e + \zeta_i^j(x, u, \theta)\partial_{u_i^j} + \zeta_{i\alpha}^j(x, u, \theta)\partial_{u_{i\alpha}^j} + \zeta_{x_i}^{\theta^k}(x, u, \theta)\partial_{\theta_{x_i}^k} + \zeta_{u^j}^{\theta^k}(x, u, \theta)\partial_{\theta_{u^j}^k}$$

The coefficients of the prolonged operator are defined by formulae (2.20). The determining equations are

$$[\overline{X}^e F_l]_{F_l=0} = 0. \tag{4.1}$$

Because the arbitrary elements θ^k are constants, one needs to append the additional equations

$$\zeta_{x_i}^{\theta^k} = 0,\ \ \zeta_{u^j}^{\theta^k} = 0;\ \ (j = 1, 2, \ldots, 6;\ i = 1, 2, \ldots, 4;\ k = 1, 2, \ldots, 10).$$

The result of the calculations is the equivalence group with the basis of the generators:

$$\begin{array}{rcl}
X_1^e &=& x\partial_x + y\partial_y + z\partial_z - \sum_{\alpha=1}^{6} u_\alpha \partial_{u_\alpha} - k_1\partial_{k_1} - k_2\partial_{k_2},\\
X_2^e &=& t\partial_t - \omega\partial_\omega,\\
X_3^e &=& \omega t\partial_x + \sum_{\alpha=1}^{3} k_\alpha \partial_{\beta_\alpha},\\
X_4^e &=& z\partial_x + \sum_{\alpha=1}^{3} \partial_{\beta_\alpha},\\
X_5^e &=& (k_1)^{-1}(k_1 x - (\beta_3-\beta_2)\omega t - (k_3\beta_2 - k_2\beta_3)z)(u_4\partial_{u_3} - u_3\partial_{u_4} +\\
&& +u_6\partial_{u_5} - u_5\partial_{u_6}) - (k_2)^{-1}\partial_{\beta_2} - (k_3)^{-1}\partial_{\beta_3},\\
X_6^e &=& 2\sigma_1\partial_{\sigma_1} - (u_3\partial_{u_3} + u_4\partial_{u_4} + u_5\partial_{u_5} + u_6\partial_{u_6}),\\
X_7^e &=& 2\sigma_2\partial_{\sigma_2} - (u_1\partial_{u_1} + u_2\partial_{u_2} + u_5\partial_{u_5} + u_6\partial_{u_6}),\\
X_8^e &=& 2\sigma_3\partial_{\sigma_3} - (u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3} + u_4\partial_{u_4}),\\
X_9^e &=& x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2\sum_{\alpha=1}^{6} u_\alpha\partial_{u_\alpha} - \sum_{\alpha=1}^{3}\beta_\alpha\partial_{\beta_\alpha}.
\end{array}$$

Remark. In the experiments, it is very important to have $\beta_1^2 + \beta_2^2 + \beta_3^2 \neq 0$. However, in this case, an admitted group is very complicated. Group equivalence makes it possible to transform to equivalent system with $\beta_i = 0$. It essentially simplifies the calculations.

In the strength of the equivalence group, we consider $\beta_i = 0$; $i = 1, 2, 3$. The admitted Lie group corresponds to the Lie algebra L_{11} with the generators :

$$\begin{array}{rcl}
X_1 &=& \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = y\partial_x - x\partial_y,\\
X_5 &=& x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2\sum_{\alpha=1}^{6} u_\alpha\partial_{u_\alpha},\\
X_6 &=& xX_{10} + z\partial_x, \quad X_7 = yX_{10} + z\partial_y, \quad X_8 = \partial_t,\\
X_9 &=& (k_1-k_2)zX_{10} + ((k_1+k_2)z - 2\omega t)X_{11},\\
X_{10} &=& k_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + k_2(u_4\partial_{u_3} - u_3\partial_{u_4}) + (k_1+k_2)(u_6\partial_{u_5} - u_5\partial_{u_6}),\\
X_{11} &=& k_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + k_2(u_4\partial_{u_3} - u_3\partial_{u_4}) + (k_1-k_2)(u_6\partial_{u_5} - u_5\partial_{u_6}).
\end{array}$$

The table of commutators is

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1	0	0	0	$-X_2$	X_1	X_{10}	0	0	0
X_2		0	0	X_1	X_2	0	X_{10}	0	0
X_3			0	0	$2X_3$	X_1	X_2	0	$\alpha_0 X_{10} + \beta_0 X_{11}$
X_4				0	0	X_7	$-X_6$	0	0
X_5					0	X_6	X_7	$-2X_8$	$2X_9$
X_6						0	0	0	0
X_7							0	0	0
X_8								0	$-\omega X_{11}$
X_9									0

Here $\alpha_0 = (k_1-k_2)/2$, $\beta_0 = (k_1+k_2)/2$. Two generators X_{10} and X_{11} constitute the center of the algebra. Inner automorphisms are constructed with the help of the table of commutators.

To construct automorphisms, one has to solve the Lie equations. For example, for the automorphism A_1, we have the system of ordinary differential equations

$$\frac{d\overline{x}_1}{da} = \overline{x}_5, \quad \frac{d\overline{x}_2}{da} = -\overline{x}_4, \quad \frac{d\overline{x}_{10}}{da} = \overline{x}_6$$

.

and the initial values at $a = 0$

$$\overline{x}_1 = x_1, \;\; \overline{x}_2 = x_2, \;\; \overline{x}_{10} = x_{10}.$$

Therefore, the automorphism A_1 only changes the coordinates x_1, x_2 and x_{10} by the formulae

$$\overline{x}_1 = x_1 + a_1 x_5, \;\; \overline{x}_2 = x_2 - a_1 x_4, \;\; \overline{x}_{10} = x_{10} + a_1 x_6.$$

Other coordinates are not changed.

By the same way, we obtain the automorphisms $A_i \quad (i = 2, \ldots, 9)$:

$$\begin{array}{ll}
A_2: & \overline{x}_1 = x_1 + a_2 x_4, \;\; \overline{x}_2 = x_2 + a_2 x_5, \;\; \overline{x}_{10} = x_{10} + a_2 x_7, \\
A_3: & \overline{x}_1 = x_1 + a_3 x_6, \;\; \overline{x}_2 = x_2 + a_3 x_7, \;\; \overline{x}_3 = x_3 + 2a_3 x_5, \\
 & \overline{x}_{10} = x_{10} + \alpha_0 a_3 x_9, \;\; \overline{x}_{11} = x_{11} + \beta_0 a_3 x_9, \\
A_4: & \overline{x}_1 = x_1 \cos(a_4) - x_2 \sin(a_4), \;\; \overline{x}_2 = x_1 \sin(a_4) + x_2 \cos(a_4), \\
 & \overline{x}_6 = x_6 \cos(a_4) - x_7 \sin(a_4), \;\; \overline{x}_7 = x_6 \sin(a_4) + x_7 \cos(a_4), \\
A_5: & \overline{x}_1 = a_5^{-1} x_1, \;\; \overline{x}_2 = a_5^{-1} x_2, \;\; \overline{x}_3 = a_5^{-2} x_3, \;\; \overline{x}_6 = a_5 x_6, \;\; \overline{x}_7 = a_5 x_7, \\
 & \overline{x}_8 = a_5^{-2} x_8, \;\; \overline{x}_9 = a_5^2 x_9, \\
A_6: & \overline{x}_1 = x_1 - a_6 x_3, \;\; \overline{x}_6 = x_6 - a_6 x_5, \;\; \overline{x}_7 = x_7 - a_6 x_4, \;\; \overline{x}_{10} = x_{10} - a_6 x_1 + \frac{a_6^2}{2} x_3, \\
A_7: & \overline{x}_2 = x_2 - a_7 x_3, \;\; \overline{x}_6 = x_6 + a_7 x_4, \;\; \overline{x}_7 = x_7 - a_7 x_5, \;\; \overline{x}_{10} = x_{10} - a_7 x_2 + \frac{a_7^2}{2} x_3, \\
A_8: & \overline{x}_8 = x_8 + 2a_8 x_5, \;\; \overline{x}_{11} = x_{11} - \omega a_8 x_9, \\
A_9: & \overline{x}_{10} = x_{10} - \alpha_0 a_9 x_3, \;\; \overline{x}_{11} = x_{11} - a_9(\omega x_8 - \beta_0 x_3), \;\; \overline{x}_9 = x_9 - 2a_9 x_5.
\end{array}$$

Also there are two involutions

$$E_1 : \overline{x}_1 = -x_1, \quad \overline{x}_6 = -x_6,$$

$$E_2 : \overline{x}_2 = -x_2, \quad \overline{x}_7 = -x_7,$$

which correspond to the change of the independent variables x on $-x$ and y on $-y$, respectively.

4.2 Decomposition of the algebras L_{11}

Before constructing an optimal system, we study an algebraic structure of the algebra L_{11}. Let us consider the vector space $\overline{L_9}$ spanned by the operators $X_1, X_2, ..., X_9$. Note that the space $\overline{L_9}$ is not a subalgebra, because, for example $[X_1, X_6] = X_{10} \notin \overline{L_9}$. Assume that $H = \{Y_1, Y_2, ..., Y_k\}$ is a k-dimensional subalgebra of the algebra L_{11}. Operators $Y_i, \;\; (i = 1, 2, \ldots, k)$ are

$$Y_i = \overline{Y}_i + \sum_{\alpha=10}^{11} x_{i\alpha} X_\alpha, \quad \overline{Y}_i = \sum_{\alpha=1}^{9} x_{i\alpha} X_\alpha,$$

where the first part belongs to the space $\overline{L_9}$, and the second part belongs to the center $\{X_{10}, X_{11}\}$. The rank of the matrix $Q = (x_{ij})$ composed of the coefficients x_{ij}, $(i = 1, 2, \dots, k;\ \ j = 1, 2, \dots, 11)$ is equal to k. The rank of the matrix $\overline{Q}$ composed of the first 9 columns of the matrix Q can be equal to either k, $k-1$ or $k-2$. If the rank is equal to $k-1$ or $k-2$, then, without loss of generality, $Y_k \in \{X_{10}, X_{11}\}$. In this case the rank of the Jacobi matrix of the universal invariant with respect to the dependent functions is less than the number of the dependent variables. The reson for this is that the process of finding the universal invariant can start from the operator Y_k which are expressed in term of the dependent variables only. Therefore, for constructing invariant solutions, we only have to consider the subalgebras of the first kind, where the rank of the matrix $\overline{Q}$ is equal to k. This means that the vectors $\overline{Y}_i$, $(i = 1, 2, \dots, k)$ are linearly independent.

Conditions for H to be a subalgebra give

$$[Y_i, Y_j] = \sum_{\alpha=1}^{k} C_{ij}^{\alpha} Y_\alpha; \quad i, j = 1, 2, \dots, k.$$

Because the operators X_{10}, X_{11} constitute a center, then

$$[Y_i, Y_j] = [\overline{Y}_i, \overline{Y}_j] = \sum_{\alpha=1}^{k} C_{ij}^{\alpha} \overline{Y}_\alpha + Y_{ij},$$

where $Y_{ij} \in \{X_{10}, X_{11}\}$. Hence the operators $\overline{Y}_i$, $(i = 1, 2, \dots, k)$ constitute k-dimensional subalgebra of the factor algebra $L_9 = L_{11}/\{X_{10}, X_{11}\}$. A basis of L_9 can be chosen consisting of the classes $X_i/\{X_{10}, X_{11}\}$, $(i = 1, 2, \dots, 9)$. Because $\{X_{10}, X_{11}\}$ is a center, then, for the classes $X_i/\{X_{10}, X_{11}\}$, one can use the same automorphisms (without considering transformations of X_{10} and X_{11}) as for the operators X_i. Therefore, this allows simplifying the process of constructing the optimal system of subalgebras of the algebra L_{11}. At first, one can construct an optimal system of subalgebras of L_9. This gives the operators $\overline{Y}_i$. Then the operators $\overline{Y}_i$ can be supplemented by the operators from the center $Y_i = \overline{Y}_i + \sum_{\alpha=10}^{11} x_{i\alpha} X_\alpha$. On the next step, one needs to check the subalgebra conditions for the operators Y_i in L_{11} and check a possibility to simplify the coefficients $x_{i\alpha}$, $(\alpha = 10, 11)$ by the automorphisms.

Let us consider the factor algebra L_9. A basis of L_9 can be chosen, consisting of the classes $X_i/\{X_{10}, X_{11}\}$, $(i = 1, 2, \dots, 9)$. Later, for the sake of simplicity, we write X_i for the basis element $X_i/\{X_{10}, X_{11}\} \in L_9$, $(i = 1, 2, \dots, 9)$. Because the zero class of L_9 consists of the operators $c_1X_{10} + c_2X_{11}$, then a table of commutators of the algebra L_9 can be obtained from the table of commutators of L_{11} by letting the element $(c_1X_{10} + c_2X_{11}) = 0 \in L_9$.

As noted before, the automorphisms of the algebra L_{11} can be used for the algebra L_9. To construct an optimal system of subalgebra of the algebra L_9, we use two-steps algorithm, which is explained earlier. According to this algorithm on the first step, the algebra L_9 is decomposed as $I_1 \oplus N_1$, where

$I_1 = \{X_1, X_2, X_3\}$ is an ideal and $N_1 = \{X_4, X_5, X_6, X_7, X_8, X_9\}$ is a subalgebra. At the same time, the subalgebra N_1 can also be decomposed as $N_1 = I_2 \oplus N_2$ with the ideal $I_2 = \{X_6, X_7\}$ and the subalgebra $N_2 = \{X_4, X_5, X_8, X_9\}$. After these decompositions, the subalgebra N_2 is four-dimensional. Its optimal system can be easily constructed.

4.3 Classification of the algebra L_4

Let us classify the algebra $L_4 = \{X_4, X_5, X_8, X_9\}$. The table of commutators of the algebra L_4 is

	X_4	X_5	X_8	X_9
X_4	0	0	0	0
X_5	0	0	$-2X_8$	$2X_9$
X_8	0	$2X_8$	0	0
X_9	0	$-2X_9$	0	0

The automorphisms are

$$\begin{aligned}
A_4 &: \ \overline{x}_4 = x_4, \ \ \overline{x}_5 = x_5, \ \ \overline{x}_8 = x_8, \ \ \overline{x}_9 = x_9, \\
A_5 &: \ \overline{x}_4 = x_4, \ \ \overline{x}_5 = x_5, \ \ \overline{x}_8 = x_8 e^{-2a}, \ \ \overline{x}_9 = x_9 e^{2a}, \\
A_8 &: \ \overline{x}_4 = x_4, \ \ \overline{x}_5 = x_5, \ \ \overline{x}_8 = x_8 + 2a x_5, \ \ \overline{x}_9 = x_9, \\
A_9 &: \ \overline{x}_4 = x_4, \ \ \overline{x}_5 = x_5, \ \ \overline{x}_8 = x_8, \ \ \overline{x}_9 = x_9 - 2a x_5.
\end{aligned}$$

There is one involution

$$E : \overline{x}_4 = -x_4, \ \ \overline{x}_5 = x_5, \ \ \overline{x}_8 = x_8, \ \ \overline{x}_9 = x_9.$$

Let us study an optimal system of the algebra L_4. For convenience, we will denote the generators X_i by $\mathbf{i}$. A short description of different cases is given in tables D.1, E.1 and F.1

4.3.1 One-dimensional subalgebras of the algebra L_4

Let $Y = x_4\mathbf{4} + x_5\mathbf{5} + x_8\mathbf{8} + x_9\mathbf{9}$ which constitutes a one-dimensional subalgebra of the algebra L_4. The process of simplification of the coefficients of the operator Y is separated in the following cases.

Case 1. Assume that $x_5 \neq 0$, then we can divide Y by x_5. Hence, without loss of generality one can consider

$$Y = x_4\mathbf{4} + \mathbf{5} + x_8\mathbf{8} + x_9\mathbf{9}.$$

By the transformation A_8 and A_9, we can transform to $x_8 = 0$ and $x_9 = 0$. Other transformations cannot change x_4. Therefore, in the case when $x_5 \neq 0$ the class of equivalent subalgebras is described by the operators $\mathbf{5} + x_4\mathbf{4}$.

Case 2. Assume that $x_5 = 0$, then we have $Y = x_4\mathbf{4} + x_8\mathbf{8} + x_9\mathbf{9}$.

Case 2.1. Let $x_4 \neq 0$. By dividing the operator Y by x_4, we obtain $Y = \mathbf{4} + x_8\mathbf{8} + x_9\mathbf{9}$.

Case 2.1.1. Let $x_8 \neq 0$. By the automorphism A_8, the operator Y is transformed to $\mathbf{4} + \mathbf{8} + x_9\mathbf{9}$.

Case 2.1.2. If $x_8 = 0$, then $Y = \mathbf{4} + x_9\mathbf{9}$.

Case 2.1.2.1. If $x_9 \neq 0$, then the operator Y can be transformed by the automorphism A_5 to $\mathbf{4} \pm \mathbf{9}$, which is transformed by involution E to the operator $\mathbf{4} + \mathbf{9}$.

Case 2.1.2.2. If $x_9 = 0$,then the representative of the class is the operator $\mathbf{4}$.

Case 2.2. Let $x_4 = 0$, this means that $Y = x_8\mathbf{8} + x_9\mathbf{9}$.

Case 2.2.1. If $x_8 \neq 0$, then Y can be divided by $x_8 : Y = \mathbf{8} + x_9\mathbf{9}$.

Case 2.2.1.1. If $x_9 \neq 0$, then by the automorphism A_5, the operator Y is transformed to $\mathbf{8} + \varepsilon\mathbf{9}$, where $\varepsilon = \pm 1$.

Case 2.2.1.2. If $x_9 = 0$, then $Y = \mathbf{8}$.

Case 2.2.2. Let $x_8 = 0$. Because $Y \neq 0$, then the representative of the class is the operator $\mathbf{9}$.

4.3.2 Two-dimensional subalgebras of the algebra L_4

Let a subalgebra be formed by the operators

$$Y_1 = a_{11}\mathbf{4} + a_{12}\mathbf{5} + a_{13}\mathbf{8} + a_{14}\mathbf{9}, \quad Y_2 = a_{21}\mathbf{4} + a_{22}\mathbf{5} + a_{23}\mathbf{8} + a_{24}\mathbf{9}$$

where a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23} and a_{24} are arbitrary constants. Note that the rank of the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$ is equal to two.

Case 1. Assume that $a_{12} \neq 0$. We can divide Y_1 by a_{12}. Hence, by subtracting the operator $(a_{22}/a_{12})Y_1$ from Y_2, one can take $a_{22} = 0$. By the automorphisms A_8 and A_9, the operator Y_1 is transformed to $Y_1 = \mathbf{5} + a_{11}\mathbf{4}$. After checking subalgebra conditions, we have

$$[a_{11}\mathbf{4} + \mathbf{5}, a_{21}\mathbf{4} + a_{23}\mathbf{8} + a_{24}\mathbf{9}] = \alpha(a_{11}\mathbf{4} + \mathbf{5}) + \beta(a_{21}\mathbf{4} + a_{23}\mathbf{8} + a_{24}\mathbf{9})$$

where α and β are arbitrary constants. By calculating the left hand side and comparing the coefficients in the left hand side with coefficients in the right hand side, we have

$$-2a_{23}\mathbf{8} + 2a_{24}\mathbf{9} = (\alpha a_{11} + \beta a_{21})\mathbf{4} + \alpha\mathbf{5} + \beta a_{23}\mathbf{8} + \beta a_{24}\mathbf{9}.$$

Therefore

$$(\beta + 2)a_{23} = 0, \quad (\beta - 2)a_{24} = 0, \quad \beta a_{21} = 0.$$

Further consideration depends on value of the coefficients a_{21}, a_{23}, a_{24}.

Case 1.1. If $a_{23} \neq 0$, then $\beta = -2$ and $a_{24} = 0, a_{21} = 0$. By the automorphism A_8 the operators Y_1 and Y_2 are transformed to $Y_1 = a_{11}\mathbf{4} + \mathbf{5}$, $Y_2 = \mathbf{8}$.

Case 1.2. If $a_{23} = 0$ and $a_{24} \neq 0$, then $\beta = 2$ and $a_{21} = 0$. Hence, the operators Y_1 and Y_2 are $Y_1 = a_{11}\mathbf{4} + \mathbf{5}, Y_2 = \mathbf{9}$.

Case 1.3. If $a_{23} = 0$ and $a_{24} = 0$, then $a_{21} \neq 0$. Then operators Y_1 and Y_2 are $Y_1 = \mathbf{5}, Y_2 = \mathbf{4}$.

Case 2. :Assume that $a_{12} = 0$. If $a_{22} \neq 0$ then by changing Y_1 and Y_2, this case corresponds to the previous case. Hence, one can take $a_{22} = 0$. Therefore, the operations are $Y_1 = a_{11}\mathbf{4} + a_{13}\mathbf{8} + a_{14}\mathbf{9}, Y_2 = a_{21}\mathbf{4} + a_{23}\mathbf{4} + a_{24}\mathbf{9}$. By checking the subalgebra condition, we have

$$[Y_1, Y_2] = 0 = \alpha(a_{11}\mathbf{4} + a_{13}\mathbf{8} + a_{14}\mathbf{9}) + \beta(a_{21}\mathbf{4} + a_{23}\mathbf{8} + a_{24}\mathbf{9}),$$

with some coefficient α, β. Hence

$$\alpha a_{11} + \beta a_{21} = 0, \quad \alpha a_{13} + \beta a_{23} = 0, \quad \alpha a_{14} + \beta a_{24} = 0.$$

Further consideration depends on value of the coefficients a_{11}, a_{21}, a_{13}, a_{23}, a_{14} and a_{24}.

Case 2.1. Let $a_{11} \neq 0$. After dividing Y_1 by a_{11}, one can take $a_{11} = 1$. Without loss of generality one can also take $a_{21} = 0$.

Case 2.1.1. Let $a_{24} \neq 0$. After dividing the operator Y_2 by a_{24}, and excluding $\mathbf{4}$ from Y_2 we obtain $Y_1 = \mathbf{4} + a_{13}\mathbf{8}, \quad Y_2 = \mathbf{9} + a_{23}\mathbf{8}$.

Case 2.1.1.1. If $a_{13} \neq 0$, then by the automorphism A_5 the coefficient a_{13} is transformed to 1

$$Y_1 = \mathbf{4} + \mathbf{8}, \quad Y_2 = \mathbf{9} + a_{23}\mathbf{8}.$$

Case 2.1.1.2. If $a_{13} = 0$ and $a_{23} \neq 0$ then by the automorphism A_5 the operators Y_1 and Y_2 are transformed to $\mathbf{4}, \varepsilon\mathbf{8} + \mathbf{9}$, where $\varepsilon = \pm 1$.

Case 2.1.1.3. If $a_{13} = 0$ and $a_{23} = 0$ then $Y_1 = \mathbf{4}, Y_2 = \mathbf{9}$.

Case 2.1.2. Let $a_{24} = 0$. In this case we have the operators $\mathbf{4} + a_{14}\mathbf{9}, \mathbf{8}$.

Case 2.1.2.1. If $a_{14} \neq 0$ then, by the automorphism A_5, the coefficient a_{14} can be transformed to 1. Hence, we get $Y_1 = \mathbf{4} + \mathbf{9}, Y_2 = \mathbf{8}$.

Case 2.1.2.2. If $a_{14} = 0$ then $Y_1 = \mathbf{4}, Y_2 = \mathbf{8}$.

Case 2.2. Let $a_{11} = 0$. We remind that $a_{12} = a_{22} = 0$. Note that if $a_{21} \neq 0$), then we have the previous case by changing Y_1 and Y_2. Hence, $a_{21} = 0$. In this case we obtain $Y_1 = \mathbf{8}, Y_2 = \mathbf{9}$.

4.3.3 Three-dimensional subalgebras of the algebra L_4

Let a subalgebra be formed by the operators

$$\begin{aligned} Y_1 &= a_{11}\mathbf{4} + a_{12}\mathbf{5} + a_{13}\mathbf{8} + a_{14}\mathbf{9} \\ Y_2 &= a_{21}\mathbf{4} + a_{22}\mathbf{5} + a_{23}\mathbf{8} + a_{24}\mathbf{9} \\ Y_3 &= a_{31}\mathbf{4} + a_{32}\mathbf{5} + a_{33}\mathbf{8} + a_{34}\mathbf{9} \end{aligned}$$

where $a_{ij},\ \ (i=1,2,3;\ j=1,2,3,4)$ are arbitrary constants. Note that rank of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

is equal to three.

Case 1.

Assume that $a_{12}\neq 0$, then by taking linear combinations one can take $a_{12}=1$ and $a_{22}=a_{32}=0$. By the automorphisms A_8 and A_9 the coefficients a_{13} and a_{14} can be transformed to zero. Hence, the operators are $Y_1=a_{11}\mathbf{4}+\mathbf{5},\ Y_2=a_{21}\mathbf{4}+a_{23}\mathbf{8}+a_{24}\mathbf{9},\ Y_3=a_{31}\mathbf{4}+a_{33}\mathbf{8}+a_{34}\mathbf{9}$.

Case 1.1. If $a_{21}\neq 0$, then we can take $a_{21}=1$ and $a_{11}=a_{31}=0$. Hence, the operators are $Y_1=\mathbf{5},\ Y_2=\mathbf{4}+a_{23}\mathbf{8}+a_{24}\mathbf{9},\ Y_3=a_{33}\mathbf{8}+a_{34}\mathbf{9}$.

Case 1.1.1. If $a_{33}\neq 0$, then we can take $a_{33}=1$ and $a_{23}=0$. Hence, the operators are $Y_1=\mathbf{5},\ Y_2=\mathbf{4}+a_{24}\mathbf{9},\ Y_3=\mathbf{8}+a_{34}\mathbf{9}$.
The operators $Y_1,\ Y_2,\ Y_3$ must constitute a subalgebra. Let us consider the commutators

$$\begin{aligned} [\mathbf{5},\,\mathbf{4}+a_{24}\mathbf{9}] &= 2a_{24}\mathbf{9} = \alpha_1\mathbf{5}+\beta_1(\mathbf{4}+a_{24}\mathbf{9})+\gamma_1(\mathbf{8}+a_{34}\mathbf{9}),\\ [\mathbf{5},\,\mathbf{8}+a_{34}\mathbf{9}] &= 2a_{34}\mathbf{9} = \alpha_2\mathbf{5}+\beta_2(\mathbf{4}+a_{22}\mathbf{9})+\gamma_2(\mathbf{8}+a_{34}\mathbf{9}),\end{aligned}$$

with some constants $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \gamma_1$and γ_2. Hence $\alpha_1=0,\ \alpha_2=0,\ \beta_1=0,\ \beta_2=0,\ \gamma_1=0,\ \gamma_2=0,\ a_{24}=0,\ a_{34}=0$. Therefore, the operators $Y_1,\ Y_2$ and Y_3 can be transformed to $Y_1=\mathbf{5},\ Y_2=\mathbf{4}$ and $Y_3=\mathbf{8}$.

Case 1.1.2. If $a_{33}=0$ then a_{34} has to be nonzero. Hence, we can take $a_{34}=1$ and $a_{24}=0$. Therefore the operators are $Y_1=\mathbf{5},\ Y_2=\mathbf{4}+a_{23}\mathbf{8}$ and $Y_3=\mathbf{8}$. By checking the subalgebra conditions, we have

$$[\mathbf{5},\,\mathbf{4}+a_{23}\mathbf{8}]=-2a_{23}\mathbf{8}=\alpha\mathbf{5}+\beta(\mathbf{4}+a_{23}\mathbf{8})+\gamma\mathbf{9},$$

with some coefficient α, β and γ. Hence $\alpha=0,\ \beta=0,\ \gamma=0,\ a_{23}=0$. Therefore, the operators Y_1, Y_2 and Y_3 can be transformed to $Y_1=\mathbf{5}$, $Y_2=\mathbf{4}$, $Y_3=\mathbf{9}$.

Case 1.2. Assume that $a_{21}=0$. If $a_{31}\neq 0$, then by changing Y_2 and Y_3 this case corresponds to the previous case. Hence, one can take $a_{31}=0$. Because the rank of the matrix

$$\begin{pmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{pmatrix}$$

is equal to 2, then by taking linear combinations the operators Y_1, Y_2 and Y_3 can be transformed to $Y_1=a_{11}\mathbf{4}+\mathbf{5},\ Y_2=\mathbf{8},\ Y_3=\mathbf{9}$.

Case 2.

Assume that $a_{12}=0$. From case 1, we can take $a_{22}=a_{32}=0$. since the rank of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

is equal to 3, then by taking linear combinations one can account that $Y_1 = \mathbf{4}$, $Y_2 = \mathbf{8}$, $Y_3 = \mathbf{9}$.

4.3.4 Four-dimensional subalgebra of the algebra L_4

Let a subalgebra be formed by the operators

$$\begin{array}{rcl} Y_1 & = & a_{11}\mathbf{4} + a_{12}\mathbf{5} + a_{13}\mathbf{8} + a_{14}\mathbf{9}, \\ Y_2 & = & a_{21}\mathbf{4} + a_{22}\mathbf{5} + a_{23}\mathbf{8} + a_{24}\mathbf{9}, \\ Y_3 & = & a_{31}\mathbf{4} + a_{32}\mathbf{5} + a_{33}\mathbf{8} + a_{34}\mathbf{9}, \\ Y_4 & = & a_{41}\mathbf{4} + a_{42}\mathbf{5} + a_{43}\mathbf{8} + a_{44}\mathbf{9}, \end{array}$$

where a_{ij}, $(i, j = 1, 2, 3, 4)$ are arbitrary constants. Note that the rank of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

is equal to four. Therefore,

$$Y_1 = \mathbf{4}, Y_2 = \mathbf{5}, Y_3 = \mathbf{8}, Y_4 = \mathbf{9},$$

The optimal system of the algebra $L_4 = \{\mathbf{4},\ \mathbf{5},\ \mathbf{8},\ \mathbf{9}\}$ is the following:

Dimension			
1	2	3	4
$\mathbf{4}$	$\mathbf{4}, \mathbf{5}$	$\mathbf{4}, \mathbf{5}, \mathbf{8}$	$\mathbf{4}, \mathbf{5}, \mathbf{8}, \mathbf{9}$
$\mathbf{8}$	$\mathbf{4}, \mathbf{8}$	$\mathbf{4}, \mathbf{5}, \mathbf{9}$	
$\mathbf{9}$	$\mathbf{4}, \mathbf{9}$	$\mathbf{4}, \mathbf{8}, \mathbf{9}$	
$\mathbf{4} + \mathbf{9}$	$\mathbf{8}, \mathbf{9}$	$\mathbf{5} + x_4\mathbf{4}, \mathbf{8}, \mathbf{9}$	
$\mathbf{5} + x_4\mathbf{4}$	$\mathbf{8}, \mathbf{4} + \mathbf{9}$		
$\mathbf{8} + \varepsilon\mathbf{9}$	$\mathbf{4}, \mathbf{9} + \varepsilon\mathbf{8}$		
$\mathbf{4} + \mathbf{8} + x_9\mathbf{9}$	$\mathbf{5} + x_4\mathbf{4}, \mathbf{8}$		
	$\mathbf{5} + x_4\mathbf{4}, \mathbf{9}$		
	$\mathbf{4} + \mathbf{8}, \mathbf{9} + x_8\mathbf{8}$		

where x_4, x_8 and x_9 are arbitrary real parameter and $\varepsilon = \pm 1$.

4.3.5 Optimal system of subalgebras of the algebra $L_6 = \{\mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$

The next step is a construction of an optimal system of the algebra $L_6 = \{\mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$. We construct it by gluing the subalgebras from the optimal system of the algebra L_4 with the ideal $I_2 = \{\mathbf{6}, \mathbf{7}\}$.

The table of commutators of the algebra L_6 is

	X_4	X_5	X_6	X_7	X_8	X_9
X_4	0	0	X_7	$-X_6$	0	0
X_5	0	0	X_6	X_7	$-2X_8$	$2X_9$
X_6	$-X_7$	$-X_6$	0	0	0	0
X_7	X_6	$-X_7$	0	0	0	0
X_8	0	$2X_8$	0	0	0	0
X_9	0	$-2X_9$	0	0	0	0

The automorphisms are

$$\begin{array}{ll} A_4: & \overline{x}_6 = x_6\cos a - x_7\sin a,\ \ \overline{x}_7 = x_6\sin a + x_7\cos a, \\ A_5: & \overline{x}_6 = x_6e^a,\ \ \overline{x}_7 = x_7e^a, \overline{x}_8 = x_8e^{-2a}, \overline{x}_9 = x_9e^{2a}, \\ A_6: & \overline{x}_6 = x_6 - ax_5,\ \ \overline{x}_7 = x_7 - ax_4, \\ A_7: & \overline{x}_6 = x_6 + ax_4,\ \ \overline{x}_7 = x_7 - ax_5, \\ A_8: & \overline{x}_8 = x_8 + 2ax_5, \\ A_9: & \overline{x}_9 = x_9 - 2ax_5 \end{array}$$

and two involutions are

$$\begin{array}{ll} E_1: & \overline{x}_4 = -x_4,\ \ \overline{x}_6 = -x_6, \\ E_2: & \overline{x}_4 = -x_4,\ \ \overline{x}_7 = -x_7. \end{array}$$

Here we give one example of the process of gluing. Other elements of the optimal system of the algebra L_6 are constructed similarly.

Let us consider the subalgebra $\{\mathbf{5} + x_4\mathbf{4}, \mathbf{9}\}$ and the ideal $I_2 = \{\mathbf{6}, \mathbf{7}\}$. To construct two-dimensional subalgebras of the algebra L_6 we use the matrix of coefficients

X_6	X_7	X_4	X_5	X_8	X_9
h_1	h_2	x_4	1	0	0
h_3	h_4	0	0	0	1

for three-dimensional subalgebras of the algebra L_6 the matrix is

X_6	X_7	X_4	X_5	X_8	X_9
h_1	h_2	x_4	1	0	0
h_3	h_4	0	0	0	1
h_5	h_6	0	0	0	0

and for a four-dimensional case, it is

X_6	X_7	X_4	X_5	X_8	X_9
h_1	h_2	x_4	1	0	0
h_3	h_4	0	0	0	1
h_5	h_6	0	0	0	0
h_7	h_8	0	0	0	0

Note that the rank of the matrix

$$\mathbf{B} = (h_5\ \ h_6)$$

in the three-dimensional case is equal to one and in the four-dimensional case the rank of the matrix

$$\mathbf{C} = \begin{pmatrix} h_5 & h_6 \\ h_7 & h_8 \end{pmatrix}$$

is equal to two. Hence, the matrix **B** is reduced to either

$$\mathbf{B} = (1 \;\; h_6) \quad \text{or} \quad \mathbf{B} = (0 \;\; 1)$$

and the matrix **C** is the unit matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us consider three-dimensional subalgebras. Assume that the matrix $\mathbf{B} = (1 \;\; h_6)$. Then without loss of generality one can take $h_1 = h_3 = 0$. By the automorphisms A_4 and A_7, the matrix of coefficients is transformed to

X_6	X_7	X_4	X_5	X_8	X_9
0	0	x_4	1	0	0
0	h_4	0	0	0	1
1	0	0	0	0	0

The subalgebra conditions give

$$[x_4\mathbf{4} + \mathbf{5},\ \mathbf{9} + h_4\mathbf{7}] = \alpha(x_4\mathbf{4} + \mathbf{5}) + \beta(\mathbf{9} + h_4\mathbf{7}) + \gamma\mathbf{6}$$

where α, β and γ are arbitrary constants. By calculating the left side, we have

$$-x_4h_4\mathbf{6} + 2\mathbf{9} + h_4\mathbf{7} = \alpha(x_4\mathbf{4} + \mathbf{5}) + \beta(\mathbf{9} + h_4\mathbf{7}) + \gamma\mathbf{6}.$$

Comparing the coefficients in the left side with the coefficients in the right side, we obtain

$$\beta = 2, \quad h_4 = 0, \quad \gamma = 0.$$

Considering another commutator, we have

$$[x_4\mathbf{4} + \mathbf{5},\ \mathbf{6}] = \alpha(x_4\mathbf{4} + \mathbf{5}) + \beta(\mathbf{9} + h_4\mathbf{7}) + \gamma\mathbf{6}.$$

By calculating the left side and comparing the coefficients, we obtain $x_4 = 0$. Therefore, in this case the subalgebra can be transformed to $\{\mathbf{5},\ \mathbf{6},\ \mathbf{9}\}$. The optimal system of the algebra L_6 is given in Appendix D.

4.3.6 Optimal system of subalgebras of the algebra L_9

After constructing the optimal system of subalgebras of the algebra L_6, the next step is a construction of an optimal system of the algebra $L_9 =$

$\{\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6},\ \mathbf{7},\ \mathbf{8},\ \mathbf{9}\}$, by gluing subalgebras from the optimal system of the algebra L_6 with the ideal $I_1 = \{\mathbf{1},\ \mathbf{2},\ \mathbf{3}\}$.

The table of commutators of the algebra L_9 is

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1	0	0	0	$-x_2$	x_1	0	0	0	0
X_2	0	0	0	x_1	x_2	0	0	0	0
X_3	0	0	0	0	$2x_3$	x_1	x_2	0	0
X_4	x_2	$-x_1$	0	0	0	x_7	$-x_6$	0	0
X_5	$-x_1$	$-x_2$	$-2x_3$	0	0	x_6	x_7	$-2x_8$	$2x_9$
X_6	0	0	$-x_1$	$-x_7$	x_6	0	0	0	0
X_7	0	0	$-x_2$	x_6	x_7	0	0	0	0
X_8	0	0	0	0	$2x_8$	0	0	0	0
X_9	0	0	0	0	$-2x_9$	0	0	0	0

The automorphisms are

$$
\begin{array}{lll}
A_1 & : & \overline{x}_1 = x_1 + ax_5, \overline{x}_2 = x_2 - ax_4, \\
A_2 & : & \overline{x}_1 = x_1 + ax_4, \overline{x}_2 = x_2 + ax_5, \\
A_3 & : & \overline{x}_1 = x_1 + ax_6, \overline{x}_2 = x_2 + ax_7, \overline{x}_3 = x_3 + 2ax_5, \\
A_4 & : & \overline{x}_1 = x_1 \cos a - x_2 \sin a, \overline{x}_2 = x_1 \sin a + x_2 \cos a, \\
 & & \overline{x}_6 = x_6 \cos a - x_7 \sin a, \overline{x}_7 = x_6 \sin a + x_7 \cos a, \\
A_5 & : & \overline{x}_1 = x_1 e^{-a}, \overline{x}_2 = x_2 e^{-a}, \overline{x}_3 = x_3 e^{-2a}, \\
 & & \overline{x}_6 = x_6 e^{a}, \overline{x}_7 = x_7 e^{a}, \overline{x}_8 = x_8 e^{-2a}, \overline{x}_9 = x_9 e^{2a} \\
A_6 & : & \overline{x}_1 = x_1 - ax_3, \overline{x}_6 = x_6 - ax_5, \overline{x}_7 = x_7 - ax_4, \\
A_7 & : & \overline{x}_2 = x_2 - ax_3, \overline{x}_6 = x_6 + ax_4, \overline{x}_7 = x_7 - ax_5, \\
A_8 & : & \overline{x}_8 = x_8 + 2ax_5, \\
A_9 & : & \overline{x}_9 = x_9 - 2ax_5
\end{array}
$$

and two involutions are

$$
\begin{array}{ll}
E_1: & \overline{x}_1 = -x_1, \quad \overline{x}_4 = -x_4, \quad \overline{x}_6 = -x_6, \\
E_2: & \overline{x}_2 = -x_2, \quad \overline{x}_4 = -x_4, \quad \overline{x}_7 = -x_7.
\end{array}
$$

As it was seen on the algebra L_6, the process of constructing an optimal system of subalgebras of the algebra L_9 by gluing the algebra L_6 and the ideal I_1 consists of the following steps. On the first step, the vectors

$$
\begin{array}{lll}
Y_i & = & \sum_{j=1}^{3} a_{ij} X_j + \sum_{j=4}^{9} b_{ij} X_j, \quad (i = 1, 2, \ldots, k), \\
Y_{i+k} & = & \sum c_{ij} X_j \qquad\qquad\qquad (i = 1, 2, \ldots, s),
\end{array}
$$

are composed. Here the vectors

$$
\sum_{i=1}^{9} b_{ij} X_j
$$

are basis elements from the one of k–dimensional subalgebras $\overline{L}_k$ of the optimal system of the algebra L_6. In matrix form, this step can be explained as a construction of the matrix

1 2 3	4 5 6 7 8 9
A	**B**
C	**0**

where the matrices **A**, **B** and **C** consist of the coefficients a_{ij}, $b_{i\alpha}$, $c_{\beta j}$, $(i = 1, 2, \ldots, k;\ j = 4, 5, \ldots, 9;\ \alpha = 4, 5, \ldots, 9;\ \beta = 1, 2, \ldots, s)$. On this step, the matrix **A** is arbitrary. The rank of the matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix}$$

is equal to $k+s$ and this is the dimension of the subalgebra from the algebra L_9. The matrix **C** is chosen to be the simplest by taking linear combinations and has to take all possible values of the given rank s. Note also that, the matrix **A** can be simplified with the help of the matrix **C**.

The process of checking the subalgebra conditions is very cumbersome. For example, for 8–dimensional subalgebra, in order to check subalgebra conditions, one needs to construct $C_{8,2} = 28$ commutators and check their linear dependence on the basis generators of the subalgebra. Therefore for this step, we need to use a computer for the calculation. In this research, we use the Reduce-program (see Appendix A).

Here we consider one example for constructing initial data for the Reduce-program. Let us take the subalgebra $\{X_4\,,\, X_9\}$ of the algebra L_6. We will glue the ideal I_2 to this subalgebra. The maximum possible dimension of subalgebra of the algebra L_9 after gluing a subalgebra to I_2 is five. In this case, the matrix **C** is a 3×3 square matrix, the rank of which is equal to 3. Hence, by linear combinations of rows, this matrix can be transformed to the identical matrix $\mathbf{C} = \mathbf{E}$:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{E} & \mathbf{O} \end{pmatrix}$$

By taking linear combinations of rows of the last matrix one can eliminate the elements of the matrix **A**, Therefore, we obtain the initial data

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0

Let us consider four-dimensional subalgebras of the algebra L_9. In this case, the rank of the matrix **C** is equal to two.

Assume that $c_{11}^2 + c_{21}^2 \neq 0$, then without loss of generality we can account that $c_{11} = 1,\ c_{21} = 0$. Because the rank of the matrix **C** is equal to two, then $c_{22}^2 + c_{23}^2 \neq 0$.

First, assume that $c_{22} \neq 0$, then one can take $c_{22} = 1,\ c_{12} = 0$. Hence, in this case the matrix **C** for the initial data is

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & c_{13} \\ 0 & 1 & c_{23} \end{pmatrix}$$

By taking linear combinations the matrix A can be transformed to the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \end{pmatrix}$$

and the initial data for the Reduce-program in this case are

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	a_{13}	1	0	0	0	0	0
0	0	a_{23}	0	0	0	0	0	1
1	0	c_{13}	0	0	0	0	0	0
0	1	c_{23}	0	0	0	0	0	0

Now assume that $c_{22} = 0$, then $c_{23} \neq 0$ and one can take $c_{23} = 1$. Note that in this case by the automorphism A_4 one can transform c_{12} to zero. The matrix **A** by taking linear combinations is transformed to the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \end{pmatrix}$$

and by the automorphism A_1 the element a_{12} can also be transformed to zero. Hence, the initial data for the Reduce-program are

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	1	0	0	0	0	0
0	a_{22}	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0

For the three-dimensional case the table of coefficients is

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
a_{11}	a_{12}	a_{13}	1	0	0	0	0	0
a_{21}	a_{22}	a_{23}	0	0	0	0	0	1
c_{11}	c_{12}	c_{13}	0	0	0	0	0	0

If $c_{11}^2 + c_{12}^2 \neq 0$, then using the automorphism A_4 we can take $c_{11} = 1,\ c_{12} = 0$. In this case by linear combinations and by the automorphism A_1 the matrix **A** is transformed to the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \end{pmatrix}$$

and the initial data for the Reduce-program are

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	a_{13}	1	0	0	0	0	0
0	a_{22}	a_{23}	0	0	0	0	0	1
1	0	c_{13}	0	0	0	0	0	0

If $c_{11}^2 + c_{12}^2 = 0$, then $c_{13} \neq 0$, without loss of generality we can take $c_{13} = 1$. After linear combinations and the automorphism A_1 and A_2 the initial data in this case are

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
a_{11}	0	0	1	0	0	0	0	0
a_{21}	a_{22}	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0

In the same way we determined matrices for all subalgebras of the algebra L_6. The result of this analysis is initial data (see in Appendix B) for the computer program, which checks the subalgebra conditions. This Reduce-program is given in Appendix A. It must be noted that a computer cannot do full analysis of checking the subalgebra conditions, because this requires analysis of a nonlinear system of equations, which are obtained after taking commutators. But many of these equations can be simplified by using computer (taking linear combinations)

After calculations and analysis of these calculations we obtain the optimal system of subalgebras of the algebra L_9. The list of subalgebras of the optimal system consists of 299 representative classes (see the table in Appendix E).

4.4 Three–dimensional subalgebras of the algebra L_{11}

As explained before, the next step in constructing the optimal system of subalgebras of the algebra L_{11} invariant solutions is to study the subalgebra conditions of the subalgebras $\{Y_i, \ldots, Y_k\}$ where

$$Y_i = \overline{Y}_i + \sum_{\alpha=10}^{11} x_{i\alpha} X_\alpha, \quad \overline{Y}_i = \sum_{\alpha=1}^{9} x_{i\alpha} X_\alpha.$$

Here $\left\{\overline{Y}_1, \ldots, \overline{Y}_k\right\}$ with

$$\overline{Y}_i = \sum_{\alpha=1}^{9} x_{i\alpha} \overline{X}_\alpha \in L_9,$$

is an element of the optimal system of the factor algebra L_9 and the rank of the matrix $\overline{Q}$ composed the coefficients $x_{i\alpha}$ $(i = 1, 2, \ldots, k;\ \alpha = 1, 2, \ldots, 9)$ is equals to k.

In this thesis, we study three-dimensional subalgebras of the algebra L_{11}. Three-dimensional subalgebras allow obtaining an invariant solutions, which reduce the initial system of partial differential equations to a system of ordinary differential equations.

The result of calculations of the optimal system of three-dimensional subalgebras of the algebra L_{11} (after running the Reduce-program and analysis of the calculations) consists of 417 classes. According to the discussion on subalgebras explained before, only 196 classes of subalgebras can have invariant solutions. The table of subalgebras is given in Appendix F. In the table, $\alpha, \beta, \varepsilon, \varepsilon_1$ and ε_1 are arbitrary constants with $\varepsilon = \pm 1, \varepsilon_1 \neq 0, \varepsilon_2 \neq 0$. All subalgebras are written in symbolical form: only number of the basis generators are given.

Chapter V

Invariant Solutions of the Three-wave Equations

In this chapter, we study invariant solutions of the three-wave equations, which reduce the initial system of partial differential equations to a system of ordinary differential equations. We are confined to considering three-dimensional subalgebras of the algebra L_{11}. The list of all essentially different (nonsimilar) three-dimensional subalgebras that can have invariant solutions is presented in Appendix F and it consists of 196 classes. Note that the subalgebras in which one of the basis elements is X_9 have to be excluded from our consideration. The reason for excluding such class of subalgebras is the same as in the case of subalgebras with the operators $c_1X_{10}+c_2X_{11}$ being in the basis of subalgebra: the rank of the Jacobi matrix of an universal invariant with respect to the dependent variables is less than number of the dependent functions. Therefore, from 417 classes, only 196 classes can have invariant solutions.

In this section, some of invariant solutions are presented. As noted before, these solutions reduce the initial system of partial differential equations to a system of ordinary differential equations. To solve the ordinary differential equations, we use the forth–order Runge–Kutta method. For the sake of complete consideration, this method is explained in Appendix G.

5.1 Subalgebra 1 : $\{4, 5, \alpha 3 + 8\}$

The basis of this subalgebra consists of the generators

$$X_4 = y\partial_x - x\partial_y,\ X_5 = x\partial_x - y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k},\ \alpha X_3 + X_8 = \alpha\partial_z + \partial_t.$$

In order to find invariant solution, one needs to find a universal invariant of this subalgebra. Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be an invariant of the generator X_4. This means that

$$yf_x - xf_y = 0.$$

The general solution of this equation is

$$f = F(t, x^2 + y^2, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into the equation

$$(\alpha X_3 + X_8)f = 0$$

we obtain the equation

$$\alpha F_z + F_t = 0.$$

The general solution of this equation is

$$f = \varphi(r, \overline{z}, u_1, u_2, u_3, u_4, u_5, u_6),$$

where $r = x^2 + y^2$ and $\overline{z} = z - \alpha t$. Substituting the last representation into the equation

$$(x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k})f = 0,$$

we obtain

$$(2r\varphi_r + 2\overline{z}\varphi_{\overline{z}} - 2u_k\varphi_{u_k}) = 0.$$

Hence, the universal invariant of this subalgebra consists of the invariants

$$\frac{x^2 + y^2}{z - \alpha t}, \ (z - \alpha t)u_k, \ \ (k = 1, 2, \ldots, 6).$$

The representation of the invariant solution of this subalgebra has the following form

$$u_k = (z - \alpha t)^{-1}\phi_k(q), \ \ (k = 1, 2, \ldots, 6)$$

with arbitrary functions $\phi_k(q)$, $(k = 1, 2, \ldots, 6)$. Here $q = (x^2 + y^2)/(z - \alpha t)$. The functions $\phi_k(q)$ have to satisfy the equations, which are obtained after substituting the representation of the invariant solution into the initial system. The system for the functions $\phi_k(q)$ is called a reduced system. Hence, the functions $\phi_k(q)$ must satisfy the equations

$$\begin{aligned}
2\omega q\frac{d^2\phi_1}{dq^2} + 2\omega\frac{d\phi_1}{dq} + k_1 q(\alpha k_1 - \omega)\frac{d\phi_2}{dq} - \sigma_1 k_1\omega(\phi_3\phi_5 + \phi_4\phi_6) + k_1(\alpha k_1 - \omega)\phi_2 = 0,\\
-2\omega q\frac{d^2\phi_2}{dq^2} - 2\omega\frac{d\phi_2}{dq} + k_1 q(\alpha k_1 - \omega)\frac{d\phi_1}{dq} + \sigma_1 k_1\omega(\phi_3\phi_6 - \phi_4\phi_5) + k_1(\alpha k_1 - \omega)\phi_1 = 0,\\
2\omega q\frac{d^2\phi_3}{dq^2} + 2\omega\frac{d\phi_3}{dq} + k_2 q(\alpha k_2 - \omega)\frac{d\phi_4}{dq} - \sigma_2 k_2\omega(\phi_1\phi_5 + \phi_2\phi_6) + k_2(\alpha k_2 - \omega)\phi_4 = 0,\\
-2\omega q\frac{d^2\phi_4}{dq^2} - 2\omega\frac{d\phi_4}{dq} + k_2 q(\alpha k_2 - \omega)\frac{d\phi_3}{dq} + \sigma_2 k_2\omega(\phi_1\phi_6 - \phi_2\phi_5) + k_2(\alpha k_2 - \omega)\phi_3 = 0,\\
2\omega q\frac{d^2\phi_5}{dq^2} + 2\omega\frac{d\phi_5}{dq} + q\frac{d\phi_6}{dq} + \sigma_3 k_3\omega(\phi_2\phi_4 - \phi_1\phi_3)k_3^2 q\frac{d\phi_2}{dq} - k_3\omega + k_3(\alpha k_3\phi_2 - \omega\phi_6) = 0,\\
-2\omega q\frac{d^2\phi_6}{dq^2} - 2\omega\frac{d\phi_6}{dq} + k_3^2 q\frac{d\phi_2}{dq} - k_3\omega q\frac{d\phi_5}{dq} + \sigma_3 k_3\omega(\phi_1\phi_4 - \phi_2\phi_3) + k_3(\alpha k_3\phi_1 - \omega\phi_5) = 0.
\end{aligned} \tag{5.1}$$

The reduced system is a system of the second order ordinary differential equations. For constructing solution of this system, we use the fourth-order Runge-Kutta method. The result of the calculations is illustrated in Figures H.1, H.2 and H.3 in Appendix H. The initial values ($q = 1.0$) for the calculations were the following :

$$\phi_1 = 0.2, \quad \phi_2 = 0.3, \quad \phi_3 = 0.4, \quad \phi_4 = 0.5, \quad \phi_5 = 0.6, \quad \phi_6 = 0.7,$$

$$\frac{d\phi_i}{dq} = 1.0, \quad (i = 1, 2, \dots, 6).$$

5.2 Subalgebra 24 : $\{3, 4, 5\}$

The basis of this subalgebra is

$$X_3 = \partial_z,\ X_4 = y\partial_x - x\partial_y,\ X_5 = x\partial_x - y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k}.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be an invariant of the generator X_3. This means that

$$f_z = 0.$$

The general solution of this equation is

$$f = F(t, x, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into the equation $X_4 f = 0$, we obtain the equation

$$yF_x - xF_y = 0$$

The general solution of this equation is

$$f = \varphi(t, x^2 + y^2, u_1, u_2, u_3, u_4, u_5, u_6),$$

Substituting the last representation into the equation

$$(x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k})f = 0,$$

we obtain

$$(2r\varphi_r + 2t\varphi_t - 2u_k\varphi_{u_k}) = 0.$$

where $r = x^2 + y^2$. Hence, the universal invariant of this subalgebra consists of the invariants

$$\frac{x^2 + y^2}{t},\ (x^2 + y^2)u_k,\ \ (k = 1, 2, \dots, 6).$$

The representation of the invariant solution is

$$u_k = (x^2 + y^2)^{-1}\phi_k(q),\ \ (k = 1, 2, \dots, 6).$$

with arbitrary functions $\phi_k(q)$, $(k = 1, 2, \ldots, 6)$ and $q = (x^2 + y^2)/t$. After substituting this representation into the initial system, we obtain the reduced system of the ordinary differential equations:

$$\begin{aligned}
2\omega q^2\frac{d^2\phi_1}{dq^2} - 2\omega q\frac{d\phi_1}{dq} - k_1^2q^2\frac{d\phi_2}{dq} - \sigma_1k_1\omega(\phi_3\phi_5 + \phi_4\phi_6) + 2\omega)\phi_1 &= 0,\\
-2\omega q^2\frac{d^2\phi_2}{dq^2} + 2\omega q\frac{d\phi_2}{dq} - k_1^2q^2\frac{d\phi_1}{dq} + \sigma_1k_1\omega(\phi_3\phi_6 - \phi_4\phi_5) - 2\omega)\phi_2 &= 0,\\
2\omega q^2\frac{d^2\phi_3}{dq^2} - 2\omega q\frac{d\phi_3}{dq} - k_2^2q^2\frac{d\phi_4}{dq} - \sigma_2k_2\omega(\phi_1\phi_5 + \phi_2\phi_6) + 2\omega)\phi_3 &= 0,\\
-2\omega q^2\frac{d^2\phi_4}{dq^2} + 2\omega q\frac{d\phi_4}{dq} - k_2^2q^2\frac{d\phi_3}{dq} + \sigma_2k_2\omega(\phi_1\phi_6 - \phi_2\phi_5) - 2\omega)\phi_4 &= 0, \quad (5.2)\\
2\omega q^2\frac{d^2\phi_5}{dq^2} - 2\omega q\frac{d\phi_5}{dq} - k_3^2q^2\frac{d\phi_2}{dq} + \sigma_3k_3\omega(\phi_2\phi_4 - \phi_1\phi_3) + 2\omega)\phi_5 &= 0,\\
-2\omega q^2\frac{d^2\phi_6}{dq^2} + 2\omega q\frac{d\phi_6}{dq} - k_3^2q^2\frac{d\phi_1}{dq} + \sigma_3k_3\omega(\phi_1\phi_4 + \phi_2\phi_3) - 2\omega)\phi_6 &= 0.
\end{aligned}$$

The result of the calculations is illustrated in Figures H.4, H.5 and H.6 in Appendix H. The initial values ($q = 1.0$) for the calculations were the following :

$$\phi_1 = 0.2, \quad \phi_2 = 0.3, \quad \phi_3 = 0.4, \quad \phi_4 = 0.5, \quad \phi_5 = 0.6, \quad \phi_6 = 0.7,$$

$$\frac{d\phi_i}{dq} = 1.0, \quad (i = 1, 2, \ldots, 6).$$

5.3 Subalgebra 25 : $\{3, 4 + \varepsilon_1 10 + \alpha 11, 5\}$

For this subalgebra, it is more convenient to use dependent variables : $\theta_1, v_1, \theta_2, v_2, \theta_3, v_3$, where

$$u_1 = v_1\cos\theta_1,\ u_3 = v_2\cos\theta_2,\ u_5 = v_3\cos\theta_3,$$

$$u_2 = v_1\sin\theta_1,\ u_4 = v_2\sin\theta_2,\ u_6 = v_3\sin\theta_3,$$

$$\tan\theta_1 = \frac{u_2}{u_1},\ \tan\theta_2 = \frac{u_4}{u_3},\ \tan\theta_3 = \frac{u_6}{u_5}.$$

The system (1.1) can be written as follows :

$$
\begin{aligned}
v_1\left\{2k_1^2\frac{\partial\theta_1}{\partial t}-\omega\left[\left(\frac{\partial\theta_1}{\partial x}\right)^2+\left(\frac{\partial\theta_1}{\partial y}\right)^2\right]+2k_1\omega\frac{\partial\theta_1}{\partial z}\right\}+\omega\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}\right)& \\
-2\sigma_1 v_2 v_3 k_1\omega\cos(\theta_3-\theta_1-\theta_2) &= 0,\\
v_1\omega\left(\frac{\partial^2\theta_1}{\partial x^2}+\frac{\partial^2\theta_1}{\partial y^2}\right)+2\left(\omega\frac{\partial\theta_1}{\partial x}\frac{\partial v_1}{\partial x}+\omega\frac{\partial\theta_1}{\partial y}\frac{\partial v_1}{\partial y}-k_1^2\frac{\partial v_1}{\partial t}-k_1\omega\frac{\partial v_1}{\partial z}\right)& \\
+2\sigma_1 v_2 v_3 k_1\omega\sin(\theta_1+\theta_2-\theta_3) &= 0,\\
v_2\left\{2k_2^2\frac{\partial\theta_2}{\partial t}-\omega\left[\left(\frac{\partial\theta_2}{\partial x}\right)^2+\left(\frac{\partial\theta_2}{\partial y}\right)^2\right]+2k_2\omega\frac{\partial\theta_2}{\partial z}\right\}+\omega\left(\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}\right)& \\
-2\sigma_2 v_1 v_3 k_2\omega\cos(\theta_3-\theta_1-\theta_2) &= 0,\\
v_2\omega\left(\frac{\partial^2\theta_2}{\partial x^2}+\frac{\partial^2\theta_2}{\partial y^2}\right)+2\left(\omega\frac{\partial\theta_2}{\partial x}\frac{\partial v_2}{\partial x}+\omega\frac{\partial\theta_2}{\partial y}\frac{\partial v_2}{\partial y}-k_2^2\frac{\partial v_2}{\partial t}-k_2\omega\frac{\partial v_2}{\partial z}\right)& \\
+2\sigma_2 v_1 v_3 k_2\omega\sin(\theta_1+\theta_2-\theta_3) &= 0,\\
v_3\left\{2k_3^2\frac{\partial\theta_3}{\partial t}-\omega\left[\left(\frac{\partial\theta_3}{\partial x}\right)^2+\left(\frac{\partial\theta_3}{\partial y}\right)^2\right]+2k_3\omega\frac{\partial\theta_3}{\partial z}\right\}+\omega\left(\frac{\partial^2 v_3}{\partial x^2}+\frac{\partial^2 v_3}{\partial y^2}\right)& \\
-2\sigma_3 v_1 v_2 k_3\omega\cos(\theta_3-\theta_1-\theta_2) &= 0,\\
v_3\omega\left(\frac{\partial^2\theta_3}{\partial x^2}+\frac{\partial^2\theta_3}{\partial y^2}\right)+2\left(\omega\frac{\partial\theta_3}{\partial x}\frac{\partial v_3}{\partial x}+\omega\frac{\partial\theta_3}{\partial y}\frac{\partial v_3}{\partial y}-k_3^2\frac{\partial v_3}{\partial t}-k_3\omega\frac{\partial v_3}{\partial z}\right)& \\
+2\sigma_3 v_1 v_2 k_3\omega\sin(\theta_1+\theta_2-\theta_3) &= 0.
\end{aligned}
\tag{5.3}
$$

The generators of the subalgebra $\{\mathbf{3},\mathbf{5},\mathbf{4}+\varepsilon_{11}\mathbf{10}\}$ in this case are

$$
\partial_z,\ x\partial_x+y\partial_y+2z\partial_z+2t\partial_t-2v_1\partial_{v_1}-2v_2\partial_{v_2}-2v_3\partial_{v_3},
$$

$$
y\partial_x-x\partial_y-\varepsilon_1\left[k_1\partial_{\theta_1}+k_2\partial_{\theta_2}+(k_1+k_2)\partial_{\theta_3}\right].
$$

Let a function

$$
f=f(t,x,y,z,\theta_1,\theta_2,\theta_3,,v_1,v_2,v_3)
$$

be an invariant of the generator X_3. This means that

$$
f_z=0.
$$

The general solution of this equation is

$$
f=F(t,y,\theta_1,\theta_2,\theta_3,,v_1,v_2,v_3).
$$

After substituting it into $X_5f=0$, we obtain the equation

$$
xF_x+yF_y+2tF_t-2v_1F_{v_1}-2v_2F_{v_2}-2v_3F_{v_3}=0.
$$

The general solution of this equation is

$$
f=\varphi(\overline{x},\overline{y},\theta_1,\theta_2,\theta_3,v_1,v_2,v_3),
$$

where $\overline{x} = x^2/t$ and $\overline{y} = y^2/t$. The function f has to satisfy one more equation

$$(y\partial_x - x\partial_y - \varepsilon_1[k_1\partial_{\theta_1} + k_2\partial_{\theta_2} + (k_1+k_2)\partial_{\theta_3}])f = 0.$$

Hence, for the function φ there is the equation

$$2\sqrt{\overline{xy}}(\varphi_{\overline{x}} - \varphi_{\overline{y}}) - \varepsilon_1[k_1\varphi_{\theta_1} + k_2\varphi_{\theta_2} + (k_1+k_2)\varphi_{\theta_3}] = 0.$$

Therefore, the universal invariant of this subalgebra consists of the invariants

$$\theta_1 - \varepsilon_1 k_1 \arcsin\left(\sqrt{\frac{x^2}{x^2+y^2}}\right), \theta_2 - \varepsilon_1 k_2 \arcsin\left(\sqrt{\frac{x^2}{x^2+y^2}}\right),$$

$$\theta_3 - \varepsilon_1 k_3 \arcsin\left(\sqrt{\frac{x^2}{x^2+y^2}}\right), \frac{x^2+y^2}{t}, \quad v_1 t, v_2 t, v_3 t$$

The representation of the invariant solution is

$$\theta_i = \phi_i\,(q) - \varepsilon_1 k_i \arcsin\left(\sqrt{\frac{x^2}{x^2+y^2}}\right), \quad (i = 1,2,3).$$

$$v_j = t^{-1}V_j\left(\frac{x^2+y^2}{t}\right), \quad (j = 1,2,3).$$

where $q = (x^2+y^2)/t$. Substituting the representation of the invariant solution of the subalgebra $\{\mathbf{3}, \mathbf{4}+\varepsilon_1\mathbf{10}+\alpha\mathbf{11}, \mathbf{5}\}$ into system (5.3), we obtain the reduced system of ordinary differential equations:

$$\begin{aligned}
4v_1\omega q\frac{d^2\phi_1}{dq^2} + 8\omega q\frac{d\phi_1}{dq}\frac{dV_1}{dq} + 4v_1\omega\frac{\phi_1}{dq} + 2k_1^2 q\frac{V_1}{dq}& \\
+2\sigma_1 k_1\omega v_2 v_3\sin(\phi_1+\phi_2+\phi_3) + 2k_1^2 v_1 &= 0,\\
4v_2\omega q\frac{d^2\phi_2}{dq^2} + 8\omega q\frac{d\phi_2}{dq}\frac{dV_2}{dq} + 4v_2\omega\frac{\phi_2}{dq} + 2k_2^2 q\frac{V_2}{dq}& \\
+2\sigma_2 k_1\omega v_1 v_3\sin(\phi_1+\phi_2+\phi_3) + 2k_2^2 v_2 &= 0,\\
4v_3\omega q\frac{d^2\phi_3}{dq^2} + 8\omega q\frac{d\phi_3}{dq}\frac{dV_3}{dq} + 4v_3\omega\frac{\phi_3}{dq} + 2k_3^2 q\frac{V_3}{dq}& \\
+2\sigma_3 k_1\omega v_1 v_2\sin(\phi_1+\phi_2+\phi_3) + 2k_3^2 v_3 &= 0, \quad (5.4)\\
4\omega q^2\frac{d^2V_1}{dq^2} - 4v_1\omega q^2\left(\frac{d\phi_1}{dq}\right)^2 - 2v_1k_1^2q^2\frac{\phi_1}{dq} + 4\omega q\frac{V_1}{dq}& \\
+2\sigma_1 k_1\omega q v_2 v_3\cos(\phi_1+\phi_2+\phi_3) - \varepsilon_1^2 k_1^2\omega v_1 &= 0,\\
4\omega q^2\frac{d^2V_2}{dq^2} - 4v_2\omega q^2\left(\frac{d\phi_2}{dq}\right)^2 - 2v_2k_2^2q^2\frac{\phi_2}{dq} + 4\omega q\frac{V_2}{dq}& \\
+2\sigma_2 k_2\omega q v_1 v_3\cos(\phi_1+\phi_2+\phi_3) - \varepsilon_1^2 k_2^2\omega v_2 &= 0,\\
4\omega q^2\frac{d^2V_3}{dq^2} - 4v_3\omega q^2\left(\frac{d\phi_3}{dq}\right)^2 - 2v_3k_2^2q^2\frac{\phi_3}{dq} + 4\omega q\frac{V_3}{dq}& \\
+2\sigma_3 k_3\omega q v_1 v_2\cos(\phi_1+\phi_2+\phi_3) - \varepsilon_1^2 k_3^2\omega v_3 &= 0.
\end{aligned}$$

The result of the calculations is illustrated in figures H.7, H.8 and H.9 in Appendix H. The initial values ($q = 1.0$) for the calculations were the following :

$$\phi_1 = 0.2,\ \phi_2 = 0.3,\ \phi_3 = 0.4,\ V_1 = 0.5,\ V_2 = 0.6,\ V_3 = 0.7,$$

$$\frac{d\phi_i}{dq} = 1.0,\ \frac{dV_i}{dq} = 1.0;\ \ (i = 1, 2, 3)$$

5.4 Subalgebra 31 : $\{1, 5, \alpha 3 + 8\}$

This subalgebra is composed of the generators

$$X_1 = \partial_x,\ \ \alpha X_3 + X_8 = \alpha\partial_z + \partial_t,\ \ X_5 = x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2\sum_{k=1}^{6} u_k\partial_{u_k}.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be an invariant of the generator X_1. This means that

$$f_x = 0.$$

The general solution of this equation is

$$f = F(t, y, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into the equation

$$(\alpha X_3 + X_8)f = 0$$

we obtain

$$\alpha F_z + F_t = 0.$$

The general solution of this equation is

$$f = \varphi(y, z - \alpha t, u_1, u_2, u_3, u_4, u_5, u_6),$$

Substituting the last representation into the equation

$$(x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k})f = 0,$$

we obtain

$$(y\varphi_y + 2\overline{z}\varphi_{\overline{z}} - 2u_k\varphi_{u_k}) = 0,$$

where $\overline{z} = z - \alpha t$. Hence, the universal invariant of this subalgebra consists of the invariants

$$\frac{z - \alpha t}{y^2},\ y^2 u_k,\ \ k = 1, 2, \ldots, 6.$$

The representation of the invariant solution is

$$u_k = y^{-2}\phi_k(q),\ \ (k = 1, 2, \ldots, 6),$$

where $q = (z - \alpha t)/y^2$. After substituting this representation into the initial system, we obtain the reduced system of ordinary differential equations:

$$
\begin{aligned}
2\omega q^2 \frac{d^2\phi_1}{dq^2} - 3\omega q \frac{\phi_1}{dq} + k_1 q^2(\alpha k_1 - \omega)\frac{\phi_2}{dq} - k_1\omega\sigma_1(\phi_3\phi_5 + \phi_4\phi_6) + 3\omega\phi_1 &= 0, \\
-2\omega q^2 \frac{d^2\phi_2}{dq^2} + 3\omega q \frac{\phi_2}{dq} + k_1 q^2(\alpha k_1 - \omega)\frac{\phi_1}{dq} + k_1\omega\sigma_1(\phi_3\phi_6 - \phi_4\phi_5) - 3\omega\phi_2 &= 0, \\
2\omega q^2 \frac{d^2\phi_3}{dq^2} - 3\omega q \frac{\phi_3}{dq} + k_2 q^2(\alpha k_2 - \omega)\frac{\phi_4}{dq} - k_2\omega\sigma_2(\phi_1\phi_5 + \phi_2\phi_6) + 3\omega\phi_3 &= 0, \\
-2\omega q^2 \frac{d^2\phi_4}{dq^2} + 3\omega q \frac{\phi_4}{dq} + k_2 q^2(\alpha k_2 - \omega)\frac{\phi_3}{dq} + k_2\omega\sigma_2(\phi_1\phi_6 - \phi_2\phi_5) - 3\omega\phi_4 &= 0, \\
2\omega q^2 \frac{d^2\phi_5}{dq^2} - 3\omega q \frac{\phi_5}{dq} + \alpha k_3^2 q^2 \frac{\phi_2}{dq} - k_3\omega q^2 \frac{\phi_6}{dq} + k_3\omega\sigma_3(\phi_2\phi_4 - \phi_1\phi_3) + 3\omega\phi_5 &= 0, \\
-2\omega q^2 \frac{d^2\phi_6}{dq^2} + 3\omega q \frac{\phi_6}{dq} + \alpha k_3^2 q^2 \frac{\phi_1}{dq} - k_3\omega q^2 \frac{\phi_5}{dq} + k_3\omega\sigma_3(\phi_1\phi_4 - \phi_2\phi_3) - 3\omega\phi_6 &= 0.
\end{aligned}
\tag{5.5}
$$

The result of the calculations is illustrated in figures H.10, H.11 and H.12 in Appendix H. The initial values ($q = 1.0$) for the calculations were the following :

$$
\phi_1 = 0.2, \quad \phi_2 = 0.3, \quad \phi_3 = 0.4, \quad \phi_4 = 0.5, \quad \phi_5 = 0.6, \quad \phi_6 = 0.7,
$$

$$
\frac{d\phi_i}{dq}, \quad (i = 1, 2, \ldots, 6).
$$

5.5 Subalgebra 34 $(\alpha = 0)$: $\{3, 5, 8\}$

This subalgebra is composed of the generators

$$
X_3 = \partial_z,\ X_5 = x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k},\ X_8 = \partial_t.
$$

Let a function

$$
f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)
$$

be invariant of the generator X_3, it means that

$$
F_z = 0.
$$

The general solution of this equation is

$$
f = F(t, x, y, u_1, u_2, u_3, u_4, u_5, u_6).
$$

After substituting it into the equation

$$
X_8 f = 0
$$

we obtain the equation

$$
F_t = 0.
$$

The general solution of this equation is

$$f = \varphi(x, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

Substituting the last representation into the equation

$$(x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k})f = 0,$$

we obtain

$$x\varphi_x + y\varphi_y - 2u_k\varphi_{u_k} = 0.$$

Hence, the universal invariant of this subalgebra consist of the invariants

$$\frac{y}{x},\ x^2u_k, \quad k = 1, 2, \ldots, 6$$

and the representation of the invariant solution of this subalgebra is

$$u_k = x^{-2}\phi_k(q) \quad k = 1, 2, \ldots, 6,$$

where $q = y/x$. After substituting this representation into the initial system, the reduced system is

$$\begin{array}{rcl}
(q^2+1)\dfrac{d^2\phi_1}{dq^2} + 6q\dfrac{d\phi_1}{dq} - 2k_1\sigma_1(\phi_3\phi_5 + \phi_4\phi_6) + 6\phi_1 &=& 0,\\
-(q^2+1)\dfrac{d^2\phi_2}{dq^2} - 6q\dfrac{d\phi_2}{dq} + 2k_1\sigma_1(\phi_3\phi_6 - \phi_4\phi_5) - 6\phi_2 &=& 0,\\
(q^2+1)\dfrac{d^2\phi_3}{dq^2} + 6q\dfrac{d\phi_3}{dq} - 2k_2\sigma_2(\phi_1\phi_5 + \phi_2\phi_6) + 6\phi_3 &=& 0,\\
-(q^2+1)\dfrac{d^2\phi_4}{dq^2} - 6q\dfrac{d\phi_4}{dq} + 2k_2\sigma_2(\phi_1\phi_6 - \phi_2\phi_5) - 6\phi_4 &=& 0,\\
(q^2+1)\dfrac{d^2\phi_5}{dq^2} + 6q\dfrac{d\phi_5}{dq} + 2k_3\sigma_3(\phi_2\phi_4 - \phi_1\phi_3) + 6\phi_5 &=& 0,\\
-(q^2+1)\dfrac{d^2\phi_6}{dq^2} - 6q\dfrac{d\phi_6}{dq} + 2k_3\sigma_3(\phi_1\phi_4 - \phi_2\phi_3) - 6\phi_6 &=& 0.
\end{array} \tag{5.6}$$

The result of the calculations is illustrated in figures H.13, H.14 and H.15 in Appendix H. The initial values ($q = 0.01$) for the calculations were the following :

$$\phi_1 = 1.0, \quad \phi_2 = 2.0, \quad \phi_3 = 3.0, \quad \phi_4 = 4.0, \quad \phi_5 = 5.0, \quad \phi_6 = 6.0,$$
$$\frac{d\phi_i}{dq} = 1.0, \quad (i = 1, 2, \ldots, 6).$$

5.6 Subalgebra 34 $(\alpha \neq 0)$: $\{3, \alpha 4 + 5, 8\}$

This subalgebra is composed of the generators

$$X_3 = \partial_z,\ X_8 = \partial_t,\ \alpha X_4 + X_5 = \alpha(y\partial_x - x\partial_y) + x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k}.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be an invariant of the generator X_3, it means that

$$f_z = 0.$$

The general solution of this equation is

$$f = F(t, x, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into $X_8 f = 0$, we obtain the equation

$$F_t = 0.$$

The general solution of this equation is

$$f = \varphi(x, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

Substituting the last representation into the equation

$$(\alpha X_4 + X_5)f = 0,$$

we obtain

$$(x - \alpha y)\varphi_x + (\alpha x + y)\varphi_y - z u_k \varphi_{u_k} = 0.$$

Hence, the universal invariant of this subalgebra consist of the invariants

$$\arctan\left(\frac{y}{x}\right) - \frac{\alpha}{2}\ln\left(x^2 + y^2\right), \quad u_k e^{\frac{2\arctan\frac{y}{x}}{\alpha}}, \quad (k = 1, 2, \ldots, 6)$$

and the representation of the invariant solution of this subalgebra is

$$u_k = e^{-\frac{2}{\alpha}\arctan\frac{y}{x}}\phi_k\left(\arctan\frac{y}{x} - \frac{\alpha}{2}\ln(x^2 + y^2)\right), \quad (k = 1, 2, \ldots, 6),$$

or

$$u_k = e^{-\frac{2}{\alpha}q}(x^2 + y^2)^{-1}\phi_k(q),$$

where $q = \arctan\left(\frac{y}{x}\right) - \frac{\alpha}{2}\ln(x^2 + y^2)$.

After substituting this representation into the initial system we obtain the reduced system

$$\begin{aligned}
2q^3(\alpha^2+1)\frac{d^2\phi_1}{dq^2} + 2q^2(\alpha^2-1)\frac{d\phi_1}{dq} - k_1\alpha^2\sigma_1(\phi_3\phi_5 + \phi_4\phi_6) + 2q\phi_1 &= 0,\\
-2q^3(\alpha^2+1)\frac{d^2\phi_2}{dq^2} + 2q^2(1-\alpha^2)\frac{d\phi_2}{dq} - k_1\alpha^2\sigma_1(\phi_3\phi_6 - \phi_4\phi_5) - 2q\phi_2 &= 0,\\
2q^3(\alpha^2+1)\frac{d^2\phi_3}{dq^2} + 2q^2(\alpha^2-1)\frac{d\phi_3}{dq} - k_2\alpha^2\sigma_2(\phi_1\phi_5 + \phi_2\phi_6) + 2q\phi_3 &= 0,\\
-2q^3(\alpha^2+1)\frac{d^2\phi_4}{dq^2} + 2q^2(1-\alpha^2)\frac{d\phi_4}{dq} - k_2\alpha^2\sigma_2(\phi_1\phi_6 - \phi_2\phi_5) - 2q\phi_4 &= 0,\\
2q^3(\alpha^2+1)\frac{d^2\phi_5}{dq^2} + 2q^2(\alpha^2-1)\frac{d\phi_5}{dq} + k_3\alpha^2\sigma_3(\phi_2\phi_4 - \phi_1\phi_3) + 2q\phi_5 &= 0,\\
-2q^3(\alpha^2+1)\frac{d^2\phi_6}{dq^2} + 2q^2(1-\alpha^2)\frac{d\phi_6}{dq} + k_3\alpha^2\sigma_3(\phi_1\phi_4 + \phi_2\phi_3) - 2q\phi_6 &= 0.
\end{aligned} \tag{5.7}$$

The result of the calculations is illustrated in figures H.16, H.17 and H.18 in Appendix H. The initial values ($q = 0.5$) for the calculations were the following :

$$\phi_1 = 1.0, \quad \phi_2 = 2.0, \quad \phi_3 = 3.0, \quad \phi_4 = 4.0, \quad \phi_5 = 5.0, \quad \phi_6 = 6.0,$$

$$\frac{d\phi_i}{dq} = 1.0, \quad (i = 1, 2, \ldots, 6).$$

5.7 Subalgebra 37 : $\{3, 4, 8\}$

Basis operators of this subalgebra are

$$X_3 = \partial_z, \; X_4 = x\partial_y - y\partial_x, \; X_8 = \partial_t.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be invariant of the generator X_3, it means that

$$f_z = 0.$$

The general solution of this equation is

$$f = F(t, x, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into $X_8 f = 0$, we obtain the equation

$$F_t = 0.$$

The general solution of this equation is

$$f = \varphi(x, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

Substituting the last representation into the equation $X_4 f = 0$, we obtain

$$y\varphi_x - x\varphi_y = 0.$$

Hence, the universal invariant of this subalgebra consists of the invariants

$$x^2 + y^2, \; u_k, \quad (k = 1, 2, \ldots, 6).$$

The representation of the invariant solution of this subalgebra is

$$u_k = \phi_k(q), \quad (k = 1, 2, \ldots, 6)$$

where $q = x^2 + y^2$. After substituting this representation into the initial system, the reduced system is

$$\begin{aligned}
2q\frac{d^2\phi_1}{dq^2}+2\frac{d\phi_1}{dq}-k_1\sigma_1(\phi_3\phi_5+\phi_4\phi_6) &= 0,\\
-2q\frac{d^2\phi_2}{dq^2}-2\frac{d\phi_2}{dq}+k_1\sigma_1(\phi_3\phi_6-\phi_4\phi_5) &= 0,\\
2q\frac{d^2\phi_3}{dq^2}+2\frac{d\phi_3}{dq}-k_2\sigma_2(\phi_1\phi_5+\phi_2\phi_6) &= 0,\\
-2q\frac{d^2\phi_4}{dq^2}-2\frac{d\phi_4}{dq}+k_2\sigma_2(\phi_1\phi_6-\phi_2\phi_5) &= 0,\\
2q\frac{d^2\phi_5}{dq^2}+2\frac{d\phi_5}{dq}-k_3\sigma_3(\phi_2\phi_4-\phi_1\phi_3) &= 0,\\
-2q\frac{d^2\phi_6}{dq^2}-2\frac{d\phi_6}{dq}+k_3\sigma_3(\phi_1\phi_4+\phi_2\phi_3) &= 0.
\end{aligned} \tag{5.8}$$

The result of the calculations is illustrated in figures H.19, H.20 and H.21 in Appendix H. The initial values ($q = 1.0$) for the calculations were the following :

$$\phi_1 = 0.2,\quad \phi_2 = 0.4,\quad \phi_3 = 0.6,\quad \phi_4 = 0.8,\quad \phi_5 = 1.0,\quad \phi_6 = 0.6,$$

$$\frac{d\phi_i}{dq} = 1.0,\quad i = 1, 2, \ldots, 6.$$

5.8 Subalgebra 137 : $\{1, 3, 5\}$

The subalgebra is composed of the generators

$$X_1 = \partial_x,\ X_3 = \partial_z,\ X_5 = x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2u_k\partial_{u_k}.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be invariant of the generator X_1, it means that

$$f_x = 0.$$

The general solution of this equation is

$$f = F(t, y, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into $X_3 f = 0$, we obtain the equation

$$F_z = 0.$$

The general solution of this equation is

$$f = \varphi(t, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

Substituting the last representation into the equation $X_5 f = 0$, we obtain

$$y\varphi_y + 2t\varphi_t - 2u_k\varphi_{u_k} = 0.$$

Hence, the universal invariant of this subalgebra consists of the invariants

$$\frac{y^2}{t},\ u_k t, \quad k = 1, 2, \ldots, 6$$

The representation of the invariant solution of this subalgebra is

$$u_k = t^{-1}\phi_k(q), \quad k = 1, 2, \ldots, 6,$$

where $q = (y^2)/t$. After substituting this representation into the initial system, the reduced system is

$$\begin{aligned}
2\omega q\frac{d^2\phi_1}{dq^2} + \omega\frac{d\phi_1}{dq} - k_1^2 q\frac{d\phi_2}{dq} - \sigma_1 k_1\omega(\phi_3\phi_5 + \phi_4\phi_6) - k_1^2\phi_2 &= 0,\\
-2\omega q\frac{d^2\phi_2}{dq^2} - \omega\frac{d\phi_2}{dq} - k_1^2 q\frac{d\phi_1}{dq} + \sigma_1 k_1\omega(\phi_3\phi_6 - \phi_4\phi_5) - k_1^2\phi_1 &= 0,\\
2\omega q\frac{d^2\phi_3}{dq^2} + \omega\frac{d\phi_3}{dq} - k_1^2 q\frac{d\phi_4}{dq} - \sigma_2 k_2\omega(\phi_1\phi_5 + \phi_2\phi_6) - k_2^2\phi_4 &= 0,\\
-2\omega q\frac{d^2\phi_4}{dq^2} - \omega\frac{d\phi_4}{dq} - k_2^2 q\frac{d\phi_3}{dq} + \sigma_2 k_2\omega(\phi_1\phi_6 - \phi_2\phi_5) - k_2^2\phi_3 &= 0, \quad (5.9)\\
2\omega q\frac{d^2\phi_5}{dq^2} + \omega\frac{d\phi_5}{dq} - k_3^2 q\frac{d\phi_2}{dq} + \sigma_3 k_3\omega(\phi_2\phi_4 - \phi_1\phi_3) - k_3^2\phi_2 &= 0,\\
-2\omega q\frac{d^2\phi_6}{dq^2} - \omega\frac{d\phi_6}{dq} - k_3^2 q\frac{d\phi_1}{dq} + \sigma_3 k_3\omega(\phi_1\phi_4 + \phi_2\phi_3) - k_3^2\phi_1 &= 0,
\end{aligned}$$

The result of the calculations is illustrated in figures H.22, H.23 and H.24 in Appendix H. The initial values ($q = 1.0$) for the calculations were the following :

$$\phi_1 = 0.2, \quad \phi_2 = 0.4, \quad \phi_3 = 0.6, \quad \phi_4 = 0.8, \quad \phi_5 = 1.0, \quad \phi_6 = 0.7,$$

$$\frac{d\phi_i}{dq} = 1.0, \quad (i = 1, 2, \ldots, 6).$$

5.9 Subalgebra 144 : $\{1, 2, \alpha 3 + 4\}$

This subalgebra is composed of the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ \alpha X_3 + X_4 = \alpha\partial_z + y\partial_x - x\partial_y.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be an invariant of the generator X_1. This means that

$$f_x = 0.$$

The general solution of this equation is

$$f = F(t, y, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into $X_2 f = 0$, we obtain the equation

$$F_y = 0.$$

The general solution of this equation is

$$f = \varphi(t, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

Substituting the last representation into the equation $\alpha X_3 + X_4$, we obtain

$$\varphi_z = 0.$$

Hence, the universal invariant of this subalgebra consists of invariants

$$t,\ u_k, \quad (k = 1, 2, \ldots, 6).$$

The representation of invariant solution of this subalgebra is

$$u_k = \phi_k(t), \quad (k = 1, 2, \ldots, 6).$$

After substitution this representation into the initial system. The reduced system is

$$\begin{aligned}
k_1 \frac{d\phi_2}{dt} - \sigma_1 \omega(\phi_3\phi_5 - \phi_4\phi_6) &= 0,\\
k_1 \frac{d\phi_1}{dt} + \sigma_1 \omega(\phi_3\phi_6 - \phi_4\phi_5) &= 0,\\
k_2 \frac{d\phi_4}{dt} - \sigma_2 \omega(\phi_1\phi_5 - \phi_2\phi_6) &= 0,\\
k_2 \frac{d\phi_3}{dt} + \sigma_2 \omega(\phi_1\phi_6 - \phi_2\phi_5) &= 0,\\
k_3 \frac{d\phi_6}{dt} + \sigma_3 \omega(\phi_2\phi_4 - \phi_1\phi_3) &= 0,\\
k_3 \frac{d\phi_5}{dt} + \sigma_3 \omega(\phi_1\phi_4 + \phi_2\phi_3) &= 0,
\end{aligned} \tag{5.10}$$

The result of the calculations is illustrated in figures H.25, H.26 and H.27 in Appendix H. The initial values ($q = 0.0$) for the calculations were the following :

$$\phi_1 = 0.1, \quad \phi_2 = 0.2, \quad \phi_3 = 0.3, \quad \phi_4 = 0.4, \quad \phi_5 = 0.5, \quad \phi_6 = 0.6,$$

5.10 Subalgebra 158 : $\{1, 2, \alpha 3 + 8\}$

This subalgebra is composed of the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ \alpha X_3 + X_8 = \alpha\partial_z + \partial_t.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be an invariant of the generator X_1. This means that

$$f_x = 0.$$

The general solution of this equation is

$$f = F(t, y, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into $X_2 f = 0$, we obtain the equation

$$F_y = 0.$$

The general solution of this equation is

$$f = \varphi(t, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

Substituting the last representation into the equation $\alpha X_3 + X_8$, we obtain

$$\alpha\varphi_z + \varphi_t = 0.$$

Hence, the universal invariant of this subalgebra consists of invariants

$$z - \alpha t,\ u_k, \quad k = 1, 2, \ldots, 6$$

The representation of invariant solution of this subalgebra is

$$u_k = \phi_k(q), \quad k = 1, 2, \ldots, 6.$$

where $q = z - \alpha t$. After substituting this representation into the initial system. The reduced system is

$$\begin{aligned}
(\omega - \alpha k_1)\frac{d\phi_2}{dq} - \sigma_1\omega(\phi_3\phi_5 + \phi_4\phi_6) &= 0,\\
(\omega - \alpha k_1)\frac{d\phi_1}{dq} + \sigma_1\omega(\phi_3\phi_6 - \phi_4\phi_5) &= 0,\\
(\omega - \alpha k_2)\frac{d\phi_4}{dq} - \sigma_2\omega(\phi_1\phi_5 + \phi_2\phi_6) &= 0,\\
(\omega - \alpha k_2)\frac{d\phi_3}{dq} + \sigma_2\omega(\phi_1\phi_6 - \phi_2\phi_5) &= 0,\\
-\alpha k_3\frac{d\phi_2}{dq} + \omega\frac{d\phi_6}{dq} + \sigma_3\omega(\phi_2\phi_4 - \phi_1\phi_3) &= 0,\\
-\alpha k_3\frac{d\phi_1}{dq} + \omega\frac{d\phi_5}{dq} + \sigma_3\omega(\phi_1\phi_4 - \phi_2\phi_3) &= 0.
\end{aligned} \tag{5.11}$$

The result of the calculations is illustrated in figures H.28, H.29 and H.30 in Appendix H. The initial values ($q = 0.0$) for the calculations were the following :

$$\phi_1 = 0.1, \quad \phi_2 = 0.2, \quad \phi_3 = 0.3, \quad \phi_4 = 0.4, \quad \phi_5 = 0.5, \quad \phi_6 = 0.6,$$

5.11 Subalgebra 171 : $\{\mathbf{1}, \mathbf{3}, \alpha\mathbf{2} + \mathbf{8}\}$

This subalgebra is composed of the generators

$$X_1 = \partial_x,\ X_3 = \partial_z,\ \alpha X_2 + X_8 = \alpha\partial_y + \partial_t.$$

Let a function

$$f = f(t, x, y, z, u_1, u_2, u_3, u_4, u_5, u_6)$$

be an invariant of the generator X_1. This means that

$$f_x = 0.$$

The general solution of this equation is

$$f = F(t, y, z, u_1, u_2, u_3, u_4, u_5, u_6).$$

After substituting it into $X_3 f = 0$, we obtain the equation

$$F_z = 0.$$

The general solution of this equation is

$$f = \varphi(t, y, u_1, u_2, u_3, u_4, u_5, u_6).$$

Substituting the last representation into the equation $\alpha X_2 + X_8$, we obtain

$$\alpha\varphi_y + \varphi_t = 0.$$

Hence, the universal invariant of this subalgebra consists of invariants

$$y - \alpha t,\ u_k, \quad k = 1, 2, \ldots, 6$$

The representation of invariant solution of this subalgebra is

$$u_k = \phi_k(q), \quad k = 1, 2, \ldots, 6.$$

where $q = y - \alpha t$. After substituting this representation into the initial system. The reduced system is

$$\begin{aligned}
\omega\frac{d^2\phi_1}{dq^2} - 2\alpha k_1^2\frac{d\phi_2}{dq} - 2\sigma_1\omega k_1(\phi_3\phi_5 + \phi_4\phi_6) &= 0,\\
-\omega\frac{d^2\phi_2}{dq^2} - 2\alpha k_1^2\frac{d\phi_1}{dq} + 2\sigma_1\omega k_1(\phi_3\phi_6 - \phi_4\phi_5) &= 0,\\
\omega\frac{d^2\phi_3}{dq^2} - 2\alpha k_2^2\frac{d\phi_4}{dq} - 2\sigma_2\omega k_2(\phi_1\phi_5 + \phi_2\phi_6) &= 0,\\
-\omega\frac{d^2\phi_4}{dq^2} - 2\alpha k_2^2\frac{d\phi_3}{dq} + 2\sigma_2\omega k_2(\phi_1\phi_6 - \phi_2\phi_5) &= 0,\\
\omega\frac{d^2\phi_5}{dq^2} - 2\alpha k_3^2\frac{d\phi_2}{dq} + 2\sigma_3\omega k_3(\phi_2\phi_4 - \phi_1\phi_3) &= 0,\\
-\omega\frac{d^2\phi_6}{dq^2} - 2\alpha k_3^2\frac{d\phi_1}{dq} + 2\sigma_3\omega k_3(\phi_1\phi_4 + \phi_2\phi_3) &= 0.
\end{aligned} \tag{5.12}$$

The result of the calculations is illustrated in figures H.31, H.32 and H.33 in Appendix H. The initial values ($q = 0.0$) for the calculations were the following :

$$\phi_1 = 0.2, \quad \phi_2 = 0.3, \quad \phi_3 = 0.4, \quad \phi_4 = 0.5, \quad \phi_5 = 0.6, \quad \phi_6 = 0.7,$$

$$\frac{d\phi_i}{dq} = 1.0, \quad (i = 1, 2, \ldots, 6).$$

Chapter VI

Conclusion

6.1 Thesis summary

In this thesis, we have considered the application of group analysis to the three–wave equations, which describe the behavior of beam lights propagating through a nonlinear medium.

6.1.1 Problems

The three–wave equation were derived from the Maxwell's equations by using the slow envelope approximation method. In the compact form, these equations can be written as follows

$$\begin{aligned} M_1A_1 &= i\sigma_1 A_3 A_2^* e^{i\Delta kz}, \\ M_2A_2 &= i\sigma_2 A_3 A_1^* e^{i\Delta kz}, \\ M_3A_3 &= i\sigma_3 A_1 A_2 e^{-i\Delta kz} \end{aligned}$$

Here

$$A_1 = u_1 + iu_2,\ A_2 = u_3 + iu_4,\ A_3 = u_5 + iu_6$$

are complex-valued amplitudes; A_1, A_2 are the amplitudes of two fundamental harmonic fields with different polarizations; A_3 is the amplitude of the second-harmonic field; M_j are linear differential operators

$$M_j = \frac{\partial}{\partial z} + \beta_j \frac{\partial}{\partial x} + \frac{i}{2k_j}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{k_j}{\omega}\frac{\partial}{\partial t};\ (j = 1, 2, 3),$$

z is the coordinate along the propagation direction; (x, y) are the transverse coordinates; t is time; k_1, k_2 are the linear wave numbers of the fundamental frequencies; k_3 is the linear wave number of the second–harmonic frequency; $\Delta k = k_3 - (k_1 + k_2)$ is the wave vector-mismatch; the symbol $*$ denotes the complex conjugation; β_j are the walk–off angles of the fundamental and second harmonic; ω is the frequency of the light and σ_j are nonlinear coupling coefficients. We have studied the case of exact phase–matched condition:$\Delta k = 0$. Mostly one–dimensional temporal or spatial solutions of these equations were considered.

Therefore, it is natural to investigate more complex representations of solutions. Group analysis can do it.

The application of group analysis consists of several steps. After finding an equivalence and admitted groups one has to construct an optimal system of subgroups (subalgebras). The admitted algebra of the three–wave equations is eleven-dimensional. The construction of an optimal system of subalgebras can be done relatively easy for small dimensions. Therefore, this problem was divided into several steps. In the first step, we classified the factor algebra $L_9 = L_{11}/\{X_{10}, X_{11}\}$, where $\{X_{10}, X_{11}\}$ is the center of the algebra L_{11}. The algebra L_9 is difficult to be classified. Its classification is divided into several steps. The main problem of the thesis was to construct all subalgebras of the algebra L_{11}, which can be the source of invariant solutions with a system of ordinary differential equations as a reduced system.

6.1.2 Results

1. For the classification of the algebra L_9, the two–steps algorithm developed by Ovsiannikov (1978) was used. The algebra L_9 was decomposed into the ideal $I_1 = \{X_1, X_2, X_3\}$ and the subalgebra $N_1 = \{X_4, X_5, X_6, X_7, X_8, X_9\}$. The subalgebra N_1 was decomposed into the ideal $I_2 = \{X_6, X_7\}$ and the subalgebra $N_2 = \{X_4, X_5, X_8, X_9\}$. Optimal systems of the algebras N_2, N_1 and then L_9 were constructed. The optimal system of N_2 consists of 21 representative classes, the optimal system of N_1 consists of 53 classes and the optimal system of the algebra L_9 consists of 299 classes.

2. The Reduce-program for checking subalgebra conditions was prepared.

3. All essentially different three-dimensional subalgebras of the algebra L_{11} were obtained.

4. Invariant solutions with respect to some of subalgebras of the algebra L_{11} were studied. For the reduced systems, the Runge-Kutta method was used.

6.2 Limitations

The present research of an application of group analysis is used for finding exact solutions of the three-wave equations (1.1) in nonlinear optics. We limit the research to invariant solutions whose reduced systems contain only one independent variables.

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Appendix

Appendix A

Reduce-program Checking for Subalgebra Conditions

```
%off echo$
share ork,orj,ssu,lak;
symbolic procedure fir(ff)$
begin scalar ssu;
ssu:=cadr ff;
return ssu;
end;
symbolic procedure seco(ff)$
begin scalar ssu;
ssu:= reverse ff;
ssu:= car ssu;
return ssu;
end;
algebraic procedure creatmat(k_mat,n)$
begin
if not (modd=m) then begin
write("\hline");
write("\hline");
write("\end{tabular}");
write("\begin{tabular}{|l|l|l|}");
write("\hline");
write("N & Generator  & Tab Norm. \\ ");
write("\hline");
write("\hline");
write("  \multicolumn{3}{|c|}{r=",m,"}   \\ ");
write("\hline");
model :=0; modd:=m;
end;
m:=k_mat;
if k_mat=8 then for k:=1:m do for j:=1:n do ama(k,j):=am8(k,j);
if k_mat=7 then for k:=1:m do for j:=1:n do ama(k,j):=am7(k,j);
if k_mat=6 then for k:=1:m do for j:=1:n do ama(k,j):=am6(k,j);
if k_mat=5 then for k:=1:m do for j:=1:n do ama(k,j):=am5(k,j);
```

```
if k_mat=4 then for k:=1:m do for j:=1:n do ama(k,j):=am4(k,j);
if k_mat=3 then for k:=1:m do for j:=1:n do ama(k,j):=am3(k,j);
if k_mat=2 then for k:=1:m do for j:=1:n do ama(k,j):=am2(k,j);
if k_mat=3 then for k:=1:m do begin
    for j:=3:n do ama(k,j-2):=am3(k,j);
    for j:=1:2 do ama(k,9+j):=am3(k,j);
end;
matrix pri(1,n);
for k:=1:m do for j:=1:n do
    if (che(ama(k,j))=1 or ama(k,j)=E) then aa(k,j):=ama(k,j) else
        if not (ama(k,j)=0) then clear aa(k,j)$
pois(m,n)$
for k:=1:m do for j:=1:n do aa(k,j):=0$
end$
algebraic procedure pois(m,n)$
begin integer mn;
symbolic operator seco$
symbolic operator fir$
for k:=1:m do oper(k):=for lj:=1:n sum aa(k,lj)*x(lj);
mn:=0;
for k:=1:(m-1) do for j:=(k+1):m do
    begin
mn:=mn+1;
op(mn):=for kl:=1:n sum for lj:=1:n sum aa(k,kl)*aa(j,lj)*com(kl,lj);
sk(mn):=sop(k,j);
    end;
mn1:=mn+m;
for k:=1:m do
    begin
op(k+mn):=for kl:=1:n sum for lj:=1:n sum z(kl)*aa(k,lj)*com(kl,lj);
    end;
for k:=1:m do begin
    for l:=1:n do if aa(k,l)=1 then ll:=l;
    for kj:=1:mn1 do op(kj):=op(kj)-df(op(kj),x(ll))*oper(k);
end;
kkn:=0 ;
for kj:=1:m do for kl:=1:n do
    if not (aa(kj,kl)=0 or aa(kj,kl)=1 or aa(kj,kl)=E ) then
begin
    kkn:=kkn+1; y(kkn):=aa(kj,kl); la(kkn):=la(kj,kl);
end;
gh:=0;
for kj:=1:mn do for l:=1:n do bk(kj,l):=df(op(kj),x(l));
for kj:=1:mn do for kl:=1:n do if not(bk(kj,kl)=0) then
    if che(bk(kj,kl))=1 then begin
```

```
    gh:=2;  opkj:=kj;
    return;
    end;
if gh=2 then go to ss1;
for kj:=1:mn do for kl:=10:11 do
    if not (bk(kj,kl)=0) then begin %   end 0
    for k:=1:kkn do begin  %    end 2
ord:=0;
    s:=df(bk(kj,kl),y(k));
    if not (s=0) then begin %   end 3
    ss:=df(s,y(k));   if not (ss=0) then ord:=2  else ord:=1;
if ord = 1 then for sl:=1:kkn do if not (df(s,y(sl)) = 0) then ord:=0;
if ord = 2 then for l:=1:kkn do if not (df(ss,y(l))=0) then  ord:=3;
    end;
    if ord = 1 then if not(che(s)=1) then ord := 0;
    if ((ord=1) or
    ((ord=2) and (2*bk(kj,kl)-df(bk(kj,kl),y(k),2)*y(k)**2=0) ) )
            then begin   %  end 4
lak:=la(k);
    ork:=fir (lak);
    orj:=seco(lak);
if k_out=1 then begin
    if ord = 1 then write("denom  s = ", s );
    if ord = 2 then write("coef. = ", bk(kj,kl)," = 0 " );
end;
if ord=1 then
    aa(ork,orj):=aa(ork,orj)-bk(kj,kl)/s
    else aa(ork,orj):=0;
clear y(k);
    end;
    end;
    end;    model:=model+1;
if k_out=3 then
 write("*****","    model =(",m,",",model,")*****");
for k:=1:mn do  if not (op(k)=0) then if
k_out=1 then
 write ("op(",k,") := commut of ",sk(k),":=",op(k)," =0");
 upor:=1; for j:=1:n do
    for k:=upor:m do
begin
    if df(oper(k),x(j))=1 then
    if k=upor then upor:=upor+1 else
    if not (k=upor) then begin
    s1:=oper(k);   s2:=oper(upor);  clear oper(k),oper(upor);
    oper(upor):=s1; oper(k):=s2;
```

```
    upor:=upor+1;
    end;
end;
if k_out=1 then <<
for k:=1:m do
write ("oper(",k,") := ",oper(k)) ;
write("____________________________________________________________");
>>
else if not (k_out=3) then begin    %   end 0
write(model);
write("  &  ");
for kj:=1:m do begin    %   end 1
kk:=1;
for k:=1:n do begin %   end 2
    s:=df(oper(kj),x(k));
    if not (s=0) then begin     %   end 3
        if kk=2 then write(" + ");
        if not (s=1) then write(s," * ",k) else write(k);
    kk:=2;
    end;    %   3
    end;    %   2
if not (kj=m) then write(" , ");
end;
write("  \\  ");
write("\hline");
end;
for k:=(1+mn):mn1 do  if not (op(k)=0) then
if k_out=4 then
for kj:=1:n do begin
    s:=df(op(k),x(kj)); if not (s=0) then
write ("for normal.:  ", s," = 0 ");
end;
go to ss2;
ss1:
if k_out=4 then
write ("commutator of ", sk(opkj),
    " contradicts to SC   op(",opkj,")= ",op(opkj));
ss2:
end;
end;
end$
```

Appendix B

Initial Data for the Program for the Algebra L_9

```
am3 :=mat((u, u, u,     1, 0, 0, 0, 0, 0 ),
          (0, 0, 0,     0, 1, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u, u,     1, 0, 0, 0, 0, 0 ),
          (0, 0, 0,     0, 1, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 1, 0 ) )
am3 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 1, 0 ),
          (u, u, u,     0, 0, 0, 0, 0, 1 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 1, 0 ),
          (u, u, u,     0, 0, 0, 0, 0, 1 ) )
am3 :=mat((0, 0, 0,     0, 1, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 0, 1 ) )
am3 :=mat((0, 0, 0,     0, 1, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u, u,     0, 0, 1, 0,  0, 0 ),
          (u, u, u,     0, 0, 0, u,  1, 0 ),
          (u, u, u,     0, 0, 0, u,  0, 1 ) )
am3 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 1, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
          (0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 1, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 0, 1 ),
          (0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 1, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 1, u ),
          (0, u, u,     0, 0, 1, 0, 0, 0 ),
```

```
          (u, u, u,     0, 0, 0, 1, 0, 0 ) )
am3 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 1, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 1, 0 ) )
am3 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 1, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 1, e ) )
am3 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 1, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 0, 1 ) )
am3 :=mat((0, u, u,     1, 0, 0, 0, 0, 0 ),
          (0, 0, 0,     0, 1, 0, 0, 0, 0 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, u, 0,     1, 0, 0, 0, 0, 0 ),
          (0, 0, 0,     0, 1, 0, 0, 0, 0 ),
          (0, 0, 1,     0, 0, 0, 0,  0, 0 ) )
am3 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (0, u, u,     0, 0, 0, 0, 0, 1 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (u, u, 0,     0, 0, 0, 0, 0, 1 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (0, u, u,     0, 0, 0, 0, 1, 0 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (u, u, 0,     0, 0, 0, 0, 1, 0 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
          (0, u, u,     0, 0, 0, 0, e, 1 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     1, 0, 0, 0, 0, 0 ),
          (u, u, 0,     0, 0, 0, 0, e, 1 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
          (0, u, u,     0, 0, 0, 0, 0, 1 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     1, 0, 0, 0, 0, 0 ),
          (u, u, 0,     0, 0, 0, 0, 0, 1 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 1, 0 ),
          (0, u, u,     0, 0, 0, 0, u, 1 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     1, 0, 0, 0, 1, 0 ),
          (u, u, 0,     0, 0, 0, 0, u, 1 ),
```

```
            (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 0, 1 ),
            (0, u, u,     0, 0, 0, 0, 1, 0 ),
            (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     1, 0, 0, 0, 0, 1 ),
            (u, u, 0,     0, 0, 0, 0, 1, 0 ),
            (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
            (0, u, u,     0, 0, 0, 0, 1, 0 ),
            (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     1, 0, 0, 0, 0, 0 ),
            (u, u, 0,     0, 0, 0, 0, 1, 0 ),
            (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, u,     0, 0, u, u, 1, 0 ),
            (0, u, u,     0, 0, u, u, 0, 1 ),
            (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, u, 0,     0, 0, u, u, 1, 0 ),
            (u, u, 0,     0, 0, u, u, 0, 1 ),
            (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     0, 1, 0, 0, 0, 0 ),
            (0, u, u,     0, 0, 1, 0, 0, 0 ),
            (1, u, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     0, 1, 0, 0, 0, 0 ),
            (u, 0, u,     0, 0, 1, 0, 0, 0 ),
            (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     0, 1, 0, 0, 0, 0 ),
            (u, u, 0,     0, 0, 1, 0, 0, 0 ),
            (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
            (0, u, u,     0, 0, 0, u, 1, e ),
            (1, u, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, 0, u,     0, 0, 1, 0, 0, 0 ),
            (u, 0, u,     0, 0, 0, u, 1, e ),
            (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, u, 0,     0, 0, 1, 0, 0, 0 ),
            (u, u, 0,     0, 0, 0, u, 1, e ),
            (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
            (0, u, u,     0, 0, 0, u, 1, 0 ),
            (1, u, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, 0, u,     0, 0, 1, 0, 0, 0 ),
            (u, 0, u,     0, 0, 0, u, 1, 0 ),
            (0, 1, u,     0, 0, 0, 0, 0, 0 ))
am3 :=mat((u, u, 0,     0, 0, 1, 0, 0, 0 ),
            (u, u, 0,     0, 0, 0, u, 1, 0 ),
```

```
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, u,      0, 0, 1, 0, 0, 0 ),
             (0, u, u,      0, 0, 0, u, 0, 1 ),
             (1, u, u,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, 0, u,      0, 0, 1, 0, 0, 0 ),
             (u, 0, u,      0, 0, 0, u, 0, 1 ),
             (0, 1, u,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, u, 0,      0, 0, 1, 0, 0, 0 ),
             (u, u, 0,      0, 0, 0, u, 0, 1 ),
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, u,      0, 0, 1, 0, 0, 0 ),
             (0, 0, u,      0, 0, 0, 1, 0, 0 ),
             (1, 0, u,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, u, 0,      0, 0, 1, 0, 0, 0 ),
             (u, 0, 0,      0, 0, 0, 1, 0, 0 ),
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,      0, 0, 1, 0, 0, 0 ),
             (1, 0, u,      0, 0, 0, 0, 0, 0 ),
             (0, 1, u,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, 0,      0, 0, 1, 0, 0, 0 ),
             (1, u, 0,      0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,      0, 0, 1, 0, 0, 0 ),
             (0, 1, 0,      0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,      u, 1, 0, 0, 0, 0 ),
             (1, 0, u,      0, 0, 0, 0, 0, 0 ),
             (0, 1, u,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,      u, 1, 0, 0, 0, 0 ),
             (1, 0, 0,      0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,      1, 0, 0, 0, 1, u ),
             (1, 0, u,      0, 0, 0, 0, 0, 0 ),
             (0, 1, u,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,      1, 0, 0, 0, 1, u ),
             (1, 0, 0,      0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,      1, 0, 0, 0, 0, 1 ),
             (1, 0, u,      0, 0, 0, 0, 0, 0 ),
             (0, 1, u,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,      1, 0, 0, 0, 0, 1 ),
             (1, 0, 0,      0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,      0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,      1, 0, 0, 0, 0, 0 ),
             (1, 0, u,      0, 0, 0, 0, 0, 0 ),
```

```
             (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, 0,     1, 0, 0, 0, 0, 0 ),
             (1, 0, 0,     0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     0, 0, u, 0, 1, e ),
             (1, 0, u,     0, 0, 0, 0, 0, 0 ),
             (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, 0,     0, 0, u, 0, 1, e ),
             (1, u, 0,     0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, 0, 0,     0, 0, u, 0, 1, e ),
             (0, 1, 0,     0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     0, 0, u, 0, 1, 0 ),
             (1, 0, u,     0, 0, 0, 0, 0, 0 ),
             (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, 0,     0, 0, u, 0, 1, 0 ),
             (1, u, 0,     0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, 0, 0,     0, 0, u, 0, 1, 0 ),
             (0, 1, 0,     0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, 0, u,     0, 0, u, 0, 0, 1 ),
             (1, 0, u,     0, 0, 0, 0, 0, 0 ),
             (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((0, u, 0,     0, 0, u, 0, 0, 1 ),
             (1, u, 0,     0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, 0, 0,     0, 0, u, 0, 0, 1 ),
             (0, 1, 0,     0, 0, 0, 0, 0, 0 ),
             (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, u, u,     1, 0, 0, 0, 0, 0 ),
             (0, 0, 0,     0, 1, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
             (u, u, u,     0, 0, 0, 0, 0, 1 ) )
am2 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
             (u, u, u,     0, 0, 0, 0, 1, 0 ) )
am2 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
             (u, u, u,     0, 0, 0, 0, e, 1 ) )
am2 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
             (u, u, u,     0, 0, 0, 0, 0, 1 ) )
am2 :=mat((0, 0, u,     1, 0, 0, 0, 1, 0 ),
             (u, u, u,     0, 0, 0, 0, u, 1 ) )
am2 :=mat((0, 0, u,     1, 0, 0, 0, 0, 1 ),
             (u, u, u,     0, 0, 0, 0, 1, 0 ) )
```

```
am2 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 0, 1, 0 ) )
am2 :=mat((u, u, u,     0, 0, u, u, 1, 0 ),
          (u, u, u,     0, 0, u, 0 ,0, 1 ) )
am2 :=mat((0, 0, 0,     0, 1, 0, 0, 0, 0 ),
          (u, u, u,     0, 0, 1, 0, 0, 0 ) )
am2 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, u, 1, e ) )
am2 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, u, 1, 0 ) )
am2 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, u, 0, 1 ) )
am2 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (u, u, u,     0, 0, 0, 1, 0, 0 ) )
am2 :=mat((0, u, u,     0, 0, 1, 0, 0, 0 ),
          (1, u, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, u,     0, 0, 1, 0, 0, 0 ),
          (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, u, 0,     0, 0, 1, 0, 0, 0 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, 0,     u, 1, 0, 0, 0, 0 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, u,     1, 0, 0, 0, 1, u ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, 0,     1, 0, 0, 0, 1, u ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, u,     1, 0, 0, 0, 0, 1 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, 0,     1, 0, 0, 0, 0, 1 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, u,     1, 0, 0, 0, 0, 0 ),
          (1, 0, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, 0, 0,     1, 0, 0, 0, 0, 0 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, u, u,     0, 0, u, 0, 1, e ),
          (1, u, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, 0, u,     0, 0, u, 0, 1, e ),
          (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, u, 0,     0, 0, u, 0, 1, e ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, u, u,     0, 0, u, 0, 1, 0 ),
          (1, u, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, 0, u,     0, 0, u, 0, 1, 0 ),
```

```
          (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, u, 0,     0, 0, u, 0, 1, 0 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((0, u, u,     0, 0, u, 0, 0, 1 ),
          (1, u, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, 0, u,     0, 0, u, 0, 0, 1 ),
          (0, 1, u,     0, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, u, 0,     0, 0, u, 0, 0, 1 ),
          (0, 0, 1,     0, 0, 0, 0, 0, 0 ) )
```

Appendix C

Initial Data for the Program for the Algebra L_{11}

```
am3 :=mat((u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     0, 0, 0, u, 1, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, 0, 0, 1, 0 ),
          (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, u, 1, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, 0, 0, 1, 0 ),
          (u, u,     0, 0, u, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 1, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 1, 1, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, u, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 0, 0, 0, 1, 0 ),
          (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
```

```
             (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, u, 1, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, u, 1, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 0, 0, e, 1 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 1, 0, 0, 0, 1, 0 ),
             (u, u,     0, 0, 0, 0, 0, 0, 0, u, 1 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 1, 0, 0, 0, 0, 1 ),
             (u, u,     0, 0, 0, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, u, u, 0, 0, u, u, 1, 0 ),
             (u, u,     0, 0, u, 0, 0, u, u, 0, 1 ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, u, u, 1, 0 ),
             (u, u,     0, u, 0, 0, 0, u, 0, 0, 1 ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, u, 0, 0, 0, u, 0, 1, 0 ),
             (u, u,     0, u, 0, 0, 0, u, 0, 0, 1 ) )
am3 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 0, 1, 0 ),
             (u, u,     u, u, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ) )
am3 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ))
am3 :=mat((u, u,     0, 0, 0, u, 1, 0, 0, 0, 0 ),
             (u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ),
```

```
             (u, u,     0, 0, 0, 0, 0, 0, 1, 0, 0 ) )
am3 :=mat((u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ),
             (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 1, 0, 0 ) )
am3 :=mat((u, u,     0, 0, 0, 1, 0, 0, 0, 0, 1 ),
             (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 1, 0, 0 ) )
am3 :=mat((u, u,     0, 0, 0, 1, 0, 0, 0, 1, u ),
             (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 1, 0, 0 ) )
am3 :=mat((u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 1, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 1, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, 0, 1, e ) )
am3 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
             (u, u,     0, u, u, 0, 0, 0, u, 1, e ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 0, u, 1, e ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
             (u, u,     0, u, 0, 0, 0, 0, 0, 1, e ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     0, u, 0, 0, 0, 0, 0, 1, e ) )
am3 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, u, 1, e ) )
am3 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
             (u, u,     0, u, u, 0, 0, 0, u, 1, 0 ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 0, u, 1, 0 ) )
am3 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, u, u, 0, 0, 1, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
             (u, u,     u, 0, 0, 0, 0, 0, u, 1, 0 ) )
am3 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
             (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
```

```
              (u, u,     0, u, u, 0, 0, 0, u, 0, 1 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
              (u, u,     0, 0, u, 0, 0, 0, u, 0, 1 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, u, u, 0, 0, 1, 0, 0, 0 ),
              (u, u,     0, 0, u, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
              (u, u,     0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,         0, 1, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
              (u, u,     u, 0, 0, 0, 0, 0, u, 0, 1 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
              (u, u,     0, 0, u, 0, 0, 0, 1, 0, 0 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, 0, u, 1, 0, 0, 0, 0 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, u, 1, 0, 0, 0, 1, 1 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, u, 1, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, u, 1, 0, 0, 0, 0, 0 ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, u, 0, 0, u, 0, 1, e ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, u, 0, 0, 0, u, 0, 1, e ) )
am3 :=mat((u, u,         1, 0, 0, 0, 0, 0, 0, 0, 0 ),
              (u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
```

```
        (u, u,    0, 0, u, 0, 0, u, 0, 1, 0 ) )
am3 :=mat((u, u,    1, 0, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, 1, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, u, 0, 0, 0, u, 0, 1, 0 ) )
am3 :=mat((u, u,    1, 0, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 1, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, u, 0, 0, u, 0, 0, 1 ) )
am3 :=mat((u, u,    1, 0, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, 1, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, u, 0, 0, 0, u, 0, 0, 1 ) )
am3 :=mat((u, u,    0, 1, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, 1, 0, 0, 0, 0, 0, 0 ),
        (u, u,    u, 0, 0, 0, 0, 0, 0, 1, e ) )
am3 :=mat((u, u,    0, 1, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, 1, 0, 0, 0, 0, 0, 0 ),
        (u, u,    u, 0, 0, 0, 0, 0, 0, 1, 0 ) )
am3 :=mat((u, u,    1, 0, u, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 1, u, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, u, 0, 0, 0, 0, 0, 1 ) )
am3 :=mat((u, u,    1, 0, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 1, u, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, u, 0, 0, u, 0, 0, 1 ) )
am3 :=mat((u, u,    0, 1, 0, 0, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, 1, 0, 0, 0, 0, 0, 0 ),
        (u, u,    u, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am2 :=mat((u, u,    0, 0, 0, 1, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, 0, 0, 1, 0, 0, 0, 0 ) )
am2 :=mat((u, u,    0, 0, 0, u, 1, 0, 0, 0, 0 ),
        (u, u,    0, 0, 0, 0, 0, 0, 0, 0, 1 ) )
am2 :=mat((u, u,    0, 0, 0, u, 1, 0, 0, 0, 0 ),
        (u, u,    0, 0, u, 0, 0, 0, 0, 1, 0 ) )
am2 :=mat((u, u,    0, 0, u, 1, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, u, 0, 0, 0, 0, e, 1 ) )
am2 :=mat((u, u,    0, 0, u, 1, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, u, 0, 0, 0, 0, 0, 1 ) )
am2 :=mat((u, u,    0, 0, u, 1, 0, 0, 0, 1, 0 ),
        (u, u,    0, 0, u, 0, 0, 0, 0, u, 1 ) )
am2 :=mat((u, u,    0, 0, u, 1, 0, 0, 0, 0, 1 ),
        (u, u,    0, 0, u, 0, 0, 0, 0, 1, 0 ) )
am2 :=mat((u, u,    0, 0, u, 1, 0, 0, 0, 0, 0 ),
        (u, u,    0, 0, u, 0, 0, 0, 0, 1, 0 ) )
am2 :=mat((u, u,    u, u, u, 0, 0, u, 0, 1, 0 ),
        (u, u,    0, 0, u, 0, 0, u, 0, 0, 1 ) )
am2 :=mat((u, u,    u, u, 0, 0, 0, u, u, 1, 0 ),
        (u, u,    0, u, 0, 0, 0, u, 0, 0, 1 ) )
```

```
am2 :=mat((u, u,     0, u, u, 0, 0, u, 0, 1, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 0, 1 ) )
am2 :=mat((u, u,     u, 0, 0, 0, 0, 0, 0, 1, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 0, 1 ) )
am2 :=mat((u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ) )
am2 :=mat((u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 1, e ) )
am2 :=mat((u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 1, e ) )
am2 :=mat((u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 1, 0 ) )
am2 :=mat((u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, u, 1, 0 ) )
am2 :=mat((u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 0, 0, 1 ) )
am2 :=mat((u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, u, 0, 1 ) )
am2 :=mat((u, u,     0, 0, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, u, 0, 0, 0, 0, 1, 0, 0 ) )
am2 :=mat((u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ),
          (u, u,     u, 0, 0, 0, 0, 0, 1, 0, 0 ) )
am2 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ) )
am2 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ) )
am2 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, u, 0, 0, 0, 1, 0, 0, 0 ) )
am2 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, 1, 0, 0, 0 ) )
am2 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 0, 1, 0, 0, 0, 0 ) )
am2 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, u, 1, 0, 0, 0, 0 ) )
am2 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 1, 0, 0, 0, 1, u ) )
am2 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 1, 0, 0, 0, 0, 1 ) )
am2 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, 0, 1, 0, 0, 0, 0, 0 ) )
am2 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, u, 0, 1, e ) )
am2 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, u, 0, 0, 0, u, 0, 1, e ) )
am2 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
```

```
          (u, u,     0, 0, u, 0, 0, u, 0, 1, e ) )
am2 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     u, 0, 0, 0, 0, 0, 0, 1, e ) )
am2 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     u, 0, 0, 0, 0, 0, 0, 1, e ) )
am2 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, u, 0, 1, 0 ) )
am2 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, u, 0, 0, 0, u, 0, 1, 0 ) )
am2 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, u, 0, 1, 0 ) )
am2 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     u, 0, 0, 0, 0, 0, 0, 1, 0 ) )
am2 :=mat((u, u,     1, u, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, u, 0, 0, 1 ) )
am2 :=mat((u, u,     1, 0, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, u, 0, 0, 0, u, 0, 0, 1 ) )
am2 :=mat((u, u,     0, 1, 0, 0, 0, 0, 0, 0, 0 ),
          (u, u,     0, 0, u, 0, 0, u, 0, 0, 1 ) )
am2 :=mat((u, u,     0, 0, 1, 0, 0, 0, 0, 0, 0 ),
          (u, u,     u, 0, 0, 0, 0, 0, 0, 0, 1 ) )
```

Appendix D

Optimal System of the Algebra L_6

Table D.1: Subalgebras of the optimal system of the algebra L_6.

6-Dimension

N	Generator
1	4, 5, 6, 7, 8, 9

5-Dimension

N	Generator	N	Generator
2	4, 5, 6, 7, 8	4	4, 6, 7, 8, 9
3	4, 5, 6, 7, 9	5	5 + $x_4$4, 6, 7, 8, 9

4-Dimension

N	Generator	N	Generator
6	4, 5, 6, 7	12	4, 6, 7, 9 + ε8
7	4, 5, 8, 9	13	4 + 9, 6, 7, 8
8	4, 6, 7, 8	14	5 + $x_4$4, 6, 7, 8
9	4, 6, 7, 9	15	5 + $x_4$4, 6, 7, 9
10	5, 6, 8, 9	16	4 + 8, 6, 7, 9 + $x_8$8
11	6, 7, 8, 9		

3-Dimension

N	Generator	N	Generator
17	4, 5, 8	24	6, 7, 9
18	4, 5, 9	25	4 + 9, 6, 7
19	4, 6, 7	26	6, 7, 8 + ε9
20	4, 8, 9	27	5 + $x_4$4, 6, 7
21	5, 6, 8	28	5 + $x_4$4, 8, 9
22	5, 6, 9	29	4 + 8 + $x_9$9, 6, 7
23	6, 7, 8	30	6, 8 + $x_7$7, 9 + $x_7$7

2-Dimension

N	Generator	N	Generator
31	4, 5	38	5 + $x_4$4, 9
32	4, 8	39	4, 8 + ε9
33	5, 9	40	4 + 8, 9 + $x_8$8
34	5, 6	41	6, 8 + $x_7$7
35	6, 7	42	6, 9 + $x_7$7
36	4 + 9, 8	43	6, 8 + 9 + $x_7$7
37	5 + $x_4$4, 8	44	8 + $x_6$6 + $x_7$7, 9 + $y_6$6

1-Dimension

N	Generator	N	Generator
45	0	50	8 + $x_6$6
46	4	51	9 + $x_6$6
47	6	52	4 + 8 +$x_9$9
48	4 + 9	53	8 + ε9 + $x_6$6
49	5 + $x_4$4		

Appendix E

Optimal System of the Algebra L_9

Table E.1: Subalgebras of the optimal system of the algebra L_9.

9-Dimension

N	Generator
1	1, 2, 3, 4, 5, 6, 7, 8, 9

8-Dimension

N	Generator
2	1, 2, 4, 5, 6, 7, α 3 + 8, 9

7-Dimension

N	Generator
3	1, 2, 4, 5, 6, 7, 9
4	1, 2, 4, 5, 6, 7, α 3 + 8
5	1, 2, α 4 + 5, 6, 7, β 3 + 8, 9
6	1, 2, α 3 + 4, 6, 7, β 3 + 8 , γ 3 + 9
7	1, 2, 4, 6, 7, β 3 + 8, 9

6-Dimension

N	Generator
8	4, 5, 6, 7, 8, 9
9	1, 5, 6, 7, 8, 9
10	1, 2, 4, 5, α 3 + 8, 9
11	1, 2, 5, 6, α 3 + 8, 9
12	1, 3, 5, 6, 8, 9
13	1, 2, 4, 5, 6, 7
14	1, 2, α 4 + 5, 6, 7, 9
15	1, 2, β 4 + 5, 6, 7, α 3 + 8
16	1, 2, α 3 + 4, 6, 7, β 3 + ε 8 + 9
17	1, 2, α 3 + 4, 6, 7, β 3 + 9
18	1, 2, 4, 6, 7, 9
19	1, 2 , α 3 + 4 + ε 8, 6, 7, γ 3 + δ 8 + 9
20	1, 2, α 3 + 4 + ε 9, 6, 7, γ 3 + 8
21	1, 2, β 3 + 4, 6, 7, α 3 + 8
22	1, 2, 4, 6, 7, α 3 + 8
23	1, 2, α 3 + 6, β 3 + 7, γ 3 + 8, δ3 + 9
24	1, 2, 6, β 3 + 7, γ 3 + 8, δ3 + 9
25	1, 2, 6, 7, γ 3 + 8, δ3 + 9

5-Dimension

N	Generator	N	Generator
26	4, 5, 6, 7, 9	44	1, 2, 5, 6, 9
27	4, 5, 6, 7, 8	45	1, 3, 5, 6, 9
28	β 4 + 5, 6, 7, 8, 9	46	1, 2, 5, 6, α 3 + 8
29	4, - α 2 + 6, α 1 + 7, 8, 9	47	1, 3, 5, 6, 8
30	4, 6, 7, 8, 9	48	1, 2, δ 3 + 6, μ 3 + α 7 + 8, ν 3 + β 7 + 9
31	3, 4, 5, 8, 9		
32	1 + α 2, 5, 6, β 3 + 8, 9	49	1, 2, 6, μ 3 + 8, 9
33	2, 5, 6, 8, 9	50	1, 3, α 2 + 6, β 2 + 8 , γ 2 + 9
34	1, 5, 6, 7, 9		
35	1, 5, 6, 7, 8	51	1, 3, 6, 8, 9
36	1, α 2 + 6, β 3 + 7, γ 2 + 8, δ 2 + 9	52	1, 2, α 4 + 5, 6, 7
		53	1, 2, α 3 + 4, 6, 7
37	1, 6, 7, 8, 9	54	1, 2, α 3 + 4 + 9, 6, 7
38	1, 2, 4, 5, 9	55	1, 2, α 9 + β 3 + 4 + 8, 6, 7
39	1, 2, 4, 5, α 3 + 8	56	1, 2, α 3 + 6, β 3 + 7, γ 3 + 8
40	1, 2, β 4 + 5, α 3 + 8, 9		
41	1, 3, 5, 8, 9	57	1, 2, α 3 + 6, β 3 + 7, γ 3 + ϵ 9 + 8
42	1, 2, α 3 + 4, β 3 + 8 , γ 3 + 9		
		58	1, 2, α 3 + 6, β 3 + 7, γ 3 + 9
43	1, 2, 4, β 3 + 8, 9		

4-Dimension

N	Generator	N	Generator
59	4, 5, α 3 + 8, 9	72	β 2 + 6, -β 1 + 7, 8, 9
60	5, 6, 8, 9	73	6, 7, 8, 9
61	4, 5, 6, 7	74	3, 4, 5, 9
62	β 4 + 5, 6, 7, 9	75	3, 4, 5, 8
63	β 4 + 5, 6, 7, 8	76	1, 5, α 3 + 8, 9
64	4, - α 2 + 6, α 1 + 7, ε 8 + 9	77	3, α 4 + 5, 8, 9
		78	3, 4, 8, 9
65	4, - α 2 + 6, α 1 + 7, 9	79	1 + α 2, 5, 6, 9
66	4, 6, 7, 9	80	2, 5, 6, 9
67	4 + ε 8, - α 2 + 6, α 1 + 7, β 8 + 9	81	1 + α 2, 5, 6, β 3 + 8
		82	2, 5, 6, 8
68	4 + ε 9, - α 2 + 6, α 1 + 7, 8	83	1 + α 2, β 3 + 6, γ 2 + μ (β 3 - α 7) + 8, δ 2 + κ (β 3 - α 7) + 9
69	4, - α 2 + 6, α 1 + 7, 8		
70	4, 6, 7, 8	84	1 + α 2, 6, 8, 9
71	α 1 + β 2 + 6, γ 1 + 7, δ 1 + 8, μ 1 + ν 2 + 9	85	1, β 2 + 6, μ (α 3 + 7) + 8, δ 2 + κ (α 3 + 7) + 9

4-Dimension

N	Generator
86	1, β 2 + 6, μ 2 + 8, κ (α 3 + 7) + 9
87	1, α 3 + 6, ν 2 + β 3 + 8, δ 2 + γ 3 + 9
88	1, μ 2 + 6, β 3 + 8, δ 2 + γ 3 + 9
89	1, 6, β 3 + 8, 9
90	1, μ 2 + 6, ν 2 + 8, γ 3 + 9
91	1, μ 2 + 6, ν 2 + 8, γ 2 + 9
92	1, 6, 8, 9
93	2, γ 3 + 6, δ 1 + α 7 + 8, μ 1 + β 7 + 9
94	2, 6, 8, 9
95	1, 2, 4, 5
96	1, 2, β 4 + 5, 9
97	1, 3, 5, 9
98	1, 2, β 4 + 5, α 3 + 8
99	1, 3, 5, 8
100	1, 2, α 3 + 4, β 3 + ε 8 + 9
101	1, 2, α 3 + 4, β 3 + 9
102	1, 2, 4, 9
103	1, 2, α 3 + 4 + ε 8, β 3 + γ 8 + 9
104	1, 2, α 3 + 4 + ε 9, β 3 + 8
105	1, 2, α 3 + 4, β 3 + 8
106	1, 2, 4, β 3 + 8

N	Generator
107	1, 2, ν 3 + α 6 + γ 7 + 8, κ 3 + β 6 + 9
108	1, 3, γ 2 + α 6 + 8, δ 2 + β 6 + 9
109	1, 3, 8, 9
110	1, 2, 5, 6
111	1, 3, 5, 6
112	1, 2, γ 3 + 6, δ 3 + α 7 + 8 + ε 9
113	1, 3, α 2 + 6, β 2 + 8 + ε 9
114	1, 2, β 3 + 6, γ 3 + α 7 + 8
115	1, 2, 6, γ 3 + 8
116	1, 3, α 2 + 6, β 2 + 8
117	1, 3, 6, 8
118	1, 2, β 3 + 6, γ 3 + α 7 + 9
119	1, 2, 6, 9
120	1, 3, α 2 + 6, β 2 + 9
121	1, 3, 6, 9
122	1, 2, α 3 + 6, β 3 + 7
123	1, 2, 6, 7
124	1, 5, 6, 7
125	1, α 2 + 6, β 3 + 7, γ 2 + 8
126	1, α 2 + 6, β 2 + 7, γ 2 + ϵ 9 + 8
127	1, α 2 + 6, β 3 + 7, γ 2 + 9

3-Dimension

N	Generator
128	4, 5, 9
129	4, 5, α 3 + 8
130	β 4 + 5, α 3 + 8, 9
131	α 3 + 4, β 3 + 8, γ 3 + 9
132	4, β 3 + 8, 9
133	5, 6, 9
134	5, 6, 8
135	α 3 + 6, β 1 + γ 2 + 8, μ 1 + ν 2 + 9
136	α 2 + 6, β 1 + γ 2 + μ 7 + 8, κ 1 + ν 2 + δ 7 + 9

N	Generator
137	6, 8, 9
138	3, 4, 5
139	1, 5, 9
140	3, α 4 + 5, 9
141	1, 5, α 3 + 8
142	3, β 4 + 5, 8
143	3, 4, ε 8 + 9
144	3, 4, 9
145	3, 4 + ε 8, α 8 + 9
146	3, 4 + ε 9, 8
147	3, 4, 8

3-Dimension

N	Generator	N	Generator
148	1, μ (γ 3 + 7) + δ 6 + 9,	178	1, α 2 + 6, 8
	α 2 + β(γ 3 + 7) + κ 6 + 8	179	1, 6, 8
149	1, α 2 + μ 3 + κ 6 + 8,	180	2, α 3 + 6, β 1 + γ 7 + 8
	γ 3 + δ 6 + 9	181	2, 6, 8
150	1, μ 6 + α(κ 3 + 7) + 8,	182	1 + α 2, γ 3 + 6,
	γ 2 + δ 6 + 9		β 2 + ν (α 7 - γ 3) + 9
151	1, α 3 + κ 6 + 8,	183	1 + α 2, 6, 9
	γ 2 + δ 6 + 9	184	1, γ 2 + 6, ν (μ 3 + 7) + 9
152	1,α 3 + 8, 9	185	1, α 2 + β 3 + 6, μ 3 + 9
153	1, α 2 + κ 6 + 8,	186	1, 6, μ 3 + 9
	γ 2 + δ 6 + 9	187	1, β 3 + 6, 9
154	1, 8, 9	188	1, α 2 + 6, 9
155	3, α 1 + 8, β 1 + γ 2 + 9	189	1, 6, 9
156	3, 8, β 1 + 9	190	2, α 3 + 6, β 1 + γ 7 + 9
157	3, 8, 9	191	2, 6, 9
158	1 + α 2, 5, 6	192	1, α 2 + 6, β 3 + 7
159	2, 5, 6	193	1, 6, 7
160	α 4 + 5, 6, 7	194	1, 2, α 3 + 6
161	4, α 2 + 6, β 1 + 7	195	1, 2, 6
162	4 + 9, α 2 + 6, β 1 + 7	196	1, 3, α 2 + 6
163	α 9 + 8 + 4, β 2 + 6,	197	1, 3, 6
	γ 1 + 7	198	1, 2, α 4 + 5
164	α 2 + 6, β 1 + 7, γ 1 + 8	199	1, 3, 5
165	α 2 + 6, β 1 + 7,	200	1, 2, α 3 + 4 + ε 8 + β 9
	γ 1 + ε 9+ 8	201	1, 2, α 3 + 4 + ε 9
166	1 + α 2, β 3 + 6,	202	1, 2, α 3 + 4
	γ 2 + μ (α 7 - β 3) + 8 + ε 9	203	1, 2, 4
167	1, β 2 + 6,	204	1, 2, α 3 + β 6 + 8 + ε 9
	μ (7 + ν 3) + 8 + ε 9	205	1, 2, α 3 + 8 + ε 9
168	1, α 2 + β 3 + 6,	206	1, 3, α 2 + β 6 + 8 + ε 9
	μ 3 + 8 + ε 9	207	1, 2, α 3 + β 6 + 8
169	1, β 3 + 6, μ 2 + 8 + ε 9	208	1, 2, α 3 + 8
170	1, α 2 + 6, μ 2 + 8 + ε 9	209	1, 3, α 2 + β 6 + 8
171	2, α 3 + 6,	210	1, 3, 8
	β 1 + γ 7 + 8 + ε 9	211	1, 2, α 3 + β 6 + 9
172	1 + α 2, γ 3 + 6,	212	1, 2, α 3 + 9
	β 2 + ν (α 7 - γ 3) + 8	213	1, 2, 9
173	1 + α 2, 6, 8	214	1, 3, α 2 + β 6 + 9
174	1, γ 2 + 6, ν (μ 3 + 7) + 8	215	1, 3, 9
175	1, α 2 + β 3 + 6, μ 3 + 8	216	α 3 + 1, β 3 + 2, γ 3 + 9
176	1, 6, μ 3 + 8	217	1, α 3 + 2, β 6 + γ 3 + 9
177	1, β 3 + 6, 8	218	1, 3, α 2 + 9

2-Dimension

N	Generator	N	Generator
219	4, 5	252	$1 + \alpha\, 2$, $\beta\, 3 + 6$
220	$\alpha\, 4 + 5$, 9	253	$1 + \alpha\, 2$, 6
221	$\alpha\, 4 + 5$, $\beta\, 3 + 8$	254	1, $\alpha\, 3 + 6$
222	$\alpha\, 3 + 4$, $\beta\, 3 + \varepsilon\, 8 + 9$	255	1, $\alpha\, 2 + 6$
223	$\alpha\, 3 + 4$, $\beta\, 3 + 9$	256	1, 6
224	4, 9	257	2, $\alpha\, 3 + 6$
225	$\alpha\, 3 + 4 + \varepsilon\, 8$, $\beta\, 3 + \gamma\, 8 + 9$	258	2, 6
226	$\alpha\, 3 + 4 + \varepsilon\, 9$, $\beta\, 3 + 8$	259	1, 5
227	$\alpha\, 3 + 4$, $\beta\, 3 + 8$	260	3, $\alpha\, 4 + 5$
228	4, $\beta\, 3 + 8$	261	3, $4 + \varepsilon\, 8 + \alpha\, 9$
229	$\alpha\, 1 + \beta\, 2 + \gamma\, (\mu\, 3 + 6) + 8$,	262	3, $4 + \varepsilon\, 9$
	$\nu\, (\mu\, 3 + 6) + 9$	263	3, 4
230	$\alpha\, 1 + \beta\, 2 + \gamma\, 6 + \delta\, 7 + 8$,	264	$1 + \alpha\, 2$,
	$\kappa\, 2 + \nu\, 6 + 9$		$\beta\, (\mu\, 3 + 6) + 8 + \varepsilon\, 9$
231	$\alpha\, 1 + \beta\, 3 + 8$, $\mu\, 3 + 9$	265	1, $\mu\, 3 + \beta\, 6 + 8 + \varepsilon\, 9$
232	$\beta\, 3 + 8$, $\mu\, 3 + 9$	266	1, $\mu\, 2 + \beta\, 6 + 8 + \varepsilon\, 9$
233	$\alpha\, 2 + \beta\, (\mu\, 3 + 6) + 8$,	267	2, $\alpha\, (\mu\, 3 + 6) + 8 + \varepsilon$
	$\gamma\, 1 + \delta\, 2 + 9$	268	2, $\beta\, 3 + 8 + \varepsilon\, 9$
234	$\alpha\, 2 + \beta\, 6 + 8$, $\gamma\, 1 + \delta\, 2 + 9$	269	2, $\alpha\, 1 + 8 + \varepsilon\, 9$
235	$\mu\, 3 + 8$, $\alpha\, 1 + 9$	270	3, $\alpha\, 1 + 8 + \varepsilon\, 9$
236	$\mu\, 3 + 8$, 9	271	3, $8 + \varepsilon\, 9$
237	$\mu\, 1 + 8$, $\alpha\, 1 + \beta\, 2 + 9$	272	$1 + \alpha\, 2$, $\beta(\mu\, 3 + 6) + 8$
238	8, $\alpha\, 1 + 9$	273	1, $\beta\, 3 + \alpha\, 6 + 8$
239	8, 9	274	1, $\beta\, 3 + 8$
240	5, 6	275	1, $\beta\, 2 + \alpha\, 6 + 8$
241	$\alpha\, 3 + 6$, $\beta\, 1 + \gamma\, 2 + 8 + \varepsilon\, 9$	276	1, 8
242	$\alpha\, 2 + 6$, $\beta\, 1 + \gamma\, 2 + 8 + \varepsilon\, 9$	277	2, $\beta\, (\mu\, 3 + 6) + 8$
243	$\alpha\, 3 + 6$, $\beta\, 1 + \gamma\, 2 + 8$	278	3, $\alpha\, 1 + 8$
244	$\alpha\, 2 + 6$, $\beta\, 1 + \gamma\, 2 + \delta\, 7 + 8$	279	3, 8
245	6, 8	280	$1 + \alpha\, 2$, $\beta(\mu\, 3 + 6) + 9$
246	$\alpha\, 3 + 6$, $\beta\, 1 + \gamma\, 2 + 9$	281	1, $\beta\, 3 + \alpha\, 6 + 9$
247	$\alpha\, 2 + 6$, $\beta\, 1 + \gamma\, 2 + \delta\, 7 + 9$	282	1, $\beta\, 2 + \alpha\, 6 + 9$
248	6, 9	283	1, 9
249	6, $\alpha\, 1 + \beta\, 2 + 7$	284	2, $\beta\, (\mu\, 3 + 6) + 9$
250	$\alpha\, 2 + 6$, $- \alpha\, 1 + 7$	285	3, $\alpha\, 1 + 9$
251	6, 7	286	3, 9

1-Dimension

N	Generator	N	Generator
287	0	294	$9 + \alpha 3$
288	1	295	$9 + \alpha 2$
289	2	296	$9 + \alpha 6 + \beta 2 + \gamma 3$
290	3	297	$4 + 8 + \alpha 3 + \beta 9$
291	$4 + \alpha 3$	298	$8 + \varepsilon 9 + \alpha 2 + \beta 3 + \gamma 6$
292	$6 + \alpha 2 + \beta 3$	299	$8 + \varepsilon 9 + \alpha 1 + \beta 2 + \gamma 3$
293	$4 + 9 + \alpha 3$		

Appendix F

Optimal System of three-dimensional Subalgebra of the Algebra L_{11}

Table F.1: Subalgebras of the optimal system of 3-dimension of the algebra L_{11}, which can have invariant solutions.

N	generators	N	generators
1	$4,\ 5,\ \alpha\ 3 + 8$	12	$4 + \alpha 10 + \varepsilon_2 11,\ \omega 3 + \beta_0 8,$
2	$4 + \varepsilon_1 10 + \alpha 11,\ 5,\ \beta 3 + 8$		$\beta 3 + 9$
3	$4,\ 5 + \varepsilon_1 10 + \alpha 11,\ \beta 3 + 8$	13	$4,\ \beta 3 + 9,$
4	$4 + \alpha 10 + \varepsilon_2 11,\ 5, \beta 3 + 8$		$\omega 3 + \beta_0 8 + \alpha 10 + \beta_0 \varepsilon_2 11$
5	$4,\ 5 + \alpha 10 + \varepsilon_2 11,\ \beta\ 3 + 8$	14	$4,\ \omega 3 + 8,\ \alpha 3 + 9 + \varepsilon_2 11$
6	$4 + \varepsilon_1 10,\ 5 + \varepsilon_2 11,\ \beta 3 + 8$	15	$4 + \varepsilon_1 10,\ \alpha\ 3{+}9,$
7	$4 + \varepsilon_2 11,\ 5 + \varepsilon_1 10,\ \beta 3 + 8$		$\omega 3 + \beta_0 8 + \beta_0 \varepsilon_2 11$
8	$4,\ \omega 3 + \beta_0 8,\ \alpha 3 + 9$	16	$4 + \varepsilon_1 10,\ \omega 3 + \beta_0 8,\ \alpha 3 + 9$
9	$4 + \varepsilon_1 10 + \alpha 11,\ \omega 3 + \beta_0 8,$	17	$4 + \varepsilon_2 11,\ \alpha 3 + 9,$
	$\beta 3 + 9$		$\omega 3 + \beta_0 8 + \beta_0 \varepsilon_1 10$
10	$4,\ \omega 3 + 8 + \beta_0 \varepsilon_1 10 + \alpha 11,$	18	$4,\ \omega 3 + \beta_0 8 + \beta_0 \varepsilon_1 10,$
	$\beta 3 + 9$		$\alpha 3 + 9$
11	$4,\ \omega 3 + 8,\ \alpha 3 + \varepsilon_1 10 + 9$		

N	generators
19	$4 + \varepsilon_2 11$, $\omega 3 + \beta_0 8$, $\alpha 3 + 9$
20	4, $\omega 3 + \beta_0 8 + \beta_0 \varepsilon_2 11$, $\alpha 3 + 9$
21	5, 6, 8
22	$5 + \varepsilon_1 10 + \alpha 11$, 6, 8
23	$5 + \alpha 10 + \varepsilon_2 11$, 6, 8
24	3, 4, 5
25	3, $4 + \varepsilon_1 10 + \alpha 11$, 5
26	3, 4, $5 + \varepsilon_1 10 + \alpha 11$
27	3, $4 + \alpha 10 + \varepsilon_2 11$, 5
28	3, 4, $5 + \alpha 10 + \varepsilon_2 11$
29	3, $4 + \varepsilon_1 10$, $5 + \varepsilon_2 11$
30	3, $4 + \varepsilon_2 11$, $5 + \varepsilon_1 10$
31	1, 5, $\alpha 3 + 8$
32	1, $5 + \varepsilon_1 10 + \alpha 11$, $\beta 3 + 8$
33	1, $5 + \alpha 10 + \varepsilon_2 11$, $\beta 3 + 8$
34	3, $5 + \alpha 4$, 8
35	3, $\alpha 4 + 5 + \varepsilon_1 10 + \beta 11$, 8
36	3, $\alpha 4 + 5 + \beta 10 + \varepsilon_2 11$, 8
37	3, 4, 8
38	$3 + \varepsilon_1 10 + \alpha 11$, 4, 8
39	3, $4 + \varepsilon_1 10 + \alpha 11$, 8
40	3, 4, $8 + \varepsilon_1 10 + \alpha 11$
41	$3 + \alpha 10 + \varepsilon_2 11$, 4, 8
42	3, $4 + \alpha 10 + \varepsilon_2 11$, 8
43	3, 4, $8 + \alpha\ 10 + \varepsilon_2 11$
44	$3 + \varepsilon_1 10$, $4{+}\varepsilon_2 11$, 8
45	$3 + \varepsilon_1 10$, 4, $8 + \varepsilon_2 11$
46	$3 + \varepsilon_2 11$, $4 + \varepsilon_1 10$, 8
47	3, $4 + \varepsilon_1 10$, $8 + \varepsilon_2 11$
48	$3 + \varepsilon_2 11$, 4, $8 + \varepsilon_1 10$
49	3, $4 + \varepsilon_2 11$, $8 + \varepsilon_1 10$
50	1, $\beta_0(8 + \alpha\ 7) + \omega 3$, $\omega 9 + \beta(\alpha 7 + 3)$
51	1, $\beta_0 8 + \omega 3$, $9 + \alpha 3$
52	$1 + \varepsilon_1 10 + \alpha 11$, $\beta_0(8 + \beta\ 7) + \omega 3$, $\omega 9 + \gamma(\beta 7 + 3)$

N	Generator
53	$1 + \varepsilon_1 10 + \alpha 11$, $\beta_0 8 + \omega 3$, $9 + \beta 3$
54	1, $\beta_0(8 + \beta 7 + \alpha 11) + \omega 3$, $\omega 9 + \gamma(\beta 7 + 3)$
55	1, $\beta_0(8 + \varepsilon_1 10 + \alpha 11) + \omega 3$, $+ \alpha 11) + \omega 3$, $9 + \beta 3$
56	1, $\beta_0(8 + \beta 7) + \omega 3$, $\omega 9 + \varepsilon_1 10 + \alpha\ (\beta 7 + 3)$
57	1, $\beta_0 8 + \omega 3$, $9 + \alpha 3 + \varepsilon_1 10$
58	$1 + \alpha 10 + \varepsilon_2 11$, $\omega 9 + \gamma(\beta 7 + 3)$, $\beta_0(8 + \beta 7) + \omega 3$
59	$1 + \alpha 10 + \varepsilon_2 11$, $\beta_0 8 + \omega 3$, $9 + \beta 3$
60	1, $\beta_0(8 + \beta 7 + \varepsilon_2 11) + \omega 3$, $\omega 9 + \alpha(\beta 7 + 3)$
61	1, $\beta_0(8 + \alpha 10 + \varepsilon_2 11)$ $+ \omega 3$, $9 + \beta 3$
62	1, $\beta_0(8 + \beta 7) + \omega 3$, $\omega 9 + \varepsilon_2 11 + \alpha(\beta 7 + 3)$
63	1, $\beta_0 8 + \omega 3$, $9 + \alpha 3 + \varepsilon_2 11$
64	$1 + \varepsilon_1 10$, $\omega 9 + \alpha(\beta 7 + 3)$, $\beta_0(8 + \beta 7 + \varepsilon_2 11) + \omega 3$
65	$1 + \varepsilon_1 10$, $\beta_0(8 + \varepsilon_2 11) + \omega 3$, $9 + \alpha 3$
66	$1 + \varepsilon_1 10$, $\beta_0(8 + \beta 7) + \omega 3$, $\omega 9 + \alpha(\beta 7 + 3)$
67	$1 + \varepsilon_1 10$, $\beta_0 8 + \omega 3$, $9 + \alpha 3$
68	$1 + \varepsilon_2 11$, $\beta_0(8 + \beta 7) + \omega 3$, $\omega 9 + \alpha(\beta 7 + 3)$
69	$1 + \varepsilon_2 11$, $\beta_0(8 + \varepsilon_1 10) + \omega 3$, $9 + \alpha 3$
70	1, $\beta_0(8 + \beta 7) + \omega 3$, $\omega 9 + \alpha(\beta 7 + 3)$
71	1, $\beta_0(8 + \varepsilon_1 10) + \omega 3$, $9 + \alpha 3$

N	Generator
72	$1 + \varepsilon_2 11$, $\beta_0 8 + \omega 3$, $9 + \alpha 3$
73	1, $\alpha\beta_0 8 + \omega(\alpha 3 + \alpha_0 7)$, $\alpha 2 + 9$
74	$1 + \varepsilon_1 10 + \alpha 11$, $\beta 2 + 9$ $\beta\beta_0 8 + \omega(\beta\ 3 + \alpha_0 7)$,
75	1, $\alpha\beta_0(8 + \beta 11)\ \alpha_0 7)$, $+\ \omega(\alpha\ 3 + \alpha 2 + 9$
76	1, $\alpha\beta_0 8 + \omega(\alpha 3 + \alpha_0 7)$, $\alpha 2 + 9 + \varepsilon_1 10$
77	$1 + \alpha 10 + \varepsilon_2 11$, $\beta 2 + 9$ $\beta\beta_0 8 + \omega(\beta 3 + \alpha_0 7)$,
78	1, $\alpha\beta_0 8 + \omega(\alpha 3 + \alpha_0 7)$, $\alpha 2 + 9 + \varepsilon_2 11$
79	$1 + \varepsilon_1 10$, $\alpha 2 + 9$, $\alpha\beta_0(8 + \varepsilon_2 11)$ $+\ \omega\ (\alpha 3 + \alpha_0 7)$
80	$1 + \varepsilon_2 11$, $\alpha\ 2{+}9$, $\alpha\beta_0 8 + \omega\ (\alpha\ 3 + \alpha_0 7)$
81	1,$\alpha\ \beta_0 8 + \omega(\alpha 3 + \alpha_0 7)$, $\alpha 2 + 9$
82	2, 5, 6
83	2, 5, $\varepsilon_1 10 + \alpha 11$, 6
84	2, $5 + \alpha 10 + \varepsilon_2 11$, 6
85	$\alpha 4 + 5$, 6, 7
86	$\alpha 4 + 5 + \varepsilon_1 10 + \beta 11$, 6, 7
87	$\alpha 4 + 5 + \beta 10 + \varepsilon_2 11$, 6, 7
88	4, 6, 7
89	$4 + \varepsilon_1 10 + \alpha 11$, 6, 7
90	$4 + \alpha 10 + \varepsilon_2 11$, 6, 7
91	4 + 9, 6, 7
92	$4 + 9 + \varepsilon_1 10 + \alpha 11$, 6, 7
93	$4 + 9 + \alpha 10 + \varepsilon_2 11$, 6, 7
94	$4 + 8 + \alpha 9$, 6, 7
95	$4 + 8 + \alpha 9 + \varepsilon_1 10 + \beta 11$, 6, 7
96	$4 + 8 + \alpha 9 + \beta 10 + \varepsilon_2 11$, 6, 7
97	$\alpha 2 + 6 + \beta 11$, $\gamma 1 + 7$, 8

N	Generator
98	$\alpha 2 + 6$, $\beta 1 + 7 + \gamma 11$, 8
99	$\alpha 2 + 6$, $\beta 1 + 7$, $8 + \varepsilon_1 10 + \gamma 11$
100	$\alpha 2 + 6$, $\beta 1 + 7$, $8 + \gamma 10 + \varepsilon_2 11$
101	$\alpha 2 + 6 + \varepsilon_2 11$, $\beta 1 + 7$, $8 + \varepsilon_1 10$
102	$\alpha 2 + 6$, $\beta 1 + 7 + \varepsilon_2 11$, $8 + \varepsilon_1 10$
103	$\alpha 2 + 6$, $\beta 1 + 7$,8 + e9
104	$\alpha 2 + 6 + \beta 11$, $\gamma 1 + 7$, 8 + e9
105	$\alpha 2 + 6$, $\beta 1 + \gamma 11 + 7$, 8 + e9
106	$\alpha 2 + 6$, $\beta 1 + 7$, 8 + e9 + $\varepsilon_1 10 + \gamma 11$
107	$\alpha 2 + 6 + \varepsilon_2 11$, $\beta 1 + 7$, 8 + e9
108	$\alpha 2 + 6$, $\beta 1 + 7$, 8 + e9 + $\gamma 10 + \varepsilon_2 11$
109	$\alpha 2 + 6$, $\beta 1 + 7$, 8 + e9 + $\varepsilon_2 11$
110	$\alpha 2 + 6 + \varepsilon_2 11$, $\beta 1 + 7$, 8 + e9 + $\varepsilon_1 10$
111	$\alpha 2 + 6$, $\beta 1 + 7 + \varepsilon_2 11$, 8 + e9 + $\varepsilon_1 10$
112	2, 6, 8 + e9
113	$2 + \varepsilon_1 10 + \alpha\ 11$, 6, 8 + e9
114	2,6 + $\alpha 11$, 8 + e9
115	2, 6, 8 + e9 + $\varepsilon_1 10{+}\alpha 11$
116	$2 + \alpha 10 + \varepsilon_2 11$, 6, 8 + e9
117	2, 6, 8 + e9 + $\alpha 10 + \varepsilon_2 11$
118	$2 + \varepsilon_1 10$, $6 + \varepsilon_2 11$, 8 + e9
119	$2 + \varepsilon_1 10$, 6, 8 + e9 + $\varepsilon_2 11$
120	$2 + \varepsilon_2 11$, 6, 8 + e9 + $\varepsilon_1 10$
121	2, $6 + \varepsilon_2 11$, 8 +e9 + $\varepsilon_1 10$
122	2, $\alpha 3 + 6 + \beta 11$, 8
123	$2 + \varepsilon_1 10 + \alpha 11$, $6 + \beta 3$, 8
124	2, $6 + \alpha 3$, $8 + \varepsilon_1 10 + \beta 11$

N	Generator
125	$2 + \alpha 10 + \varepsilon_2 11$, $6 + \beta 3$, 8
126	2, $6 + \alpha 3 + \varepsilon_2 11$, 8
127	2, $6 + \alpha 3$, $8 + \beta 10 + \varepsilon_2 11$
128	$2 + \varepsilon_1 10$, $6 + \alpha 3 + \varepsilon_2 11$, 8
129	$2 + \varepsilon_1 10$, $6 + \alpha 3$, $8 + \varepsilon_2 11$
130	$2 + \varepsilon_2 11$, $6 + \alpha 3$, 8
131	2, $6 + \alpha 3$, $8 + \varepsilon_2 11$
132	$2 + \varepsilon_2 11$, $6 + \alpha 3$, $8 + \varepsilon_1 10$
133	2, $6 + \alpha 3 + \varepsilon_2 11$, $8 + \varepsilon_1 10$
134	1, 2, $\alpha 4 + 5$
135	1, 2, $\alpha 4 + 5 + \varepsilon_1 10 + \beta 11$
136	1, 2, $\alpha 4 + 5 + \beta 10 + \varepsilon_2 11$
137	1, 3, 5
138	1, 3, $5 + \varepsilon_1 10 + \beta 11$
139	1, 3, $5 + \beta 10 + \varepsilon_2 11$
140	1, 2, $\alpha 3 + 4 + 8 + \beta\ 9$
141	1, 2, $\alpha 3 + 4 + 8$ $+ 9 + \varepsilon_2 11$
142	1, 2, $\alpha 3 + 4 + \beta 9$
143	1, 2, $\alpha 3 + 4 + 9 + \varepsilon_2 11$
144	1, 2, $\alpha 3 + 4$
145	1, 2, $\alpha 3 + 4 + \varepsilon_1 10 + \beta 11$
146	1, 2, $\alpha 3 + 4 + \beta 10 + \varepsilon_2 11$
147	1, 2, $\alpha 3 + 8 + \mathrm{e}9$
148	$1 + \varepsilon_1 10 + \beta 11$, 2, $\alpha 3 + 8 + \mathrm{e}9$
149	1, $2 + \varepsilon_1 10 + \beta 11$, $\alpha 3 + 8 + \mathrm{e}9$
150	1, 2, $\alpha 3 + 8 + \mathrm{e}9 + \beta 11$
151	$1 + \beta 10 + \varepsilon_2 11$, 2, $\alpha 3 + 8 + \mathrm{e}9$
152	1, $2 + \beta 10 + \varepsilon_2 11$, $\alpha 3 + 8 + \mathrm{e}9$
153	$1 + \varepsilon_1 10$, $2 + \varepsilon_2 11$, $\alpha 3 + 8 + \mathrm{e}9$
154	$1 + \varepsilon_2 11$, $2 + \varepsilon_1 10$, $\alpha 3 + 8 + \mathrm{e}9$
155	1, $2 + \varepsilon_1 10$, $\alpha 3 + 8 + \mathrm{e}9$
156	$1 + \varepsilon_2 11$, 2, $\alpha 3 + 8 + \mathrm{e}9$
157	1, $2 + \varepsilon_2 11$, $\alpha 3 + 8 + \mathrm{e}9$
158	1, 2, $\alpha 3 + 8$
159	$1 + \varepsilon_1 10 + \beta 11$, 2, $\alpha 3 + 8$
160	1, $2 + \varepsilon_1 10 + \beta 11$, $\alpha 3 + 8$
161	1, 2, $\alpha 3 + 8 + \varepsilon_1 10 + \beta 11$
162	$1 + \beta 10 + \varepsilon_2 11$, 2, $\alpha 3 + 8$
163	1, $2 + \beta 10 + \varepsilon_2 11$, $\alpha 3 + 8$
164	1, 2, $\alpha 3 + 8 + \beta 10 + \varepsilon_2 11$
165	$1 + \varepsilon_1 10$, $2 + \varepsilon_2 11$, $\alpha 3 + 8$
166	$1 + \varepsilon_1 10$, 2, $\alpha 3 + 8 + \varepsilon_2 11$
167	$1+\varepsilon_2 11$, $2 + \varepsilon_1 10$,$\alpha 3 + 8$
168	1, $2 + \varepsilon_1 10$, $\alpha 3 + 8 + \varepsilon_2 11$
169	$1 + \varepsilon_2 11$, 2, $\alpha 3 + 8 + \varepsilon_1 10$
170	1, $2 + \varepsilon_2 11$, $\alpha 3 + 8 + \varepsilon_1 10$
171	1, 3, $\alpha 2 + 8$
172	$1 + \varepsilon_1 10 + \beta 11$, 3, $\alpha 2 + 8$
173	1, $3 + \varepsilon_1 10+\beta 11$, $\alpha 2 + 8$
174	1, 3, $\alpha 2 + 8 + \varepsilon_1 10 + \beta 11$
175	$1 + \beta 10 + \varepsilon_2 11$, 3, $\alpha 2 + 8$
176	1, $3 + \beta 10 + \varepsilon_2 11$, $\alpha 2 + 8$
177	1, 3, $\alpha 2 + 8 + \beta 10 + \varepsilon_2 11$
178	$1 + \varepsilon_1 10$, $3 + \varepsilon_2 11$, $\alpha 2 + 8$
179	$1 + \varepsilon_1 10$, 3, $\alpha 2 + 8 + \varepsilon_2 11$
180	$1 + \varepsilon_2 11$, $3 + \varepsilon_1 10$, $\alpha 2 + 8$
181	1, $3 + \varepsilon_1 10$, $\alpha 2 + 8 + \varepsilon_2 11$
182	$1 + \varepsilon_2 11$, 3, $\alpha 2 + 8 + \varepsilon_1 10$
183	1, $3 + \varepsilon_2 11$, $\alpha 2 + 8 + \varepsilon_1 10$
184	1, 2, $\alpha 3 + 9$
185	$1 + \varepsilon_1 10 + \beta 11$, 2, $\alpha 3 + 9$
186	1, $2 + \varepsilon_1 10 + \beta 11$, $\alpha 3 + 9$
187	1, 2, $\alpha 3 + 9 + \beta 11$
188	$1 + \beta 10 + \varepsilon_2 11$, 2, $\alpha 3 + 9$
189	1, $2 + \beta 10 + \varepsilon_2 11$, $\alpha 3 + 9$
190	1, 2, $\alpha 3 + 9 + \varepsilon_2 11$
191	$1 + \varepsilon_1 10$, $2 + \varepsilon_2 11$, $\alpha 3 + 9$
192	$1 + \varepsilon_1 10$, 2, $\alpha 3 + 9$
193	$1 + \varepsilon_2 11$, $2 + \varepsilon_1 10$, $\alpha 3 + 9$
194	1, $2 + \varepsilon_1 10$, $\alpha 3 + 9$
195	$1 + \varepsilon_2 11$, 2, $\alpha 3 + 9$
196	1, $2 + \varepsilon_2 11$, $\alpha 3 + 9$

Appendix G

Fortran-program of the Fourth-order Runge-Kutta Method

For solving ordinary differential equations we use fourth-order Runge-Kutta method. Here for the sake of complete consideration we explain it.

Let

$$\frac{dy}{dx} = F(x,y),\ y(x_0) = y_0$$

be a Cauchy problem. One of the Runge-Kutta methods with step size h is given by the formula

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where $k_1 = F(x_i, y_i),\ k_2 = F(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1),\ k_3 = F(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2),\ k_4 = F(x_i + h, y_i + hk_3)$. We apply this algorithm for reduced systems. Some of these systems are second order. For the second order system of equations one has to reduce this system to a first order. We use the following way for reducing.

Let

$$\begin{cases} \frac{d^2\phi}{dx^2} &= F(x,\phi,\phi'), \\ \phi(x_0) &= \phi_0, \quad \phi'(x_0) = \phi_1 \end{cases}$$

be a Cauchy problem. This problem can be rewritten as the following

$$\begin{cases} \frac{d\phi}{dx} &= z, & \frac{dz}{dx} &= f(x,\phi,z), \\ \phi(x_0) &= \phi_0, & z(x_0) &= \phi_1. \end{cases}$$

For the obtained system of first order equations one can use Runge-Kutta method described above with

$$y = (\phi, z)^*, \quad \mathbf{F} = (z, f)^*.$$

Example of program in Fortran

```
REAL P(12),PP(12),SP(4,12),QQ,Q,H,T1,T2,T3,T4,S1,S2,S3,S4
INTEGER N
OPEN (12,FILE='ORG1')
OPEN (21,FILE='FH1')
```

```
      OPEN (22,FILE='FH2')
      OPEN (23,FILE='FH3')
      OPEN (24,FILE='FH4')
      OPEN (25,FILE='FH5')
      OPEN (26,FILE='FH6')
      OPEN (27,FILE='FH7')
      DATA Q,P,N,H/1.0,
     =    .2,.3,.4,.5,.6,.7,6*1.0,10,0.1/
      DATA SI1,SI2,SI3,K1,K2,K3,AL,OM/7*1.,2./
      WRITE (*,100) H
100   FORMAT('STEP SIZE H =',E10.4)
      WRITE (*,150)
        WRITE (12,150)
150     FORMAT(4X,'Q',6X,'F1',9X,'F2',9X,'F3',9X,'F4',9X
     =          ,'F5',9X,'F6')
            T1= 0.5*K1*(AL*K1-OM)/OM
            T2= 0.5*K2*(AL*K2-OM)/OM
            T3= 0.5*(AL*K3**2)/OM
          T4= 0.5*K3
          S1=(0.5*K1*SI1)/Q
          S2=(0.5*K2*SI2)/Q
          S3=(0.5*K3*SI3)/Q
          S4=1.0/Q
      DO 600 I=1,N
             DO 300 J=1,12
                PP(J)=P(J)
300          CONTINUE
             QQ=Q
             DO 400 J=1,4
                DO 320 K=1,12
                   SP(J,K)=FUNC(K,PP,T1,T2,T3,T4,S1,S2,S3,S4)
320                CONTINUE
                   IF (J.LT.3) THEN
                      QQ=Q+0.5*H
                      DO 340 K=1,12
                         PP(K)=P(K)+0.5*H*SP(J,K)
340                   CONTINUE
                   ELSE
                      IF (J.EQ.3) THEN
                         QQ=Q+H
                         DO 360 K=1,12
                            PP(K)=P(K)+H*SP(J,K)
360                      CONTINUE
                      ELSE
                         DO 380 K=1,12
```

```
                            P(K)=P(K)+0.1667*H*(SP(1,K)+2.*SP(2,K)
     =                           +2.*SP(3,K)+SP(4,K))
380                         CONTINUE
                         ENDIF
                      ENDIF
400         CONTINUE
            Q=Q+H
                  WRITE(*,450) Q,(P(K),K=1,6)
                  WRITE(12,450) Q,(P(K),K=1,6)
450               FORMAT(F5.2,6E11.4)

            WRITE(21,480) P(1)
            WRITE(22,480) P(2)
            WRITE(23,480) P(3)
            WRITE(24,480) P(4)
            WRITE(25,480) P(5)
            WRITE(26,480) P(6)
            WRITE(27,490) Q
480         FORMAT(E11.4)
490         FORMAT(F5.2)
600      CONTINUE
             CLOSE (12)
           CLOSE (21)
           CLOSE (22)
           CLOSE (23)
           CLOSE (24)
           CLOSE (25)
           CLOSE (26)
           CLOSE (27)

         STOP
         END
c*****************************************************************
             FUNCTION FUNC(K,PP,T1,T2,T3,T4,S1,S2,S3,S4)
               REAL PP(12)
               IF (K.LT.7) THEN
                  FUNC = PP(K+6)
                  RETURN
               ENDIF
                   IF (K.EQ.7) THEN
                      FUNC=-S4*PP(7)-T1*PP(8)
     =                   +S1*(PP(3)*PP(5)+PP(4)*PP(6))-(T1*S4*PP(1))
                      RETURN
                   ENDIF
                   IF (K.EQ.8) THEN
```

```
              FUNC=-S4*PP(8)+T1*PP(7)
=                +S1*(PP(3)*PP(6)-PP(4)*PP(5))+(T1*S4*PP(2))
      RETURN
           ENDIF
           IF (K.EQ.9) THEN
              FUNC=-S4*PP(9)-T2*PP(10)
=                +S2*(PP(1)*PP(5)+PP(2)*PP(6))-(T2*S4*PP(3))
              RETURN
           ENDIF
           IF (K.EQ.10) THEN
              FUNC=-S4*PP(10)+T2*PP(9)
=                +S2*(PP(1)*PP(6)-PP(2)*PP(5))+(T2*S4*PP(4))
              RETURN
           ENDIF
           IF (K.EQ.11) THEN
              FUNC=-S4*PP(11)-T3*PP(8)+T4*PP(12)
=       +S3*(PP(1)*PP(3)-PP(2)*PP(4))-T3*S4*PP(2)+T4*S4*PP(6)
              RETURN
           ENDIF
              FUNC=-S4*PP(12)+T3*PP(7)-T4*PP(11)
=       +S3*(PP(1)*PP(4)+PP(2)*PP(3))+T3*S4*PP(6)-T4*S4*PP(5)
              RETURN
 END
```

Appendix H

Figures

The Figure H.1, H.2 and H.3 show the result of calculation of system (5.1) for the Runge-Kutta method ($h = 0.1$)

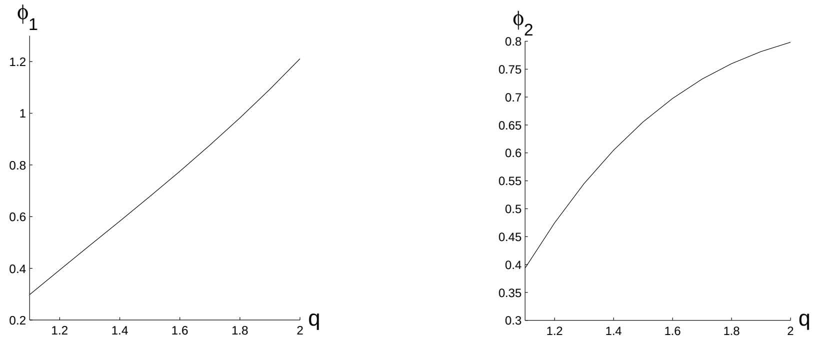

Figure H.1: Graphs of $\phi_1,\ \phi_2$

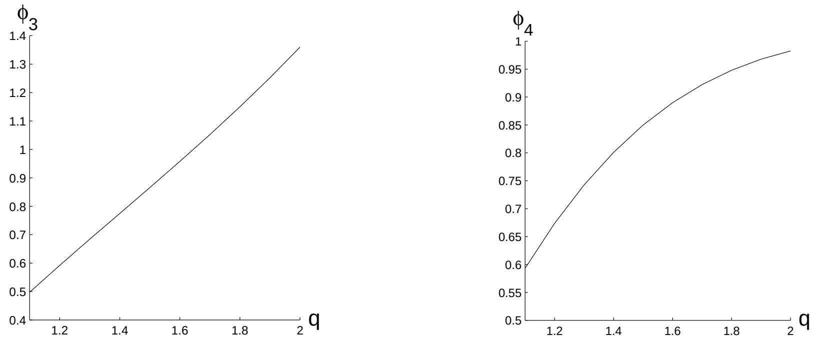

Figure H.2: Graphs of $\phi_3,\ \phi_4$

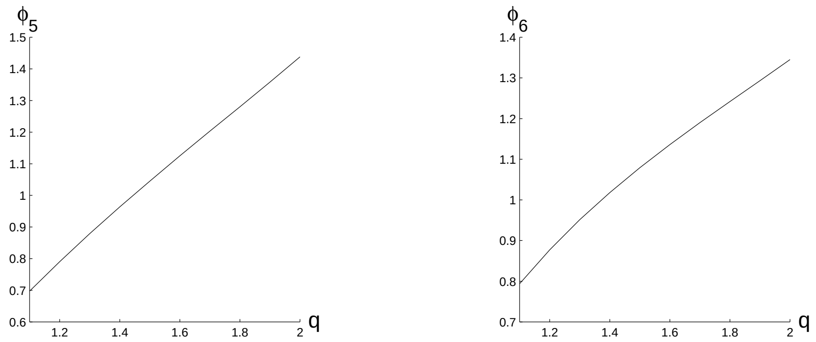

Figure H.3: Graphs of $\phi_5,\ \phi_6$

The Figure H.4, H.5 and H.6 show the result of calculation of system (5.2) for the Runge-Kutta method ($h = 0.1$)

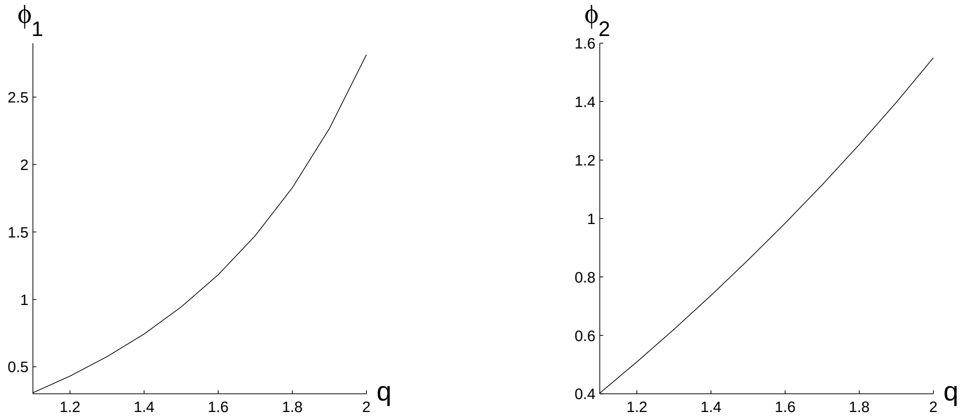

Figure H.4: Graphs of $\phi_1,\ \phi_2$

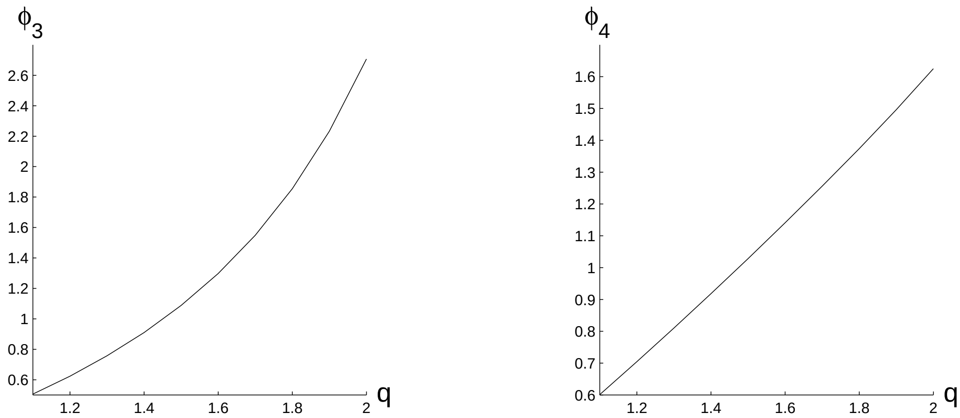

Figure H.5: Graphs of $\phi_3,\ \phi_4$

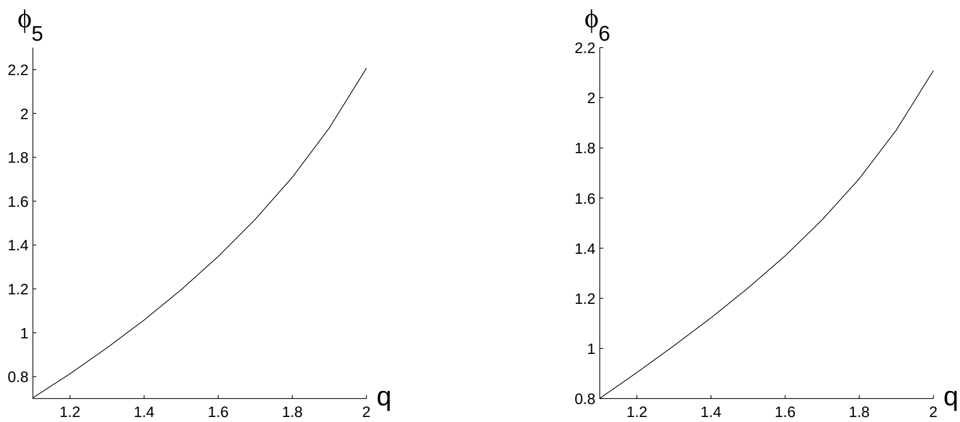

Figure H.6: Graphs of $\phi_5,\ \phi_6$

The Figure H.7, H.8 and H.9 show the result of calculation of system (5.4) for the Runge-Kutta method ($h = 0.05$)

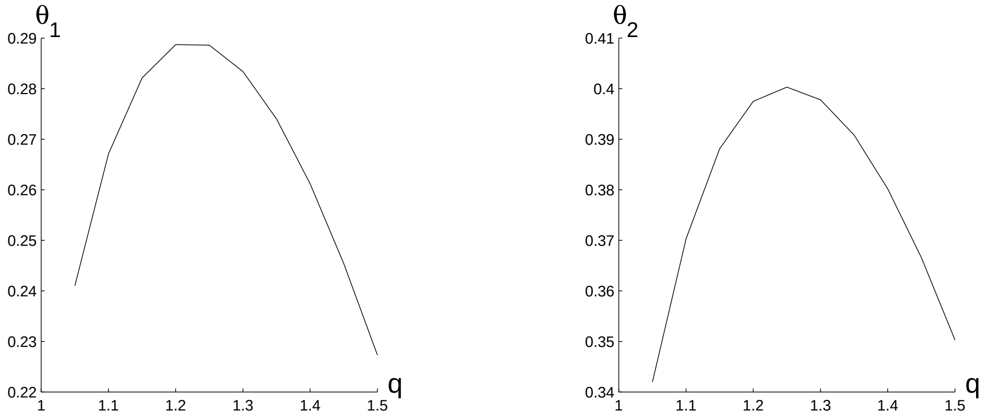

Figure H.7: Graphs of $\phi_1,\ \phi_2$

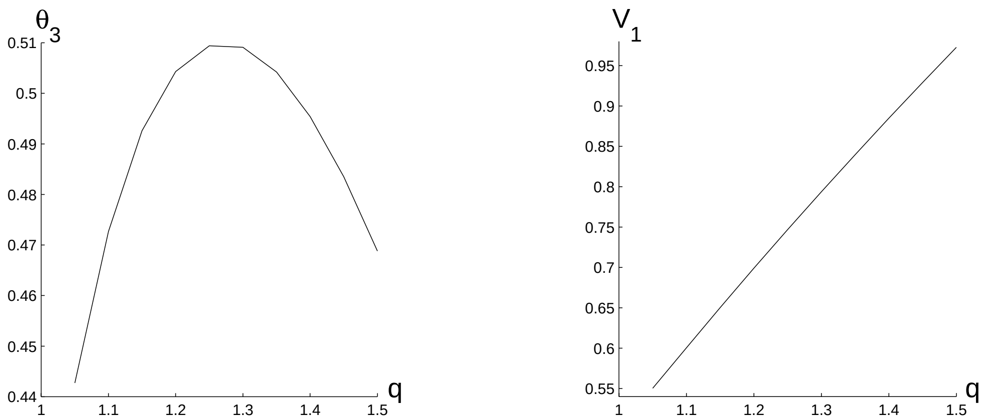

Figure H.8: Graphs of $\phi_3,\ V_1$

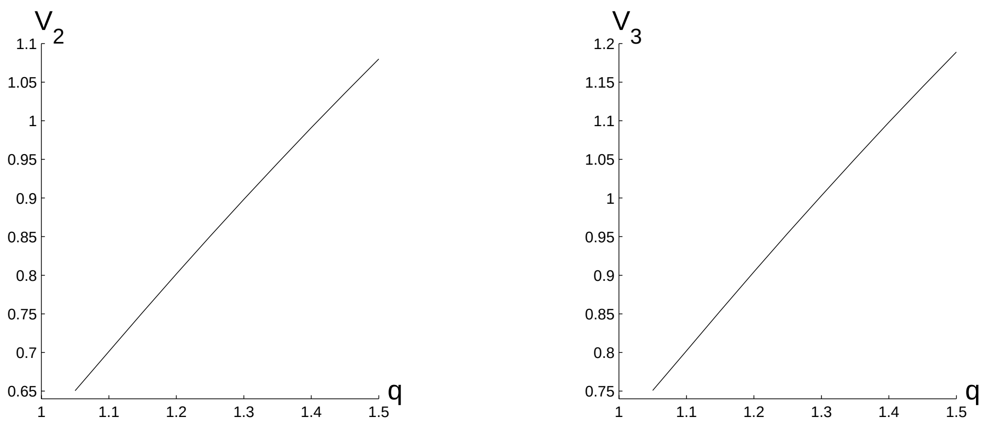

Figure H.9: Graphs of $V_2,\ V_3$

The Figure H.10, H.11 and H.12 show the result of calculation of system (5.5) for the Runge-Kutta method ($h = 0.1$)

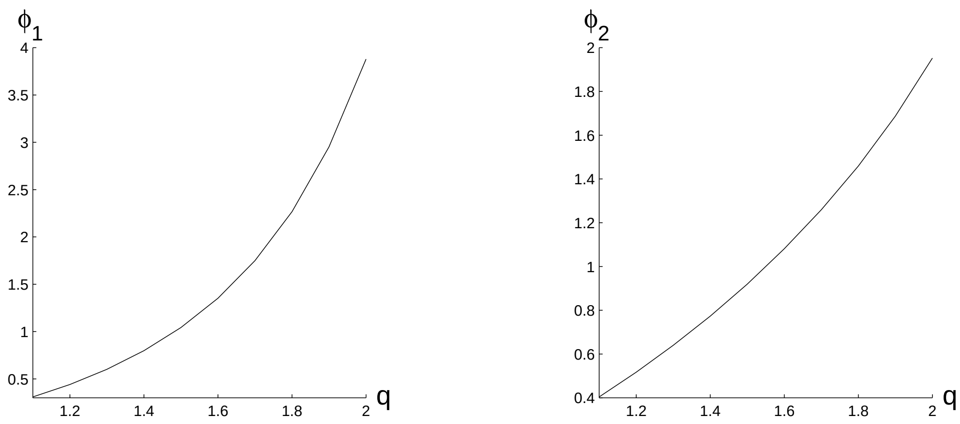

Figure H.10: Graphs of $\phi_1,\ \phi_2$

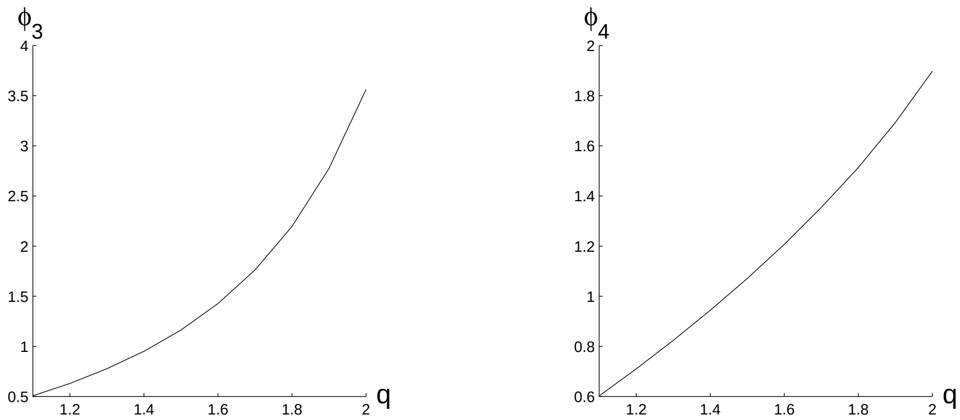

Figure H.11: Graphs of $\phi_3,\ \phi_4$

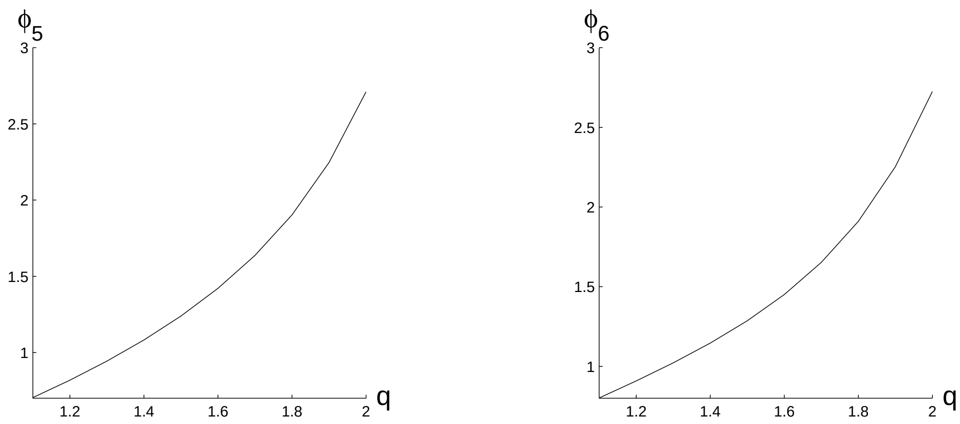

Figure H.12: Graphs of $\phi_5,\ \phi_6$

The Figure H.13, H.14 and H.15 show the result of calculation of system (5.6) for the Runge-Kutta method ($h = 0.01$)

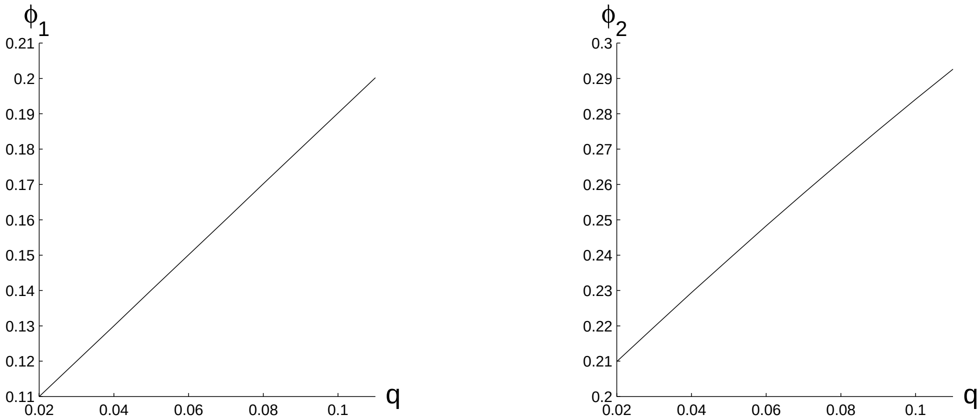

Figure H.13: Graphs of $\phi_1,\ \phi_2$

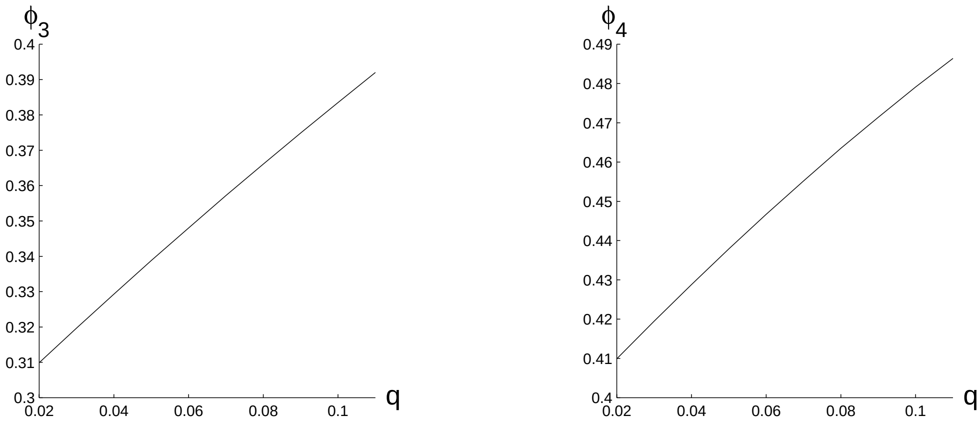

Figure H.14: Graphs of $\phi_3,\ \phi_4$

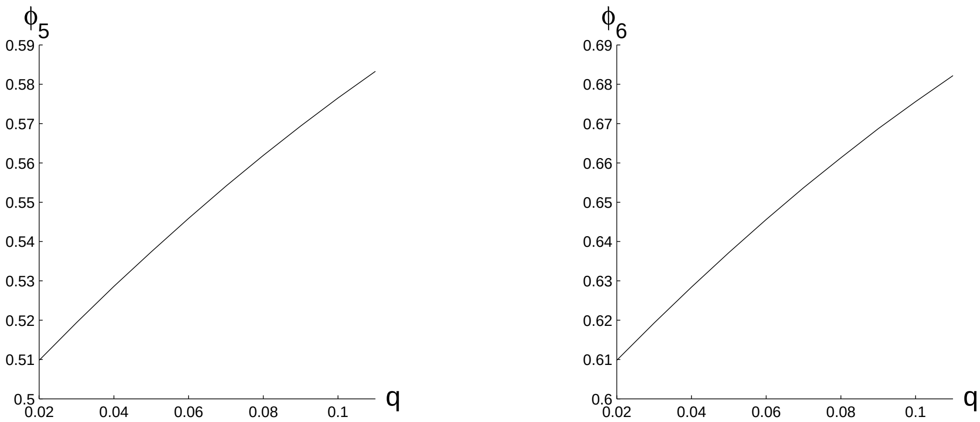

Figure H.15: Graphs of $\phi_5,\ \phi_6$

The Figure H.16, H.17 and H.18 show the result of calculation of system (5.7) for the Runge-Kutta method ($h = 0.05$)

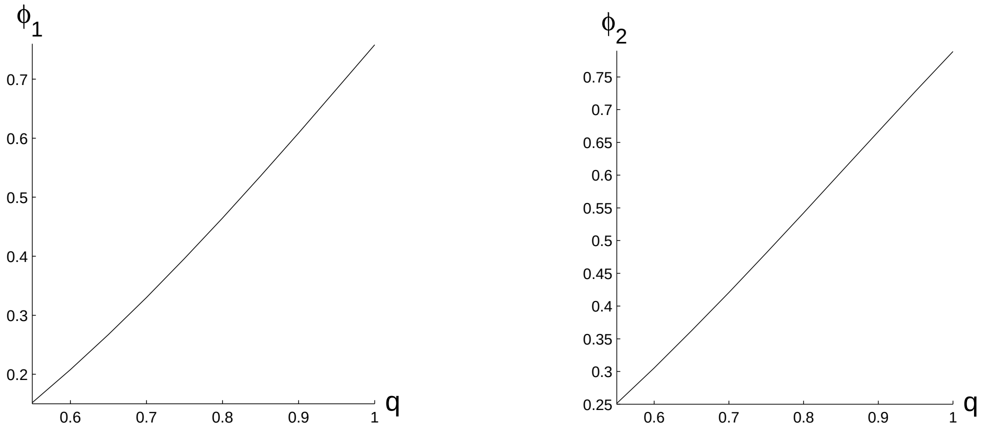

Figure H.16: Graphs of $\phi_1,\ \phi_2$

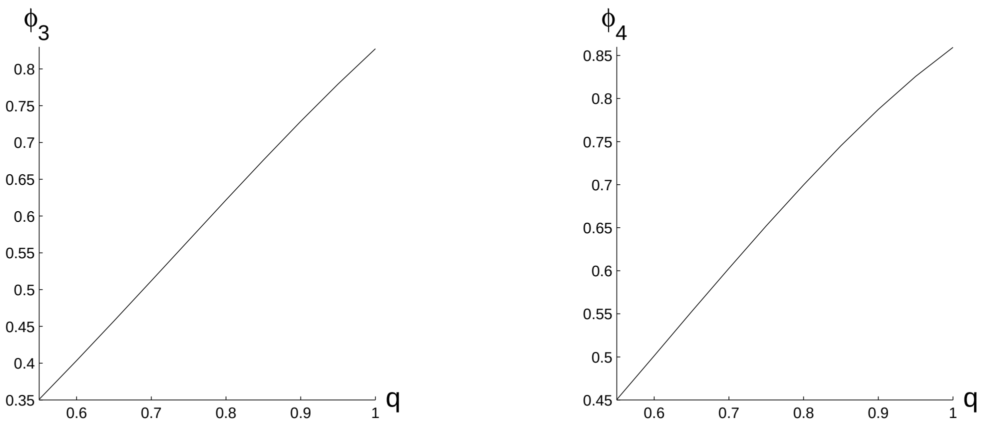

Figure H.17: Graphs of $\phi_3,\ \phi_4$

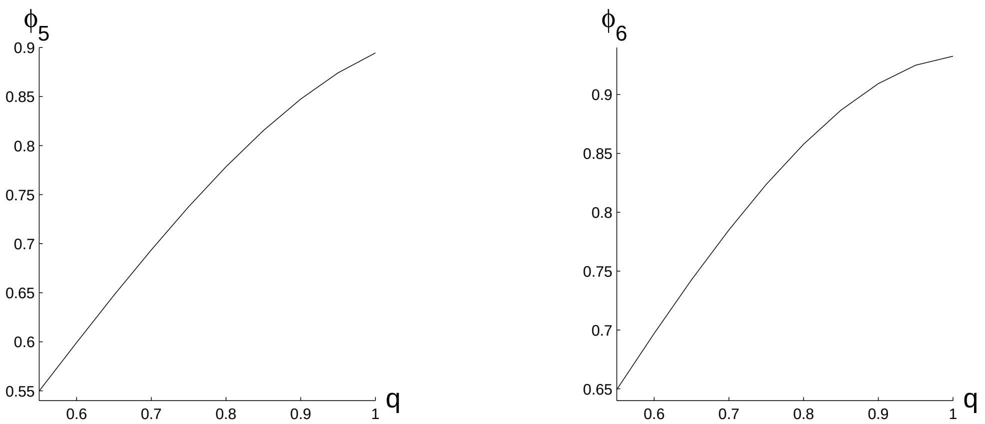

Figure H.18: Graphs of $\phi_5,\ \phi_6$

The Figure H.19, H.20, H.21 show the result of system (5.8) for the Range-Kutta method ($h = 0.1$)

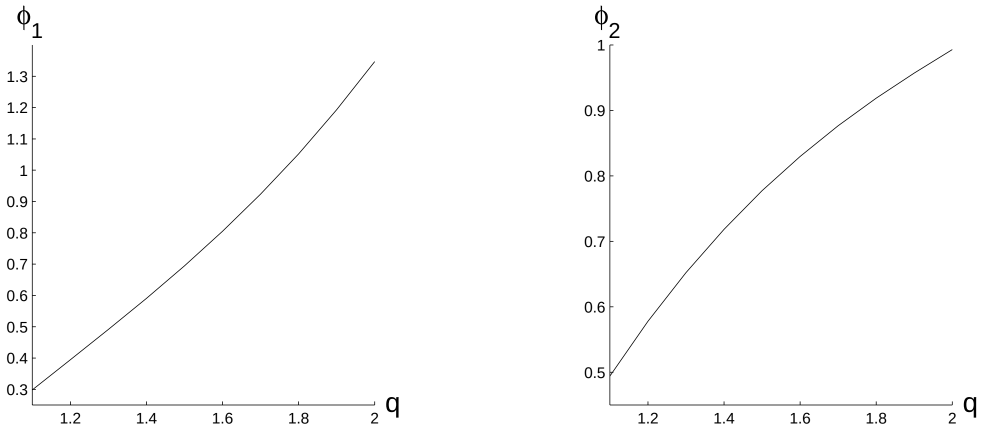

Figure H.19: Graphs of $\phi_1,\ \phi_2$

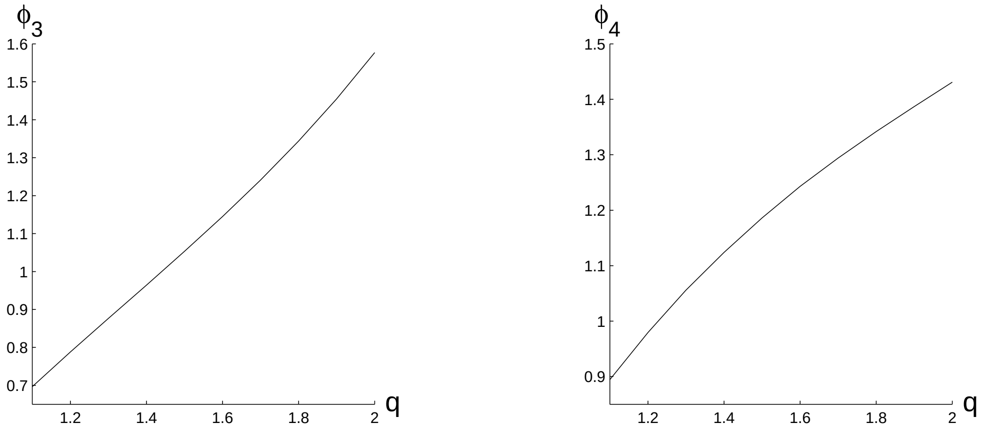

Figure H.20: Graphs of $\phi_3,\ \phi_4$

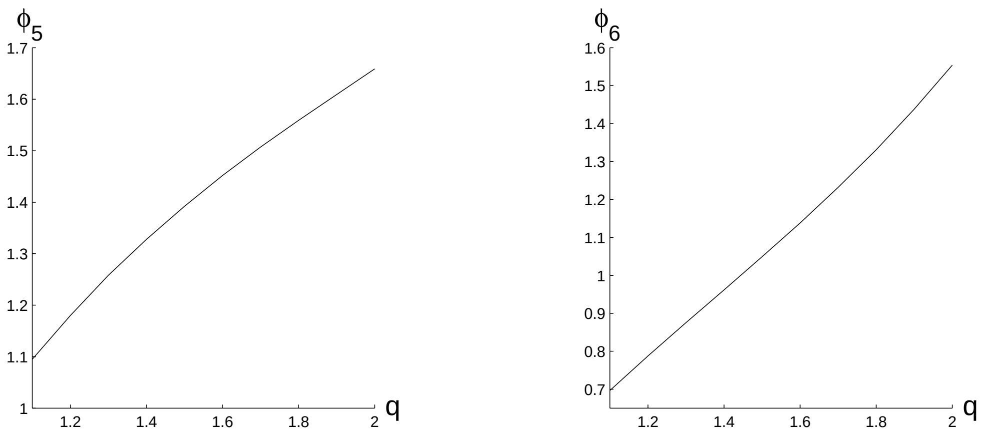

Figure H.21: Graphs of $\phi_5,\ \phi_6$

The Figure H.22, H.23 and H.24 show the result of calculation of system (5.9) for the Runge-Kutta method ($h = 0.01$)

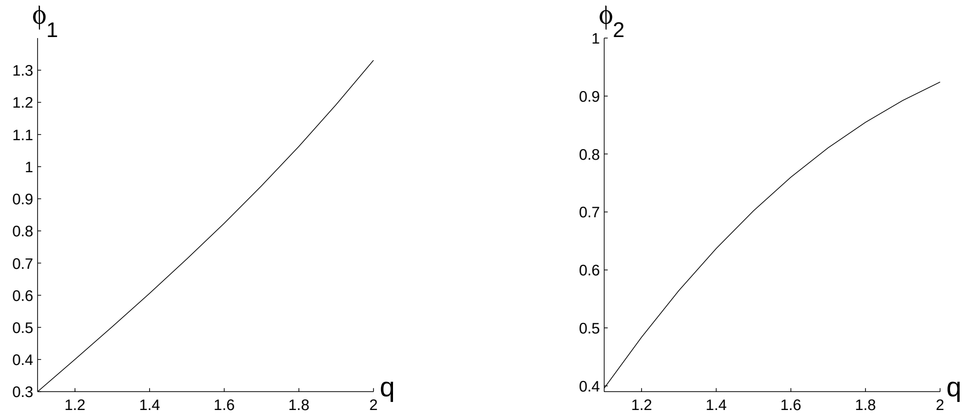

Figure H.22: Graphs of $\phi_1,\ \phi_2$

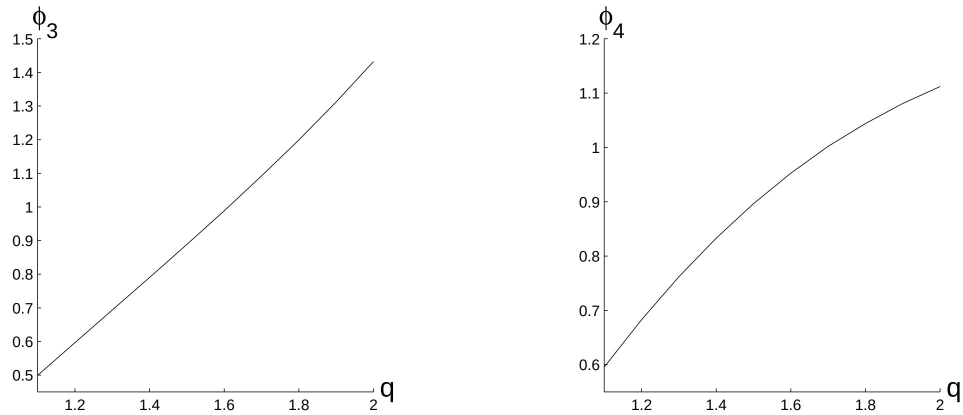

Figure H.23: Graphs of $\phi_3,\ \phi_4$

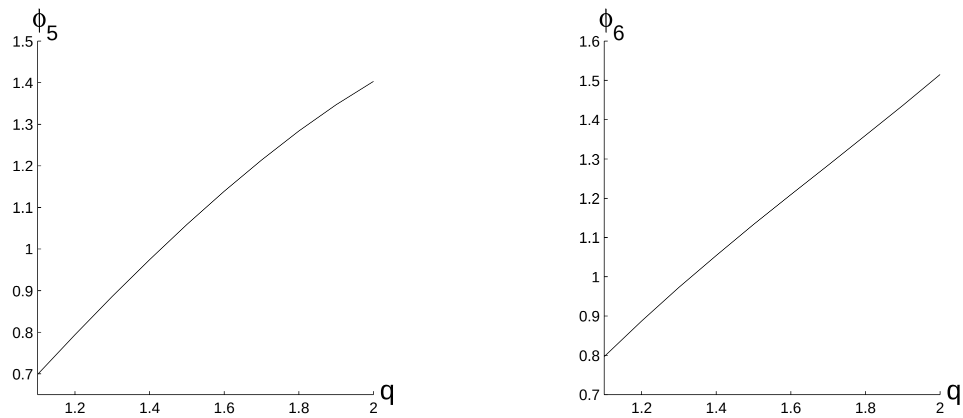

Figure H.24: Graphs of $\phi_5,\ \phi_6$

The Figure H.25, H.26 and H.27 show the result of calculation of system (5.10) for the Runge-Kutta method ($h = 0.01$)

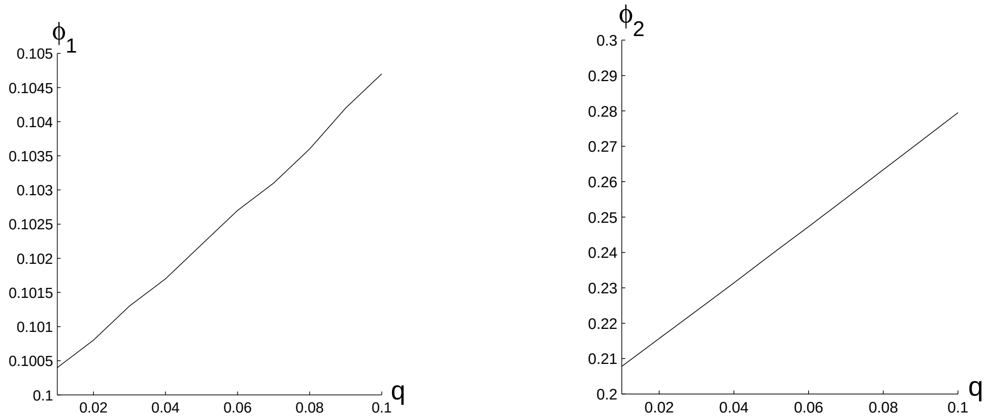

Figure H.25: Graphs of $\phi_1,\ \phi_2$

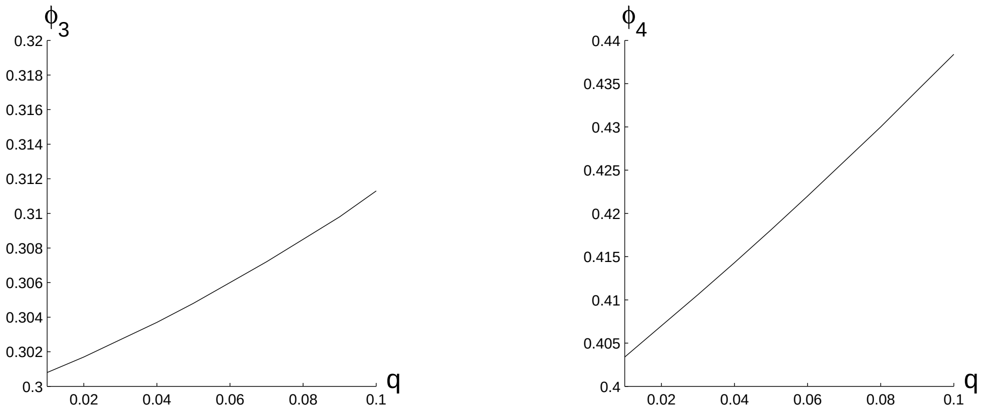

Figure H.26: Graphs of $\phi_3,\ \phi_4$

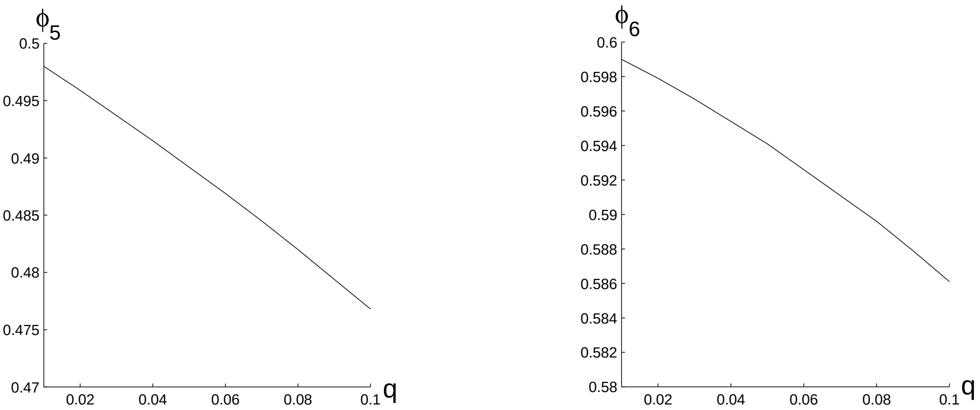

Figure H.27: Graphs of $\phi_5,\ \phi_6$

The Figure H.28, H.29 and H.30 show the result of calculation of system (5.11) for the Runge-Kutta method ($h = 0.01$)

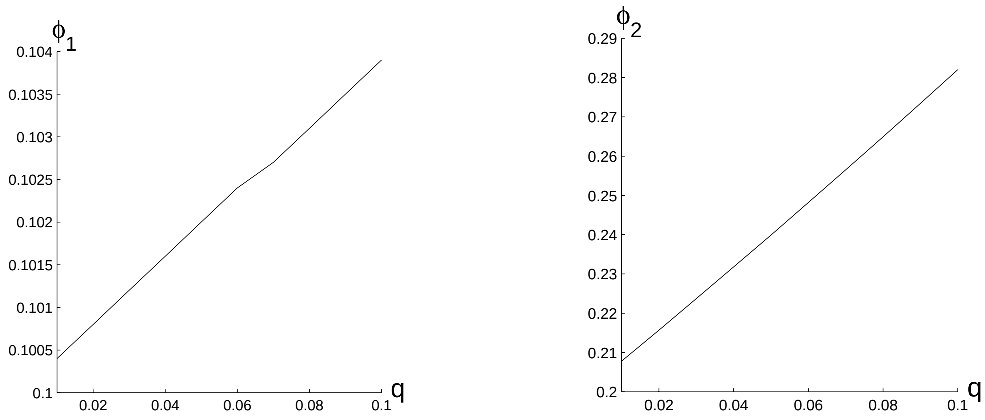

Figure H.28: Graphs of $\phi_1,\ \phi_2$

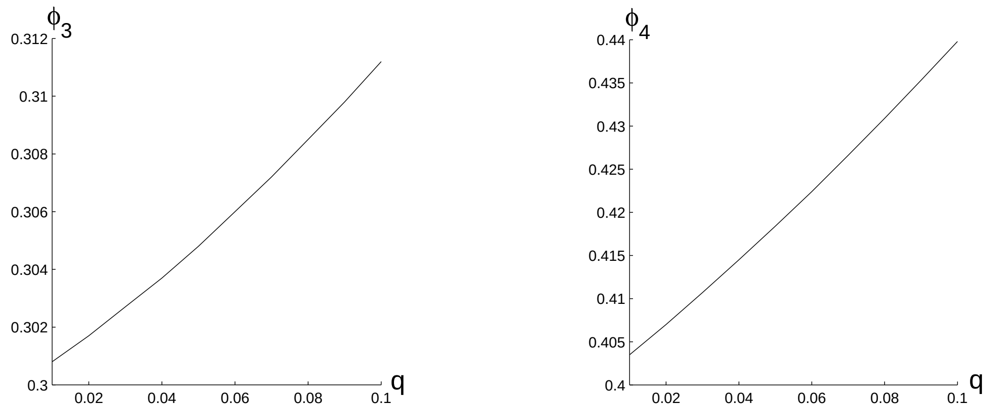

Figure H.29: Graphs of $\phi_3,\ \phi_4$

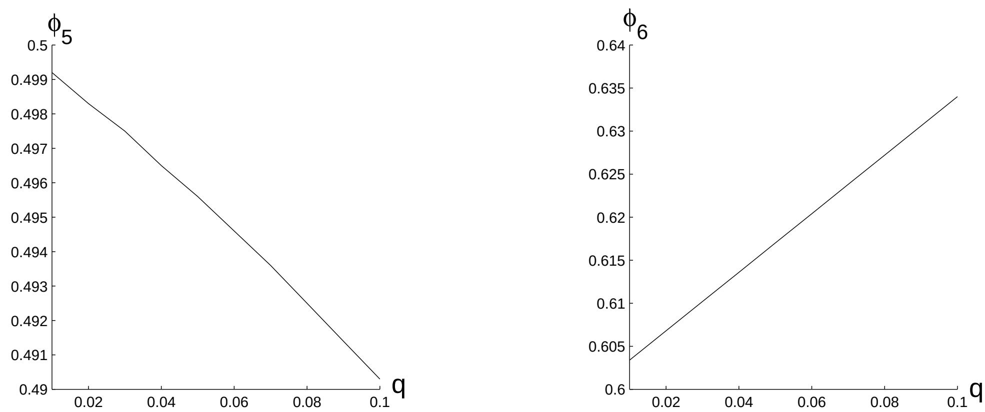

Figure H.30: Graphs of $\phi_5,\ \phi_6$

The Figure H.31, H.32 and H.33 show the result of calculation of system (5.12) for the Runge-Kutta method ($h = 0.05$)

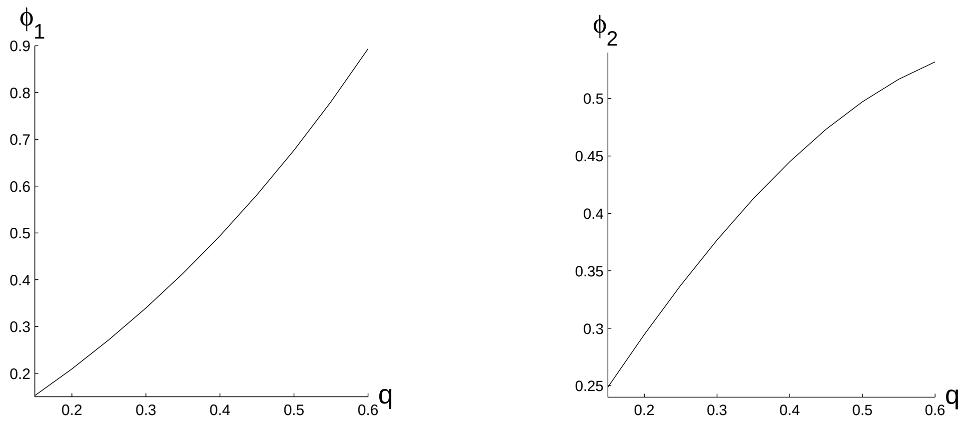

Figure H.31: Graphs of $\phi_1,\ \phi_2$

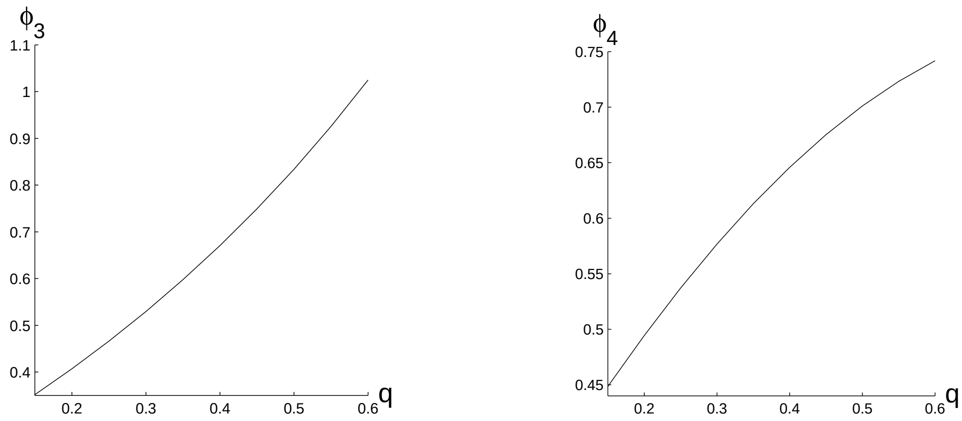

Figure H.32: Graphs of $\phi_3,\ \phi_4$

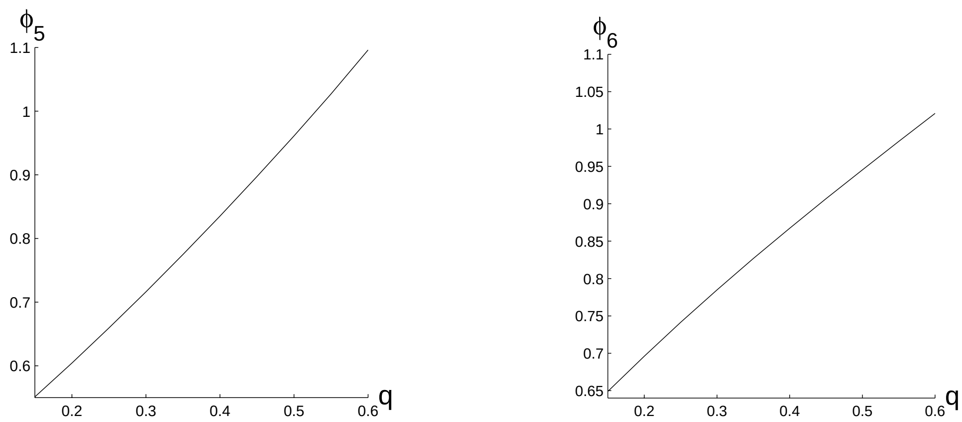

Figure H.33: Graphs of $\phi_5,\ \phi_6$

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