PARTICLES AND STRINGS CORRELATIONS IN QUANTUM FIELD THEORY

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สหสัมพันธ์ของอนุภาคและสตริงในทฤษฎีสนามควอนตัม

นายนัฏพงษ์ ยงรัมย์

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ได้แสดงการหาสมการสำหรับคำนวณค่าสหสัมพันธ์โพลาไรเซชันของสปิน 2 ตัวที่วัดได้พร้อมกันและเกิดในกระบวนการต่างๆ ที่อธิบายได้โดยใช้ทฤษฎี สนามควอนตัมซึ่งได้จากการขยายจากทฤษฎีควอนตัมฟิสิกส์ไปครอบคลุมขอบเขตสัมพัทธภาพเพื่ออธิบายอนุภาคมูลฐานที่มีพลังงานสูง กระบวนการที่พิจารณาประกอบด้วยการรวมตัวกันของอิเล็กตรอน-โพสิตรอนแล้วกลายเป็นโฟตอน 2 ตัว การสร้างคู่อิเล็กตรอน–โพสิตรอนจากการชนกันของรังสีแกมมา 2 โฟตอน การกระเจิงเนื่องจากการชนกันเองของอิเล็กตรอน กระบวนการทั้งหมดนี้พิจารณาโดยใช้พลศาสตร์ไฟฟ้าควอนตัม (คิวอีดี) การผลิตโฟตอน 2 ตัวในพลศาสตร์ไฟฟ้าสเกลาร์ การผลิตมิวออน–ปฏิมิวออนจากการรวมกันของอิเล็กตรอน–โพสิตรอน โดยใช้อันตรกิริยาไฟฟ้าอย่างอ่อนของวายน์เบิร์ก–ซาลาม สมการของสหสัมพันธ์โพลาไรเซชันเหล่านี้ ได้มาจากการคำนวณเชิงพลวัต โดยไม่มีการคาดการณ์และไม่มีการใช้สมมุติฐาน พบว่าสมการที่ได้ขึ้นอยู่กับอัตราเร็ว และสำหรับกระบวนการท้ายสุด พบว่าสมการขึ้นอยู่กับการคู่ควบด้วย สิ่งที่ได้เหล่านี้ต่างจากผลการพิจารณาโดยการรวมสปินของอนุภาคโดยตรงซึ่งมักใช้ในการศึกษาเชิงจลนศาสตร์ สำหรับอัตราเร็วต่ำสมการจากคิวอีดีให้ผลเหมือนผลจากการรวมสปินอนุภาคโดยตรง กรณีค่าพลังงานขีดเริ่มที่ต้องใช้ในการสร้างคู่ มิวออน–ปฏิมิวออน อัตราเร็วขีดเริ่มที่ต้องใช้ในการทำให้เกิดอนุภาคที่มีอัตราเร็วต่ำใกล้ศูนย์ที่ได้จากความสัมพันธ์เป็นศูนย์ไม่ได้และถ้าใช้วิธีการรวมสปินอย่างเดียวเป็นหลัก เหมือนกับที่มีการใช้ในกระบวนการอื่นมานานหลายปี ปรากฏว่าผลที่ได้ล้มเหลวโดยสิ้นเชิง สมการสหสัมพันธ์โพลาไรเซชันที่ได้แสดงให้เห็นความขัดแย้งอย่างชัดเจนเมื่อทดสอบตามวิธีการทดสอบของเบลล์ เราหวังว่าสมการสำหรับคำนวณค่าสหสัมพันธ์โพลาไรเซชันที่เราได้เหล่านี้จะชี้นำทำให้เกิดแนวความคิดของการทดสอบแบบใหม่คล้ายการทดสอบของเบลล์ ซึ่งจะทำหน้าที่เป็นเครื่องมือในการชี้วัดอัตราเร็วและในการสำรวจขอบเขตที่ต้องใช้สัมพัทธภาพพลังงานสูง สุดท้ายเนื่องจากเมื่อเร็วๆนี้เราได้พยายามขยายทฤษฎีเดิมที่ใช้กับอนุภาคที่เป็นจุดออกไปใช้กับระบบที่มีขนาดใหญ่ขึ้น เช่น สตริง การวิเคราะห์ทำได้ในทำนองเดียวกันกับที่กล่าวไว้

ข้างต้น เพื่อให้การวิเคราะห์ในกรณีนี้สำเร็จอย่างสมบูรณ์เราจึงพิจารณาการผลิตอิเล็กตรอน-โพสิตรอน จากเส้นประจุและเส้นที่ประกอบด้วยอนุภาคที่เป็นกลางและเป็นเส้นปิดที่เรียกว่า นัมบุสตริง โดยกรณีหลังนี้เราพิจารณาว่าเกิดการแลกเปลี่ยนกราวิตอน ในกรณีที่อนุภาคในระบบมีพลังงานมากและเป็นไปตามทฤษฎีสัมพัทธภาพได้ค่าสหสัมพันธ์โพลาไรเซชันที่ตรงกันทั้งสองกรณี แต่โดยทั่วไปกรณีที่พลังงานไม่สูงจะได้ค่าสหสัมพันธ์โพลาไรเซชันที่ต่างกันโดยขึ้นกับว่าเส้นอนุภาคที่พิจารณานั้นมีประจุหรือไม่ สมการทั้งหมดของสหสัมพันธ์โพลาไรเซชันที่ได้เป็นสิ่งใหม่และเราได้ตีพิมพ์แล้วเมื่อเร็วๆนี้

สาขาวิชาฟิสิกส์ ลายมือชื่อนักศึกษา ______________

ปีการศึกษา 2548 ลายมือชื่ออาจารย์ที่ปรึกษา ______________

NATTAPONG YONGRAM : PARTICLES AND STRINGS CORRELATIONS IN QUANTUM FIELD THEORY. THESIS ADVISOR : PROF. EDOUARD B. MANOUKIAN, Ph. D. 391 PP. ISBN 974-533-508-8

QUANTUM FIELD THEORY/ HIGH-ENERGY PHYSICS/ GAUGE THEORIES: QUANTUM ELECTRODYNAMICS, WEINBERG–SALAM UNIFIED ELECTROWEAK THEORY/ STRINGS/ FUNDAMENTAL PROCESSES/ POLARIZATION CORRELATIONS/ BELL-LIKE EXPERIMENTS.

Explicit computations are carried out of polarization correlations of simultaneous measurements of spins of two particles produced in fundamental processes directly from quantum field theory where the latter emerges from extending quantum physics to the high-energy relativistic regime of elementary particles. The processes considered are that of e^+e^- annihilation into two photons, e^+e^- production from $\gamma\gamma$ collision, $e^-e^- \to e^-e^-$ scattering, all in quantum electrodynamics (QED), two photons production in scalar electrodynamics, as well as of $\mu^+\mu^-$ production in e^+e^- annihilation in the Weinberg–Salam electro-weak interaction. The explicit expressions of these polarization correlations, follow from these *dynamical* computations are non-speculative involving no arbitrary input assumptions, are seen to depend on speed, and for the latter process on the couplings as well. These are unlike naïve considerations of simply combining the spins of the particles in question which are of kinematical nature. In the limit of zero speeds, the QED expressions are shown to reduce to the naïve ones just mentioned. As a threshold energy is needed to create the $\mu^+\mu^-$ pair, the speed zero limit of the corresponding expression *cannot* be taken to zero and formal arguments based on combining spins only, as done for other processes for years, completely fail. It is remarkable that the remarkable that these expressions for the polarization correlations show clear violations of Bell's test. As we have *explicit* expressions for the correlations, we hope that they will lead to new experiments in the light of Bell-like tests which mon-

itor speed and explore the high-energy relativistic regime. Finally due to recent attempts to generalize point particles to extended ones, such as strings, similar anal are carried as above, for completeness, for e^+e^- production from charged and neutral Nambu strings with graviton exchange occurring for the latter case. In the extreme relativistic case the corresponding polarizations correlations for both cases coincide, but, in general, are different and inquiries about polarization correlations alone, indicate whether the string is charged or uncharged. All of the expressions of the polarization correlations derived are novel and have been recently published.

School of Physics Student's Signature ______

Academic Year 2005 Advisor's Signature ______

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CHAPTER I

INTRODUCTION

The theory which emerges from extending quantum physics to the high-energy relativistic regime is called "Quantum Field Theory". When the energies and momenta of particles to be described are too high for a non-relativistic treatment, as well as for the creation of an unlimited number of particles, which is not necessarily conserved as is observed in high-energy collisions, the need of such a theory necessarily arises. Quantum field theory provides the non-phenomenological consistent approach to high-energy physics [cf. Bejorken and Drell, 1966; Itzykson and Zuber, 1980; Sokolov, Ternov, Zhukovski and Borisov, 1988; Field, 1989; Altarelli, 1994; Iofffe, 2001]. All quantum field theory interactions must necessarily be renormalizable [cf. Manoukian, 1983; 't Hooft and Veltman, 1973; 't Hooft, 1999; Veltman, 1999; Weinberg, 1980; Salam, 1980; Glashow, 1980; Gross *et al.*, 2004] so that computations, such as of radiative corrections, may be consistently and unambiguously carried out and the fundamental parameters of the theory, such as coupling constants and masses, may be also consistently defined which go right to the heart of renormalization theory. All of the fundamental interactions in physics dealing with elementary particles are gauge theories which remain invariant under gauge transformations associated with gauge fields such as the photon. The present thesis is necessarily restricted to renormalizable gauge theories such as quantum electrodynamics based on the quantized Dirac-Maxwell theory, and the standard Weinberg-Salam electro-weak theory. As Feynman (1985) put it, quantum electrodynamics is the most precise theory devised by man, and the so-called unified gauge theories provide promising candidates for more general interactions. Due to the importance of the generalization of point particles to extended objects, consistent with relativity, such as strings, particle production from strings as well is also considered in this work specifically via photon and graviton emissions as the case may be.

The purpose of the present thesis is to carry out a systematic analysis of polarization correlations of simultaneous measurements of spins of two particles emerging from basic processes applicable to any energies available which may be as high as necessary. The fundamental processes are then necessarily computed from quantum field theory interactions and are dynamical of origin which are non-speculative non-phenomenological with no room for handwaving arguments and arbitrary assumptions. All of the corresponding polarization correlations probabilities based on dynamical analyses *following* from quantum field theory, as encountered in the present work, share the interesting property that they depend on the speeds of the colliding particles due to the mere fact that typically the latter carry speeds in order to collide. Such analyses are unlike kinematical considerations based on formal arguments of simplify combining spins [see e.g., Clauser and Shimoney, 1978]. Here it is worth recalling that the total spin of a two-particle system each with spin, such as two of spin $1/2$, is obtained not only from combining the spins of the latter but also from any orbital angular momentum residing in their center of mass system. For low speeds, one expects that the argument based simply on combining the spins of the colliding particles should provide an accurate description of the polarization correlations sought. The idea is very simple. Suppose one may naïvely neglect any orbital angular momentum, and hence of any speed dependence, of a two-particle system. Consider two particles each of spin $1/2$, such as the e^-e^- system, emerging from the scattering process $e^-e^- \to e^-e^-$ which are in the singlet state, i.e., have a total spin 0. Such a singlet state is given by

$$|0\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]. \tag{1.1}$$

Suppose we make simultaneous measurements of spin of the emerging electron e^-e^- along unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$ making respectively angles χ_1, χ_2 with respect to the x-axis.

The corresponding normalized spinors are given by

$$|\xi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi_1/2} \\ \mathrm{e}^{\mathrm{i}\chi_1/2} \end{pmatrix}, \quad |\xi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi_2/2} \\ \mathrm{e}^{\mathrm{i}\chi_2/2} \end{pmatrix}. \tag{1.2}$$

Due to the spin $1/2$ character of e^-e^-, the possible outcomes of the measurements of spins for the pair are given by

$$(\mathbf{n}_1, \mathbf{n}_2),\ (-\mathbf{n}_1, \mathbf{n}_2),\ (\mathbf{n}_1, -\mathbf{n}_2),\ (-\mathbf{n}_1, -\mathbf{n}_2), \tag{1.3}$$

where $(-\mathbf{n}_1, \mathbf{n}_2)$, for example, denotes one of the particle's spin is in the *opposite* direction to $\mathbf{n}_1$, while for the other, the spin is *along* the unit vector $\mathbf{n}_2$. We have exactly *four* possible outcomes for the simultaneous measurements of spins along $\mathbf{n}_1$, $\mathbf{n}_2$ for the emerging electrons. These are pictorially represented by

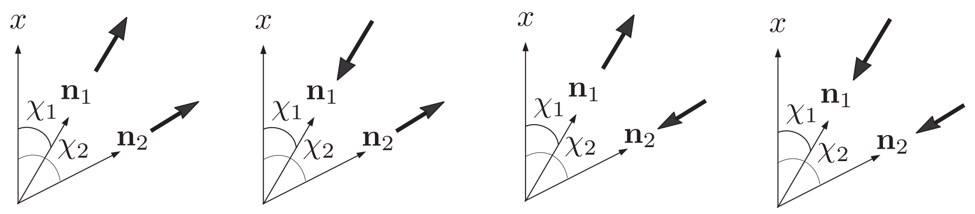

Figure 1.1 Possible outcomes of simultaneous measurement of spins along $\mathbf{n}_1$, $\mathbf{n}_2$ of the two electrons. The thick arrows denote the directions of spins. There are exactly four possible outcomes.

The joint probability of the emerging electrons (e^-e^-) polarizations correlations is then given by the simple quantum mechanical rule:

$$\begin{aligned} P[\chi_1, \chi_2] &= \| \langle\xi_1| \langle\xi_2| \,|0\rangle \|^2 \\ &= \frac{1}{2}\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right). \end{aligned} \tag{1.4}$$

The above method for computing polarization correlations has been used for years [cf.

Clauser and Shimoney, 1978] and is purely kinematical of origin. The quantum field theory computation of $P[\chi_1, \chi_2]$ shows *a clear speed dependence of the electrons (see Eq. (4.82)) and is of dynamical* origin. In the limit of low energies (i.e., of small speeds), we recover the expression Eq. (1.4) (see below Eq. (4.83)) as expected. We will, however, also consider the case of $\mu^+\mu^-$ production from e^+e^- scattering in the Weinberg–Salam electroweak interaction, where due to the threshold energy needed to create the pair $\mu^+\mu^-$ the low energy limit *cannot* be taken and all arguments based simply on combining spins, without dynamical considerations, completely fail. In this latter case, we do not only encounter *speed dependence but dependence on coupling constants as well.*

The fundamental processes considered [Chapter II] are e^+e^- pair annihilation into two photons, e^+e^- production from $\gamma\gamma$ collision, e^-e^- scattering, the so-called Møller scattering, all in quantum electrodynamics, i.e., spin $1/2$ electrodynamics, as well as of two photons production from pair annihilation in scalar, i.e., spin 0 electrodynamics. We then consider the fundamental process of $\mu^+\mu^-$ production, mentioned above, by e^+e^- pair annihilation in the Weinberg–Salam electro-weak theory. *All* of the polarization correlations probabilities recorded in the thesis are published [Yongram and Manoukian, 2003; Manoukian and Yongram, 2004; Manoukian and Yongram, 2005; Yongram, Manoukian and Siranan, 2006]. As there is ample support of the dependence of polarizations correlations on speeds, as we have shown by explicit computations in quantum field theory in the electro-weak interaction as well as in the quantum electrodynamics ones, we hope that some new experiments will be carried out in determining these polarization correlations in the fundamental processes studied in this work by *monitoring* speed and exploring, in particular, the high-energy relativistic regime. Due to the importance of extending the point-like property of a particle, perhaps, to an extended object, such as a string, we have also investigated e^+e^- production from a Nambu, i.e., closed, strings [Nambu, 1973, 1974, 1977, 1979, 1981; Scherk, 1975; Vielenkin, 1981; Kibble and Turok, 1982; Thorn, 1986; Brink and Henneaux, 1988;

Albrecht and Turok, 1989; Sakellariadou, 1990; Manoukian, 1991a, 1991b, 1992b, 1997; Manoukian and Caramanlian, 1994b; Hatfield, 1992] and their corresponding polarization correlations have been computed [Chapter VI]. Such polarization correlations are further generalizations of angular correlations of momenta in fundamental processes carried out in recent years in quantum field theory [Manoukian, 1992a, 1994a, 1998].

All of the above polarization correlations computed are then used to test against the so-called Local Hidden Variables Theories, which are referred to as Bell's test. This provides us the opportunity to bring together quantum field theory and basic quantum physics under the same umbrella of investigation. The idea of Local Hidden Variables Theories [Appendix E] is simple. It states that under exactly the same experimental conditions surrounding the experiment itself, represented by a parameter λ in Eqs. (E.2)–(E.3), the probability of simultaneous measurements of spins, for example, along the unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, making angles χ_1, χ_2 with the x-axis as discussed earlier, is given by the *product* of the two probabilities of measuring the spin of only one particle at a time (Eq. (E.4)). That is, under the same experimental situations, the events for the measurements, respectively, of the spins of the two particles along the unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are independent as, one may argue, invoking locality (relativity of signals) implying that a measurement of spin of one particle cannot instantly influence the outcome for the spin of the other. Our quantum field theory computations, which are necessarily dynamical of origin and, re-iterating, combine quantum physics and relativity, *violate* the Local Hidden Variables prediction as established in the bulk of this thesis. Several experiments have been performed over the years on particles' polarizations correlations [Irby, 2003; Osuch, Popkiewicz, Szeflinski and Wilhelmi, 1996; Kaday, Ulman and Wu, 1975; Fry, 1995] in the light of Bell's test and many more have been proposed [Go, 2004; Bertlman, Bramon, Garbarino and Hiesmayr, 2004; Abel, Dittmar and Dreiner, 1992; Privitera, 1992; Lednický and Lyuboshitz, 2001; Genovese, Novero and Predazzi, 2001]. As we have *explicit* expressions of the polarization correlations for the basic processes in quantum electrodynamics and the electro-weak theory, we hope that these will open the way of testing these expressions as they follow *directly* from quantum field theory with *no* speculations and arbitrary assumptions, and monitor, in the processes, their dependence on speed and coupling constants, in general.

In Chapter II, we carry out a detailed study of all the processes involved in this work in view of applications to polarization correlations in later chapters. Such polarization correlations are then computed for some basic processes in quantum and scalar electrodynamics in Chapter III. The concept of entanglement and further quantum electrodynamics analyses of polarization correlations are given in Chapter IV. The polarization correlation of $\mu^+\mu^-$ production in pair annihilation e^+e^- in the Weinberg–Salam theory is worked out in Chapter V. Finally polarization correlation of e^+e^- production from charged and neutral Nambu strings are given in Chapter VI. For the neutral string graviton emission from the string which further decays in e^+e^- is encountered. All of the expressions for the polarization correlations obtained in this work are *novel* and have been recently published [Yongram and Manoukian, 2003; Manoukian and Yongram, 2004; Manoukian and Yongram, 2005; Yongram, Manoukian and Siranan, 2006]. They all lead to a clear violation of the Local Hidden Variables theory prediction referred to as Bell's test as established in the bulk of the thesis. Chapter VII deals with our conclusion summarizing, in the process, our basic findings. Eight appendices, with various subsections, are given dealing with rather technical details needed in this work and, together, they constitute an important part of this thesis.

CHAPTER II

FUNDAMENTAL PROCESSES IN VIEW OF APPLICATIONS TO POLARIZATION CORRELATIONS

In this chapter, we consider all the fundamental processes analyzed *in this work in view of applications to polarization correlations*. I provide the details corresponding to these processes which will be needed in the subsequent chapters. Computations are carried to the leading orders since the couplings involved are weak dealing with the electromagnetic, the electro-weak ones as well as the gravitational ones. The fundamental processes considered are e^+e^- pair annihilation into two photons (§2.1), e^+e^- pair production by two photons annihilation (§2.2), $e^-e^- \rightarrow e^-e^-$ scattering, the so-called Møller scattering, all in quantum electrodynamics, i.e., spin $1/2$ electrodynamics, as well as two photons production in scalar (§2.4), i.e., spin 0 electrodynamics. We then consider the fundamental process of $\mu^+\mu^-$ pair production (§2.5) by e^+e^- annihilation in the Weinberg-Salam electro-weak theory whose cross section is in excellent agreement with experiments. Finally, we consider pair productions (e^+e^-) from some charged (§2.6) and neutral (§2.7) Nambu strings which are closed circularly oscillating strings. For the neutral string gravitational interactions are necessarily considered. Needless to say quantum field theoretical studies associated with strings have become quite fashionable in recent years.

2.1 e^+e^- Pair Annihilation in Quantum Electrodynamics

The first interesting process in QED that we shall study in this section, deals with the annihilation of a particle and an anti-particle. Here we study the process by

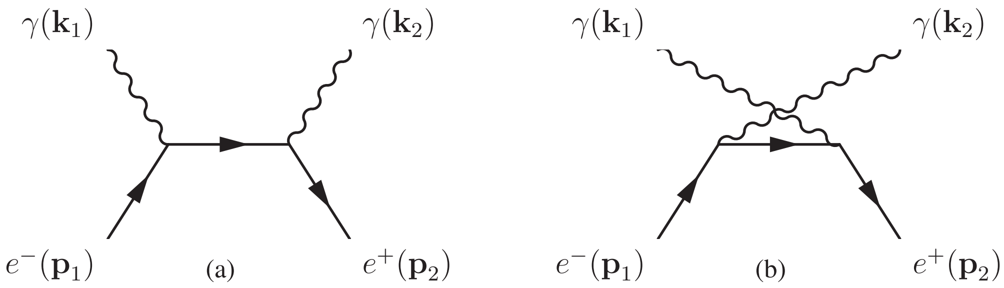

Figure 2.1 Feynman diagrams of the process of pair annihilation into two photons ($e^+e^- \to \gamma\gamma$).

considering the example of the annihilation of an electron-positron pair into photons, so-called "e^-e^+ pair annihilation". The two corresponding Feynman diagram are shown in figure **2.1** .

The amplitude corresponding to the processes in figure 2.1(a), can be easily written down by applying the vacuum-to-vacuum transition amplitude, derived in Appendix B. This is one of a pair annihilation process that we may write from the vacuum-to-vacuum transition amplitude in coordinate space (see in diagram of figure 2.1(a)) as

$$\mathrm{i}e^2 \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z') J^\alpha(z) J^\rho(z') D_{\mu\alpha}(z,y) D_{\nu\rho}(z',y')$$

$$\times\, \overline{\eta}(x') S_+(x',y') \gamma^\nu S_+(y',y) \gamma^\mu S_+(y,x) \eta(x), \tag{2.1}$$

where $\eta(x)$, $\overline{\eta}(x')$, $J^\alpha(z)$ and $J^\rho(z')$ are the presence of external sources (electron and photon sources), and denoting the propagator of electron is

$$S_+(x,x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p(x-x')}(-\gamma p + m)}{p^2 + m^2 - \mathrm{i}\varepsilon}, \quad \varepsilon \to +0 \tag{2.2}$$

and denoting the propagator of photon is

$$D_{\mu\nu}(x,x') = \int \frac{(\mathrm{d}q)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}q(x-x')}}{q^2 - \mathrm{i}\varepsilon} g_{\mu\nu}, \quad \varepsilon \to +0, \tag{2.3}$$

where $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. With these conventions in hand, we will rewrite the vacuum-to-vacuum transition amplitude in Eq. (2.1) in momentum space. We then use the properties of the Fourier transform

$$F(p)=\int(\mathrm{d}x)\mathrm{e}^{-\mathrm{i}px}F(x), \tag{2.4}$$

where $F(p)$ denoting the arbitrary function in momentum space and $F(x)$ denoting the arbitrary function in coordinate space. By using conventional integration, keep in mind,

$$\int(\mathrm{d}x')\overline{\eta}(x')S_+(x',x)=\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}px}}{2p^0}\overline{\eta}(p)(-\gamma p+m)\quad\text{when } x'^0>x^0, \tag{2.5}$$

$$\int(\mathrm{d}x')S_+(x,x')\eta(x')=\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}px}}{2p^0}(-\gamma p+m)\eta(p)\quad\text{when } x^0>x'^0, \tag{2.6}$$

$$\int(\mathrm{d}x')J^\mu(x')D_{\mu\nu}(x',x)=\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}kx}}{2k^0}J^*_\nu(k)\quad\text{when } x'^0>x^0, \tag{2.7}$$

to transform Eq. (2.1), and see the diagram in figure 2.1(a). We then write the vacuum-to-vacuum transition amplitude in momentum space of a process in figure 2.1(a) as:

$$\begin{aligned}&\mathrm{i}e^2\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-k_1)y}\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-k_2)y'}\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J^*_\nu(k_1)\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J^*_\mu(k_2)\\&\times\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}\overline{\eta}(-p_2)(\gamma p_2+m)\gamma^\nu\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p(y'-y)}(-\gamma p+m)}{p^2+m^2}\gamma^\mu\\&\times\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}(-\gamma p_1+m)\eta(p_1).\end{aligned} \tag{2.8}$$

The y- and y'-integrations can be performed immediately yielding

$$\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-k_1-p)y}=(2\pi)^4\delta^4(p_1-k_1-p), \tag{2.9}$$

$$\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-k_2+p)y'}=(2\pi)^4\delta^4(p_2-k_2+p), \tag{2.10}$$

and now the p-integration is easily done:

$$\int \frac{(\mathrm{d}p)}{(2\pi)^4} \frac{(-\gamma p + m)}{p^2 + m^2} (2\pi)^4 \delta^4(p_1 - k_1 - p)(2\pi)^4 \delta^4(p_2 - k_2 + p)$$

$$= (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) \frac{(-\gamma(p_1 - k_1) + m)}{(p_1 - k_1)^2 + m^2}, \tag{2.11}$$

and the properties of the summing over all spin of the initial electron and the initial positron, may be written in the standard form :

$$2m \sum_{\sigma_-} u(\mathbf{p}, \sigma_-)\overline{u}(\mathbf{p}, \sigma_-) = (-\gamma p + m), \tag{2.12}$$

$$(-2m) \sum_{\sigma_+} v(\mathbf{p}, \sigma_+)\overline{v}(\mathbf{p}, \sigma_+) = (\gamma p + m). \tag{2.13}$$

Hence we can rewrite the vacuum-to-vacuum transition amplitude in momentum space of the process in figure 2.1(a) by substituting Eqs. (2.9)–(2.10), Eq. (2.11) and Eqs. (2.12)–(2.13) in Eq. (2.8), and change integral form to summation form, written

$$\begin{aligned} &- \mathrm{i}e^2 (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) \sum \mathrm{i} \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3} \frac{1}{2k_1^0} J_\nu^*(k_1) \mathrm{i} \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3} \frac{1}{2k_2^0} J_\mu^*(k_2) \\ &\times \mathrm{i} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3} \frac{m}{p_2^0} \mathrm{i} \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3} \frac{m}{p_1^0} \overline{v}(\mathbf{p}_2, \sigma_2) v(\mathbf{p}_2, \sigma_2) \overline{\eta}(-p_2) \left[\gamma^\nu \frac{(-\gamma(p_1 - k_1) + m)}{(p_1 - k_1)^2 + m^2} \gamma^\mu \right] \\ &\times u(\mathbf{p}_1, \sigma_1) \overline{u}(\mathbf{p}_1, \sigma_1) \eta(p_1). \end{aligned} \tag{2.14}$$

Next we introduce the convenient notation for the emission source (the electron and the positron), are represented by

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{m}{p^0}} \overline{u}(\mathbf{p}, \sigma_-)\eta(p) = \mathrm{i}\eta_{\mathbf{p}\sigma_-}; \quad e^- \text{ emission}, \tag{2.15}$$

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{m}{p^0}} \overline{\eta}(-p) v(\mathbf{p}, \sigma_+) = \mathrm{i}\eta_{\mathbf{p}\sigma_+}; \quad e^+ \text{ emission}, \tag{2.16}$$

where the signature σ_- corresponding to a spin of electron (e^-), and σ_+ to a spin of positron (e^+). And for the detection source (photon) are represented by

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3}\frac{1}{2k_i^0}}e^{\mu}(\mathbf{k}_i,\lambda)J_{\mu}^*(k_i)=\mathrm{i}J^*_{\mathbf{k}_i\lambda}, \tag{2.17}$$

where $e^{\mu}(\mathbf{k}_i,\lambda)$ are the polarization vector, $\lambda, i = 1, 2$.

By substituting Eqs. (2.16)–(2.17) in Eq. (2.14). Finally, we obtain the vacuum-to-vacuum transition amplitude in momentum space of a diagram in figure 2.1(a), replacing $\sigma_1 \to \sigma_-$ and $\sigma_2 \to \sigma_+$, be written as

$$\begin{aligned}&-\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\\&\times J^*_{k_1\lambda}J^*_{k_2\lambda}\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\overline{v}(\mathbf{p}_2,\sigma_2)\left[e^*_{\nu}\gamma^{\nu}\frac{(-\gamma(p_1-k_1)+m)}{(p_1-k_1)^2+m^2}e^*_{\mu}\gamma^{\mu}\right]u(\mathbf{p}_1,\sigma_1). \quad (2.18)\end{aligned}$$

In similarly, we consider one of a pair annihilation process in figure 2.1(b). We start with the vacuum-to-vacuum transition amplitude in coordinate space, given by

$$\begin{aligned}&\mathrm{i}e^2\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')J^{\alpha}(z)J^{\rho}(z')D_{\nu\alpha}(z,y')D_{\mu\rho}(z',y)\\&\times\overline{\eta}(x')S_+(x',y')\gamma^{\mu}S_+(y',y)\gamma^{\nu}S_+(y,x)\eta(x). \quad (2.19)\end{aligned}$$

where $\eta(z)$, $\overline{\eta}(z')$, $J^{\alpha}(x)$ and $J^{\rho}(x')$ are the presence of external sources, $S_+(x,x')$ denoting the propagator of electron, defined in Eq. (2.2) and $D_+(x,x')$ denoting the Feynman propagator of photon, defined in Eq. (2.3).

We then transform Eq. (2.19) by using the properties in Eqs.(2.5)–(2.7) and see the diagram in figure 2.1(b). We can write the vacuum-to-vacuum transition amplitude in momentum space of a process in figure 2.1(b) as:

$$\mathrm{i}e^2\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-k_2)y}\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-k_1)y'}\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J^*_{\nu}(k_1)\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J^*_{\mu}(k_2)$$

$$\times\, \mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}\overline{\eta}(-p_2)(\gamma p_2+m)\gamma^\mu \mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p(y'-y)}(-\gamma p+m)}{p^2+m^2}\gamma^\nu$$

$$\times\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}(-\gamma p_1+m)\eta(p_1). \tag{2.20}$$

Hence we can rewrite the vacuum-to-vacuum transition amplitude in momentum space of the process in figure 2.1(b) by substituting Eqs. (2.9)–(2.10), Eq. (2.11) and Eqs. (2.12)–(2.13) in Eq. (2.19), and change integral form to summation form, be written as

$$-\,\mathrm{i}\mathrm{e}^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sum \mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J_\nu^*(k_1)\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J_\mu^*(k_2)$$

$$\times\, \mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}\overline{v}(\mathbf{p}_2,\sigma_2)v(\mathbf{p}_2,\sigma_2)\overline{\eta}(-p_2)\left[\gamma^\nu\frac{(-\gamma(p_1-k_2)+m)}{(p_1-k_2)^2+m^2}\gamma^\mu\right]$$

$$\times\, u(\mathbf{p}_1,\sigma_1)\overline{u}(\mathbf{p}_1,\sigma_1)\eta(p_1). \tag{2.21}$$

By substituting Eqs. (2.16)–(2.17) in Eq. (2.21). Finally, we obtain the vacuum-to-vacuum transition amplitude in momentum space of a diagram in figure 2.1(a), replacing $\sigma_1\to\sigma_-$ and $\sigma_2\to\sigma_+$, be written as

$$-\,\mathrm{i}\mathrm{e}^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}$$

$$\times\, J_{k_1\lambda}^*J_{k_2\lambda}^*\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\overline{v}(\mathbf{p}_2,\sigma_2)\left[e_\mu^*\gamma^\mu\frac{(-\gamma(p_1-k_2)+m)}{(p_1-k_2)^2+m^2}e_\nu^*\gamma^\nu\right]u(\mathbf{p}_1,\sigma_1). \tag{2.22}$$

From the vacuum-to-vacuum transition amplitude that are derived in Eq. (2.18) and Eq. (2.22), are the transition vacuum-to-vacuum transition amplitude of a pair annihilation process, to lowest order in the fine-structure constant α, the amplitude for the process $e^+e^-\to\gamma\gamma$ is, up to unimportant factors for the problem at hand, in a standard notation ($e_\mu(k_1,\lambda) \equiv e^*_\mu$ and $e_\nu(k_2,\lambda) \equiv e^*_\nu$)

$$
\begin{aligned}
&- \mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}} \\
&\times J^*_{k_1\lambda}J^*_{k_2\lambda}\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\overline{v}(\mathbf{p}_2,\sigma_2)\left[e_\nu(k_2,\lambda)\gamma^\nu\frac{(-\gamma(p_1-k_1)+m)}{(p_1-k_1)^2+m^2}e_\mu(k_1,\lambda)\gamma^\mu\right. \\
&\qquad\left. + e_\mu(k_1,\lambda)\gamma^\mu\frac{(-\gamma(p_1-k_2)+m)}{(p_1-k_2)^2+m^2}e_\nu(k_2,\lambda)\gamma^\nu\right]u(\mathbf{p}_1,\sigma_1).
\end{aligned}
\tag{2.23}
$$

We introduce the new amplitude $\mathcal{A}$ as follows:

$$
\begin{aligned}
\mathcal{A} = \alpha\overline{v}(\mathbf{p}_2,\sigma_2)&\left[e_\nu(k_2,\lambda)\gamma^\nu\frac{(-\gamma(p_1-k_1)+m)}{(p_1-k_1)^2+m^2}e_\mu(k_1,\lambda)\gamma^\mu\right. \\
&\left. + e_\mu(k_1,\lambda)\gamma^\mu\frac{(-\gamma(p_1-k_2)+m)}{(p_1-k_2)^2+m^2}e_\nu(k_2,\lambda)\gamma^\nu\right]u(\mathbf{p}_1,\sigma_1),
\end{aligned}
\tag{2.24}
$$

where we neglected an unimportant multiplicative factor

$$
\begin{aligned}
&-\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}} \\
&\times J^*_{k_1\lambda}J^*_{k_2\lambda}\eta_{p_2\sigma_2}\eta_{p_1\sigma_1},
\end{aligned}
\tag{2.25}
$$

it is real, and disappear when we normalize the square of the vacuum-to-vacuum transition amplitude.

The term in the bracket $[\cdot]$ can be discarded since it is orthogonal to the energy projection operator in Eq. (2.24). Therefore, We can simplify Eq. (2.24), by using the mass shell condition, $p^2+m^2=0$, $(-\gamma p+m)\psi=0$ and $k^2=0$. The Dirac matrices in $[\cdot]$ are still complicated expressions. The following calculations can be siplified considerably, however, if we choose a convenient gauge in which the polarization vectors are orthogonal to initial electron momentum p_i where $i = 1, 2$:

$$e_\mu(k_1, \lambda) \cdot p_i = 0, \quad e_\nu(k_2, \lambda) \cdot p_i = 0. \tag{2.26}$$

We obtain

$$\mathcal{A} = \alpha e_\mu(k_1, \lambda) e_\nu(k_2, \lambda) \overline{v}(\mathbf{p}_2, \sigma_2) \left[\frac{\gamma^\nu \gamma k_1 \gamma^\mu}{2p_1 k_1} + \frac{\gamma^\mu \gamma k_2 \gamma^\nu}{2p_1 k_2} + \frac{\gamma^\mu p_1^\nu}{p_1 k_1} + \frac{\gamma^\nu p_1^\mu}{p_1 k_2} \right] u(\mathbf{p}_1, \sigma_1). \tag{2.27}$$

In the center of mass motion (c.m.) where $p_1 = (p^0, \mathbf{p}) = -p_2$ this amounts to the "radiation gauge" in which the electromagnetic potential has no 0-component, i.e. $e_1^\mu(\lambda) = (0, \mathbf{e}_1(\lambda))$. However, the condition in Eq. (2.26) can be imposed in any given frame of reference. Starting from an arbitrary set of polarization vectors $e_1^\mu(\lambda)$, $e_2^\nu(\lambda)$ we can perform a gauge transformation

$$\epsilon_{1\lambda}^\mu = \left(\delta_\nu^\mu - \frac{p_{1\nu} k_1^\mu}{p_1 k_1} \right) e_1^\nu(\lambda), \quad k_1 e_1(\lambda) = 0, \tag{2.28}$$

$$\epsilon_{2\lambda'}^\mu = \left(\delta_\nu^\mu - \frac{p_{1\nu} k_2^\mu}{p_1 k_1} \right) e_2^\nu(\lambda'), \quad k_2 e_2(\lambda') = 0. \tag{2.29}$$

So that the new polarization vectors $\epsilon_{1\lambda}^\mu$ are orthogonal to p_i. The normalization and transversality conditions $\epsilon^\mu \epsilon_\mu = -1$, $k^\mu \epsilon_\mu = 0$ are not affected by the transformation Eqs. (2.28)–(2.29)

$$\epsilon_{1\lambda} \cdot \epsilon_{1\lambda} = \epsilon_{2\lambda'} \cdot \epsilon_{2\lambda'} = -1, \tag{2.30}$$

$$\epsilon_{1\lambda} \cdot k_1 = \epsilon_{2\lambda'} \cdot k_2 = 0, \tag{2.31}$$

which immediately follows from $k_1^2 = k_2^2 = 0$. Thus without restricting the generality of our calculation we will impose the condition in Eq. (2.26).

Using Eq. (2.26) and Eqs. (2.29)–(2.30), we finally have to evaluate the matrix element in $[\cdot]$ with

$$\Gamma = \left[\frac{\gamma^\nu \gamma k_1 \gamma^\mu}{2p_1k_1} + \frac{\gamma^\mu \gamma k_2 \gamma^\nu}{2p_1k_2}\right]. \tag{2.32}$$

We rewrite Eq. (2.27), here $\lambda = \lambda'$ (not need to write down in equation), as

$$\mathcal{A} = \alpha\epsilon_{1\mu}\epsilon_{2\nu}\overline{v}(p_2,\sigma_2)\left[\frac{\gamma^\nu \gamma k_1 \gamma^\mu}{2p_1k_1} + \frac{\gamma^\mu \gamma k_2 \gamma^\nu}{2p_1k_2}\right]u(p_1,\sigma_1). \tag{2.33}$$

Here we will discuss polarization effects of two photons but not discuss polarization effects of the electron (positron), and therefore will average over the initial polarization state of the electron (positron) but not sum over the final polarization state of two photons in the reaction.

In accord with the results, if the initial state is unpolarized, we must also average the result over the initial spin states.

To carry out the these operations, let us write the amplitude of the process as

$$\mathcal{A} = \alpha\epsilon_{1\mu}\epsilon_{2\nu}\overline{v}(p_2,\sigma_2)\Gamma u(p_1,\sigma_1). \tag{2.34}$$

The square of the modulus ($|\mathcal{A}|^2$) is

$$|\mathcal{A}|^2 = \alpha^2\epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}\overline{u}(p_1,\sigma_1)\overline{\Gamma}v(p_2,\sigma_2)\overline{v}(p_2,\sigma_2)\Gamma u(p_1,\sigma_1), \tag{2.35}$$

where $\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$ that can be constructed by means of the easily verifiable formula

$$\overline{\gamma^\mu\gamma^\nu\ldots\gamma^\lambda\gamma^\rho} = \gamma^0(\gamma^\mu\ldots\gamma^\rho)^\dagger\gamma^0 = \gamma^\rho\gamma^\lambda\ldots\gamma^\nu\gamma^\mu,$$

and, in particular,

$$\overline{\gamma^\mu} = \gamma^\mu,$$

we obtain

$$\overline{\Gamma} = \left[\frac{\gamma^\mu \gamma k_1 \gamma^\nu}{2p_1k_1} + \frac{\gamma^\nu \gamma k_2 \gamma^\mu}{2p_1k_2}\right]. \tag{2.36}$$

If we write out all the bispinor indices in Eq. (2.35), we have

$$|\mathcal{A}|^2 = \alpha^2 \epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}(\overline{u})_{\alpha\beta}(\overline{\Gamma})_{\beta\sigma}(v)_{\sigma\mu}(\overline{v})_{\mu\nu}(\Gamma)_{\nu\rho}(u)_{\rho\alpha}, \tag{2.37}$$

which is the trace of a matrix product:

$$|\mathcal{A}|^2 = \alpha^2 \epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}\,\mathrm{Tr}[\overline{u}(p_1,\sigma_1)\overline{\Gamma}v(p_2,\sigma_2)\overline{v}(p_2,\sigma_2)\Gamma u(p_1,\sigma_1)]. \tag{2.38}$$

By virtue of a property of a trace, the cofactors in Eq. (2.37) allow for cyclic permutations:

$$|\mathcal{A}|^2 = \alpha^2 \epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}\,\mathrm{Tr}[u(p_1,\sigma_1)\overline{u}(p_1,\sigma_1)\overline{\Gamma}v(p_2,\sigma_2)\overline{v}(p_2,\sigma_2)\Gamma]. \tag{2.39}$$

Let us now recall the identity Eqs. (2.12)–(2.13). Thus, if the initial state have one electron and one positron, then summing over the initial polarization yield

$$|\mathcal{A}|^2 = \alpha^2 \epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}\left(-\frac{1}{4m^2}\right)\mathrm{Tr}[(-\gamma p_1 + m)\overline{\Gamma}(\gamma p_2 + m)\Gamma], \tag{2.40}$$

and combining Eq. (2.40) with the conservation law $p_1 + p_2 = k_1 + k_2$ yields

$$|\mathcal{A}|^2 = \alpha^2 \epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}\left(-\frac{1}{4m^2}\right)\mathrm{Tr}[(-\gamma p_1 + m)\overline{\Gamma}(\gamma[k_1 + k_2 - p_1] + m)\Gamma]. \tag{2.41}$$

For further calculations it is expedient to recall the following properties of γ-matrices ($g^{\mu\nu} = \mathrm{diag}[-1,1,1,1]$)

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu},$$

$$\mathrm{Tr}[\gamma^\mu] = 0,$$

$$\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n+1}}] = 0,$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = -4g^{\mu\nu}.$$

Using the properties of γ-matrices and the fact that $\gamma^\mu(-\gamma p+m) = (\gamma p+m)\gamma^\mu + 2p^\mu$, we can shift $(-\gamma p_1 + m)$ on the right-hand side of Eq. (2.40) to the left as far as the factor $\gamma(k_1 + k_2 - p_1) + m$ and then return it to its former place via the identity:

$$\gamma^\mu\gamma k_1\gamma^\nu(-\gamma p_1 + m) = (\gamma p_1 + m)\gamma^\mu\gamma k_1\gamma^\nu - 2(p_1 \cdot k_1)\gamma^\mu\gamma^\nu, \tag{2.42a}$$

$$\gamma^\mu\gamma k_1\gamma^\nu(-\gamma p_1 + m) = (\gamma p_1 + m)\gamma^\nu\gamma k_2\gamma^\mu - 2(p_1 \cdot k_2)\gamma^\nu\gamma^\mu. \tag{2.42b}$$

After this, using the well-known property of trace, $\mathrm{Tr}[AB\ldots X]\ldots\mathrm{Tr}[XAB\ldots]$, we take $-\gamma p_1 + m$ to the left-hand side and move it to the right as far as the factor $\gamma(k_1 + k_2 - p_1) + m$ and then return it to its former place via the identity in Eqs. (2.42a)–(2.42b). As a result, we obtain

$$\begin{aligned}|\mathcal{A}|^2 &= \alpha^2\epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}\left(-\frac{1}{4m^2}\right)\mathrm{Tr}[(-\gamma p_1 + m)\overline{\Gamma}(\gamma[k_1 + k_2 - p_1] + m)\Gamma] \\ &= \alpha^2\epsilon_{1\mu}\epsilon_{1\mu}\epsilon_{2\nu}\epsilon_{2\nu}\left(-\frac{1}{4m^2}\right)[T_1 + T_2]. \end{aligned} \tag{2.43}$$

Here we have introduced the notation

$$T_1 = \mathrm{Tr}[(-\gamma p_1 + m)\overline{\Gamma}(-\gamma p_1 + m)\Gamma], \tag{2.44}$$

$$T_2 = \mathrm{Tr}[(-\gamma p_1 + m)\overline{\Gamma}\gamma(k_1 + k_2)\Gamma]. \tag{2.45}$$

To calculate T_1, we shift the right $-\gamma p_1 + m$ to the left:

$$\begin{aligned}T_1 &= \mathrm{Tr}[(-\gamma p_1 + m)\overline{\Gamma}(-\gamma p_1 + m)\Gamma] \\ &= \mathrm{Tr}[(-\gamma p_1 + m)[(\gamma p_1 + m)\overline{\Gamma} - (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)]\Gamma]\end{aligned}$$

$$= 2g^{\mu\nu}\,\mathrm{Tr}[(-\gamma p_1 + m)\Gamma]$$

$$= -2g^{\mu\nu}\,\mathrm{Tr}[\gamma p_1 \Gamma]$$

$$= 8g^{\mu\nu}g^{\mu\nu}, \tag{2.46}$$

where we have used the relation $(-\gamma p + m)(\gamma p + m) = p^2 + m^2 = 0$, $\mathrm{Tr}[\Gamma] = 0$

Next we calculate T_2 by writing it out explicitly:

$$T_2 = \mathrm{Tr}[(-\gamma p_1 + m)\overline{\Gamma}\gamma(k_1 + k_2)\Gamma]$$

$$= \mathrm{Tr}\left\{(-\gamma p_1)\left[\frac{\gamma^\mu\gamma k_1\gamma^\nu}{2p_1k_1} + \frac{\gamma^\nu\gamma k_2\gamma^\mu}{2p_1k_2}\right]\gamma(k_1+k_2)\left[\frac{\gamma^\nu\gamma k_1\gamma^\mu}{2p_1k_1} + \frac{\gamma^\mu\gamma k_2\gamma^\nu}{2p_1k_2}\right]\right\}$$

$$= \mathrm{Tr}\left\{\frac{(-\gamma p_1)\gamma^\mu\gamma k_1\gamma^\nu\gamma k_2\gamma^\nu\gamma k_1\gamma^\mu}{(2p_1k_1)^2} + \frac{(-\gamma p_1)\gamma^\nu\gamma k_2\gamma^\mu\gamma k_1\gamma^\mu\gamma k_2\gamma^\nu}{(2p_1k_2)^2}\right\}$$

$$= -g^{\mu\mu}g^{\nu\nu}\frac{2(k_1\cdot k_2)}{(p_1k_1)} - g^{\mu\mu}g^{\nu\nu}\frac{2(k_1\cdot k_2)}{(p_1k_2)}$$

$$= -2g^{\mu\mu}g^{\nu\nu}(k_1\cdot k_2)\left[\frac{1}{(p_1k_1)} + \frac{1}{(p_1k_2)}\right]. \tag{2.47}$$

Here we have allowed for $k_1k_2 = p_1(k_1 + k_2)$, we can rewrite Eq. (2.47) as

$$T_2 = -2g^{\mu\mu}g^{\nu\nu}\frac{(k_1\cdot k_2)^2}{(p_1k_1)(p_1k_2)}. \tag{2.48}$$

The square of the modulus ($|\mathcal{A}|^2$) of this process is

$$|\mathcal{A}|^2 = \alpha^2\left(\frac{8}{4m^2}\right)\left[\frac{1}{4}\frac{(k_1\cdot k_2)^2}{(p_1k_1)(p_1k_2)} - (\epsilon_{1\mu}\cdot\epsilon_{2\nu})^2\right], \tag{2.49}$$

that we use this in subsequent chapter.

2.2 e^+e^- Pair Production in Quantum Electrodynamics

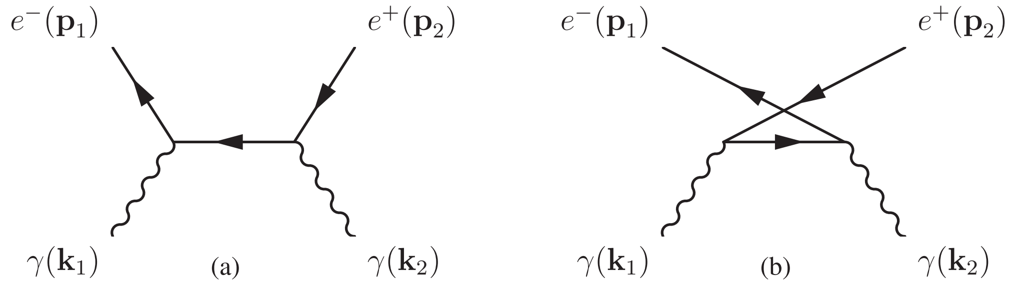

Figure 2.2 Feynman diagrams of e^-e^+ pair production ($\gamma\gamma \to e^+e^-$).

We have studied the e^+e^- Pair Annihilation in §2.1. Now we will study e^+e^- Pair Production that have the two corresponding Feynman diagram are shown in figure **2.2** .

The amplitude corresponding to the processes in figure 2.2(a), can be easily written down by applying the vacuum-to-vacuum transition amplitude, derived in Appendix B. These is one of a pair production process that we may write from the vaccum-to-vacuum transition amplitude in coordinate space (see in diagram of figure 2.2(a))

$$\mathrm{i}e^2 \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z') J^\alpha(x) J^\rho(x') D_{\mu\alpha}(y,x) D_{\nu\rho}(y',x')$$

$$\times\, \overline{\eta}(z) S_+(z,y)\gamma^\nu S_+(y,y')\gamma^\mu S_+(y',z')\eta(z'), \tag{2.50}$$

where $\eta(z)$, $\overline{\eta}(z')$, $J^\alpha(x)$ and $J^\rho(x')$ are the presence of external sources, $S_+(x,x')$ denoting the propagator of electron, defined in Eq. (2.2) and $D_+(x,x')$ denoting the Feynman propagator of photon, defined in Eq. (2.3).

As in e^+e^- pair annihilation process in §2.1 with using conventional integration, keep in mind, given by

$$\int (\mathrm{d}x) J^\mu(x) D_{\mu\nu}(x',x) = \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}kx'}}{2k^0} J_\nu(k) \quad \text{when } x'^0 > x^0. \tag{2.51}$$

To transform Eq. (2.50) by using the properties in Eq. (2.5)–(2.6) and Eq. (2.51), and then substituting them in Eq. (2.50). To do this, we write the amplitude in momentum space as

$$
\begin{aligned}
&\mathrm{i}e^2\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(k_1-p_1)y}\int\mathrm{d}y'\mathrm{e}^{\mathrm{i}(k_2-p_2)y'}\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J_\nu(k_1)\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J_\mu(k_2)\\
&\times\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}\overline{\eta}(p_1)(-\gamma p_1+m)\gamma^\nu\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p(y-y')}(-\gamma p+m)}{p^2+m^2}\gamma^\mu\\
&\times\int\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}(\gamma p_2+m)\eta(-p_2). \qquad (2.52)
\end{aligned}
$$

The y- and y'-integrations can be performed immediately yielding

$$\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(k_1-p_1+p)y}=(2\pi)^4\delta^4(k_1-p_1+p), \qquad (2.53)$$

$$\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(k_2-p_2-p)y'}=(2\pi)^4\delta^4(k_2-p_2-p). \qquad (2.54)$$

Now the p-integration is easily done :

$$
\begin{aligned}
&\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{(-\gamma p+m)}{p^2+m^2}(2\pi)^4\delta^4(k_1-p_1+p)(2\pi)^4\delta^4(k_2-p_2-p)\\
&=(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\frac{(-\gamma(k_2-p_2)+m)}{(k_2-p_2)^2+m^2}. \qquad (2.55)
\end{aligned}
$$

By using the properties in Eqs. (2.12)–(2.13), Eqs. (2.53)–(2.54) and Eq. (2.55), and substitute them in Eq. (2.52). We then write the amplitude in momentum space of this processes in simply form, in summation form, be written as

$$
\begin{aligned}
&-\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sum\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J_\nu(k_1)\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J_\mu(k_2)\\
&\times\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}\overline{u}(\mathbf{p}_2,\sigma_2)u(\mathbf{p}_2,\sigma_2)\overline{\eta}(p_1)\left[\gamma^\nu\frac{(-\gamma(k_2-p_2)+m)}{(k_2-p_2)^2+m^2}\gamma^\mu\right]\\
&\times v(\mathbf{p}_2,\sigma_2)\overline{v}(\mathbf{p}_2,\sigma_2)\eta(-p_2), \qquad (2.56)
\end{aligned}
$$

and for the detection source (the electron and the positron) are represented by

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{m}{p^0}}\overline{v}(\mathbf{p},\sigma_+)\eta(-p) = \mathrm{i}\eta^*_{\mathbf{p}\sigma_+}; \quad e^+ \text{ detection,} \tag{2.57}$$

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{m}{p^0}}\overline{\eta}(p)u(\mathbf{p},\sigma_-) = \mathrm{i}\eta^*_{\mathbf{p}\sigma_-}; \quad e^- \text{ detection,} \tag{2.58}$$

for the emission source (photon) are represented by

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3}\frac{1}{2k_i^0}}e^\mu(\mathbf{k}_i,\lambda)J_\mu(k_i) = \mathrm{i}J_{\mathbf{k}_i\lambda}, \tag{2.59}$$

where $e^\mu(\mathbf{k}_i,\lambda)$ are the polarization vector, $\lambda, i = 1,2$.

We obtain the amplitude of diagram in figure 2.2(a), be written as

$$\mathrm{i}e^2(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}$$

$$\times J_{k_1\lambda}J_{k_2\lambda}\eta^*_{p_2\sigma_2}\eta^*_{p_1\sigma_1}\overline{u}(\mathbf{p}_1,\sigma_1)\left[e_\nu\gamma^\nu\frac{(-\gamma(k_2-p_2)+m)}{(k_2-p_2)^2+m^2}e_\mu\gamma^\mu\right]v(\mathbf{p}_2,\sigma_2). \tag{2.60}$$

Another one of a e^+e^- pair productions process in figure 2.2(b). We again start with the vacuum-to-vacuum transition amplitude in coordinate space, given by

$$\mathrm{i}e^2\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')J^\alpha(x)J^\rho(x')D_{\nu\alpha}(y,x)D_{\mu\rho}(y',x')$$

$$\times\,\overline{\eta}(z)S_+(z,y')\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,z')\eta(z'). \tag{2.61}$$

where $\eta(z)$, $\overline{\eta}(z')$, $J^\alpha(x)$ and $J^\rho(x')$ are the presence of external sources, $S_+(x,x')$ denoting the propagator of electron, defined in Eq. (2.2) and $D_+(x,x')$ denoting the Feynman propagator of photon, defined in Eq. (2.3). We then transform Eq. (2.61) by using the properties in Eqs. (2.5)–(2.6),and substitute them in Eq. (2.61). To do this, we then write the amplitude in momentum space of this processes as

$$\mathrm{i}e^2\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(k_1-p_2)y}\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(k_2-p_1)y'}\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J_\nu(k_1)\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J_\mu(k_2)$$

$$\times\,\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_2^0}\overline{\eta}(p_1)(-\gamma p_1+m)\gamma^\mu\mathrm{i}\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p(y'-y)}(-\gamma p+m)}{p^2+m^2}\gamma^\nu$$

$$\times\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}(\gamma p_2+m)\eta(-p_2). \tag{2.62}$$

The p-integration is easily done :

$$\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{(-\gamma p+m)}{p^2+m^2}(2\pi)^4\delta^4(k_1-p_2-p)(2\pi)^4\delta^4(k_2-p_1+p)$$

$$=(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\frac{(-\gamma(k_1-p_2)+m)}{(k_1-p_2)^2+m^2}. \tag{2.63}$$

By using the properties in Eqs. (2.12)–(2.13), Eqs. (2.53)–(2.54) and Eq. (2.63), and substitute them in Eq. (2.63). we can rewrite the amplitude in momentum space of this processes in simply form, in summation form, be written as

$$-\,\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sum\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J_\nu(k_1)\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J_\mu(k_2)$$

$$\times\,\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}\overline{u}(\mathbf{p}_1,\sigma_1)u(\mathbf{p}_1,\sigma_1)\overline{\eta}(p_1)\left[\gamma^\nu\frac{(-\gamma(k_1-p_2)+m)}{(k_1-p_2)^2+m^2}\gamma^\mu\right]$$

$$\times\,v(\mathbf{p}_2,\sigma_2)\overline{v}(\mathbf{p}_2,\sigma_2)\eta(-p_2). \tag{2.64}$$

By substituting Eq. (2.57)–(2.59) in Eq. (2.64). It is easily checked for figure 2.2(b), by substituting $k_1\leftrightarrow k_2$ and $\nu\leftrightarrow\mu$. We finally obtain the vacuum-to-vacuum transition amplitude in momentum space of diagram in figure 2.2(b), be written as

$$-\,\mathrm{i}e^2(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}$$

$$\times J_{k_1\lambda}J_{k_2\lambda}\eta^*_{p_2\sigma_2}\eta^*_{p_1\sigma_1}\overline{u}(\mathbf{p}_1,\sigma_1)\left[e_\mu\gamma^\mu\frac{(-\gamma(k_1-p_2)+m)}{(k_1-p_2)^2+m^2}e_\nu\gamma^\nu\right]v(\mathbf{p}_2,\sigma_2). \qquad (2.65)$$

From the amplitudes that are derived in Eq. (2.60) and Eq. (2.65), the transition amplitude of a process $\gamma\gamma \to e^+e^-$, denote as $\mathcal{A}$, to lowest order in the fine-structure constant α, the amplitude for a process $\gamma\gamma \to e^+e^-$ is, up to unimportant factors for the problem at hand, in a standard notation

$$\mathcal{A} = -\mathrm{i}e^2(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}$$
$$\times J_{k_1\lambda}J_{k_2\lambda}\eta^*_{p_2\sigma_2}\eta^*_{p_1\sigma_1}\overline{u}(\mathbf{p}_1,\sigma_1)\left[e_\nu(k_2,\lambda)\gamma^\nu\frac{(-\gamma(k_2-p_2)+m)}{(k_2-p_2)^2+m^2}e_\mu(k_1,\lambda)\gamma^\mu\right.$$
$$\left.+\,e_\mu(k_1,\lambda)\gamma^\mu\frac{(-\gamma(k_1-p_2)+m)}{(k_1-p_2)^2+m^2}e_\nu(k_2,\lambda)\gamma^\nu\right]v(\mathbf{p}_2,\sigma_2). \qquad (2.66)$$

To lowest order in the fine-structure constant α, the amplitude for the process $\gamma\gamma \to e^+e^-$ is, up to unimportant factors for the problem at hand, in a standard notation,

$$\mathcal{A} = \alpha\overline{u}(\mathbf{p}_1,\sigma_1)\left[e_\nu(k_2,\lambda)\gamma^\nu\frac{(-\gamma(k_2-p_2)+m)}{(k_2-p_2)^2+m^2}e_\mu(k_1,\lambda)\gamma^\mu\right.$$
$$\left.+\,e_\mu(k_1,\lambda)\gamma^\mu\frac{(-\gamma(k_1-p_2)+m)}{(k_1-p_2)^2+m^2}e_\nu(k_2,\lambda)\gamma^\nu\right]v(\mathbf{p}_2,\sigma_2), \qquad (2.67)$$

where we neglected an unimportant multiplicative factor

$$-\,\delta^4(k_1+k_2-p_1-p_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}$$
$$\times J_{k_1\lambda}J_{k_2\lambda}\eta^*_{p_2\sigma_2}\eta^*_{p_1\sigma_1}. \qquad (2.68)$$

The term in the bracket $[\cdot]$ can be discarded since it is orthogonal to the energy projection operator in Eq. (2.67). Therefore, We simplify Eq. (2.67), by using the mass shell condition, $p^2 + m^2 = 0$, $(-\gamma p + m)\psi = 0$ and $k^2 = 0$. The Dirac matrices in $[\cdot]$ are still complicated expressions. The following calculations can be simplified considerably, however, if we choose a convenient gauge in which the polarization vectors are orthogonal to initial electron momentum p_i where $i = 1, 2$. We finally obtain

$$\mathcal{A} = \alpha e_\mu(k_1, \lambda) e_\nu(k_2, \lambda) \overline{u}(\mathbf{p}_1, \sigma_1) \left[\frac{\gamma^\nu \gamma k_1 \gamma^\mu}{2p_1 k_1} + \frac{\gamma^\mu \gamma k_2 \gamma^\nu}{2p_1 k_2} + \frac{\gamma^\mu p_1^\nu}{p_1 k_1} + \frac{\gamma^\nu p_1^\mu}{p_1 k_2} \right] v(\mathbf{p}_2, \sigma_2). \tag{2.69}$$

2.3 Møller ($e^- e^- \to e^- e^-$) Scattering in Quantum Electrodynamics

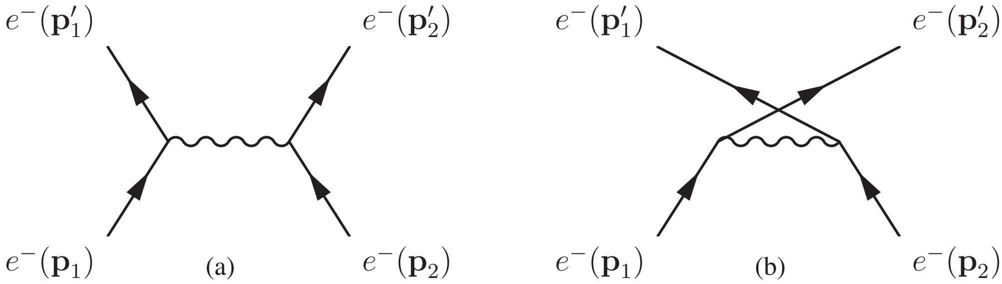

Figure 2.3 Feynman diagrams of $e^- e^- \to e^- e^-$, so-called Møller scattering.

In this section, we will consider a electron-electron scattering process in QED, $e^- e^- \to e^- e^-$, so-called Møller scattering. There are two Feynman diagrams corresponding to this process, shown in figure **2.3** .

The amplitude corresponding to the processes in figure 2.3(a), can be easily written down by applying the vacuum-to-vacuum transition amplitude, derived in Appendix B. These is one of a pair production process that we may write from the vaccum-to-vacuum transition amplitude in coordinate space (see in diagram of figure 2.3(a))

$$\mathrm{i}e^2 \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z') \overline{\eta}(z) S_+(z, y) \gamma^\mu S_+(y, x) \eta(x) D_{\mu\nu}(y', y)$$
$$\times \overline{\eta}(z') S_+(z', y') \gamma^\nu S_+(y', x') \eta(x'), \tag{2.70}$$

where $\eta(z)$ and $\overline{\eta}(z')$ are the presence of external sources, $S_+(x, x')$ denoting the propagator of electron, defined in Eq. (2.2) and $D_{\mu\nu}(x,x')$ denoting the Feynman propagator of photon, defined in Eq. (2.3). By using conventional integration, keep in mind, given by

$$\int (\mathrm{d}x')\overline{\eta}(x')S_+(x',x) = \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}px}}{2p^0}\overline{\eta}(p)(-\gamma p+m) \quad \text{when } x'^0 > x^0, \tag{2.71}$$

$$\int (\mathrm{d}x')S_+(x,x')\eta(x') = \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}px}}{2p^0}(-\gamma p+m)\eta(p) \quad \text{when } x^0 > x'^0. \tag{2.72}$$

We then transform Eq. (2.70) by using the properties in Eqs. (2.71)–(2.72),and substitute them in Eq. (2.70). To do this, we then write the amplitude in momentum space of this processes as

$$\begin{aligned} &\mathrm{i}e^2\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-p_1')y}\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-p_2')y'}\Bigg[\,\mathrm{i}\!\int\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{1}{2p_1'^0}\overline{\eta}(p_1')(-\gamma p_1'+m)\gamma^\mu \\ &\times\mathrm{i}\!\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}(-\gamma p_1+m)\eta(p_1)\;\mathrm{i}\!\int\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{1}{2p_2'^0}\overline{\eta}(p_2')(-\gamma p_2'+m)\gamma^\nu \\ &\times\mathrm{i}\!\int\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}(-\gamma p_2+m)\eta(p_2)\Bigg]D_{\mu\nu}(y',y). \end{aligned} \tag{2.73}$$

In these two terms, the roles of initial and final electron are reversed. Therefore on the right-hand side in Eq. (2.73) invariant under the interchange electron. This symmetry, so called crossing symmetry, is automatically incorporated in the source theory, and is simply a consequence of particle statistics.

Now, the integration region is far away from the sources, and since the interaction certainly occurs later in times than the emissions and earlier than the detections, we have, in the interaction region,

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{m}{p^0}}\,\overline{\eta}(p)u(\mathbf{p},\sigma) = \mathrm{i}\eta^*_{\mathbf{p}\sigma}; \quad e^- \text{ detection}, \tag{2.74}$$

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{m}{p^0}}\overline{u}(\mathbf{p},\sigma)\eta(p) = \mathrm{i}\eta_{\mathbf{p}\sigma}; \quad e^- \text{ emission}, \tag{2.75}$$

and from standard notation of sum over all spin of electron, read as

$$(2m)\sum_{\sigma} u(\mathbf{p},\sigma)\overline{u}(\mathbf{p},\sigma) = (-\gamma p + m). \tag{2.76}$$

By substituting Eqs. (2.74)–(2.76) in Eq. (2.73). To do this, we then have

$$\begin{aligned}
&\mathrm{i}e^2 \int \frac{(\mathrm{d}k)}{(2\pi)^4}\frac{1}{k^2}\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-p_1'-k)y}\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-p_2'+k)y'} \\
&\times \sum_{\sigma_1,\sigma_2}\Bigg[\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{2m}{2p_1'^0}\overline{\eta}(p_1')u(\mathbf{p}_1,\sigma_1)\overline{u}(\mathbf{p}_1,\sigma_1)\gamma^\mu \mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{2m}{2p_1^0}u(\mathbf{p},\sigma_1)\overline{u}(\mathbf{p}_1,\sigma_1)\eta(p_1) \\
&\times \mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{2m}{2p_2'^0}\overline{\eta}(p_2')u(\mathbf{p}_2,\sigma_2)\overline{u}(\mathbf{p}_2,\sigma_2)\gamma^\nu \mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{2m}{2p_2^0}u(\mathbf{p}_2,\sigma_2)\overline{u}(\mathbf{p}_2,\sigma_2)\eta(p_2)\Bigg]g_{\mu\nu}.
\end{aligned} \tag{2.77}$$

The y- and y'-integrations can be performed immediately yielding

$$\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-p_1'-k)y} = (2\pi)^4\delta^4(p_1-p_1'-k), \tag{2.78}$$

$$\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-p_2'+k)y'} = (2\pi)^4\delta^4(p_2-p_2'+k). \tag{2.79}$$

Now the k-integration is easily done :

$$\begin{aligned}
&\int\frac{(\mathrm{d}k)}{(2\pi)^4}\frac{1}{k^2}(2\pi)^4\delta^4(p_1-p_1'-k)(2\pi)^4\delta^4(p_2-p_2'+k) \\
&\qquad = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\frac{1}{(p_1-p_1')^2}.
\end{aligned} \tag{2.80}$$

Therefore we obtain the amplitude of diagram in figure (**2.4** a), be written as

$$\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{m}{p_1'^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{m}{p_2'^0}}$$

$$\times\,\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\eta^*_{p_2'\sigma_2'}\eta^*_{p_1'\sigma_1'}[\overline{u}(p_1',\sigma_1')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_2',\sigma_2')\gamma^\nu u(p_2,\sigma_2)]\frac{g_{\mu\nu}}{(p_1-p_1')^2},\quad (2.81)$$

or

$$\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{m}{p_1'^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{m}{p_2'^0}}$$

$$\times\,\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\eta^*_{p_2'\sigma_2'}\eta^*_{p_1'\sigma_1'}[\overline{u}(p_1',\sigma_1')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_2',\sigma_2')\gamma_\mu u(p_2,\sigma_2)]\frac{1}{(p_1'-p_1)^2}.\quad (2.82)$$

Another one of a $e^-e^- \rightarrow e^-e^-$ process in figure 2.3(b). We again start with the vacuum-to-vacuum transition amplitude in coordinate space, given by

$$\mathrm{i}e^2\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(z')S_+(z',y)\gamma^\mu S_+(y,x)\eta(x)D_{\mu\nu}(y,y')$$

$$\times\,\overline{\eta}(z)S_+(z,y')\gamma^\nu S_+(y',x')\eta(x'),\quad (2.83)$$

where $\eta(z)$ and $\overline{\eta}(z')$ are the presence of external sources, $S_+(x,x')$ denoting the propagator of electron, defined in Eq. (2.2) and $D_{\mu\nu}(x,x')$ denoting the Feynman propagator of photon, defined in Eq. (2.3). We then transform Eq. (2.83) by using the properties in Eqs. (2.71)–(2.72),and substitute them in Eq. (2.83). To do this, we then write the amplitude in momentum space of this processes as

$$\mathrm{i}e^2\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-p_2')y}\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-p_1')y'}\Bigg[\,\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{1}{2p_2'^0}\overline{\eta}(p_2')(-\gamma p_2'+m)\gamma^\mu$$

$$\times\,\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}(-\gamma p_1+m)\eta(p_1)\ \mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{1}{2p_1'^0}\overline{\eta}(p_1')(-\gamma p_1'+m)\gamma^\nu$$

$$\times\, \mathrm{i}\!\int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}(-\gamma p_2 + m)\eta(p_2)\Bigg] D_{\mu\nu}(y,y') \tag{2.84}$$

By substituting Eqs. (2.74)–(2.76) in Eq. (2.84). To do this, we then have

$$\begin{aligned}
&\mathrm{i}e^2 \int \frac{(\mathrm{d}k)}{(2\pi)^4}\frac{1}{k^2}\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-p_2'+k)y}\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-p_1'-k)y'}\\
&\times \sum_{\sigma_1,\sigma_2}\Bigg[\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{2m}{2p_2'^0}\overline{\eta}(p_2')u(\mathbf{p}_2,\sigma_2)\overline{u}(\mathbf{p}_2,\sigma_2)\gamma^\mu \mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{2m}{2p_1^0}u(\mathbf{p}_1,\sigma_1)\overline{u}(\mathbf{p}_1,\sigma_1)\eta(p_1)\\
&\times \mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{2m}{2p_1'^0}\overline{\eta}(p_1')u(\mathbf{p}_1,\sigma_1)\overline{u}(\mathbf{p}_1,\sigma_1)\gamma^\nu \mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{2m}{2p_2^0}u(\mathbf{p}_2,\sigma_2)\overline{u}(\mathbf{p}_2,\sigma_2)\eta(p_2)\Bigg] g_{\mu\nu},
\end{aligned} \tag{2.85}$$

The y- and y'-integrations can be performed immediately yielding

$$\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-p_2'+k)y} = (2\pi)^4\delta^4(p_1 - p_2' + k), \tag{2.86}$$

$$\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-p_1'-k)y'} = (2\pi)^4\delta^4(p_2 - p_1' - k). \tag{2.87}$$

Now the k-integration is easily done :

$$\begin{aligned}
&\int \frac{(\mathrm{d}k)}{(2\pi)^4}\frac{1}{k^2}(2\pi)^4\delta^4(p_1 - p_2' + k)(2\pi)^4\delta^4(p_2 - p_1' - k)\\
&\qquad = (2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2')\frac{1}{(p_2 - p_1')^2}.
\end{aligned} \tag{2.88}$$

Therefore we obtain the amplitude of diagram in figure 2.3(b), be written as

$$\begin{aligned}
&\mathrm{i}e^2(2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2')\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{m}{p_1'^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{m}{p_2'^0}}\\
&\times \eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\eta^*_{p_2'\sigma_2'}\eta^*_{p_1'\sigma_1'}[\overline{u}(p_2',\sigma_2')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_1',\sigma_1')\gamma^\nu u(p_2,\sigma_2)]\frac{g_{\mu\nu}}{(p_1 - p_2')^2},
\end{aligned} \tag{2.89}$$

or

$$\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{m}{p_1'^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{m}{p_2'^0}}$$

$$\times\,\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\eta^*_{p_2'\sigma_2'}\eta^*_{p_1'\sigma_1'}[\overline{u}(p_2',\sigma_2')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_1',\sigma_1')\gamma_\mu u(p_2,\sigma_2)]\frac{1}{(p_2-p_1')^2}. \quad (2.90)$$

From the amplitudes that are derived in Eq. (2.82) and Eq. (2.90), and the property of the Fermion statistics, the transition amplitude of a process $e^-e^-\to e^-e^-$, denote as $\mathcal{A}$ equal to Eq. (2.82)-Eq. (2.90), to lowest order in the fine-structure constant α, the amplitude for a process $e^-e^-\to e^-e^-$ is, up to unimportant factors for the problem at hand, in a standard notation

$$\mathcal{A}=\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{m}{p_1'^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{m}{p_2'^0}}$$

$$\times\,\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\eta^*_{p_2'\sigma_2'}\eta^*_{p_1'\sigma_1'}\left\{[\overline{u}(p_1',\sigma_1')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_2',\sigma_2')\gamma_\mu u(p_2,\sigma_2)]\frac{1}{(p_1-p_1')^2}\right.$$

$$\left.-[\overline{u}(p_2',\sigma_2')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_1',\sigma_1')\gamma_\mu u(p_2,\sigma_2)]\frac{1}{(p_2-p_1')^2}\right\}. \quad (2.91)$$

Note that we introduce the new amplitude for the this process is easily extracted to be

$$\mathcal{A}=\mathrm{i}e^2(2\pi)^4\left\{[\overline{u}(p_1',\sigma_1')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_2',\sigma_2')\gamma_\mu u(p_2,\sigma_2)]\frac{1}{(p_1'-p_1)^2}\right.$$

$$\left.-[\overline{u}(p_2',\sigma_2')\gamma^\mu u(p_1,\sigma_1)][\overline{u}(p_1',\sigma_1')\gamma_\mu u(p_2,\sigma_2)]\frac{1}{(p_2'-p_1)^2}\right\}, \quad (2.92)$$

where we neglected an unimportant multiplicative factor

$$\delta^4(p_1+p_2-p_1'-p_2')\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3}\frac{m}{p_1'^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^3}\frac{m}{p_2'^0}}$$

$$\times\,\eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\eta^*_{p_2'\sigma_2'}\eta^*_{p_1'\sigma_1'}. \quad (2.93)$$

2.4 Photons Production in Pair Annihilation in Scalar Electrodynamics

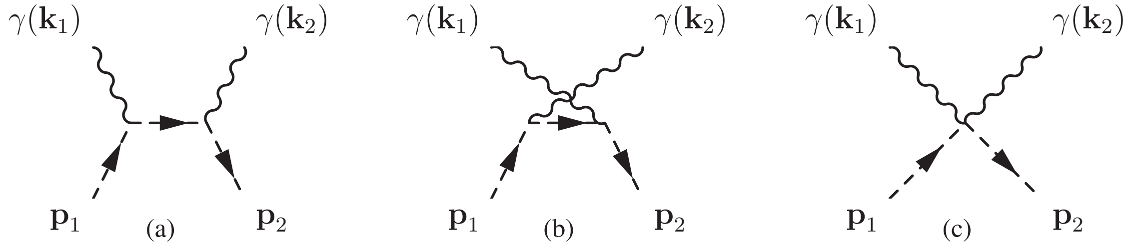

Figure 2.4 Feynman diagrams of spin 0 pair annihilation into two photons.

The annihilation of the particle and anti-particle which are spin $1/2$ particle, is a conceptually very interesting process, e^+e^- pair annihilation, studied in §2.1. In this section, we shall study the process by considering the example of the annihilation of pair spin 0 particles into photons. The three corresponding Feynman diagram are shown in figure **2.4** .

The amplitude corresponding to the processes in figure 2.4(a), can be easily written down by applying the vacuum-to-vacuum transition amplitude, derived in Appendix D. These is one of a pair annihilation process that we may write from the vacuum-to-vacuum transition amplitude in coordinate space (see in diagram of figure 2.4(a)) as

$$-\,\mathrm{i}e^2\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')J^{\alpha}(z)J^{\rho}(z')D_{\mu\alpha}(z,y)D_{\nu\rho}(z',y')$$
$$\times\, K^{\dagger}(x')[\partial_{\nu}^{y'}\Delta_+(x',y')]\Delta_+(y',y)[\partial_{\mu}^{y}\Delta_+(y,x)]K(x), \tag{2.94}$$

where $K(x)$, $K^{\dagger}(x')$, $J^{\alpha}(z)$ and $J^{\rho}(z')$ are the presence of external sources, denoting the Feynman propagator of spin 0 particle is

$$\Delta_+(x,x')=\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2+m^2-\mathrm{i}\varepsilon},\quad \varepsilon\to+0, \tag{2.95}$$

and denoting the Feynman propagator of photon is

$$D_{\mu\nu}(x,x') = \int \frac{(\mathrm{d}q)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}q(x-x')}}{q^2 - \mathrm{i}\varepsilon} g_{\mu\nu}, \quad \varepsilon \to +0. \tag{2.96}$$

With these conventions in hand, we will rewrite the vacuum-to-vacuum transition amplitude in Eq. (2.1) in momentum space. We then use the properties of the Fourier transform in Eq. (2.4). By using conventional integration, keep in mind, given by

$$\int (\mathrm{d}x') K^{\dagger}(x') [\partial_x^{\nu} \Delta_+(x',x)] = \mathrm{i} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{p^{\nu} \mathrm{e}^{\mathrm{i}px}}{2p^0} K^{\dagger}(-p) \quad \text{when } x'^0 > x^0, \tag{2.97}$$

$$\int (\mathrm{d}x') [\partial_x^{\mu} \Delta_+(x,x')] K(x') = \mathrm{i} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{p^{\mu} \mathrm{e}^{\mathrm{i}px}}{2p^0} K(p) \quad \text{when } x^0 > x'^0, \tag{2.98}$$

$$\int (\mathrm{d}x') J^{\mu}(x') D_{\mu\nu}(x',x) = \mathrm{i} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\mathrm{e}^{-\mathrm{i}kx}}{2k^0} J_{\nu}^{*}(k) \quad \text{when } x'^0 > x^0, \tag{2.99}$$

where $K(x)$ denoting the external source in the coordinate space for spin 0 particles, may be written in the momentum space:

$$K(p) = \int (\mathrm{d}x) \mathrm{e}^{-\mathrm{i}px} K(x), \tag{2.100}$$

and $K^{\dagger}(x')$ denoting the external source in the coordinate space for anti-spin 0 particles, may be written in the momentum space:

$$K^{\dagger}(-p) = \int (\mathrm{d}x) \mathrm{e}^{-\mathrm{i}px} K^{\dagger}(x). \tag{2.101}$$

To transform Eq. (2.94), we substitute Eqs. (2.97)–(2.101) in Eq. (2.94) and see the diagram in figure 2.4(a). We then write the vacuum-to-vacuum transition amplitude in momentum space of a process in figure 2.4(a) as:

$$-\mathrm{i}e^2 \int (\mathrm{d}y) \mathrm{e}^{\mathrm{i}(p_1-k_1)y} \int (\mathrm{d}y') \mathrm{e}^{\mathrm{i}(p_2-k_2)y'} \mathrm{i} \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3} \frac{1}{2k_1^0} J_{\mu}^{*}(k_1) \mathrm{i} \int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3} \frac{1}{2k_2^0} J_{\nu}^{*}(k_2)$$

$$\times\,\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{p_2^\nu}{2p_2^0}K^\dagger(p_2)\mathrm{i}\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p(y'-y)}}{p^2+m^2}\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{p_1^\mu}{2p_1^0}K(p_1). \tag{2.102}$$

The y- and y'-integrations can be performed immediately yielding

$$\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-k_1-p)y}=(2\pi)^4\delta^4(p_1-k_1-p), \tag{2.103a}$$

$$\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-k_2+p)y'}=(2\pi)^4\delta^4(p_2-k_2+p). \tag{2.103b}$$

Now the p-integration is easily done :

$$\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{1}{p^2+m^2}(2\pi)^4\delta^4(p_1-k_1-p)(2\pi)^4\delta^4(p_2-k_2+p)$$

$$=(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\frac{1}{(p_1-k_1)^2+m^2}. \tag{2.104}$$

Hence we can rewrite the vacuum-to-vacuum transition amplitude in momentum space of the process in figure 2.4(a) by substituting Eqs. (2.103a)–(2.103b), Eq. (2.104) and Eqs. (2.12)–(2.13) in Eq. (2.102), and change integral form to summation form, be written as

$$-\,\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sum\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J_\mu^*(k_1)\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J_\nu^*(k_2)$$

$$\times\,\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}K^\dagger(-p_2)K(p_1)\left[e_\nu^*\frac{p_2^\nu p_1^\mu}{(p_1-k_1)^2+m^2}e_\mu^*\right] \tag{2.105}$$

and for the emission source (the spin 0 particle and the anti-spin 0 particle) are represented by

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2p^0}}K(p)=\mathrm{i}K_{\mathbf{p}};\quad\text{spin 0 particle emission,} \tag{2.106a}$$

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2p^0}}K^\dagger(-p)=\mathrm{i}K_{\mathbf{p}}^*;\quad\text{anti-spin 0 particle emission,} \tag{2.106b}$$

for the detection source (photon) are represented by

$$\mathrm{i}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3}\frac{1}{2k_i^0}}e^{\mu}(\mathbf{k}_i,\lambda)J_{\mu}^*(k_i) = \mathrm{i}J_{\mathbf{k}_i\lambda}^*, \tag{2.107}$$

where $e^{\mu}(\mathbf{k}_i,\lambda)$ are the polarization vector, $\lambda, i = 1, 2$.

From Eqs. (2.106a)–(2.106b) and Eq. (2.107) we obtain the amplitude of diagram in figure 2.4(a), be written as

$$-\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}}$$

$$\times J_{k_1\lambda}^* J_{k_2\lambda}^* \eta_{p_2\sigma_2}\eta_{p_1\sigma_1}\left[e_{\nu}^*\frac{p_2^{\nu}p_1^{\mu}}{(p_1-k_1)^2+m^2}e_{\mu}^*\right]. \tag{2.108}$$

From figure 2.4(b) that is a one of the three diagram of spin 0 pair annihilation process. We start with the vacuum-to-vacuum transition amplitude in coordinate space, given by

$$-\mathrm{i}e^2\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')J^{\alpha}(z)J^{\rho}(z')D_{\nu\alpha}(z,y')D_{\mu\rho}(z',y)$$

$$\times K^{\dagger}(x')[\partial_{\nu}^{y'}\Delta_+(x',y')]\Delta_+(y',y)[\partial_{\mu}^{y}\Delta_+(y,x)]K(x), \tag{2.109}$$

where $K(x)$, $K^{\dagger}(x')$, $J^{\alpha}(z)$ and $J^{\rho}(z')$ are the presence of external sources, $\Delta_+(x,x')$ denoting the Feynman propagator of spin 0 particle, defined in Eq. (2.95) and $D_{\mu\nu}(x,x')$ denoting the Feynman propagator of anti-spin 0 particle, defined in Eq. (2.96). Form figure 2.4(b) and by substituting Eqs. (2.106a)–(2.106b) and Eq. (2.107) in Eq. (2.109). We then rewrite the amplitude in momentum space as

$$-\mathrm{i}e^2\int(\mathrm{d}y)e^{\mathrm{i}(p_1-k_2)y}\int(\mathrm{d}y')e^{\mathrm{i}(p_2-k_1)z'}\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J_{\mu}^*(k_1)\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J_{\nu}^*(k_2)$$

$$\times\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{p_2^{\mu}}{2p_2^0}K^{\dagger}(-p_2)\mathrm{i}\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{e^{\mathrm{i}p(y'-y)}}{p^2+m^2}\mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{p_1^{\nu}}{2p_1^0}K(p_1). \tag{2.110}$$

The y- and z'-integrations can be performed immediately yielding

$$\int(\mathrm{d}y)\mathrm{e}^{\mathrm{i}(p_1-k_2+p)y} = (2\pi)^4\delta^4(p_1-k_2+p), \tag{2.111a}$$

$$\int(\mathrm{d}y')\mathrm{e}^{\mathrm{i}(p_2-k_1-p)z'} = (2\pi)^4\delta^4(p_2-k_1-p). \tag{2.111b}$$

The p-integration is easily done :

$$\int\frac{(\mathrm{d}p)}{(2\pi)^4}\frac{1}{p^2+m^2}(2\pi)^4\delta^4(p_2-k_1-p)(2\pi)^4\delta^4(p_1-k_2+p)$$
$$= (2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\frac{1}{(p_2-k_1)^2+m^2}. \tag{2.112}$$

From Eqs. (2.111a)–(2.111b) and Eq. (2.112), we can rewrite the amplitude in momentum space of this processes in simply form, in summation form, be written

$$-\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sum\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}J^*_\mu(k_1)\mathrm{i}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}J^*_\nu(k_2)$$
$$\times\,\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}\mathrm{i}\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}K^\dagger(-p_2)K(p_1)\left[e^*_\mu\frac{p_2^\mu p_1^\nu}{(p_1-k_1)^2+m^2}e^*_\nu\right]. \tag{2.113}$$

where $e^\mu(\mathbf{k}_i,\lambda)$ are the polarization vector, $\lambda, i = 1,2$.

From Eqs. (2.106a)–(2.106b) and Eq. (2.107) we obtain the amplitude of diagram in figure 2.4(b), be written as

$$-\mathrm{i}e^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}}$$
$$\times\,J^*_{k_1\lambda}J^*_{k_2\lambda}K_{p_2}K_{p_1}\left[e^*_\mu\frac{p_2^\mu p_1^\nu}{(p_1-k_1)^2+m^2}e^*_\nu\right]. \tag{2.114}$$

Now we consider figure 2.4(c) that is the last diagram of spin 0 pair annihilation process. We again start with the vacuum-to-vacuum transition amplitude in coordinate space, given by

$$
\begin{aligned}
&- \mathrm{i}\mathrm{e}^2 \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}z') J^{\alpha}(z) J^{\rho}(z') D_{\nu\alpha}(z', y) D_{\mu\rho}(z, y) \\
&\quad \times K^{\dagger}(x') \Delta_{+}(x', y) \Delta_{+}(y, x) K(x)
\end{aligned} \tag{2.115}
$$

where $K(x)$, $K^{\dagger}(x')$, $J^{\alpha}(z)$ and $J^{\rho}(z')$ are the presence of external sources, $\Delta_{+}(x, x')$ denoting the Feynman propagator of spin 0 particle, defined in Eq. (2.95) and $D_{\mu\nu}(x, x')$ denoting the Feynman propagator of anti-spin 0 particle, defined in Eq. (2.96). Form figure 2.4(c) and by substituting Eqs. (2.106a)–(2.106b) and Eq. (2.107) in Eq. (2.115). We then rewrite the amplitude in momentum space as

$$
\begin{aligned}
&- \mathrm{i}e^2 \int (\mathrm{d}y) \mathrm{e}^{\mathrm{i}(p_1 + p_2 - k_1 - k_2)y} \mathrm{i} \int \frac{\mathrm{d}^3 \mathbf{k}_1}{(2\pi)^3} \frac{1}{2k_1^0} J_{\nu}^{*}(k_1) \mathrm{i} \int \frac{\mathrm{d}^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{2k_2^0} J_{\mu}^{*}(k_2) \\
&\times \mathrm{i} \int \frac{\mathrm{d}^3 \mathbf{p}_2}{(2\pi)^3} \frac{1}{2p_2^0} K^{\dagger}(-p_2) \mathrm{i} \int \frac{\mathrm{d}^3 \mathbf{p}_1}{(2\pi)^3} \frac{1}{2p_1^0} K(p_1)
\end{aligned} \tag{2.116}
$$

The y-integration is easily done :

$$
\int (\mathrm{d}y) \mathrm{e}^{\mathrm{i}(p_1 + p_2 - k_1 - k_2)y} = (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2). \tag{2.117}
$$

From Eq. (2.117), we can rewrite the amplitude in momentum space of this processes in simply form, in summation form, be written

$$
\begin{aligned}
&- \mathrm{i}\mathrm{e}^2 (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) \sum \mathrm{i} \frac{\mathrm{d}^3 \mathbf{k}_1}{(2\pi)^3} \frac{1}{2k_1^0} J_{\mu}^{*}(k_1) \mathrm{i} \frac{\mathrm{d}^3 \mathbf{k}_2}{(2\pi)^3} \frac{1}{2k_2^0} J_{\nu}^{*}(k_2) \\
&\times \mathrm{i} \frac{\mathrm{d}^3 \mathbf{p}_2}{(2\pi)^3} \frac{1}{2p_2^0} \mathrm{i} \frac{\mathrm{d}^3 \mathbf{p}_1}{(2\pi)^3} \frac{1}{2p_1^0} K^{\dagger}(-p_2) K(p_1) \left[e_{\mu}^{*} e_{\nu}^{*} \right].
\end{aligned} \tag{2.118}
$$

From Eqs. (2.106a)–(2.106b) and Eq. (2.107) in Eq. (2.118), we then rewrite the amplitude in momentum space as

$$- \mathrm{ie}^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}}$$

$$\times J^*_{k_1\lambda}J^*_{k_2\lambda}K_{p_2}K_{p_1}\left[e^*_\mu e^*_\nu\right]. \tag{2.119}$$

From the amplitudes that are derived in Eq. (2.108), Eq. (2.114) and Eq. (2.119), the transition amplitude of this process, denote as $\mathcal{A}$, to lowest order in the fine-structure constant α, the amplitude for the spin 0 pair annihilation into two photons process is, up to unimportant factors for the problem at hand, in a standard notation ($e_\mu(k_1,\lambda) \equiv e^*_\mu$ and $e_\nu(k_2,\lambda) \equiv e^*_\nu$)

$$\mathcal{A} = -\mathrm{ie}^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}}$$

$$\times J^*_{k_1\lambda}J^*_{k_2\lambda}K_{p_2}K_{p_1}\Bigg[e_\nu(k_2,\lambda)\frac{p_2^\nu p_1^\mu}{(p_1-k_1)^2+m^2}e_\mu(k_1,\lambda)$$

$$+ e_\mu(k_1,\lambda)\frac{p_2^\mu p_1^\nu}{(p_1-k_2)^2+m^2}e_\nu(k_2,\lambda) + e_\mu(k_1,\lambda)e_\nu(k_2,\lambda)\Bigg]. \tag{2.120}$$

To lowest order in fine-structure constant, the amplitude for the this process is easily extracted to be

$$\mathcal{A} = \alpha\Bigg[e_\nu(k_2,\lambda)\frac{p_2^\nu p_1^\mu}{(p_1-k_1)^2+m^2}e_\mu(k_1,\lambda)$$

$$+ e_\mu(k_1,\lambda)\frac{p_2^\mu p_1^\nu}{(p_1-k_2)^2+m^2}e_\nu(k_2,\lambda) + e_\mu(k_1,\lambda)e_\nu(k_2,\lambda)\Bigg], \tag{2.121}$$

where we neglected an unimportant multiplicative factor

$$- \delta^4(p_1+p_2-k_1-k_2)\sqrt{\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{1}{2k_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2k_2^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}}$$

$$\times J^*_{k_1\lambda} J^*_{k_2\lambda} K_{p_2} K_{p_1}. \tag{2.122}$$

We simplify Eq. (2.122), by using the mass shell condition, $p^2 + m^2 = 0$ and $k^2 = 0$. The Dirac matrices in $[\cdot]$ are still complicated expressions. The following calculations can be simplified considerably, we obtain

$$\mathcal{A} = \alpha e_\mu(k_1, \lambda) e_\nu(k_2, \lambda) \left[-\frac{p_2^\nu p_1^\mu}{2p_1 k_1} - \frac{p_2^\mu p_1^\nu}{2p_1 k_2} + g^{\mu\nu} \right] \tag{2.123}$$

in simplify, we take out $-1/2$ that is some constant, and be neglected.

$$\mathcal{A} \propto \alpha e_\mu(k_1, \lambda) e_\nu(k_2, \lambda) \left[\frac{p_2^\nu p_1^\mu}{p_1 k_1} + \frac{p_2^\mu p_1^\nu}{p_1 k_2} - 2g^{\mu\nu} \right] \tag{2.124}$$

where $e_1^\mu(\lambda)$ and $e_2^\nu(\lambda)$ are the polarization vectors

2.5 $e^+e^- \to \mu^+\mu^-$ in The Weinberg-Salam Electro-Weak Theory

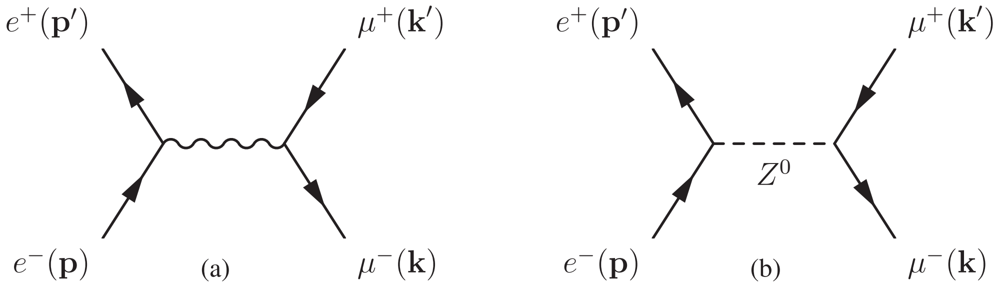

Figure 2.5 Feynman diagrams of $e^+e^- \to \mu^+\mu^-$ process.

Earlier we discussed the elastic scattering process for the electron-positron pair. If the energy of the incoming electron-positron pair is high enough, the scattering need not be elastic. In the final state, we can obtain a particle-anti-particle pair corresponding to a higher mass. Among the known elementary charged particles, the electron is the lightest, and the next lightest is the muon.

The Lagrangian for the process in figure **2.5** is

$$\mathscr{L} = -\sum_i \psi_i \left(\mathrm{i}\gamma\partial - m_i - \frac{g m_i H}{2M_{\mathrm{W}}} \right) \psi_i - e \sum_i q_i \overline{\psi}_i \gamma^\mu \psi_i A_\mu$$

$$-\frac{g}{2\cos\theta_{\rm W}}\sum_i \overline{\psi}_i\gamma^\mu(g_V^i - g_A^i\gamma^5)\psi_i Z_\mu \tag{2.125}$$

where $\theta_{\rm W} \equiv \tan^{-1}(g'/g)$ is the Weinberg angle; $e = g\sin\theta_{\rm W}$ is the positron electric charge; and $A \equiv B\cos\theta_{\rm W} + W^3\sin\theta_{\rm W}$ is the (massless) photon field. $Z \equiv -B\sin\theta_{\rm W} + W^3\cos\theta_{\rm W}$ is the neutral weak boson fields. The vector and axial-vector coupling are

$$g_V^i \equiv t_{3L}(i) - 2q_i\sin^2\theta_{\rm W}, \tag{2.126}$$

$$g_A^i \equiv t_{3L}(i), \tag{2.127}$$

where $t_{3L}(i)$ is the weak isospin of fermion i ($+1/2$ for u_i and ν_i; $-1/2$ for d_i and e_i) and q_i is the charge of ψ_i in units of e.

For momenta small compared to $M_{\rm W}$, this term gives rise to the effective four-fermion interaction with fermion constant given (at tree level, i.e., lowest order in perturbation theory) by $G_F/\sqrt{2} = g^2/8M_{\rm W}^2$.

The Feynman amplitude can then be written as

$$\begin{aligned}\mathcal{A} &= e^2\overline{U}(\mathbf{p}',\sigma')\gamma^\mu V(\mathbf{k}',s')\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2}\overline{v}(\mathbf{k},s)\gamma^\nu u(\mathbf{p},\sigma)\\ &+ \frac{g^2}{16\cos^2\theta_{\rm W}}\overline{U}(\mathbf{p}',\sigma')\gamma^\mu\left((1-4\sin^2\theta_{\rm W}) - \gamma_5\right)V(\mathbf{k}',s')\\ &\times\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)\frac{1}{q^2+M_Z^2}\overline{v}(\mathbf{k},s)\gamma^\nu\left((1-4\sin^2\theta_{\rm W}) - \gamma_5\right)u(\mathbf{p},\sigma)\\ &+ \frac{g^2}{e^2}\frac{m_\mu m_e}{4M_{\rm W}^2}\overline{U}(\mathbf{p}',\sigma')\gamma^\mu V(\mathbf{k}',s')\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2}\overline{v}(\mathbf{k},s)\gamma^\nu u(\mathbf{p},\sigma)\end{aligned} \tag{2.128}$$

We first compare the relative magnitudes of the coupling constants in the three matrix element

$$\mathcal{A}_y \sim e^2 \sim g^2\sin^2\theta_{\rm W} \tag{2.129}$$

$$\mathcal{A}_Z \sim \frac{g^2}{16\cos^2\theta_{\rm W}} \tag{2.130}$$

$$\mathcal{A}_H \sim \frac{g^2}{e^2}\frac{m_\mu m_e}{4M_{\rm W}^2} \sim 10^{-8}\frac{g^2}{e^2} \tag{2.131}$$

Obviously the contribution from the Higgs particle is totally negligible, whereas the matrix elements $\mathcal{A}_y$, $\mathcal{A}_Z$ are of the same order of magnitude at least at scattering energies in the range of mass of the intermediate boson. The reason for this is simple: we have used the Higgs field to generate masses of the intermediate bosons as well as of the leptons. Since the strength of the coupling between the Higgs particle and intermediate bosons is given

$$\mathcal{M} = \overline{U}(\mathbf{p}',\sigma')\gamma^\mu V(\mathbf{k}',s')\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2}\overline{v}(\mathbf{k},s)\gamma^\nu u(\mathbf{p},\sigma)$$
$$+ \frac{a}{q^2+M_Z^2}\overline{U}(\mathbf{p}',\sigma')\gamma^\mu\left(b-\gamma_5\right)V(\mathbf{k}',s')\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)\overline{v}(\mathbf{k},s)\gamma^\nu\left(b-\gamma_5\right)u(\mathbf{p},\sigma) \tag{2.132}$$

where $a = \dfrac{g^2}{16e^2\cos^2\theta_{\rm W}}$ and $b = 1 - 4\sin^2\theta_{\rm W}$.

2.6 e^+e^- Production from a Charged Nambu String

We have studied pair productions such as photons and electron-positron that are produced by interacting parent sources (point-like particles) in §2.1–§2.5. Here we generalize point-like particle sources to strings.

In this section deal with the particularly interesting situation of e^+e^- emission from a circularly oscillating Nambu string. We consider e^+e^- production by a Nambu string. We analytically calculate the amplitude of e^+e^- production from a closed string arising from the Nambu action as solution of a circularly oscillating closed string as perhaps the simplest object generalizing emissions from point-like particles within the framework of quantum electrodynamics. *Given* that such a process has occurred producing a mono-energetic pair e^+e^- with a given energy, we first calculate the amplitude of the e^+e^- produced in figure **2.6** , during one period of oscillation of the string, to lowest order in the fine-structure constant. The amplitude of this process (shown in

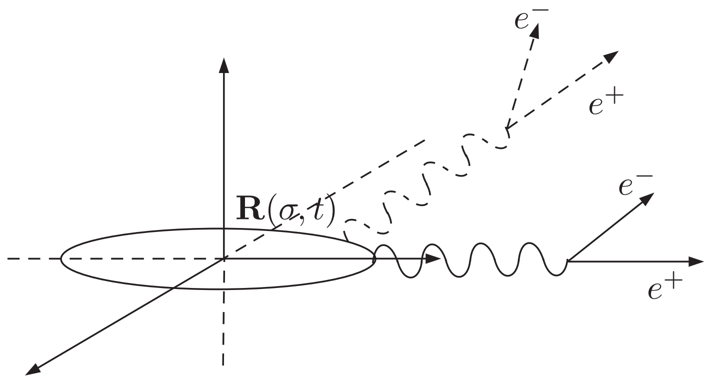

Figure 2.6 e^+e^- production from a charged Nambu string.

figure **2.6**), is given by

$$\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')J^{\nu}_{\mathrm{string}}(x)D_{\mu\nu}(x,x')J^{\mu}_{e^+e^-}(x'), \tag{2.133}$$

where $J^{\nu}_{\mathrm{string}}$ is the electromagnetic current associated with the string, and $J^{\mu}_{e^+e^-}$ is the electromagnetic current associated with the e^+e^- pair. These will be determined in subsequent chapters.

Form figure **2.6** , we consider photon annihilate into electron-positron, by using vacuum-to-vacuum transition amplitude in QED, we obtain

$$\mathcal{A} = -\mathrm{ie}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)J^{\nu}(x')D_{\mu\nu}(x,x')\overline{\eta}(y)S_+(y,x)\gamma^{\mu}S_+(x,z)\eta(z), \tag{2.134}$$

and we use the properties of the propagator in Eqs. (2.2)–(2.3), so that

$$\int(\mathrm{d}y)\overline{\eta}(y)S_+(y,x) = \mathrm{i}\int\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p^0_1}\mathrm{e}^{-\mathrm{i}p_1x}\overline{\eta}(p_1)(-\gamma p_1+m), \tag{2.135}$$

$$\int (\mathrm{d}z) S_+(x,z)\eta(z) = \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}\mathrm{e}^{-\mathrm{i}p_2 x}(\gamma p_2 + m)\eta(-p_2), \tag{2.136}$$

$$\int (\mathrm{d}x')\mathrm{e}^{-\mathrm{i}kx'} J^\nu(x') = J^\nu(k). \tag{2.137}$$

Then we obtain the momentum-space amplitude as

$$\begin{aligned}
\mathcal{A} = -\mathrm{i}e \int &(\mathrm{d}x)\mathrm{e}^{\mathrm{i}(k-(p_1+p_2))x}\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}\overline{\eta}(p_1)(-\gamma p_1 + m)\gamma^\mu \\
&\times \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}(\gamma p_2 + m)\eta(-p_2)\int \frac{\mathrm{d}k}{(2\pi)^4}\frac{J^\nu(k) g_{\mu\nu}}{k^2} \\
&= -\mathrm{i}e(2\pi)^4\delta^4(k-(p_1+p_2))\mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2p_1^0}\overline{\eta}(p_1)(-\gamma p_1 + m)\gamma^\mu \\
&\times \mathrm{i}\int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2p_2^0}(\gamma p_2 + m)\eta(-p_2)\int \frac{(\mathrm{d}k)}{(2\pi)^4}\frac{J_\mu(k)}{k^2}.
\end{aligned} \tag{2.138}$$

And we then consider the propagator of photon, we have

$$(2\pi)^4 \int \frac{(\mathrm{d}k)}{(2\pi)^4}\frac{J_\mu(k)}{k^2}\delta^4(k-(p_1+p_2)) = \frac{J_\mu(p_1+p_2)}{(p_1+p_2)^2}, \tag{2.139}$$

we use the properties of the propagator in Eqs. (2.2)–(2.3), giving

$$\mathcal{A} = \mathrm{i}(2\pi)e\sqrt{\frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{m}{p_1^0}}\sqrt{\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{m}{p_2^0}}\eta^*_{\mathbf{p}_1\sigma_1}\eta^*_{\mathbf{p}_2\sigma_2}J_\mu(p_1+p_2)\overline{u}(\mathbf{p}_1,\sigma_1)\frac{\gamma^\mu}{(p_1+p_2)^2}v(\mathbf{p}_2,\sigma_2) \tag{2.140}$$

We obtain the amplitude of this process delete removing the external source part ($\overline{\eta}$, η), as

$$\mathcal{A} = eJ_\mu(p_1+p_2)\overline{u}(\mathbf{p}_1,\sigma_1)\frac{\gamma^\mu}{(p_1+p_2)^2}v(\mathbf{p}_2,\sigma_2). \tag{2.141}$$

2.7 e^+e^- Production from a Neutral Nambu String

In this section we deal with the particularly interesting situation of e^+e^- emission from a circularly oscillating Nambu string. We consider e^+e^- production by a neutral Nambu string. We analytically calculate the amplitude of e^+e^- production from a closed string arising from the Nambu action as solution of a circularly oscillating closed string as perhaps the simplest object generalizing emissions from point-like particles within the framework of quantum electrodynamics. *Given* that such a process has occurred producing a mono-energetic pair e^+e^- with a given energy, we first calculate the amplitude of the e^+e^- produced in figure **2.7** , during one period of oscillation of the string, to lowest order in the fine-structure constant. The amplitude of this process

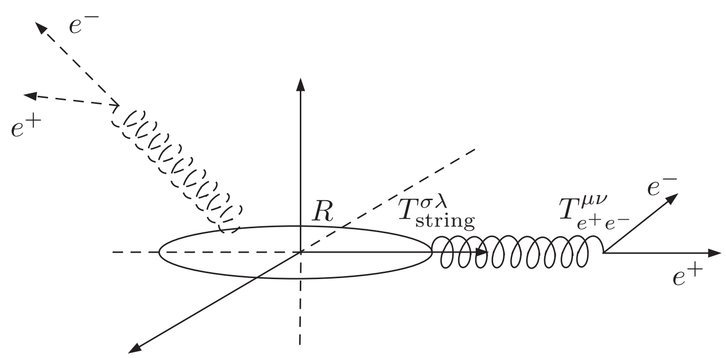

Figure 2.7 e^+e^- production from a neutral Nambu string.

(shown in figure **2.7**), is given by

$$\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')T^{\sigma\lambda}_{\rm string}(x)D_{\sigma\lambda\mu\nu}(x,x')T^{\mu\nu}_{e^+e^-}(x'), \tag{2.142}$$

where $T^{\mu\nu}_{e^+e^-}$ is the energy-momentum tensor associated with electron-positron, written as

$$T^{\mu\nu}_{e^+e^-} \propto \overline{u}[\gamma^\mu(p^\nu - p'^\nu) + \gamma^\nu(p^\mu - p'^\mu)]v, \tag{2.143}$$

p^μ and p'^μ are the momenta of electron and positron, $T^{\sigma\lambda}_{\rm string}$ the energy-momentum tensor associated with the string. The latter will be determined in Chapter VI.

CHAPTER III

EXPLICIT EXPRESSIONS FOR POLARIZATION CORRELATIONS OF SOME PROCESSES IN QUANTUM AND SCALAR ELECTRODYNAMICS

The purpose of this chapter is to derive and study explicit expressions of simultaneous measurements of two photons polarizations, so-called photon polarization correlations, with the two photons produced in e^+e^- annihilation in quantum electrodynamics. These polarization correlations are critical in investigations dealing with Bell's inequalities. In view of this, we carry out an analysis of our expressions for the correlations in the light of Bell's inequality. The analysis carried out in this chapter sets up the stage for the study of such concepts as entanglement in Chapter IV as well as considering the more complicated processes studied in Chapters V and VI. For completeness, we also investigate, in this chapter, photons productions in scalar electrodynamics in the framework of the above study. The Clauser-Horne (C-H) inequality for Bell's test, which is also critical in our study, is derived in the Appendix E to this chapter.

3.1 Polarization Correlations in e^+e^- Annihilation in QED

The purpose of this section is to derive the explicit joint probability distributions of photon ($\gamma\gamma$) polarization *correlations* in e^+e^- annihilation, in *flight* [Manoukian and Ungkitchanukit, 1994], in QED, as well as to obtain the corresponding probabilities when only one of the photon's polarization is measured. This provides clear cut *dynamical*, rather than kinematical, descriptions of photon polarizations correlations as follow directly from this monumental and experimentally reliable QED theory. Particle correlations have been systematically studied earlier [e.g., Manoukian, 1992, 1994, 1998; Manoukian and Ungkitchanukit, 1994] emphasizing, however, different experimental situations and aspects, polarizations phenomenae, but not correlation, were studied many years ago [McMaster, 1961], but we are, however, interested in correlations aspects that have been quite important experimentally in recent years [Clauser and Horne, 1974; Clauser and Shimoney, 1998; Fry, 1995; Selleri, 1988] in the light of the foundations of quantum physics vis-à-vis Bell-like inequalities. Two types of collisions are considered for e^+e^- annihilation in flight in the c.m. (center of mass) motion. The first one in which a e^+ and e^- in c.m., initially prepared to be moving along a specific axis, annihilate each other and two photons are observed to be moving along a given specific axis. Given that this process has occurred, we compute the conditional joint probabilities distributions of photon polarizations as well as the probabilities corresponding to the measurement of only one of the photon's polarization. The second one is involved with all repeated experiments corresponding to all orientations of the axis of motion of e^+e^- pairs in the c.m. initially prepared with the same speeds, and a pair of photons is observed moving along a given axis in each case after the annihilation process, given that these collisions mentioned above have occurred. In this latter case we must average over the initial orientations of axis along which the e^+e^- pair may initially move before annihilation occurs. With the explicit expressions for the probabilities derived from this quantum dynamical analysis, we finally show a clear violation of the relevant Bell-like inequality [Clauser and Horne, 1974; Clauser and Shimoney, 1998; Fry, 1995; Selleri, 1988] as a function of the speed of e^+ (or of e^-). Our convention for the metric tensor is $[g_{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$ (see Appendix A).

3.1.1 Computations of The Probability Distributions of Correlations

The transition probability of $e^+(p_1)e^-(p_2) \to \gamma(k_1)\gamma(k_2)$ (see in §2.1) to the leading order in the fine-structure constant α is, up to an important multiplicative factor for the problem at hand, given by (e.g., Itzykson and Zuber, 1980; Sokolov *et al.*, 1988)

$$\text{Prob} \propto \left[\frac{1}{4}\frac{(k_1 \cdot k_2)^2}{(p_1k_1)(p_1k_2)} - (\epsilon_{1\lambda} \cdot \epsilon_{2\lambda'})^2\right], \quad (3.1)$$

where

$$\epsilon^{\mu}_{1\lambda} = \left(\delta^{\mu}_{\nu} - \frac{p_{1\nu}k^{\mu}_{1}}{p_1k_1}\right) e^{\nu}_{1}(\lambda), \quad k_1e_1(\lambda) = 0, \quad (3.2)$$

$$\epsilon^{\mu}_{2\lambda'} = \left(\delta^{\mu}_{\nu} - \frac{p_{1\nu}k^{\mu}_{2}}{p_1k_2}\right) e^{\nu}_{2}(\lambda'), \quad k_2e_2(\lambda') = 0, \quad (3.3)$$

$e^{\nu}_{1,2}(\lambda)$ denote the polarization vectors of the photons satisfying the completeness relation

$$\sum_{\lambda} e^{\mu}_{i}(\lambda)e^{\nu}_{i}(\lambda) = g^{\mu\nu} - \frac{k^{\mu}_{i}\overline{k}^{\nu}_{i} + \overline{k}^{\mu}_{i}k^{\nu}_{i}}{k_ik_i}, \quad (3.4)$$

(no sum over i), where $k = (k^0, \mathbf{k})$, $\overline{k} = (k^0, -\mathbf{k})$. We note that $\epsilon^{\nu}_{i\lambda}$ are invariant under the gauge transformations $e^{\nu}_{i}(\lambda) \to e^{\nu}_{i}(\lambda) + k^{\nu}_{i}b_{\lambda}(k_i)$.

In the c.m. of a pair e^+e^-

$$\left.\begin{array}{c} \mathbf{p}_2 = -\mathbf{p}_1 \equiv \mathbf{p},\ \mathbf{k}_2 = -\mathbf{k}_1 \equiv \mathbf{k},\ p^0_1 = p^0_2 = k^0_1 = k^0_2 \equiv p^0 \\ k^0 = |\mathbf{k}|,\ p^0 = \sqrt{\mathbf{p}^2 + m^2} \end{array}\right\}. \quad (3.5)$$

In Fig. **3.1** we show how to introduce the polarization $e^{\mu}_{1}(\lambda) = (0, \mathbf{e}_1(\lambda))$ in reference to the vector k^{μ}. If $\mathbf{k}$ is chosen to lie in the x-z plane, then

$$\mathbf{k} := |\mathbf{k}|(\sin\theta, 0, \cos\theta), \quad (3.6)$$

and from the figure, with $\mathbf{e}_1(\lambda) \equiv \mathbf{e}_1$

$$\mathbf{e}_1 := (-\cos\theta\cos\chi_1, \sin\chi_1, \sin\theta\cos\chi_1), \quad (3.7)$$

where, here, $\mathbf{p} = |\mathbf{p}|(0, 0, 1)$. For a general orientation of $\mathbf{k}$ and $\mathbf{e}_1$, we must rotate the $x - y - z$ coordinate system c.w. (clockwise) about the z-axis by an angle ϕ. This

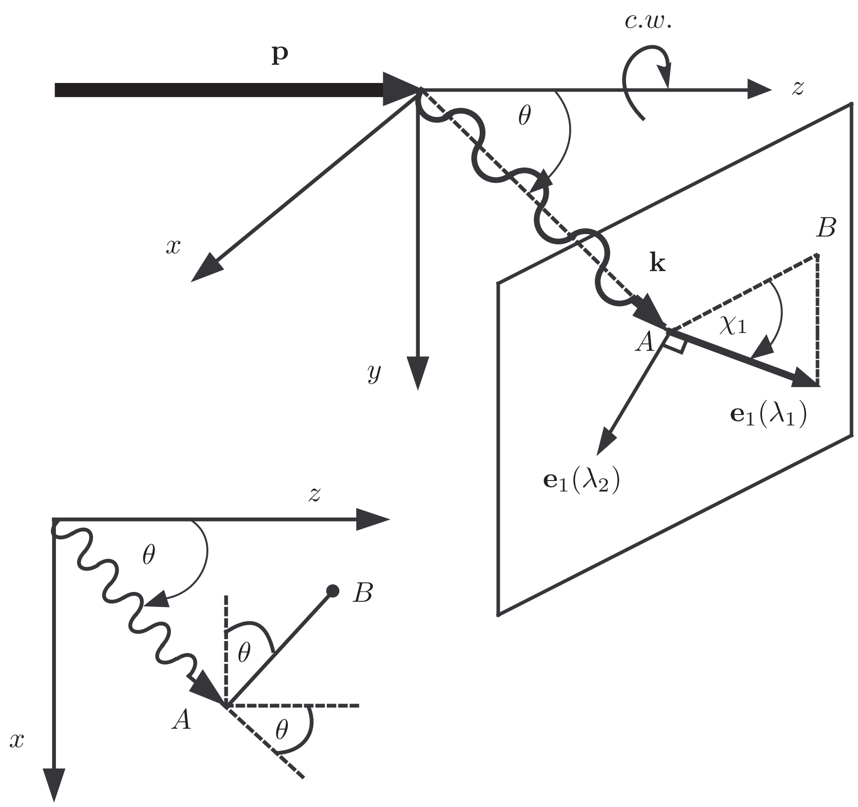

Figure 3.1 In this figure $\mathbf{k}$ lies in the x-z plane and $\mathbf{p}$ is along the z-axis. The polarizations vectors $\mathbf{e}_1(\lambda_1)$, $\mathbf{e}_1(\lambda_2)$ are orthogonal to each other and are orthogonal to $\mathbf{k}$. The line segment AB, of length $|\cos\chi_1|$, lies in the x-z plane. By rotating the coordinate system clockwise (c.w.) about the z-axis, by an angle ϕ, the vectors $\mathbf{k}$, $\mathbf{e}_1(\lambda_1)$, $\mathbf{e}_1(\lambda_2)$ will have general orientations in the resulting coordinate system.

accomplished by the rotation matrix R (see in Appendix F) with matrix elements:

$$R^{il} = \delta^{il} - \epsilon^{ijl}\frac{p^j}{|\mathbf{p}|}\sin\phi + \left(\delta^{il} - \frac{p^j p^l}{|\mathbf{p}|^2}\right)(\cos\phi - 1), \tag{3.8}$$

and from the property of the rotation of the vector $\mathbf{r}$ about the vector $\mathbf{n}$ with angle ϕ, $(r')^i = R^{il}r^i$ (see in Appendix F),where R^{il} is the rotation matrix (see in Eq. (3.8)), give the result as

$$\mathbf{k}' = \mathbf{k} - \sin\phi\left(\frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{k}\right) + \left[\mathbf{k} - \left(\mathbf{k}\cdot\frac{\mathbf{p}}{|\mathbf{p}|}\right)\frac{\mathbf{p}}{|\mathbf{p}|}\right](\cos\phi - 1), \tag{3.9}$$

and from figure **3.1** , $\mathbf{p} = |\mathbf{p}|(0, 0, 1)$, we have

$$\frac{\mathbf{p}}{|\mathbf{p}|} = \hat{z}, \tag{3.10}$$

where $\hat{z}$ denoting the unit vector in z-axis. Hence we can rewrite $\mathbf{k}$, after rotating about z-axis, as (we set $\mathbf{k} := (k^{(1)}, k^{(2)}, k^{(3)})$, compare with Eq. (3.6))

$$\begin{aligned} k'^{(1)} &= k^{(1)} - \sin\phi \left(\frac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{k}\right)^{(1)} + \left[k^{(1)} - \left(k^{(1)}(\hat{x} \cdot \hat{z})\right)\hat{z}\right](\cos\phi - 1) \\ &= k^{(1)} + k^{(1)}(\cos\phi - 1). \end{aligned} \tag{3.11}$$

By substituting $k^{(1)} \equiv |\mathbf{k}| \sin\theta$ from Eq. (3.6) in Eq. (3.11), gives

$$k'^{(1)} = k^{(1)} \cos\phi = |\mathbf{k}| \sin\theta \cos\phi, \tag{3.12}$$

where $\dfrac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{k} := (0, |\mathbf{k}| \sin\theta, 0)$, $\hat{x}$ denoting the unit vector in x-axis, For $k^{(2)}$ component that is rotated about z-axis with angle ϕ, using properties of Eq. (3.9) and $k^{(2)} = 0$ from Eq. (3.6), we have

$$\begin{aligned} k'^{(2)} &= k^{(2)} - \sin\phi \left(\frac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{k}\right)^{(2)} + \left[k^{(2)} - \left(k^{(2)}(\hat{y} \cdot \hat{z})\right)\hat{z}\right](\cos\phi - 1) \\ &= -|\mathbf{k}| \sin\theta \sin\phi, \end{aligned} \tag{3.13}$$

where $\hat{y}$ denoting the unit vector in y-axis. Finally, $k^{(3)}$ component that is rotated about z-axis with angle ϕ, using properties of Eq. (3.9) and $k^{(3)} \equiv |\mathbf{k}| \cos\theta$ from Eq. (3.6), gives

$$\begin{aligned} k'^{(3)} &= k^{(3)} - \sin\phi \left(\frac{\mathbf{p}}{|\mathbf{p}|} \times \mathbf{k}\right)^{(3)} + \left[k^{(3)} - \left(k^{(3)}(\hat{z} \cdot \hat{z})\right)\hat{z}\right](\cos\phi - 1) \\ &= k^{(3)} = |\mathbf{k}| \cos\theta. \end{aligned} \tag{3.14}$$

From Eqs. (3.12)–(3.14) of above computation, we transform $\mathbf{k}$ by rotating about z-axis with angle ϕ, be written as:

$$\mathbf{k}' := |\mathbf{k}|(\cos\phi\sin\theta, -\sin\phi\sin\theta, \cos\theta). \tag{3.15}$$

To convenient, by substituting $\mathbf{k}' \to \mathbf{k}$, gives

$$\mathbf{k} := |\mathbf{k}|(\cos\phi\sin\theta, -\sin\phi\sin\theta, \cos\theta). \tag{3.16}$$

As expected, and similarly, for the rotation of $\mathbf{e}_1$ about z-axis with angle ϕ, defined $\mathbf{e}_1 = (e_1^{(1)}, e_1^{(2)}, e_1^{(3)})$, we have

$$\begin{aligned} e_1'^{(1)} &= e_1^{(1)} - \sin\phi\left(\frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{e}_1\right)^{(1)} + \left[e_1^{(1)} - \left(e_1^{(1)}(\hat{x}\cdot\hat{z})\right)\hat{z}\right](\cos\phi - 1) \\ &= e_1^{(1)} + \sin\phi\sin\chi_1 + e_1^{(1)}(\cos\phi - 1) \end{aligned} \tag{3.17}$$

By substituting $e_1^{(1)} \equiv -\cos\theta\cos\chi_1$ from Eq. (3.7) in Eq. (3.17), gives

$$e_1'^{(1)} = -\cos\theta\cos\chi_1\cos\phi + \sin\phi\sin\chi_1, \tag{3.18}$$

where $\dfrac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{e}_1 := (-\sin\chi_1, -\cos\theta\cos\chi_1, 0)$. For $e_1^{(2)}$ component that is rotated about z-axis with angle ϕ, using properties of Eq. (3.9) and $e_1^{(2)} = \sin\chi_1$ from Eq. (3.7), we have

$$\begin{aligned} e_1'^{(2)} &= e_1^{(2)} - \sin\phi\left(\frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{e}_1\right)^{(2)} + \Big[e_1^{(2)} + \sin\phi\cos\theta\cos\chi_1 \\ &\qquad - \left(e_1^{(2)}(\hat{y}\cdot\hat{z})\right)\hat{z}\Big](\cos\phi - 1) \\ &= \sin\chi_1\cos\phi + \sin\phi\cos\theta\cos\chi_1. \end{aligned} \tag{3.19}$$

Final component $e_1^{(3)}$ that is rotated about z-axis with angle ϕ, using properties of Eq. (3.9) and $e_1^{(3)} \equiv \sin\theta\cos\chi_1$ from Eq. (3.7), giving

$$e_1'^{(3)} = e_1^{(3)} - \sin\phi\left(\frac{\mathbf{p}}{|\mathbf{p}|}\times\mathbf{e}_1\right)^{(3)} + \left[e_1^{(3)} - \left(e_1^{(3)}(\hat{z}\cdot\hat{z})\right)\hat{z}\right](\cos\phi - 1)$$

$$= \sin\theta\cos\chi_1. \tag{3.20}$$

From Eqs. (3.18)–(3.20) of above computation, we transform $\mathbf{e}_1$ by rotating about z-axis with angle ϕ, be written as:

$$\mathbf{e}_1 := (-\cos\theta\cos\chi_1\cos\phi + \sin\chi_1\sin\phi, \sin\chi_1\cos\phi + \cos\theta\cos\chi_1\sin\phi, \sin\theta\cos\chi_1), \tag{3.21}$$

in the resulting coordinate system. A similar expression for $\mathbf{e}_2(\lambda') \equiv \mathbf{e}_2$ is obtained by replacing χ_1 by χ_2. With $\mathbf{e}_1 \equiv \mathbf{e}_1(\lambda_1)$, $\mathbf{e}_1(\lambda_2)$ is obtained from $\mathbf{e}_1$ by the substitution $\chi_1 \to \chi_1 + \pi/2$.

Now we consider the probability of this process (see Eq. (3.1)), have derived in §2.1, in the convenient form. By using the properties in Eqs. (3.2)–(3.3), we have

$$\epsilon_1\epsilon_2 = \left(e_1 - \frac{(e_1\cdot p_1)k_1}{p_1k_1}\right)\left(e_2 - \frac{(e_2\cdot p_1)k_2}{p_1k_2}\right)$$

$$= e_1\cdot e_2 - \frac{(e_1\cdot p_1)(e_2\cdot p_1)(k_1k_2)}{(p_1k_1)(p_1k_2)}, \tag{3.22}$$

where $k_1e_2 = k_2e_1 = 0$, and $e_1\cdot e_2 = \mathbf{e}_1\cdot\mathbf{e}_2$, $e_1\cdot p_1 = \mathbf{e}_1\cdot\mathbf{p} = -e_2\cdot p_1$.

In the c.m. of e^+e^-, Eq. (3.1) may be rewritten in the convenient from

$$\text{Prob} \propto \left[\frac{1}{4}\frac{(k_1k_2)^2}{(p_1k_1)(p_1k_2)} - \left(\mathbf{e}_1\cdot\mathbf{e}_2 + \frac{\mathbf{e}_1\cdot\mathbf{p}\ \mathbf{e}_2\cdot\mathbf{p}\ (k_1k_2)}{(p_1k_1)(p_1k_2)}\right)^2\right]. \tag{3.23}$$

We treat two processes of annihilation associated with the relative probability given in Eq. (3.23).

3.1.2 Process 1: e^+e^- Moving Along z-axis

We consider the annihilation of e^+e^- pairs in flight in the c.m. (located at the origin of the coordinate system) initially prepared to be moving along the z-axis, as in the figure **3.1** , each moving with speed $v = \beta c$, prior to their annihilation into pairs of photons, and place detectors for the latter at opposite ends of the x-axis.

Using the scalar products

$$\mathbf{e}_i \cdot \mathbf{p} = |\mathbf{p}| \sin\theta \cos\chi_i, \quad \mathbf{p} \cdot \mathbf{k}_1 = |\mathbf{p}||\mathbf{k}| \cos\theta = -\mathbf{p} \cdot \mathbf{k}_2. \tag{3.24}$$

From Eq. (3.23), we rewrite the first term in bracket $[\cdot]$ in term of the speed (β) of initial particles (e^+, e^-) and by using properties in Eq. (3.5). Then we have scalar product

$$\begin{aligned} k_1 k_2 &= -|\mathbf{k}|^2 - (k^0)^2 \\ &= -(k^0)^2 - (k^0)^2 \\ &= -2(k^0)^2, \end{aligned} \tag{3.25}$$

and also using the property $|\mathbf{p}|/p^0 = \beta$ which is the speed, β, of initially particles, e^-, e^+, we have

$$\begin{aligned} p_1 k_1 &= \mathbf{p} \cdot \mathbf{k}_1 - p^0 k_1^0 \\ &= |\mathbf{p}||\mathbf{k}| \cos\theta - p^0 k^0 \\ &= -p^0 k^0 \left(1 - \frac{|\mathbf{p}|}{p^0} \cos\theta\right) \\ &= -(k^0)^2 (1 - \beta\cos\theta) \\ pk_1 &= -(k^0)^2 (1 - \beta\cos\theta). \end{aligned} \tag{3.26}$$

Similarly, we have finally scalar product to use in Eq. (3.23) as

$$
\begin{aligned}
p_1 k_2 &= \mathbf{p}\cdot\mathbf{k}_2 - p^0 k_2^0 \\
&= -|\mathbf{p}||\mathbf{k}|\cos\theta - p^0 k^0 \\
&= -p^0 k^0 (1 + \frac{|\mathbf{p}|}{p^0}\cos\theta) \\
&= -(k^0)^2 (1 + \beta\cos\theta) \\
pk_2 &= -(k^0)^2 (1 + \beta\cos\theta). \qquad (3.27)
\end{aligned}
$$

So that, we rewrite the first term in Eq. (3.23) in term of the speed (β) of initial particles (e^+, e^-), by substituting with Eqs. (3.25)–(3.27), be written as

$$
\frac{1}{4}\frac{(k_1 k_2)^2}{(p_1 k_1)(p_1 k_2)} = \frac{1}{1-\beta^2\cos^2\theta}. \qquad (3.28)
$$

After we rotate the system with $\phi = 2\pi$ about z-axis, from Eq. (3.21), we found that the system is same as the initial system ($\phi = 0$). We have

$$
\mathbf{e}_1 = (-\cos\theta\cos\chi_1, \sin\chi_1, \sin\theta\cos\chi_1), \qquad (3.29)
$$

$$
\mathbf{e}_2 = (-\cos\theta\cos\chi_2, \sin\chi_2, \sin\theta\cos\chi_2), \qquad (3.30)
$$

$$
\mathbf{p} = |\mathbf{p}|(0,0,1), \qquad (3.31)
$$

$$
\mathbf{k} = |\mathbf{k}|(\sin\theta, 0, \cos\theta), \qquad (3.32)
$$

and the scalar product of the two polarizations of photon, we write as:

$$
\mathbf{e}_1\cdot\mathbf{e}_2 = \cos^2\theta\cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2 + \sin^2\theta\cos\chi_1\cos\chi_2
$$

$$= (\cos^2\theta + \sin^2\theta)\cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2$$

$$= \cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2.$$

The latter works out, we obtain

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos(\chi_1 - \chi_2). \tag{3.33}$$

So that, we can rewrite Eq. (3.23) in the convenient form as

$$\text{Prob} \propto \frac{1}{(1-\beta^2\cos^2\theta)} - \left(\cos(\chi_1-\chi_2) - \frac{2|\mathbf{p}|^2\sin^2\theta\cos\chi_1\cos\chi_2(k^0)^2}{(k^0)^4(1-\beta^2\cos^2\theta)}\right)^2. \tag{3.34}$$

We consider the second term in Eq. (3.34) by using

$$\frac{|\mathbf{p}|}{k^0} = \frac{|\mathbf{p}|}{p^0} = \beta, \tag{3.35}$$

$$\beta^2\sin^2\theta = -(1-\beta^2) + (1-\beta^2\cos^2\theta). \tag{3.36}$$

Next we expand the second term of Eq. (3.34), given by

$$\left(\cos(\chi_1-\chi_2) - \frac{2\beta^2\sin^2\theta\cos\chi_1\cos\chi_2}{(1-\beta^2\cos^2\theta)}\right)^2$$

$$= \left(\cos(\chi_1-\chi_2) - \frac{2[-(1-\beta^2)+(1-\beta^2\cos^2\theta)]\cos\chi_1\cos\chi_2}{(1-\beta^2\cos^2\theta)}\right)^2$$

$$= \left(\cos(\chi_1-\chi_2) + \frac{2(1-\beta^2)}{1-\beta^2\cos^2\theta}\cos\chi_1\cos\chi_2 - 2\cos\chi_1\cos\chi_2\right)^2$$

$$= \cos^2(\chi_1-\chi_2) - 4\cos\chi_1\cos\chi_2 + 4\cos^2\chi_1\cos^2\chi_2$$

$$+ \frac{4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)^2} - \frac{8(1-\beta^2)\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)}$$

$$+\frac{4(1-\beta^2)\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2}{(1-\beta^2\cos^2\theta)}. \tag{3.37}$$

Therefore, we write the probability of this process in term of speed (β) of the initial particle (e^-, e^+) as

$$\begin{aligned}\text{Prob} \propto\ & \frac{1}{(1-\beta^2\cos^2\theta)} - \frac{4(1-\beta^2)\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2}{(1-\beta^2\cos^2\theta)} \\ & + \frac{8(1-\beta^2)\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)} - \frac{4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)^2} \\ & - (\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2)^2. \end{aligned} \tag{3.38}$$

A direct evaluation but tedious computation of the corresponding probability of occurrence with initial polarized electron and positron, be written as

$$\begin{aligned}\text{Prob} \propto\ & \frac{[1-4(1-\beta^2)\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)]}{(1-\beta^2\cos^2\theta)} \\ & - \frac{4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)^2} - [\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2]^2. \end{aligned} \tag{3.39}$$

where $\beta = |\mathbf{p}|/p^0$ is the speed of e^+ (or of e^-) divided by the speed of light, and θ is the angle between $\mathbf{k}$ and $\mathbf{p}$. We note that the angles χ_1, χ_2 have given fixed values when the vector $\mathbf{k}$ is made to rotate in the coordinate system.

Since θ is a continuous variable, we may integrate the expression in Eq. (3.39) over θ from $\pi/2-\delta$ to $\pi/2+\delta$ and then rigorously take the limit $\delta \to 0$ in evaluating the normalized probabilities in question. The ϕ-integral, here, is not important in evaluating these normalized probabilities since it leads to overall multiplicative factors that cancel out in the final expressions.

Upon using the integrals

$$\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{\sin\theta}{(1-\beta^2\cos^2\theta)}\,\mathrm{d}\theta = \frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right), \tag{3.40}$$

$$\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{\sin\theta}{(1-\beta^2\cos^2\theta)^2}\,\mathrm{d}\theta = \frac{1}{\beta}\left[\frac{\beta\sin\delta}{(1-\beta^2\sin^2\delta)} + \frac{1}{2}\ln\!\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right], \tag{3.41}$$

and we define

$$F_\delta(\chi_1,\chi_2) \equiv \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \mathrm{Prob}\sin\theta\mathrm{d}\theta. \tag{3.42}$$

Since the expression in Eq. (3.39) depend on speed (β) of initial particle, the angles χ_1, χ_2 and θ, but we integrate it over θ. Therefore another term are the constant, giving

$$\begin{aligned} F_\delta(\chi_1,\chi_2) = &\Big[1-4(1-\beta^2)\cos\chi_1\cos\chi_2\big(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2\big)\Big] \\ &\times \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{\sin\theta}{(1-\beta^2\cos^2\theta)}\,\mathrm{d}\theta \\ &- 4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2 \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \frac{\sin\theta}{(1-\beta^2\cos^2\theta)^2}\,\mathrm{d}\theta \\ &- \big(\cos\big(\chi_1-\chi_2\big)-2\cos\chi_1\cos\chi_2\big)^2 \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \sin\theta\,\mathrm{d}\theta. \end{aligned} \tag{3.43}$$

By using the integral from Eqs. (3.40)–(3.42), we obtain

$$\begin{aligned} &F_\delta(\chi_1,\chi_2) \\ &= \frac{[1-4(1-\beta^2)\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)]}{\beta}\ln\!\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\ &\quad - 4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)} + \frac{1}{2\beta}\ln\!\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right] \\ &\quad - 2\sin\delta(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)^2. \end{aligned} \tag{3.44}$$

To normalized the expression in Eq. (3.44), we have to sum $F_\delta(\chi_1,\chi_2)$ over the polarizations directions specified by the pairs of angles:

$$(\chi_1, \chi_2), \quad (\chi_1 + \frac{\pi}{2}, \chi_2), \quad (\chi_1, \chi_2 + \frac{\pi}{2}), \quad (\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}). \tag{3.45}$$

That is, we have to find the normalization factor

$$\begin{aligned} N_\delta = {} & F_\delta(\chi_1, \chi_2) + F_\delta(\chi_1 + \frac{\pi}{2}, \chi_2) \\ & + F_\delta(\chi_1, \chi_2 + \frac{\pi}{2}) + F_\delta(\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}). \end{aligned} \tag{3.46}$$

By considering each term in Eq. (3.46), for $(\chi_1 + \pi/2, \chi_2)$, substituting $\chi_1 \to \chi_1 + \pi/2$, we have

$$\begin{aligned} & F_\delta(\chi_1 + \frac{\pi}{2}, \chi_2) \\ & = \left[1 + 4(1-\beta^2)\sin\chi_1\cos\chi_2(-\sin(\chi_1 - \chi_2) + 2\sin\chi_1\cos\chi_2)\right]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\ & \quad - 4(1-\beta^2)^2\sin^2\chi_1\cos^2\chi_2\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)} + \frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right] \\ & \quad - 2\sin\delta(-\sin(\chi_1 - \chi_2) + 2\sin\chi_1\cos\chi_2)^2, \end{aligned} \tag{3.47}$$

for $(\chi_1 + \pi/2, \chi_2 + \pi/2)$, substituting $\chi_2 \to \chi_2 + \pi/2$, we have

$$\begin{aligned} & F_\delta(\chi_1, \chi_2 + \frac{\pi}{2}) \\ & = \left[1 + 4(1-\beta^2)\cos\chi_1\sin\chi_2(\sin(\chi_1 - \chi_2) + 2\cos\chi_1\sin\chi_2)\right]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\ & \quad - 4(1-\beta^2)^2\cos^2\chi_1\sin^2\chi_2\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)} + \frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right] \\ & \quad - 2\sin\delta(\sin(\chi_1 - \chi_2) + 2\cos\chi_1\sin\chi_2)^2. \end{aligned} \tag{3.48}$$

Finally, for $(\chi_1+\pi/2, \chi_2+\pi/2)$, substituting $\chi_1 \to \chi_1+\pi/2$, $\chi_2 \to \chi_2+\pi/2$, we have

$$
\begin{aligned}
&F_\delta(\chi_1+\frac{\pi}{2}, \chi_2+\frac{\pi}{2}) \\
&= \left[1-4(1-\beta^2)\sin\chi_1\sin\chi_2(\cos(\chi_1-\chi_2)-2\sin\chi_1\sin\chi_2)\right]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\
&\quad -4(1-\beta^2)^2\sin^2\chi_1\sin^2\chi_2\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)}+\frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right] \\
&\quad -2\sin\delta(\cos(\chi_1-\chi_2)-2\sin\chi_1\sin\chi_2)^2. \qquad (3.49)
\end{aligned}
$$

We can rewrite N_δ by substituting Eq. (3.44) and Eqs. (3.47)–(3.49) in Eq. (3.46), be written as

$$
\begin{aligned}
&= \Big\{[1-4(1-\beta^2)\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)] \\
&\quad +[1+4(1-\beta^2)\sin\chi_1\cos\chi_2(-\sin(\chi_1-\chi_2)+2\sin\chi_1\cos\chi_2)] \\
&\quad +[1+4(1-\beta^2)\cos\chi_1\sin\chi_2(\sin(\chi_1-\chi_2)+2\cos\chi_1\sin\chi_2)] \\
&\quad +[1-4(1-\beta^2)\sin\chi_1\sin\chi_2(\cos(\chi_1-\chi_2)-2\sin\chi_1\sin\chi_2)] \\
&\qquad \Big\}\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\
&\quad -4(1-\beta^2)^2\left[\frac{\sin\delta}{1-\beta^2\sin^2\delta}+\frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right] \\
&\qquad\qquad \times\{\cos^2\chi_1\cos^2\chi_2+\sin^2\chi_1\cos^2\chi_2+\cos^2\chi_1\sin^2\chi_2+\sin^2\chi_1\sin^2\chi_2\} \\
&\quad -2\sin\delta\Big\{(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)^2+(-\sin(\chi_1-\chi_2)+2\sin\chi_1\cos\chi_2)^2
\end{aligned}
$$

$$+ (\sin(\chi_1 - \chi_2) + 2\cos\chi_1 \sin\chi_2)^2 + (\cos(\chi_1 - \chi_2) - 2\sin\chi_1 \sin\chi_2)^2 \Big\}. \tag{3.50}$$

To simplify Eq. (3.50), we calculate each term as: the first term be written as

$$\begin{aligned}
&= 4 - 4(1-\beta^2)\Big\{ \cos\chi_1 \cos\chi_2(\cos(\chi_1 - \chi_2) - 2\cos\chi_1 \cos\chi_2)] \\
&\qquad - \sin\chi_1 \cos\chi_2(-\sin(\chi_1 - \chi_2) + 2\sin\chi_1 \cos\chi_2)] \\
&\qquad - \cos\chi_1 \sin\chi_2(\sin(\chi_1 - \chi_2) + 2\cos\chi_1 \sin\chi_2)] \\
&\qquad + \sin\chi_1 \sin\chi_2(\cos(\chi_1 - \chi_2) - 2\sin\chi_1 \sin\chi_2)]\Big\} \\
&= 4 + 4(1-\beta^2),
\end{aligned} \tag{3.51}$$

and the second term be written as

$$\cos^2\chi_1 \cos^2\chi_2 + \sin^2\chi_1 \cos^2\chi_2 + \cos^2\chi_1 \sin^2\chi_2 + \sin^2\chi_1 \sin^2\chi_2 = 1, \tag{3.52}$$

and the final term be written as

$$\begin{aligned}
&= (\cos(\chi_1 - \chi_2) - 2\cos\chi_1 \cos\chi_2)^2 + (-\sin(\chi_1 - \chi_2) + 2\sin\chi_1 \cos\chi_2)^2 \\
&\quad + (\sin(\chi_1 - \chi_2) + 2\cos\chi_1 \sin\chi_2)^2 + (\cos(\chi_1 - \chi_2) - 2\sin\chi_1 \sin\chi_2)^2.
\end{aligned}$$

To simplify above term on the right-hand side, we distribute square term

$$\begin{aligned}
&= \cos^2(\chi_1 - \chi_2) - 4\cos(\chi_1 - \chi_2)\cos\chi_1 \cos\chi_2 + 4\cos^2\chi_1 \cos^2\chi_2 \\
&\quad + \sin^2(\chi_1 - \chi_2) - 4\sin(\chi_1 - \chi_2)\sin\chi_1 \cos\chi_2 + 4\sin^2\chi_1 \cos^2\chi_2 \\
&\quad + \sin^2(\chi_1 - \chi_2) + 4\sin(\chi_1 - \chi_2)\cos\chi_1 \sin\chi_2 + 4\cos^2\chi_1 \sin^2\chi_2
\end{aligned}$$

$$+\cos^2(\chi_1-\chi_2) - 4\cos(\chi_1-\chi_2)\sin\chi_1\sin\chi_2 + 4\sin^2\chi_1\sin^2\chi_2,$$

and we then rearrange them as

$$\begin{aligned}
&= \cos^2(\chi_1-\chi_2) + \sin^2(\chi_1-\chi_2) + \sin^2(\chi_1-\chi_2) + \cos^2(\chi_1-\chi_2) \\
&- 4\{\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2 + \sin(\chi_1-\chi_2)\sin\chi_1\cos\chi_2 \\
&- \sin(\chi_1-\chi_2)\cos\chi_1\sin\chi_2 + \cos(\chi_1-\chi_2)\sin\chi_1\sin\chi_2\} \\
&+ 4\{\cos^2\chi_1\cos^2\chi_2) + \sin^2\chi_1\cos^2\chi_2 + \cos^2\chi_1\sin^2\chi_2 + \sin^2\chi_1\sin^2\chi_2)\} \\
&= 2 - 4\{\cos^2(\chi_1-\chi_2) + \sin^2(\chi_1-\chi_2)\} + 4 = 2. \qquad (3.53)
\end{aligned}$$

The latter works out to be

$$\begin{aligned}
N_\delta &= \left[4 + 4(1-\beta^2) - 2(1-\beta^2)^2\right]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\
&- 4(1-\beta^2)^2\frac{\sin\delta}{(1-\beta^2\sin^2\delta)} - 4\sin\delta. \qquad (3.54)
\end{aligned}$$

Therefore, given that the process has occurred as described above, with two photons moving (back-to-back) along the x-axis, the conditional joint probability of the photon polarizations, specified by the angles χ_1, χ_2, is rigorously given by

$$P(\chi_1,\chi_2) = \lim_{\delta\to 0}\frac{F_\delta(\chi_1,\chi_2)}{N_\delta}. \qquad (3.55)$$

For all $0 \leqslant \beta \leqslant 1$, we use the limit

$$\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \underset{\delta\to 0}{\sim} 2\delta. \qquad (3.56)$$

Here we take the limit $\delta \to 0$ that the physical meaning of δ is very small. Since

$\sin\delta \approx \delta$, and the expansion of $\dfrac{1}{1-\beta^2\sin^2\delta} \approx \dfrac{1}{1-\beta^2\delta^2} = 1-\beta^2\delta^2+\beta^4\delta^4-\ldots$ that we consider only first order of δ, because of the condition of the limit $\delta \to 0$. Therefore we obtain $\dfrac{1}{1-\beta^2\sin^2\delta} \approx 1$. The following result, we have

$$\begin{aligned}\lim_{\delta\to 0} F_\delta(\chi_1,\chi_2) &\approx [1-4(1-\beta^2)\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)] \\ &\quad -4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2-\big(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2\big)^2 \\ &= 1-[\cos(\chi_1-\chi_2)-2\beta^2\cos\chi_1\cos\chi_2]^2, \end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\lim_{\delta\to 0} N_\delta &\approx [4+4(1-\beta^2)]-4(1-\beta^2)^2-2 \\ &= [(4-2)+4(1-\beta^2)(1-(1-\beta^2))] \\ &= [2+4\beta^2(1-\beta^2)], \end{aligned} \tag{3.58}$$

where we neglect 2δ.

To obtain from Eqs. (3.57)–(3.58), the conditional joint probability of the photon polarizations is given by

$$P(\chi_1,\chi_2) = \frac{1-[\cos(\chi_1-\chi_2)-2\beta^2\cos\chi_1\cos\chi_2]^2}{2[1+2\beta^2(1-\beta^2)]}, \tag{3.59}$$

for all $0 \leqslant \beta \leqslant 1$.

If only one of the polarizations is measured, then we have to evaluate $F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1,\chi_2+\pi/2)$ and $F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1+\pi/2,\chi_2)$. To this end, Eq. (3.44) and Eq. (3.47)–(3.48) gives

$$F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1,\chi_2+\frac{\pi}{2}) = [2+2(1-\beta^4)\cos^2\chi_1]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$

$$-4(1-\beta^2)^2\cos^2\chi_1\frac{\sin\delta}{(1-\beta^2\sin^2\delta)}-2\sin\delta, \quad (3.60)$$

$$F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1+\frac{\pi}{2},\chi_2)=[2+2(1-\beta^4)\cos^2\chi_2]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$

$$-4(1-\beta^2)^2\cos^2\chi_2\frac{\sin\delta}{(1-\beta^2\sin^2\delta)}-2\sin\delta. \quad (3.61)$$

This is, the conditional probabilities associated with the measurement of only of the polarizations are given by

$$P(\chi_1,-)=\lim_{\delta\to 0}\frac{F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1,\chi_2+\pi/2)}{N_\delta}, \quad (3.62)$$

$$P(-,\chi_2)=\lim_{\delta\to 0}\frac{F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1+\pi/2,\chi_2)}{N_\delta}. \quad (3.63)$$

Similarly in Eq. (3.57), by taking the limit $\delta\to 0$, we have

$$\lim_{\delta\to 0}F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1,\chi_2+\frac{\pi}{2})=[2+2(1-\beta^4)\cos^2\chi_1]-4(1-\beta^2)^2\cos^2\chi_1-1$$

$$=1+4\beta^2(1-\beta^2)\cos^2\chi_1, \quad (3.64)$$

$$\lim_{\delta\to 0}F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1+\frac{\pi}{2},\chi_2)=[2+2(1-\beta^4)\cos^2\chi_2]-4(1-\beta^2)^2\cos^2\chi_2-1$$

$$=1+4\beta^2(1-\beta^2)\cos^2\chi_2. \quad (3.65)$$

From Eqs. (3.62)–(3.65), these work out to be simply given by

$$P(\chi_1,-)=\frac{1+4\beta^2(1-\beta^2)\cos^2\chi_1}{2[1+2\beta^2(1-\beta^2)]}, \quad (3.66)$$

$$P(-,\chi_2)=\frac{1+4\beta^2(1-\beta^2)\cos^2\chi_2}{2[1+2\beta^2(1-\beta^2)]}, \quad (3.67)$$

for *all* $0\leqslant\beta\leqslant 1$, and are, respectively, *dependent* on χ_1, χ_2.

We note the important statistical property that

$$P(\chi_1, \chi_2) \neq P(\chi_1, -)P(-, \chi_2) \tag{3.68}$$

in general.

In the notation of Local Hidden Variables (LHV) theory [Clauser and Horne, 1974; Clauser and Shimoney, 1978; Fry, 1995; Selleri, 1988], we have the identifications

$$P(\chi_1, \chi_2) = \frac{P_{12}(a_1, a_2)}{P_{12}(\infty, \infty)}, \tag{3.69}$$

$$P(\chi_1, -) = \frac{P_{12}(a_1, \infty)}{P_{12}(\infty, \infty)}, \tag{3.70}$$

$$P(-, \chi_2) = \frac{P_{12}(\infty, a_2)}{P_{12}(\infty, \infty)}, \tag{3.71}$$

where a_1, a_2 specify directions for measurements of polarizations. Defining (see Appendix E)

$$\begin{aligned} S = {} & P(\chi_1, \chi_2) - P(\chi_1, \chi_2') + P(\chi_1', \chi_2) \\ & + P(\chi_1', \chi_2') - P(\chi_1', -) - P(-, \chi_2), \end{aligned} \tag{3.72}$$

for four angles χ_1, χ_2, χ_1', χ_2', LHV theory gives the Bell-like bound (Clauser and Horne, 1974; Clauser and Shimoney, 1978):

$$-1 \leqslant S \leqslant 0. \tag{3.73}$$

It is sufficient to realize one experimental situation that violates the bounds in Eq. (3.73).

For example, for $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, Eq. (3.59), Eqs. (3.66)–(3.67), as obtained from QED, gives $S = 0.207$ for $\beta = 0$ that violates Eq. (3.73) from above. For $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi'_1 = 45°$, $\chi'_2 = 67°$, we obtain $S = -1.207$ for $\beta = 0$ violating Eq. (3.73) from below. Both bounds are violated for all $\beta \leqslant 0.2$ for these same angles, respectively.

3.1.3 Process 2: Two Photons Moving Along z-axis

Here we put the two detectors on opposite sides of the z-axis. We consider all repeated experiments with pairs e^+e^- produced in flight in the c.m. (located at the origin), each particle moving with speed $v = \beta c$, corresponding to all possible orientations of the axis along which a given pair moves. Here we must average over all angles θ, ϕ of the vector $\mathbf{p}$, with $\mathbf{k}$ along the z-axis.

In the present case

$$\mathbf{k} = |\mathbf{k}|(0, 0, 1), \tag{3.74}$$

$$\mathbf{p} = |\mathbf{p}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta), \tag{3.75}$$

$$\mathbf{e}_1 = (-\cos\chi_1, \sin\chi_1, 0), \tag{3.76}$$

$$\mathbf{e}_2 = (-\cos\chi_2, \sin\chi_2, 0). \tag{3.77}$$

We then obtain the scalar product:

$$\mathbf{e}_1 \cdot \mathbf{p} = -|\mathbf{p}|\sin\theta\cos(\phi + \chi_1), \tag{3.78}$$

$$\mathbf{e}_2 \cdot \mathbf{p} = -|\mathbf{p}|\sin\theta\cos(\phi + \chi_2), \tag{3.79}$$

thus obtaining for Eq. (3.29)

$$\text{Prob} \propto \frac{1}{(1-\beta^2\cos^2\theta)} + \frac{8(1-\beta^2)}{(1-\beta^2\cos^2\theta)}\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)$$

$$- \cos^2(\chi_1 - \chi_2) - \frac{4(1-\beta^2)\cos(\chi_1-\chi_2)\cos(\chi_1+\phi)\cos(\chi_2+\phi)}{(1-\beta^2\cos^2\theta)}$$

$$- 4\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi) + 4\cos(\chi_1-\chi_2)\cos(\chi_1+\phi)\cos(\chi_2+\phi)$$

$$- \frac{4(1-\beta^2)^2\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)}{(1-\beta^2\cos^2\theta)^2}, \tag{3.80}$$

where $\beta = |\mathbf{p}|/p^0$ is the speed of e^+ (or of e^-) divided by the speed of light, θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, and ϕ is the angle between $\mathbf{p}$ and x-axis. We note that the angles χ_1, χ_2 have given fixed values when the vector $\mathbf{k}$ is made to rotate in the coordinate system.

Since θ is a continuous variable, we may integrate the expression in Eq. (3.80) over θ from 0 to π, with the latter two deduced from Eqs. (3.40)–(3.41) by replacing δ by $\pi/2$. The ϕ-integral, here, is important in evaluating, not same as Process 1, we also integrate the expression in Eq. (3.80) over ϕ from 0 to 2π.

Upon using the integrals

$$\int_0^{2\pi} \mathrm{d}\phi \cos(\chi_1+\phi)\cos(\chi_2+\phi) = \pi\cos(\chi_1-\chi_2), \tag{3.81}$$

$$\int_0^{2\pi} \mathrm{d}\phi \cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi) = \frac{\pi}{4}[1+2\cos^2(\chi_1-\chi_2)], \tag{3.82}$$

and

$$\int_0^{\pi} \frac{\sin\theta \mathrm{d}\theta}{(1-\beta^2\cos^2\theta)} = \frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right), \tag{3.83}$$

$$\int_0^{\pi} \frac{\sin\theta \mathrm{d}\theta}{(1-\beta^2\cos^2\theta)^2} = \frac{1}{\beta}\left[\frac{\beta}{1-\beta^2} + \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]. \tag{3.84}$$

Since the integration of Eq. (3.80), $\iint \mathrm{Prob}\sin\theta \mathrm{d}\theta \mathrm{d}\phi$, be written as

$$\iint \mathrm{Prob}\sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

$$
\begin{aligned}
&\propto \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \frac{\sin\theta}{(1-\beta^2\cos^2\theta)}\,\mathrm{d}\theta - \cos^2(\chi_1-\chi_2)\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\mathrm{d}\theta \\
&+ 8(1-\beta^2)\int_0^{\pi}\frac{\sin\theta}{(1-\beta^2\cos^2\theta)}\,\mathrm{d}\theta\int_0^{2\pi}\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)\mathrm{d}\phi \\
&- 4(1-\beta^2)\cos(\chi_1-\chi_2)\int_0^{\pi}\frac{\sin\theta}{(1-\beta^2\cos^2\theta)}\,\mathrm{d}\theta\int_0^{2\pi}\cos(\chi_1+\phi)\cos(\chi_2+\phi)\mathrm{d}\phi \\
&- 4\int_0^{\pi}\sin\theta\mathrm{d}\theta\int_0^{2\pi}\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)\mathrm{d}\phi \\
&+ 4\int_0^{\pi}\sin\theta\mathrm{d}\theta\int_0^{2\pi}\cos(\chi_1-\chi_2)\cos(\chi_1+\phi)\cos(\chi_2+\phi)\mathrm{d}\phi \\
&- 4(1-\beta^2)^2\int_0^{\pi}\frac{\sin\theta}{(1-\beta^2\cos^2\theta)^2}\,\mathrm{d}\theta\int_0^{2\pi}\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)\mathrm{d}\phi. \quad (3.85)
\end{aligned}
$$

By using properties of the integral in Eqs. (3.81)–(3.84) and we set $F_\beta(\chi_1,\chi_2) \equiv \iint \text{Prob}\sin\theta\mathrm{d}\theta\mathrm{d}\phi$, gives

$$
\begin{aligned}
&F_\beta(\chi_1,\chi_2) \\
&= \frac{2\pi}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 4(1-\beta^2)\cos^2(\chi_1-\chi_2)(\frac{\pi}{\beta})\ln\left(\frac{1+\beta}{1-\beta}\right) - 4\pi\cos^2(\chi_1-\chi_2) \\
&+ 8(1-\beta^2)\left(\frac{\pi}{4}\right)[1+2\cos^2(\chi_1-\chi_2)]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) \\
&- 4(\frac{\pi}{4})(1-\beta^2)^2[1+2\cos^2(\chi_1-\chi_2)]\left[\frac{1}{1-\beta^2}+\frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right] \\
&+ 4\pi\cos^2(\chi_1-\chi_2)(2) - 4(\frac{\pi}{4})[1+2\cos^2(\chi_1-\chi_2)](2). \quad (3.86)
\end{aligned}
$$

We neglect 2π because it is the constant that cancel out after normalization, the above equation be rewritten as:

$$F_\beta(\chi_1,\chi_2)$$

$$\propto \left[1 - 2(1-\beta^2)\cos^2(\chi_1-\chi_2) + (1-\beta^2)[1+2\cos^2(\chi_1-\chi_2)]\right] \frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

$$- \frac{1}{2}(1-\beta^2)^2[1+2\cos^2(\chi_1-\chi_2)]\left[\frac{1}{1-\beta^2} + \frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]$$

$$+ 2\cos^2(\chi_1-\chi_2) - [1+2\cos^2(\chi_1-\chi_2)], \tag{3.87}$$

and to simplify the above term, we rewrite as

$$= \left[1 + (1-\beta^2)\right]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{1}{2}(1-\beta^2)\left[1+2\cos^2(\chi_1-\chi_2)\right] - 1$$

$$- \frac{1}{2}(1-\beta^2)^2\left[1+2\cos^2(\chi_1-\chi_2)\right]\left[\frac{1}{(1-\beta^2)} + \frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]$$

$$= \left[1 + (1-\beta^2) - \frac{1}{4}(1-\beta^2)^2\right]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{1}{2}(1-\beta^2) - 1$$

$$- \left[(1-\beta^2) + (1-\beta^2)^2\frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\cos^2(\chi_1-\chi_2)$$

$$= \frac{1}{4}\left[4 + 4(1-\beta^2) - (1-\beta^2)^2\right]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{3}{2} + \frac{\beta^2}{2}$$

$$- (1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\cos^2(\chi_1-\chi_2).$$

Finally, we have the convenient probability, be written as:

$$F_\beta(\chi_1,\chi_2) \propto \frac{[4+4(1-\beta^2)-(1-\beta^2)^2]}{4\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{3}{2} + \frac{\beta^2}{2}$$

$$- (1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\cos^2(\chi_1-\chi_2). \tag{3.88}$$

We introduce the new probability, given by

$$F_\beta(\chi_1,\chi_2) \propto [A(\beta) + B(\beta)\cos^2(\chi_1-\chi_2)], \tag{3.89}$$

where

$$A(\beta) = \frac{[4 + 4(1-\beta^2) - (1-\beta^2)^2]}{4\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{3}{2} + \frac{\beta^2}{2}, \tag{3.90}$$

$$B(\beta) = -(1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right)\right]. \tag{3.91}$$

To normalized the expression in Eq. (3.89), we have to sum $F_\delta(\chi_1, \chi_2)$ over the polarizations directions specified by the pairs of angles (see as Eq. (3.45)):

$$(\chi_1, \chi_2), \quad (\chi_1 + \frac{\pi}{2}, \chi_2), \quad (\chi_1, \chi_2 + \frac{\pi}{2}), \quad (\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}). \tag{3.92}$$

That is, we have to find the normalization factor

$$\begin{aligned} N_\beta = {} & F_\beta(\chi_1, \chi_2) + F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2) \\ & + F_\beta(\chi_1, \chi_2 + \frac{\pi}{2}) + F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}). \end{aligned} \tag{3.93}$$

The explicit expression of above term, by replacing χ_1 by $\chi_1 + \pi/2$, χ_2 by $\chi_2 + \pi/2$, gives

$$F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2) = A(\beta) + B(\beta)\sin^2(\chi_1 - \chi_2), \tag{3.94}$$

$$F_\beta(\chi_1, \chi_2 + \frac{\pi}{2}) = A(\beta) + B(\beta)\sin^2(\chi_1 - \chi_2), \tag{3.95}$$

$$F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}) = A(\beta) + B(\beta)\cos^2(\chi_1 - \chi_2). \tag{3.96}$$

The following computation for the normalization factor we have upon summing over the set in Eq. (3.92), by substituting Eq. (3.89), Eqs. (3.94)–(3.96) in Eq. (3.93).

Giving

$$N_\beta = \frac{[4(1-\beta^2) - 2(1-\beta^2)^2]}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 8 + 4\beta^2$$
$$\equiv 2[2A(\beta) + B(\beta)]. \tag{3.97}$$

According, for the joint conditional probabilities, using the properties in Eq. (3.69), we have

$$P_\beta(\chi_1, \chi_2) = \frac{A(\beta) + B(\beta)\cos^2(\chi_1 - \chi_2)}{2[2A(\beta) + B(\beta)]}, \tag{3.98}$$

given that the two photons have emerged (back-to-back) along the z-axis.

For the measurement of only one of the polarizations, Eqs. (3.70)–(3.71) leads to

$$P_\beta(\chi_1, -) = \frac{[2A(\beta) + B(\beta)]}{2[2A(\beta) + B(\beta)]} = \frac{1}{2} = P_\beta(-, \chi_2), \tag{3.99}$$

for all $0 \leqslant \beta \leqslant 1$, and the latter are, respectively, *independent* of χ_1, χ_2.

Again we have the important statistical property

$$P_\beta(\chi_1, \chi_2) \neq P_\beta(\chi_1, -)P_\beta(-, \chi_2), \tag{3.100}$$

in general. It is interesting to note that an equality in Eq. (3.101) holds in the extreme relativistic case $\beta \to 1$, where each side is equal to $1/4$.

Only in the limiting case $\beta \to 0$, the joint probability in Eq. (3.98) for this process coincides with that in Eq. (3.59) for the first process.

As in Eq. (3.72), we define

$$S_\beta = P_\beta(\chi_1, \chi_2) - P_\beta(\chi_1, \chi_2') + P_\beta(\chi_1', \chi_2)$$
$$+ P_\beta(\chi_1', \chi_2') - P_\beta(\chi_1', -) - P_\beta(-, \chi_2), \tag{3.101}$$

for four angles χ_1, χ_2, χ_1', χ_2', LHV theory gives [Clauser and Horne, 1974; Clauser and Shimoney, 1978]

$$-1 \leqslant S_\beta \leqslant 0. \tag{3.102}$$

For $\beta \to 1$, an equality holds in Eq. (3.101), $S_\beta \to -1/2$, and this process, to be useful for testing the violation of Eq. (3.102), should not be conducted at very high speeds. For $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, we have $S_\beta = 0.120, 0.184, 0.201, 0.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (3.102) from above. For $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi_1' = 45°$, $\chi_2' = 67°$, we have $S_\beta = -1.120, -1.184, -1.201, -1.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (3.102) from below. For β larger than 0.2 but close to it, S_β already turns out to be too close to the critical interval given in Eq. (3.102) to be relevant experimentally.

3.2 Polarization Correlations in Pair Photons Production in Scalar Electrodynamics

We now compute the explicit joint probability distributions of photon ($\gamma\gamma$) polarization correlations in pair photons production in scalar electrodynamics, electrodynamics of charged spin-zero particles. Similar to QED, we compute the conditional joint probabilities distributions of photons as well as the probabilities corresponding to the measurement of only one of the photon's polarization. The second application is involved with all repeated experiments corresponding to all orientations of the axis of motion of pair of spin-zero particles in the c.m. initially prepared with the same speeds, and a pair of photons is observed moving along a given axis in each case after the annihilation process, given that these collisions mentioned above have occurred. In this later case we must average over the initial orientations of axis along a pair of spin-zero particles may initially move before annihilation occurs. With the explicit expressions for the probabilities derived from this quantum dynamical analysis, we finally show a clear violation of the relevant Bell-like inequality as a function of the speed of pair of spin-zero particle.

3.2.1 Computations of The Probability Distributions of Correlation

We start from the transition amplitude of pair photons production in scalar electrodynamics, derived in §2.2, given (see Eq. 2.87):

$$\mathcal{A} \propto \left[\frac{p_1^\mu p_2^\nu}{p_1 k_1} + \frac{p_1^\nu p_2^\mu}{p_1 k_2} - g^{\mu\nu}\right] e_\mu(\lambda_1) e_\nu(\lambda_2), \tag{3.103}$$

where $e_\mu(\lambda)$ is the polarization vectors of photons.

So that, we obtain the transition probability of this process ($|\mathcal{A}|^2 = \mathcal{A}^{\mu\nu}\mathcal{A}^{\sigma\beta}$), to the leading order in the fine-structure constant, up to an important multiplicative factor for the problem at hand, written as

$$\text{Prob} \propto \left[\frac{p_1^\mu p_2^\nu}{p_1 k_1} + \frac{p_1^\nu p_2^\mu}{p_1 k_2} - g^{\mu\nu}\right]\left[\frac{p_1^\sigma p_2^\beta}{p_1 k_1} + \frac{p_1^\beta p_2^\sigma}{p_1 k_2} - g^{\sigma\beta}\right] e_\mu(\lambda_1) e_\nu(\lambda_2) e_\sigma(\lambda_1) e_\beta(\lambda_2), \tag{3.104}$$

where $[\cdot][\cdot]$, we rewrite as

$$\begin{aligned}
[\cdot][\cdot] &= \left[\frac{p_1^\mu p_2^\nu}{p_1 k_1} + \frac{p_1^\nu p_2^\mu}{p_1 k_2} - g^{\mu\nu}\right]\left[\frac{p_1^\sigma p_2^\beta}{p_1 k_1} + \frac{p_1^\beta p_2^\sigma}{p_1 k_2} - g^{\sigma\beta}\right] \\
&= \frac{p_1^\mu p_2^\nu p_1^\sigma p_2^\beta}{(p_1 k_1)^2} + \frac{p_1^\mu p_2^\nu p_1^\beta p_2^\sigma}{(p_1 k_1)(p_1 k_2)} - g^{\sigma\beta}\frac{p_1^\mu p_2^\nu}{(p_1 k_1)} + \frac{p_1^\nu p_2^\mu p_2^\sigma p_1^\beta}{(p_1 k_1)(p_1 k_2)} \\
&\quad + \frac{p_1^\nu p_2^\mu p_1^\beta p_2^\sigma}{(p_1 k_2)^2} - g^{\sigma\beta}\frac{p_1^\nu p_2^\mu}{(p_1 k_2)} - g^{\mu\nu}\frac{p_1^\sigma p_2^\beta}{(p_1 k_1)} - g^{\mu\nu}\frac{p_1^\beta p_2^\sigma}{(p_1 k_2)} + g^{\mu\nu} g^{\sigma\beta}.
\end{aligned} \tag{3.105}$$

After the directly computation of above term, we obtain

$$\begin{aligned}
\text{Prob} \propto \Bigg\{ & \frac{p_1^\mu p_2^\nu p_1^\sigma p_2^\beta}{(p_1 k_1)^2} + \frac{p_1^\mu p_2^\nu p_1^\beta p_2^\sigma}{(p_1 k_1)(p_1 k_2)} - g^{\sigma\beta}\frac{p_1^\mu p_2^\nu}{(p_1 k_1)} + \frac{p_1^\nu p_2^\mu p_2^\sigma p_1^\beta}{(p_1 k_1)(p_1 k_2)} + \frac{p_1^\nu p_2^\mu p_1^\beta p_2^\sigma}{(p_1 k_2)^2} \\
& - g^{\sigma\beta}\frac{p_1^\nu p_2^\mu}{(p_1 k_2)} - g^{\mu\nu}\frac{p_1^\sigma p_2^\beta}{(p_1 k_1)} - g^{\mu\nu}\frac{p_1^\beta p_2^\sigma}{(p_1 k_2)} + g^{\mu\nu} g^{\sigma\beta} \Bigg\} e_\mu^{(1)} e_\nu^{(2)} e_\sigma^{(1)} e_\beta^{(2)},
\end{aligned} \tag{3.106}$$

where $e^{(i)}$, $i = 1, 2$, is the polarization vectors of photons.

After we multiply the polarization vectors of photons in Eq. (3.106). Note that our problem want to study the correlations polarization of emerging photons in this process. Therefore we not some over all polarization of emerging photons, then we obtain the transition probability of this process, written as

$$\begin{aligned}
\text{Prob} \propto & \frac{(p_1 \cdot e_1)^2 (p_2 \cdot e_2)^2}{(p_1 k_1)^2} + \frac{(p_1 \cdot e_1)(p_2 \cdot e_2)(p_1 \cdot e_2)(p_2 \cdot e_1)}{(p_1 k_1)(p_1 k_2)} \\
& - \frac{(e_1 \cdot e_2)(p_1 \cdot e_1)(p_2 \cdot e_2)}{(p_1 k_1)} - \frac{(e_1 \cdot e_2)(p_1 \cdot e_2)(p_2 \cdot e_1)}{(p_1 k_1)} \\
& + \frac{(p_1 \cdot e_2)^2 (p_2 \cdot e_1)^2}{(p_1 k_2)^2} + \frac{(p_1 \cdot e_1)(p_2 \cdot e_2)(p_1 \cdot e_2)(p_2 \cdot e_1)}{(p_1 k_1)(p_1 k_2)} \\
& - \frac{(e_1 \cdot e_2)(p_1 \cdot e_1)(p_2 \cdot e_2)}{(p_1 k_1)} - \frac{(e_1 \cdot e_2)(p_1 \cdot e_2)(p_2 \cdot e_1)}{(p_1 k_2)} \\
& + (e_1 \cdot e_2)^2. \qquad (3.107)
\end{aligned}$$

The simply transition probability of this process, given by

$$\begin{aligned}
\text{Prob} \propto & \frac{(p_1 \cdot e_1)^2 (p_2 \cdot e_2)^2}{(p_1 k_1)^2} + 2\frac{(p_1 \cdot e_1)(p_2 \cdot e_2)(p_1 \cdot e_2)(p_2 \cdot e_1)}{(p_1 k_1)(p_1 k_2)} \\
& - 2\frac{(e_1 \cdot e_2)(p_1 \cdot e_1)(p_2 \cdot e_2)}{(p_1 k_1)} - 2\frac{(e_1 \cdot e_2)(p_1 \cdot e_2)(p_2 \cdot e_1)}{(p_1 k_2)} \\
& + \frac{(p_1 \cdot e_2)^2 (p_2 \cdot e_1)^2}{(p_1 k_2)^2} + (e_1 \cdot e_2)^2. \qquad (3.108)
\end{aligned}$$

In the c.m. of a pair spin-zero particles

$$\left.\begin{array}{c}
\mathbf{p}_2 = -\mathbf{p}_1 \equiv \mathbf{p},\ \mathbf{k}_2 = -\mathbf{k}_1 \equiv \mathbf{k},\ p_1^0 = p_2^0 = k_1^0 = k_2^0 \equiv p^0 \\
k^0 = |\mathbf{k}|,\ p^0 = \sqrt{\mathbf{p}^2 + m^2}
\end{array}\right\}. \qquad (3.109)$$

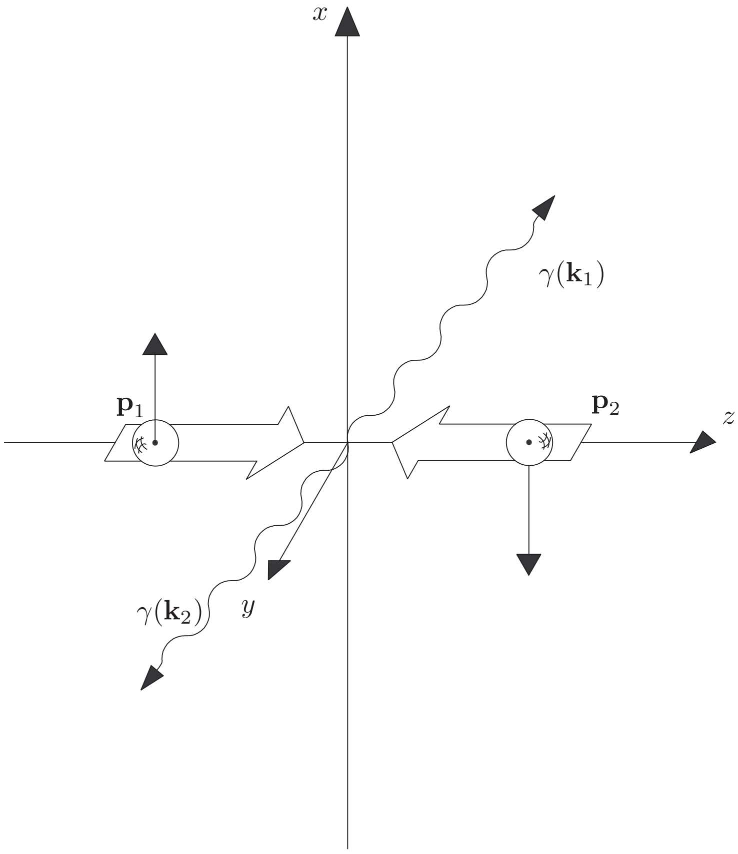

Figure 3.2 The figure depicts pair photons production in scalar electrodynamics, with initially spin-zero particles moving along the z-axis, while the emerging photons moving along the x-axis.

In figure **3.2** we show how to introduce the polarization $e_1^\mu(\lambda) = (0, \mathbf{e}_1(\lambda))$ in reference to the vector k^μ. If $\mathbf{k}$ chosen to lie in x-z plane, then

$$\mathbf{k} := |\mathbf{k}|(\sin\theta, 0, \cos\theta), \tag{3.110}$$

and the polarization vector that specify same as in §3.1.1, given by

$$\mathbf{e}_1 := (-\cos\theta\cos\chi_1, \sin\chi_1, \sin\theta\cos\chi_1), \tag{3.111}$$

where, here, $\mathbf{p} = |\mathbf{p}|(0, 0, 1)$. For a general orientation of $\mathbf{k}$ and $\mathbf{e}_1$, we must rotate the $x - y - z$ coordinate system c.w. (clockwise) about the z-axis by an angle ϕ.

After we rotate the system with ϕ about z-axis, by using the rotation matrix in Eq. (3.8), we have

$$\mathbf{k} := |\mathbf{k}|(\sin\theta, -\sin\phi\sin\theta, \cos\theta), \tag{3.112}$$

and

$$\mathbf{e}_1 := (-\cos\theta\cos\chi_1\cos\phi + \sin\chi_1\sin\phi, \sin\chi_1\cos\phi + \cos\theta\cos\chi_1\sin\phi, \sin\theta\cos\chi_1), \tag{3.113}$$

in the resulting coordinate system. A similar expression for $\mathbf{e}_2(\lambda') \equiv \mathbf{e}_2$ is obtained by replacing χ_1 by χ_2. With $\mathbf{e}_1 \equiv \mathbf{e}_1(\lambda_1)$, $\mathbf{e}_1(\lambda_2)$ is obtained from $\mathbf{e}_1$ by the substitution $\chi_1 \to \chi_1 + \pi/2$

Now we consider the probability of this process (see in Eq. (3.108)) in the convenient form. By using the properties of dot product of $e_i = (0, \mathbf{e}_i)$, $i = 1, 2$ and $p_1 = (0, \mathbf{p})$. In the c.m. of spin-zero particles, may be written in the convenient form

$$\text{Prob} \propto \frac{(\mathbf{e}_1\cdot\mathbf{p})^2(\mathbf{e}_2\cdot\mathbf{p})^2}{(pk_1)^2} + 2\frac{(\mathbf{e}_1\cdot\mathbf{p})^2(\mathbf{e}_2\cdot\mathbf{p})^2}{(pk_1)(pk_2)} + 2\frac{(\mathbf{e}_1\cdot\mathbf{e}_2)(\mathbf{e}_1\cdot\mathbf{p})(\mathbf{e}_2\cdot\mathbf{p})}{(pk_1)}$$
$$+ 2\frac{(\mathbf{e}_1\cdot\mathbf{e}_2)(\mathbf{e}_1\cdot\mathbf{p})(\mathbf{e}_2\cdot\mathbf{p})}{(pk_2)} + \frac{(\mathbf{e}_1\cdot\mathbf{p})^2(\mathbf{e}_2\cdot\mathbf{p})^2}{(pk_2)^2} + (\mathbf{e}_1\cdot\mathbf{e}_2)^2. \tag{3.114}$$

We treat two processes of annihilation associated with the relative probability given in Eq. (3.114)

3.2.2 Process 1: Pair Spin-Zero Particles Moving Along the z-axis

We consider the annihilation of spin 0 particle pairs in flight in the c.m. (Located at the origin of the coordinate system) initially prepared to be moving along the z-axis, as in the figure **3.2** , each moving with speed $v = \beta c$, prior to annihilation into pairs of photons, and place detectors for the latter at opposite ends of the x-axis.

With using the properties in Eq. (3.109) we note that

$$
\begin{aligned}
k_1 k_2 &= -|\mathbf{k}|^2 - (k^0)^2 \\
&= -(k^0)^2 - (k^0)^2 \\
&= -2(k^0)^2,
\end{aligned}
\tag{3.115}
$$

and also using the property $|\mathbf{p}|/p^0 = \beta$ which is the speed, β, of initially particles, spin 0 particle, anti-spin 0 particle, we have

$$
\begin{aligned}
p_1 k_1 &= \mathbf{p} \cdot \mathbf{k}_1 - p^0 k_1^0 \\
&= |\mathbf{p}||\mathbf{k}| \cos\theta - p^0 k^0 \\
&= -p^0 k^0 \left(1 - \frac{|\mathbf{p}|}{p^0} \cos\theta\right) \\
&= -(k^0)^2 (1 - \beta\cos\theta) \\
pk_1 &= -(k^0)^2 (1 - \beta\cos\theta),
\end{aligned}
\tag{3.116}
$$

Similarly, we have finally scalar product to use in Eq. (3.114) as

$$
\begin{aligned}
p_1 k_2 &= \mathbf{p} \cdot \mathbf{k}_2 - p^0 k_2^0 \\
&= -|\mathbf{p}||\mathbf{k}| \cos\theta - p^0 k^0 \\
&= -p^0 k^0 (1 + \frac{|\mathbf{p}|}{p^0} \cos\theta) \\
&= -(k^0)^2 (1 + \beta\cos\theta) \\
pk_2 &= -(k^0)^2 (1 + \beta\cos\theta).
\end{aligned}
\tag{3.117}
$$

After we rotate the system with $\phi = 2\pi$ about z-axis, from Eq. 3.10, we found that the system is same as the initial system ($\phi = 0$). We have

$$\mathbf{e}_1 = (-\cos\theta\cos\chi_1, \sin\chi_1, \sin\theta\cos\chi_1), \tag{3.118}$$

$$\mathbf{e}_2 = (-\cos\theta\cos\chi_2, \sin\chi_2, \sin\theta\cos\chi_2), \tag{3.119}$$

$$\mathbf{p} = |\mathbf{p}|(0, 0, 1), \tag{3.120}$$

$$\mathbf{k} = |\mathbf{k}|(\sin\theta, 0, \cos\theta), \tag{3.121}$$

and the scalar product of the two polarizations of photon, we write as:

$$\begin{aligned}\mathbf{e}_1 \cdot \mathbf{e}_2 &= \cos^2\theta\cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2 + \sin^2\theta\cos\chi_1\cos\chi_2 \\ &= (\cos^2\theta + \sin^2\theta)\cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2 \\ &= \cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2\end{aligned}$$

The latter works out, gives

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos(\chi_1 - \chi_2). \tag{3.122}$$

So that, we can rewrite the probability in Eq. (3.108) in term of θ, speed (β) of the initial particles, χ_1 and χ_2, given by

$$\begin{aligned}\text{Prob} \propto\ &\frac{|\mathbf{p}|^2\sin^2\theta\cos^2\chi_1|\mathbf{p}|^2\sin^2\theta\cos^2\chi_2}{(k^0)^4(1-\beta\cos\theta)^2} + \frac{|\mathbf{p}|^2\sin^2\theta\cos^2\chi_1|\mathbf{p}|^2\sin^2\theta\cos^2\chi_2}{(k^0)^4(1-\beta^2\cos^2\theta)} \\ &+ \frac{2\cos(\chi_1-\chi_2)|\mathbf{p}|\sin\theta\cos\chi_1|\mathbf{p}|\sin\theta\cos\chi_2}{-(k^0)^2(1-\beta\cos\theta)} \\ &+ \frac{2\cos(\chi_1-\chi_2)|\mathbf{p}|\sin\theta\cos\chi_1|\mathbf{p}|\sin\theta\cos\chi_2}{-(k^0)^2(1+\beta\cos\theta)}\end{aligned}$$

$$+\frac{|\mathbf{p}|^2\sin^2\theta\cos^2\chi_1|\mathbf{p}|^2\sin^2\theta\cos^2\chi_2}{(k^0)^4(1+\beta\cos\theta)^2}+\cos^2(\chi_1-\chi_2). \tag{3.123}$$

Using the properties in Eq. (3.19), then we simplify Eq. (3.123) as

$$\begin{aligned}\text{Prob}\propto\ &\frac{\beta^4\sin^4\theta\cos^2\chi_1\cos^2\chi_2}{(1-\beta\cos\theta)^2}-\frac{2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos\chi_1\cos\chi_2}{(1-\beta\cos\theta)}\\ &+\frac{2\beta^4\sin^4\theta\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)}-\frac{2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos\chi_1\cos^2\chi_2}{(1+\beta\cos\theta)}\\ &+\frac{\beta^4\sin^4\theta\cos\chi_1\cos^2\chi_2}{(1+\beta\cos\theta)^2}+\cos^2(\chi_1-\chi_2).\end{aligned} \tag{3.124}$$

We consider the right hand side of Eq. (3.124) that can arrange each term as:

$$\begin{aligned}&=\beta^4\sin^4\theta\cos^2\chi_1\cos^2\chi_2\left[\frac{1}{(1-\beta\cos\theta)^2}+\frac{1}{(1+\beta\cos\theta)^2}+\frac{2}{(1-\beta^2\cos^2\theta)}\right]\\ &-2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos\chi_1\cos\chi_2\left[\frac{1}{(1-\beta\cos\theta)}+\frac{1}{(1+\beta\cos\theta)}\right]\\ &+\cos^2(\chi_1-\chi_2),\end{aligned}$$

and the directly computation of above term, we have

$$\begin{aligned}&=\beta^4\sin^4\theta\cos^2\chi_1\cos^2\chi_2\left[\frac{1+2\beta\cos\theta+\beta^2\cos^2\theta+1-2\beta\cos\theta+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^2}\right]\\ &+\beta^4\sin^4\theta\cos^2\chi_1\cos^2\chi_2\left[\frac{2}{(1-\beta^2\cos^2\theta)}\right]\\ &-2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos\chi_1\cos\chi_2\left[\frac{1+\beta\cos\theta+1-\beta\cos\theta}{(1-\beta^2\cos^2\theta)}\right]\\ &+\cos^2(\chi_1-\chi_2).\end{aligned}$$

where the fist term is so complicate, but it become simple term after some term cancel out. We then obtain

$$= \beta^4 \sin^4\theta \cos^2\chi_1 \cos^2\chi_2 \left[\frac{2 + 2\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^2} + \frac{2}{(1-\beta^2\cos^2\theta)}\right]$$

$$- 2\beta^2 \cos(\chi_1 - \chi_2) \sin^2\theta \cos\chi_1 \cos\chi_2 \left[\frac{2}{(1-\beta^2\cos^2\theta)}\right]$$

$$+ \cos^2(\chi_1 - \chi_2)$$

$$= 2\beta^4 \sin^4\theta \cos^2\chi_1 \cos^2\chi_2 \left[\frac{(1+\beta^2\cos^2\theta) + (1-\beta^2\cos^2\theta)}{(1-\beta^2\cos^2\theta)^2}\right]$$

$$- 2\beta^2 \cos(\chi_1 - \chi_2) \sin^2\theta \cos\chi_1 \cos\chi_2 \left[\frac{2}{(1-\beta^2\cos^2\theta)}\right]$$

$$+ \cos^2(\chi_1 - \chi_2).$$

Therefore we write down the probability of this process as:

$$\text{Prob} = \frac{4\beta^4 \sin^4\theta \cos^2\chi_1 \cos^2\chi_2}{(1-\beta^2\cos^2\theta)^2} - \frac{4\beta^2 \cos(\chi_1-\chi_2) \sin^2\theta \cos\chi_1 \cos\chi_2}{(1-\beta^2\cos^2\theta)}$$

$$+ \cos^2(\chi_1 - \chi_2). \tag{3.125}$$

By using the properties of the trigonometry to specify

$$\beta^2 \sin^2\theta = -(1-\beta^2) + (1-\beta^2\cos^2), \tag{3.126}$$

$$\beta^4 \sin^4\theta = (1-\beta^2)^2 - 2(1-\beta^2)(1-\beta^2\cos^2) + (1-\beta^2\cos^2)^2, \tag{3.127}$$

to simply Eq. (3.125), by replacing with Eqs. (3.126)–(3.127), we obtain

$$\text{Prob} = \frac{4(1-\beta^2)^2 \cos^2\chi_1 \cos^2\chi_2}{(1-\beta^2\cos^2\theta)^2} - \frac{8(1-\beta^2)\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)} + 4\cos^2\chi_1\cos^2\chi_2$$

$$+\frac{4(1-\beta^2)\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2}{(1-\beta^2\cos^2\theta)} - 4\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2$$

$$+\cos^2(\chi_1-\chi_2). \tag{3.128}$$

After the tedious computation, we obtain the probability of this process that use to study the polarized photon, given by

$$\text{Prob} = \frac{4(1-\beta^2)\cos\chi_1\cos\chi_2[\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2]}{(1-\beta^2\cos^2\theta)}$$

$$+\frac{4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2}{(1-\beta^2\cos^2\theta)^2} + (\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)^2. \tag{3.129}$$

where $\beta = |\mathbf{p}|/p^0$ is the speed of e^+ (or of e^-) divided by the speed of light, θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, and ϕ is the angle between $\mathbf{p}$ and x-axis. We note that the angles χ_1, χ_2 have given fixed values when the vector $\mathbf{k}$ is made to rotate in the coordinate system.

Similarity in the process in §3.1.2, we use the integration in Eqs. (3.40)–(3.41), and define

$$F_\delta(\chi_1,\chi_2) \equiv \int_0^{2\pi}\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} \text{Prob}\sin\theta \mathrm{d}\theta\mathrm{d}\phi. \tag{3.130}$$

Since θ is a continuous variable, we may integrate the expression in Eq. (3.130) over θ from $\pi/2-\delta$ to $\pi/2+\delta$ and then rigorously take the limit $\delta \to 0$ in evaluating the normalized probabilities in question. The ϕ-integral, here, is not important in evaluating these normalized probabilities since it leads to overall multiplicative factors that cancel out in the final expressions. We have

$$F_\delta(\chi_1,\chi_2) = 4(1-\beta^2)\cos\chi_1\cos\chi_2[\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2]\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta}\frac{\sin\theta\mathrm{d}\theta}{(1-\beta^2\cos^2\theta)}$$

$$+\,4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta}\frac{\sin\theta\mathrm{d}\theta}{(1-\beta^2\cos^2\theta)^2}$$

$$+\,(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)^2\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta}\sin\theta\mathrm{d}\theta. \tag{3.131}$$

The latter works out, we obtain

$$F_\delta(\chi_1,\chi_2)=4(1-\beta^2)\cos\chi_1\cos\chi_2[\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2]\frac{1}{\beta}\ln\!\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$

$$+\,4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2\frac{1}{\beta}\left[\frac{\beta\sin\delta}{(1-\beta^2\sin^2\delta)}+\frac{1}{2}\ln\!\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right]$$

$$+\,2\sin\delta(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)^2. \tag{3.132}$$

Then we have to find the normalization factor

$$N_\delta=F_\delta(\chi_1,\chi_2)+F_\delta(\chi_1+\frac{\pi}{2},\chi_2)$$

$$+\,F_\delta(\chi_1,\chi_2+\frac{\pi}{2})+F_\delta(\chi_1+\frac{\pi}{2},\chi_2+\frac{\pi}{2}). \tag{3.133}$$

By considering each term in Eq. (3.133), for $(\chi_1+\pi/2,\chi_2)$, substituting $\chi_1\to\chi_1+\pi/2$, we have

$$F_\delta(\chi_1+\frac{\pi}{2},\chi_2)$$

$$=-4(1-\beta^2)\sin\chi_1\cos\chi_2\,[-\sin(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2]\,\frac{1}{\beta}\ln\!\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$

$$+\,4(1-\beta^2)^2\sin^2\chi_1\cos^2\chi_2\frac{1}{\beta}\left[\frac{\beta\sin\delta}{(1-\beta^2\sin^2\delta)}+\frac{1}{2}\ln\!\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right]$$

$$+\,2\sin\delta(-\sin(\chi_1-\chi_2)+2\sin\chi_1\cos\chi_2)^2, \tag{3.134}$$

for $(\chi_1+\pi/2, \chi_2+\pi/2)$, substituting $\chi_2 \to \chi_2+\pi/2$, we have

$$F_\delta(\chi_1, \chi_2+\frac{\pi}{2})$$
$$= -4(1-\beta^2)\cos\chi_1\sin\chi_2\left[\sin(\chi_1-\chi_2)+2\cos\chi_1\sin\chi_2\right]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$
$$+ 4(1-\beta^2)^2\cos^2\chi_1\sin^2\chi_2\frac{1}{\beta}\left[\frac{\beta\sin\delta}{(1-\beta^2\sin^2\delta)}+\frac{1}{2}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right]$$
$$+ 2\sin\delta(\sin(\chi_1-\chi_2)+2\cos\chi_1\sin\chi_2)^2, \quad (3.135)$$

Finally, for $(\chi_1+\pi/2, \chi_2+\pi/2)$, substituting $\chi_1 \to \chi_1+\pi/2$, $\chi_2 \to \chi_2+\pi/2$, we have

$$F_\delta(\chi_1+\frac{\pi}{2}, \chi_2+\frac{\pi}{2})$$
$$= 4(1-\beta^2)\sin\chi_1\sin\chi_2\left[\cos(\chi_1-\chi_2)-2\sin\chi_1\sin\chi_2\right]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$
$$+ 4(1-\beta^2)^2\sin^2\chi_1\sin^2\chi_2\frac{1}{\beta}\left[\frac{\beta\sin\delta}{(1-\beta^2\sin^2\delta)}+\frac{1}{2}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right]$$
$$+ 2\sin\delta(\cos(\chi_1-\chi_2)-2\sin\chi_1\sin\chi_2)^2. \quad (3.136)$$

We can rewrite N_δ by substituting Eq. (3.132) and Eqs. (3.134)–(3.136) in Eq. (3.133), given by

$$= \Big\{[4(1-\beta^2)\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)]$$
$$+ [4(1-\beta^2)\sin\chi_1\cos\chi_2(-\sin(\chi_1-\chi_2)+2\sin\chi_1\cos\chi_2)]$$
$$+ [4(1-\beta^2)\cos\chi_1\sin\chi_2(\sin(\chi_1-\chi_2)+2\cos\chi_1\sin\chi_2)]$$

$$+ [4(1-\beta^2)\sin\chi_1\sin\chi_2(\cos(\chi_1-\chi_2) - 2\sin\chi_1\sin\chi_2)]\Big\}\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$

$$+ 4(1-\beta^2)^2\{\cos^2\chi_1\cos^2\chi_2 + \sin^2\chi_1\cos^2\chi_2 + \cos^2\chi_1\sin^2\chi_2$$

$$+ \sin^2\chi_1\sin^2\chi_2\}\left[\frac{\sin\delta}{1-\beta^2\sin^2\delta} + \frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right]$$

$$+ 2\sin\delta\Big\{(\cos(\chi_1-\chi_2 - 2\cos\chi_1\cos\chi_2)^2 + (-\sin(\chi_1-\chi_2 + 2\sin\chi_1\cos\chi_2)^2$$

$$+ (\sin(\chi_1-\chi_2 + 2\cos\chi_1\sin\chi_2)^2 + (\cos(\chi_1-\chi_2 - 2\sin\chi_1\sin\chi_2)^2\Big\}. \tag{3.137}$$

To simplify Eq. (3.137), we calculate each term as: the first term be written as

$$= 4(1-\beta^2)\{\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2 + \sin(\chi_1-\chi_2)\sin\chi_1\cos\chi_2$$

$$- \sin(\chi_1-\chi_2)\cos\chi_1\sin\chi_2 + \cos(\chi_1-\chi_2)\sin\chi_1\sin\chi_2\}$$

$$+ 4(1-\beta^2)\{-2\cos^2\chi_1\cos^2\chi_2 - 2\sin^2\chi_1\cos^2\chi_2 - 2\cos^2\chi_1\sin^2\chi_2$$

$$- 2\sin^2\chi_1\sin^2\chi_2\}$$

$$= 4(1-\beta^2)\{\cos^2\chi_1\cos^2\chi_2 + \sin\chi_1\sin\chi_2\cos\chi_1\cos\chi_2 + \sin^2\chi_1\cos^2\chi_2$$

$$- \cos\chi_1\sin\chi_2\sin\chi_1\cos\chi_2 - \sin\chi_1\cos\chi_2\cos\chi_1\sin\chi_2$$

$$+ \cos^2\chi_1\sin^2\chi_2 + \cos\chi_1\cos\chi_2\sin\chi_1\sin\chi_2 + \sin^2\chi_1\sin^2\chi_2\}$$

$$- 8(1-\beta^2)\{\cos^2\chi_1\cos^2\chi_2 + \sin^2\chi_1\cos^2\chi_2 + \cos^2\chi_1\sin^2\chi_2 + \sin^2\chi_1\sin^2\chi_2\}$$

$$= 4(1-\beta^2)\{\cos^2\chi_1\cos^2\chi_2 + \sin^2\chi_1\cos^2\chi_2 + \cos^2\chi_1\sin^2\chi_2 + \sin^2\chi_1\sin^2\chi_2\}$$

$$- 8(1-\beta^2) = -4(1-\beta^2), \tag{3.138}$$

and the second term be written as

$$\cos^2\chi_1\cos^2\chi_2+\sin^2\chi_1\cos^2\chi_2+\cos^2\chi_1\sin^2\chi_2+\sin^2\chi_1\sin^2\chi_2=1, \tag{3.139}$$

and the final term be written as

$$\begin{aligned}&=(\cos(\chi_1-\chi_2)-2\cos\chi_1\cos\chi_2)^2+(-\sin(\chi_1-\chi_2)+2\sin\chi_1\cos\chi_2)^2\\&\quad+(\sin(\chi_1-\chi_2)+2\cos\chi_1\sin\chi_2)^2+(\cos(\chi_1-\chi_2)-2\sin\chi_1\sin\chi_2)^2.\end{aligned}$$

To simplify above term on the right-hand side, we distribute square term

$$\begin{aligned}&=\cos^2(\chi_1-\chi_2)-4\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2+4\cos^2\chi_1\cos^2\chi_2\\&\quad+\sin^2(\chi_1-\chi_2)-4\sin(\chi_1-\chi_2)\sin\chi_1\cos\chi_2+4\sin^2\chi_1\cos^2\chi_2\\&\quad+\sin^2(\chi_1-\chi_2)+4\sin(\chi_1-\chi_2)\cos\chi_1\sin\chi_2+4\cos^2\chi_1\sin^2\chi_2\\&\quad+\cos^2(\chi_1-\chi_2)-4\cos(\chi_1-\chi_2)\sin\chi_1\sin\chi_2+4\sin^2\chi_1\sin^2\chi_2,\end{aligned}$$

and we then rearrange them as

$$\begin{aligned}&=\cos^2(\chi_1-\chi_2)+\sin^2(\chi_1-\chi_2)+\sin^2(\chi_1-\chi_2)+\cos^2(\chi_1-\chi_2)\\&\quad-4\{\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2+\sin(\chi_1-\chi_2)\sin\chi_1\cos\chi_2\\&\quad-\sin(\chi_1-\chi_2)\cos\chi_1\sin\chi_2+\cos(\chi_1-\chi_2)\sin\chi_1\sin\chi_2\}\\&\quad+4\left[\cos^2\chi_1\cos^2\chi_2+\sin^2\chi_1\cos^2\chi_2+\cos^2\chi_1\sin^2\chi_2+\sin^2\chi_1\sin^2\chi_2)\right]\\&=2-4\left[\cos^2(\chi_1-\chi_2)+\sin^2(\chi_1-\chi_2)\right]+4=2.\end{aligned} \tag{3.140}$$

The latter work out to be

$$N_\delta = [2(1-\beta^2)^2 - 4(1-\beta^2)]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) + 4(1-\beta^2)^2\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)}\right]$$

$$+ 4\sin\delta. \tag{3.141}$$

Therefore, given that the process has occurred as described above, with two photons moving (back-to-back) along x-axis, the conditional joint probability of the photons, specified by the angles χ_1, χ_2, is rigorously given by

$$P(\chi_1, \chi_2) = \lim_{\delta\to 0}\frac{F_\delta(\chi_1, \chi_2)}{N_\delta}. \tag{3.142}$$

We use similarly method and the condition in the process in §3.1.2, gives

$$\lim_{\delta\to 0} F_\delta(\chi_1, \chi_2) \approx [-4(1-\beta^2)\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2)]$$

$$+ 4(1-\beta^2)^2\cos^2\chi_1\cos^2\chi_2$$

$$+ (\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2)^2$$

$$= (\cos(\chi_1-\chi_2) - 2\beta^2\cos\chi_1\cos\chi_2)^2, \tag{3.143}$$

and

$$\lim_{\delta\to 0} N_\delta \approx [-4(1-\beta^2)] + 4(2\delta)(1-\beta^2)^2 + 2(2\delta)$$

$$= [-4(1-\beta^2)(1-(1-\beta^2)) + 2]$$

$$= [2 - 4\beta^2(1-\beta^2)], \tag{3.144}$$

where we neglect 2δ.

To obtain from Eqs. (3.143)–(3.144), the conditional joint probability of the photon polarizations is given by

$$P(\chi_1, \chi_2) = \frac{[\cos(\chi_1 - \chi_2) - 2\beta^2 \cos\chi_1 \cos\chi_2]^2}{2[1 - 2\beta^2(1-\beta^2)]}, \tag{3.145}$$

for all $0 \leq \beta \leq 1$.

If we take limit $\beta \to 0$. We obtain

$$P(\chi_1, \chi_2) \underset{\beta\to 0}{\sim} \frac{1}{2}\cos^2(\chi_1 - \chi_2). \tag{3.146}$$

If only one of the polarizations is measured, then we have to evaluate $F_\delta(\chi_1, \chi_2) + F_\delta(\chi_1, \chi_2 + \pi/2)$ and $F_\delta(\chi_1, \chi_2) + F_\delta(\chi_1, \chi_2 + \pi/2)$. To do this, Eq. (3.142) and Eqs. (3.134)–(3.135) gives

$$\begin{aligned}
&F_\delta(\chi_1, \chi_2) + F_\delta(\chi_1, \chi_2 + \frac{\pi}{2}) \\
&= \Big\{4(1-\beta^2)\,[\cos\chi_1 \cos\chi_2(\cos(\chi_1 - \chi_2) - 2\cos\chi_1\cos\chi_2) \\
&\quad - \cos\chi_1 \sin\chi_2(\sin(\chi_1 - \chi_2) + 2\cos\chi_1 \sin\chi_2)]\Big\}\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\
&\quad + 4(1-\beta^2)^2[\cos^2\chi_1\cos^2\chi_2 + \cos^2\chi_1\sin^2\chi_2]\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)}\right. \\
&\qquad\qquad \left. + \frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right] \\
&\quad + 2\sin\delta\{(\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2)^2 + (\sin(\chi_1-\chi_2) + 2\cos\chi_1\sin\chi_2)^2\}.
\end{aligned} \tag{3.147}$$

The below expressions are defined to compute Eq. (3.147) :

$$\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2) - \cos\chi_1\sin\chi_2(\sin(\chi_1-\chi_2) + 2\cos\chi_1\sin\chi_2)$$

$$= \cos\chi_1 \cos\chi_2(-\cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2) - \cos\chi_1\sin\chi_2$$

$$\times (\sin\chi_1\cos\chi_2 + \cos\chi_1\sin\chi_2)$$

$$= -\cos^2\chi_1\cos^2\chi_2 + \sin\chi_1\sin\chi_2\cos\chi_1\cos\chi_2 - \cos^2\chi_1\sin^2\chi_2$$

$$- \sin\chi_1\sin\chi_2\cos\chi_1\cos\chi_2 = -\cos^2\chi_1, \tag{3.148}$$

and

$$(\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2)^2 + (\sin(\chi_1-\chi_2) + 2\cos\chi_1\sin\chi_2)^2$$

$$= \cos^2(\chi_1-\chi_2) + \sin^2(\chi_1-\chi_2) - 4\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2$$

$$+ 4\sin(\chi_1-\chi_2)\cos\chi_1\sin\chi_2 + 4(\cos^2\chi_1\cos^2\chi_2 + \cos^2\chi_1\sin^2\chi_2)$$

$$= 1 - 4\cos^2\chi_1 + 4\cos^2\chi_1 = 1. \tag{3.149}$$

From above computation, Eqs. (3.148)–(3.149), we can rewrite Eq. (3.147) as

$$F_\delta(\chi_1,\chi_2) + F_\delta(\chi_1,\chi_2+\frac{\pi}{2}) = -4\left(1-\beta^2\right)\cos^2\chi_1 \frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$

$$+ 4(1-\beta^2)^2\cos^2\chi_1\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)} + \frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right]$$

$$+ 2\sin\delta. \tag{3.150}$$

This is, the conditional probabilities associated with the measurement of only of the polarizations are given by

$$P(\chi_1,-) = \lim_{\delta\to 0}\frac{F_\delta(\chi_1,\chi_2) + F_\delta(\chi_1,\chi_2+\frac{\pi}{2})}{N_\delta}$$

$$= \frac{-4(1-\beta^2)\cos^2\chi_1 + 4(1-\beta^2)^2\cos^2\chi_1 + 1}{2[1-2\beta^2(1-\beta^2)]}$$

$$= \frac{1-4(1-\beta^2)\cos^2\chi_1(1-(1-\beta^2))}{[2+4\beta^2(1-\beta^2)]}$$

$$= \frac{1+4\beta^2(1-\beta^2)\cos^2\chi_1}{[2+4\beta^2(1-\beta^2)]}. \tag{3.151}$$

where $0 \leq \beta \leq 1$.

If $\beta \to 0$, we can rewrite Eq. (3.151) as

$$P(\chi_1, -) \underset{\beta\to 0}{\sim} \frac{1}{2}. \tag{3.152}$$

Similarly,

$$\begin{aligned} &F_\delta(\chi_1,\chi_2) + F_\delta(\chi_1 + \frac{\pi}{2}, \chi_2) \\ &= 4(1-\beta^2)\left[\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2) \right. \\ &\quad \left. - \sin\chi_1\cos\chi_2(-\sin(\chi_1-\chi_2) + 2\sin\chi_1\cos\chi_2)\right]\frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right) \\ &\quad + 4(1-\beta^2)^2\left[\cos^2\chi_1\cos^2\chi_2 + \sin^2\chi_1\cos^2\chi_2\right]\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)} \right. \\ &\qquad\qquad \left. + \frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right] \\ &+ 2\sin\delta\left\{(\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2)^2\,(-\sin(\chi_1-\chi_2) + 2\sin\chi_1\cos\chi_2)^2\right\} \end{aligned} \tag{3.153}$$

The below expressions are defined to compute Eq. (3.153) :

$$\cos\chi_1\cos\chi_2(\cos(\chi_1-\chi_2) - 2\cos\chi_1\cos\chi_2) - \sin\chi_1\cos\chi_2(-\sin(\chi_1-\chi_2) + 2\cos\chi_1\sin\chi_2)$$

$$= \cos\chi_1 \cos\chi_2(-\cos\chi_1\cos\chi_2 + \sin\chi_1\sin\chi_2) - \sin\chi_1\cos\chi_2$$

$$\times (\sin\chi_1\cos\chi_2 + \cos\chi_1\sin\chi_2)$$

$$= -\cos^2\chi_1\cos^2\chi_2 + \sin\chi_1\sin\chi_2\cos\chi_1\cos\chi_2 - \sin^2\chi_1\cos^2\chi_2$$

$$- \sin\chi_1\sin\chi_2\cos\chi_1\cos\chi_2 = -\cos^2\chi_2, \quad (3.154)$$

and

$$(\cos(\chi_1 - \chi_2) - 2\cos\chi_1\cos\chi_2)^2 + (-\sin(\chi_1-\chi_2) + 2\sin\chi_1\cos\chi_2)^2$$

$$= \cos^2(\chi_1-\chi_2) + \sin^2(\chi_1-\chi_2) - 4\cos(\chi_1-\chi_2)\cos\chi_1\cos\chi_2$$

$$+ 4\sin(\chi_1-\chi_2)\sin\chi_1\cos\chi_2 + 4(\cos^2\chi_1\cos^2\chi_2 + \sin^2\chi_1\cos^2\chi_2)$$

$$= 1 - 4\cos^2\chi_2 + 4\cos^2\chi_2 = 1. \quad (3.155)$$

From above computation, Eqs. (3.154)–(3.155), we can rewrite Eq. (3.153) as

$$F_\delta(\chi_1,\chi_2) + F_\delta(\chi_1 + \frac{\pi}{2}, \chi_2) = -4(1-\beta^2)\cos^2\chi_2 \frac{1}{\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)$$

$$+ 4(1-\beta^2)^2\cos^2\chi_2\left[\frac{\sin\delta}{(1-\beta^2\sin^2\delta)} + \frac{1}{2\beta}\ln\left(\frac{1+\beta\sin\delta}{1-\beta\sin\delta}\right)\right]$$

$$+ 2\sin\delta. \quad (3.156)$$

This is, the conditional probabilities associated with the measurement of only of the polarizations are given by

$$P(-,\chi_2) = \lim_{\delta\to 0}\frac{F_\delta(\chi_1,\chi_2) + F_\delta(\chi_1 + \frac{\pi}{2}, \chi_2)}{N_\delta}$$

$$= \frac{-4(1-\beta^2)\cos^2\chi_2 + 4(1-\beta^2)^2\cos^2\chi_2 - 1}{2[1-2\beta^2(1-\beta^2)]}$$

$$= \frac{1 - 4(1-\beta^2)\cos^2\chi_2(1-(1-\beta^2))}{2[1-2\beta^2(1-\beta^2)]}$$

$$= \frac{1 - 4\beta^2(1-\beta^2)\cos^2\chi_2}{2[1-2\beta^2(1-\beta^2)]} \tag{3.157}$$

where $0 \leq \beta \leq 1$.

If $\beta \to 0$, we can rewrite Eq. (3.157) as

$$P(-,\chi_2) = \frac{1}{2} \tag{3.158}$$

In the notation of Local Hidden Variables (LHV) theory [Clauser and Horne, 1974; Clauser and Shimoney, 1978; Fry, 1995; Selleri, 1988], we have the identifications

$$P(\chi_1,\chi_2) = \frac{P_{12}(a_1,a_2)}{P_{12}(\infty,\infty)}, \tag{3.159}$$

$$P(\chi_1,-) = \frac{P_{12}(a_1,\infty)}{P_{12}(\infty,\infty)}, \tag{3.160}$$

$$P(-,\chi_2) = \frac{P_{12}(\infty,a_2)}{P_{12}(\infty,\infty)}, \tag{3.161}$$

where a_1, a_2 specify directions for measurements of polarizations. Defining (see Appendix E)

$$S = P(\chi_1,\chi_2) - P(\chi_1,\chi_2') + P(\chi_1',\chi_2)$$

$$+ P(\chi_1',\chi_2') - P(\chi_1',-) - P(-,\chi_2), \tag{3.162}$$

for four angles χ_1, χ_2, χ_1', χ_2', LHV theory gives the Bell-like bound [Clauser and

Horne, 1974; Clauser and Shimoney, 1978]:

$$-1 \leqslant S \leqslant 0. \tag{3.163}$$

It is sufficient to realize one experimental situation that violates the bounds in Eq. (3.163).

For example, for $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi_1' = 45°$, $\chi_2' = 67°$, Eq. (3.145), Eq. (3.151), Eq. (3.157), as obtained from scalar electrodynamics, gives $S = 0.207$ for $\beta = 0$ that violates Eq. (3.163) from above. For $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, we obtain $S = -1.207$ for $\beta = 0$ violating Eq. (3.163) from below. Both bounds are violated for all $\beta \leqslant 0.2$ for these same angles, respectively.

3.2.3 Process 2: Two Photons Moving Along z-axis

Here we put the two detects on opposite sides of the z-axis. We consider all repeated experiments with pair spin 0 particle produced in fight in the C.M.(located at the origin),each particle moving with speed $v = \beta c$,corresponding to all possible orientations of the axis along which a given pair moves. Here we must average over all angles θ, ϕ of the vector $\mathbf{p}$, with $\mathbf{k}$ along the z-axis.

In the present case

$$\mathbf{k} = |\mathbf{k}|(0, 0, 1), \tag{3.164}$$

$$\mathbf{p} = |\mathbf{p}|(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \tag{3.165}$$

$$\mathbf{e}_1 = (-\cos\chi_1, \sin\chi_1, 0), \tag{3.166}$$

$$\mathbf{e}_2 = (-\cos\chi_2, \sin\chi_2, 0). \tag{3.167}$$

Since we have computed the scar product, same as Process 2 in QED, given by

$$pk_1 = -(k^0)^2(1-\beta\cos\theta), \tag{3.168}$$

$$pk_1 = -(k^0)^2(1+\beta\cos\theta), \tag{3.169}$$

$$\mathbf{e}_1\cdot\mathbf{p} = -|\mathbf{p}|\sin\theta\cos(\phi+\chi_1), \tag{3.170}$$

$$\mathbf{e}_2\cdot\mathbf{p} = -|\mathbf{p}|\sin\theta\cos(\phi+\chi_2), \tag{3.171}$$

$$\mathbf{e}_1\cdot\mathbf{e}_2 = \cos\chi_1\cos\chi_2+\sin\chi_1\sin\chi_2 = \cos(\chi_1-\chi_2). \tag{3.172}$$

So that, we can rewrite the probability in Eq. (3.108) in term of θ, speed (β) of the initial particles, χ_1 and χ_2, given by

$$\begin{aligned}\text{Prob} \propto\ & \frac{|\mathbf{p}|^2\sin^2\theta\cos^2(\phi+\chi_1)|\mathbf{p}|^2\sin^2\theta\cos^2(\phi+\chi_2)}{(k^0)^4(1-\beta\cos\theta)^2}\\ &+\frac{|\mathbf{p}|^2\sin^2\theta\cos^2(\phi+\chi_1)|\mathbf{p}|^2\sin^2\theta\cos^2(\phi+\chi_2)}{(k^0)^4(1-\beta^2\cos^2\theta)}\\ &+\frac{2\cos(\chi_1-\chi_2)|\mathbf{p}|\sin\theta\cos(\phi+\chi_1)|\mathbf{p}|\sin\theta\cos(\phi+\chi_2)}{-(k^0)^2(1-\beta\cos\theta)}\\ &+\frac{2\cos(\chi_1-\chi_2)|\mathbf{p}|\sin\theta\cos(\phi+\chi_1)|\mathbf{p}|\sin\theta\cos(\phi+\chi_2)}{-(k^0)^2(1+\beta\cos\theta)}\\ &+\frac{|\mathbf{p}|^2\sin^2\theta\cos^2(\phi+\chi_1)|\mathbf{p}|^2\sin^2\theta\cos^2(\phi+\chi_2)}{(k^0)^4(1+\beta\cos\theta)^2}+\cos^2(\chi_1-\chi_2)\end{aligned} \tag{3.173}$$

Using the properties in Eq. (3.19), then we simplify Eq. (3.173) as

$$\begin{aligned}\text{Prob} \propto\ & \frac{\beta^4\sin^4\theta\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)}{(1-\beta\cos\theta)^2}+\frac{2\beta^4\sin^4\theta\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)}{(1-\beta^2\cos^2\theta)}\\ &-\frac{2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos(\phi+\chi_1)\cos(\phi+\chi_2)}{(1-\beta\cos\theta)}+\cos^2(\chi_1-\chi_2)\end{aligned}$$

$$
- \frac{2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos(\phi+\chi_1)\cos^2(\phi+\chi_2)}{(1+\beta\cos\theta)}
+ \frac{\beta^4\sin^4\theta\cos(\phi+\chi_1)\cos^2(\phi+\chi_2)}{(1+\beta\cos\theta)^2} \tag{3.174}
$$

We consider the right hand side of Eq. (3.174) that can arrange each term as:

$$
\begin{aligned}
&= \beta^4\sin^4\theta\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)\left[\frac{1}{(1-\beta\cos\theta)^2}+\frac{1}{(1+\beta\cos\theta)^2}\right. \\
&\left. + \frac{2}{(1-\beta^2\cos^2\theta)}\right] - 2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos(\phi+\chi_1)\cos(\phi+\chi_2) \\
&\times \left[\frac{1}{(1-\beta\cos\theta)}+\frac{1}{(1+\beta\cos\theta)}\right] + \cos^2(\chi_1-\chi_2)
\end{aligned}
$$

and the directly computation of above term, we have

$$
\begin{aligned}
&= \beta^4\sin^4\theta\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)\left[\frac{1+2\beta\cos\theta+\beta^2\cos^2\theta+1-2\beta\cos\theta+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^2}\right] \\
&+ \beta^4\sin^4\theta\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)\left[\frac{2}{(1-\beta^2\cos^2\theta)}\right] \\
&- 2\cos(\chi_1-\chi_2)\beta^2\sin^2\theta\cos(\phi+\chi_1)\cos(\phi+\chi_2)\left[\frac{1+\beta\cos\theta+1-\beta\cos\theta}{(1-\beta^2\cos^2\theta)}\right] \\
&+ \cos^2(\chi_1-\chi_2),
\end{aligned}
$$

where the fist term is so complicate, but it become simple term after some term cancel out. We then obtain

$$
\begin{aligned}
&= \beta^4\sin^4\theta\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)\left[\frac{2+2\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^2}+\frac{2}{(1-\beta^2\cos^2\theta)}\right] \\
&- 2\beta^2\cos(\chi_1-\chi_2)\sin^2\theta\cos(\phi+\chi_1)\cos(\phi+\chi_2)\left[\frac{2}{(1-\beta^2\cos^2\theta)}\right] \\
&+ \cos^2(\chi_1-\chi_2)
\end{aligned}
$$

$$= 2\beta^4 \sin^4\theta \cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)\left[\frac{(1+\beta^2\cos^2\theta)+(1-\beta^2\cos^2\theta)}{(1-\beta^2\cos^2\theta)^2}\right]$$

$$-\,2\beta^2\cos(\chi_1-\chi_2)\sin^2\theta\cos(\phi+\chi_1)\cos(\phi+\chi_2)\left[\frac{2}{(1-\beta^2\cos^2\theta)}\right]$$

$$+\cos^2(\chi_1-\chi_2).$$

Therefore we write down the probability of this process as:

$$\text{Prob} = \frac{4\beta^4\sin^4\theta\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)}{(1-\beta^2\cos^2\theta)^2} + \cos^2(\chi_1-\chi_2)$$
$$-\frac{4\beta^2\cos(\chi_1-\chi_2)\sin^2\theta\cos(\phi+\chi_1)\cos(\phi+\chi_2)}{(1-\beta^2\cos^2\theta)}. \qquad (3.175)$$

By using the properties of the trigonometry to specify

$$\beta^2\sin^2\theta = -(1-\beta^2)+(1-\beta^2\cos^2), \qquad (3.176)$$

$$\beta^4\sin^4\theta = (1-\beta^2)^2 - 2(1-\beta^2)(1-\beta^2\cos^2)+(1-\beta^2\cos^2)^2, \qquad (3.177)$$

to simply Eq. (3.175), by replacing with Eqs. (3.176)–(3.177), we obtain

$$\text{Prob} = \frac{4(1-\beta^2)^2\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)}{(1-\beta^2\cos^2\theta)^2}$$
$$-\frac{8(1-\beta^2)\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2)}{(1-\beta^2\cos^2\theta)}$$
$$-\frac{4(1-\beta^2)\cos(\chi_1-\chi_2)\cos(\phi+\chi_1)\cos(\phi+\chi_2)}{(1-\beta^2\cos^2\theta)}$$
$$-\,4\cos(\chi_1-\chi_2)\cos(\phi+\chi_1)\cos(\phi+\chi_2)$$
$$+\cos^2(\chi_1-\chi_2)+4\cos^2(\phi+\chi_1)\cos^2(\phi+\chi_2). \qquad (3.178)$$

Using the integrals in Eqs. (3.81)–(3.84). Therefore, we can find

$$F_\beta(\chi_1,\chi_2) \equiv \int_0^{2\pi}\int_0^{\pi} \mathrm{Prob}\sin\theta \mathrm{d}\theta \mathrm{d}\phi, \tag{3.179}$$

given by

$$\begin{aligned}
F_\beta(\chi_1,\chi_2) = {} & 4\int_0^{2\pi} \cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)\mathrm{d}\phi \int_0^{\pi}\sin\theta\mathrm{d}\theta \\
& + 4(1-\beta^2)\cos(\chi_1-\chi_2)\int_0^{2\pi}\mathrm{d}\phi\cos(\chi_1+\phi)\cos(\chi_2+\phi)\int_0^{\pi}\frac{\sin\theta\mathrm{d}\theta}{(1-\beta^2\cos^2\theta)} \\
& - 8(1-\beta^2)\int_0^{2\pi}\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)\mathrm{d}\phi\int_0^{\pi}\frac{\sin\theta\mathrm{d}\theta}{(1-\beta^2\cos^2\theta)} \\
& + 4(1-\beta^2)^2\int_0^{2\pi}\cos^2(\chi_1+\phi)\cos^2(\chi_2+\phi)\mathrm{d}\phi\int_0^{\pi}\frac{\sin\theta\mathrm{d}\theta}{(1-\beta^2\cos^2\theta)^2} \\
& - 4\cos(\chi_1-\chi_2)\int_0^{2\pi}\cos(\chi_1+\phi)\cos(\chi_2+\phi)\mathrm{d}\phi\int_0^{\pi}\sin\theta\mathrm{d}\theta \\
& + 2\pi\cos^2(\chi_1-\chi_2)\int_0^{\pi}\sin\theta\mathrm{d}\theta.
\end{aligned}$$

To simplify above equation, we note that

$$\begin{aligned}
= {} & -8(1-\beta^2)(\frac{\pi}{4})[1+2\cos^2(\chi_1-\chi_2)](\frac{1}{\beta})\ln\left(\frac{1+\beta}{1-\beta}\right) \\
& + 4(\frac{\pi}{4})(1-\beta^2)^2[1+2\cos^2(\chi_1-\chi_2)]\left[\frac{1}{1-\beta^2}+\frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right] \\
& - 4\pi\cos^2(\chi_1-\chi_2)(2) + 4(\frac{\pi}{4})[1+2\cos^2(\chi_1-\chi_2)](2) \\
= {} & [4\pi(1-\beta^2)\cos^2(\chi_1-\chi_2) - 2\pi(1-\beta^2)[1+2\cos^2(\chi_1-\chi_2)]]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) \\
& + \pi(1-\beta^2)^2[1+2\cos^2(\chi_1-\chi_2)]\left[\frac{1}{1-\beta^2}+\frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&- 4\pi\cos^2(\chi_1 - \chi_2) + 2\pi[1 + 2\cos^2(\chi_1 - \chi_2)] \\
&= [4\pi(1-\beta^2)\cos^2(\chi_1 - \chi_2) - 2\pi(1-\beta^2) - 4\pi(1-\beta^2)\cos^2(\chi_1 - \chi_2)]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) \\
&+ \pi(1-\beta^2)^2[1 + 2\cos^2(\chi_1 - \chi_2)]\left[\frac{1}{(1-\beta^2)} + \frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right] \\
&- 4\pi\cos^2(\chi_1 - \chi_2) + 2\pi[1 + 2\cos^2(\chi_1 - \chi_2)] \\
&= -2\pi(1-\beta^2)\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) + 2\pi \\
&+ \pi(1-\beta^2)^2[1 + 2\cos^2(\chi_1 - \chi_2)]\left[\frac{1}{(1-\beta^2)} + \frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right] \\
&= [-2\pi(1-\beta^2) + \frac{\pi}{2}(1-\beta^2)^2]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) + \pi(1-\beta^2) + 2\pi \\
&+ \left[2\pi(1-\beta^2) + 2\pi(1-\beta^2)^2\frac{1}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\cos^2(\chi_1 - \chi_2) \\
&= \frac{\pi}{2}[-4(1-\beta^2) + (1-\beta^2)^2]\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) + 3\pi - \pi\beta^2 \\
&+ 2\pi(1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\cos^2(\chi_1 - \chi_2),
\end{aligned}
$$

finally, we obtain

$$
\begin{aligned}
F_\beta(\chi_1, \chi_2) = &\frac{[-4(1-\beta^2) + (1-\beta^2)^2]}{4\beta}\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{3}{2} + \frac{\beta^2}{2} \\
&+ (1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\cos^2(\chi_1 - \chi_2), \qquad (3.180)
\end{aligned}
$$

where we neglect 2π because it is canceled after normalized. We next introduce the convenient $F_\beta(\chi_1, \chi_2)$, given by

$$
F_\beta(\chi_1, \chi_2) = A(\beta) + B(\beta)\cos^2(\chi_1 - \chi_2), \qquad (3.181)
$$

where

$$A(\beta) = \frac{[-4(1-\beta^2) + (1-\beta^2)^2]}{4\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) + \frac{3}{2} - \frac{\beta^2}{2}, \tag{3.182}$$

$$B(\beta) = (1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right)\right]. \tag{3.183}$$

To normalized the expression in Eq. (3.181), we have to sum $F_\beta(\chi_1, \chi_2)$ over the polarizations directions specified by the pairs of angles (see as Eq. (3.45)):

$$(\chi_1, \chi_2), \quad (\chi_1 + \frac{\pi}{2}, \chi_2), \quad (\chi_1, \chi_2 + \frac{\pi}{2}), \quad (\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}). \tag{3.184}$$

That is, we have to find the normalization factor

$$\begin{aligned} N_\beta = F_\beta(\chi_1, \chi_2) + F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2) \\ + F_\beta(\chi_1, \chi_2 + \frac{\pi}{2}) + F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}). \end{aligned} \tag{3.185}$$

The explicit expression of above term, by replacing χ_1 by $\chi_1 + \pi/2$, χ_2 by $\chi_2 + \pi/2$, gives

$$F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2) = A(\beta) + B(\beta)\sin^2(\chi_1 - \chi_2), \tag{3.186}$$

$$F_\beta(\chi_1, \chi_2 + \frac{\pi}{2}) = A(\beta) + B(\beta)\sin^2(\chi_1 - \chi_2), \tag{3.187}$$

$$F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2 + \frac{\pi}{2}) = A(\beta) + B(\beta)\cos^2(\chi_1 - \chi_2). \tag{3.188}$$

The following computation for the normalization factor we have upon summing over the set in Eq. (3.184), by substituting Eq. (3.181), Eqs. (3.186)–(3.188) in Eq. (3.185). Giving

$$N_\beta = 4A(\beta) + B(\beta)[\cos^2(\chi_1 - \chi_2) + \sin^2(\chi_1 - \chi_2) + \sin^2(\chi_1 - \chi_2) + \cos^2(\chi_1 - \chi_2)]$$

$$= 2[2A(\beta) + B(\beta)]. \tag{3.189}$$

Accordingly, for the joint conditional probabilities, using the properties in Eq. (3.69), we have

$$P_\beta(\chi_1, \chi_2) = \frac{A(\beta) + B(\beta)\cos^2(\chi_1 - \chi_2)}{2[2A(\beta) + B(\beta)]}, \tag{3.190}$$

For the measurement of only one of the polarizations (χ_1), using the properties in Eq. (3.70), and we compute

$$F_\beta(\chi_1, \chi_2) + F_\beta(\chi_1, \chi_2 + \frac{\pi}{2}) = 2A(\beta) + B(\beta)[\cos^2(\chi_1 - \chi_2) + \sin^2(\chi_1 - \chi_2)]$$

$$= [2A(\beta) + B(\beta)]. \tag{3.191}$$

Therefore, we obtain the measurement of only one of the polarizations (χ_1) as:

$$P_\beta(\chi_1, \_) = \frac{[2A(\beta) + B(\beta)]}{2[2A(\beta) + B(\beta)]} = \frac{1}{2}, \tag{3.192}$$

for all $0 \leqslant \beta \leqslant 1$ and *independent* χ_1.

For the measurement of only one of the polarizations (χ_2), using the properties in Eq. (3.71), and we compute

$$F_\beta(\chi_1, \chi_2) + F_\beta(\chi_1 + \frac{\pi}{2}, \chi_2) = 2A(\beta) + B(\beta)[\cos^2(\chi_1 - \chi_2) + \sin^2(\chi_1 - \chi_2)]$$

$$= [2A(\beta) + B(\beta)]. \tag{3.193}$$

Therefore, we obtain the measurement of only one of the polarizations (χ_2) as:

$$P_\beta(\_, \chi_2) = \frac{[2A(\beta) + B(\beta)]}{2[2A(\beta) + B(\beta)]} = \frac{1}{2}, \tag{3.194}$$

for all $0 \leqslant \beta \leqslant 1$ and *independent* χ_2.

Again we have the important statistical property

$$P_\beta(\chi_1, \chi_2) \neq P_\beta(\chi_1, -)P_\beta(-, \chi_2), \tag{3.195}$$

in general. It is interesting to note that an equality in Eq. (3.190) holds in the extreme relativistic case $\beta \to 1$, where each side is equal to $1/4$.

Only in the limiting case $\beta \to 0$, the joint probability in Eq. (3.190) for this process coincides with that in Eq. (3.145) for the first process.

As in Eq. (3.72), we define

$$S_\beta = P_\beta(\chi_1, \chi_2) - P_\beta(\chi_1, \chi_2') + P_\beta(\chi_1', \chi_2) + P_\beta(\chi_1', \chi_2') - P_\beta(\chi_1', -) - P_\beta(-, \chi_2), \tag{3.196}$$

for four angles χ_1, χ_2, χ_1', χ_2', LHV theory gives [Clauser and Horne, 1974; Clauser and Shimoney, 1978]

$$-1 \leqslant S_\beta \leqslant 0. \tag{3.197}$$

For $\beta \to 1$, an equality holds in Eq. (3.190), $S_\beta \to -1/2$, and this process, to be useful for testing the violation of Eq. (3.197), should not be conducted at very high speeds. For $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, we have $S_\beta = 0.120, 0.184, 0.201, 0.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (3.197) from above. For $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi_1' = 45°$, $\chi_2' = 67°$, we have $S_\beta = -1.120, -1.184, -1.201, -1.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (3.197) from below. For β larger than 0.2 but close to it, S_β already turns out to be too close to the critical interval given in Eq. (3.197) to be relevant experimentally.

CHAPTER IV

SPEED DEPENDENT POLARIZATION CORRELATIONS IN QED AND ENTANGLEMENT

In this chapter, we carry out a detailed investigation of all the electrodynamic processes outlined in Chapter II. We put much emphasis on the concept of entanglement. Explicit polarization correlations for the particles produced in all of these processes are derived. We show clear violations with Bell's inequality of Local Hidden Variables (LHV) theories. The concept of entanglement is outlined in the Appendix to this chapter.

4.1 Introduction

We carry out exact computations of joint probabilities of particle polarizations correlations in QED, to the leading order, for initially *polarized* and *unpolarized* particles. The interesting lesson we have learnt from such studies is that the mere fact that particles emerging from a process have non-zero speeds to reach detectors implies, in general, that their polarizations correlations probabilities *depend* on speed [Yongram and Manoukian, 2003]. The present extended, and needless to say, dynamical analysis shows that this is true, in general. This is unlike formal arguments based simply on combining spins only. As a byproduct of this work, we obtain clear violations with Bell's inequality [Clauser and Horne, 1974; Clauser and Shimoney, 1978; Selleri, 1988; Aspect, Dalibard and Roger, 1982] of LHV theories. We will also see how QED generates speed dependent entangled states.

Several experiments have been performed in recent years [Aspect, Dalibard and Roger, 1982; Fry, 1995; Kaday, Ulman and Wu, 1975; Osuch, Popkiewicz, Szeflinski and Wilhelmi, 1996; Irby, 2003] on particles' polarizations correlations. And, it is expected that the novel properties recorded here by explicit calculations following directly from field theory, which is based on the principle of relativity and quantum theory, will lead to new experiments on polarization correlations monitoring speed in the light of Bell's theorem. We hope that theses computations will be also useful in such areas of physics as quantum teleportation and quantum information in general.

The relevant quantity of interest here in testing Bell's inequality of LHV [Clauser and Horne, 1974] theories is, in a standard notation,

$$S = \frac{p_{12}(a_1, a_2)}{p_{12}(\infty, \infty)} - \frac{p_{12}(a_1, a_2')}{p_{12}(\infty, \infty)} + \frac{p_{12}(a_1', a_2)}{p_{12}(\infty, \infty)} + \frac{p_{12}(a_1', a_2')}{p_{12}(\infty, \infty)} - \frac{p_{12}(a_1', \infty)}{p_{12}(\infty, \infty)} - \frac{p_{12}(\infty, a_2)}{p_{12}(\infty, \infty)} \tag{4.1}$$

as is *computed from* QED. Here a_1, a_2 (a_1', a_2') specify directions along which the polarizations of two particles are measured, with $p_{12}(a_1, a_2)/p_{12}(\infty, \infty)$ denoting the joint probability, and $p_{12}(a_1, \infty)/p_{12}(\infty, \infty)$, $p_{12}(\infty, a_2)/p_{12}(\infty, \infty)$ denoting the probabilities when the polarization of only one of the particles is measured. [$p_{12}(\infty, \infty)$ is normalization factor.] The corresponding probabilities as computed from QED will be denoted by $P[\chi_1, \chi_2]$, $P[\chi_1, -]$, $P[-, \chi_2]$ with χ_1, χ_2 denoting angles the polarization vectors make with certain axes spelled out in the bulk of the paper. To show that QED is in violation with Bell's inequality of LHV, it is sufficient to find one set of angles χ_1, χ_2, χ_1', χ_2' and speed β, such that S, as computed in QED, leads to a value of S with $S > 0$ or $S < -1$. In this work, it is implicitly assumed that the polarization parameters in the particle states are directly observable and may be used for Bell-type measurements as discussed.

The need of a relativistic treatment based on explicit quantum field *dynamical* calculations in testing Bell-like inequalities is critically important. An intriguing and very recent reference [Peres and Terno, 2004], which appeared after our relevant 2003 paper publication [Yongram and Manoukian, 2003], however, discusses the role of relativity in quantum information, in general, and traces the historical development of its role, and most importantly, in the light of our present investigations, emphasizes the need of quantum field theory as necessary for a consistent description of interactions. Most earlier analyses dealing with relativistic aspects, relevant to information theory and Bell-like tests are kinematical of nature or deal with basic general properties of local operators associated with bounded regions of spacetime setting limits on measurements and localizability of quantum systems. These probabilities are well documented in some of the recent monographs [Haag, 1996; Araki, 1999; Bratelli and Robinson, 1987] on the subject. Notable important other recent references on such general aspects which are, however, non-dynamical of nature are [Summers and Werner, 1987; Landau, 1987; Alsing and Milburn, 2002; Gingrich and Adami, 2002; Bartlett and Terno, 2004; Bergou, Gingrich and Adami, 2003; Terno, 2003], and a paper by Czachor [Czachor, 1997] indicating how a possible decrease in violation of Bell's inequalities may occur. In the present work, we are interested in dynamical aspects and related uniquely determined probabilities (intensities) of correlations based on QED, as a fully relativistic quantum field theory (i.e., encompassing quantum theory and relativity) that meet the verdict of experiments. QED is a non-speculative theory and as Feynman [Feynman, 1985] puts it, it is the most precise theory we have in fundamental physics. The closest investigation to our own is that of reference [Pachos and Solono, 2002], a reference we encountered after the submission of our relevant paper for publication, which considers spin-spin interactions, in a QED setting, for *non*-relativistic electrons and, unfortunately, does *not* compute their polarizations correlations which are much relevant experimentally. In the present paper, exact fully relativistic QED, computations, to the leading order, of polarizations *correlations* are explicitly carried out for initially polarized and unpolarized particles. The importance of also considering unpolarized spin stems from the fact that we discover the existence of non-trivial correlations, in the outcome of the processes, even for such mixed states (since one averages over spin) and not only for pure states arising from polarized spins, leading, in particular, in both cases to speed dependent probabilities. The main results of this chapter are given in Eq. (4.82), Eq. (4.89), Eq. (4.92), Eq. (4.142), Eqs. (4.148)–(4.149), Eq. (4.159), Eq. (4.162), Eq. (4.168), Eq. (4.181), Eqs. (4.185)–(4.190). All of these probabilities lead to a violation of Bell's inequality of LHV theories. As the computations are based on the fully relativistic QED, it is of some urgency that relevant experiments are carried out by monitoring speed.

4.2 Polarizations Correlations: Initially Polarized Particles

We now study explicit expressions of simultaneous measurements of two particles polarizations. One of expressions of simultaneous measurements of two particles polarizations is two electron polarizations, so-called spin polarization correlations, in process $e^-e^- \to e^-e^-$, in §4.2.1. An another one is two photon polarizations, so-called photon polarization correlations, in process $e^+e^- \to \gamma\gamma$, in §4.2.2. In this case, we consider the initially polarized particles, not summing over all polarization.

4.2.1 The Initially Polarized Electrons in $e^-e^- \to e^-e^-$

In this section, we consider the process $e^-e^- \to e^-e^-$, in the center of mass (c.m.), with initially polarized electrons with one spin up, along the z-axis, and one spin down with $\mathbf{p}_1 = -\mathbf{p}_2$ denoting the momenta of the initial electrons. We consider momenta of the emerging electrons with $\mathbf{p}'_1 = -\mathbf{p}'_2$ (Shown as in figure **4.1**). The expression for the amplitude of this process ($e^-e^- \to e^-e^-$) is well known (see in §2.4):

$$\mathcal{A} \propto \frac{\overline{u}(p'_1)\gamma^\mu u(p_1)\overline{u}(p'_2)\gamma_\mu u(p_2)}{(p'_1 - p_1)^2} - \frac{\overline{u}(p'_2)\gamma^\mu u(p_1)\overline{u}(p'_1)\gamma_\mu u(p_2)}{(p'_2 - p_1)^2}. \tag{4.2}$$

We simplify the amplitude of this process to convenient calculation, by using the properties of the gamma matrices $\gamma_i = \gamma^i$ and $\gamma_0 = -\gamma^0$ (see as in Appendix A), we obtain

$$\mathcal{A} = -\frac{\overline{u}(p'_1)\gamma^0 u(p_1)\overline{u}(p'_2)\gamma^0 u(p_2)}{(p'_1 - p_1)^2} + \frac{\overline{u}(p'_1)\gamma^j u(p_1)\overline{u}(p'_2)\gamma^j u(p_2)}{(p'_1 - p_1)^2}$$

$$+\frac{\overline{u}(p_2')\gamma^0 u(p_1)\overline{u}(p_1')\gamma^0 u(p_2)}{(p_2'-p_1)^2}-\frac{\overline{u}(p_2')\gamma^j u(p_1)\overline{u}(p_1')\gamma^j u(p_2)}{(p_2'-p_1)^2}. \tag{4.3}$$

For the four-spinors of the initial electrons, we have

$$u(p_1)=\left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\uparrow\\ \mathrm{i}\rho\downarrow\end{pmatrix}, \tag{4.4}$$

$$u(p_2)=\left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\downarrow\\ \mathrm{i}\rho\uparrow\end{pmatrix}, \tag{4.5}$$

where $\uparrow\equiv\begin{pmatrix}1\\0\end{pmatrix}$ denoting the spin up and $\downarrow\equiv\begin{pmatrix}0\\1\end{pmatrix}$ denoting the spin down, $\rho=\dfrac{p}{p^0+m}=\dfrac{\gamma\beta}{\gamma+1}=\dfrac{\beta}{1+\sqrt{1-\beta^2}}$, $p^0=m\gamma$, $p=m\gamma\beta$ and for the four-spinors of the emerging electrons

$$u(p_1')=\left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\xi_1\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\xi_1\end{pmatrix}, \tag{4.6}$$

$$u(p_2')=\left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\xi_2\\ -\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\xi_2\end{pmatrix}, \tag{4.7}$$

where the two-spinors ξ_1, ξ_2 will be specified later. From the amplitude in Eqs. (4.2)–(4.3), we need the adjoint four-spinors of the emerging electrons, $\overline{u}(p_1')$ and $\overline{u}(p_2')$, by using the property of the adjoint four-spinors $\overline{u}=u^\dagger\gamma^0$ (see as in Appendix A), $\gamma^0=\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix}$ denoting 2×2 unit matrix, we have

$$\overline{u}(p_1')=\left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\xi_1^\dagger & -\xi_1^\dagger\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}, \tag{4.8}$$

$$\overline{u}(p_2') = \left(\frac{p^0 + m}{2m}\right)^{1/2} \left(\xi_2^\dagger \quad \xi_2^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}_1'}{p^0 + m}\right). \tag{4.9}$$

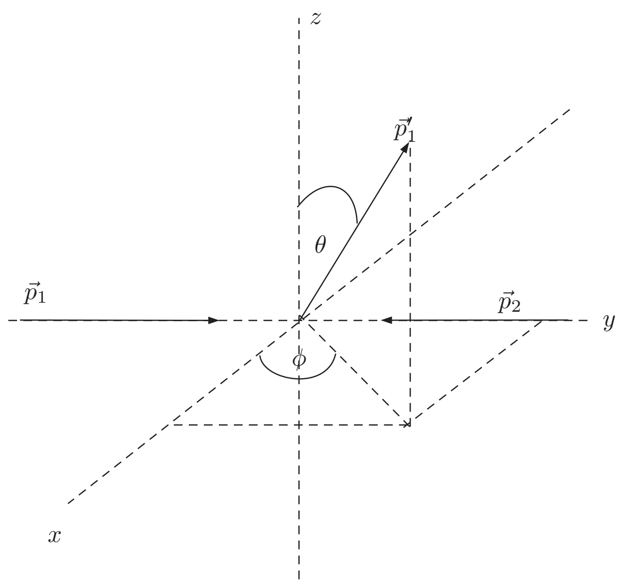

Figure 4.1 The figure depicts direction of momenta of initial electrons with along y-axis and direction of momenta of emerging electrons with the arbitrary direction.

From figure **4.1** , we write momenta of the initial and emerging electrons in x-, y-, z-axis as:

$$\mathbf{p}_1 = p(0, 1, 0) = -\mathbf{p}_2, \tag{4.10}$$

$$\mathbf{p}_1' = p(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) = -\mathbf{p}_2'. \tag{4.11}$$

From the momentum conservation, $p_1 + p_2 = p_1' + p_2'$. Note that each electron has the same energy $p^0 \equiv p_1^0 = p_2^0 = p_1'^0 = p_2'^0$ and the same momentum $p \equiv |\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{p}_1'| = |\mathbf{p}_2'|$, we express $(p_1' - p_1)^2$ and $(p_2' - p_1)^2$ in Eqs. (**4.2**)–(**4.3**) in term of speed (β), $\gamma = 1/\sqrt{1-\beta^2}$, the following expressions:

$$(p_1' - p_1)^2 = (\mathbf{p}_1' - \mathbf{p}_1)^2 - (p_1'^0 - p_1^0)^2$$

$$= |\mathbf{p}_1'|^2 + |\mathbf{p}_1|^2 - 2\mathbf{p}_1' \cdot \mathbf{p}_1$$

$$= p^2 + p^2 - 2p^2 \sin\phi \cos\theta$$

$$(p_1' - p_1)^2 = 2p^2(1 - \sin\phi\cos\theta) = 2\gamma^2 m^2 \beta^2 (1 - \sin\phi\cos\theta), \tag{4.12}$$

and

$$(p_2' - p_1)^2 = (\mathbf{p}_2' - \mathbf{p}_1)^2 - (p_2'^0 - p_1^0)^2$$

$$= |\mathbf{p}_2'|^2 + |\mathbf{p}_1|^2 - 2\mathbf{p}_2' \cdot \mathbf{p}_1$$

$$= p^2 + p^2 + 2p^2 \sin\phi \cos\theta$$

$$(p_2' - p_1)^2 = 2p^2(1 + \sin\phi\cos\theta) = 2\gamma^2 m^2 \beta^2 (1 + \sin\phi\cos\theta). \tag{4.13}$$

We will now calculate the matrix elements in Eq. (**4.3**). We start with the computation of part of the matrix elements: the expression of $\overline{u}(p_1')\gamma^0 u(p_1)$

$$\overline{u}(p_1')\gamma^0 u(p_1) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \uparrow \\ \mathrm{i}\rho \downarrow \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} \uparrow \\ -\mathrm{i}\rho \downarrow \end{pmatrix}$$

$$\overline{u}(p_1')\gamma^0 u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\xi_1^\dagger \uparrow + \mathrm{i}\rho\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \downarrow\right], \tag{4.14}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_1')\gamma^0 u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\xi_1^\dagger \begin{pmatrix}1\\0\end{pmatrix} + \mathrm{i}\rho\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\begin{pmatrix}0\\1\end{pmatrix}\right], \tag{4.15}$$

and the expression of $\overline{u}(p_2')\gamma^0 u(p_2)$

$$\overline{u}(p_2')\gamma^0 u(p_2) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\downarrow\\ \mathrm{i}\rho\uparrow\end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}\downarrow\\ -\mathrm{i}\rho\uparrow\end{pmatrix}$$

$$\overline{u}(p_2')\gamma^0 u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\xi_2^\dagger\downarrow - \mathrm{i}\rho\xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\uparrow\right], \tag{4.16}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_2')\gamma^0 u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\xi_2^\dagger \begin{pmatrix}0\\1\end{pmatrix} - \mathrm{i}\rho\xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\begin{pmatrix}1\\0\end{pmatrix}\right], \tag{4.17}$$

and the expression of $\overline{u}(p_2')\gamma^0 u(p_1)$

$$\overline{u}(p_2')\gamma^0 u(p_1) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\uparrow\\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}\uparrow\\ -\mathrm{i}\rho\downarrow\end{pmatrix}$$

$$\overline{u}(p_2')\gamma^0 u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\xi_2^\dagger\uparrow - \mathrm{i}\rho\xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\downarrow\right], \tag{4.18}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_2')\gamma^0 u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\xi_2^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathrm{i}\rho\xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right], \tag{4.19}$$

and the expression of $\overline{u}(p_1')\gamma^0 u(p_2)$

$$\overline{u}(p_1')\gamma^0 u(p_2) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \downarrow \\ \mathrm{i}\rho\uparrow \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} \downarrow \\ -\mathrm{i}\rho\uparrow \end{pmatrix}$$

$$\overline{u}(p_1')\gamma^0 u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\xi_1^\dagger\downarrow + \mathrm{i}\rho\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\uparrow\right], \tag{4.20}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_1')\gamma^0 u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\xi_1^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{i}\rho\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right], \tag{4.21}$$

and the expression of $\overline{u}(p_1')\gamma^j u(p_1)$

$$\overline{u}(p_1')\gamma^j u(p_1) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} \uparrow \\ \mathrm{i}\rho\downarrow \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} \mathrm{i}\rho\sigma^j\downarrow \\ -\sigma^j\uparrow \end{pmatrix}$$

$$\overline{u}(p_1')\gamma^j u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_1^\dagger\sigma^j\downarrow + \xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j\uparrow\right], \tag{4.22}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_1')\gamma^j u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_1^\dagger\sigma^j \begin{pmatrix}0\\1\end{pmatrix} + \xi_1^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j \begin{pmatrix}1\\0\end{pmatrix}\right], \tag{4.23}$$

and the expression of $\overline{u}(p_2')\gamma^j u(p_2)$

$$\overline{u}(p_2')\gamma^j u(p_2) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix}\begin{pmatrix}\downarrow\\ \mathrm{i}\rho\uparrow\end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}\mathrm{i}\rho\sigma^j\uparrow\\ -\sigma^j\downarrow\end{pmatrix}$$

$$\overline{u}(p_2')\gamma^j u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_2^\dagger\sigma^j\uparrow - \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j\downarrow\right], \tag{4.24}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_2')\gamma^j u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_2^\dagger\sigma^j \begin{pmatrix}1\\0\end{pmatrix} - \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j \begin{pmatrix}0\\1\end{pmatrix}\right], \tag{4.25}$$

and the expression of $\overline{u}(p_2')\gamma^j u(p_1)$

$$\overline{u}(p_2')\gamma^j u(p_1) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix}\begin{pmatrix}\uparrow\\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix}\xi_2^\dagger & \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\end{pmatrix}\begin{pmatrix}\mathrm{i}\rho\sigma^j\downarrow\\ -\sigma^j\uparrow\end{pmatrix}$$

$$\overline{u}(p_2')\gamma^j u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_2^\dagger\sigma^j\downarrow - \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j\uparrow\right], \tag{4.26}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_2')\gamma^j u(p_1) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_2^\dagger\sigma^j \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right], \tag{4.27}$$

finally, the expression of $\overline{u}(p_1')\gamma^j u(p_2)$

$$\overline{u}(p_1')\gamma^j u(p_2) = \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} \downarrow \\ \mathrm{i}\rho\uparrow \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}\begin{pmatrix} \mathrm{i}\rho\sigma^j\uparrow \\ -\sigma^j\downarrow \end{pmatrix}$$

$$\overline{u}(p_1')\gamma^j u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_1^\dagger\sigma^j\uparrow + \xi_1^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j\downarrow\right], \tag{4.28}$$

the explicit expression of above matrix elements is written as

$$\overline{u}(p_1')\gamma^j u(p_2) = \left(\frac{p^0+m}{2m}\right)\left[\mathrm{i}\rho\xi_1^\dagger\sigma^j \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]. \tag{4.29}$$

We will now consider the process $e^-e^- \to e^-e^-$ in vary case that depend on direction of the emerging electrons.

In this case, we consider direction of the emerging electrons in flight with the arbitrary direction in $x-z$ plane. So that, we set direction of the flight of the emerging electron from the x-axis with the angle (ϕ) is equal to zero ($\phi = 0$), shown as in figure **4.2** . So that, we have $(p_1' - p_1)^2 = 2m^2\gamma^2\beta^2 = (p_2' - p_1)^2$,and consider $\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}$ in term of the pauli matrices (see Appendix A) and θ. It is written as:

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} = \frac{1}{p^0+m}\{\sigma^1 p_1' + \sigma^2 p_2' + \sigma^3 p_3'\}$$

$$= \frac{1}{p^0+m}\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} p\sin\theta + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} p\cos\theta\right\}$$

$$= \frac{p}{p^0+m}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} = \rho\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} = \rho\mathbb{M}, \tag{4.30}$$

where $\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma^2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$, $\sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\mathbb{M} \equiv \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$

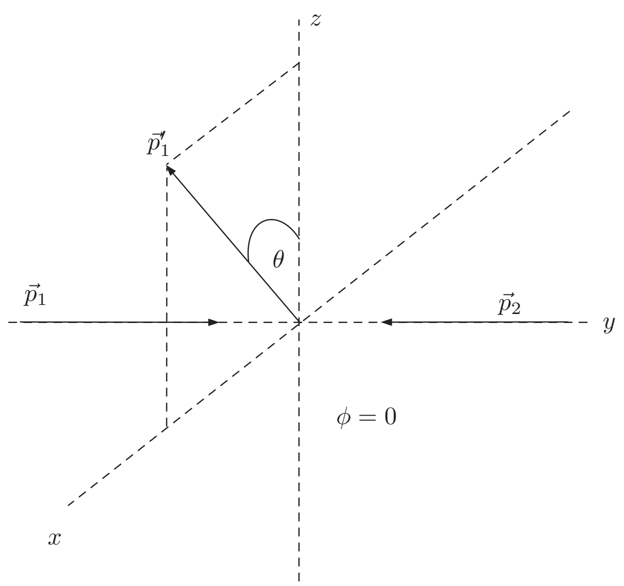

Figure 4.2 The figure depicts direction of momenta of initial electrons with along y-axis and direction of momenta of emerging electrons with the arbitrary direction in $x - z$ plane.

Here we can calculate the matrix element in Eq. (4.3), γ^0, by using properties in Eq. (4.15), Eq. (4.17), Eq. (4.19) and Eq. (4.21), neglected $(p^0+m)/2m$ that cancel out in the normalization, to find the expression of $\overline{u}(p_1')\gamma^0 u(p_1)\overline{u}(p_2')\gamma^0 u(p_2)$, given by

$$\overline{u}(p_1')\gamma^0 u(p_1)\overline{u}(p_2')\gamma^0 u(p_2) \sim \Big[\xi_1^\dagger \uparrow + \mathrm{i}\rho\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \downarrow\Big]\Big[\xi_2^\dagger \downarrow - \mathrm{i}\rho\xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \uparrow\Big]$$

$$= \Big[\xi_1^\dagger \uparrow + \mathrm{i}\rho^2\xi_1^\dagger \mathbb{M} \downarrow\Big]\Big[\xi_2^\dagger \downarrow - \mathrm{i}\rho^2\xi_2^\dagger \mathbb{M} \uparrow\Big]. \tag{4.31}$$

To carry out exact expression of above term, we multiply matrix elements, following:

$$\mathbb{M}\uparrow = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \text{ or } \cos\theta\begin{pmatrix}1\\0\end{pmatrix} + \sin\theta\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.32}$$

$$\mathbb{M}\downarrow = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix} \text{ or } \sin\theta\begin{pmatrix}1\\0\end{pmatrix} - \cos\theta\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.33}$$

the latter works out to

$$\begin{aligned}\overline{u}(p_1')\gamma^0 u(p_1)\overline{u}(p_2')\gamma^0 u(p_2) = \xi_1^\dagger\xi_2^\dagger \Bigg\{ &\Bigg[\begin{pmatrix}1\\0\end{pmatrix}_1 + \mathrm{i}\rho^2\sin\theta\begin{pmatrix}1\\0\end{pmatrix}_1 - \mathrm{i}\rho^2\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_1\Bigg]\\ &\times\Bigg[\begin{pmatrix}0\\1\end{pmatrix}_2 - \mathrm{i}\rho^2\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_2 - \mathrm{i}\rho^2\sin\theta\begin{pmatrix}0\\1\end{pmatrix}_2\Bigg]\Bigg\}\\ = \xi_1^\dagger\xi_2^\dagger \Bigg\{ &\Bigg[(1+\mathrm{i}\rho^2\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_1 - \mathrm{i}\rho^2\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_1\Bigg]\\ &\times\Bigg[(1-\mathrm{i}\rho^2\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_2 - \mathrm{i}\rho^2\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_2\Bigg]\Bigg\},\end{aligned} \tag{4.34}$$

where subscript 1 and 2 denoting the representation spin matrix multiply $\xi_1^\dagger$, $\xi_2^\dagger$, respectively.

The another matrix element in Eq. (4.3), γ^j, $j = 1, 2, 3$, by using properties in Eq. (4.23), Eq. (4.25), Eq. (4.27) and Eq. (4.29), neglected $(p^0+m)/2m$ that cancel out in the normalization, find the expression of $\overline{u}(p_1')\gamma^j u(p_1)\overline{u}(p_2')\gamma^j u(p_2)$, given by

$$\overline{u}(p_1')\gamma^j u(p_1)\overline{u}(p_2')\gamma^j u(p_2) \sim \left[\mathrm{i}\rho\xi_1^\dagger\sigma^j\downarrow + \xi_1^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j\uparrow\right]\left[\mathrm{i}\rho\xi_2^\dagger\sigma^j\uparrow - \xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j\downarrow\right]$$

$$= \left[\mathrm{i}\rho\xi_1^\dagger\sigma^j\downarrow + \rho\xi_1^\dagger\mathbb{M}\sigma^j\uparrow\right]\left[\mathrm{i}\rho\xi_2^\dagger\sigma^j\uparrow - \rho\xi_2^\dagger\mathbb{M}\sigma^j\downarrow\right]. \tag{4.35}$$

To simply above term, we collect $\mathbb{M}\sigma^j$ as:

$$\text{when } j=1;\ \mathbb{M}\sigma^1 = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}\sin\theta & \cos\theta\\ -\cos\theta & \sin\theta\end{pmatrix}, \tag{4.36}$$

$$\text{when } j=2;\ \mathbb{M}\sigma^2 = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} = \mathrm{i}\begin{pmatrix}\sin\theta & -\cos\theta\\ -\cos\theta & -\sin\theta\end{pmatrix}, \tag{4.37}$$

$$\text{when } j=3;\ \mathbb{M}\sigma^3 = \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \tag{4.38}$$

and $\sigma^j\downarrow$

$$\text{when } j=1;\ \sigma^1\downarrow = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} = \uparrow, \tag{4.39}$$

$$\text{when } j=2;\ \sigma^2\downarrow = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = -\mathrm{i}\begin{pmatrix}1\\ 0\end{pmatrix} = -\mathrm{i}\uparrow, \tag{4.40}$$

$$\text{when } j=3;\ \sigma^3\downarrow = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = -\begin{pmatrix}0\\ 1\end{pmatrix} = -\downarrow, \tag{4.41}$$

and $\sigma^j \uparrow$

$$\text{when } j=1;\ \sigma^1\uparrow = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} =\downarrow, \tag{4.42}$$

$$\text{when } j=2;\ \sigma^2\uparrow = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{i}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{i}\downarrow, \tag{4.43}$$

$$\text{when } j=3;\ \sigma^3\uparrow = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} =\uparrow, \tag{4.44}$$

The expression the matrix elements in Eq. (4.3) are calculated as:

$$\begin{aligned} \overline{u}(p_1')\gamma^j u(p_1)\overline{u}(p_2')\gamma^j u(p_2) = & \Big[\mathrm{i}\rho\xi_1^\dagger\uparrow + \rho\xi_1^\dagger\mathbb{A}\uparrow\Big]\Big[\mathrm{i}\rho\xi_2^\dagger\downarrow - \rho\xi_2^\dagger\mathbb{A}\downarrow\Big] \\ & + \Big[\rho\xi_1^\dagger\uparrow + \mathrm{i}\rho\xi_1^\dagger\mathbb{B}\uparrow\Big]\Big[-\rho\xi_2^\dagger\downarrow - \mathrm{i}\rho\xi_2^\dagger\mathbb{B}\downarrow\Big] \\ & + \Big[-\mathrm{i}\rho\xi_1^\dagger\downarrow + \rho\xi_1^\dagger\mathbb{C}\uparrow\Big]\Big[\mathrm{i}\rho\xi_2^\dagger\uparrow - \rho\xi_2^\dagger\mathbb{C}\downarrow\Big], \end{aligned} \tag{4.45}$$

where $\mathbb{A} \equiv \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}, \mathbb{B} \equiv \begin{pmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}, \mathbb{C} \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$

To carry out exact expression of above term, we multiply matrix elements, following:

$$\mathbb{A}\uparrow = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \text{ or } \sin\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \cos\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.46}$$

$$\mathbb{A}\downarrow = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ or } \cos\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.47}$$

$$\mathbb{B}\uparrow = \begin{pmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \text{ or } \sin\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \cos\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.48}$$

$$\mathbb{B}\downarrow = \begin{pmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} \text{ or } -\cos\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.49}$$

$$\mathbb{C}\uparrow = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ or } \cos\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.50}$$

$$\mathbb{C}\downarrow = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \text{ or } -\sin\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.51}$$

the latter works out to

$$\begin{aligned}
\overline{u}(p_1')\gamma^j u(p_1)\overline{u}(p_2')\gamma^j u(p_2) = \rho^2\xi_1^\dagger\xi_2^\dagger \Bigg\{ & \left[\mathrm{i}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \sin\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \cos\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \\
& \times \left[\mathrm{i}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \cos\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \sin\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \\
& + \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \mathrm{i}\sin\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \mathrm{i}\cos\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \\
& \times \left[-\begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 + \mathrm{i}\cos\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \mathrm{i}\sin\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \\
& + \left[\mathrm{i}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 + \cos\theta\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \sin\theta\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right]
\end{aligned}$$

$$\times \left[\mathrm{i} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \sin\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \Bigg\} .$$

After we simplify above term, gives

$$\begin{aligned} \overline{u}(p_1')\gamma^j u(p_1)\overline{u}(p_2')\gamma^j u(p_2) = \rho^2 \xi_1^\dagger \xi_2^\dagger \Bigg\{ & \left[(\mathrm{i} + \sin\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \\ & \times \left[(\mathrm{i} - \sin\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \\ & + \left[(1 + \mathrm{i}\sin\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \mathrm{i}\cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \\ & \times \left[(-1 + \mathrm{i}\sin\theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 + \mathrm{i}\cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \\ & + \left[(-\mathrm{i} + \sin\theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 + \cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right] \\ & \times \left[(\mathrm{i} + \sin\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \Bigg\} . \end{aligned} \tag{4.52}$$

Similarly, by using properties in Eq. (4.15), Eq. (4.17), Eq. (4.19) and Eq. (4.21), neglected $(p^0 + m)/2m$ that cancel out in the normalization, to find the expression of $\overline{u}(p_2')\gamma^0 u(p_1)\overline{u}(p_1')\gamma^0 u(p_2)$, given by

$$\overline{u}(p_2')\gamma^0 u(p_1)\overline{u}(p_1')\gamma^0 u(p_2) \sim \left[\xi_2^\dagger \uparrow - \mathrm{i}\rho\xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0 + m} \downarrow \right] \left[\xi_1^\dagger \downarrow + \mathrm{i}\rho\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0 + m} \uparrow \right]$$

$$= \Big[\xi_2^\dagger \uparrow -\mathrm{i}\rho^2\xi_2^\dagger \mathbb{M} \downarrow\Big]\Big[\xi_1^\dagger \downarrow +\mathrm{i}\rho^2\xi_1^\dagger \mathbb{M} \uparrow\Big]\,. \tag{4.53}$$

By using properties in Eq. (4.30), Eqs. (4.32)–(4.33) gives

$$\begin{aligned}
\overline{u}(p_2')\gamma^0 u(p_1)\overline{u}(p_1')\gamma^0 u(p_2) &= \left[\xi_2^\dagger \begin{pmatrix}1\\0\end{pmatrix}_2 - \mathrm{i}\rho^2\xi_2^\dagger \sin\theta \begin{pmatrix}1\\0\end{pmatrix}_2 + \mathrm{i}\rho^2\xi_2^\dagger \cos\theta \begin{pmatrix}0\\1\end{pmatrix}_2\right] \\
&\times \left[\xi_1^\dagger \begin{pmatrix}0\\1\end{pmatrix}_1 + \mathrm{i}\rho^2\xi_1^\dagger \cos\theta \begin{pmatrix}1\\0\end{pmatrix}_1 + \mathrm{i}\rho^2\xi_1^\dagger \sin\theta \begin{pmatrix}0\\1\end{pmatrix}_1\right] \\
&= \xi_2^\dagger\xi_1^\dagger \left\{\left[\begin{pmatrix}1\\0\end{pmatrix}_2 - \mathrm{i}\rho^2 \sin\theta \begin{pmatrix}1\\0\end{pmatrix}_2 + \mathrm{i}\rho^2 \cos\theta \begin{pmatrix}0\\1\end{pmatrix}_2\right]\right. \\
&\times \left.\left[\begin{pmatrix}0\\1\end{pmatrix}_1 + \mathrm{i}\rho^2 \cos\theta \begin{pmatrix}1\\0\end{pmatrix}_1 + \mathrm{i}\rho^2 \sin\theta \begin{pmatrix}0\\1\end{pmatrix}_1\right]\right\} \\
&= \xi_2^\dagger\xi_1^\dagger \left\{\left[(1 - \mathrm{i}\rho^2 \sin\theta) \begin{pmatrix}1\\0\end{pmatrix}_2 + \mathrm{i}\rho^2 \cos\theta \begin{pmatrix}0\\1\end{pmatrix}_2\right]\right. \\
&\times \left.\left[(1 + \mathrm{i}\rho^2) \sin\theta \begin{pmatrix}0\\1\end{pmatrix}_1 + \mathrm{i}\rho^2 \cos\theta \begin{pmatrix}1\\0\end{pmatrix}_1\right]\right\}.
\end{aligned} \tag{4.54}$$

Similarly, by using properties in Eq. (4.23), Eq. (4.25), Eq. (4.27) and Eq. (4.29), neglected $(p^0 + m)/2m$ that cancel out in the normalization, to find the expression of $\overline{u}(p_2')\gamma^j u(p_1)\overline{u}(p_1')\gamma^j u(p_2)$, given by

$$\begin{aligned}
\overline{u}(p_2')\gamma^j u(p_1)\overline{u}(p_1')\gamma^j u(p_2) &\sim \left[\mathrm{i}\rho\xi_2^\dagger\sigma^j \downarrow -\xi_2^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j \uparrow\right]\left[\mathrm{i}\rho\xi_1^\dagger\sigma^j \uparrow +\xi_1^\dagger \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\sigma^j \downarrow\right] \\
&= \Big[\mathrm{i}\rho\xi_2^\dagger\sigma^j \downarrow -\rho\xi_2^\dagger\mathbb{M}\sigma^j \uparrow\Big]\Big[\mathrm{i}\rho\xi_1^\dagger\sigma^j \uparrow +\rho\xi_1^\dagger\mathbb{M}\sigma^j \downarrow\Big]\,.
\end{aligned} \tag{4.55}$$

By using properties in Eqs. (4.39)–(4.44) gives

$$\overline{u}(p_2')\gamma^j u(p_1)\overline{u}(p_1')\gamma^j u(p_2) = \rho^2 \xi_2^\dagger \xi_2^\dagger \Bigg\{ \left[\mathrm{i} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \sin\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right]$$

$$\times \left[\mathrm{i} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \sin\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right]$$

$$+ \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \mathrm{i}\sin\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \mathrm{i}\cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right]$$

$$\times \left[- \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 - \mathrm{i}\cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \mathrm{i}\sin\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right]$$

$$+ \left[-\mathrm{i} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \sin\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right]$$

$$\times \left[\mathrm{i} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \sin\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \Bigg\}.$$

After we simplify above term, gives

$$\overline{u}(p_2')\gamma^j u(p_1)\overline{u}(p_1')\gamma^j u(p_2) = \rho^2 \xi_1^\dagger \xi_2^\dagger \Bigg\{ \left[(\mathrm{i} - \sin\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right]$$

$$\times \left[(\mathrm{i} + \sin\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 + \cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right]$$

$$+ \left[(1 - \mathrm{i}\sin\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \mathrm{i}\cos\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right]$$

$$\times \left[-(1+\mathrm{i}\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_1 - \mathrm{i}\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_1 \right]$$

$$+ \left[-(\mathrm{i}+\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_2 - \cos\theta\begin{pmatrix}1\\0\end{pmatrix}_2 \right]$$

$$\times \left[(\mathrm{i}-\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_1 + \cos\theta\begin{pmatrix}0\\1\end{pmatrix}_1 \right]\Bigg\} . \qquad (4.56)$$

After the tedious calculation of the matrix elements, we rewrite the amplitude of this process as:

$$\mathcal{A} \sim -\xi_1^\dagger \xi_2^\dagger \Bigg\{ \left[(1+\mathrm{i}\rho^2\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_1 - \mathrm{i}\rho^2\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_1 \right]$$

$$\times \left[(1-\mathrm{i}\rho^2\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_2 - \mathrm{i}\rho^2\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_2 \right]\Bigg\}$$

$$+ \xi_2^\dagger \xi_1^\dagger \Bigg\{ \left[(1+\mathrm{i}\rho^2\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_1 + \mathrm{i}\rho^2\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_1 \right]$$

$$\times \left[(1-\mathrm{i}\rho^2\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_2 + \mathrm{i}\rho^2\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_2 \right]\Bigg\}$$

$$+ \rho^2 \xi_1^\dagger \xi_2^\dagger \Bigg\{ \left[(\mathrm{i}+\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_1 - \cos\theta\begin{pmatrix}0\\1\end{pmatrix}_1 \right]$$

$$\times \left[(\mathrm{i}-\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_2 - \cos\theta\begin{pmatrix}1\\0\end{pmatrix}_2 \right]\Bigg\}$$

$$
\begin{aligned}
&+\rho^2\xi_1^\dagger\xi_2^\dagger\Bigg\{\left[(1+\mathrm{i}\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_1-\mathrm{i}\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_1\right]\\
&\qquad\times\left[(-1+\mathrm{i}\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_2+\mathrm{i}\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_2\right]\Bigg\}\\
&+\rho^2\xi_1^\dagger\xi_2^\dagger\Bigg\{\left[(-\mathrm{i}+\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_1+\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_1\right]\\
&\qquad\times\left[(\mathrm{i}+\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_2-\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_2\right]\Bigg\}\\
&-\rho^2\xi_1^\dagger\xi_2^\dagger\Bigg\{\left[(\mathrm{i}+\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_1+\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_1\right]\\
&\qquad\times\left[(\mathrm{i}-\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_2+\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_2\right]\Bigg\}\\
&+\rho^2\xi_1^\dagger\xi_2^\dagger\Bigg\{\left[(1+\mathrm{i}\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_1+\mathrm{i}\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_1\right]\\
&\qquad\times\left[(1-\mathrm{i}\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_2+\mathrm{i}\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_2\right]\Bigg\}\\
&+\rho^2\xi_1^\dagger\xi_2^\dagger\Bigg\{\left[(\mathrm{i}-\sin\theta)\begin{pmatrix}1\\0\end{pmatrix}_1+\cos\theta\begin{pmatrix}0\\1\end{pmatrix}_1\right]\\
&\qquad\times\left[(\mathrm{i}+\sin\theta)\begin{pmatrix}0\\1\end{pmatrix}_2+\cos\theta\begin{pmatrix}1\\0\end{pmatrix}_2\right]\Bigg\}.
\end{aligned}
\tag{4.57}
$$

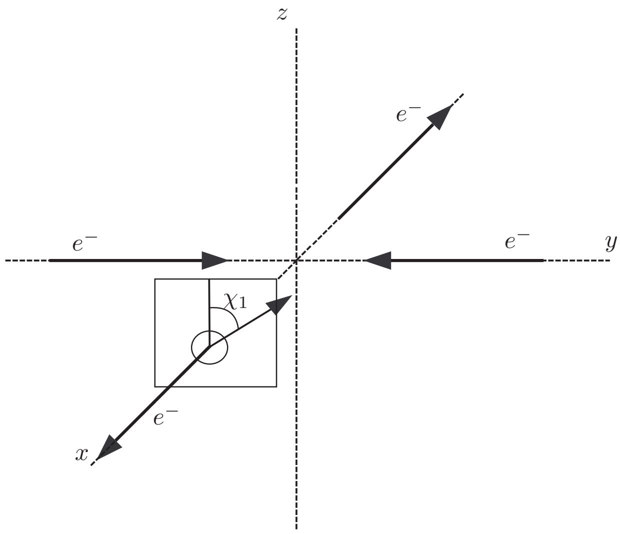

Figure 4.3 The figure depicts e^-e^- scattering, with the electrons initially moving along the y-axis, while the emerging electrons moving along the x-axis. The angle χ_1, measured relative to the z-axis, denotes the orientation of spin of one of the emerging electrons may make.

CASE I: $\theta = \pi/2$ **and** $\phi = 0$

In this case, we set $\theta = \pi/2$ and $\phi = 0$, see in figure **4.3** , given by

$$\mathcal{A} \sim -\xi_1^\dagger \xi_2^\dagger \left\{ \left[(1 + \mathrm{i}\rho^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right] \left[(1 - \mathrm{i}\rho^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \right\}$$

$$+ \xi_2^\dagger \xi_1^\dagger \left\{ \left[(1 + \mathrm{i}\rho^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \left[(1 - \mathrm{i}\rho^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right\}$$

$$+ \rho^2 \xi_1^\dagger \xi_2^\dagger \left\{ \left[(\mathrm{i} + 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right] \left[(\mathrm{i} - 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right\}$$

$$+ \rho^2 \xi_1^\dagger \xi_2^\dagger \left\{ \left[(1+\mathrm{i}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right] \left[(-1+\mathrm{i}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \right\}$$

$$+ \rho^2 \xi_1^\dagger \xi_2^\dagger \left\{ \left[(-\mathrm{i}+1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \left[(\mathrm{i}+1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right\}$$

$$- \rho^2 \xi_1^\dagger \xi_2^\dagger \left\{ \left[(\mathrm{i}+1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \left[(\mathrm{i}-1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right\}$$

$$+ \rho^2 \xi_1^\dagger \xi_2^\dagger \left\{ \left[(1+\mathrm{i}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \left[(1-\mathrm{i}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \right\}$$

$$+ \rho^2 \xi_1^\dagger \xi_2^\dagger \left\{ \left[(\mathrm{i}-1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right] \left[(\mathrm{i}+1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \right\}. \tag{4.58}$$

After simplify above term, we have

$$\mathcal{A} \sim \xi_1^\dagger \xi_2^\dagger (1+\rho^4) \left\{ - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right\}$$

$$+ \rho^2 \xi_1^\dagger \xi_2^\dagger \left\{ -2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right.$$

$$\left. +2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right\}, \tag{4.59}$$

this give

$$\mathcal{A} \sim \xi_1^\dagger \xi_2^\dagger \left\{ -(1+\rho^4) \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - 6\rho^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right.$$

$$+(1+\rho^4)\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+6\rho^2\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\Bigg\}$$

$$=\xi_1^\dagger\xi_2^\dagger\left\{-(1+6\rho^2+\rho^4)\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+(1+6\rho^2+\rho^4)\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\}.$$

The latter work out to gives

$$\mathcal{A}\propto\xi_1^\dagger\xi_2^\dagger(1+6\rho^2+\rho^4)\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]. \tag{4.60}$$

Introducing a entangled state of the emerging electrons that operate with two-spinors, corresponding to χ_1, χ_2, we then have to form the state

$$\mathcal{A}=\xi_1^\dagger\xi_2^\dagger\left|\psi\right\rangle, \tag{4.61}$$

where

$$\left|\psi\right\rangle=C\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right], \tag{4.62}$$

C denoting some constant that specify latter and neglected $(1+6\rho^2+\rho^4)$ that is just constant.

Here the entangled state is normalized, using $||\left|\psi\right\rangle||^2=1$, that is given by

$$|C|^2\left[\begin{pmatrix}0&1\end{pmatrix}_1\begin{pmatrix}1&0\end{pmatrix}_2-\begin{pmatrix}1&0\end{pmatrix}_1\begin{pmatrix}0&1\end{pmatrix}_2\right]\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]=1,$$

to obtaining

$$C=\frac{1}{\sqrt{2}}, \tag{4.63}$$

generating the (normalized) entangled state of the emerging electrons

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]. \tag{4.64}$$

The joint probability of the electrons polarizations correlations is then given by

$$P[\chi_1, \chi_2] = ||\ \xi_1^\dagger \xi_2^\dagger\, |\psi\rangle||^2. \tag{4.65}$$

With the measurements of spin relative to the z-axis. We then specify two-spinors as $\xi_j = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\chi_j/2}\\ \mathrm{e}^{\mathrm{i}\chi_j/2}\end{pmatrix}$, $j = 1, 2$, see appendix A. Therefore, we have

$$\begin{aligned}\xi_1^\dagger \xi_2^\dagger\, |\psi\rangle &= \frac{1}{2\sqrt{2}}\left[\mathrm{e}^{-\mathrm{i}\chi_1/2}\mathrm{e}^{\mathrm{i}\chi_2/2} - \mathrm{e}^{\mathrm{i}\chi_1/2}\mathrm{e}^{-\mathrm{i}\chi_2/2}\right]\\ &= -\frac{\mathrm{i}}{\sqrt{2}}\sin\!\left(\frac{\chi_1-\chi_2}{2}\right),\end{aligned} \tag{4.66}$$

Eq. (4.65) leads to the joint probability of the electrons polarizations correlations

$$P[\chi_1, \chi_2] = \frac{1}{2}\sin^2\left(\frac{\chi_1-\chi_2}{2}\right), \tag{4.67}$$

for all $0 \leqslant \beta \leqslant 1$, leading to rather familiar expression $P[\chi_1, \chi_2] = \sin^2[(\chi_1 - \chi_2)/2]/2$.

If only one of the spins is measured, say, corresponding to χ_1, the probability $P[\chi_1, -]$ may be *equivalently* obtained by summing $P[\chi_1, \chi_2]$ over the two angles

$$\chi_2,\ \chi_2 + \pi, \tag{4.68}$$

for any arbitrarily chosen fixed χ_2, i.e.,

$$P[\chi_1, -] = P[\chi_1, \chi_2] + P[\chi_1, \chi_2 + \pi]. \tag{4.69}$$

Eqs. (4.67)–(4.68) leads to the corresponding probability

$$
\begin{aligned}
P[\chi_1, -] &= \frac{1}{2}\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\chi_1 - (\chi_2 + \pi)}{2}\right) \\
&= \frac{1}{2}\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \frac{1}{2}\cos^2\left(\frac{\chi_1 - \chi_2}{2}\right) \\
&= \frac{1}{2},
\end{aligned}
\tag{4.70}
$$

and similarly for $P[-, \chi_2]$, Eqs. (4.67)–(4.68), replacing $\chi_1 \to \chi_1 + \pi$, leads to the corresponding probability

$$
\begin{aligned}
P[-, \chi_2] &= \frac{1}{2}\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \frac{1}{2}\sin^2\left(\frac{(\chi_1 + \pi) - \chi_2}{2}\right) \\
&= \frac{1}{2}\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \frac{1}{2}\cos^2\left(\frac{\chi_1 - \chi_2}{2}\right) \\
&= \frac{1}{2}.
\end{aligned}
\tag{4.71}
$$

CASE II: $\theta = 0$ **and** $\phi = 0$

In this case, we set $\theta = 0$ and $\phi = 0$, see in figure **4.4** , given by

$$
\begin{aligned}
\mathcal{A} \sim -\xi_1^\dagger \xi_2^\dagger &\left\{ \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \mathrm{i}\rho^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \mathrm{i}\rho^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right\} \\
+ \xi_2^\dagger \xi_1^\dagger &\left\{ \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 + \mathrm{i}\rho^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \right] \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 + \mathrm{i}\rho^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \right\} \\
+ \rho^2 \xi_1^\dagger \xi_2^\dagger &\left\{ \left[\mathrm{i} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \right] \left[\mathrm{i} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right\}
\end{aligned}
$$

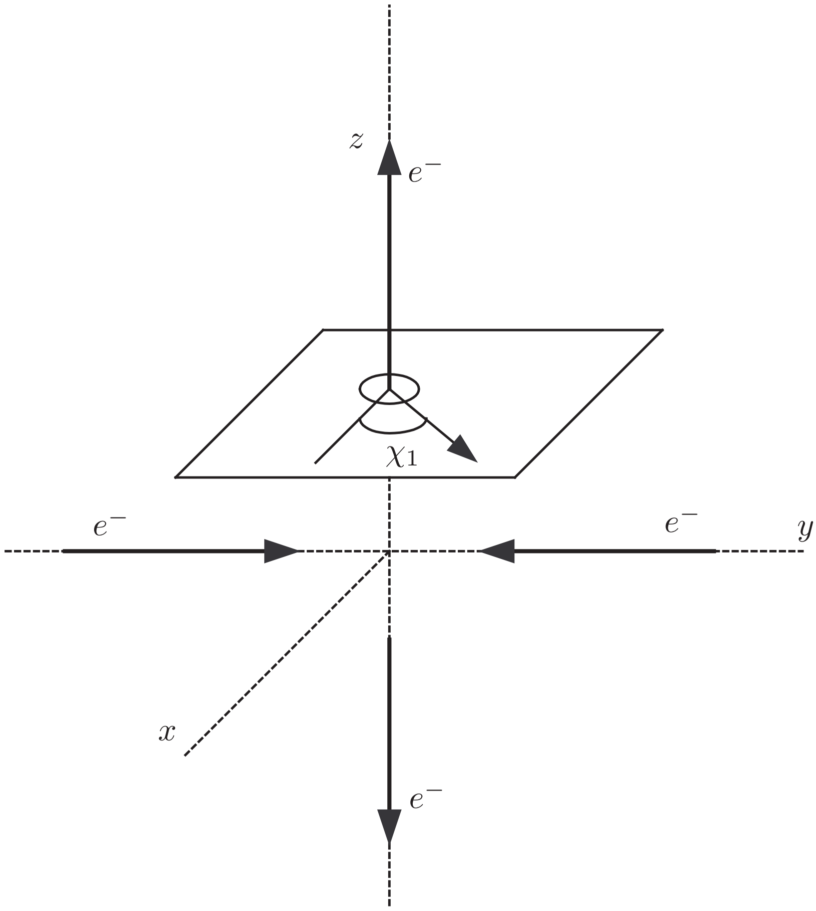

Figure 4.4 The figure depicts e^-e^- scattering, with the electrons initially moving along the y-axis, while the emerging electrons moving along the z-axis. The angle χ_1, measured relative to the x-axis, denotes the orientation of spin of one of the emerging electrons may make.

$$+\,\rho^2\xi_1^\dagger\xi_2^\dagger\left\{\left[\begin{pmatrix}1\\0\end{pmatrix}_1 - \mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\right]\left[-\begin{pmatrix}0\\1\end{pmatrix}_2 + \mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\}$$

$$+\,\rho^2\xi_1^\dagger\xi_2^\dagger\left\{\left[\begin{pmatrix}1\\0\end{pmatrix}_1 - \mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\right]\left[-\begin{pmatrix}0\\1\end{pmatrix}_2 + \mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\}$$

$$-\,\rho^2\xi_1^\dagger\xi_2^\dagger\left\{\left[\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1 + \begin{pmatrix}0\\1\end{pmatrix}_1\right]\left[\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_2\right]\right\}$$

$$+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{\left[\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1+\begin{pmatrix}0\\1\end{pmatrix}_1\right]\left[\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\}$$

$$+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{\left[\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1+\begin{pmatrix}0\\1\end{pmatrix}_1\right]\left[\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\} \tag{4.72}$$

So that, we have $\mathcal{A}$

$$\sim -\xi_1^\dagger\xi_2^\dagger\left\{\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\mathrm{i}\rho^2\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\mathrm{i}\rho^2\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\rho^4\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\}$$

$$+\xi_2^\dagger\xi_1^\dagger\left\{\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\rho^2\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\rho^2\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\rho^4\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right\}$$

$$+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\}$$

$$+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\}$$

$$+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right\}$$

$$-\rho^2\xi_1^\dagger\xi_2^\dagger\left\{-\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right\}$$

$$+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right\}$$

$$+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\} \tag{4.73}$$

We have simplify the amplitude ($\mathcal{A}$) of this process as

$$\begin{aligned}\sim -\xi_1^\dagger\xi_2^\dagger&\left\{\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]+\mathrm{i}\rho^2\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]\right.\\&\left.+\mathrm{i}\rho^2\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]+\rho^4\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]\right\}\\&+\rho^2\xi_1^\dagger\xi_2^\dagger\left\{-6\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+2\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2+2\mathrm{i}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right.\\&\left.+6\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\}\end{aligned} \tag{4.74}$$

The latter works out to gives

$$\begin{aligned}\mathcal{A}=\xi_1^\dagger\xi_2^\dagger&\left\{(1+6\rho^2+\rho^4)\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2-\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]\right.\\&\left.+4\mathrm{i}\rho^2\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\}.\end{aligned} \tag{4.75}$$

Similarly for this case, we introduce a entangled state of the emerging electrons that operate with two-spinors, corresponding to χ_1, χ_2, we then have to form the entangled state

$$\mathcal{A}=\xi_1^\dagger\xi_2^\dagger\,|\phi\rangle\,, \tag{4.76}$$

where

$$|\phi\rangle = N\left\{(1+6\rho^2+\rho^4)\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]\right.$$
$$\left. + 4\mathrm{i}\rho^2\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\}, \tag{4.77}$$

Here the entangled state is normalized, using $||\,|\phi\rangle\,||^2 = 1$, that is given by

$$|N|^2\left\{(1+6\rho^2+\rho^4)^2\left[\begin{pmatrix}0 & 1\end{pmatrix}_1\begin{pmatrix}1 & 0\end{pmatrix}_2 - \begin{pmatrix}1 & 0\end{pmatrix}_1\begin{pmatrix}0 & 1\end{pmatrix}_2\right]\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]\right.$$
$$\left. + 16\rho^4\left[\begin{pmatrix}0 & 1\end{pmatrix}_1\begin{pmatrix}0 & 1\end{pmatrix}_2 + \begin{pmatrix}1 & 0\end{pmatrix}_1\begin{pmatrix}1 & 0\end{pmatrix}_2\right]\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\} = 1$$

to obtaining

$$N = \frac{1}{\sqrt{2}\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}}, \tag{4.78}$$

generating the speed dependent (normalized) entangled state of the emerging electrons we obtain

$$|\phi\rangle = \frac{1}{\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}}\left\{\frac{(1+6\rho^2+\rho^4)}{\sqrt{2}}\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]\right.$$
$$\left. + \frac{4\mathrm{i}\rho^2}{\sqrt{2}}\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]\right\}. \tag{4.79}$$

The joint probability of the electrons polarizations correlations is then given by

$$P[\chi_1, \chi_2] = ||\ \xi_1^\dagger \xi_2^\dagger \,|\phi\rangle ||^2. \tag{4.80}$$

With the measurements of spin relative to the x-axis. We then specify two-spinors as $\xi_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi_j/2} \\ \mathrm{e}^{\mathrm{i}\chi_j/2} \end{pmatrix}$, $j = 1, 2$, see appendix A. Therefore, we have

$$\begin{aligned}
\xi_1^\dagger \xi_2^\dagger \,|\phi\rangle =& \frac{\xi_1^\dagger \xi_2^\dagger}{\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}} \Bigg\{ \frac{(1+6\rho^2+\rho^4)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \\
&+ \frac{4\mathrm{i}\rho^2}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \Bigg\} \\
=& \frac{1}{\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}} \left\{ \frac{(1+6\rho^2+\rho^4)}{2\sqrt{2}} \left[\mathrm{e}^{-\mathrm{i}\chi_1/2}\mathrm{e}^{\mathrm{i}\chi_2/2} - \mathrm{e}^{\mathrm{i}\chi_1/2}\mathrm{e}^{-\mathrm{i}\chi_2/2} \right] \right. \\
&\left. + \frac{4\mathrm{i}\rho^2}{2\sqrt{2}} \left[\mathrm{e}^{-\mathrm{i}\chi_1/2}\mathrm{e}^{-\mathrm{i}\chi_2/2} + \mathrm{e}^{\mathrm{i}\chi_1/2}\mathrm{e}^{\mathrm{i}\chi_2/2} \right] \right\} \\
=& \frac{-\mathrm{i}}{\sqrt{2}\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}} \left\{ (1+6\rho^2+\rho^4) \sin\left(\frac{\chi_1-\chi_2}{2}\right) \right. \\
&\left. - 4\rho^2 \cos\left(\frac{\chi_1+\chi_2}{2}\right) \right\},
\end{aligned} \tag{4.81}$$

Eq. (4.80) leads to the joint probability of the electrons polarizations correlations

$$P[\chi_1, \chi_2] = \frac{1}{2N(\rho)} \left[(1+6\rho^2+\rho^4) \sin\left(\frac{\chi_1-\chi_2}{2}\right) - 4\rho^2 \cos\left(\frac{\chi_1+\chi_2}{2}\right) \right]^2, \tag{4.82}$$

where

$$N(\rho) = [(1+6\rho^2+\rho^4)^2 + 16\rho^4], \tag{4.83}$$

ρ is defined in Eq. (4.5). [For $\beta \to 0$, one obtains a rather familiar expression $P[\chi_1, \chi_2] = \sin^2[(\chi_1 - \chi_2)/2]/2$.]

If only one of the spins is measured, say, corresponding to χ_1, we then have to form the state

$$\xi_1^\dagger |\phi\rangle = \frac{\xi_1^\dagger}{\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}} \left\{ \frac{(1+6\rho^2+\rho^4)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \right] \right.$$

$$\left. + \frac{4\mathrm{i}\rho^2}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 \right] \right\}$$

$$= \frac{1}{2\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}} \left\{ (1+6\rho^2+\rho^4) \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi_1/2} \\ -\mathrm{e}^{\mathrm{i}\chi_1/2} \end{pmatrix}_2 + 4\mathrm{i}\rho^2 \begin{pmatrix} \mathrm{e}^{\mathrm{i}\chi_1/2} \\ \mathrm{e}^{-\mathrm{i}\chi_1/2} \end{pmatrix}_2 \right\}. \tag{4.84}$$

Eq. (4.84) leads to

$$\| \xi_1^\dagger |\phi\rangle \|^2 = \frac{1}{4N(\rho)} \left\{ (1+6\rho^2+\rho^4) \begin{pmatrix} \mathrm{e}^{\mathrm{i}\chi_1/2} & -\mathrm{e}^{-\mathrm{i}\chi_1/2} \end{pmatrix}_2 - 4\mathrm{i}\rho^2 \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi_1/2} & \mathrm{e}^{\mathrm{i}\chi_1/2} \end{pmatrix}_2 \right\}$$

$$\times \left\{ (1+6\rho^2+\rho^4) \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi_1/2} \\ \mathrm{e}^{\mathrm{i}\chi_1/2} \end{pmatrix}_2 + 4\mathrm{i}\rho^2 \begin{pmatrix} \mathrm{e}^{\mathrm{i}\chi_1/2} \\ -\mathrm{e}^{-\mathrm{i}\chi_1/2} \end{pmatrix}_2 \right\}$$

$$= \frac{1}{4N(\rho)} \left\{ (1+6\rho^2+\rho^4)^2 + 16\rho^4 + 4\mathrm{i}\rho^2(1+6\rho^2+\rho^4)[\mathrm{e}^{\mathrm{i}\chi_1} - \mathrm{e}^{-\mathrm{i}\chi_1}] \right.$$

$$\left. -4\mathrm{i}\rho^2(1+6\rho^2+\rho^4)[\mathrm{e}^{-\mathrm{i}\chi_1} - \mathrm{e}^{\mathrm{i}\chi_1}] \right\}$$

$$= \frac{1}{2} - \frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4} \sin\chi_1, \tag{4.85}$$

from which we obtain the corresponding probability

$$P[\chi_1, -] = \frac{1}{2} - \frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4}\sin\chi_1. \tag{4.86}$$

The probability $P[\chi_1, -]$ may be *equivalently* obtained by summing $P[\chi_1, \chi_2]$ over the two angles

$$\chi_2,\ \chi_2+\pi, \tag{4.87}$$

for any arbitrarily chosen fixed χ_2, i.e.,

$$P[\chi_1, -] = P[\chi_1, \chi_2] + P[\chi_1, \chi_2+\pi]. \tag{4.88}$$

Eqs. (4.86)–(4.88) leads to the corresponding probability

$$\begin{aligned} P[\chi_1, -] &= \frac{1}{2N(\rho)}\left[(1+6\rho^2+\rho^4)\sin\left(\frac{\chi_1-\chi_2}{2}\right) - 4\rho^2\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right]^2 \\ &+ \frac{1}{2N(\rho)}\left[(1+6\rho^2+\rho^4)\sin\left(\frac{\chi_1-(\chi_2+\pi)}{2}\right) - 4\rho^2\cos\left(\frac{\chi_1+(\chi_2+\pi)}{2}\right)\right]^2 \\ &= \frac{1}{2} - \frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4}\sin\chi_1, \end{aligned} \tag{4.89}$$

as is easily checked, and similarly, for only one of the spins is measured, say, corresponding to χ_2, we then have to form the state

$$\begin{aligned} \xi_2^\dagger\,|\phi\rangle = \frac{\xi_2^\dagger}{\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}} \Bigg\{ & \frac{(1+6\rho^2+\rho^4)}{\sqrt{2}}\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right] \\ & + \frac{4i\rho^2}{\sqrt{2}}\left[\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]\Bigg\} \end{aligned}$$

$$= \frac{1}{2\sqrt{(1+6\rho^2+\rho^4)^2+16\rho^4}}\left\{(1+6\rho^2+\rho^4)\begin{pmatrix}-\mathrm{e}^{-\mathrm{i}\chi_2/2}\\ \mathrm{e}^{\mathrm{i}\chi_2/2}\end{pmatrix}_1 + 4\mathrm{i}\rho^2\begin{pmatrix}\mathrm{e}^{\mathrm{i}\chi_2/2}\\ \mathrm{e}^{-\mathrm{i}\chi_2/2}\end{pmatrix}_1\right\}. \tag{4.90}$$

Eq. (4.90) leads to

$$\begin{aligned}\|\,\xi_2^\dagger\,|\phi\rangle\,\|^2 &= \frac{1}{4N(\rho)}\left\{(1+6\rho^2+\rho^4)\begin{pmatrix}-\mathrm{e}^{\mathrm{i}\chi_2/2} & \mathrm{e}^{-\mathrm{i}\chi_2/2}\end{pmatrix}_1 - 4\mathrm{i}\rho^2\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\chi_2/2} & \mathrm{e}^{\mathrm{i}\chi_2/2}\end{pmatrix}_1\right\}\\
&\qquad\times\left\{(1+6\rho^2+\rho^4)\begin{pmatrix}-\mathrm{e}^{-\mathrm{i}\chi_2/2}\\ \mathrm{e}^{\mathrm{i}\chi_2/2}\end{pmatrix}_1 + 4\mathrm{i}\rho^2\begin{pmatrix}\mathrm{e}^{\mathrm{i}\chi_2/2}\\ \mathrm{e}^{-\mathrm{i}\chi_2/2}\end{pmatrix}_1\right\}\\
&= \frac{1}{4N(\rho)}\left\{(1+6\rho^2+\rho^4)^2+16\rho^4+4\mathrm{i}\rho^2(1+6\rho^2+\rho^4)[-\mathrm{e}^{\mathrm{i}\chi_2}+\mathrm{e}^{-\mathrm{i}\chi_2}]\right.\\
&\qquad\left. -4\mathrm{i}\rho^2(1+6\rho^2+\rho^4)[\mathrm{e}^{\mathrm{i}\chi_2}-\mathrm{e}^{-\mathrm{i}\chi_2}]\right\}\\
&= \frac{1}{2}+\frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4}\sin\chi_2,\end{aligned} \tag{4.91}$$

from which we obtain the corresponding probability

$$P[-,\chi_2] = \frac{1}{2}+\frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4}\sin\chi_2. \tag{4.92}$$

The probability $P[-,\chi_2]$ may be *equivalently* obtained by summing $P[\chi_1,\chi_2]$ over the two angles

$$\chi_1,\ \chi_1+\pi, \tag{4.93}$$

for any arbitrarily chosen fixed χ_2, i.e.,

$$P[-,\chi_2] = P[\chi_1,\chi_2]+P[\chi_1+\pi,\chi_2]. \tag{4.94}$$

Eqs. (4.92)–(4.94) leads to the corresponding probability

$$P[-,\chi_2] = \frac{1}{2N(\rho)}\left[(1+6\rho^2+\rho^4)\sin\left(\frac{\chi_1-\chi_2}{2}\right) - 4\rho^2\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right]^2$$
$$+\frac{1}{2N(\rho)}\left[(1+6\rho^2+\rho^4)\sin\left(\frac{(\chi_1+\pi)-\chi_2}{2}\right)-4\rho^2\cos\left(\frac{(\chi_1+\pi)+\chi_2}{2}\right)\right]^2$$
$$=\frac{1}{2}+\frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4}\sin\chi_2. \tag{4.95}$$

For all $0 \leqslant \beta \leqslant 1$, angles χ_1, χ_2, χ_1', χ_2' are readily found leading to a violation of Bell's inequality of LHV theories. For example, for $\beta = 0.3$, $\chi_1 = 0°$, $\chi_2 = 137°$, $\chi_1' = 12°$, $\chi_2' = 45°$, $S = -1.79$ violating the inequality from below.

4.2.2 The Initially Polarized Electron and Positron in $e^+e^- \to \gamma\gamma$

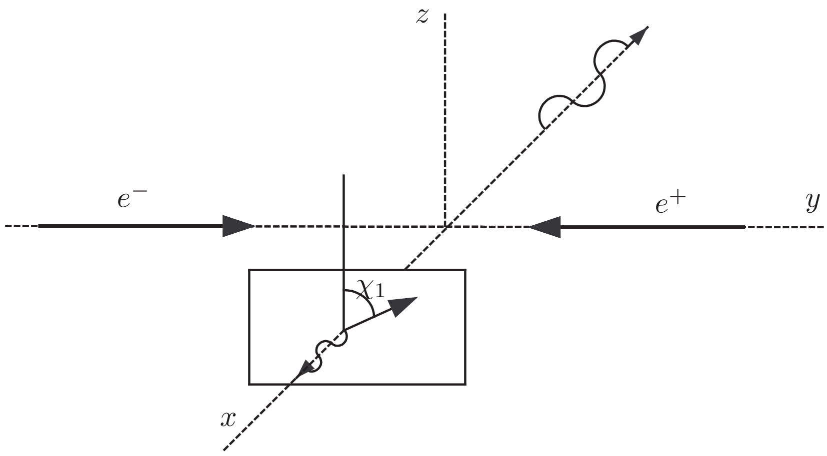

Figure 4.5 The figure depicts e^+e^- annihilation into 2γ, with e^+, e^- moving along the y-axis, and the emerging photons moving along the x-axis. χ_1 denotes the angle the polarization vector of one of the photons may make with the z-axis.

Now we consider the process $e^+e^- \to 2\gamma$, in the c.m. of e^-, e^+ with spins up, along the z-axis, and down, respectively. With $\mathbf{p}_1 = \mathbf{p}(e^-) = \gamma m\beta(0,1,0) =$

$-\mathbf{p}(e^+) = -\mathbf{p}_2$, we have the amplitude of this process (see Eq. (2.21) in §2.2):

$$\mathcal{A} \propto \mathrm{i}e^2 \overline{v}(\mathbf{p}_2, \sigma_2) \left[\frac{\gamma^\mu \gamma k_1 \gamma^\nu}{2p_1 k_1} + \frac{\gamma^\nu \gamma k_2 \gamma^\mu}{2p_1 k_2} + \frac{\gamma^\mu p_1^\nu}{p_1 k_1} + \frac{\gamma^\nu p_1^\mu}{p_1 k_2} \right] u(\mathbf{p}_1, \sigma_1) e_1^\nu e_2^\mu, \tag{4.96}$$

where $e_1^\mu = (0, \mathbf{e}_1)$, $e_2^\mu = (0, \mathbf{e}_2)$ are the polarizations of the photons with $(j = 1, 2)$

$$\mathbf{e}_j = (-\cos\theta \cos\chi_j, \sin\chi_j, \sin\theta\cos\chi_j) \equiv \left(e_j^{(1)}, e_j^{(2)}, e_j^{(3)} \right), \tag{4.97}$$

and for e^-, e^+ the four-spinors given by

$$u = \left(\frac{p^0 + m}{2m} \right)^{1/2} \begin{pmatrix} \uparrow \\ \mathrm{i}\rho \downarrow \end{pmatrix}, \tag{4.98}$$

$$v = \left(\frac{p^0 + m}{2m} \right)^{1/2} \begin{pmatrix} \mathrm{i}\rho \downarrow \\ \uparrow \end{pmatrix}, \tag{4.99}$$

with $\rho = \dfrac{\gamma\beta}{\gamma + 1} = \dfrac{\beta}{1 + \sqrt{1 - \beta^2}}$, and we consider momenta of the photons

$$\mathbf{k}_1 = \gamma m (\sin\theta, 0, \cos\theta) = -\mathbf{k}_2 \tag{4.100}$$

where we have used the facts that

$$|\mathbf{k}_1| = |\mathbf{k}_2| = k_1^0 = k_2^0 = p^0(e^\pm) \equiv p^0 = \gamma m. \tag{4.101}$$

We need the adjoint four-spinors of the emerging prositron, $\overline{v}(\mathbf{p}_2, \sigma_2)$, by using the property of the adjoint four-spinors $\overline{v} = v^\dagger \gamma^0$ (see as in Appendix A), $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ denoting 2×2 unit matrix, we have

$$\overline{v} \sim \left(\frac{p^0 + m}{2m} \right)^{1/2} \left(\mathrm{i}\rho \begin{pmatrix} 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \right), \tag{4.102}$$

where neglected minus sign.

We have computed the joint probability polarization of photons in Chapter III that is the initially particles (e^-, e^+) unpolarized, summing over all spin of initial particles. In this case we will compute joint probability polarization of photons that is the initially particles (e^-, e^+) polarized. We then start with the computation of the amplitude ($\mathcal{A}$) of the process $e^+e^- \rightarrow 2\gamma$, in the c.m. of e^+, e^- with spins up, along the z-axis, and down, respectively. With

$$\mathbf{p}_1 = \mathbf{p}(e^-) = \gamma m\beta(0,1,0) = -\mathbf{p}(e^+) = -\mathbf{p}_2. \tag{4.103}$$

To start the computation of the amplitude of this process, let us introduce the new amplitude that is convenient, given by

$$\begin{aligned}\mathcal{A} \sim \overline{v}\left(\frac{\gamma^i\gamma^m\gamma^j}{2p_1k_1} - \frac{\gamma^j\gamma^m\gamma^i}{2p_1k_2}\right) uk_m e_1^{(j)}e_2^{(i)} \\ - \overline{v}\left(\frac{\gamma^i\gamma^0\gamma^j}{2p_1k_1} + \frac{\gamma^j\gamma^0\gamma^i}{2p_1k_2}\right) uk^0 e_1^{(j)}e_2^{(i)} \\ + \overline{v}\left(\frac{\gamma^i\delta^{j2}}{p_1k_1} - \frac{\gamma^j\delta^{i2}}{p_1k_2}\right) upe_1^{(j)}e_2^{(i)},\end{aligned} \tag{4.104}$$

where $e_1^{(i)}$, $e_2^{(j)}$ are defined in Eq. (4.97), k_m, $m = 1, 2, 3$ is the momentum of photon, k^0 is the energy, and $\overline{v}$, u are denoted the four-spinors of positron, electron, respectively.

To extract the singular terms in Eq. (4.104) we first compute each term in $(\cdot)$ in right-hand side of Eq. (4.104), start with

$$\begin{aligned}\gamma^i\gamma^m\gamma^j &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_m \\ -\sigma_m & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} -\sigma_m\sigma_j & 0 \\ 0 & -\sigma_m\sigma_j \end{pmatrix}.\end{aligned} \tag{4.105}$$

Therefore

$$\gamma^i\gamma^m\gamma^j = \begin{pmatrix} 0 & -\sigma_i\sigma_m\sigma_j \\ \sigma_i\sigma_m\sigma_j & 0 \end{pmatrix}. \tag{4.106}$$

This result that express in generally form by using the sigma properties, given by

$$\sigma_m\sigma_j = \delta_{mj} + \mathrm{i}\epsilon_{mjk}\sigma_k, \tag{4.107}$$

from this gives

$$\begin{aligned}
\sigma_i\sigma_m\sigma_j &= \sigma_i(\delta_{mj} + \mathrm{i}\epsilon_{mjk}\sigma_k) \\
&= \sigma_i\delta_{mj} + \mathrm{i}\epsilon_{mjk}\sigma_i\sigma_k \\
&= \sigma_i\delta_{mj} + \mathrm{i}\epsilon_{mjk}(\delta_{ik} + \mathrm{i}\epsilon_{ikl}\sigma_l) \\
&= \sigma_i\delta_{mj} + \mathrm{i}\epsilon_{mjk}\delta_{ik} - \epsilon_{mjk}\epsilon_{ikl}\sigma_l \\
&= \sigma_i\delta_{mj} + \mathrm{i}\epsilon_{mji} + \epsilon_{mjk}\epsilon_{ilk}\sigma_l \\
&= \sigma_i\delta_{mj} + \mathrm{i}\epsilon_{mji}\delta_{ik} + (\delta_{mi}\delta_{jl} - \delta_{ml}\delta_{ji})\sigma_l \\
\sigma_i\sigma_m\sigma_j &= (\sigma_i\delta_{mj} + \sigma_j\delta_{mi} - \sigma_m\delta_{ji}) + \mathrm{i}\epsilon_{mji}. \qquad (4.108)
\end{aligned}$$

To do this, replace Eq. (4.104) by Eq. (4.108). Therefore we obtain the new form of Eq. (4.104), be written as

$$\begin{aligned}
\gamma^i\gamma^m\gamma^j &= -\sigma_i\sigma_m\sigma_j \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \\
&= -[(\sigma_i\delta_{mj} + \sigma_j\delta_{mi} - \sigma_m\delta_{ji}) + \mathrm{i}\epsilon_{mji}] \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}
\end{aligned}$$

$$= -\sigma_i\delta_{mj}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} - \sigma_j\delta_{mi}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} + \sigma_m\delta_{ji}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$$- \mathrm{i}\epsilon_{mji}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{4.109}$$

the latter works out to give

$$\gamma^i\gamma^m\gamma^j = -\delta_{mj}\gamma_i - \delta_{mi}\gamma_j + \delta_{ij}\gamma_m - \mathrm{i}\epsilon_{mji}\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{4.110}$$

Similarly, to calculate one of Eq. (4.104), by setting $m = 0$, we then have $\gamma^i\gamma^0\gamma^j$. Therefore we write

$$\gamma^i\gamma^0\gamma^j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix}$$

$$= \sigma_i\sigma_j\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$= (\delta_{ij} + \mathrm{i}\epsilon_{ijk}\sigma_k)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{4.111}$$

to obtain

$$\gamma^i\gamma^0\gamma^j = \delta_{ij}\begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix} + \mathrm{i}\epsilon_{ijk}\begin{pmatrix}\sigma_k & 0 \\ 0 & -\sigma_k\end{pmatrix}. \tag{4.112}$$

After we calculate each term in $(\cdot)$ in Eq. (4.104) that is written as the matrix element. Since we use above computation to find $\overline{v}(\gamma^i\gamma^0\gamma^j)u$, be written as

$$\begin{aligned}
\overline{v}(\gamma^i\gamma^0\gamma^j)u = &\left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)\delta_{ij}\begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\uparrow \\ \mathrm{i}\rho\downarrow\end{pmatrix} \\
&+ \mathrm{i}\epsilon_{ijk}\left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)\begin{pmatrix}\sigma_k & 0 \\ 0 & -\sigma_k\end{pmatrix}\begin{pmatrix}\uparrow \\ \mathrm{i}\rho\downarrow\end{pmatrix} \\
&= \delta_{ij}\left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)\begin{pmatrix}\uparrow \\ -\mathrm{i}\rho\downarrow\end{pmatrix} \\
&+ \mathrm{i}\epsilon_{ijk}\left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)\begin{pmatrix}\sigma_k\uparrow \\ -\mathrm{i}\rho\sigma_k\downarrow\end{pmatrix} \\
&= \delta_{ij}\mathrm{i}\rho\left(\frac{p^0+m}{2m}\right)\left[\begin{pmatrix}0 & 1\end{pmatrix}\uparrow - \begin{pmatrix}1 & 0\end{pmatrix}\downarrow\right] \\
&- \epsilon_{ijk}\rho\left(\frac{p^0+m}{2m}\right)\left[\begin{pmatrix}0 & 1\end{pmatrix}\sigma_k\uparrow - \begin{pmatrix}1 & 0\end{pmatrix}\sigma_k\downarrow\right]
\end{aligned}$$

$$\overline{v}(\gamma^i\gamma^0\gamma^j)u = -\epsilon_{ijk}\rho\left(\frac{p^0+m}{2m}\right)\left[\begin{pmatrix}0 & 1\end{pmatrix}\sigma_k\uparrow - \begin{pmatrix}1 & 0\end{pmatrix}\sigma_k\downarrow\right], \tag{4.113}$$

or the explicit above term can rewrite as

$$\overline{v}(\gamma^i\gamma^0\gamma^j)u = -\epsilon_{ijk}\rho\left(\frac{p^0+m}{2m}\right)\left[\begin{pmatrix}0 & 1\end{pmatrix}\sigma_k\begin{pmatrix}1 \\ 0\end{pmatrix} - \begin{pmatrix}1 & 0\end{pmatrix}\sigma_k\begin{pmatrix}0 \\ 1\end{pmatrix}\right]. \tag{4.114}$$

In Eq. (4.114) that we can express it in the exact term. With calculation of the matrix element in the right-hand side of Eq. (4.114), $\begin{pmatrix} 0 & 1 \end{pmatrix} \sigma_k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, express as

- $k = 1$; $\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$ or δ^{k1},
- $k = 2$; $\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{i} \end{pmatrix} = \mathrm{i}$ or $\mathrm{i}\delta^{k2}$,
- $k = 3$; $\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ or $(0)\delta^{k3}$,

and for $\begin{pmatrix} 1 & 0 \end{pmatrix} \sigma_k \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ express as

- $k = 1$; $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ or δ^{k1},
- $k = 2$; $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{i} \\ 0 \end{pmatrix} = -\mathrm{i}$ or $-\mathrm{i}\delta^{k2}$,
- $k = 3$; $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0$ or $(0)\delta^{k3}$.

From above result we obtain

$$\overline{v}(\gamma^i \gamma^0 \gamma^j) u = -\epsilon_{ijk} \rho \left(\frac{p^0 + m}{2m} \right) \left[(1-1)\delta^{k1} + (\mathrm{i} - (-\mathrm{i}))\delta^{k2} + (0-0)\delta^{k3} \right]. \quad (4.115)$$

The latter work out to give

$$\overline{v}(\gamma^i \gamma^0 \gamma^j) u = -(2\mathrm{i}) \rho \epsilon_{ij2} \left(\frac{p^0 + m}{2m} \right), \quad (4.116)$$

where we note that consider only $k = 2$, because another component $k = 1$ and $k = 3$ make the matrix element equal to zero. Consider $\overline{v}\gamma^i u$

$$\overline{v}\gamma^i u = \left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)\begin{pmatrix}0 & \sigma_i \\ -\sigma_i & 0\end{pmatrix}\begin{pmatrix}\uparrow \\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)\begin{pmatrix}\mathrm{i}\rho\sigma_i\downarrow \\ -\sigma_i\uparrow\end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\left[-\rho^2\begin{pmatrix}0 & 1\end{pmatrix}\sigma_i\downarrow - \begin{pmatrix}1 & 0\end{pmatrix}\sigma_i\uparrow\right]$$

$$\overline{v}\gamma^i u = -\left(\frac{p^0+m}{2m}\right)\left[\rho^2\begin{pmatrix}0 & 1\end{pmatrix}\sigma_i\downarrow + \begin{pmatrix}1 & 0\end{pmatrix}\sigma_i\uparrow\right], \tag{4.117}$$

and the explicit above term rewrite as

$$\overline{v}\gamma^i u = -\left(\frac{p^0+m}{2m}\right)\left[\rho^2\begin{pmatrix}0 & 1\end{pmatrix}\sigma_i\begin{pmatrix}0 \\ 1\end{pmatrix} + \begin{pmatrix}1 & 0\end{pmatrix}\sigma_i\begin{pmatrix}1 \\ 0\end{pmatrix}\right]. \tag{4.118}$$

In Eq. (4.118) that we can express it in the exact term. With calculation of the matrix element in the right-hand side of Eq. (4.118), $\begin{pmatrix}0 & 1\end{pmatrix}\sigma_i\begin{pmatrix}0 \\ 1\end{pmatrix}$

- $i = 1$; $\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} = 0$ or $(0)\delta^{i1}$,
- $i = 2$; $\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i} \\ \mathrm{i} & 0\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}-\mathrm{i} \\ 0\end{pmatrix} = 0$ or $(0)\delta^{i2}$,
- $i = 3$; $\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 \\ -1\end{pmatrix} = -1$ or $-\delta^{i3}$,

and for $\begin{pmatrix} 1 & 0 \end{pmatrix} \sigma_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ express as

- $i = 1;\ \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ or $(0)\delta^{i1}$,
- $i = 2;\ \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \mathrm{i} \end{pmatrix} = 0$ or $(0)\delta^{i2}$,
- $i = 3;\ \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ or δ^{i3}.

By substituting the above matrix element in Eq. (4.118) we obtain

$$\overline{v}\gamma^i u = -\left(\frac{p^0 + m}{2m}\right)\left[(\rho^2(0) + 0)\delta^{i1} + (\rho^2(0) + 0)\delta^{i2} + (\rho^2 - 1)\delta^{i3}\right]. \tag{4.119}$$

The latter work out to give

$$\overline{v}\gamma^i u = -\left(\frac{p^0 + m}{2m}\right)(\rho^2 - 1)\delta^{i3} \tag{4.120}$$

where we note that consider only $i = 3$, because another component $i = 1$ and $i = 2$ make the matrix element equal to zero. For the final matrix element, $\overline{v}(\gamma^i\gamma^m\gamma^j)u$ that is needed in the calculation of the expression of the amplitude in Eq. (4.104), write as

$$\begin{aligned}
&\overline{v}(\gamma^i\gamma^m\gamma^j)u \\
&= \left(\frac{p^0 + m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix} 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\right)\left[-\delta_{mj}\gamma_i - \delta_{mi}\gamma_j + \delta_{ij}\gamma_m\right]\begin{pmatrix} \uparrow \\ \mathrm{i}\rho \downarrow \end{pmatrix} \\
&\quad - \mathrm{i}\epsilon_{mji}\left(\frac{p^0 + m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix} 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \uparrow \\ \mathrm{i}\rho \downarrow \end{pmatrix}
\end{aligned}$$

$$= \left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)[-\delta_{mj}\gamma_i - \delta_{mi}\gamma_j + \delta_{ij}\gamma_m]\begin{pmatrix}\uparrow \\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$- \mathrm{i}\epsilon_{mji}\left(\frac{p^0+m}{2m}\right)\left(\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\end{pmatrix}\right)\begin{pmatrix}0 & \mathbb{1} \\ -\mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\uparrow \\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)[\delta_{mj}(\rho^2-1)\delta_{i3} + \delta_{mi}(\rho^2-1)\delta_{j3} - \delta_{ij}(\rho^2-1)\delta_{m3}]$$
$$+ \mathrm{i}\epsilon_{mji}\left(\frac{p^0+m}{2m}\right)[\rho^2+1]. \tag{4.121}$$

The latter work out to give

$$\overline{v}(\gamma^i\gamma^m\gamma^j)u = \left(\frac{p^0+m}{2m}\right)(\rho^2-1)[\delta_{mj}\delta_{i3}+\delta_{mi}\delta_{j3} - \delta_{ij}\delta_{m3}] + \mathrm{i}\epsilon_{mji}\left(\frac{p^0+m}{2m}\right)[\rho^2+1]. \tag{4.122}$$

We recall all matrix element, calculated in Eq. (4.116), Eq. (4.118), Eq. (4.120) and Eq. (4.122), to apply in the computation of the amplitude, $\mathcal{A}$, in Eq. (4.104). For the first term in the right-hand side of Eq. (4.104) rewrite as

$$\overline{v}\left(\frac{\gamma^i\gamma^m\gamma^j}{2p_1k_1} - \frac{\gamma^j\gamma^m\gamma^i}{2p_1k_2}\right)uk_m e_1^{(j)}e_2^{(i)}$$

$$= \left[\frac{1}{2p_1k_1} - \frac{1}{2p_1k_2}\right]\overline{v}\gamma^j\gamma^m\gamma^i uk_m e_1^j e_2^i$$

$$= \left[\frac{1}{2p_1k_1} - \frac{1}{2p_1k_2}\right]\left(\frac{p^0+m}{2m}\right)(\rho^2-1)[\delta_{mj}\delta_{i3} + \delta_{mi}\delta_{j3} - \delta_{ij}\delta_{m3}]k_m e_1^j e_2^i$$

$$+ \mathrm{i}\epsilon_{mji}\left(\frac{p^0+m}{2m}\right)[\rho^2+1]k_m e_1^j e_2^i\left[\frac{1}{2p_1k_1} - \frac{1}{2p_1k_2}\right]$$

$$= -\left(\frac{p^0+m}{2m}\right)(\rho^2-1)k_3(\mathbf{e}_1\cdot\mathbf{e}_2)\left[\frac{1}{2p_1k_1} - \frac{1}{2p_1k_2}\right]$$

$$+ \mathrm{i}\left(\frac{p^0+m}{2m}\right)(\rho^2+1)\mathbf{k}\cdot(\mathbf{e}_1\times\mathbf{e}_2)\left[\frac{1}{2p_1k_1} - \frac{1}{2p_1k_2}\right], \tag{4.123}$$

with negligence $\left(\frac{p^0+m}{2m}\right)$ that is some constant, and cancel out after the normalization of the amplitude. We then rewrite above term as

$$\overline{v}\left(\frac{\gamma^i\gamma^m\gamma^j}{2p_1k_1}-\frac{\gamma^j\gamma^m\gamma^i}{2p_1k_2}\right)uk_m e_1^{(j)}e_2^{(i)} \sim -(\rho^2-1)k_3(\mathbf{e}_1\cdot\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]$$
$$+\,\mathrm{i}(\rho^2+1)\mathbf{k}\cdot(\mathbf{e}_1\times\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]. \tag{4.124}$$

For the second term in the right-hand side of Eq. (4.104) rewrite as

$$\begin{aligned}
&-\overline{v}\left(\frac{\gamma^i\gamma^0\gamma^j}{2p_1k_1}+\frac{\gamma^j\gamma^0\gamma^i}{2p_1k_2}\right)uk^0e_1^{(j)}e_2^{(i)}\\
&=-\left(\frac{1}{2p_1k_1}+\frac{1}{2p_1k_2}\right)\overline{v}\gamma^i\gamma^0\gamma^j uk^0e_1^je_2^i\\
&=\left(\frac{1}{2p_1k_1}+\frac{1}{2p_1k_2}\right)(2\mathrm{i})\rho\epsilon_{ij2}\left(\frac{p^0+m}{2m}\right)k^0e_1^je_2^i\\
&=-\left(\frac{1}{2p_1k_1}+\frac{1}{2p_1k_2}\right)(2\mathrm{i})\rho\left(\frac{p^0+m}{2m}\right)k^0(\mathbf{e}_1\times\mathbf{e}_2)_2,
\end{aligned} \tag{4.125}$$

with negligence $\left(\frac{p^0+m}{2m}\right)$ that is some constant, and cancel out after the normalization of the amplitude. We then rewrite above term as

$$-\overline{v}\left(\frac{\gamma^i\gamma^0\gamma^j}{2p_1k_1}+\frac{\gamma^j\gamma^0\gamma^i}{2p_1k_2}\right)uk^0e_1^{(j)}e_2^{(i)} \sim -\left(\frac{1}{2p_1k_1}+\frac{1}{2p_1k_2}\right)(2\mathrm{i})\rho k^0(\mathbf{e}_1\times\mathbf{e}_2)_2 \tag{4.126}$$

For the final term in the right-hand side of Eq. (4.104) rewrite as

$$\overline{v}\left(\frac{\gamma^i\delta^{j2}}{p_1k_1}-\frac{\gamma^j\delta^{i2}}{p_1k_2}\right)upe_1^{(j)}e_2^{(i)} \sim (\rho^2-1)\left(\frac{\delta^{i3}\delta^{j2}}{p_1k_1}-\frac{\delta^{j3}\delta^{i2}}{p_1k_2}\right)pe_1^je_2^i. \tag{4.127}$$

By replacing Eq. (4.124), Eqs. (4.126)–(4.127) in Eq. (4.104) we obtain the amplitude of the process $e^+e^-\to\gamma\gamma$, in the c.m. of e^+, e^- with spin up, along the z-axis and down, respectively, and two emerging photons moving along $\mathbf{n}$ with the angle ϕ respect x-axis and the angle θ z-axis. We then have

$$\mathcal{A} \sim -(\rho^2-1)k_3(\mathbf{e}_1\cdot\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]$$
$$+\,\mathrm{i}(\rho^2+1)\mathbf{k}\cdot(\mathbf{e}_1\times\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]$$
$$-\left(\frac{1}{2p_1k_1}+\frac{1}{2p_1k_2}\right)(2\mathrm{i})\rho k^0(\mathbf{e}_1\times\mathbf{e}_2)_2$$
$$+\,(\rho^2-1)\left(\frac{\delta^{i3}\delta^{j2}}{p_1k_1}-\frac{\delta^{j3}\delta^{i2}}{p_1k_2}\right)pe_1^je_2^i. \tag{4.128}$$

Using

$$\frac{\mathbf{k}_1}{|\mathbf{k}_1|}=\mathbf{n}, \tag{4.129}$$

the amplitude $\mathcal{A}$ is then given by

$$\mathcal{A} \sim (1-\rho^2)m\gamma n_3(\mathbf{e}_1\times\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]$$
$$+\,\mathrm{i}(1+\rho^2)m\gamma\mathbf{n}\cdot(\mathbf{e}_1\times\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]$$
$$-\,m\gamma\left(\frac{1}{2p_1k_1}+\frac{1}{2p_1k_2}\right)(2\mathrm{i})\rho(\mathbf{e}_1\times\mathbf{e}_2)_2$$
$$+\,m\gamma\beta(\rho^2-1)\left(\frac{\delta^{i3}\delta^{j2}}{p_1k_1}-\frac{\delta^{j3}\delta^{i2}}{p_1k_2}\right)e_1^je_2^i. \tag{4.130}$$

By dividing Eq. (4.130) by $m\gamma$ we obtain the amplitude $\mathcal{A}$ as

$$\mathcal{A} \sim (1-\rho^2)n_3(\mathbf{e}_1\cdot\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]$$
$$+\,\mathrm{i}(1+\rho^2)\mathbf{n}\cdot(\mathbf{e}_1\times\mathbf{e}_2)\left[\frac{1}{2p_1k_1}-\frac{1}{2p_1k_2}\right]$$

$$
-\,(2\mathrm{i})\rho\left(\frac{1}{2p_1k_1}+\frac{1}{2p_1k_2}\right)(\mathbf{e}_1\times\mathbf{e}_2)_2
$$

$$
+\,\beta(\rho^2-1)\left(\frac{\delta^{i3}\delta^{j2}}{p_1k_1}-\frac{\delta^{j3}\delta^{i2}}{p_1k_2}\right)e_1^je_2^i. \tag{4.131}
$$

Here the scalar product that use in the computation of the exact amplitude, by using property in Eq. (4.100) and $\mathbf{p}_1 = m\beta\gamma(0,1,0)$, given by

$$
p_1k_1 = m^2\gamma^2\beta\sin\phi\sin\theta - p^0k^0 = m^2\gamma^2(\beta\sin\phi\sin\theta-1)
$$

$$
= -m^2\gamma^2(1-\beta\sin\phi\sin\theta), \tag{4.132}
$$

$$
p_1k_2 = m^2\gamma^2(-\beta\sin\phi\sin\theta - p^0k^0) = -m^2\gamma^2(\beta\sin\phi\sin\theta+1)
$$

$$
= -m^2\gamma^2(1+\beta\sin\phi\sin\theta). \tag{4.133}
$$

Hence we can rewrite the amplitude, with multiply by $m^2\gamma^2$, as

$$
\mathcal{A} \sim \frac{(1-\rho^2)}{2}n_3(\mathbf{e}_1\cdot\mathbf{e}_2)\left[\frac{1}{(1-\beta\sin\phi\sin\theta)}-\frac{1}{(1+\beta\sin\phi\sin\theta)}\right]
$$

$$
+\,\mathrm{i}\frac{(1+\rho^2)}{2}\mathbf{n}\cdot(\mathbf{e}_1\times\mathbf{e}_2)\left[\frac{1}{(1-\beta\sin\phi\sin\theta)}-\frac{1}{(1+\beta\sin\phi\sin\theta)}\right]
$$

$$
-\,\mathrm{i}\rho(\mathbf{e}_1\times\mathbf{e}_2)_2\left[\frac{1}{(1-\beta\sin\phi\sin\theta)}+\frac{1}{(1+\beta\sin\phi\sin\theta)}\right]
$$

$$
+\,\beta(\rho^2-1)\left[\frac{\delta^{i3}\delta^{j2}}{(1-\beta\sin\phi\sin\theta)}-\frac{\delta^{j3}\delta^{i2}}{(1+\beta\sin\phi\sin\theta)}\right]e_1^je_2^i. \tag{4.134}
$$

Here we study two emerging photons moving in x-z plane. We then set $\phi = 0$. To do this Eq. (4.134) is reduced as

$$
\mathcal{A} \sim -\mathrm{i}(1+\rho^2)\mathbf{n}\cdot(\mathbf{e}_1\times\mathbf{e}_2)+\beta(1-\rho^2)\left[e_1^{(2)}e_2^{(3)}+e_1^{(3)}e_2^{(2)}\right]. \tag{4.135}
$$

With the above amplitude $\mathcal{A}$ we can study the photon polarization in two cases:
CASE I: when we set $\theta = 0$.

In this case one of two photon move along z-axis and one move in opposite direction. Using the property in Eq. (4.97), we have

$$\mathbf{e}_1 = (-\cos\chi_1, \sin\chi_1, 0), \tag{4.136}$$

$$\mathbf{e}_2 = (-\cos\chi_2, \sin\chi_2, 0), \tag{4.137}$$

$$\mathbf{n}\cdot(\mathbf{e}_1 \times \mathbf{e}_2) = e_2^{(1)}e_1^{(2)} - e_1^{(1)}e_2^{(2)}. \tag{4.138}$$

Hence

$$\mathcal{A} \sim -\mathrm{i}(1+\rho^2)(-\cos\chi_1, \sin\chi_1, 0)_1(-\cos\chi_2, \sin\chi_2, 0)_2 \left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right]. \tag{4.139}$$

Introducing a entangled state of the emerging photons that operate with the polarizations of photons, corresponding to χ_1, χ_2, we then have to form the state

$$\mathcal{A} = -\mathrm{i}(-\cos\chi_1, \sin\chi_1, 0)_1(-\cos\chi_2, \sin\chi_2, 0)_2 \left|\psi\right\rangle, \tag{4.140}$$

where

$$\left|\psi\right\rangle = C\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right], \tag{4.141}$$

C denoting some constant that specify latter and neglected $-\mathrm{i}(1+\rho^2)$ that is just constant.

Here the entangled state is normalized, using $|| \, |\psi\rangle \, ||^2 = 1$, that is given by

$$|C|^2\Big[\begin{pmatrix}0 & 1 & 0\end{pmatrix}_1\begin{pmatrix}0 & 0 & 1\end{pmatrix}_2-\begin{pmatrix}0 & 0 & 1\end{pmatrix}_1\begin{pmatrix}0 & 1 & 0\end{pmatrix}_2\Big]\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2-\begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right]=1,$$

to obtaining

$$C = \frac{1}{\sqrt{2}}, \tag{4.142}$$

generating the (normalized) entangled state of the emerging photons

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2-\begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right]. \tag{4.143}$$

The joint probability of the electrons polarizations correlations is then given by

$$P[\chi_1, \chi_2] = ||(-\cos\chi_1, \sin\chi_1, 0)_1(-\cos\chi_2, \sin\chi_2, 0)_2 \, |\psi\rangle \, ||^2, \tag{4.144}$$

to obtaining

$$P[\chi_1, \chi_2] = \frac{1}{2}\sin^2(\chi_1 - \chi_2) \tag{4.145}$$

ρ is defined in Eq. (4.5). [For $\beta \to 0$, one obtains a rather familiar expression $P[\chi_1, \chi_2] = \sin^2[(\chi_1 - \chi_2)/2]/2$.]

If only one of the spins is measured, say, corresponding to χ_1, the probability $P[\chi_1, -]$ may be *equivalently* obtained by summing $P[\chi_1, \chi_2]$ over the two angles

$$\chi_2, \; \chi_2 + \frac{\pi}{2}, \tag{4.146}$$

for any arbitrarily chosen fixed χ_2, i.e.,

$$P[\chi_1, -] = P[\chi_1, \chi_2] + P[\chi_1, \chi_2 + \frac{\pi}{2}]. \tag{4.147}$$

Eqs. (4.146)–(4.147) leads to the corresponding probability

$$\begin{aligned} P[\chi_1, -] &= \frac{1}{2}\sin^2(\chi_1 - \chi_2) + \frac{1}{2}\sin^2\Big(\chi_1 - \Big(\chi_2 + \frac{\pi}{2}\Big)\Big) \\ &= \frac{1}{2}\sin^2(\chi_1 - \chi_2) + \frac{1}{2}\cos^2(\chi_1 - \chi_2) \\ &= \frac{1}{2}, \end{aligned} \tag{4.148}$$

and similarly for $P[-, \chi_2]$, Eqs. (4.146)–(4.147), replacing $\chi_1 \to \chi_1 + (\pi/2)$, leads to the corresponding probability

$$\begin{aligned} P[-, \chi_2] &= \frac{1}{2}\sin^2(\chi_1 - \chi_2) + \frac{1}{2}\sin^2\Big(\Big(\chi_1 + \frac{\pi}{2} - \chi_2\Big)\Big) \\ &= \frac{1}{2}\sin^2(\chi_1 - \chi_2) + \frac{1}{2}\cos^2(\chi_1 - \chi_2) \\ &= \frac{1}{2}. \end{aligned} \tag{4.149}$$

CASE II: when we set $\theta = \pi/2$.

In this case one of two photon move along x-axis and one move in opposite direction. Using the property in Eq. (4.97), we have

$$\mathbf{e}_1 = (0, \sin\chi_1, \cos\chi_1), \tag{4.150}$$

$$\mathbf{e}_2 = (0, \sin\chi_2, \cos\chi_2), \tag{4.151}$$

$$\mathbf{n} \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = e_1^{(2)} e_2^{(3)} - e_2^{(2)} e_3^{(2)}. \tag{4.152}$$

Hence

$$\mathcal{A} \sim -\mathrm{i}(1+\rho^2)\left[e_1^{(2)}e_2^{(3)} - e_2^{(2)}e_1^{(2)}\right] + \beta(1-\rho^2)\left[e_1^{(2)}e_2^{(3)} + e_2^{(2)}e_1^{(2)}\right]. \tag{4.153}$$

Similarly, Introducing a entangled state of the emerging photons that operate with the polarizations of photons, corresponding to χ_1, χ_2, we then have to form the state

$$\mathcal{A} \propto -(0, \sin\chi_1, \cos\chi_1)_1(0, \sin\chi_2, \cos\chi_2)_2 \left|\phi\right\rangle, \tag{4.154}$$

where

$$\begin{aligned}\left|\phi\right\rangle =& N\left\{\mathrm{i}(1+\rho^2)\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right]\right.\\ &\left. - \beta(1-\rho^2)\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right]\right\},\end{aligned} \tag{4.155}$$

N denoting some constant that specify latter. Here the entangled state is normalized, using $||\left|\phi\right\rangle||^2 = 1$, that is given by

$$\begin{aligned}|N|^2&\left\{\mathrm{i}(1+\rho^2)\left[\begin{pmatrix}0&1&0\end{pmatrix}_1\begin{pmatrix}0&0&1\end{pmatrix}_2 - \begin{pmatrix}0&0&1\end{pmatrix}_1\begin{pmatrix}0&1&0\end{pmatrix}_2\right]\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right]\right.\\ &\left.- \beta(1-\rho^2)\left[\begin{pmatrix}0&1&0\end{pmatrix}_1\begin{pmatrix}0&0&1\end{pmatrix}_2 + \begin{pmatrix}0&0&1\end{pmatrix}_1\begin{pmatrix}0&1&0\end{pmatrix}_2\right]\left[\begin{pmatrix}0\\1\\0\end{pmatrix}_1\begin{pmatrix}0\\0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\\0\end{pmatrix}_2\right]\right\} = 1,\end{aligned}$$

to obtaining

$$N = \frac{1}{\sqrt{2}\sqrt{(1+\rho^2)^2 + \beta^2(1-\rho^2)^2}}, \tag{4.156}$$

generating the speed dependent (normalized) entangled state of the emerging electrons we obtain

$$|\phi\rangle = \frac{1}{\sqrt{(1+\rho^2)^2 + \beta^2(1-\rho^2)^2}} \left\{ \frac{\mathrm{i}(1+\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_2 \right] \right.$$
$$\left. - \beta \frac{(1-\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_2 \right] \right\}. \tag{4.157}$$

The joint probability of the electrons polarizations correlations is then given by

$$P[\chi_1, \chi_2] = ||\ (0, \sin\chi_1, \cos\chi_1)_1 (0, \sin\chi_2, \cos\chi_2)_2\ |\phi\rangle||^2, \tag{4.158}$$

to obtain

$$P[\chi_1, \chi_2] = \frac{(1+\rho^2)^2 \sin^2(\chi_1 - \chi_2) + \beta^2(1-\rho^2)^2 \cos^2(\chi_1 + \chi_2)}{2[(1+\rho^2)^2 + \beta^2(1-\rho^2)^2]}. \tag{4.159}$$

ρ is defined in Eq. (4.5). [For $\beta \to 0$, one obtains a rather familiar expression $P[\chi_1, \chi_2] = \sin^2[(\chi_1 - \chi_2)/2]/2$.]

If only one of the spins is measured, say, corresponding to χ_1, we then have to form the state

$$(0, \sin\chi_1, \cos\chi_1)_1\ |\phi\rangle$$

$$= \frac{(0, \sin\chi_1, \cos\chi_1)_1}{\sqrt{(1+\rho^2)^2 + \beta^2(1-\rho^2)^2}} \left\{ \frac{\mathrm{i}(1+\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_2 \right] \right.$$

$$\left. - \beta \frac{(1-\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_2 \right] \right\}$$

$$= \frac{1}{\sqrt{(1+\rho^2)^2 + \beta^2(1-\rho^2)^2}} \left\{ \frac{\mathrm{i}(1+\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 0 \\ \sin\chi_1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ \cos\chi_1 \\ 0 \end{pmatrix}_2 \right] \right.$$

$$\left. - \beta \frac{(1-\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 0 \\ \sin\chi_1 \end{pmatrix}_2 + \begin{pmatrix} 0 \\ \cos\chi_1 \\ 0 \end{pmatrix}_2 \right] \right\}. \tag{4.160}$$

Eq. (4.160) leads to

$$\| (0, \sin\chi_1, \cos\chi_1)_1 |\phi\rangle \|^2 = \frac{1}{2[(1+\rho^2)^2 + \beta^2(1-\rho^2)^2]} \left\{ (1+\rho^2)[\sin^2\chi_1 + \cos^2\chi_1] \right.$$

$$\left. + \beta^4(1-\rho^2)^2[\sin^2\chi_1 + \cos^2\chi_1] \right\}, \tag{4.161}$$

from which we obtain the corresponding probability

$$P[\chi_1, -] = \frac{1}{2}. \tag{4.162}$$

The probability $P[\chi_1, -]$ may be *equivalently* obtained by summing $P[\chi_1, \chi_2]$ over the two angles

$$\chi_2,\ \chi_2 + \frac{\pi}{2}, \tag{4.163}$$

for any arbitrarily chosen fixed χ_2, i.e.,

$$P[\chi_1, -] = P[\chi_1, \chi_2] + P[\chi_1, \chi_2 + \frac{\pi}{2}]. \tag{4.164}$$

Eqs. (4.86)–(4.88) leads to the corresponding probability

$$\begin{aligned} P[\chi_1, -] &= \frac{\big[(1+\rho^2)^2 \sin^2(\chi_1 - \chi_2) + \beta^2(1-\rho^2)^2 \cos^2(\chi_1 + \chi_2)\big]}{2[(1+\rho^2)^2 + \beta^2(1-\rho^2)^2]} \\ &+ \frac{\big[(1+\rho^2)^2 \sin^2\left(\chi_1 - \chi_2 - \frac{\pi}{2}\right) + \beta^2(1-\rho^2)^2 \cos^2\left(\chi_1 + \chi_2 + \frac{\pi}{2}\right)\big]}{2[(1+\rho^2)^2 + \beta^2(1-\rho^2)^2]} \\ &= \frac{1}{2}, \end{aligned} \tag{4.165}$$

as is easily checked, and similarly, for only one of the spins is measured, say, corresponding to χ_2, we then have to form the state

$$\begin{aligned} &(0, \sin\chi_2, \cos\chi_1)_2 \, |\phi\rangle \\ &= \frac{(0, \sin\chi_2, \cos\chi_2)_2}{\sqrt{(1+\rho^2)^2 + \beta^2(1-\rho^2)^2}} \left\{ \frac{\mathrm{i}(1+\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_2 \right] \right. \\ &\qquad \left. - \beta \frac{(1-\rho^2)}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_2 \right] \right\} \end{aligned}$$

$$= \frac{1}{\sqrt{(1+\rho^2)^2+\beta^2(1-\rho^2)^2}}\Bigg\{\frac{\mathrm{i}(1+\rho^2)}{\sqrt{2}}\left[\begin{pmatrix}0\\ \cos\chi_2\\ 0\end{pmatrix}_1-\begin{pmatrix}0\\ 0\\ \sin\chi_2\end{pmatrix}_1\right]$$

$$-\beta\frac{(1-\rho^2)}{\sqrt{2}}\left[\begin{pmatrix}0\\ \cos\chi_2\\ 0\end{pmatrix}_1+\begin{pmatrix}0\\ 0\\ \sin\chi_2\end{pmatrix}_1\right]\Bigg\}. \tag{4.166}$$

Eq. (4.166) leads to

$$\|\,(0,\sin\chi_2,\cos\chi_2)_2\,|\phi\rangle\,\|^2 = \frac{1}{2[(1+\rho^2)^2+\beta^2(1-\rho^2)^2]}\Bigg\{(1+\rho^2)[\sin^2\chi_2+\cos^2\chi_2]$$

$$+\beta^4(1-\rho^2)^2[\sin^2\chi_2+\cos^2\chi_2]\Bigg\}, \tag{4.167}$$

from which we obtain the corresponding probability

$$P[-,\chi_2] = \frac{1}{2}. \tag{4.168}$$

The probability $P[-,\chi_2]$ may be *equivalently* obtained by summing $P[\chi_1,\chi_2]$ over the two angles

$$\chi_1,\ \chi_1+\frac{\pi}{2}, \tag{4.169}$$

for any arbitrarily chosen fixed χ_2, i.e.,

$$P[-,\chi_2] = P[\chi_1,\chi_2]+P[\chi_1+\frac{\pi}{2},\chi_2]. \tag{4.170}$$

Eqs. (4.169)–(4.170) leads to the corresponding probability

$$P[-,\chi_2] = \frac{\big[(1+\rho^2)^2\sin^2(\chi_1-\chi_2)+\beta^2(1-\rho^2)^2\cos^2(\chi_1+\chi_2)\big]}{2[(1+\rho^2)^2+\beta^2(1-\rho^2)^2]}$$
$$+\frac{\big[(1+\rho^2)^2\sin^2\left(\chi_1-\chi_2+\frac{\pi}{2}\right)+\beta^2(1-\rho^2)^2\cos^2\left(\chi_1+\chi_2+\frac{\pi}{2}\right)\big]}{2[(1+\rho^2)^2+\beta^2(1-\rho^2)^2]}$$
$$=\frac{1}{2}. \tag{4.171}$$

For all $0\leqslant\beta\leqslant 1$, angles χ_1, χ_2, χ'_1, χ'_2 are readily found leading to a violation of Bell's inequality of LHV theories. For example, for $\beta=0.2$, $\chi_1=0°$, $\chi_2=23°$, $\chi'_1=45°$, $\chi'_2=67°$, $S=-1.187$ violating the inequality from below.

4.3 Polarizations Correlations: Initially Unpolarized Particles

For the process $e^-e^-\rightarrow e^-e^-$, in the c.m., with initially unpolarized spins, with momenta $\mathbf{p}_1=\gamma m\beta(0,1,0)=-\mathbf{p}_2$, we take for the final electrons

$$\mathbf{p'}_1=\gamma m\beta(1,0,0)=-\mathbf{p}'_2 \tag{4.172}$$

and for the four-spinors

$$u(p'_1)=\left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\xi_1\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'_1}{p^0+m}\xi_1\end{pmatrix},\quad \xi_1=\begin{pmatrix}-\mathrm{i}\cos\chi_1/2\\ \sin\chi_1/2\end{pmatrix}, \tag{4.173}$$

$$u(p'_2)=\left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\xi_2\\ -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'_1}{p^0+m}\xi_2\end{pmatrix},\quad \xi_2=\begin{pmatrix}-\mathrm{i}\cos\chi_2/2\\ \sin\chi_2/2\end{pmatrix}. \tag{4.174}$$

A straightforward but tedious computation of the corresponding probability of occurrence with initially unpolarized electrons, (2.7) leads to

$$\begin{aligned}\text{Prob} \propto\ & [\overline{u}(p_1')\gamma^\mu(-\gamma p_1 + m)\gamma^\sigma u(p_1')]\, [\overline{u}(p_2')\gamma_\mu(-\gamma p_2 + m)\gamma_\sigma u(p_2')] \\ & - [\overline{u}(p_1')\gamma^\mu(-\gamma p_1 + m)\gamma^\sigma u(p_2')]\, [\overline{u}(p_2')\gamma_\mu(-\gamma p_2 + m)\gamma_\sigma u(p_1')] \\ & - [\overline{u}(p_2')\gamma^\mu(-\gamma p_1 + m)\gamma^\sigma u(p_1')]\, [\overline{u}(p_1')\gamma_\mu(-\gamma p_2 + m)\gamma_\sigma u(p_2')] \\ & + [\overline{u}(p_2')\gamma^\mu(-\gamma p_1 + m)\gamma^\sigma u(p_2')]\, [\overline{u}(p_1')\gamma_\mu(-\gamma p_2 + m)\gamma_\sigma u(p_1')]\,, \end{aligned} \tag{4.175}$$

which after simplification and of collecting terms reduces to

$$\begin{aligned}\text{Prob} \propto\ & (1-\beta^2)(1+3\beta^2)\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \beta^4\cos^2\left(\frac{\chi_1 + \chi_2}{2}\right) + 4\beta^4 \\ & \equiv F[\chi_1, \chi_2], \end{aligned} \tag{4.176}$$

where we have used the expressions for the spinors in Eq. (4.2), Eq. (4.3).

Given that the process has occurred, the conditional probability that the spins of the emerging electrons make angles χ_1, χ_2 with the z-axis, is directly obtained from Eq. (3.5) to be

$$P[\chi_1, \chi_2] = \frac{F[\chi_1, \chi_2]}{C}. \tag{4.177}$$

The normalization constant C is obtained by summing over the polarizations of the emerging electrons. This is equivalent to summing of $F[\chi_1, \chi_2]$ over the pairs of angles

$$(\chi_1, \chi_2),\ (\chi_1 + \pi, \chi_2),\ (\chi_1, \chi_2 + \pi)\ (\chi_1 + \pi, \chi_2 + \pi), \tag{4.178}$$

for any arbitrarily chosen fixed χ_1, χ_2, corresponding to the orthonormal spinors

$$\begin{pmatrix} -\mathrm{i}\cos\chi_j/2 \\ \sin\chi_j/2 \end{pmatrix}, \quad \begin{pmatrix} -\mathrm{i}\cos(\chi_j+\pi)/2 \\ \sin(\chi_j+\pi)/2 \end{pmatrix} = \begin{pmatrix} \mathrm{i}\sin\chi_j/2 \\ \cos\chi_j/2 \end{pmatrix}, \tag{4.179}$$

providing a complete set, for each $j = 1, 2$, in reference to Eq. (4.2), Eq. (4.3). This is,

$$C = F[\chi_1,\chi_2] + F[\chi_1+\pi,\chi_2] + F[\chi_1,\chi_2+\pi] + F[\chi_1+\pi,\chi_2+\pi]$$

$$= 2(1 + 2\beta^2 + 6\beta^4), \tag{4.180}$$

which as expected is independent of χ_1, χ_2, giving

$$P[\chi_1,\chi_2] = \frac{(1-\beta^2)(1+3\beta^2)\sin^2\left(\frac{\chi_1-\chi_2}{2}\right) + \beta^4\cos^2\left(\frac{\chi_1+\chi_2}{2}\right) + 4\beta^4}{2(1+2\beta^2+6\beta^4)}. \tag{4.181}$$

By summing over

$$\chi_2, \quad \chi_2 + \pi, \tag{4.182}$$

for any arbitrarily fixed χ_2, we obtain

$$P[\chi_1, -] = \frac{1}{2}, \tag{4.183}$$

and similarly,

$$P[-, \chi_2] = \frac{1}{2}, \tag{4.184}$$

for the probabilities when only one of the photons polarizations is measured.

A clear violation of Bell's inequality of LHV theories was obtained for all $0 \leqslant \beta \leqslant 0.45$. For example, for $\beta = 0.3$, with $\chi_1 = 0°$, $\chi_2 = 45°$, $\chi_1' = 90°$, $\chi_2' = 135°$ give $S = -1.165$ violating the inequality from below. For larger β values, alone, one cannot discriminate between LHV theories and quantum theory for this process. A violation of Bell's inequality for at least some β values, as seen, however, automatically violates LHV theories.

The probability of photon polarizations correlations in $e^+e^- \rightarrow 2\gamma$ with initially unpolarized e^+, e^-, has been given in work of N. Yongram and E. B. Manoikian (2003) to be

$$P[\chi_1, \chi_2] = \frac{1 - [\cos(\chi_1 - \chi_2) - 2\beta^2 \cos\chi_1 \cos\chi_2]^2}{2[1 + 2\beta^2(1 - \beta^2)]} \tag{4.185}$$

$$P[\chi_1, -] = \frac{1 + 4\beta^2(1 - \beta^2)\cos^2\chi_1}{2[1 + 2\beta^2(1 - \beta^2)]} \tag{4.186}$$

$$P[-, \chi_2] = \frac{1 + 4\beta^2(1 - \beta^2)\cos^2\chi_2}{2[1 + 2\beta^2(1 - \beta^2)]} \tag{4.187}$$

and a clear violation of Bell's inequality of LHV theories was obtained for all $0 \leqslant \beta \leqslant 0.2$. Again, for larger values of β, alone, one cannot discriminate between LHV theories and quantum theory for this process. A violation of Bell's inequality for at least some β values, as seen, however, automatically occurs violating LHV theories.

For completeness, we mention that for the annihilation of the spin 0 pair into 2γ the following probabilities are similarly worked out:

$$P[\chi_1, \chi_2] = \frac{(\cos(\chi_1 - \chi_2) - 2\beta^2 \cos\chi_1 \cos\chi_2)^2}{2[1 - 2\beta^2(1 - \beta^2)]} \tag{4.188}$$

$$P[\chi_1, -] = \frac{1 - 4\beta^2(1 - \beta^2)\cos^2\chi_1}{2[1 - 2\beta^2(1 - \beta^2)]} \tag{4.189}$$

$$P[-, \chi_2] = \frac{1 - 4\beta^2(1 - \beta^2)\cos^2\chi_2}{2[1 - 2\beta^2(1 - \beta^2)]} \tag{4.190}$$

and violates Bell's inequality of LHV theories for all $0 \leqslant \beta \leqslant 1$.

We have seen by explicit dynamical computations based on QED, that the polarizations correlations probabilities of particles emerging in processes *depend* on speed, for initially *polarized* as well as *unpolarized* particles, in general. We have also seen how QED leads directly to speed dependent entangled states. For processes with initially polarized particles (as well as for spin 0 pairs annihilation into 2γ), a clear violation of Bell's inequality of LHV theories was obtained for all speeds. This clear violation was also true for several speeds for processes with initially unpolarized particles, but the tests are more sensitive on the speed for such processes. The main results of this chapter are given in Eq. (4.82), Eq. (4.89), Eq. (4.92), Eq. (4.142), Eqs. (4.148)–(4.149), Eq. (4.159), Eq. (4.162), Eq. (4.168), Eq. (4.181), Eqs. (4.185)–(4.190). We feel that it is a matter of some urgency that the relevant experiments are carried out by monitoring speed.

CHAPTER V

MUON PAIR PRODUCTION IN THE WEINBERG-SALAM ELECTROWEAK THEORY

In the present chapter, we study the process $e^-e^+ \to \mu^-\mu^+$ for muon pair production in the Weinberg-Salam standard electro-weak theory and we encounter completely novel properties not encountered in our QED calculations in Chapters III and IV. The reasons for considering this process are many. For one thing the differential cross section is in excellent agreement with experiments unlike its QED counterpart. The main reason, within the framework, however, of our study is that due to the threshold energy needed to create the $\mu^-\mu^+$ pair, the limit of the speed (β) of the colliding particles cannot be taken to go to zero. Therefore all arguments based on simply combining the spins of e^-, e^+, without dynamical considerations fail. Accordingly, *a quantum field theoretical calculation of the polarization correlations of* $\mu^-\mu^+$ *is necessary as a dynamical treatment*. We show that the polarizations correlations depend not only *speed but also have explicit dependence on the underlying couplings*. The latter is a completely novel property not encountered in our QED computations in Chapter III, IV. Finally we show a clear violation with Bell's inequality.

5.1 General Survey

Several experiments have been performed over the years on particles' polarizations correlations [Irby, 2003; Osuch, Popkiewicz, Szeflinski and Wilhelmi, 1996; Kaday, Ulman and Wu, 1975; Fry, 1995; Aspect, Dalibard and Roger, 1982] in the light of Bell's inequality and many Bell–like experiments have been proposed recently in high energy physics [Go, 2004; Bertlman, Bramon, Garbarino and Hiesmayr, 2004; Abel, Dittmar and Dreiner, 1992; Privitera, 1992; Lednický and Lyuboshitz, 2001; Genovese, Novero and Predazzi, 2001]. We have been particularly interested in actual quantum field theory computations of polarizations correlations probabilities of particles produced in basic processes because of novelties encountered in dynamical calculations as opposed to kinematical considerations to be discussed. Here it is worth recalling that quantum field theory originates from the combination of quantum physics *and* relativity and involve non-trivial dynamics. Many such computations have been done in QED [Yongram and Manoukian, 2003; Manoukian and Yongram, 2004] in our earlier chapters as well as in e^-e^+ pair production from some charged and neutral strings in Chapter VI. All of these polarizations correlations probabilities based on dynamical analyses following from field theory share the interesting property that they depend on the energy (speeds) of the colliding particles due to the mere fact that typically the latter carry speeds in order to collide. Such analyses are unlike considerations based on formal arguments of simply combining spins, as is usually done, and are of kinematical nature, void of dynamical considerations. Here it is worth recalling that the total spin of a two-particle system each with spin [Clauser and Shimoney, 1978], such as of two spin $1/2$'s, is obtained not only from combining the spins of the latter but also from any orbital angular momentum residing in their center of mass system. For low speeds, one expects that the argument based simply on combining the spins of the colliding particles should provide an accurate description of the polarization correlations sought and all of our QED computations [Yongram and Manoukian, 2003; Manoukian and Yongram, 2004] show the correctness of such an argument in the limit of low speeds. Needless to say, we are interested in the *relativistic* regime as well, and the formal arguments just mentioned fail to provide the correct expressions for the correlations. As a byproduct of the work, our computations of the joint polarizations correlations carried out in a full quantum field theory setting show a clear violation of Bell's inequality. For the Lagrangian density of the underlying theory see §2.5.

In the present communication we encounter additional *completely novel proper-*

ties not encountered in our earlier QED [Yongram and Manoukian, 2003; Manoukian and Yongram, 2004] calculations. We consider the process $e^-e^+ \to \mu^-\mu^+$ as described in the standard electroweak (EW) model. It is well known that this process [Althoff et al., 1984] as computed in the EW model is in much better agreement with experiments than that of a QED computation. The reasons for considering such a process in the EW model are many, one of which is the high precision of the differential cross section obtained as just discussed. Reasons which are, however, more directly relevant to our anylyses are the following. *Due to the theshold energy needed to create the $\mu^-\mu^+$ pair, the limit of the speed β of the colliding particles cannot be taken to go to zero*. This is unlike processes treated by the authors in QED such as in $e^-e^- \to e^-e^-$, $e^+e^- \to 2\gamma$, Therefore all arguments based simply on combining the spins of e^-, e^+, without dynamical considerations, *fail*. [As a matter of fact the latter argument would lead for the joint probability *in* Eq. (5.73) we are seeking, the incorrect result $(1/2)\sin^2\left(\left(\chi_1 - \chi_2\right)/2\right)$ — an expression which has been used for years.] Another novelty we encounter in the present investigation is that the polarization correlations *not only depend on speed but have also an explicit dependence on the underlying couplings*. Again this latter explicit dependence is unlike the situation arising in QED [Yongram and Manoukian, 2003; Manoukian and Yongram, 2004].

The relevant quantity of interest here in testing Bell's inequality [Clauser and Horne, 1974; Clauser and Shimoney, 1978] in Eq. (4.1) as is *computed from* the electroweak model. Here a_1, a_2 (a_1', a_2') specify directions along which the polarizations of two particles are measured, with $p_{12}(a_1, a_2)/p_{12}(\infty, \infty)$ denoting the joint probability, and $p_{12}(a_1, \infty)/p_{12}(\infty, \infty)$, $p_{12}(\infty, a_2)/p_{12}(\infty, \infty)$ denoting the probabilities when the polarization of only one of the particles is measured. [$p_{12}(\infty, \infty)$ is a normalization factor.] The corresponding probabilities as computed from the electroweak model will be denoted by $P(\chi_1, \chi_2)$, $P(\chi_1, -)$, $P(-, \chi_2)$ with χ_1, χ_2 denoting angles specifying directions along which spin measurements are carried out with respect to certain axes spelled out in the bulk of the paper. To show that the electroweak model is in violation with Bell's inequality of LHV, it is sufficient to find one set of angles χ_1, χ_2, χ_1', χ_2', such that S, as computed in the electroweak model, leads to a value of S outside the interval $[-1, 0]$. In this work, it is implicitly assumed that the polarization parameters in the particle states are directly observable and may be used for Bell–type measurements as discussed.

5.2 Explicit Probability Expression

In this section, we consider the process $e^+e^- \to \mu^+\mu^-$, in center of mass (c.m.) frame, shown in figure **5.1** , with initially polarized electron with spin up, along the z-axis, and initially polarized positron with spin down. $\mathbf{p}$ and $\mathbf{k}$ denote the momenta of initial electron and initial positron, respectively. Similarly, $\mathbf{p}'$ and $\mathbf{k}'$ denote the momenta of final muon and final anti-muon, respectively. The transition amplitude of this process ($e^+e^- \to \mu^+\mu^-$) is well known (see §2.5) :

$$\mathcal{M} = \overline{U}(\mathbf{p}', \sigma')\gamma^\mu V(\mathbf{k}', s') \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{1}{q^2} \overline{v}(\mathbf{k}, s)\gamma^\nu u(\mathbf{p}, \sigma)$$

$$+ \frac{a}{q^2 + M_Z^2} \overline{U}(\mathbf{p}', \sigma')\gamma^\mu (b - \gamma_5) V(\mathbf{k}', s') \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2} \right) \overline{v}(\mathbf{k}, s)\gamma^\nu (b - \gamma_5) u(\mathbf{p}, \sigma), \tag{5.1}$$

where e=charge, g=weak coupling constant, θ_{W}=Weinberg angle, M_Z = a mass of boson (Z^0), $a = \dfrac{g^2}{16e^2 \cos^2 \theta_{\mathrm{W}}}$ and $b = 1 - 4\sin^2\theta_{\mathrm{W}}$

We first simplify the expression for the amplitude of this process by expanding the various terms in Eq. (5.1). Let

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2, \tag{5.2}$$

where

$$\mathcal{M}_1 = \overline{U}\gamma^\mu V \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{1}{q^2} \overline{v}\gamma^\nu u$$

$$= \frac{1}{q^2}\overline{U}\gamma^\mu V g_{\mu\nu}\overline{v}\gamma^\nu u - \frac{q_\mu q_\nu}{q^4}\overline{U}\gamma^\mu V\overline{v}\gamma^\nu u$$

$$= \frac{1}{q^2}\overline{U}\gamma^\mu V\overline{v}\gamma_\mu u - \frac{q_\mu q_\nu}{q^4}\overline{U}\gamma^\mu V\overline{v}\gamma^\nu u$$

$$= \frac{1}{q^2}\left\{\overline{U}\gamma^0 V\overline{v}\gamma_0 u + \overline{U}\gamma^i V\overline{v}\gamma_i u\right\} - \frac{q_0 q_0}{q^4}\overline{U}\gamma^0 V\overline{v}\gamma^0 u$$

$$= \frac{1}{(2p^0)^2}\left\{\overline{U}\gamma^i V\overline{v}\gamma_i u - \overline{U}\gamma^0 V\overline{v}\gamma^0 u\right\} - \frac{(2p^0)^2}{(2p^0)^4}\overline{U}\gamma^0 V\overline{v}\gamma^0 u$$

$$\mathcal{M}_1 = \frac{1}{(2p^0)^2}\left\{\overline{U}\gamma^i V\overline{v}\gamma^i u - 2\overline{U}\gamma^0 V\overline{v}\gamma^0 u, \right\} \tag{5.3}$$

and

$$\mathcal{M}_2 = a\overline{U}\gamma^\mu(b-\gamma_5)V\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}\right)\frac{1}{q^2+M_Z^2}\overline{v}\gamma^\nu(b-\gamma_5)u$$

$$= \frac{a}{M_Z^2 - 4(p^0)^2}\overline{U}\gamma^\mu(b-\gamma_5)V g_{\mu\nu}\overline{v}\gamma^\nu(b-\gamma_5)u$$

$$+ \frac{a q_\mu q_\nu}{(M_Z^2 - 4(p^0)^2)M_Z^2}\overline{U}\gamma^\mu(b-\gamma_5)V\overline{v}\gamma^\nu(b-\gamma_5)u$$

$$= \frac{a}{M_Z^2 - 4(p^0)^2}\overline{U}\gamma^\mu(b-\gamma_5)V\overline{v}\gamma_\mu(b-\gamma_5)u$$

$$+ \frac{a q_\mu q_\nu}{(M_Z^2 - 4(p^0)^2)M_Z^2}\overline{U}\gamma^\mu(b-\gamma_5)V\overline{v}\gamma^\nu(b-\gamma_5)u$$

$$= \frac{a}{M_Z^2 - 4(p^0)^2}\left\{\overline{U}\gamma^0(b-\gamma_5)V\overline{v}\gamma_0(b-\gamma_5)u\right.$$

$$\left. + \overline{U}\gamma^i(b-\gamma_5)V\overline{v}\gamma_i(b-\gamma_5)u\right\}$$

$$+ \frac{a q_0 q_0}{M_Z^2 - 4(p^0)^2}\overline{U}\gamma^0(b-\gamma_5)V\overline{v}\gamma^0(b-\gamma_5)u$$

$$= \frac{a}{M_Z^2 - 4(p^0)^2}\left\{\overline{U}\gamma^i(b-\gamma_5)V\overline{v}\gamma_i(b-\gamma_5)u - \overline{U}\gamma^0(b-\gamma_5)V\overline{v}\gamma^0(b-\gamma_5)u\right\}$$

$$+\frac{a(2p^0)^2}{((2p^0)^2+M_Z^2)M_Z^2}\overline{U}\gamma^0(b-\gamma_5)V\overline{v}\gamma^0(b-\gamma_5)u$$

$$=\frac{a}{M_Z^2-4(p^0)^2}\left\{\overline{U}\gamma^i(b-\gamma_5)V\overline{v}\gamma^i(b-\gamma_5)u\right.$$

$$\left.+\left(\frac{(2p^0)^2}{M_Z^2}-1\right)\overline{U}\gamma^0(b-\gamma_5)V\overline{v}\gamma^0(b-\gamma_5)u\right\}$$

$$\mathcal{M}_2=\frac{a}{M_Z^2-4(p^0)^2}\left\{\overline{U}\gamma^i(b-\gamma_5)V\overline{v}\gamma^i(b-\gamma_5)u\right\}$$

$$-\left(\frac{a}{M_Z^2}\right)\left\{\overline{U}\gamma^0(b-\gamma_5)V\overline{v}\gamma^0(b-\gamma_5)u.\right\} \tag{5.4}$$

where $q^2=\mathbf{q}^2-(q^0)^2$, $q=(2p^0,0)$, and $q^2+M_Z^2=M_Z^2-4(p^0)^2$

For the four-spinors of the electron and the positron, we have

$$u(\mathbf{p},\sigma)=\sqrt{\frac{p^0+m_e}{2m_e}}\begin{pmatrix}\xi_\sigma\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m_e}\xi_\sigma\end{pmatrix}, \tag{5.5}$$

$$v(\mathbf{k},s)=\sqrt{\frac{k^0+m_e}{2m_e}}\begin{pmatrix}\dfrac{\boldsymbol{\sigma}\cdot\mathbf{k}}{k^0+m_e}\xi_s\\ \xi_s\end{pmatrix}. \tag{5.6}$$

In our case, we choose $\mathbf{p}=-\mathbf{k}$, $p^0=k^0$, so that we can rewrite Eq. 5.6 as

$$v(\mathbf{k},s)=\sqrt{\frac{p^0+m_e}{2m_e}}\begin{pmatrix}-\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m_e}\xi_s\\ \xi_s\end{pmatrix}. \tag{5.7}$$

where the two-spinors ξ_σ, ξ_s, we will be specified later.

And for the four-spinors of μ^- and μ^+, we have

$$U(\mathbf{p}',\sigma')=\sqrt{\frac{p'^0+m_\mu}{2m_\mu}}\begin{pmatrix}\xi_{\sigma'}\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{p'^0+m_\mu}\xi_{\sigma'}\end{pmatrix}, \tag{5.8}$$

$$V(\mathbf{k}', s') = \sqrt{\frac{k'^0 + m_\mu}{2m_\mu}} \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \mathbf{k}'}{k'^0 + m_\mu} \xi_{s'} \\ \xi_{s'} \end{pmatrix}. \tag{5.9}$$

In our case, we choose $\mathbf{p}' = -\mathbf{k}'$, $p^0 = k^0$, so that we can rewrite Eq. (5.9) as

$$V(\mathbf{k}', s') = \sqrt{\frac{p'^0 + m_\mu}{2m_\mu}} \begin{pmatrix} -\dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}'}{p'^0 + m_\mu} \xi_{s'} \\ \xi_{s'} \end{pmatrix} \tag{5.10}$$

where the two-spinors $\xi_{\sigma'}$, $\xi_{s'}$, we will be specified later.

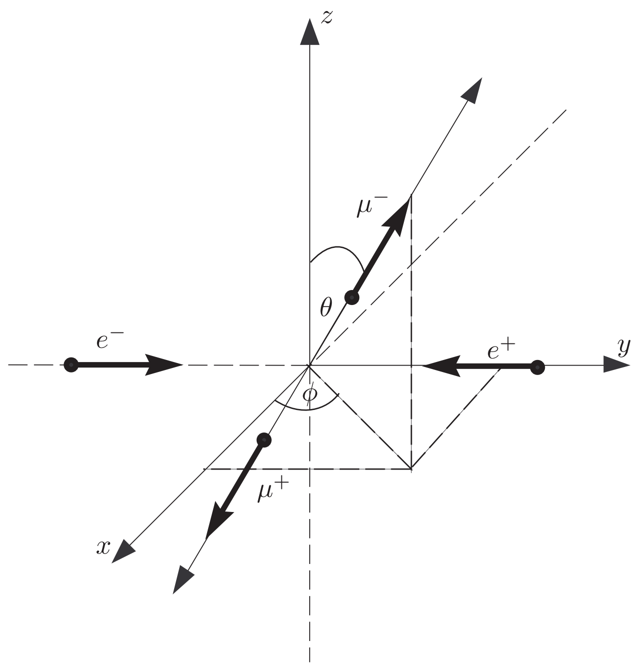

Figure 5.1 Diagram of the process $e^+e^- \to \mu^+\mu^-$ in c.m. frame.

From conservation of energy momentum : $p + k = p' + k'$, so that we obtain

$$p = (p^0, \mathbf{p}), \quad p' = (p'^0, \mathbf{p}'), \tag{5.11a}$$

$$k = (p^0, -\mathbf{p}), \quad k' = (p'^0, -\mathbf{p}'), \tag{5.11b}$$

$$q = (2p^0, \mathbf{0}), \quad p^0 = \sqrt{\mathbf{p}^2 + m_e^2}, \quad p'^0 = \sqrt{\mathbf{p}'^2 + m_\mu^2}, \tag{5.11c}$$

$$\sqrt{\mathbf{p}^2 + m_e^2} = \sqrt{\mathbf{p}'^2 + m_\mu^2}, \tag{5.11d}$$

$$\mathbf{p}^2 + m_e^2 = \mathbf{p}'^2 + m_\mu^2, \tag{5.11e}$$

$$p^0 = p'^0. \tag{5.11f}$$

In this process, we will study the spin correlation of two final particles that depend on speed (β) of two initial particles. If we define β' is the speed of two final particles. Therefore, relation between β and β'. From Eq. (5.11d), we have

$$p = k = (m_e\gamma, m_e\gamma\beta), \tag{5.12}$$

$$p' = k' = (m_\mu\gamma', m_\mu\gamma'\beta'), \tag{5.13}$$

and using the properties of the conservation of energy in this process, read as:

$$\mathbf{p}^2 + m_e^2 = \mathbf{p}'^2 + m_\mu^2, \tag{5.14}$$

by replacing the momenta of initially particles with the speed (β and β') of emerging particles, we have

$$m_e^2\gamma^2\beta^2 + m_e^2 = m_\mu^2\gamma'^2\beta'^2 + m_\mu^2$$

$$m_e^2\left[\frac{\beta^2}{1-\beta^2} + 1\right] = m_\mu^2\left[\frac{\beta'^2}{1-\beta'^2} + 1\right]$$

$$\frac{m_e^2}{1-\beta^2}\left[\beta^2 + (1-\beta^2)\right] = \frac{m_\mu^2}{1-\beta'^2}\left[\beta'^2 + (1-\beta'^2)\right]$$

$$\frac{m_e^2}{1-\beta^2} = \frac{m_\mu^2}{1-\beta'^2}$$

$$m_\mu^2(1-\beta^2) = m_e^2(1-\beta'^2)$$

$$(1-\beta'^2) = \frac{m_\mu^2}{m_e^2}(1-\beta^2)$$

$$\beta'^2 = 1 - \frac{m_\mu^2}{m_e^2}(1-\beta^2).$$

We obtain relation between β and β', written as

$$\beta' = \sqrt{1 - \frac{m_\mu^2}{m_e^2}(1-\beta^2)}. \tag{5.15}$$

Give $m \equiv m_\mu/m_e$, so that

$$\beta' = \sqrt{1-m^2+m^2\beta^2} = \sqrt{1-m^2(1-\beta^2)} = \sqrt{1-\frac{m^2}{\gamma^2}}, \tag{5.16}$$

where $\gamma' = \dfrac{\gamma}{m}$.

Form figure **5.1** , we rewrite the momenta of the particles in term of θ, ϕ, we have

$$\mathbf{p} = |\mathbf{p}|(0,1,0), \quad \mathbf{k} = -|\mathbf{p}|(0,1,0), \tag{5.17a}$$

$$\mathbf{p}' = |\mathbf{p}'|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{5.17b}$$

$$\mathbf{k}' = -|\mathbf{p}'|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{5.17c}$$

Consider the four-spinors of two initial and final particles, write in term of speed (β or β'), we have (initial particle)

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m_e} = \frac{|\mathbf{p}|}{p^0+m_e}(\sigma^1\cdot 0 + \sigma^2\cdot 1 + \sigma^3\cdot 0)$$

$$= \frac{m_e\gamma\beta}{m_e\gamma+m_e}\sigma^2$$

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m_e} = \frac{\gamma\beta}{\gamma+1}\sigma^2, \tag{5.18}$$

and (final particle)

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{p'^0+m_\mu} = \frac{|\mathbf{p}'|}{p'^0+m_\mu}(\sigma^1\sin\theta\cos\phi+\sigma^2\sin\theta\sin\phi+\sigma^3\cos\theta)$$

$$= \frac{m_\mu\gamma'\beta'}{m_\mu\gamma'+m_\mu}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin\theta\cos\phi+\mathrm{i}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sin\theta\sin\phi\right.$$

$$\left.+\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cos\theta\right]$$

$$= \frac{\gamma'\beta'}{\gamma'+1}\begin{pmatrix}\cos\theta & \sin\theta(\cos\phi-\mathrm{i}\sin\phi)\\ \sin\theta(\cos\phi+\mathrm{i}\sin\phi) & -\cos\theta\end{pmatrix}$$

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{p'^0+m_\mu} = \frac{\gamma'\beta'}{\gamma'+1}\begin{pmatrix}\cos\theta & \sin\theta\mathrm{e}^{-\mathrm{i}\phi}\\ \sin\theta\mathrm{e}^{\mathrm{i}\phi} & -\cos\theta\end{pmatrix}. \tag{5.19}$$

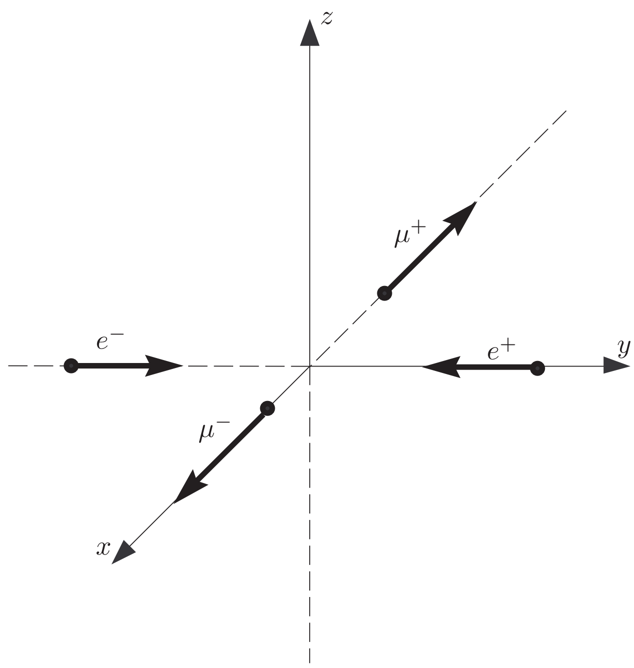

Figure 5.2 Diagram of the process $e^+e^- \to \mu^+\mu^-$ in c.m. frame.

Our case, we chose $\theta = \frac{\pi}{2}$ & $\phi = 0$. We rewrite Eq. (5.19) as

$$\frac{\boldsymbol{\sigma} \cdot \mathbf{p}'}{p'^0 + m_\mu} = \frac{\gamma' \beta'}{\gamma' + 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.20}$$

and Eqs. (5.17a)–(5.17c) are written as

$$\mathbf{p} = |\mathbf{p}|(0,1,0), \quad \mathbf{k} = -|\mathbf{p}|(0,1,0), \tag{5.21a}$$

$$\mathbf{p}' = |\mathbf{p}'|(1,0,0), \tag{5.21b}$$

$$\mathbf{k}' = -|\mathbf{p}'|(1,0,0). \tag{5.21c}$$

We consider the process $e^- e^+ \to \mu^- \mu^+$ in the center of mass frame (see figure **5.2**) with the momentum of, say, e^- chosen to be $\mathbf{p} = \gamma\beta m_e\big(0,1,0\big) = -\mathbf{k}$, m_e denoting its mass and $\gamma = 1/\sqrt{1-\beta^2}$. The momentum of the emerging μ^- will be taken to be $\mathbf{p}' = \gamma'\beta' m_\mu\big(1,0,0\big) = -\mathbf{k}'$, $\gamma' = 1/\sqrt{1-\beta'^2}$, and m_μ is the mass of $\mu^-(\mu^+)$, the spinors of e^-, e^+ are chosen as

$$\xi_\sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and we obtain the four-spinors of e^- and e^+, given by

$$u(\mathbf{p},\uparrow) = \sqrt{\frac{\gamma+1}{2}} \begin{pmatrix} \uparrow \\ \mathrm{i}\rho \downarrow \end{pmatrix}, \tag{5.22}$$

$$v(\mathbf{k},\downarrow) = \sqrt{\frac{\gamma+1}{2}} \begin{pmatrix} \mathrm{i}\rho \downarrow \\ \downarrow \end{pmatrix}, \tag{5.23}$$

and

$$\overline{v}(\mathbf{k},\downarrow) = -\sqrt{\frac{\gamma+1}{2}}\left(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\right), \tag{5.24}$$

where $\rho \equiv \left(\dfrac{\gamma\beta}{\gamma+1}\right)$

For the four-spinors of μ^- and μ^+

$$U(\mathbf{p}',\sigma') = \sqrt{\frac{m_\mu\gamma' + m_\mu}{2m_\mu}}\begin{pmatrix}\xi_{\sigma'} \\ \rho'\sigma^1\xi_{\sigma'}\end{pmatrix}$$

$$= \sqrt{\frac{\gamma'+1}{2}}\begin{pmatrix}\xi_{\sigma'} \\ \rho'\sigma^1\xi_{\sigma'}\end{pmatrix}, \tag{5.25}$$

$$V(\mathbf{k}',s') = \sqrt{\frac{\gamma'+1}{2}}\begin{pmatrix}-\rho'\sigma^1\xi_{s'} \\ \xi_{s'}\end{pmatrix}, \tag{5.26}$$

and

$$\overline{U}(\mathbf{p}',\sigma') = \sqrt{\frac{\gamma'+1}{2}}\begin{pmatrix}\xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1\end{pmatrix}, \tag{5.27}$$

where $\xi^\dagger_\sigma$ and $\xi^\dagger_{\sigma'}$ will be specified latter and $\rho' \equiv \left(\dfrac{\gamma'\beta'}{\gamma'+1}\right)$.

To calculate the transition amplitude of this process, by using

$$\mathbb{1} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \tag{5.28}$$

$$\gamma^0 = \begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}, \quad \gamma^i = \begin{pmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}, \tag{5.29}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.30}$$

then, the necessary matrix element for this calculation, we consider each the matrix element in the amplitude of this process.

For the first needed matrix element, $\overline{U}\gamma^0 V$, on the right-side of Eq. (5.3), be written as

$$\begin{aligned}
\overline{U}\gamma^0 V &= \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix} \xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} -\rho'\sigma^1\xi_{s'} \\ \xi_{s'} \end{pmatrix} \\
&= \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix} \xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1 \end{pmatrix}\begin{pmatrix} -\rho'\sigma^1\xi_{s'} \\ -\xi_{s'} \end{pmatrix} \\
&= \left(\frac{\gamma'+1}{2}\right)\Big[-\rho'\xi^\dagger_{\sigma'}\sigma^1\xi_{s'} + \rho'\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\Big] \\
\overline{U}\gamma^0 V &= 0.
\end{aligned} \tag{5.31}$$

and the second needed matrix element, $\overline{U}\gamma^i V$, on the right-side of Eq. (5.3), be written as

$$\begin{aligned}
\overline{U}\gamma^i V &= \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix} \xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} -\rho'\sigma^1\xi_{s'} \\ \xi_{s'} \end{pmatrix} \\
&= \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix} \xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1 \end{pmatrix}\begin{pmatrix} \sigma^i\xi_{s'} \\ \rho'\sigma^i\sigma^1\xi_{s'} \end{pmatrix} \\
\overline{U}\gamma^i V &= \left(\frac{\gamma'+1}{2}\right)\Big[\xi^\dagger_{\sigma'}\sigma^i\xi_{s'} - \rho'^2\xi^\dagger_{\sigma'}\sigma^1\sigma^i\sigma^1\xi_{s'}\Big].
\end{aligned} \tag{5.32}$$

On the right-hand side of Eq. (5.32), we define $\sigma^1\sigma^i\sigma^1 =?$ as

$$i = 1; \quad \sigma^1\sigma^1\sigma^1 = \sigma^1 \text{ or } \sigma^1\delta^{i1},$$

$$i = 2; \quad \sigma^1\sigma^2\sigma^1 = -\sigma^2 \text{ or } -\sigma^2\delta^{i2},$$

$$i = 3; \quad \sigma^1\sigma^3\sigma^1 = -\sigma^3 \text{ or } -\sigma^3\delta^{i3}.$$

Since

$$\begin{aligned}
\overline{U}\gamma^i V = &\left(\frac{\gamma' + 1}{2}\right)\Big[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\delta^{i1} - \rho'^2\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\delta^{i1} + \xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i2} + \rho'^2\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i2} \\
&+ \xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i3} + \rho'^2\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i3}\Big] \\
= &\left(\frac{\gamma' + 1}{2}\right)\Big[\left(1 - \rho'^2\right)\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\delta^{i1} + \left(1 + \rho'^2\right)\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i2} \\
&+ \left(1 + \rho'^2\right)\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i3}\Big]. \qquad (5.33)
\end{aligned}$$

With using the property that is defined by

$$\begin{aligned}
1 - \rho'^2 &= 1 - \left(\frac{\gamma'\beta'}{\gamma' + 1}\right)^2 \\
&= [(\gamma' + 1)^2 - \gamma'^2\beta'^2]\frac{1}{(\gamma' + 1)^2} \\
&= [\gamma'^2(1 - \beta'^2) + 2\gamma' + 1]\frac{1}{(\gamma' + 1)^2} \\
1 - \rho'^2 &= \frac{2}{(\gamma' + 1)}. \qquad (5.34)
\end{aligned}$$

where $\gamma'^2(1 - \beta'^2) = 1$, and

$$1 + \rho'^2 = 1 + \left(\frac{\gamma'\beta'}{\gamma' + 1}\right)^2$$

$$= [(\gamma' + 1)^2 + \gamma'^2\beta'^2]\frac{1}{(\gamma' + 1)^2}$$

$$= [\gamma'^2(1 + \beta'^2) + 2\gamma' + 1]\frac{1}{(\gamma' + 1)^2}$$

$$1 + \rho'^2 = \frac{2\gamma'}{(\gamma' + 1)} \tag{5.35}$$

where $1 + \beta'^2 = \dfrac{1}{\gamma'^2}(2\gamma'^2 - 1)$, we then rewrite Eq. (5.33) as

$$\overline{U}\gamma^i V = \xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\delta^{i1} + \gamma'\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i2} + \gamma'\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i3}. \tag{5.36}$$

and the third needed matrix element, $\overline{v}\gamma^0 u$, on the right-side of Eq. (5.3), be written as

$$\overline{v}\gamma^0 u = -\left(\frac{\gamma+1}{2}\right)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\uparrow\\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= -\left(\frac{\gamma+1}{2}\right)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix}\uparrow\\ -\mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= -\left(\frac{\gamma+1}{2}\right)\left[\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix}\uparrow - \mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix}\downarrow\right]$$

$$\overline{v}\gamma^0 u = 0 \tag{5.37}$$

and the finally needed matrix element, $\overline{v}\gamma^i u$, on the right-side of Eq. (5.3), be written as

$$\overline{v}\gamma^i u = -\left(\frac{\gamma+1}{2}\right)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\uparrow\\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= -\left(\frac{\gamma+1}{2}\right)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix}\mathrm{i}\rho\sigma^i\downarrow\\ -\sigma^i\uparrow\end{pmatrix}$$

$$= -\left(\frac{\gamma+1}{2}\right)\left[-\rho^2\begin{pmatrix}1 & 0\end{pmatrix}\sigma^i\downarrow - \begin{pmatrix}0 & 1\end{pmatrix}\sigma^i\uparrow\right]$$

$$\overline{v}\gamma^i u = \left(\frac{\gamma+1}{2}\right)\left[\rho^2\begin{pmatrix}1 & 0\end{pmatrix}\sigma^i\downarrow + \begin{pmatrix}0 & 1\end{pmatrix}\sigma^i\uparrow\right]. \tag{5.38}$$

The expression of $\begin{pmatrix}0 & 1\end{pmatrix}\sigma_i\begin{pmatrix}0\\1\end{pmatrix}$ on the right-hand side of Eq. (5.38), be defined by

- $i=1;\ \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$ or $(0)\delta^{i1}$
- $i=2;\ \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i}\\\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}-\mathrm{i}\\0\end{pmatrix} = 0$ or $(0)\delta^{i2}$
- $i=3;\ \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} = -1$ or $-\delta^{i3}$

and the expression of $\begin{pmatrix}1 & 0\end{pmatrix}\sigma_i\begin{pmatrix}1\\0\end{pmatrix}$ on the right-hand side of Eq. (5.38), be defined by

- $i=1;\ \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0$ or $(0)\delta^{i1}$
- $i=2;\ \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & -\mathrm{i}\\\mathrm{i} & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\\mathrm{i}\end{pmatrix} = 0$ or $(0)\delta^{i2}$
- $i=3;\ \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1$ or δ^{i3}

With two matrix elements from above, we obtain

$$\overline{v}\gamma^i u = \left(\frac{\gamma+1}{2}\right)\left[\rho^2\left(\delta^{i1}-\mathrm{i}\delta^{i2}\right) + \left(\delta^{i1}+\mathrm{i}\delta^{i2}\right)\right]$$

$$= \left(\frac{\gamma+1}{2}\right)\left[\left(1+\rho^2\right)\delta^{i1} + \mathrm{i}\left(1-\rho^2\right)\delta^{i2}\right]$$

$$\overline{v}\gamma^i u = \left[\gamma\delta^{i1} + \mathrm{i}\delta^{i2}\right]. \tag{5.39}$$

In the next our considering, we will express the needed matrix element of Eq. (5.4). With start from the exact calculation of $\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

$$\gamma^0(b-\gamma_5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\left[\begin{pmatrix} b\mathbb{1} & 0 \\ 0 & b\mathbb{1} \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} b\mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & b\mathbb{1} \end{pmatrix}$$

$$\gamma^0(b-\gamma_5) = \begin{pmatrix} b\mathbb{1} & -\mathbb{1} \\ \mathbb{1} & -b\mathbb{1} \end{pmatrix}, \tag{5.40}$$

and

$$\gamma^i(b-\gamma_5) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\left[\begin{pmatrix} b\mathbb{1} & 0 \\ 0 & b\mathbb{1} \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} b\mathbb{1} & -\mathbb{1} \\ -\mathbb{1} & b\mathbb{1} \end{pmatrix}$$

$$\gamma^i(b-\gamma_5) = \begin{pmatrix} -\sigma^i & b\sigma^i \\ -b\sigma^i & \sigma^i \end{pmatrix}. \tag{5.41}$$

Form Eqs. (5.40)–(5.41), the first needed matrix element, $\overline{U}\gamma^0(b-\gamma_5)V$, on the right-side of Eq. (5.4), be written as

$$\overline{U}\gamma^0(b-\gamma_5)V = \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix}\xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1\end{pmatrix}\begin{pmatrix} b\mathbb{1} & -\mathbb{1} \\ \mathbb{1} & -b\mathbb{1}\end{pmatrix}\begin{pmatrix}-\rho'\sigma^1\xi_{s'} \\ \xi_{s'}\end{pmatrix}$$

$$= \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix}\xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1\end{pmatrix}\begin{pmatrix}-b\rho'\sigma^1\xi_{s'} - \xi_{s'} \\ -\rho'\sigma^1\xi_{s'} - b\xi_{s'}\end{pmatrix}$$

$$= \left(\frac{\gamma'+1}{2}\right)\Big[-b\rho'\xi^\dagger_{\sigma'}\sigma^1\xi_{s'} - \xi^\dagger_{\sigma'}\xi_{s'} + \rho'^2\xi^\dagger_{\sigma'}\sigma^1\sigma^1\xi_{s'}$$

$$+b\left(\frac{\gamma'\beta'}{\gamma'+1}\right)\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\Big]$$

$$\overline{U}\gamma^0(b-\gamma_5)V = -\left(\frac{\gamma'+1}{2}\right)\left(1-\rho'^2\right)\left[\xi^\dagger_{\sigma'}\xi_{s'}\right]$$

$$= -\xi^\dagger_{\sigma'}\xi_{s'}. \tag{5.42}$$

and the second needed matrix element, $\overline{U}\gamma^i(b-\gamma_5)V$, on the right-side of Eq. (5.4), be written as

$$\overline{U}\gamma^i(b-\gamma_5)V = \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix}\xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1\end{pmatrix}\begin{pmatrix} -\sigma^i & b\sigma^i \\ -b\sigma^i & \sigma^i\end{pmatrix}\begin{pmatrix}-\rho'\sigma^1\xi_{s'} \\ \xi_{s'}\end{pmatrix}$$

$$= \left(\frac{\gamma'+1}{2}\right)\begin{pmatrix}\xi^\dagger_{\sigma'} & -\rho'\xi^\dagger_{\sigma'}\sigma^1\end{pmatrix}\begin{pmatrix}\rho'\sigma^i\sigma^1\xi_{s'} + b\sigma^i\xi_{s'} \\ b\rho'\sigma^i\sigma^1\xi_{s'} + \sigma^i\xi_{s'}\end{pmatrix}$$

$$= \left(\frac{\gamma'+1}{2}\right)\Big[\rho'\xi^\dagger_{\sigma'}\sigma^i\sigma^1\xi_{s'} + b\xi^\dagger_{\sigma'}\sigma^i\xi_{s'} - b\rho'^2\xi^\dagger_{\sigma'}\sigma^1\sigma^i\sigma^1\xi_{s'}$$

$$-\rho'\xi^\dagger_{\sigma'}\sigma^1\sigma^i\xi_{s'}\Big].$$

With the expression of $\sigma^i\sigma^1$ on the right-side of an above term, where $\sigma^i\sigma^1 + \sigma^1\sigma^i =$

$2\delta^{i1}$, be defined by

- $i = 1;\qquad \sigma^1\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$ or δ^{i1}
- $i = 2;\qquad \sigma^2\sigma^1 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathrm{i}\sigma^3$ or $-\mathrm{i}\sigma^3\delta^{i2}$
- $i = 3;\qquad \sigma^3\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma^2$ or $\mathrm{i}\sigma^2\delta^{i3}$

and the expression of $\sigma^1\sigma^i$, be defined by

- $i = 1;\qquad \sigma^1\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$ or δ^{i1}
- $i = 2;\qquad \sigma^1\sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{i}\sigma^3$ or $\mathrm{i}\sigma^3\delta^{i2}$
- $i = 3;\qquad \sigma^1\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\sigma^2$ or $-\mathrm{i}\sigma^2\delta^{i3}$

With two matrix elements from above, we obtain

$$\overline{U}\gamma^i(b-\gamma_5)V = \left(\frac{\gamma'+1}{2}\right)\Big[2\mathrm{i}\rho'\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i3} - 2\mathrm{i}\rho'\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i2} + b\left(1-\rho'^2\right)\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\delta^{i1}$$
$$+ b\left(1+\rho'^2\right)\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i2} + b\left(1+\rho'^2\right)\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i3}\Big]$$

$$\overline{U}\gamma^i(b-\gamma_5)V = \Big[\mathrm{i}\left(\gamma'\beta'\right)\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i3} - \mathrm{i}\left(\gamma'\beta'\right)\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i2} + b\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\delta^{i1}$$
$$+b\gamma'\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i2} + \gamma' b\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i3}\Big] \tag{5.43}$$

and the third needed matrix element, $\overline{v}\gamma^0(b-\gamma_5)u$, on the right-side of Eq. (5.4), be written as

$$\overline{v}\gamma^0(b-\gamma_5)u = -\Big(\frac{\gamma+1}{2}\Big)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix} b\mathbb{1} & -\mathbb{1} \\ \mathbb{1} & -b\mathbb{1}\end{pmatrix}\begin{pmatrix}\uparrow \\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= -\Big(\frac{\gamma+1}{2}\Big)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix} b\uparrow -\mathrm{i}\rho\downarrow \\ \uparrow -\mathrm{i}b\rho\downarrow\end{pmatrix}$$

$$= -\Big(\frac{\gamma+1}{2}\Big)\Big[\mathrm{i}b\rho\begin{pmatrix}1 & 0\end{pmatrix}\uparrow -\rho^2\begin{pmatrix}1 & 0\end{pmatrix}\downarrow + \begin{pmatrix}0 & 1\end{pmatrix}\uparrow$$
$$-\mathrm{i}b\rho\begin{pmatrix}0 & 1\end{pmatrix}\downarrow\Big]$$

$$\overline{v}\gamma^0(b-\gamma_5)u = 0. \tag{5.44}$$

and the finally needed matrix element, $\overline{v}\gamma^i(b-\gamma_5)u$, on the right-side of Eq. (5.4), be written as

$$\overline{v}\gamma^i(b-\gamma_5)u = -\Big(\frac{\gamma+1}{2}\Big)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix} -\sigma^i & b\sigma^i \\ -b\sigma^i & \sigma^i\end{pmatrix}\begin{pmatrix}\uparrow \\ \mathrm{i}\rho\downarrow\end{pmatrix}$$

$$= -\Big(\frac{\gamma+1}{2}\Big)\Big(\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\end{pmatrix}\Big)\begin{pmatrix} -\sigma^i\uparrow +\mathrm{i}b\rho\sigma^i\downarrow \\ -b\sigma^i\uparrow +\mathrm{i}\rho\sigma^i\downarrow\end{pmatrix}$$

$$= -\Big(\frac{\gamma+1}{2}\Big)\Big[-\mathrm{i}\rho\begin{pmatrix}1 & 0\end{pmatrix}\sigma^i\uparrow -b\rho^2\begin{pmatrix}1 & 0\end{pmatrix}\sigma^i\downarrow$$
$$-b\begin{pmatrix}0 & 1\end{pmatrix}\sigma^i\uparrow +\mathrm{i}\rho\begin{pmatrix}0 & 1\end{pmatrix}\sigma^i\downarrow\Big].$$

Since we have $\begin{pmatrix}1 & 0\end{pmatrix}\sigma^i\begin{pmatrix}0 \\ 1\end{pmatrix} = \delta^{i1}-\mathrm{i}\delta^{i2}$ and $\begin{pmatrix}0 & 1\end{pmatrix}\sigma^i\begin{pmatrix}1 \\ 0\end{pmatrix} = \delta^{i1}+\mathrm{i}\delta^{i2}$. Therefore we will calculate the expression of $\begin{pmatrix} 1 & 0 \end{pmatrix} \sigma^i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- $i = 1$; $\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ or $(0)\delta^{i1}$
- $i = 2$; $\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \mathrm{i} \end{pmatrix} = 0$ or $(0)\delta^{i2}$
- $i = 3$; $\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ or δ^{i3}

and $\begin{pmatrix} 0 & 1 \end{pmatrix} \sigma^i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $i = 1$; $\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ or $(0)\delta^{i1}$
- $i = 2$; $\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} -\mathrm{i} \\ 0 \end{pmatrix} = 0$ or $(0)\delta^{i2}$
- $i = 3$; $\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1$ or $-\delta^{i3}$

From do these, we obtain

$$\begin{aligned}
\overline{v}\gamma^i(b-\gamma_5)u &= -\left(\frac{\gamma+1}{2}\right)\bigg[-\mathrm{i}\rho\delta^{i3} - b\rho^2\left(\delta^{i1} - \mathrm{i}\delta^{i2}\right) - b\left(\delta^{i1} + \mathrm{i}\delta^{i2}\right) \\
&\qquad - \mathrm{i}\rho\delta^{i3}\bigg] \\
&= -\left(\frac{\gamma+1}{2}\right)\bigg[-b\left(1+\rho^2\right)\delta^{i1} - b\mathrm{i}\rho\delta^{i2} - 2\mathrm{i}\rho\delta^{i3}\bigg]
\end{aligned}$$

$$\overline{v}\gamma^i(b-\gamma_5)u = b\gamma\delta^{i1} + \mathrm{i}b\delta^{i2} + \mathrm{i}\gamma\beta\delta^{i3}. \tag{5.45}$$

From Eqs. (5.42)–(5.45), we obtain

$$\overline{U}\gamma^0(b-\gamma_5)V\overline{v}\gamma^0(b-\gamma_5)u = 0,$$

$$\overline{U}\gamma^i(b-\gamma_5)V\overline{v}\gamma^i(b-\gamma_5)u = \gamma b^2[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}] + \gamma'[\mathrm{i}b^2 - \gamma\beta\beta'][\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}]$$
$$+ \gamma' b[\beta' + \mathrm{i}\gamma\beta][\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}]. \tag{5.46}$$

From Eqs. (5.34)–(5.37), we can rewrite $\mathcal{M}_1$ as

$$\mathcal{M}_1 = \frac{1}{(2p^0)^2}\left\{ \left[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\delta^{i1} + \gamma'\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\delta^{i2} + \gamma'\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\delta^{i3}\right]\left[\gamma\delta^{i1} + \mathrm{i}\delta^{i2}\right] \right\},$$

where $i = 1, 2$ we obtain

$$\mathcal{M}_1 = \frac{1}{(2p^0)^2}\left\{\gamma\left[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\right] + \mathrm{i}\gamma'\left[\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\right]\right\}. \tag{5.47}$$

From Eqs. (5.42)–(5.46), we can rewrite $\mathcal{M}_2$ as

$$\mathcal{M}_2 = \frac{a}{M_Z^2 - 4(p^0)^2}\left\{\gamma b^2[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}] + \gamma'[\mathrm{i}b^2 - \gamma\beta\beta'][\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}]\right.$$
$$\left. + \gamma' b[\beta' + \mathrm{i}\gamma\beta][\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}]\right\}. \tag{5.48}$$

After a tedious calculation we finally have the amplitude, $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$, of this process as

$$\mathcal{M} = \frac{a}{M_Z^2 - 4(p^0)^2}\left\{\gamma b^2[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}] + \gamma'[\mathrm{i}b^2 - \gamma\beta\beta'][\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}] + \gamma' b[\beta' - \mathrm{i}\gamma\beta][\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}]\right\}$$
$$+ \frac{1}{(2p^0)^2}\left\{\gamma\left[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\right] + \mathrm{i}\gamma'\left[\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\right]\right\}$$

$$= \gamma\left(\frac{1}{4(p^0)^2} + \frac{ab^2}{M_Z^2 - 4(p^0)^2}\right)\left[\xi^\dagger_{\sigma'}\sigma^1\xi_{s'}\right] + \frac{\gamma' ab}{M_Z^2 - 4(p^0)^2}[\beta' + \mathrm{i}\gamma\beta]\left[\xi^\dagger_{\sigma'}\sigma^3\xi_{s'}\right]$$

$$+ \gamma'\left(\frac{\mathrm{i}}{4(p^0)^2} + \frac{a}{M_Z^2 - 4(p^0)^2}[\mathrm{i}b^2 - \gamma\beta\beta']\right)\left[\xi^\dagger_{\sigma'}\sigma^2\xi_{s'}\right]. \tag{5.49}$$

The measurement of the spin projection of the muon in our process is specified by using properties of the representation of the spin operator **S** along an arbitrary **n** for spin $1/2$, shown in figure **5.2** , as it is derived in Appendix A. For the two-spinors, we

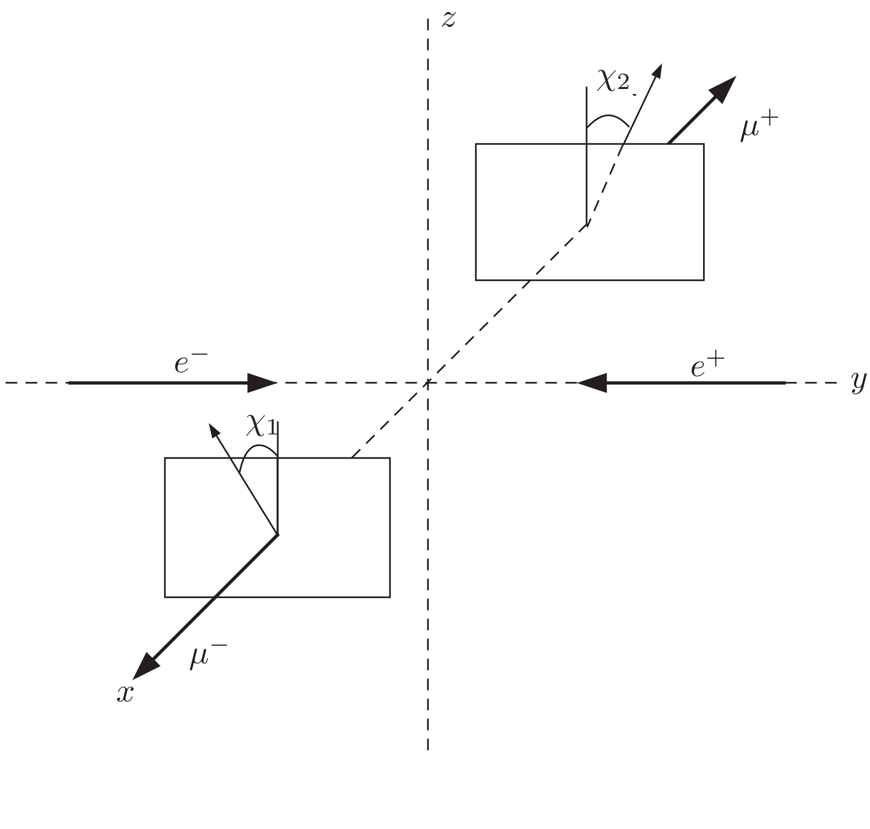

Figure 5.3 The figure depicts the process $e^-e^+ \to \mu^-\mu^+$, with e^-, e^+ moving along the y–axis, and the emerging muons moving along the x–axis. χ_1 and χ_2 denote the angles with the z–axis specifying the directions of measurements of the spins of μ^- and μ^+, respectively.

have (see in Eq. (A.83) and Eq. (A.97))

$$\xi_{\sigma'} = \begin{pmatrix} \cos\dfrac{\chi_1}{2} \\ \sin\dfrac{\chi_1}{2} \end{pmatrix}, \tag{5.50}$$

$$\xi^\dagger_{\sigma'} = \begin{pmatrix} \cos\dfrac{\chi_1}{2} & \sin\dfrac{\chi_1}{2} \end{pmatrix}, \tag{5.51}$$

$$\xi_{s'} = \begin{pmatrix} \cos\dfrac{\chi_2}{2} \\ -\sin\dfrac{\chi_2}{2} \end{pmatrix}. \tag{5.52}$$

Therefore we can calculate the exact $\xi^\dagger_{\sigma'}\sigma^i\xi_{s'}$ where $i = 1, 2, 3$. For $i = 1$, we have

$$\xi^\dagger_{\sigma'}\sigma^1\xi_{s'} = \begin{pmatrix} \cos\dfrac{\chi_1}{2} & \sin\dfrac{\chi_1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\dfrac{\chi_2}{2} \\ -\sin\dfrac{\chi_2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\dfrac{\chi_1}{2} & \sin\dfrac{\chi_1}{2} \end{pmatrix}\begin{pmatrix} -\sin\dfrac{\chi_2}{2} \\ \cos\dfrac{\chi_2}{2} \end{pmatrix}$$

$$= \left[-\cos\frac{\chi_1}{2}\sin\frac{\chi_2}{2} + \cos\frac{\chi_2}{2}\sin\frac{\chi_1}{2}\right]$$

$$\xi^\dagger_{\sigma'}\sigma^1\xi_{s'} = \sin\left(\frac{\chi_1 - \chi_2}{2}\right), \tag{5.53}$$

and for $i = 2$, we have

$$\xi^\dagger_{\sigma'}\sigma^2\xi_{s'} = \begin{pmatrix} \cos\dfrac{\chi_1}{2} & \sin\dfrac{\chi_1}{2} \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \cos\dfrac{\chi_2}{2} \\ -\sin\dfrac{\chi_2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\dfrac{\chi_1}{2} & \sin\dfrac{\chi_1}{2} \end{pmatrix}\begin{pmatrix} \mathrm{i}\sin\dfrac{\chi_2}{2} \\ \mathrm{i}\cos\dfrac{\chi_2}{2} \end{pmatrix}$$

$$= \mathrm{i}\left[\cos\frac{\chi_1}{2}\sin\frac{\chi_2}{2} + \cos\frac{\chi_2}{2}\sin\frac{\chi_1}{2}\right]$$

$$\xi^{\dagger}_{\sigma'}\sigma^2\xi_{s'} = \mathrm{i}\sin\left(\frac{\chi_1+\chi_2}{2}\right), \tag{5.54}$$

finally, for $i = 3$, we have

$$\xi^{\dagger}_{\sigma'}\sigma^3\xi_{s'} = \begin{pmatrix}\cos\frac{\chi_1}{2} & \sin\frac{\chi_1}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\chi_2}{2}\\ -\sin\frac{\chi_2}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\chi_1}{2} & \sin\frac{\chi_1}{2}\end{pmatrix}\begin{pmatrix}\cos\frac{\chi_2}{2}\\ \sin\frac{\chi_2}{2}\end{pmatrix}$$

$$= \left[\cos\frac{\chi_1}{2}\cos\frac{\chi_2}{2} + \sin\frac{\chi_2}{2}\sin\frac{\chi_1}{2}\right]$$

$$\xi^{\dagger}_{\sigma'}\sigma^3\xi_{s'} = \cos\left(\frac{\chi_1-\chi_2}{2}\right). \tag{5.55}$$

The following above calculation, giving

$$\begin{aligned}\mathcal{M} = {}& \gamma\left(\frac{1}{4(p^0)^2} + \frac{ab^2}{M_Z^2 - 4(p^0)^2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right)\\ &+ \gamma'\left(\frac{\mathrm{i}}{4(p^0)^2} + \frac{a}{M_Z^2 - 4(p^0)^2}[\mathrm{i}b^2 - \gamma\beta\beta']\right)\mathrm{i}\sin\left(\frac{\chi_1+\chi_2}{2}\right)\\ &+ \frac{\gamma' ab}{M_Z^2 - 4(p^0)^2}[\beta' + \mathrm{i}\gamma\beta]\cos\left(\frac{\chi_1-\chi_2}{2}\right).\end{aligned} \tag{5.56}$$

After we simplify Eq. (5.56), given

$$\begin{aligned}\mathcal{M} = {}& \gamma\left(\frac{1}{4(p^0)^2} + \frac{ab^2}{M_Z^2 - 4(p^0)^2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right)\\ &+ \gamma'\mathrm{i}\left(\frac{1}{4(p^0)^2} + \frac{ab^2}{M_Z^2 - 4(p^0)^2}\right)\mathrm{i}\sin\left(\frac{\chi_1+\chi_2}{2}\right)\\ &+ \frac{\gamma'\beta' ab}{M_Z^2 - 4(p^0)^2}\cos\left(\frac{\chi_1-\chi_2}{2}\right)\end{aligned}$$

$$- \mathrm{i}\frac{\gamma\beta\gamma'\beta' a}{M_Z^2 - 4(p^0)^2}\sin\left(\frac{\chi_1+\chi_2}{2}\right) + \mathrm{i}\frac{\gamma'\gamma\beta ab}{M_Z^2 - 4(p^0)^2}\cos\left(\frac{\chi_1-\chi_2}{2}\right). \quad (5.57)$$

where we note that $\gamma' = \left(\frac{\gamma}{m}\right)$, $m = \frac{m_\mu}{m_e}$, $\beta' = \sqrt{1-m^2(1-\beta^2)}$, $p^0 = m_e\gamma$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. So that, we rewrite the amplitude in Eq. (5.56) of this process as

$$\begin{aligned}
\mathcal{M} = &\left(\frac{1}{\sqrt{1-\beta^2}}\right)\left(\frac{1}{M_Z^2 - 4m^2\gamma^2}\right)\Bigg\{ -\left(\frac{M_Z^2(1-\beta^2)}{4m_e^2} + ab^2 - 1\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right) \\
&- \left(\frac{1}{m}\right)\left(\frac{M_Z^2(1-\beta^2)}{4m_e^2} + ab^2 - 1\right)\sin\left(\frac{\chi_1+\chi_2}{2}\right) \\
&+ \left(\frac{1}{m}\right) ab\sqrt{1-m^2(1-\beta^2)}\cos\left(\frac{\chi_1+\chi_2}{2}\right) + \mathrm{i}\left(\frac{\beta ab}{m\sqrt{1-\beta^2}}\right)\cos\left(\frac{\chi_1+\chi_2}{2}\right) \\
&- \mathrm{i}\left(\frac{\beta a}{m\sqrt{1-\beta^2}}\right)\sqrt{1-m^2(1-\beta^2)}\sin\left(\frac{\chi_1+\chi_2}{2}\right)\Bigg\}
\end{aligned} \quad (5.58)$$

We will approximate $\mathcal{M}$ because $\left(\frac{1}{\sqrt{1-\beta^2}}\right)\left(\frac{1}{M_Z^2 - 4m^2\gamma^2}\right)$, we rewrite Eq. (5.58) as

$$\begin{aligned}
\mathcal{M} \propto &\left(\frac{M_Z^2(1-\beta^2)}{4m_e^2} + ab^2 - 1\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right) \\
&- \left(\frac{1}{m}\right)\left(\frac{M_Z^2(1-\beta^2)}{4m_e^2} + ab^2 - 1\right)\sin\left(\frac{\chi_1+\chi_2}{2}\right) \\
&+ \left(\frac{1}{m}\right) ab\sqrt{1-m^2(1-\beta^2)}\cos\left(\frac{\chi_1-\chi_2}{2}\right) \\
&- \mathrm{i}\left(\frac{\beta a}{m\sqrt{1-\beta^2}}\right)\left[\sqrt{1-m^2(1-\beta^2)}\sin\left(\frac{\chi_1+\chi_2}{2}\right) - b\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]
\end{aligned} \quad (5.59)$$

Obviously, there is a non-zero probability of occurrence of the above process. Given that such a process has occurred, we compute the conditional joint probability of spins measurements of μ^-, μ^+ along directions specified by the angles χ_1, χ_2 as shown in figure **5.1** . Here we have considered the so-called singlet state. The triplet state leads to an expression similar to the one in Eq. (5.77) for the probability in question with different coefficients $A(\mathcal{E})$, ..., $E(\mathcal{E})$, $N(\mathcal{E})$ and leads again to a violation of Bell's inequality. The corresponding details may be obtained from the authors by the interested reader.

A fairly tedious computation for the invariant amplitude of the process in figure **5.1** leads to

$$\mathcal{M} \propto \left[A(\mathcal{E}) \sin\left(\frac{\chi_1 - \chi_2}{2}\right) + B(\mathcal{E}) \sin\left(\frac{\chi_1 + \chi_2}{2}\right) + C(\mathcal{E}) \cos\left(\frac{\chi_1 - \chi_2}{2}\right)\right] - \mathrm{i}\left[D(\mathcal{E}) \sin\left(\frac{\chi_1 + \chi_2}{2}\right) + E(\mathcal{E}) \cos\left(\frac{\chi_1 - \chi_2}{2}\right)\right], \tag{5.60}$$

where

$$A(\mathcal{E}) = \left(\frac{M_Z^2}{4\mathcal{E}^2} + ab^2 - 1\right), \tag{5.61a}$$

$$B(\mathcal{E}) = -\left(\frac{m_e}{m_\mu}\right)\left(\frac{M_Z^2}{4\mathcal{E}^2} + ab^2 - 1\right), \tag{5.61b}$$

$$C(\mathcal{E}) = \frac{abm_e}{\mathcal{E}m_\mu}\sqrt{\mathcal{E}^2 - m_\mu^2}, \tag{5.61c}$$

$$D(\mathcal{E}) = \frac{a}{m_\mu \mathcal{E}}\sqrt{\mathcal{E}^2 - m_\mu^2}\,\sqrt{\mathcal{E}^2 - m_e^2}, \tag{5.61d}$$

$$E(\mathcal{E}) = -\frac{ab}{m_\mu}\sqrt{\mathcal{E}^2 - m_e^2}, \tag{5.61e}$$

and

$$a \equiv \frac{g^2}{16e^2\cos^2\theta_\mathrm{W}} \cong 0.353, \qquad b \equiv 1 - 4\sin^2\theta_\mathrm{W} \cong 0.08, \tag{5.62}$$

g denotes the weak coupling constant, θ_W is the Weinberg angle, e denotes the electric charge, and $\mathcal{E}$ denotes the energy that depend on the speed (β) of the initial electron (positron), $\mathcal{E} = m_e/\sqrt{1-\beta^2}$. The contribution of the Higgs particles turns out to be too small and is negligible.

The probability density of this process is given $F(\chi_1, \chi_2) \equiv |\mathcal{M}|^2$, be written as:

$$F(\chi_1,\chi_2) = \left[A(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right) + B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right) + C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 + \left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right) + E(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right]^2. \tag{5.63}$$

To normalize the expression in Eq. (5.63), we have to sum $F(\chi_1, \chi_2)$ over the spin polarization directions specified by the pair angles:

$$(\chi_1,\chi_2), \quad (\chi_1+\pi,\chi_2), \quad (\chi_1,\chi_2+\pi), \quad (\chi_1+\pi,\chi_2+\pi). \tag{5.64}$$

That is, we have to find the normalization factor

$$N(\mathcal{E}) = F(\chi_1,\chi_2) + F(\chi_1+\pi,\chi_2) + F(\chi_1,\chi_2+\pi) + F(\chi_1+\pi,\chi_2+\pi). \tag{5.65}$$

The first one, we rotate angle χ_1 with π, by replacing $\chi_1 \to \chi_1 + \pi$, given

$$\begin{aligned} F(\chi_1+\pi,\chi_2) = &\left[A(\mathcal{E})\sin\left(\frac{\chi_1+\pi-\chi_2}{2}\right) + B(\mathcal{E})\sin\left(\frac{\chi_1+\pi+\chi_2}{2}\right)\right. \\ &\left. +C(\mathcal{E})\cos\left(\frac{\chi_1+\pi-\chi_2}{2}\right)\right]^2 \\ &+\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\pi+\chi_2}{2}\right) + E(\mathcal{E})\cos\left(\frac{\chi_1+\pi-\chi_2}{2}\right)\right]^2 \\ = &\left[A(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right) + B(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right. \\ &\left. -C(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \end{aligned}$$

$$+\left[D(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)-E(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2. \tag{5.66}$$

The second one, we rotate angle χ_2 with π, by replacing $\chi_2 \to \chi_2 + \pi$, given

$$F(\chi_1, \chi_2+\pi) = \left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2-\pi}{2}\right)+B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2+\pi}{2}\right)\right.$$

$$\left.+C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2-\pi}{2}\right)\right]^2$$

$$+\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2+\pi}{2}\right)+E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2-\pi}{2}\right)\right]^2$$

$$=\left[-A(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right.$$

$$\left.+C(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2$$

$$+\left[D(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2. \tag{5.67}$$

Finally, we rotate angle χ_1, χ_2 with π, by replacing $\chi_1 \to \chi_1 + \pi$, $\chi_2 \to \chi_2 + \pi$, given

$$F(\chi_1+\pi, \chi_2+\pi) = \left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}+\pi\right)\right.$$

$$\left.+C(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right]^2$$

$$+\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}+\pi\right)+E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2$$

$$=\left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)-B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)\right.$$

$$\left.+C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2$$

$$+\left[-D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2. \quad (5.68)$$

The latter works out to be

$$\begin{aligned}
N(\mathcal{E}) = &\left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[A(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)-C(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[D(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)-E(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[-A(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)+C(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[+D(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)-B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[-D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2. \quad (5.69)
\end{aligned}$$

To convenient calculation of the normalization, we separate $N(\mathcal{E})$ into two term, $N(\mathcal{E}) = N_1 + N_2$. Therefore the first term, we write as:

$$\begin{aligned}
N_1 = &\left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[A(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)-C(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2 \\
&+\left[-A(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)+B(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)+C(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2
\end{aligned}$$

$$+\left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)-B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2. \tag{5.70}$$

We expand the square term in Eq. (5.70) and then simplify it. Giving

$$\begin{aligned}
&=[A(\mathcal{E})]^2\Bigg[\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1-\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1-\chi_2}{2}\right)\\
&\qquad+\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)\Bigg]\\
&+[B(\mathcal{E})]^2\Bigg[\sin^2\left(\frac{\chi_1+\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)\\
&\qquad+\sin^2\left(\frac{\chi_1+\chi_2}{2}\right)\Bigg]\\
&+[C(\mathcal{E})]^2\Bigg[\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1-\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1-\chi_2}{2}\right)\\
&\qquad+\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)\Bigg]\\
&+2A(\mathcal{E})B(\mathcal{E})\Bigg\{\sin\left(\frac{\chi_1-\chi_2}{2}\right)\sin\left(\frac{\chi_1+\chi_2}{2}\right)+\cos\left(\frac{\chi_1-\chi_2}{2}\right)\cos\left(\frac{\chi_1+\chi_2}{2}\right)\\
&\qquad-\cos\left(\frac{\chi_1-\chi_2}{2}\right)\cos\left(\frac{\chi_1+\chi_2}{2}\right)-\sin\left(\frac{\chi_1-\chi_2}{2}\right)\sin\left(\frac{\chi_1+\chi_2}{2}\right)\Bigg\}\\
&+2A(\mathcal{E})C(\mathcal{E})\Bigg\{\sin\left(\frac{\chi_1-\chi_2}{2}\right)\cos\left(\frac{\chi_1-\chi_2}{2}\right)-\cos\left(\frac{\chi_1-\chi_2}{2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right)\\
&\qquad-\cos\left(\frac{\chi_1-\chi_2}{2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right)+\sin\left(\frac{\chi_1-\chi_2}{2}\right)\cos\left(\frac{\chi_1-\chi_2}{2}\right)\Bigg\}\\
&+2B(\mathcal{E})C(\mathcal{E})\Bigg\{\sin\left(\frac{\chi_1+\chi_2}{2}\right)\cos\left(\frac{\chi_1-\chi_2}{2}\right)-\cos\left(\frac{\chi_1+\chi_2}{2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right)
\end{aligned}$$

$$+\cos\left(\frac{\chi_1+\chi_2}{2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right)-\sin\left(\frac{\chi_1+\chi_2}{2}\right)\cos\left(\frac{\chi_1-\chi_2}{2}\right)\Bigg\}. \tag{5.71}$$

This is, we have to find the first term of the normalization factor

$$N_1 = 2\Big\{[A(\mathcal{E})]^2+[B(\mathcal{E})]^2+[C(\mathcal{E})]^2\Big\}.$$

For the second term N_2, we have

$$\begin{aligned}
N_2 = &\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2\\
&+\left[D(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)-E(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2\\
&+\left[D(\mathcal{E})\cos\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2\\
&+\left[-D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right)+E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2.
\end{aligned} \tag{5.72}$$

We expand the square term in Eq. (5.72) and then simplify it. Giving

$$\begin{aligned}
N_2 = [D(\mathcal{E})]^2&\Bigg[\sin^2\left(\frac{\chi_1+\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)+\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)\\
&+\sin^2\left(\frac{\chi_1+\chi_2}{2}\right)\Bigg]\\
+[E(\mathcal{E})]^2&\Bigg[\cos^2\left(\frac{\chi_1-\chi_2}{2}\right)+\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)+\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)\\
&+\cos^2\left(\frac{\chi_1-\chi_2}{2}\right)\Bigg]
\end{aligned}$$

$$+ 2D(\mathcal{E})E(\mathcal{E})\Bigg\{ \sin\left(\frac{\chi_1+\chi_2}{2}\right)\cos\left(\frac{\chi_1-\chi_2}{2}\right) - \cos\left(\frac{\chi_1+\chi_2}{2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right)$$

$$+ \cos\left(\frac{\chi_1+\chi_2}{2}\right)\sin\left(\frac{\chi_1-\chi_2}{2}\right) - \sin\left(\frac{\chi_1+\chi_2}{2}\right)\cos\left(\frac{\chi_1-\chi_2}{2}\right)\Bigg\}. \tag{5.73}$$

This is, we have to find the second term of the normalization factor

$$N_2 = 2\{[D(\mathcal{E})]^2 + [E(\mathcal{E})]^2\}. \tag{5.74}$$

The latter works out to be

$$N(\mathcal{E}) = 2\Big\{\big[A(\mathcal{E})\big]^2 + \big[B(\mathcal{E})\big]^2 + \big[C(\mathcal{E})\big]^2 + \big[D(\mathcal{E})\big]^2 + \big[E(\mathcal{E})\big]^2\Big\}. \tag{5.75}$$

Using the notation $F(\chi_1, \chi_2)$ for the absolute value squared of the right-hand side of Eq. (5.3), the conditional joint probability distribution of spin measurements along the directions specified by the angles χ_1, χ_2 is given by

$$P(\chi_1, \chi_2) = \frac{F(\chi_1, \chi_2)}{N(\mathcal{E})} \tag{5.76}$$

giving

$$P(\chi_1, \chi_2)$$

$$= \frac{1}{N(\mathcal{E})}\left[A(\mathcal{E})\sin\left(\frac{\chi_1-\chi_2}{2}\right) + B(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right) + C(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2$$

$$+ \frac{1}{N(\mathcal{E})}\left[D(\mathcal{E})\sin\left(\frac{\chi_1+\chi_2}{2}\right) + E(\mathcal{E})\cos\left(\frac{\chi_1-\chi_2}{2}\right)\right]^2. \tag{5.77}$$

The probabilities associated with the measurement of only one of the polarizations are given respectively, by

$$P(\chi_1, -) = \frac{F(\chi_1, \chi_2) + F(\chi_1, \chi_2 + \pi)}{N(\mathcal{E})}, \tag{5.78}$$

$$P(-, \chi_2) = \frac{F(\chi_1, \chi_2) + F(\chi_1 + \pi, \chi_2)}{N(\mathcal{E})}. \tag{5.79}$$

For $F(\chi_1, \chi_2) + F(\chi_1, \chi_2 + \pi)$, we have

$$F(\chi_1, \chi_2) + F(\chi_1, \chi_2 + \pi)$$

$$= [A(\mathcal{E})]^2 \left[\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \cos^2\left(\frac{\chi_1 - \chi_2}{2}\right) \right]$$

$$+ [B(\mathcal{E})]^2 \left[\sin^2\left(\frac{\chi_1 + \chi_2}{2}\right) + \cos^2\left(\frac{\chi_1 + \chi_2}{2}\right) \right]$$

$$+ [C(\mathcal{E})]^2 \left[\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \cos^2\left(\frac{\chi_1 - \chi_2}{2}\right) \right]$$

$$+ 2A(\mathcal{E})B(\mathcal{E}) \left\{ \sin\left(\frac{\chi_1 - \chi_2}{2}\right) \sin\left(\frac{\chi_1 + \chi_2}{2}\right) - \cos\left(\frac{\chi_1 - \chi_2}{2}\right) \cos\left(\frac{\chi_1 + \chi_2}{2}\right) \right\}$$

$$+ 2A(\mathcal{E})C(\mathcal{E}) \left\{ \sin\left(\frac{\chi_1 - \chi_2}{2}\right) \cos\left(\frac{\chi_1 - \chi_2}{2}\right) - \cos\left(\frac{\chi_1 - \chi_2}{2}\right) \sin\left(\frac{\chi_1 - \chi_2}{2}\right) \right\}$$

$$+ 2B(\mathcal{E})C(\mathcal{E}) \left\{ \sin\left(\frac{\chi_1 + \chi_2}{2}\right) \cos\left(\frac{\chi_1 - \chi_2}{2}\right) + \cos\left(\frac{\chi_1 + \chi_2}{2}\right) \sin\left(\frac{\chi_1 - \chi_2}{2}\right) \right\}$$

$$= [A(\mathcal{E})]^2 + [B(\mathcal{E})]^2 + [C(\mathcal{E})]^2 + [D(\mathcal{E})]^2 + [E(\mathcal{E})]^2$$

$$- 2A(\mathcal{E})B(\mathcal{E}) \cos\chi_1 + 0 - 2B(\mathcal{E})C(\mathcal{E}) \sin\chi_1$$

$$= \frac{N}{2} - 2B(\mathcal{E})[A(\mathcal{E}) \cos\chi_1 + C(\mathcal{E}) \sin\chi_1]. \tag{5.80}$$

The probabilities associated with the measurement of only one of the polarizations is

$$P(\chi_1, -) = \frac{1}{2} - \frac{2B(\mathcal{E})}{N(\mathcal{E})}\big[A(\mathcal{E})\cos\chi_1 + C(\mathcal{E})\sin\chi_1\big]. \tag{5.81}$$

And similarly for $P(-, \chi_2)$, for $F(\chi_1, \chi_2) + F(\chi_1 + \pi, \chi_2)$

$$F(\chi_1, \chi_2) + F(\chi_1 + \pi, \chi_2)$$

$$= [A(\mathcal{E})]^2\left[\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \cos^2\left(\frac{\chi_1 - \chi_2}{2}\right)\right]$$

$$+ [B(\mathcal{E})]^2\left[\sin^2\left(\frac{\chi_1 + \chi_2}{2}\right) + \cos^2\left(\frac{\chi_1 + \chi_2}{2}\right)\right]$$

$$+ [C(\mathcal{E})]^2\left[\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \cos^2\left(\frac{\chi_1 - \chi_2}{2}\right)\right]$$

$$+ 2A(\mathcal{E})B(\mathcal{E})\left\{\sin\left(\frac{\chi_1 - \chi_2}{2}\right)\sin\left(\frac{\chi_1 + \chi_2}{2}\right) + \cos\left(\frac{\chi_1 - \chi_2}{2}\right)\cos\left(\frac{\chi_1 + \chi_2}{2}\right)\right\}$$

$$+ 2A(\mathcal{E})C(\mathcal{E})\left\{\sin\left(\frac{\chi_1 - \chi_2}{2}\right)\cos\left(\frac{\chi_1 - \chi_2}{2}\right) - \cos\left(\frac{\chi_1 - \chi_2}{2}\right)\sin\left(\frac{\chi_1 - \chi_2}{2}\right)\right\}$$

$$+ 2B(\mathcal{E})C(\mathcal{E})\left\{\sin\left(\frac{\chi_1 + \chi_2}{2}\right)\cos\left(\frac{\chi_1 - \chi_2}{2}\right) - \cos\left(\frac{\chi_1 + \chi_2}{2}\right)\sin\left(\frac{\chi_1 - \chi_2}{2}\right)\right\}$$

$$= [A(\mathcal{E})]^2 + [B(\mathcal{E})]^2 + [C(\mathcal{E})]^2 + [D(\mathcal{E})]^2 + [E(\mathcal{E})]^2$$

$$+ 2A(\mathcal{E})B(\mathcal{E})\cos\chi_2 + 0 + 2B(\mathcal{E})C(\mathcal{E})\sin\chi_2$$

$$= \frac{N}{2} + 2B(\mathcal{E})[A(\mathcal{E})\cos\chi_2 + C(\mathcal{E})\sin\chi_2]. \tag{5.82}$$

The probabilities associated with the measurement of only one of the polarizations is

$$P(-,\chi_2)=\frac{1}{2}+\frac{2B(\mathcal{E})}{N(\mathcal{E})}\big[A(\mathcal{E})\cos\chi_2+C(\mathcal{E})\sin\chi_2\big]. \tag{5.83}$$

It is important to note that $P(\chi_1,\chi_2)\neq P(\chi_1,-)P(-,\chi_2)$, in general, showing the obvious correlations occurring between the two spins.

The indicator S in (5.1) computed according to the probabilities $P(\chi_1,\chi_2)$, $P(\chi_1,-)$, $P(-,\chi_2)$ in (5.7), (5.8), (5.9) may be readily evaluated. To show violation of Bell's inequality, it is sufficient to find four angles χ_1, χ_2, χ_1', χ_2' at accessible energies, for which S falls outside the interval $[-1,0]$. For $\mathcal{E}=105.656$ MeV, i.e., near threshold, an optimal value of S is obtained equal to -1.28203, for $\chi_1=0°$, $\chi_2=45°$, $\chi_1'=90°$, $\chi_2'=135°$, clearly violating Bell's inequality. For the energies originally carried out in the experiment on the differential cross section at $\mathcal{E}\sim 34$ GeV, an optimal value of S is obtained equal to -1.22094 for $\chi_1=0°$, $\chi_2=45°$, $\chi_1'=51.13°$, $\chi_2'=170.85°$.

As mentioned in the introductory part of the chapter, one of the reasons for this investigation arose from the fact that the limit of the speed β of e^-e^+ cannot be taken to go to zero due to the threshold energy needed to create the $\mu^-\mu^+$ pair and methods used for years by simply combining the spins of the particles in question completely fail. The present computations are expected to be relevant near the threshold energy for measuring the spins of the $\mu^-\mu^+$ pair. Near the threshold, the indicator S_{QED} computed within QED coincides with that of S given above in the electroweak model, and varies slightly at higher energies, thus confirming that the weak effects are negligible. Due to the persistence of the dependence of the indicator S on speed, as seen above, in a non-trivial way, it would be interesting if any experiments may be carried out to assess the accuracy of the indicator S as computed within (relativistic) quantum field theory. As there is ample support of the dependence of polarizations correlations, as we have shown by explicit computations in quantum field theory in the electroweak interaction as well as QED ones, on speed, we hope that some new experiments will be carried out in the light of Bell–like tests which monitor speed as further practical tests of quantum physics in the relativistic regime.

CHAPTER VI

POLARIZATION CORRELATIONS IN PAIR PRODUCTION FROM CHARGED AND NEUTRAL STRINGS

6.1 Introduction and General Survey

In this chapter, we investigate the polarizations correlations of e^+e^- pair productions from charged and neutral Nambu strings via processes of photon and graviton emissions, respectively. We consider circularly oscillating closed strings as a cylindrical symmetric solution [Vilenkin, 1981; Gott, 1985; Larse, 1994; Manoukian, 1991, 1994, 1997, 1998; Manoukian and Ungkitchanukit, 1994] arising from the Nambu action [Manoukian, 1991, 1994, 1997, 1998; Manoukian and Ungkitchanukit, 1994; Goddard, Goldstone, Rebbi and Thorne, 1973; Kibble and Turok, 1982; Sakellariadou, 1990; Albrecht and Turok, 1989], as perhaps the simplest string structures in field theory studies. Explicit expressions are derived for their corresponding correlations probabilities and are found to be *speed* dependent. In particular, due to the difference of these probabilities, in general, inquiries about such correlations, would indicate whether *the string is charged or uncharged*. In the extreme relativistic case, however, these probabilities are shown to coincide. The study of such polarization correlations are carried out in the spirit of classic experiments [cf. Clauser and Horne, 1974; Clauser and Shimoney, 1978], to discriminate against Local Hidden Variable (LHV) theories. In this respect alone, it is remarkable that our explicit expressions of polarizations correlations, as obtained from dynamical relativistic quantum field theories, are found to be in clear violation with LHV theories. The speed dependence of polarizations correlations is a common feature of dynamical computations in quantum field theory.

The trajectory of the closed string is described by a vector function $\mathbf{R}(\sigma, t)$, where σ parameterizes the string, satisfying [Manoukian, 1991, 1994, 1997, 1998; Manoukian and Ungkitchanukit, 1994; Goddard, Goldstone, Rebbi and Thorne, 1973; Kibble and Turok, 1982; Sakellariadou, 1990; Albrecht and Turok, 1989]

$$\ddot{\mathbf{R}} - \mathbf{R}'' = 0, \tag{6.1}$$

with constraints

$$\dot{\mathbf{R}} \cdot \mathbf{R}' = 0, \quad \dot{\mathbf{R}}^2 + \mathbf{R}'^2 = 1, \quad \mathbf{R}(\sigma + \frac{2\pi}{m}, t) = \mathbf{R}(\sigma, t), \tag{6.2}$$

where the mass scale m is taken to be the mass of the electron, $\dot{\mathbf{R}} = \partial\mathbf{R}/\partial t$, $\mathbf{R}' = \partial\mathbf{R}/\partial\sigma$, with the general solution

$$\mathbf{R}(\sigma, t) = \frac{1}{2}\left[\mathbf{A}(\sigma - t) + \mathbf{B}(\sigma + t)\right], \tag{6.3}$$

where $\mathbf{A}$ and $\mathbf{B}$ satisfy, in particular, the normalization conditions $\mathbf{A}'^2 = \mathbf{B}'^2 = 1$. For the system in Eqs. (6.1–6.2), we consider a solution of the form (Gott, 1985; Larse, 1994; Manoukian, 1991, 1994, 1997, 1998; Manoukian and Ungkitchanukit, 1994):

$$\mathbf{R}(\sigma, t) = \frac{1}{m}(\cos m\sigma, \sin m\sigma, 0) \sin mt, \tag{6.4}$$

with the z-axis perpendicular to the plane of oscillations.

6.2 Pair Production from a Charged String

In this section, we consider e^+e^- production by Nambu string. We analytical calculate the polarization correlation of e^+e^- production from a closed charged string arising from the Nambu action as a solution of a circularly oscillating closed charged string as perhaps the simplest object generalizing emissions from point-like particles within the framework of quantum electrodynamics. The charged string, during one period of oscillation, to lowest order in the fine-structure constant, emits a virtual photon which in turn decays into e^+e^- pair with momenta $\mathbf{p}_1$, $\mathbf{p}_2$ and spins σ_1, σ_2, respectively. From this we study the polarization correlation of the e^+e^- pair.

For a string of total charge Q, this generates a current density [Manoukian, 1994] $J^\mu(x)$ with structure $(x = (t, \mathbf{r}, z))$

$$J^\mu(x) = \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{(2\pi)} \int_{-\infty}^{\infty} \frac{\mathrm{d}P^0}{(2\pi)} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \mathrm{e}^{\mathrm{i}qz} \mathrm{e}^{-\mathrm{i}P^0 t} J^\mu(P^0, \mathbf{p}), \tag{6.5}$$

$$J^\mu(P^0, \mathbf{p}) = 2\pi \sum_N \delta(P^0 - mN) B^\mu(\mathbf{p}, N), \tag{6.6}$$

summing over integers,

$$B^0(\mathbf{p}, N) = a_N J^2_{N/2}\left(\frac{|\mathbf{p}|}{2m}\right), \tag{6.7}$$

$$\mathbf{B}(\mathbf{p}, N) = \frac{mN}{|\mathbf{p}|^2} \mathbf{p} B^0(\mathbf{p}, N), \tag{6.8}$$

$$a_N = Q(-1)^{N/2} \cos\left(\frac{N\pi}{2}\right), \tag{6.9}$$

where $J_{N/2}$ are the ordinary Bessel functions of order $N/2$.

We consider the process of e^+e^- pair production via a photon emission, given by the amplitude [Manoukian, 1991, 1994, 1997, 1998; Manoukian and Ungkitchanukit, 1994], [cf. Wichoski, 2002; Transchen, 1988], up to an overall multiplicative factor irrelevant for the problem at hand (see in Eq. (2.125)).

$$A \propto eJ^\mu(2p^0, \mathbf{p}_1 + \mathbf{p}_2) \frac{1}{(p_1 + p_2)^2} \left[\overline{u}(\mathbf{p}_1, \sigma_1)\gamma_\mu v \mathbf{p}_2, \sigma_2)\right] \tag{6.10}$$

with the four momenta of e^-, e^+, respectively given by

$$\mathbf{p}_1 = k(0, 1, 0), \quad \mathbf{p}_2 = k(1, 0, 0), \quad k = m\gamma\beta \tag{6.11}$$

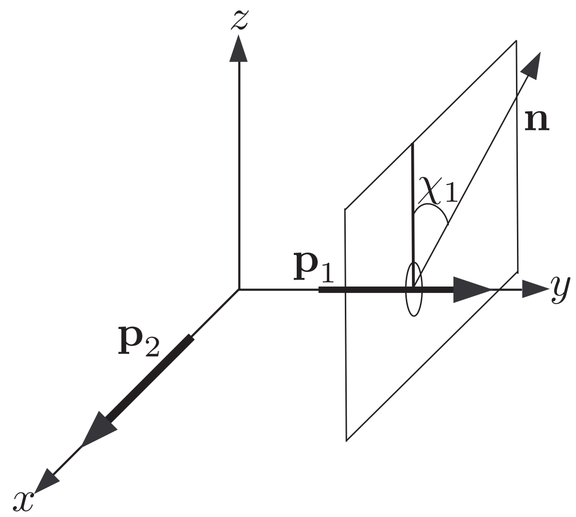

Figure 6.1 The measurement of the spin projection of the electron is taken along an axis making an angle χ_1 with the z-axis and lying in a plane parallel to the x-z plane.

$$p_1^0 = p_2^0 = (\mathbf{k} + m^2)^{1/2} \equiv p^0 = m\gamma \tag{6.12}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor. The measurement of the spin projection of the electron is taken along an axis making an angle χ_1 with the z-axis and lying in a plane parallel to the x-z plane (see in figure **6.1**),

$$u = \sqrt{\frac{p^0+m}{2m}} \begin{pmatrix} \xi_1 \\ \frac{k\sigma_2}{p^0+m}\xi_1 \end{pmatrix}, \qquad v = \sqrt{\frac{p^0+m}{2m}} \begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix} \tag{6.13}$$

where the direction of the spin of the positron lies in a plane parallel to the y-z plane, and the two-spinors χ_1, χ_2 will be specified later (see in Appendix A).

The expression for the amplitude of this process be written:

$$\begin{aligned} A \propto\ & - eJ^0(2p^0, \mathbf{p}_1 + \mathbf{p}_2)\frac{1}{(p_1+p_2)^2}[\overline{u}(p_1)\gamma^0 v(p_2)] \\ & + e\mathbf{J}(2p^0, \mathbf{p}_1 + \mathbf{p}_2)\frac{1}{(p_1+p_2)^2}[\overline{u}(p_1)\gamma^i v(p_2)]. \end{aligned} \tag{6.14}$$

In our process, we will study the polarization correlation of e^+e^-. Therefore, we not sum over all spins of e^+e^- and by using the four spinors of e^+e^- in Eq. (6.13). We have

$$\bar{u}\gamma^0 v = \sqrt{\frac{p^0+m}{2m}} \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{k\sigma_2}{p^0+m} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \sqrt{\frac{p^0+m}{2m}} \begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right) \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{k\sigma_2}{p^0+m} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right) \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{k\sigma_2}{p^0+m} \end{pmatrix} \begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ -\xi_2 \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right) \left[-\frac{k}{p^0+m}\xi_1^\dagger \sigma_1 \xi_2 + \frac{k}{p^0+m}\xi_1^\dagger \sigma_2 \xi_2 \right]$$

$$= \left(\frac{p^0+m}{2m}\right) \frac{k}{p^0+m} [\xi_1^\dagger \sigma_1 \xi_2 - \xi_1^\dagger \sigma_2 \xi_2].$$

By using properties Eqs. (6.11)–(6.12), we rewrite above term as:

$$\overline{u}\gamma^0 v = \frac{k}{2m}[-\xi_1^\dagger \sigma_1 \xi_2 + \xi_1^\dagger \sigma_2 \xi_2]$$

$$= \frac{\gamma\beta}{2}[-\xi_1^\dagger \sigma_1 \xi_2 + \xi_1^\dagger \sigma_2 \xi_2]. \tag{6.15}$$

For $\overline{u}\gamma^i v$, we have

$$\overline{u}\gamma^i v = \left(\frac{p^0+m}{2m}\right) \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{k\sigma_2}{p^0+m} \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right) \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{k\sigma_2}{p^0+m} \end{pmatrix} \begin{pmatrix} \sigma_i \xi_2 \\ \frac{k\sigma_1}{p^0+m}\sigma_i \sigma_1 \xi_2 \end{pmatrix}$$

$$\overline{u}\gamma^i v = \left(\frac{p^0+m}{2m}\right)[\xi_1^\dagger \sigma_i \xi_2 - (\frac{k}{p^0+m})^2 \xi_1^\dagger \sigma_2 \sigma_i \sigma_1 \xi_2]. \tag{6.16}$$

To simplify above term, we change $\sigma_2\sigma_i\sigma_1$ by using the properties of the Pauli matrix (see in Appendix A). Given by

$$i = 1 \qquad \sigma_2\sigma_1\sigma_1 = \sigma_2\sigma_1^2 = \sigma_2,$$

$$i = 2 \qquad \sigma_2\sigma_2\sigma_1 = \sigma_2^2\sigma_1 = \sigma_1.$$

We note that $i = 3$ is not considered because we consider the momenta of e^+e^- in x-y plane.

In this case, we obtain

$$\overline{u}\gamma^1 v = \left(\frac{p^0+m}{2m}\right)\left[\xi_1^\dagger \sigma_1 \xi_2 - \left(\frac{k}{p^0+m}\right)^2 \xi_1^\dagger \sigma_2 \xi_2\right],$$

$$\overline{u}\gamma^2 v = \left(\frac{p^0+m}{2m}\right)\left[\xi_2^\dagger \sigma_2 \xi_2 - \left(\frac{k}{p^0+m}\right)^2 \xi_1^\dagger \sigma_1 \xi_2\right].$$

The matrix elements are needed to use in Eq. (6.14), be written:

$$\overline{u}\gamma^i v = \left(\frac{p^0+m}{2m}\right)\left(1 - \left(\frac{k}{p^0+m}\right)^2\right)[\xi_1^\dagger \sigma_1 \xi_2 + \xi_1^\dagger \sigma_2 \xi_2], \tag{6.17}$$

where $i = 1, 2$.

To simplify Eq. (6.17), let

$$\left(\frac{p^0+m}{2m}\right)\left(1 - \left(\frac{k}{p^0+m}\right)^2\right) = \left(\frac{p^0+m}{2m}\right)\left(\frac{1}{p^0+m}\right)^2\left(\left(p^0+m\right)^2 - k^2\right)$$

$$= \left(\frac{1}{2m}\right)\left(\frac{1}{p^0+m}\right)\left(\left(p^0\right)^2 + 2mp^0 + m^2 - k^2\right),$$

by using properties in Eqs. (6.11)–(6.12), we have

$$= \left(\frac{1}{2m}\right)\left(\frac{1}{p^0+m}\right)\left(m\left(2p^0+m\right)+m^2\gamma^2(1-\beta^2)\right)$$

$$= \left(\frac{1}{2m}\right)\left(\frac{1}{m\gamma+m}\right)\left(m\left(2m\gamma+m\right)+m^2\gamma^2(1-\beta^2)\right)$$

$$= \left(\frac{1}{2m}\right)\left(\frac{1}{m}\right)\left(\frac{1}{\gamma+1}\right)\left(m^2\left(2\gamma+1\right)+m^2\gamma^2(1-\beta^2)\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\gamma+1}\right)\left((2\gamma+1)+\gamma^2(1-\beta^2)\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\gamma+1}\right)\left((2\gamma+1)+1\right)$$

$$\left(\frac{p^0+m}{2m}\right)\left(1-\left(\frac{k}{p^0+m}\right)^2\right) = 1. \tag{6.18}$$

The matrix elements are needed to use in Eq. (6.14), be written:

$$\overline{u}\gamma^i v = [\xi_1^\dagger\sigma_1\xi_2 + \xi_1^\dagger\sigma_2\xi_2], \tag{6.19}$$

where $i = 1, 2$.

We consider the electromagnetic current

$$J^\mu(P^0, \mathbf{p}_1+\mathbf{p}_2) = 2\pi\sum_N \delta(P^0 - mN)B^\mu(\mathbf{p}_1+\mathbf{p}_2, N),$$

where

$$\mathbf{B}(\mathbf{p}_1+\mathbf{p}_2, N) = \frac{mN}{|\mathbf{p}_1+\mathbf{p}_2|^2}(\mathbf{p}_1+\mathbf{p}_2)B^0(\mathbf{p}_1+\mathbf{p}_2, N)$$

$$B^0(\mathbf{p}_1+\mathbf{p}_2, N) = a_N J^2_{N/2}(|\mathbf{p}_1+\mathbf{p}_2|/2m).$$

Or write in term of the summation of $B(\mathbf{p}_1+\mathbf{p}_2,N)$

$$J^{\nu}(2p^0,|\mathbf{p}_1+\mathbf{p}_2|)=2\pi\sum_N\delta(2p^0-mN)B^{\nu}(\mathbf{p}_1+\mathbf{p}_2,N), \tag{6.20}$$

when $mN=2p^0$ and $\mathbf{p}_1+\mathbf{p}_2=k(1,1,0)$, therefor, $|\mathbf{p}_1+\mathbf{p}_2|=\sqrt{2}k$, we have

$$\mathbf{B}(\mathbf{p}_1+\mathbf{p}_2,N)=\frac{1}{\beta}B^0(\mathbf{p}_1+\mathbf{p}_2,N). \tag{6.21}$$

By replacing Eq. (6.15), Eq. (6.17), Eq. (6.20) and Eq. (6.21) in Eq. (6.14), we obtain

$$\begin{aligned}\mathcal{A}\propto\; & -2\pi e\sum_N\delta(2p^0-mN)B^0(\mathbf{p}_1+\mathbf{p}_2,N)\frac{1}{(p_1+p_2)^2}\frac{\gamma\beta}{2}[-\xi_1^{\dagger}\sigma_1\xi_2+\xi_1^{\dagger}\sigma_2\xi_2]\\ & +2\pi e\sum_N\delta(2p^0-mN)\mathbf{B}(\mathbf{p}_1+\mathbf{p}_2,N)\frac{1}{(p_1+p_2)^2}[\xi_1^{\dagger}\sigma_1\xi_2+\xi_1^{\dagger}\sigma_2\xi_2],\\ =& -2\pi e\sum_N\delta(2p^0-mN)B^0(\mathbf{p}_1+\mathbf{p}_2,N)\frac{1}{(p_1+p_2)^2}\frac{\gamma\beta}{2}[-\xi_1^{\dagger}\sigma_1\xi_2+\xi_1^{\dagger}\sigma_2\xi_2]\\ & +2\pi e\sum_N\delta(2p^0-mN)\frac{1}{\beta}B^0(\mathbf{p},N)\frac{1}{(p_1+p_2)^2}[\xi_1^{\dagger}\sigma_1\xi_2+\xi_1^{\dagger}\sigma_2\xi_2].\end{aligned}$$

After we simplify above term. We obtain

$$\begin{aligned}\mathcal{A}\propto 2\pi e\sum_N\delta(2p^0-mN)B^0(\mathbf{p}_1+\mathbf{p}_2,N)\frac{1}{(p_1+p_2)^2}\\ \times\left\{\left[\frac{1}{\beta}+\frac{\gamma\beta}{2}\right][\xi_1^{\dagger}\sigma_1\xi_2]+\left[\frac{1}{\beta}-\frac{\gamma\beta}{2}\right][\xi_1^{\dagger}\sigma_2\xi_2]\right\},\end{aligned} \tag{6.22}$$

where the two-spinors χ_1, χ_2 will be specified later.

The measurement of the spin projection of the electron (positron) in our process is specified by using properties of the representation of the spin operator $\mathbf{S}$ along an arbitrary unit vector $\mathbf{n}$ for spin $1/2$, as it is derived in Appendix A. For the 2-spinors, we have (see in Eqs. (A.67)–(A.68))

$$\xi_1 = \begin{pmatrix} \cos\left(\frac{\chi_1}{2}\right) \\ -\sin\left(\frac{\chi_1}{2}\right) \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} \sin\left(\frac{\chi_2}{2}\right) \\ \cos\left(\frac{\chi_2}{2}\right) \end{pmatrix}. \tag{6.23}$$

Then we calculate the exact $[\xi_1^\dagger \sigma_i \xi_2]$ where $i = 1, 2$. For $i = 1$, we have

$$\begin{aligned} \xi_1^\dagger \sigma_1 \xi_2 &= \begin{pmatrix} \cos\left(\frac{\chi_1}{2}\right) & -\sin\left(\frac{\chi_1}{2}\right) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin\left(\frac{\chi_2}{2}\right) \\ \cos\left(\frac{\chi_2}{2}\right) \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\frac{\chi_1}{2}\right) & -\sin\left(\frac{\chi_1}{2}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\chi_2}{2}\right) \\ \sin\left(\frac{\chi_2}{2}\right) \end{pmatrix} \\ &= \left[\cos\left(\frac{\chi_1}{2}\right)\cos\left(\frac{\chi_2}{2}\right) - \sin\left(\frac{\chi_1}{2}\right)\sin\left(\frac{\chi_2}{2}\right)\right] \\ \xi_1^\dagger \sigma_1 \xi_2 &= \cos\left(\frac{\chi_1 + \chi_2}{2}\right), \end{aligned} \tag{6.24}$$

and for $i = 2$, we have

$$\begin{aligned} \xi_1^\dagger \sigma_2 \xi_2 &= \begin{pmatrix} \cos\left(\frac{\chi_1}{2}\right) & -\sin\left(\frac{\chi_1}{2}\right) \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \sin\left(\frac{\chi_2}{2}\right) \\ \cos\left(\frac{\chi_2}{2}\right) \end{pmatrix} \\ &= \mathrm{i} \begin{pmatrix} \cos\left(\frac{\chi_1}{2}\right) & -\sin\left(\frac{\chi_1}{2}\right) \end{pmatrix} \begin{pmatrix} -\cos\left(\frac{\chi_2}{2}\right) \\ \sin\left(\frac{\chi_2}{2}\right) \end{pmatrix} \\ &= -\mathrm{i}\left[\cos\left(\frac{\chi_1}{2}\right)\cos\left(\frac{\chi_2}{2}\right) + \sin\left(\frac{\chi_1}{2}\right)\sin\left(\frac{\chi_2}{2}\right)\right] \\ \xi_1^\dagger \sigma_2 \xi_2 &= -\mathrm{i}\cos\left(\frac{\chi_1 - \chi_2}{2}\right). \end{aligned} \tag{6.25}$$

A tedious but straightforward computation gives

$$A \propto \frac{\left[-\mathrm{i}\left(1-\frac{\gamma\beta^2}{2}\right)\cos\left(\frac{\chi_1-\chi_2}{2}\right)+\left(1+\frac{\gamma\beta^2}{2}\right)\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right]}{(\gamma^2\beta^2-2)}\sum_N \delta(2p^0-mN)B^0, \tag{6.26}$$

where we note that $2p^0/m = 2\gamma$ is quantized. *Given* that the above process has occurred, a standard computation. We can neglect $\sum_N \delta(2p^0-mN)B^0\frac{1}{(\gamma^2\beta^2-2)}$, disappear after we normalize probability. So that, we rewrite the amplitude of this process as:

$$A \propto \left[1+\frac{\gamma\beta^2}{2}\right]\cos\left(\frac{\chi_1+\chi_2}{2}\right)-\mathrm{i}\left[1-\frac{\gamma\beta^2}{2}\right]\cos\left(\frac{\chi_1-\chi_2}{2}\right). \tag{6.27}$$

The probability of this process is given $F(\chi_1,\chi_2)\equiv|A|^2$, be written as:

$$F(\chi_1,\chi_2)=\left[1+\frac{\gamma\beta^2}{2}\right]^2\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)+\left[1-\frac{\gamma\beta^2}{2}\right]^2\cos^2\left(\frac{\chi_1-\chi_2}{2}\right). \tag{6.28}$$

To normalize the expression in Eq. (6.28), we have to sum $F(\chi_1,\chi_2)$ over the spin polarization directions specified by the pairs of angles:

$$(\chi_1,\chi_2),\quad (\chi_1+\pi,\chi_2),\quad (\chi_1,\chi_2+\pi),\quad (\chi_1+\pi,\chi_2+\pi) \tag{6.29}$$

That is, we have to find the normalization factor

$$N = F(\chi_1,\chi_2)+F(\chi_1+\pi,\chi_2)+F(\chi_1,\chi_2+\pi)+F(\chi_1+\pi,\chi_2+\pi). \tag{6.30}$$

The first one, we rotated angle χ_1 with π, by replacing $\chi_1\to\chi_1+\pi$, given

$$F(\chi_1+\pi,\chi_2)=\left[1+\frac{\gamma\beta^2}{2}\right]^2\sin^2\left(\frac{\chi_1+\chi_2}{2}\right)+\left[1-\frac{\gamma\beta^2}{2}\right]^2\sin^2\left(\frac{\chi_1-\chi_2}{2}\right). \tag{6.31}$$

The second one, we rotated angle χ_2 with π, by replacing $\chi_2 \to \chi_2 + \pi$, given

$$F(\chi_1, \chi_2+\pi) = \left[1 + \frac{\gamma\beta^2}{2}\right]^2 \sin^2\left(\frac{\chi_1 + \chi_2}{2}\right) + \left[1 - \frac{\gamma\beta^2}{2}\right]^2 \sin^2\left(\frac{\chi_1 - \chi_2}{2}\right). \quad (6.32)$$

Finally, we rotated angle χ_1, χ_2 with π, by replacing $\chi_1 \to \chi_1 + \pi$, $\chi_2 \to \chi_1 + \pi$, given

$$F(\chi_1 + \pi, \chi_2 + \pi) = \left[1 + \frac{\gamma\beta^2}{2}\right]^2 \cos^2\left(\frac{\chi_1 + \chi_2}{2}\right) + \left[1 - \frac{\gamma\beta^2}{2}\right]^2 \cos^2\left(\frac{\chi_1 - \chi_2}{2}\right). \quad (6.33)$$

The latter works out to be

$$N = 2\left[\left[1 + \frac{\gamma\beta^2}{2}\right]^2 + \left[1 - \frac{\gamma\beta^2}{2}\right]^2\right]. \quad (6.34)$$

where

$$\left[1 + \frac{\gamma\beta^2}{2}\right]^2 = \left[1 + \frac{\beta^2}{2\sqrt{1-\beta^2}}\right]^2$$

$$= \frac{1}{4(1-\beta^2)}\left[2\sqrt{1-\beta^2} + \beta^2\right]^2$$

$$\left[1 - \frac{\gamma\beta^2}{2}\right]^2 = \left[1 - \frac{\beta^2}{2\sqrt{1-\beta^2}}\right]^2$$

$$= \frac{1}{4(1-\beta^2)}\left[2\sqrt{1-\beta^2} - \beta^2\right]^2$$

$$\left[1 + \frac{\gamma\beta^2}{2}\right]^2 + \left[1 - \frac{\gamma\beta^2}{2}\right]^2 = \frac{1}{4(1-\beta^2)}\left\{\left[2\sqrt{1-\beta^2} + \beta^2\right]^2 + \left[2\sqrt{1-\beta^2} - \beta^2\right]^2\right\}$$

$$= \frac{1}{4(1-\beta^2)}\left\{8(1-\beta^2) + 2\beta^4\right\}$$

$$\left[1 + \frac{\gamma\beta^2}{2}\right]^2 + \left[1 - \frac{\gamma\beta^2}{2}\right]^2 = \frac{2}{4(1-\beta^2)}\left(2 - \beta^2\right)^2. \quad (6.35)$$

Therefore, given that the process has occurred as expressed in figure **6.1** , with electron moving along y-axis and positron moving along x-axis, the probability of the spin polarizations, specified by the angles χ_1, χ_2, is rigorously given by

$$P(\chi_1, \chi_2) = \frac{F(\chi_1, \chi_2)}{N}. \tag{6.36}$$

Therefore, we obtain probabilities as

$$P(\chi_1, \chi_2) = \frac{\left[1 + \frac{\gamma\beta^2}{2}\right]^2 \cos^2\left(\frac{\chi_1+\chi_2}{2}\right) + \left[1 - \frac{\gamma\beta^2}{2}\right]^2 \cos^2\left(\frac{\chi_1-\chi_2}{2}\right)}{2\left[\left[1 + \frac{\gamma\beta^2}{2}\right]^2 + \left[1 - \frac{\gamma\beta^2}{2}\right]^2\right]}, \tag{6.37}$$

as in Yongram and Manoukian (2003) and Manoukian and Yongram (2004), given the following explicit expression for the probability of the simultaneous measurements of the spins of e^-, e^+, with angles χ_1, χ_2, as specified above,

$$P(\chi_1, \chi_2) = \frac{\left(2\sqrt{1-\beta^2} - \beta^2\right)^2 \cos^2\left(\frac{\chi_1-\chi_2}{2}\right) + \left(2\sqrt{1-\beta^2} + \beta^2\right)^2 \cos^2\left(\frac{\chi_1+\chi_2}{2}\right)}{4\left(2-\beta^2\right)^2}. \tag{6.38}$$

the so-called probability of the polarizations correlations of the emitted pair and is *speed* dependent. If the spin of only one of the particles, say, that of e^-, is measured, then we have to sum Eq. (6.38) over the two possible outcomes for $e^+ : \chi_2, \chi_2 + \pi$, for a given χ_2, i.e., for the probability of measuring the spin of e^- only, we have

$$P(\chi_1, -) = P(\chi_1, \chi_2) + P(\chi_1, \chi_2 + \pi) = 1/2. \tag{6.39}$$

Similarly, for the probability $P(-, \chi_2)$, where only a measurement of the spin of e^+ is made, we obtain

$$P(-, \chi_2) = 1/2. \tag{6.40}$$

In the extreme relativistic case $\beta \to 1$, Eq. (6.38) gives

$$P[\chi_1, \chi_2] \to \frac{1}{4}\left[\cos^2\left(\frac{\chi_1 - \chi_2}{2}\right) + \cos^2\left(\frac{\chi_1 + \chi_2}{2}\right)\right] \tag{6.41}$$

6.3 Pair Production from a Neutral String

In this section, we consider e^+e^- production by Nambu string. We analytical calculate the polarization correlation of e^+e^- production from a closed neutral string arising from the Nambu action as a solution of a circularly oscillating closed neutral string as perhaps the simplest object generalizing emissions from point-like particles within the framework of quantum electrodynamics. The neutral string, during its oscillation, to lowest order in the gravitational coupling constant, emits a graviton which in turn decays into e^+e^- pair with momenta $\mathbf{p}_1$, $\mathbf{p}_2$ and spins σ_1, σ_2, respectively. From this we study the polarization correlation of the e^+e^- pair.

The neutral string, of a given mass M, generates an energy-momentum tensor density $T^{\mu\nu}(x)$ with structure [Manoukian, 1998]

$$T^{\mu\nu}(x) = \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{(2\pi)} \int_{-\infty}^{\infty} \frac{\mathrm{d}P^0}{(2\pi)} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \mathrm{e}^{\mathrm{i}qz} \mathrm{e}^{-\mathrm{i}P^0 t} T^{\mu\nu}(P^0, \mathbf{p}) \tag{6.42}$$

$$T^{\mu\nu}(P^0, \mathbf{p}) = 2\pi \sum_{N=-\infty}^{\infty} \delta(2p^0 - mN) B^{\mu\nu}(\mathbf{p}, N) \tag{6.43}$$

$$B^{00}(\mathbf{p}, N) = \beta_N J^2_{N/2}(z), \quad z = \frac{|\mathbf{p}|}{2m} \tag{6.44}$$

$$B^{0a}(\mathbf{p}, N) = \beta_N \frac{P^0 p^a}{|\mathbf{p}|^2} J^2_{N/2}(z), \quad a = 1, 2 \tag{6.45}$$

$$B^{ab}(\mathbf{p}, N) = \beta_N \left(A_N \delta^{ab} + E_N \frac{p^a p^b}{|\mathbf{p}|^2}\right), \quad a, b = 1, 2 \tag{6.46}$$

$$B^{\mu 3}(\mathbf{p}, N) = 0, \quad \mu = 0, 1, 2, 3 \tag{6.47}$$

$$A_N = \frac{1}{4}\left[J_{\frac{N}{2}+1}(z) - J_{\frac{N}{2}-1}(z)\right]^2, \tag{6.48}$$

$$E_N = J_{\frac{N}{2}+1}(z) J_{\frac{N}{2}-1}(z), \tag{6.49}$$

$$\beta_N = M(-1)^{N/2} \cos\left(\frac{N\pi}{2}\right). \tag{6.50}$$

Here J_ν denotes a Bessel function of order ν.

For e^+e^- pair production via the emission of a graviton, the amplitude of the process is given by

$$A \propto T^{\sigma\lambda}(2p^0, \mathbf{p}_1 + \mathbf{p}_2) \frac{\left[g_{\sigma\mu}g_{\lambda\nu} - \frac{1}{2}g_{\sigma\lambda}g_{\mu\nu}\right]}{(p_1+p_2)^2} T^{\mu\nu}_{e^+e^-}, \tag{6.51}$$

where $T^{\mu\nu}_{e^+e^-}$ is the energy-momentum tensor density associated with the pair, see in Eq. (2.127), with the proportionally constant depending linearly on the gravitational coupling G:

$$T^{\mu\nu}_{e^+e^-} \propto \overline{u}\left[\gamma^\mu\left(p^\nu - p'^\nu\right) + \gamma^\nu\left(p^\mu - p'^\mu\right)\right]. \tag{6.52}$$

From Eqs. (6.43)–(6.44), Eq. (6.11), Eq. (6.22), this simplifies to

$$A \propto \frac{1}{(p_1+p_2)^2}\left\{-2m\overline{u}vT^{00} + 2\left[(\overline{u}\gamma_a v)\left(p^1_b - p^2_b\right) + m\delta_{ab}\overline{u}v\right]T^{ab}\right\}. \tag{6.53}$$

The recurrence relation

$$J_{\frac{N}{2}-1}(z) = \frac{2\sqrt{2}}{\beta} J_{\frac{N}{2}}(z) - J_{\frac{N}{2}+1}(z), \tag{6.54}$$

allows one to express A_N, E_N in terms of $J_{N/2} + 1$, $J_{N/2}$ which differ by one order only, and for sufficiently high energies, they may be expressed in terms of $J_{N/2}$.

So that, we can rewrite the amplitude of this process, with replace p_1, p_2 by p,

p', respectively, as

$$A \propto \frac{1}{(p+p')^2}\{T^{00}\bar{u}[2\gamma_0(p_0-p'_0)-2m]v+2T^{0i}\bar{u}[\gamma_0(p_i-p'_i)+\gamma_i(p_0-p'_0)]v$$

$$+T^{ij}\bar{u}[\gamma_i(p_j-p'_j)+\gamma_j(p_i-p'_i)+2m\delta_{ij}]v\}, \tag{6.55}$$

where $p_\mu B^\mu = 0$, $p^0 = mN/2 = p'^0$.

Therefor we can rewrite Eq. (6.55) as

$$A \propto \frac{1}{(p+p')^2}\left\{T^{00}\bar{u}[-2m]v+2T^{0i}\bar{u}[\gamma_0(p_i-p'_i)]v\right.$$

$$\left.+T^{ij}\bar{u}[\gamma_i(p_j-p'_j)+\gamma_j(p_i-p'_i)+2m\delta_{ij}]v\right\}. \tag{6.56}$$

Consider $T^{00}\bar{u}[-2m]v$ in Eq. (6.56) with $T^{00} = 2\pi\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)B^{00}$. In our process, we will study the spin polarization correlation of e^+e^-. Therefore, we do not sum over all spins of e^+e^- and by using the four spinors of e^+e^- in Eq. (6.13). We have

$$\bar{u}v = \sqrt{\frac{p^0+m}{2m}}\begin{pmatrix}\xi_1^\dagger & -\xi_1^\dagger\frac{k\sigma_2}{p^0+m}\end{pmatrix}\sqrt{\frac{p^0+m}{2m}}\begin{pmatrix}-\frac{k\sigma_1}{p^0+m}\xi_2\\ \xi_2\end{pmatrix}.$$

$$= \frac{p^0+m}{2m}\begin{pmatrix}\xi_1^\dagger & -\xi_1^\dagger\frac{k\sigma_2}{p^0+m}\end{pmatrix}\begin{pmatrix}-\frac{k\sigma_1}{p^0+m}\xi_2\\ \xi_2\end{pmatrix}.$$

By using properties Eqs. (6.11)–(6.12), we rewrite above term as:

$$= \left(\frac{p^0+m}{2m}\right)\left[-\frac{k}{p^0+m}\xi_1^\dagger\sigma_1\xi_2-\frac{k}{p^0+m}\xi_1^\dagger\sigma_2\xi_2\right]$$

$$= -\left(\frac{p^0+m}{2m}\right)\frac{k}{p^0+m}[\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]$$

$$= -\frac{k}{2m}[\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]$$

$$= -\frac{\gamma\beta}{2}[\xi_1^\dagger \sigma_1 \xi_2 + \xi_1^\dagger \sigma_2 \xi_2].$$

The matrix elements are needed to use in $T^{00}\bar{u}[-2m]v$ of Eq. (6.56), be written:

$$\bar{u}v = -\frac{\gamma\beta}{2}[\xi_1^\dagger \sigma_1 \xi_2 + \xi_1^\dagger \sigma_2 \xi_2]. \tag{6.57}$$

To this, we obtain

$$(-2m)T^{00}\bar{u}v = (-2m)2\pi \sum_{N=-\infty}^{\infty} \delta(2p^0 - mN)B^{00}\left(-\frac{\gamma\beta}{2}\right)[\xi_1^\dagger \sigma_1 \xi_2 + \xi_1^\dagger \sigma_2 \xi_2]$$

$$= k2\pi \sum_{N=-\infty}^{\infty} \delta(2p^0 - mN)B^{00}[\xi_1^\dagger \sigma_1 \xi_2 + \xi_1^\dagger \sigma_2 \xi_2]. \tag{6.58}$$

And the second term $(2T^{0i}\bar{u}[\gamma_0(p_i - p_i')]v)$ in Eq. (6.56), we rewrite as

$$2T^{0i}\bar{u}[\gamma_0(p_i - p_i')]v = 2T^{0i}(\mathbf{p} - \mathbf{p}')^i \bar{u}\gamma_0 v$$

$$= 2\{T^{01}(\mathbf{p} - \mathbf{p}')^1 + T^{02}(\mathbf{p} - \mathbf{p}')^2\}\bar{u}\gamma_0 v$$

$$= 2\{-kT^{01} + kT^{02}\}\bar{u}\gamma_0 v$$

$$= 2k\{-T^{01} + T^{02}\}\bar{u}\gamma_0 v \tag{6.59}$$

where

$$(\mathbf{p} - \mathbf{p}')^i \equiv (p_i - p_i')$$

$$= k(-1, 1, 0).$$

To express $-T^{01} + T^{02}$ in Eq. (6.59). We use the properties: $\mathbf{p} + \mathbf{p}' = k(1, 1, 0)$,

$k = m\gamma\beta$, $p^0 = m\gamma$ and $|\mathbf{p}+\mathbf{p}'|^2 = 2k^2$. We have

$$-T^{01}+T^{02} = 2\pi\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)B^{00}[-\frac{2p^0(\mathbf{p}+\mathbf{p}')^1}{|\mathbf{p}+\mathbf{p}'|^2}+\frac{2p^0(\mathbf{p}+\mathbf{p}')^2}{|\mathbf{p}+\mathbf{p}'|^2}]$$

$$= 2\pi\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)B^{00}[-\frac{2m^2\gamma^2\beta}{2m^2\gamma^2\beta^2}+\frac{2m^2\gamma^2\beta}{2m^2\gamma^2\beta^2}]$$

$$=0.$$

And we compute $\bar{u}\gamma_0 v$ in Eq. (6.59), we not sum over all spins of e^+e^- and by using the four spinors of e^+e^- in Eq. (6.13), and using

$$\gamma^0 = -\gamma_0; \quad \gamma_0 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{6.60}$$

We have

$$\bar{u}\gamma_0 v = \sqrt{\frac{p^0+m}{2m}}\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger\frac{k\sigma_2}{p^0+m} \end{pmatrix}\begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\sqrt{\frac{p^0+m}{2m}}\begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger\frac{k\sigma_2}{p^0+m} \end{pmatrix}\begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$= \left(\frac{p^0+m}{2m}\right)\begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger\frac{k\sigma_2}{p^0+m} \end{pmatrix}\begin{pmatrix} \frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix}$$

By using properties Eqs. (6.11)–(6.12), we rewrite above term as:

$$=(\frac{p^0+m}{2m})[\frac{k}{p^0+m}\xi_1^\dagger\sigma_1\xi_2 - \frac{k}{p^0+m}\xi_1^\dagger\sigma_2\xi_2]$$

$$=(\frac{p^0+m}{2m})\frac{k}{p^0+m}[\xi_1^\dagger\sigma_1\xi_2 - \xi_1^\dagger\sigma_2\xi_2]$$

$$=\frac{k}{2m}[\xi_1^\dagger\sigma_1\xi_2 - \xi_1^\dagger\sigma_2\xi_2]$$

$$=\frac{\gamma\beta}{2}[\xi_1^\dagger\sigma_1\xi_2 - \xi_1^\dagger\sigma_2\xi_2]$$

The matrix elements are needed to use in $2\{-kT^{01} + kT^{02}\}\bar{u}\gamma_0 v$ of Eq. (6.56), be written:

$$\bar{u}\gamma_0 v = \frac{\gamma\beta}{2}[\xi_1^\dagger\sigma_1\xi_2 - \xi_1^\dagger\sigma_2\xi_2]. \tag{6.61}$$

To this, we obtain

$$2T^{0i}\bar{u}[\gamma_0(p_i - p_i')]v = 0. \tag{6.62}$$

Finally, the last term $(T^{ij}\bar{u}[\gamma_i(p_j - p_j') + \gamma_j(p_i - p_i') + 2m\delta_{ij}]v)$ in Eq. (6.56), we rewrite as

$$T^{ij}\bar{u}[\gamma_i(p_j - p_j') + \gamma_j(p_i - p_i') + 2m\delta_{ij}]v$$

$$= T^{ij}\bar{u}[\gamma_i(\mathbf{p} - \mathbf{p}')^j + \gamma_j(\mathbf{p} - \mathbf{p}')^i + 2m\delta_{ij}]v. \tag{6.63}$$

To expression Eq. (6.62), we will calculate the energy-momentum tensor T^{ij} where $T^{\mu\nu} = 2\pi\sum_{N=-\infty}^{\infty}\delta(2p^0 - mN)B^{\mu\nu}(\mathbf{p}+\mathbf{p}', N)$ and $B^{\mu\nu}(\mathbf{p}+\mathbf{p}', N)$. Therefore, we have (for $i = j = 1$)

$$B^{11} = \beta_N A_N + \beta_N E_N \frac{(\mathbf{p} + \mathbf{p}')^1(\mathbf{p} + \mathbf{p}')^1}{|\mathbf{p} + \mathbf{p}'|^2}$$

$$= \beta_N A_N + \beta_N E_N \frac{(k)(k)}{2k^2}$$

$$B^{11} = \beta_N A_N + \frac{1}{2}\beta_N E_N$$

and (for $i = j = 2$)

$$\begin{aligned} B^{22} =& \beta_N A_N + \beta_N E_N \frac{(\mathbf{p}+\mathbf{p}')^2(\mathbf{p}+\mathbf{p}')^2}{|\mathbf{p}+\mathbf{p}'|^2} \\ =& \beta_N A_N + \beta_N E_N \frac{(k)(k)}{2k^2} \\ B^{22} =& \beta_N A_N + \frac{1}{2}\beta_N E_N. \end{aligned}$$

From above, we found $B^{11} = B^{22}$. For when $i = 1$ and $j = 2$

$$\begin{aligned} B^{12} =& \beta_N E_N \frac{(\mathbf{p}+\mathbf{p}')^1(\mathbf{p}+\mathbf{p}')^2}{|\mathbf{p}+\mathbf{p}'|^2} \\ =& \beta_N E_N \frac{(k)(k)}{2k^2} \\ B^{12} =& \frac{1}{2}\beta_N E_N. \end{aligned}$$

For when $j = 2$ and $j = 1$

$$\begin{aligned} B^{21} =& \beta_N E_N \frac{(\mathbf{p}+\mathbf{p}')^2(\mathbf{p}+\mathbf{p}')^1}{|\mathbf{p}+\mathbf{p}'|^2} \\ =& \beta_N E_N \frac{(k)(k)}{2k^2} \\ B^{21} =& \frac{1}{2}\beta_N E_N. \end{aligned}$$

From above, we found $B^{12} = B^{21}$.

To this end, we can rewrite Eq. (6.63) as

$$T^{ij}\bar{u}[\gamma_i(p_j - p'_j) + \gamma_j(p_i - p'_i) + 2m\delta_{ij}]v$$

$$= T^{11}\bar{u}[\gamma_1(\mathbf{p}-\mathbf{p}')^1 + \gamma_1(\mathbf{p}-\mathbf{p}')^1 + 2m]v + T^{12}\bar{u}[\gamma_1(\mathbf{p}-\mathbf{p}')^2 + \gamma_2(\mathbf{p}-\mathbf{p}')^1]v$$

$$+T^{21}\bar{u}[\gamma_2(\mathbf{p}-\mathbf{p}')^1+\gamma_1(\mathbf{p}-\mathbf{p}')^2]v+T^{22}\bar{u}[\gamma_2(\mathbf{p}-\mathbf{p}')^2+\gamma_2(\mathbf{p}-\mathbf{p}')^2+2m]v.$$

By using properties Eqs. (6.11)–(6.12), we rewrite above term as:

$$=T^{11}\bar{u}[-k\gamma_1-k\gamma_1+2m]v+T^{12}\bar{u}[k\gamma_1-k\gamma_2]v+T^{22}\bar{u}[k\gamma_2+k\gamma_2+2m]v$$

$$+T^{21}\bar{u}[-k\gamma_2+k\gamma_1]v$$

$$=-kT^{11}\bar{u}\left[\gamma_1+\left(\frac{2m}{k}\right)\right]v+kT^{12}\bar{u}[\gamma_1-\gamma_2]v+kT^{22}\bar{u}\left[\gamma_2+\gamma_2+\left(\frac{2m}{k}\right)\right]v$$

$$+kT^{21}\bar{u}[-\gamma_2+\gamma_1]v,$$

with arranging above term $\bar{u}\gamma_1 v$, $\bar{u}\gamma_2 v$, $\bar{u}v$, and by using properties $T^{12}=T^{21}$, we rewrite above term as:

$$=-kT^{11}\bar{u}\left[2\gamma_1-\left(\frac{2m}{k}\right)\right]v+kT^{12}\bar{u}[\gamma_1-\gamma_2]v+kT^{21}\bar{u}[-\gamma_2+\gamma_1]v$$

$$+kT^{22}\bar{u}\left[2\gamma_2+\left(\frac{2m}{k}\right)\right]v$$

$$=k[-2T^{11}+T^{12}+T^{21}][\bar{u}\gamma_1 v]+k[2T^{22}-T^{12}-T^{21}][\bar{u}\gamma_2 v]+2m[T^{11}+T^{22}][\bar{u}v]$$

$$=2k[T^{12}-T^{11}][\bar{u}\gamma_1 v]+2k[T^{22}-T^{12}][\bar{u}\gamma_2 v]+4mT^{11}\bar{u}v. \tag{6.64}$$

Hence

$$T^{12}-T^{11}=2\pi\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\{B^{12}-B^{11}\}$$

$$=2\pi\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\{\frac{1}{2}\beta_N E_N-(\beta_N A_N+\frac{1}{2}\beta_N E_N)\}$$

$$= -2\pi \sum_{N=-\infty}^{\infty} \delta(2p^0 - mN)[\beta_N A_N], \tag{6.65}$$

and

$$T^{22} - T^{12} = 2\pi \sum_{N=-\infty}^{\infty} \delta(2p^0 - mN)\{B^{22} - B^{12}\}$$

$$= 2\pi \sum_{N=-\infty}^{\infty} \delta(2p^0 - mN)\{(\beta_N A_N + \frac{1}{2}\beta_N E_N) - \frac{1}{2}\beta_N E_N\}, \tag{6.66}$$

or Eq. (6.66) be written as

$$T^{22} - T^{12} = 2\pi \sum_{N=-\infty}^{\infty} \delta(2p^0 - mN)[\beta_N A_N]. \tag{6.67}$$

For $\bar{u}\gamma_i v$, we have

$$\bar{u}\gamma_i v = \left(\frac{p^0 + m}{2m}\right) \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{k\sigma_2}{p^0+m} \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} -\frac{k\sigma_1}{p^0+m}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$= \left(\frac{p^0 + m}{2m}\right) \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger \frac{k\sigma_2}{p^0+m} \end{pmatrix} \begin{pmatrix} \sigma_i \xi_2 \\ \frac{k\sigma_1}{p^0+m}\sigma_i \sigma_1 \xi_2 \end{pmatrix}$$

$$\bar{u}\gamma_i v = \left(\frac{p^0 + m}{2m}\right) \left[\xi_1^\dagger \sigma_i \xi_2 - \left(\frac{k}{p^0 + m}\right)^2 \xi_1^\dagger \sigma_2 \sigma_i \sigma_1 \xi_2\right].$$

To simplify above term, we change $\sigma_2\sigma_i\sigma_1$ by using the properties of the Pauli matrix (see in Appendix A). Given by

$$i = 1 \qquad \sigma_2\sigma_1\sigma_1 = \sigma_2\sigma_1^2 = \sigma_2$$

$$i = 2 \qquad \sigma_2\sigma_2\sigma_1 = \sigma_2^2\sigma_1 = \sigma_1$$

We note that $i = 3$ is not considered because we consider the momenta of e^+e^- in x-y

plane.

In this case, we obtain

$$\bar{u}\gamma_1 v = \left(\frac{p^0+m}{2m}\right)\left[\xi_1^\dagger\sigma_1\xi_2 - \left(\frac{k}{p^0+m}\right)^2 \xi_1^\dagger\sigma_2\xi_2\right], \tag{6.68}$$

$$\bar{u}\gamma_2 v = \left(\frac{p^0+m}{2m}\right)\left[\xi_2^\dagger\sigma_2\xi_2 - \left(\frac{k}{p^0+m}\right)^2 \xi_1^\dagger\sigma_1\xi_2\right]. \tag{6.69}$$

By replacing Eqs. (6.65)–(6.69) in Eq. (6.64), we obtain

$$T^{ij}\bar{u}[\gamma_i(p_j-p'_j)+\gamma_j(p_i-p'_i)+2m\delta_{ij}]v$$

$$= 2k(2\pi)\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)[\beta_N A_N]\left(\frac{p^0+m}{2m}\right)\Bigg\{-\xi_1^\dagger\sigma_1\xi_2 - \left(\frac{k}{p^0+m}\right)^2\xi_1^\dagger\sigma_1\xi_2$$

$$+\xi_1^\dagger\sigma_2\xi_2 + \left(\frac{k}{p^0+m}\right)^2\xi_1^\dagger\sigma_2\xi_2\Bigg\} + 4m(2\pi)\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)$$

$$\times[\beta_N A_N + \frac{1}{2}\beta_N E_N]\left(-\frac{\gamma\beta}{2}\right)[\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]$$

$$= 2k(2\pi)\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)[\beta_N A_N]\left(\frac{p^0+m}{2m}\right)\left(1+\left(\frac{k}{p^0+m}\right)^2\right)$$

$$\times\{-\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2\}$$

$$-2\overbrace{m\gamma\beta}^{=k}(2\pi)\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)[\beta_N A_N+\frac{1}{2}\beta_N E_N][\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2].$$

On the right-hand side of an above term we will rewrite $\left(\frac{p^0+m}{2m}\right)\times\left(1+\left(\frac{k}{p^0+m}\right)^2\right)$ in term of speed(β), defined by:

$$=(\frac{p^0+m}{2m})(\frac{1}{p^0+m})^2[(p^0+m)^2+k^2]$$

$$= [\frac{1}{2m(p^0 + m)}][(p^0)^2 + 2mp^0 + m^2 + k^2]$$

$$= [\frac{1}{2m(m\gamma + m)}][(m\gamma)^2 + 2m^2\gamma + m^2 + (m\gamma\beta)^2],$$

and by replacing $p^0 = m\gamma$, $k = m\gamma\beta$ in an above term, we obtain

$$= \left[\frac{1}{2(\gamma + 1)}\right][\gamma^2 + 2\gamma + 1 + (\gamma\beta)^2]$$

$$= \left[\frac{1}{2(\gamma + 1)}\right][(2\gamma + 1) + \gamma^2(1 + \beta^2)]$$

$$= \left[\frac{1}{2(\frac{1}{\sqrt{1-\beta^2}} + 1)}\right][(2\frac{1}{\sqrt{1-\beta^2}} + 1) + \frac{(1+\beta^2)}{(1-\beta^2)}],$$

and then replacing $\gamma = \sqrt{1-\beta^2}$, we finally obtain

$$= \left[\frac{\sqrt{1-\beta^2}}{2(\sqrt{1-\beta^2} + 1)}\right]\left[(\frac{2 + \sqrt{1-\beta^2}}{\sqrt{1-\beta^2}}) + \frac{(1+\beta^2)}{(1-\beta^2)}\right]$$

$$= \left[\frac{\sqrt{1-\beta^2}}{2(\sqrt{1-\beta^2} + 1)}\right]\left[(\frac{2\sqrt{1-\beta^2} + (1-\beta^2)}{1-\beta^2}) + \frac{(1+\beta^2)}{(1-\beta^2)}\right]$$

$$= \left[\frac{\sqrt{1-\beta^2}}{2(\sqrt{1-\beta^2} + 1)}\right]\left[\frac{2(\sqrt{1-\beta^2} + 1)}{1-\beta^2})\right]$$

$$= \frac{1}{\sqrt{1-\beta^2}} = \gamma.$$

The latter works out to be

$$T^{ij}\bar{u}[\gamma_i(p_j - p'_j) + \gamma_j(p_i - p'_i) + 2m\delta_{ij}]v$$

$$=2k(2\pi)\gamma\sum_{N=-\infty}^{\infty}\delta(2p^0 - mN)[\beta_N A_N]\{-\xi_1^\dagger\sigma_1\xi_2 + \xi_1^\dagger\sigma_2\xi_2\}$$

$$-2k(2\pi)\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\left[\beta_N A_N+\frac{1}{2}\beta_N E_N\right][\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]$$

$$=2k(2\pi)\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\beta_N\Big\{\gamma A_N[-\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]$$

$$-\left[A_N+\frac{1}{2}E_N\right][\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]\Big\}.$$

A tedious but straightforward computation gives

$$A\propto\frac{1}{(p+p')^2}\Big\{(k)(2\pi)\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\beta_N J_{N/2}^2(x)[\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]$$

$$+2k(2\pi)\gamma\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\beta_N\Big\{\gamma A_N[-\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]$$

$$-\{[A_N+\frac{1}{2}E_N][\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]\}\Big\}\Big\}.$$

By using properties Eqs. (6.11)–(6.12), we rewrite above term as:

$$A\propto\frac{4k\beta_N\pi}{(p+p')^2}\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\left\{\frac{J_{N/2}^2(x)}{2}[\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]\right.$$

$$\left.+(\gamma A_N)[-\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]-\left(A_N+\frac{1}{2}E_N\right)[\xi_1^\dagger\sigma_1\xi_2+\xi_1^\dagger\sigma_2\xi_2]\right\}$$

$$=\frac{4k\beta_N\pi}{(p+p')^2}\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\left\{\left[\frac{J_{N/2}^2(x)}{2}-\gamma A_N-\left(A_N+\frac{1}{2}E_N\right)\right][\xi_1^\dagger\sigma_1\xi_2]\right.$$

$$\left.+\left[\frac{J_{N/2}^2(x)}{2}+\gamma A_N-\left(A_N+\frac{1}{2}E_N\right)\right][\xi_1^\dagger\sigma_2\xi_2]\right\},$$

and we have $(p+p')^2=2m^2\gamma^2(\beta^2-2)$. Therefore, giving

$$A\propto\frac{4k\beta_N\pi}{2m^2\gamma^2(\beta^2-2)}\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)\left\{\left[\frac{J_{N/2}^2(x)}{2}-\gamma A_N-\left(A_N+\frac{1}{2}E_N\right)\right][\xi_1^\dagger\sigma_1\xi_2]\right.$$

$$+\left[\frac{J_{N/2}^2(x)}{2}+\gamma A_N-\left(A_N+\frac{1}{2}E_N\right)\right][\xi_1^\dagger\sigma_2\xi_2]\Bigg\}, \tag{6.70}$$

where

$$A_N=\frac{1}{4}\left[J_{\frac{N}{2}+1}^2(x)+J_{\frac{N}{2}-1}^2(x)-2J_{\frac{N}{2}+1}(x)J_{\frac{N}{2}-1}(x)\right]$$

$$E_N=J_{\frac{N}{2}+1}(x)J_{\frac{N}{2}-1}(x)$$

$$A_N+\frac{1}{2}E_N=\frac{1}{4}\left[J_{\frac{N}{2}+1}^2(x)+J_{\frac{N}{2}-1}^2(x)\right].$$

In full form of the amplitude of this process may be rewritten as:

$$A\propto\frac{4k\beta_N\pi}{2m^2\gamma^2(\beta^2-2)}\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)$$

$$\times\Bigg\{\left[\frac{J_{N/2}^2(x)}{2}-\gamma A_N-\frac{1}{4}\left[J_{\frac{N}{2}+1}^2(x)+J_{\frac{N}{2}-1}^2(x)\right]\right][\xi_1^\dagger\sigma_1\xi_2]$$

$$+\left[\frac{J_{N/2}^2(x)}{2}+\gamma A_N-\frac{1}{4}\left[J_{\frac{N}{2}+1}^2(x)+J_{\frac{N}{2}-1}^2(x)\right]\right][\xi_1^\dagger\sigma_2\xi_2]\Bigg\}.$$

And using Eqs. (6.24)–(6.25), we obtain

$$A\propto\overbrace{\frac{4\beta_N\beta\pi}{m\gamma(\beta^2-2)}\sum_{N=-\infty}^{\infty}\delta(2p^0-mN)}^{Not\ important}$$

$$\times\Bigg\{\left[\frac{J_{N/2}^2(x)}{2}-\gamma A_N-\frac{1}{4}\left[J_{\frac{N}{2}+1}^2(x)+J_{\frac{N}{2}-1}^2(x)\right]\right]\cos(\chi_1+\chi_2)$$

$$-\,\mathrm{i}\left[\frac{J_{N/2}^2(x)}{2}+\gamma A_N-\frac{1}{4}\left[J_{\frac{N}{2}+1}^2(x)+J_{\frac{N}{2}-1}^2(x)\right]\right]\cos(\chi_1-\chi_2)\Bigg\}.$$

So that, we rewrite the amplitude of this process as:

$$A \propto \left[\frac{J^2_{N/2}(x)}{2} - \gamma A_N - \frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x)\right]\right]\cos(\chi_1+\chi_2)$$

$$- \mathrm{i}\left[\frac{J^2_{N/2}(x)}{2} + \gamma A_N - \frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x)\right]\right]\cos(\chi_1-\chi_2). \tag{6.71}$$

The probability of this process is given $F(\chi_1,\chi_2) \equiv |A|^2$, be written as:

$$F(\chi_1,\chi_2) = \left[\frac{J^2_{N/2}(x)}{2} - \gamma A_N - \frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x)\right]\right]^2\cos^2(\chi_1+\chi_2)$$

$$+ \left[\frac{J^2_{N/2}(x)}{2} + \gamma A_N - \frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x)\right]\right]^2\cos^2(\chi_1-\chi_2). \tag{6.72}$$

Let

$$A(\beta) = \left[\frac{J^2_{N/2}(x)}{2} - \gamma A_N - \frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x)\right]\right], \tag{6.73}$$

$$B(\beta) = \left[\frac{J^2_{N/2}(x)}{2} + \gamma A_N - \frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x)\right]\right]. \tag{6.74}$$

And we use identity of Bessel Functions

$$J_{N/2-1}(x) = \frac{N}{x}J_{N/2}(x) - J_{N/2+1}(x) \tag{6.75}$$

$$x = \frac{|\mathbf{p}+\mathbf{p}'|^2}{2m} = \frac{\sqrt{2}k}{2m} = \frac{\gamma\beta}{\sqrt{2}} \tag{6.76}$$

$$2p^0 = mN \to N = \frac{2p^0}{m} = \frac{2m\gamma}{m} = 2\gamma \tag{6.77}$$

Therefore

$$J_{\frac{N}{2}-1}(x) = \frac{2\sqrt{2}}{\beta} J_{\frac{N}{2}}(x) - J_{\frac{N}{2}+1}(x)$$

We obtain $A(\beta)$ as

$$A(\beta) = \frac{J^2_{\frac{N}{2}}(x)}{2} - \gamma \frac{1}{4} \left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x) - 2J_{\frac{N}{2}-1}(x) J_{\frac{N}{2}+1}(x) \right]$$

$$- \frac{1}{4} \left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x) \right]$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2} - \gamma \frac{1}{4} \left(J_{\frac{N}{2}+1}(x) - J_{\frac{N}{2}-1}(x) \right)^2 - \frac{1}{4} \left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x) \right]$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2} - \gamma \frac{1}{4} \left(J_{\frac{N}{2}+1}(x) - \left[\frac{2\sqrt{2}}{\beta} J_{\frac{N}{2}}(x) - J_{\frac{N}{2}+1}(x) \right] \right)^2$$

$$- \frac{1}{4} \left[J^2_{\frac{N}{2}+1}(x) + \left(\frac{2\sqrt{2}}{\beta} J_{\frac{N}{2}}(x) - J_{\frac{N}{2}+1}(x) \right)^2 \right]. \tag{6.78}$$

Now $N/2$ differ by <u>one</u> unit here $\therefore$ for sufficiently high energies $p^0 \rightarrow$ LARGE, $: J_{N/2+1} \rightarrow J_{N/2}$

$$A(\beta) = \frac{J^2_{\frac{N}{2}}(x)}{2} - \gamma \frac{1}{4} \left(J_{\frac{N}{2}}(x) - \left[\frac{2\sqrt{2}}{\beta} J_{\frac{N}{2}}(x) - J_{\frac{N}{2}}(x) \right] \right)^2$$

$$- \frac{1}{4} \left[J^2_{\frac{N}{2}}(x) + \left(\frac{2\sqrt{2}}{\beta} J_{\frac{N}{2}}(x) - J_{\frac{N}{2}}(x) \right)^2 \right]$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2} \left\{ 1 - \frac{\gamma}{2} \left(1 - \left[\frac{2\sqrt{2}}{\beta} - 1 \right] \right)^2 - \frac{1}{2} \left(1 + \left[\frac{2\sqrt{2}}{\beta} - 1 \right]^2 \right) \right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2} \left\{ \frac{1}{2} \left(1 - \left[\frac{2\sqrt{2}}{\beta} - 1 \right]^2 \right) - \frac{\gamma}{2} \left(1 - \left[\frac{2\sqrt{2}}{\beta} - 1 \right] \right)^2 \right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{\frac{1}{2}\left(\left[1-\left[\frac{2\sqrt{2}}{\beta}-1\right]\right]\left[1+\left[\frac{2\sqrt{2}}{\beta}-1\right]\right]\right)-\frac{4\gamma}{2}\left(1-\frac{\sqrt{2}}{\beta}\right)^2\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{\frac{1}{2}\left(\left[2-\frac{2\sqrt{2}}{\beta}\right]\times\frac{2\sqrt{2}}{\beta}\right)-2\gamma\left(1-\frac{\sqrt{2}}{\beta}\right)^2\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{\left(\frac{2\sqrt{2}}{\beta}\left[1-\frac{\sqrt{2}}{\beta}\right]\right)-2\gamma\left(1-\frac{\sqrt{2}}{\beta}\right)^2\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left(1-\frac{\sqrt{2}}{\beta}\right)\left\{\frac{2\sqrt{2}}{\beta}-2\gamma\left(1-\frac{\sqrt{2}}{\beta}\right)\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left(1-\frac{\sqrt{2}}{\beta}\right)\left(\frac{2}{\beta}\right)\left\{\sqrt{2}-\gamma\beta\left(1-\frac{\sqrt{2}}{\beta}\right)\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left(1-\frac{\sqrt{2}}{\beta}\right)\left(\frac{2}{\beta}\right)\left\{\sqrt{2}-\gamma\beta+\gamma\sqrt{2}\right\}, \tag{6.79}$$

and $B(\beta)$ as

$$B(\beta) = \frac{J^2_{\frac{N}{2}}(x)}{2}+\gamma\frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x)+J^2_{\frac{N}{2}-1}(x)-2J_{\frac{N}{2}-1}(x)J_{\frac{N}{2}+1}(x)\right]$$

$$-\frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x)+J^2_{\frac{N}{2}-1}(x)\right]$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}+\gamma\frac{1}{4}\left(J_{\frac{N}{2}+1}(x)-J_{\frac{N}{2}-1}(x)\right)^2-\frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x)+J^2_{\frac{N}{2}-1}(x)\right]$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}+\gamma\frac{1}{4}\left(J_{\frac{N}{2}+1}(x)-\left[\frac{2\sqrt{2}}{\beta}J_{\frac{N}{2}}(x)-J_{\frac{N}{2}+1}(x)\right]\right)^2$$

$$-\frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x)+\left(\frac{2\sqrt{2}}{\beta}J_{\frac{N}{2}}(x)-J_{\frac{N}{2}+1}(x)\right)^2\right]. \tag{6.80}$$

Now $N/2$ differ by *one* unit here $\therefore$ for sufficiently high energies $p^0 \to$ large, $: J_{N/2+1} \to$

$J_{N/2}$

$$B(\beta) = \frac{J^2_{\frac{N}{2}}(x)}{2} + \gamma\frac{1}{4}\left(J_{\frac{N}{2}}(x) - \left[\frac{2\sqrt{2}}{\beta}J_{\frac{N}{2}}(x) - J_{\frac{N}{2}}(x)\right]\right)^2$$

$$-\frac{1}{4}\left[J^2_{\frac{N}{2}}(x) + \left(\frac{2\sqrt{2}}{\beta}J_{\frac{N}{2}}(x) - J_{\frac{N}{2}}(x)\right)^2\right]$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{1 + \frac{\gamma}{2}\left(1 - \left[\frac{2\sqrt{2}}{\beta} - 1\right]\right)^2 - \frac{1}{2}\left(1 + \left[\frac{2\sqrt{2}}{\beta} - 1\right]^2\right)\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{\frac{1}{2}\left(1 - \left[\frac{2\sqrt{2}}{\beta} - 1\right]^2\right) + \frac{\gamma}{2}\left(1 - \left[\frac{2\sqrt{2}}{\beta} - 1\right]\right)^2\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{\frac{1}{2}\left(\left[1 - \left[\frac{2\sqrt{2}}{\beta} - 1\right]\right]\left[1 + \left[\frac{2\sqrt{2}}{\beta} - 1\right]\right]\right) + \frac{4\gamma}{2}\left(1 - \frac{\sqrt{2}}{\beta}\right)^2\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{\frac{1}{2}\left(\left[\frac{2\sqrt{2}}{\beta}\right] \times \frac{2\sqrt{2}}{\beta}\right) + 2\gamma\left(1 - \frac{\sqrt{2}}{\beta}\right)^2\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left\{\left(\frac{2\sqrt{2}}{\beta}\left[1 - \frac{\sqrt{2}}{\beta}\right]\right) + 2\gamma\left(1 - \frac{\sqrt{2}}{\beta}\right)^2\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left(1 - \frac{\sqrt{2}}{\beta}\right)\left\{\frac{2\sqrt{2}}{\beta} + 2\gamma\left(1 - \frac{\sqrt{2}}{\beta}\right)\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left(1 - \frac{\sqrt{2}}{\beta}\right)\left(\frac{2}{\beta}\right)\left\{\sqrt{2} + \gamma\beta\left(1 - \frac{\sqrt{2}}{\beta}\right)\right\}$$

$$= \frac{J^2_{\frac{N}{2}}(x)}{2}\left(1 - \frac{\sqrt{2}}{\beta}\right)\left(\frac{2}{\beta}\right)\left\{\sqrt{2} + \gamma\beta - \gamma\sqrt{2}\right\}. \tag{6.81}$$

The exact probability of this process is given

$$F(\chi_1,\chi_2) = A^2(\beta)\cos^2\left(\frac{\chi_1+\chi_2}{2}\right) + B^2(\beta)\cos^2\left(\frac{\chi_1-\chi_2}{2}\right). \tag{6.82}$$

To normalize the expression in Eq. (6.82), we have to sum $F(\chi_1,\chi_2)$ over the spin polarization directions specified by the pairs of angles same as in Eq. (6.29). That is, we have to find the normalization factor same as in Eq. (6.30). The first one, we rotated angle χ_1 with π, by replacing $\chi_1 \to \chi_1 + \pi$, given

$$F(\chi_1+\pi,\chi_2) = A^2(\beta)\sin^2\left(\frac{\chi_1+\chi_2}{2}\right) + B^2(\beta)\sin^2\left(\frac{\chi_1-\chi_2}{2}\right) \tag{6.83}$$

The second one, we rotated angle χ_2 with π, by replacing $\chi_2 \to \chi_2 + \pi$, given

$$F(\chi_1,\chi_2+\pi) = A^2(\beta)\sin^2\left(\frac{\chi_1+\chi_2}{2}\right) + B^2(\beta)\sin^2\left(\frac{\chi_1-\chi_2}{2}\right) \tag{6.84}$$

Finally, we rotated angle χ_1, χ_2 with π, by replacing $\chi_1 \to \chi_1 + \pi$, $\chi_2 \to \chi_1 + \pi$, given

$$F(\chi_1+\pi,\chi_2+\pi) = A^2(\beta)\cos^2\left(\frac{\chi_1+\chi_2}{2}\right) + B^2(\beta)\cos^2\left(\frac{\chi_1-\chi_2}{2}\right) \tag{6.85}$$

The latter works out to be

$$N = 2[A^2(\beta) + B^2(\beta)]. \tag{6.86}$$

Therefore, given that the process has occurred as expressed in figure **6.1** , with electron moving along y-axis and positron moving along x-axis, the probability of the spin polarizations, specified by the angles χ_1, χ_2, is rigorously given by

$$P(\chi_1,\chi_2) = \frac{A^2(\beta)\cos^2\left(\frac{\chi_1+\chi_2}{2}\right) + B^2(\beta)\cos^2\left(\frac{\chi_1-\chi_2}{2}\right)}{2[A^2(\beta)+B^2(\beta)]} \tag{6.87}$$

as in Yongram and Manoukian (2003) and Manoukian and Yongram (2004), given the following explicit expression for the probability of the simultaneous measurements of the spins of e^-, e^+, with angles χ_1, χ_2, as specified above,

$$P(\chi_1,\chi_2)=\frac{[\sqrt{2}-\gamma\beta+\gamma\sqrt{2}]^2\cos^2(\chi_1+\chi_2)+[\sqrt{2}+\gamma\beta-\gamma\sqrt{2}]^2\cos^2(\chi_1-\chi_2)}{2\Big[[\sqrt{2}-\gamma\beta+\gamma\sqrt{2}]^2+[\sqrt{2}+\gamma\beta-\gamma\sqrt{2}]^2\Big]}. \tag{6.88}$$

At high energy $\beta\to 1$, $\gamma\to\infty$. We obtain

$$\begin{aligned} A(\beta) &= \sqrt{2}-\gamma\beta+\gamma\sqrt{2} \\ &\simeq (\sqrt{2}-1)\gamma, \end{aligned} \tag{6.89}$$

$$\begin{aligned} B(\beta) &= \sqrt{2}+\gamma\beta-\gamma\sqrt{2} \\ &\simeq -(\sqrt{2}-1)\gamma, \end{aligned} \tag{6.90}$$

to obtain

$$P(\chi_1,\chi_2)\to\frac{\cos^2(\chi_1+\chi_2)+\cos^2(\chi_1-\chi_2)}{4}. \tag{6.91}$$

Same as for charged string (for extreme relativistic case). Consider $A(\beta)$ and $B(\beta)$, we have $\gamma=1/\sqrt{1-\beta^2}$

$$\begin{aligned} A(\beta) &= \sqrt{2}-\frac{1}{\sqrt{1-\beta^2}}\beta+\frac{1}{\sqrt{1-\beta^2}}\sqrt{2} \\ &= \frac{1}{\sqrt{1-\beta^2}}\left(\sqrt{2}(\sqrt{1-\beta^2})-\beta+\sqrt{2}\right), \\ B(\beta) &= \sqrt{2}+\frac{1}{\sqrt{1-\beta^2}}\beta-\frac{1}{\sqrt{1-\beta^2}}\sqrt{2} \\ &= \frac{1}{\sqrt{1-\beta^2}}\left(\sqrt{2}(\sqrt{1-\beta^2})+\beta-\sqrt{2}\right), \end{aligned}$$

and we then square above term, be written as

$$A^2(\beta) = \left(\frac{1}{\sqrt{1-\beta^2}}\right)^2 \left(\sqrt{2}(\sqrt{1-\beta^2}) - \beta + \sqrt{2}\right)^2 \tag{6.92}$$

$$B^2(\beta) = \left(\frac{1}{\sqrt{1-\beta^2}}\right)^2 \left(\sqrt{2}(\sqrt{1-\beta^2}) + \beta - \sqrt{2}\right)^2. \tag{6.93}$$

Here

$$\left(\sqrt{2}(\sqrt{1-\beta^2}) + (\sqrt{2}-\beta)\right)^2 = 4 - \beta^2 - 2\sqrt{2}\beta + 2(\sqrt{2}-\beta)\sqrt{2(1-\beta^2)}, \tag{6.94}$$

$$\left(\sqrt{2}(\sqrt{1-\beta^2}) - (\sqrt{2}-\beta)\right)^2 = 4 - \beta^2 - 2\sqrt{2}\beta - 2(\sqrt{2}-\beta)\sqrt{2(1-\beta^2)}. \tag{6.95}$$

Add Eq. (6.92)–(6.93), we obtain

$$A^2(\beta) + B^2(\beta) = 2\left(\frac{1}{\sqrt{1-\beta^2}}\right)^2 (4 - \beta^2 - 2\sqrt{2}\beta). \tag{6.96}$$

All told, given that the above process has occurred, a direct straightforward simplification of the expression in Eq. (6.56) leads to the following expression for the polarizations correlations probability of the pair

$$P[\chi_1,\chi_2] = \frac{1}{4\left(4-\beta^2-2\sqrt{2}\beta\right)}\Bigg\{\left(\sqrt{2(1-\beta^2)} - \sqrt{2} + \beta\right)^2 \cos^2\left(\frac{\chi_1-\chi_2}{2}\right) + \left(\sqrt{2(1-\beta^2)} + \sqrt{2} - \beta\right)^2 \cos^2\left(\frac{\chi_1+\chi_2}{2}\right)\Bigg\} \tag{6.97}$$

and again is speed dependent, $P[\chi_1,-] = 1/2 = P[-,\chi_2]$ for a measurement of the spin of only one of the particles. The fact that the polarizations correlations probabilities of the e^+e^- pair emitted from the charged and neutral strings are different in general, such inquiries indicate, in principle, whether the string is charged or uncharged. In the extreme relativistic case, the probability in Eq. (6.91) coincides with the expression on the right-hand side of Eq. (6.41) for a charged string.

Finally we note that these dynamical relativistic quantum field theory calculations lead to a violation of LHV theories. To this end, define:

$$S = P[\chi_1, \chi_2] - P[\chi_1, \chi_2'] + P[\chi_1', \chi_2] + P[\chi_1', \chi_2'] - P[\chi_1', -] - P[-, \chi_2] \quad (6.98)$$

for two pairs of angles (χ_1, χ_2), (χ_1', χ_2'). To show violation with LHV theories, it is sufficient to consider one experimental situation which gives for S a value outside [Clauser and Horne, 1974; Clauser and Shimoney, 1978] the interval $[-1, 0]$. To this end for $\beta = 0.8$, $\chi_1 = 0°$, $\chi_2 = 160°$, $\chi_1' = 100°$, $\chi_2' = 10°$, we obtain $S = -1.088$, $S = -1.103$ for the charged and neutral strings, respectively, leading to a clear violation of LHV theories.

CHAPTER VII

CONCLUSION

A systematic dynamical analysis of polarization correlations of simultaneous measurements of spins of two particles produced in fundamental processes has been carried out based on our explicit computations emerging from present celebrated gauge theories of elementary particles in quantum field theory and hence are applicable, in particular, at high energies. Here we recall that quantum field theory is the non-phenomenological theory which results in extending quantum physics to the high-energy relativistic regime. Given that the processes in question have occurred, we have computed the corresponding explicit joint probabilities, or probability counts, of spin measurements along given directions for the two particles observed, as well as for the measurement carried out on the spin of only one of the two particles. Such probabilities are referred to as conditional probabilities by probabilists and rely on the fact that the processes have occurred with non-zero probabilities. We have encountered that these dynamical calculations following *directly* from quantum field theory lead to non-trivial *speed dependence of the polarization correlations* due to the mere fact that particles have non-zero speeds in order to collide or to travel to the detection region and, in some cases, due to a threshold energy needed to create given particles. This is unlike naïve arguments based simply on combining spins which are of kinematical nature void of dynamical considerations. In all of our QED computations, we will see below, that for the zero speed limits, our expressions reduce to the naïve ones just mentioned. In particular, their $\beta \to 0$ limits violate Bell's inequality–a result known for years. On the other hand, due to the threshold energy needed to create $\mu^+\mu^-$ from e^+e^- scattering in the Weinberg-Salam electro-weak theory, the speed zero limit *cannot* be taken and formal arguments based simply on combining the spins of μ^+, μ^-, without dynamical quantum field theory considerations, completely fail. Here we encounter the novel property of *coupling dependence as well in addition to speed dependence*. Due to recent overwhelming interest in extending the point-like property of a particle to an extended one, such as of a string, similar analyses as above has been carried out for pair e^+e^- creation by circularly oscillating charged and neutral Nambu strings. In particular, inquiries about polarization correlations alone, indicate whether the string in uncharged or neutral. In the extreme relativistic case the corresponding correlations are found to coincide. For the neutral case, the production of a graviton by the string is encountered which in turn decays to the e^+e^- pair. All of our correlation probabilities computed are novel and have been published. It is remarkable that they all, with no exception, lead to a *violation of Bell's inequality of tests against Local Hidden Variables theories*.

We summarize our findings giving expressions of the probability counts of spin measurements of the processes considered in this work and provide some additional comments.

The probability of photon polarizations correlations in $e^+e^- \to 2\gamma$ with initially unpolarized e^+, e^-, in Process 1, is given by

$$P[\chi_1, \chi_2] = \frac{1 - [\cos(\chi_1 - \chi_2) - 2\beta^2 \cos\chi_1 \cos\chi_2]^2}{2[1 + 2\beta^2(1 - \beta^2)]}, \tag{7.1}$$

for all $0 \leqslant \beta \leqslant 1$.

For the measurement of only one of the polarizations the corresponding probabilities are given by

$$P[\chi_1, -] = \frac{1 + 4\beta^2(1 - \beta^2)\cos^2\chi_1}{2[1 + 2\beta^2(1 - \beta^2)]}, \tag{7.2}$$

$$P[-, \chi_2] = \frac{1 + 4\beta^2(1 - \beta^2)\cos^2\chi_2}{2[1 + 2\beta^2(1 - \beta^2)]}. \tag{7.3}$$

We note the important statistical property that

$$P[\chi_1, \chi_2] \neq P[\chi_1, -]P[-, \chi_2], \tag{7.4}$$

in general. showing a non-trivial dependence between the two spin measurements.

In the notation of Local Hidden Variables (LHV) theory (Clauser and Horne, 1974; Clauser and Shimoney, 1978; Fry, 1995; Selleri, 1988), we have the identifications

$$P[\chi_1, \chi_2] = \frac{P_{12}(a_1, a_2)}{P_{12}(\infty, \infty)}, \tag{7.5}$$

given that the two photons have emerged (back-to-back) along the z-axis.

For the measurement of only one of the polarizations

$$P[\chi_1, -] = \frac{P_{12}(a_1, \infty)}{P_{12}(\infty, \infty)}, \tag{7.6}$$

$$P[-, \chi_2] = \frac{P_{12}(\infty, a_2)}{P_{12}(\infty, \infty)}, \tag{7.7}$$

where a_1, a_2 specify directions for measurements of polarizations. Defining (see Appendix E)

$$\begin{aligned} S = {} & P[\chi_1, \chi_2] - P[\chi_1, \chi_2'] + P[\chi_1', \chi_2] \\ & + P[\chi_1', \chi_2'] - P[\chi_1', -] - P[-, \chi_2], \end{aligned} \tag{7.8}$$

for four angles χ_1, χ_2, χ_1', χ_2', LHV theory gives the Bell-like bound (Clauser and Horne, 1974; Clauser and Shimoney, 1978):

$$-1 \leqslant S \leqslant 0. \tag{7.9}$$

It is sufficient to realize one experimental situation that violates the bounds in Eq. (7.9).

For example, for $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, Eq. (7.1), Eqs. (7.2)–(7.3), as obtained from QED, gives $S = 0.207$ for $\beta = 0$ that violates Eq. (7.9) from above. For $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi_1' = 45°$, $\chi_2' = 67°$, we obtain $S = -1.207$ for $\beta = 0$ violating Eq. (7.9) from below. Both bounds are violated for all $\beta \leqslant 0.2$ for these same angles, respectively.

The probability of photon polarizations correlations in $e^+e^- \to 2\gamma$ with initially unpolarized e^+, e^-, in Process 2, we have

$$P_\beta[\chi_1, \chi_2] = \frac{A(\beta) + B(\beta)\cos^2(\chi_1 - \chi_2)}{2[2A(\beta) + B(\beta)]}, \tag{7.10}$$

where

$$A(\beta) = \frac{[4 + 4(1-\beta^2) - (1-\beta^2)^2]}{4\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{3}{2} + \frac{\beta^2}{2}, \tag{7.11}$$

$$B(\beta) = -(1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)\right], \tag{7.12}$$

given that the two photons have emerged (back-to-back) along the z-axis.

For the measurement of only one of the polarizations be written as

$$P_\beta[\chi_1, -] = \frac{[2A(\beta) + B(\beta)]}{2[2A(\beta) + B(\beta)]} = \frac{1}{2} = P_\beta[-, \chi_2], \tag{7.13}$$

for all $0 \leqslant \beta \leqslant 1$, and the latter are, respectively, *independent* of χ_1, χ_2.

Again we have the important statistical property

$$P_\beta[\chi_1, \chi_2] \neq P_\beta[\chi_1, -]P_\beta[-, \chi_2], \tag{7.14}$$

in general. It is interesting to note that an equality in Eq. (7.4) holds in the extreme relativistic case $\beta \to 1$, where each side is equal to $1/4$.

Only in the limiting case $\beta \to 0$, the joint probability in Eq. (7.10) for this process coincides with that in Eq. (7.1) for the first process.

As in Eq. (7.8), we define

$$S_\beta = P_\beta[\chi_1, \chi_2] - P_\beta[\chi_1, \chi_2'] + P_\beta[\chi_1', \chi_2]$$

$$+ P_\beta[\chi_1', \chi_2'] - P_\beta[\chi_1', -] - P_\beta[-, \chi_2], \tag{7.15}$$

for four angles χ_1, χ_2, χ_1', χ_2', LHV theory gives (Clauser and Horne, 1974; Clauser and Shimoney, 1978)

$$-1 \leqslant S_\beta \leqslant 0. \tag{7.16}$$

For $\beta \to 1$, an equality holds in Eq. (7.14), $S_\beta \to -1/2$, and this process, to be useful for testing the violation of Eq. (7.16), should not be conducted at very high speeds. For $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, we have $S_\beta = 0.120, 0.184, 0.201, 0.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (7.16) from above. For $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi_1' = 45°$, $\chi_2' = 67°$, we have $S_\beta = -1.120, -1.184, -1.201, -1.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (7.16) from below. For β larger than 0.2 but close to it, S_β already turns out to be too close to the critical interval given in Eq. (7.16) to be relevant experimentally.

For completeness, we mention that for the annihilation of the spin 0 pair into 2γ, Process 1, the following probabilities are similarly worked out:

$$P[\chi_1, \chi_2] = \frac{\left(\cos(\chi_1 - \chi_2) - 2\beta^2 \cos\chi_1 \cos\chi_2\right)^2}{2[1 - 2\beta^2(1-\beta^2)]} \tag{7.17}$$

for all $0 \leqslant \beta \leqslant 1$.

For the measurement of only one of the polarizations be written as

$$P[\chi_1, -] = \frac{1 - 4\beta^2(1-\beta^2)\cos^2\chi_1}{2[1 - 2\beta^2(1-\beta^2)]} \tag{7.18}$$

$$P[-, \chi_2] = \frac{1 - 4\beta^2(1-\beta^2)\cos^2\chi_2}{2[1 - 2\beta^2(1-\beta^2)]} \tag{7.19}$$

and violates Bell's inequality of LHV theories for all $0 \leqslant \beta \leqslant 1$.

For example, for $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi_1' = 45°$, $\chi_2' = 67°$, Eq. (7.17), Eq. (7.18), Eq. (7.19), as obtained from scalar electrodynamics, gives $S = 0.207$ for $\beta = 0$ that violates Eq. (7.9) from above. For $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, we obtain $S = -1.207$ for $\beta = 0$ violating Eq. (7.9) from below. Both bounds are violated for all $\beta \leqslant 0.2$ for these same angles, respectively.

The probability of photon polarizations correlations in spin 0 pair into 2γ, Process 2, accordingly, for the joint conditional probabilities, we have

$$P_\beta[\chi_1, \chi_2] = \frac{A(\beta) + B(\beta)\cos^2(\chi_1 - \chi_2)}{2[2A(\beta) + B(\beta)]}, \tag{7.20}$$

where

$$A(\beta) = \frac{[-4(1-\beta^2) + (1-\beta^2)^2]}{4\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) + \frac{3}{2} - \frac{\beta^2}{2}, \tag{7.21}$$

$$B(\beta) = (1-\beta^2)\left[1 + \frac{(1-\beta^2)}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right)\right]. \tag{7.22}$$

the measurement of only one of the polarizations as:

$$P_\beta[\chi_1, -] = \frac{1}{2} = P_\beta[-, \chi_2], \tag{7.23}$$

for all $0 \leqslant \beta \leqslant 1$ and *independent* χ_1, χ_2.

Only in the limiting case $\beta \to 0$, the joint probability in Eq. (7.20) for this process coincides with that in Eq. (7.17) for the first process.

As in Eq. (7.8), we define

$$\begin{aligned} S_\beta = {} & P_\beta[\chi_1, \chi_2] - P_\beta[\chi_1, \chi_2'] + P_\beta[\chi_1', \chi_2] \\ & + P_\beta[\chi_1', \chi_2'] - P_\beta[\chi_1', -] - P_\beta[-, \chi_2], \end{aligned} \tag{7.24}$$

for four angles χ_1, χ_2, χ_1', χ_2', LHV theory gives [Clauser and Horne, 1974; Clauser and Shimoney, 1978]

$$-1 \leqslant S_\beta \leqslant 0. \tag{7.25}$$

For $\beta \to 1$, an equality holds in Eq. (7.20), $S_\beta \to -1/2$, and this process, to be useful for testing the violation of Eq. (7.25), should not be conducted at very high speeds. For $\chi_1 = 0°$, $\chi_2 = 67°$, $\chi_1' = 135°$, $\chi_2' = 23°$, we have $S_\beta = 0.120, 0.184, 0.201, 0.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (7.25) from above. For $\chi_1 = 0°$, $\chi_2 = 23°$, $\chi_1' = 45°$, $\chi_2' = 67°$, we have $S_\beta = -1.120, -1.184, -1.201, -1.207$ for $\beta = 0.2, 0.1, 0.05, 0.01$, respectively, violating Eq. (7.25) from below. For β larger than 0.2 but close to it, S_β already turns out to be too close to the critical interval given in Eq. (7.25) to be relevant experimentally.

The probability of photon polarizations correlations in $e^-e^- \to e^-e^-$ with initially polarized e^-, e^-, so-called Case I, $\theta = 0$ and $\phi = \pi/2$, has been given in work of E. B. Manoikian and N. Yongram (2004) to be

$$P[\chi_1, \chi_2] = \frac{1}{2}\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right), \tag{7.26}$$

for all $0 \leqslant \beta \leqslant 1$.

For the measurement of only one of the polarizations we have

$$P[\chi_1, -] = \frac{1}{2} \tag{7.27}$$

$$P[-, \chi_2] = \frac{1}{2} \tag{7.28}$$

The probability of photon polarizations correlations in $e^-e^- \to e^-e^-$ with initially polarized e^-, e^-, so-called Case II, $\theta = 0$ and $\phi = 0$, has been given in work of

E. B. Manoukian and N. Yongram (2004) to be

$$P[\chi_1,\chi_2]=\frac{1}{2N(\rho)}\left[(1+6\rho^2+\rho^4)\sin\left(\frac{\chi_1-\chi_2}{2}\right)-4\rho^2\cos\left(\frac{\chi_1+\chi_2}{2}\right)\right]^2, \tag{7.29}$$

where

$$N(\rho)=[(1+6\rho^2+\rho^4)^2+16\rho^4], \tag{7.30}$$

and

$$P[\chi_1,-]=\frac{1}{2}-\frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4}\sin\chi_1, \tag{7.31}$$

$$P[-,\chi_2]=\frac{1}{2}+\frac{4\rho^2(1+6\rho^2+\rho^4)}{(1+6\rho^2+\rho^4)^2+16\rho^4}\sin\chi_2. \tag{7.32}$$

Now we make the very important observation that in the formal limiting case $\beta\to 0$, Eq. (7.29), Eqs. (7.31)–(7.32) show that $P[\chi_1,\chi_2]\to\left(\sin^2[(\chi_1-\chi_2)/2]\right)/2$, $P[\chi_1,-]\to 1/2$, $P[-,\chi_2]\to 1/2$ coinding with fromal arguments based simply on combining the spins of e^-e^- [see page 3 of Chapter I, Introduction], expressions which have been used for years [cf. Clauser and Shimoney, 1978] showing the incorrectness of such arguments for $\beta\neq 0$ due to the simple fact that the electrons move upon collision to the detection region.

For all $0\leqslant\beta\leqslant 1$, angles χ_1, χ_2, χ_1', χ_2' are readily found leading to a violation of Bell's inequality of LHV theories. For example, for $\beta=0.2$, $\chi_1=0°$, $\chi_2=23°$, $\chi_1'=45°$, $\chi_2'=67°$, $S=-1.187$ violating the inequality from below.

The probability of photon polarizations correlations in $e^+e^-\to 2\gamma$ with initially polarized e^+, e^-, so-called Case I, $\theta=0$, has been given in work of E. B. Manoukian and N. Yongram (2004) to be

$$P[\chi_1, \chi_2] = \frac{1}{2}\sin^2\left(\frac{\chi_1 - \chi_2}{2}\right), \tag{7.33}$$

for all $0 \leqslant \beta \leqslant 1$.

For the measurement of only one of the polarizations we have

$$P[\chi_1, -] = \frac{1}{2}, \tag{7.34}$$

$$P[-, \chi_2] = \frac{1}{2}. \tag{7.35}$$

The probability of photon polarizations correlations in $e^+e^- \to 2\gamma$ with initially polarized e^+, e^-, so-called Case II, $\theta = \pi/2$, has been given in work of E. B. Manoukian and N. Yongram (2004) to be

$$P[\chi_1, \chi_2] = \frac{(1+\rho^2)^2\sin^2(\chi_1 - \chi_2) + \beta^2(1-\rho^2)^2\cos^2(\chi_1 + \chi_2)}{2[(1+\rho^2)^2 + \beta^2(1-\rho^2)^2]}, \tag{7.36}$$

for all $0 \leqslant \beta \leqslant 1$.

For the measurement of only one of the polarizations we have

$$P[\chi_1, -] = \frac{1}{2}, \tag{7.37}$$

$$P[-, \chi_2] = \frac{1}{2}. \tag{7.38}$$

A clear violation of Bell's inequality of LHV theories was obtained for all $0 \leqslant \beta \leqslant 0.45$. For example, for $\beta = 0.3$, with $\chi_1 = 0°$, $\chi_2 = 45°$, $\chi_1' = 90°$, $\chi_2' = 135°$ give $S = -1.165$ violating the inequality from below. For larger β values, alone, one cannot discriminate between LHV theories and quantum theory for this process. A violation of Bell's inequality for at least some β values, as seen, however, automatically violates LHV theories.

The probability of photon polarizations correlations in $e^-e^- \to e^-e^-$ with initially unpolarized e^-, e^- has been given in work of E. B. Manoukian and N. Yongram (2004) to be

$$P[\chi_1,\chi_2] = \frac{(1-\beta^2)(1+3\beta^2)\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)+\beta^4\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)+4\beta^4}{2(1+2\beta^2+6\beta^4)}, \tag{7.39}$$

for all $0 \leqslant \beta \leqslant 1$.

For the measurement of only one of the polarizations we have

$$P[\chi_1,-] = \frac{1}{2}, \tag{7.40}$$

$$P[-,\chi_2] = \frac{1}{2}. \tag{7.41}$$

The probability of photon polarizations correlations in $e^+e^- \to \mu^+\mu^-$ with initially unpolarized e^+, e^- has been given in work of N. Yongram, E. B. Manoukian and S. Sirinan (2006) to be

$$P[\chi_1,\chi_2] = \frac{1}{N(\mathcal{E})}\big[A(\mathcal{E})\sin\big(\tfrac{\chi_1-\chi_2}{2}\big)+B(\mathcal{E})\sin\big(\tfrac{\chi_1+\chi_2}{2}\big)+C(\mathcal{E})\cos\big(\tfrac{\chi_1-\chi_2}{2}\big)\big]^2$$
$$+\frac{1}{N(\mathcal{E})}\big[D(\mathcal{E})\sin\big(\tfrac{\chi_1+\chi_2}{2}\big)+E(\mathcal{E})\cos\big(\tfrac{\chi_1-\chi_2}{2}\big)\big]^2. \tag{7.42}$$

where

$$N(\mathcal{E}) = 2\Big\{\big[A(\mathcal{E})\big]^2+\big[B(\mathcal{E})\big]^2+\big[C(\mathcal{E})\big]^2+\big[D(\mathcal{E})\big]^2+\big[E(\mathcal{E})\big]^2\Big\}. \tag{7.43}$$

and

$$A(\mathcal{E}) = \left(\frac{M_Z^2}{4\mathcal{E}^2}+ab^2-1\right),$$

$$B(\mathcal{E}) = -\left(\frac{m_e}{m_\mu}\right)\left(\frac{M_Z^2}{4\mathcal{E}^2}+ab^2-1\right),$$

$$C(\mathcal{E}) = \frac{abm_e}{\mathcal{E}m_\mu}\sqrt{\mathcal{E}^2 - m_\mu^2},$$

$$D(\mathcal{E}) = \frac{a}{m_\mu \mathcal{E}}\sqrt{\mathcal{E}^2 - m_\mu^2}\,\sqrt{\mathcal{E}^2 - m_e^2},$$

$$E(\mathcal{E}) = -\frac{ab}{m_\mu}\sqrt{\mathcal{E}^2 - m_e^2},$$

The probabilities associated with the measurement of only one of the polarizations are given respectively, by

$$P[\chi_1, -] = \frac{1}{2} - \frac{2B(\mathcal{E})}{N(\mathcal{E})}\big[A(\mathcal{E})\cos\chi_1 + C(\mathcal{E})\sin\chi_1\big], \tag{7.44}$$

$$P[-, \chi_2] = \frac{1}{2} + \frac{2B(\mathcal{E})}{N(\mathcal{E})}\big[A(\mathcal{E})\cos\chi_2 + C(\mathcal{E})\sin\chi_2\big]. \tag{7.45}$$

Here we note the non-trivial speed as well as couplings dependence [see Chapter V for more details] neither of which may be taken to go to zero and hence formal arguments of simply combining the spins of μ^+, μ^- completely fail.

To show violation of Bell's inequality, it is sufficient to find four angles χ_1, χ_2, χ_1', χ_2' at accessible energies, for which S falls outside the interval $[-1, 0]$. For $\mathcal{E} = 105.656$ MeV, i.e., near threshold, an optimal value of S is obtained equal to -1.28203, for $\chi_1 = 0°$, $\chi_2 = 45°$, $\chi_1' = 90°$, $\chi_2' = 135°$, clearly violating Bell's inequality. For the energies originally carried out in the experiment on the differential cross section at $\mathcal{E} \sim 34$ GeV, an optimal value of S is obtained to be equal to -1.22094 for $\chi_1 = 0°$, $\chi_2 = 45°$, $\chi_1' = 51.13°$, $\chi_2' = 170.85°$.

The probability of photon polarizations correlations in e^+e^- pair productions from a charged Nambu strings with initially unpolarized e^+, e^- has been given in work of E. B. Manoukian and N. Yongram (2005) to be

$$P[\chi_1,\chi_2] = \frac{\left(2\sqrt{1-\beta^2}-\beta^2\right)^2\cos^2\left(\frac{\chi_1-\chi_2}{2}\right) + \left(2\sqrt{1-\beta^2}+\beta^2\right)^2\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)}{4\left(2-\beta^2\right)^2}. \tag{7.46}$$

The probabilities associated with the measurement of only one of the polarizations are given respectively, by

$$P[\chi_1,-] = \frac{1}{2}, \tag{7.47}$$

$$P[-,\chi_2] = \frac{1}{2}. \tag{7.48}$$

The probability of photon polarizations correlations in e^+e^- pair productions from a Neutral strings with initially unpolarized e^+, e^- has been given in work of E. B. Manoukian and N. Yongram (2005) to be

$$P[\chi_1,\chi_2] = \frac{1}{4\left(4-\beta^2-2\sqrt{2}\beta\right)}\Bigg\{\left(\sqrt{2(1-\beta^2)}-\sqrt{2}+\beta\right)^2\cos^2\left(\frac{\chi_1-\chi_2}{2}\right) + \left(\sqrt{2(1-\beta^2)}+\sqrt{2}-\beta\right)^2\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)\Bigg\}. \tag{7.49}$$

The probabilities associated with the measurement of only one of the polarizations are given respectively, by

$$P[\chi_1,-] = \frac{1}{2}, \tag{7.50}$$

$$P[-,\chi_2] = \frac{1}{2}. \tag{7.51}$$

To show violation with LHV theories, it is sufficient to consider one experimental situation which gives for S a value outside (Clauser and Horne, 1974; Clauser and Shimoney,

1978) the interval $[-1, 0]$. To this end for $\beta = 0.8$, $\chi_1 = 0°$, $\chi_2 = 160°$, $\chi_1' = 100°$, $\chi_2' = 10°$, we obtain $S = -1.088$, $S = -1.103$ for the charged and neutral strings, respectively, leading to a clear violation of LHV theories.

All of the above explicit expressions of our probability counts are novel. They provide ample support of the dependence of polarization correlations on speed as we have seen in our computations in quantum field theory in the electro-weak interaction as well as in the QED ones. We hope that some new experiments will be carried out in testing these expressions following unambiguously from our present ever reliable gauge theories which have been so far in excellent agreement with experiments and that will monitor speed and lead also to some new Bell-like experiments exploring quantum physics, i.e., quantum field theory, further in the high-energy relativistic regime.

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APPENDICES

APPENDIX A

NOTATION

In this Appendix we summarize the notation for relativistic four-vectors, for Dirac matrices and for spinors.

A.1 Vectors and Tensors

In the natural system of units, the speed of light, c, is equal to unity, so that the space-time four-vector is denoted ($\hbar = c = 1$)

$$x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x})$$

$$x_\mu = (-x^0, \mathbf{x}) = (-t, \mathbf{x}), \tag{A.1}$$

where the Greek index μ varies from 0 to 3 and the Roman indices on three-vectors vary from 1 to 3. Other frequently encountered four-vectors are

energy-momentum: $p^\mu = (p^0, \mathbf{p})$

four-gradient: $\partial_\mu = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$ $\partial^\mu = \left(-\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$

four-current: $J^\mu = (J^0, \mathbf{J})$

four-vector potential: $A^\mu = (A^0, \mathbf{A}) = (\phi, \mathbf{A})$

The invariant length of the four-vectors is written

$$x^2 = x \cdot x = g_{\mu\nu} x^\mu x^\nu = x_\mu x^\mu = \mathbf{x}^2 - (x^0)^2, \tag{A.2}$$

where a sum over repeated indices is always assumed and $\mathbf{g} = (g_{\mu\nu}) = (\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$ is the *metric tensor*

$$\mathbf{g} = (g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The symbol p is used to denote either the energy-momentum four-vector or the magnitude of the momentum three-vector (they can be distinguished from each other by the context in which they are used). The four-divergence of a vector field is

$$\partial_\mu J^\mu = \frac{\partial}{\partial t} J^0 + \boldsymbol{\nabla} \cdot \mathbf{J}. \tag{A.3}$$

In manipulating three-vectors and tensors, we use the Kronecker δ_{ij} function and the antisymmetric symbol ϵ_{ijk}, which is antisymmetric in any pair of indices and normalized to $\epsilon_{123} = \epsilon^{123} = 1$. Useful identities are

$$\epsilon_{ijk}\epsilon_{jki'} = 2\delta_{ii'}$$

$$\epsilon_{ijk}\epsilon_{i'j'k} = \delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'}. \tag{A.4}$$

When manipulating four-vectors and tensors, we will sometimes need the four-dimensional antisymmetric symbol $\epsilon_{\mu\nu\lambda\delta}$, which, when *all indices are down*, is antisymmetric under the interchange of any pair of indices and normalized to $\epsilon_{0123} = 1$ and

$\epsilon^{0123} = -1$. Useful identities are

$$\epsilon_{\mu\nu\lambda\delta}\epsilon^{\mu\nu\lambda\delta} = 24$$

$$\epsilon_{\mu\nu\lambda\delta}\epsilon_{\mu'}{}^{\nu\lambda\delta} = 6g_{\mu\mu'}. \tag{A.5}$$

A.2 Dirac Matrices

The Dirac matrices γ^μ satisfy the following anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}\mathbb{1}_4 \tag{A.6}$$

where

$$\mathbb{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbb{1}_4 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The 4×4 representation of these matrices used in this thesis ($\gamma^0 = -\gamma_0$)

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{A.7}$$

where the $\boldsymbol{\sigma}$ are the Pauli matrices

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.8}$$

Note that

$$\begin{aligned} \gamma^{0\dagger} &= \gamma^0 \\ \gamma^{i\dagger} &= -\gamma^i \\ \gamma^0\gamma^{\mu\dagger}\gamma^0 &= \gamma^\mu. \end{aligned} \tag{A.9}$$

Other matrices which are related to or constructed from the γ-matrices, are $\beta = \gamma^0$, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]$, the charge conjugation matrix $\mathcal{C} = -\mathrm{i}\alpha_2$, and time inversion matrix $\mathcal{T} = \mathcal{C}\gamma^5$. Their explicit 2×2 block form is

$$\begin{aligned} \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma} &= \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \\ \sigma^{0i} = \mathrm{i}\alpha_i = \mathrm{i}&\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 0 & -\mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & 0 \end{pmatrix}, \\ \sigma^{ij} = \gamma^5\alpha_k &= \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} -\mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix}, \end{aligned} \tag{A.10}$$

where ijk are in cyclic order. Note that $\mathcal{C} = -\mathcal{C}^\dagger = -\mathcal{C}^{-1}$, $(\gamma^5)^2 = 1$, and that

$$\begin{aligned} \mathcal{C}\gamma^\mu\mathcal{C}^{-1} &= -\gamma^{\mu\top} \\ \gamma^5\gamma^\mu &= -\gamma^\mu\gamma^5. \end{aligned} \tag{A.11}$$

Using the notation

$$\not{p} \equiv \gamma^\mu p_\mu \equiv \gamma p,$$

the following identities hold for the γ-matrices:

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= -4 \times \mathbb{1}_4 \\
\gamma^\mu \not{a} \gamma_\mu &= 2\not{a} \\
\gamma^\mu \not{a}\not{b} \gamma_\mu &= 4a \cdot b \\
\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu &= 2\not{c}\not{b}\not{a}.
\end{aligned} \tag{A.12}$$

In d dimensions, these identities generalize to

$$\begin{aligned}
\gamma^\alpha\gamma_\alpha &= d \times \mathbb{1}_d \\
\gamma^\alpha \not{a} \gamma_\alpha &= (2-d)\not{a} \\
\gamma^\alpha \not{a}\not{b} \gamma_\alpha &= 4a \cdot b - (4-d)\not{a}\not{b} \\
\gamma^\alpha \not{a}\not{b}\not{c} \gamma_\alpha &= -2\not{c}\not{b}\not{a} + (4-d)\not{a}\not{b}\not{c} \\
&= -(6-d)\not{c}\not{b}\not{a} + 2(4-d)[\not{a} b \cdot c - \not{b} c \cdot a + \not{c} a \cdot b].
\end{aligned} \tag{A.13}$$

A.3 Trace Theorems

The trace of an odd number of γ-matrices is zero. Other traces are

$$\begin{aligned}
\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} &= -4g^{\mu\nu} \\
\mathrm{Tr}\{\not{a}\not{b}\} &= -4a \cdot b \\
\mathrm{Tr}\{\not{a}\not{b}\not{c}\not{d}\} &= 4(a \cdot bc \cdot d - a \cdot c\, b \cdot d + a \cdot d\, b \cdot c)
\end{aligned}$$

$$\mathrm{Tr}\{\gamma^5 \not{a}\not{b}\} = 0$$

$$\mathrm{Tr}\{\gamma^5 \not{a}\not{b}\not{c}\not{d}\} = 4\mathrm{i}\epsilon_{\mu\nu\lambda\rho} a^\mu b^\nu c^\lambda d^\rho. \tag{A.14}$$

A.4 Dirac Spinors

The four-component Dirac particle u and antiparticle v spinors are defined by the relations

$$u(\mathbf{p}, s) = \sqrt{E_p + m} \begin{pmatrix} 1 \\ \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \end{pmatrix} \chi^s$$

$$v(\mathbf{p}, s) = \sqrt{E_p + m} \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \\ 1 \end{pmatrix} [-\mathrm{i}\sigma_2 \chi^s], \tag{A.15}$$

where $E_p = \sqrt{m^2 + \mathbf{p}^2}$ is the relativistic energy of the particle and the two-component spinors χ are

$$\chi^{(1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^{(-1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A.16}$$

The antiparticle two-component spinor is sometimes denoted by $\eta^{(s)}$, where $\eta^{(-s)} = -\mathrm{i}\sigma_2 \chi^{(s)}$. Hence

$$\eta^{(-1/2)} = -\mathrm{i}\sigma_2 \chi^{(+1/2)} \begin{pmatrix} 0 \\ +1 \end{pmatrix} \qquad \eta^{(+1/2)} = -\mathrm{i}\sigma_2 \chi^{(-1/2)} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{A.17}$$

Note the sign (phase) of $\eta^{(+1/2)}$. This phase convention is introduced so that the spinors are charge conjugates of the another (see below).

The adjoint Dirac spinors are

$$\overline{u}(\mathbf{p}, s) = u^{\dagger}(\mathbf{p}, s)\gamma^0$$

$$\overline{v}(\mathbf{p}, s) = v^{\dagger}(\mathbf{p}, s)\gamma^0. \tag{A.18}$$

With this definition, the spinors satisfy the following normalization and orthogonality relations:

$$\overline{u}(\mathbf{p}, s)u(\mathbf{p}, s') = 2m\delta_{ss'} \qquad \overline{v}(\mathbf{p}, s)u(\mathbf{p}, s') = 0$$
$$\overline{v}(\mathbf{p}, s)v(\mathbf{p}, s') = -2m\delta_{ss'} \qquad \overline{u}(\mathbf{p}, s)u(\mathbf{p}, s') = 0. \tag{A.19}$$

The completeness relations are expressed in terms of the positive and negative energy projection operators

$$\sum_s u(\mathbf{p}, s)\overline{u}(\mathbf{p}, s) = \frac{-\gamma p + m}{2m}$$

$$\sum_s v(\mathbf{p}, s)\overline{v}(\mathbf{p}, s) = -\frac{(\gamma p + m)}{2m}. \tag{A.20}$$

The u and v spinors are related by conjugation:

$$C\overline{v}^{\top}(\mathbf{p}, s) = u(\mathbf{p}, s) \qquad C\overline{u}^{\top}(\mathbf{p}, s) = v(\mathbf{p}, s). \tag{A.21}$$

A.5 Two-Spinor

Let the unit vector

$$\mathbf{n} = (\cos\chi, \sin\chi, 0), \tag{A.22}$$

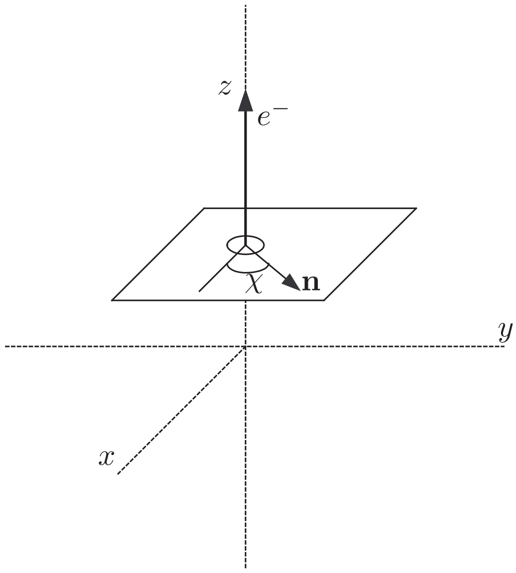

Figure A.1 The figure depicts e^-e^- scattering, with the electrons initially moving along the y-axis, while the emerging electrons moving along the z-axis. The angle χ, measured relative to the x-axis, denotes the orientation of spin of one of the emerging electrons may make.

and

$$\mathbf{S} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{A.23}$$

So that

$$\mathbf{S}\cdot\mathbf{n} = \frac{\hbar}{2}\left[\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\cos\chi + \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\sin\chi\right], \tag{A.24}$$

we approximate $\mathbf{S}\cdot\mathbf{n}$, therefore, above is written as

$$\mathbf{S}\cdot\mathbf{n} \approx \begin{pmatrix} \sigma_1\cos\chi + \sigma_2\sin\chi & 0 \\ 0 & \sigma_1\cos\chi + \sigma_2\sin\chi \end{pmatrix}. \tag{A.25}$$

To specify the two spinors ξ, for the pour-spinor is written as

$$u(p) = \left(\frac{p^0 + m}{2m}\right)^{1/2}\begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi \end{pmatrix}, \tag{A.26}$$

from

$$(\mathbf{S}\cdot\mathbf{n})u(p) = u(p). \tag{A.27}$$

Therefore

$$\begin{pmatrix} \sigma_1\cos\chi+\sigma_2\sin\chi & 0 \\ 0 & \sigma_1\cos\chi+\sigma_2\sin\chi \end{pmatrix}\begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi \end{pmatrix} = \begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi \end{pmatrix}, \tag{A.28}$$

where $\left(\dfrac{p^0+m}{2m}\right)^{1/2}$ be canceled out.

We have

$$[\sigma_1\cos\chi+\sigma_2\sin\chi]\,\xi = \xi, \tag{A.29}$$

$$[\sigma_1\cos\chi+\sigma_2\sin\chi]\,\xi = \xi, \tag{A.30}$$

where $\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}$ be canceled out.

Let $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$, where a, b are the real number, and we chose Eq. A.29, given by

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cos\chi + \begin{pmatrix} 0 & -\mathrm{i} \\ 1 & 0 \end{pmatrix}\sin\chi\right]\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & \cos\chi-\mathrm{i}\sin\chi \\ \cos\chi+\mathrm{i}\sin\chi & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\chi} \\ \mathrm{e}^{\mathrm{i}\chi} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{A.31}$$

we obtain

$$b\mathrm{e}^{-\mathrm{i}\chi} = a \tag{A.32}$$

$$a\mathrm{e}^{\mathrm{i}\chi} = b. \tag{A.33}$$

From Eqs. A.32 and A.33, we obtain the collected solution as

$$a = C\mathrm{e}^{-\mathrm{i}\chi/2} \tag{A.34}$$

$$b = C\mathrm{e}^{\mathrm{i}\chi/2}, \tag{A.35}$$

where C is the constant.

Therefore

$$\xi = C \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi/2} \\ \mathrm{e}^{\mathrm{i}\chi/2} \end{pmatrix}. \tag{A.36}$$

To normalized the two-spinor, $\xi^{\dagger}\xi = 1$, given

$$\xi^{\dagger} = C^{\dagger} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\chi/2} & \mathrm{e}^{-\mathrm{i}\chi/2} \end{pmatrix}, \tag{A.37}$$

we have

$$|C|^2 \begin{pmatrix} \mathrm{e}^{\mathrm{i}\chi/2} & \mathrm{e}^{-\mathrm{i}\chi/2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi/2} \\ \mathrm{e}^{\mathrm{i}\chi/2} \end{pmatrix} = 1$$

$$|C|^2 = \frac{1}{2}$$

$$C = \frac{1}{\sqrt{2}}. \tag{A.38}$$

Therefore, we obtain

$$\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\chi/2} \\ e^{i\chi/2} \end{pmatrix}. \tag{A.39}$$

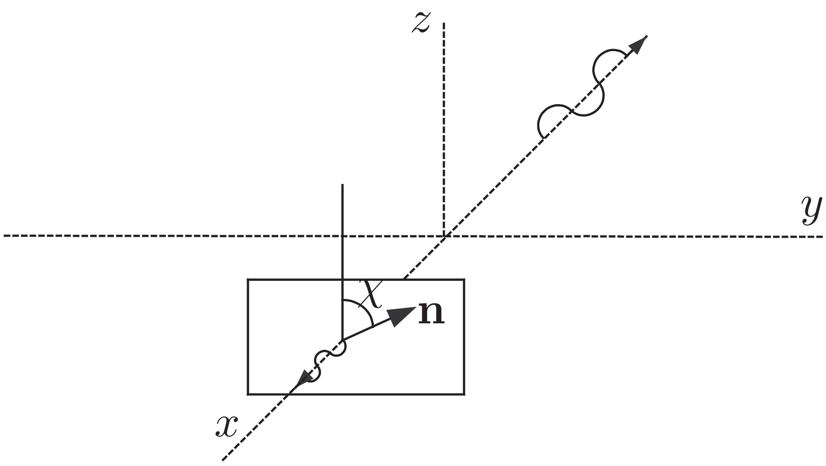

Figure A.2 The figure depicts e^+e^- annihilation into 2γ, with e^+, e^- moving along the y-axis, and the emerging photons moving along the x-axis. χ denotes the angle the polarization vector of one of the photons may make with the z-axis.

Let the unit vector

$$\mathbf{n} = (0, \sin\chi, \cos\chi). \tag{A.40}$$

So that

$$\mathbf{S}\cdot\mathbf{n} = \frac{\hbar}{2}\left[-\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\sin\chi + \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\cos\chi\right], \tag{A.41}$$

we approximate $\mathbf{S}\cdot\mathbf{n}$, therefore, above is written as

$$\mathbf{S}\cdot\mathbf{n} \approx \begin{pmatrix} \sigma_2\sin\chi + \sigma_3\cos\chi & 0 \\ 0 & \sigma_2\sin\chi + \sigma_3\cos\chi \end{pmatrix}. \tag{A.42}$$

To specify the two spinors ξ, written as

$$\begin{pmatrix} \sigma_2 \sin\chi + \sigma_3 \cos\chi & 0 \\ 0 & \sigma_2 \sin\chi + \sigma_3 \cos\chi \end{pmatrix} \begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi \end{pmatrix} = \begin{pmatrix} \xi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi \end{pmatrix}, \tag{A.43}$$

where $\left(\dfrac{p^0+m}{2m}\right)^{1/2}$ be canceled out.

We have

$$[\sigma_2 \sin\chi + \sigma_3 \cos\chi]\,\xi = \xi, \tag{A.44}$$

$$[\sigma_2 \sin\chi + \sigma_3 \cos\chi]\,\xi = \xi, \tag{A.45}$$

where $\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}$ be canceled out.

Let $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$, where a, b are the real number, and we chose Eq. A.29, given by

$$\left[\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \sin\chi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\chi\right] \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \cos\chi & -\mathrm{i}\sin\chi \\ \mathrm{i}\sin\chi & -\cos\chi \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{A.46}$$

we obtain

$$a\cos\chi - b\mathrm{i}\sin\chi = a, \tag{A.47}$$

$$a\mathrm{i}\sin\chi - b\cos\chi = b, \tag{A.48}$$

we chose Eq. A.47, so that, we get

$$a = \frac{b\mathrm{i}\sin\chi}{1-\cos\chi}. \tag{A.49}$$

Therefore

$$\xi = \begin{pmatrix} -\frac{b\mathrm{i}\sin\chi}{1-\cos\chi} \\ b \end{pmatrix} = b \begin{pmatrix} -\mathrm{i}\frac{\sin\chi}{1-\cos\chi}. \\ 1 \end{pmatrix} \tag{A.50}$$

To normalized the two-spinor, $\xi^\dagger\xi = 1$, given

$$\xi^\dagger = b^\dagger \begin{pmatrix} \mathrm{i}\frac{\sin\chi}{1-\cos\chi} & 1 \end{pmatrix}, \tag{A.51}$$

we have

$$|b|^2 \begin{pmatrix} \mathrm{i}\frac{\sin\chi}{1+\cos\chi} & 1 \end{pmatrix} \begin{pmatrix} -\mathrm{i}\frac{\sin\chi}{1-\cos\chi} \\ 1 \end{pmatrix} = 1,$$

$$|b|^2 \left[1 + \frac{\sin^2\chi}{(1-\cos\chi)^2}\right] = 1,$$

$$|b|^2 \frac{\left[1 - 2\cos\chi + \cos^2\chi + \sin^2\chi\right]}{(1-\cos\chi)^2} = 1,$$

$$|b|^2 = \frac{(1-\cos\chi)^2}{2\left[1-\cos\chi\right]},$$

$$|b|^2 = \frac{1}{2}(1-\cos\chi),$$

$$b = \sqrt{\frac{1}{2}(1-\cos\chi)} = \sin\frac{\chi}{2}. \tag{A.52}$$

Therefore, we obtain

$$\xi = \begin{pmatrix} -\mathrm{i}\cos\frac{\chi}{2} \\ \sin\frac{\chi}{2} \end{pmatrix}. \tag{A.53}$$

Let the unit vector

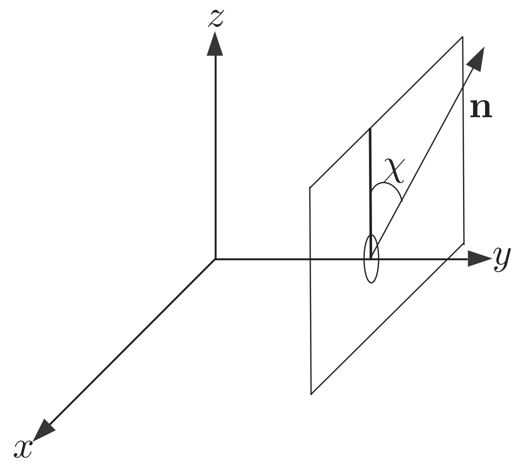

Figure A.3 The angle χ, measured relative to the z-axis, denotes the orientation of spin of one of the particles may make.

$$\mathbf{n} = (-\sin\chi, 0, \cos\chi). \tag{A.54}$$

So that

$$\mathbf{S}\cdot\mathbf{n} = \frac{\hbar}{2}\left[-\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}\sin\chi + \begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}\cos\chi\right], \tag{A.55}$$

we approximate $\mathbf{S}\cdot\mathbf{n}$, therefore, above is written as

$$\mathbf{S}\cdot\mathbf{n} \approx \begin{pmatrix}-\sigma_1\sin\chi + \sigma_3\cos\chi & 0\\ 0 & -\sigma_1\sin\chi + \sigma_3\cos\chi\end{pmatrix}. \tag{A.56}$$

To specify the two spinors ξ, for the pour-spinor is written as

$$u(p) = \left(\frac{p^0 + m}{2m}\right)^{1/2}\begin{pmatrix}\xi\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi\end{pmatrix}. \tag{A.57}$$

Therefore

$$\begin{pmatrix}-\sigma_1\sin\chi + \sigma_3\cos\chi & 0\\ 0 & -\sigma_1\sin\chi + \sigma_3\cos\chi\end{pmatrix}\begin{pmatrix}\xi\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi\end{pmatrix} = \begin{pmatrix}\xi\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi\end{pmatrix}, \tag{A.58}$$

where $\left(\frac{p^0+m}{2m}\right)^{1/2}$ be canceled out.

We have

$$[-\sigma_1 \sin\chi + \sigma_3 \cos\chi]\,\xi = \xi, \tag{A.59}$$

$$[-\sigma_1 \sin\chi + \sigma_3 \cos\chi]\,\xi = \xi, \tag{A.60}$$

where $\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}$ be canceled out.

Let $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$, where a, b are the real number, and we chose Eq. A.59, given by

$$\left[-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\chi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\chi\right]\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \cos\chi & -\sin\chi \\ -\sin\chi & -\cos\chi \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{A.61}$$

we obtain

$$a\cos\chi - b\sin\chi = a, \tag{A.62}$$

$$-a\sin\chi - b\cos\chi = b, \tag{A.63}$$

we chose Eq. A.63, so that, we get

$$b = -\frac{a\sin\chi}{1+\cos\chi}. \tag{A.64}$$

Therefore

$$\xi = \begin{pmatrix} a \\ -\frac{a\sin\chi}{1+\cos\chi} \end{pmatrix} = a\begin{pmatrix} 1 \\ -\frac{\sin\chi}{1+\cos\chi} \end{pmatrix}. \tag{A.65}$$

To normalized the two-spinor, $\xi^\dagger\xi = 1$, given

$$\xi^\dagger = a^\dagger \begin{pmatrix} 1 & -\frac{\sin\chi}{1+\cos\chi} \end{pmatrix}, \tag{A.66}$$

we have

$$|a|^2 \begin{pmatrix} 1 & -\frac{\sin\chi}{1+\cos\chi} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\sin\chi}{1+\cos\chi} \end{pmatrix} = 1,$$

$$|a|^2 \left[1 + \frac{\sin^2\chi}{(1+\cos\chi)^2}\right] = 1,$$

$$|a|^2 \frac{\left[1 + 2\cos\chi + \cos^2\chi + \sin^2\chi\right]}{(1+\cos\chi)^2} = 1,$$

$$|a|^2 = \frac{(1+\cos\chi)^2}{2\left[1+\cos\chi\right]},$$

$$|a|^2 = \frac{1}{2}(1+\cos\chi),$$

$$a = \sqrt{\frac{1}{2}(1+\cos\chi)} = \cos\frac{\chi}{2}. \tag{A.67}$$

Therefore, we obtain

$$\xi = \begin{pmatrix} \cos\frac{\chi}{2} \\ -\sin\frac{\chi}{2} \end{pmatrix}. \tag{A.68}$$

For two spinor of correlation particles, in Chapter VI, by substituting $\chi \to \chi' - \pi$, we have

$$\xi' = \begin{pmatrix} \sin\frac{\chi'}{2} \\ \cos\frac{\chi'}{2} \end{pmatrix}. \tag{A.69}$$

Let the unit vector

$$\mathbf{n} = (\sin\chi, 0, \cos\chi). \tag{A.70}$$

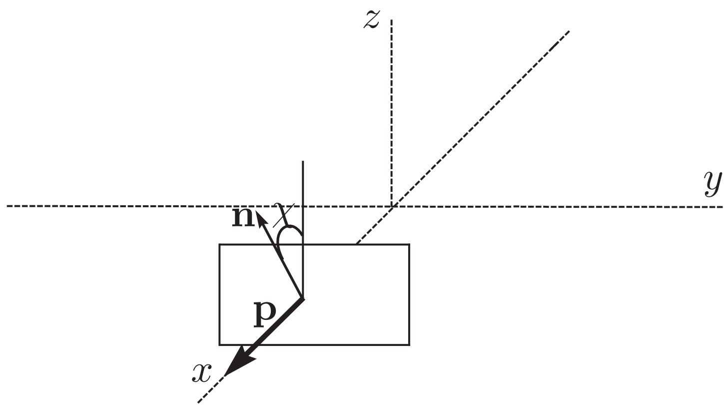

Figure A.4 The angle χ, measured relative to the z-axis, denotes the orientation of spin of one of the particles may make.

So that

$$\mathbf{S}\cdot\mathbf{n} = \frac{\hbar}{2}\left[\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}\sin\chi + \begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}\cos\chi\right], \tag{A.71}$$

we approximate $\mathbf{S}\cdot\mathbf{n}$, therefore, above is written as

$$\mathbf{S}\cdot\mathbf{n} \approx \begin{pmatrix}\sigma_1\sin\chi + \sigma_3\cos\chi & 0\\ 0 & \sigma_1\sin\chi + \sigma_3\cos\chi\end{pmatrix}. \tag{A.72}$$

Therefore

$$\begin{pmatrix}\sigma_1\sin\chi + \sigma_3\cos\chi & 0\\ 0 & \sigma_1\sin\chi + \sigma_3\cos\chi\end{pmatrix}\begin{pmatrix}\xi\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi\end{pmatrix} = \begin{pmatrix}\xi\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi\end{pmatrix}, \tag{A.73}$$

where $\left(\dfrac{p^0+m}{2m}\right)^{1/2}$ be canceled out.

We have

$$[\sigma_1\sin\chi + \sigma_3\cos\chi]\,\xi = \xi, \tag{A.74}$$

$$[\sigma_1\sin\chi + \sigma_3\cos\chi]\,\xi = \xi, \tag{A.75}$$

where $\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}$ be canceled out.

Let $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$, where a, b are the real number, and we chose Eq. A.75, given by

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\chi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\chi\right]\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \cos\chi & \sin\chi \\ \sin\chi & -\cos\chi \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{A.76}$$

we obtain

$$a\cos\chi + b\sin\chi = a, \tag{A.77}$$

$$a\sin\chi - b\cos\chi = b, \tag{A.78}$$

we chose Eq. A.78, so that, we get

$$b = \frac{a\mathrm{i}\sin\chi}{1+\cos\chi}. \tag{A.79}$$

Therefore

$$\xi = \begin{pmatrix} a \\ \frac{a\sin\chi}{1+\cos\chi} \end{pmatrix} = a\begin{pmatrix} 1 \\ \frac{\sin\chi}{1+\cos\chi} \end{pmatrix}. \tag{A.80}$$

To normalized the two-spinor, $\xi^\dagger\xi = 1$, given

$$\xi^\dagger = a^\dagger\begin{pmatrix} 1 & \frac{\sin\chi}{1+\cos\chi} \end{pmatrix}, \tag{A.81}$$

we have

$$|a|^2 \begin{pmatrix} 1 & \frac{\sin\chi}{1+\cos\chi} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\sin\chi}{1+\cos\chi} \end{pmatrix} = 1,$$

$$|a|^2 \left[1 + \frac{\sin^2\chi}{(1+\cos\chi)^2}\right] = 1,$$

$$|a|^2 \frac{\left[1 + 2\cos\chi + \cos^2\chi + \sin^2\chi\right]}{(1+\cos\chi)^2} = 1,$$

$$|a|^2 = \frac{(1+\cos\chi)^2}{2\left[1+\cos\chi\right]},$$

$$|a|^2 = \frac{1}{2}(1+\cos\chi),$$

$$a = \sqrt{\frac{1}{2}(1+\cos\chi)} = \cos\frac{\chi}{2}. \tag{A.82}$$

Therefore, we obtain

$$\xi = \begin{pmatrix} \cos\frac{\chi}{2} \\ \sin\frac{\chi}{2} \end{pmatrix}. \tag{A.83}$$

For two spinor of correlation particles, in Chapter V, by choosing the unit vector

$$\mathbf{n}' = (-\sin\chi', 0, \cos\chi'). \tag{A.84}$$

So that

$$\mathbf{S}\cdot\mathbf{n}' = \frac{\hbar}{2}\left[-\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\sin\chi' + \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\cos\chi'\right], \tag{A.85}$$

we approximate $\mathbf{S}\cdot\mathbf{n}'$, therefore, above is written as

$$\mathbf{S}\cdot\mathbf{n}' \approx \begin{pmatrix} -\sigma_1\sin\chi' + \sigma_3\cos\chi' & 0 \\ 0 & -\sigma_1\sin\chi' + \sigma_3\cos\chi' \end{pmatrix}. \tag{A.86}$$

Therefore

$$\begin{pmatrix} -\sigma_1 \sin\chi' + \sigma_3 \cos\chi' & 0 \\ 0 & -\sigma_1 \sin\chi' + \sigma_3 \cos\chi' \end{pmatrix} \begin{pmatrix} \xi' \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi' \end{pmatrix} = \begin{pmatrix} \xi' \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}\xi' \end{pmatrix}, \quad \text{(A.87)}$$

where $\left(\dfrac{p^0+m}{2m}\right)^{1/2}$ be canceled out.

We have

$$[-\sigma_1 \sin\chi' + \sigma_3 \cos\chi']\,\xi' = \xi' \quad \text{(A.88)}$$

$$[-\sigma_1 \sin\chi' + \sigma_3 \cos\chi']\,\xi' = \xi' \quad \text{(A.89)}$$

where $\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p^0+m}$ be canceled out.

Let $\xi' = \begin{pmatrix} a' \\ b' \end{pmatrix}$, where a', b' are the real number, and we chose Eq. A.89, given by

$$\left[-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\chi' + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\chi'\right] \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$$

$$\begin{pmatrix} \cos\chi' & -\sin\chi' \\ -\sin\chi & -\cos\chi' \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}, \quad \text{(A.90)}$$

we obtain

$$a' \cos\chi' + b' \sin\chi' = a', \quad \text{(A.91)}$$

$$-a' \sin\chi' - b' \cos\chi' = b', \quad \text{(A.92)}$$

we chose Eq. A.92, so that, we get

$$b' = -\frac{a' \sin\chi'}{1+\cos\chi'}. \tag{A.93}$$

Therefore

$$\xi' = \begin{pmatrix} a' \\ -\frac{a' \sin\chi'}{1+\cos\chi'} \end{pmatrix} = a' \begin{pmatrix} 1 \\ -\frac{\sin\chi'}{1+\cos\chi'} \end{pmatrix}. \tag{A.94}$$

To normalized the two-spinor, $\xi'^{\dagger}\xi' = 1$, given

$$\xi'^{\dagger} = a'^{\dagger} \begin{pmatrix} 1 & -\frac{\sin\chi'}{1+\cos\chi'} \end{pmatrix}, \tag{A.95}$$

we have

$$|a'|^2 \begin{pmatrix} 1 & -\frac{\sin\chi'}{1+\cos\chi'} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\sin\chi'}{1+\cos\chi'} \end{pmatrix} = 1,$$

$$|a'|^2 \left[1 + \frac{\sin^2\chi'}{(1+\cos\chi')^2}\right] = 1,$$

$$|a'|^2 \frac{\left[1 + 2\cos\chi' + \cos^2\chi' + \sin^2\chi'\right]}{(1+\cos\chi')^2} = 1,$$

$$|a'|^2 = \frac{(1+\cos\chi')^2}{2\left[1+\cos\chi'\right]},$$

$$|a'|^2 = \frac{1}{2}(1+\cos\chi'),$$

$$a' = \sqrt{\frac{1}{2}(1+\cos\chi')} = \cos\frac{\chi'}{2}. \tag{A.96}$$

Therefore, we obtain

$$\xi' = \begin{pmatrix} \cos\frac{\chi'}{2} \\ -\sin\frac{\chi'}{2} \end{pmatrix}. \tag{A.97}$$

APPENDIX B

VACUUM-TO-VACUUM TRANSITION AMPLITUDE IN QED

B.1 Vacuum-to-Vacuum Transition Amplitude

The Lagrangian density in QED is defined by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left[\frac{\partial_\mu}{\mathrm{i}}\overline{\psi}\gamma^\mu\psi - \overline{\psi}\gamma^\mu\frac{\partial_\mu}{\mathrm{i}}\psi\right] - m_0\overline{\psi}\psi$$

$$+ e_0\overline{\psi}\gamma^\mu A_\mu\psi + \overline{\psi}\eta + \overline{\eta}\psi + J^\mu A_\mu, \tag{B.1}$$

where e is charge,

ψ, $\overline{\psi}$ are the matter fields,

η, $\overline{\eta}$, J^μ are external sources,

and A^μ is the vector potential.

The fields equations are formally

$$\left[\gamma^\mu\left(\frac{\partial_\mu}{\mathrm{i}} - e_0A_\mu\right) + m_0\right]\psi = \eta, \tag{B.2}$$

$$\overline{\psi}\left[\gamma^\mu\left(\frac{\partial_\mu}{\mathrm{i}} + e_0A_\mu\right) - m_0\right] = -\overline{\eta}, \tag{B.3}$$

and in Feynman gauge, we have

$$-\partial_\mu F^{\mu\nu} = \left(g^{\nu\sigma} - g^{\nu k} \frac{\partial_k \partial^\sigma}{\nabla^2} \right) \left(e_0 \overline{\psi} \gamma_\sigma \psi + J_\sigma \right). \tag{B.4}$$

The momentum canonical are

$$\pi(A^0) = 0, \tag{B.5}$$

$$\pi(A^i) = -\partial_3^{-1} \left(\partial^i F^{03} - \partial^3 F^{0i} \right) \equiv \pi^i, \tag{B.6}$$

$$\pi(\psi) = \mathrm{i}\psi^\dagger, \tag{B.7}$$

$$[A^a(x), \pi^b(x)] = \mathrm{i}\delta^{ab} \delta^3(\mathbf{x} - \mathbf{x}'), \tag{B.8}$$

where $a, b = 1, 2$ and $x^0 = x'^0$.

B.2 Dynamical Principle I and II

Dynamical principle I

$$\delta \langle at \,|\, a't' \rangle = \mathrm{i} \left\langle at \middle| \int_{t'}^{t} (\mathrm{d}x) \delta \mathcal{L}(x) \middle| a't' \right\rangle. \tag{B.9}$$

Dynamical principle II gives the relation for

$$\frac{\delta}{\delta \lambda'} \langle aT;\ \lambda | Q(x) | a'T';\ \lambda \rangle, \quad T' < x^0 < T,$$

where we may write

$$\langle aT;\ \lambda|Q(x)|a'T';\ \lambda\rangle = \sum_{b,b'} \langle aT;\ \lambda\,|\,bt;\ \lambda\rangle\,\langle bt;\ \lambda|Q(x)|b't;\ \lambda\rangle\,\langle b't;\ \lambda\,|\,a'T';\ \lambda\rangle \tag{B.10}$$

$$\begin{aligned}
-\mathrm{i}\frac{\delta}{\delta\lambda'}\langle\ \ \rangle = \sum_{b,b'} \Bigg\{ &\mathrm{i}\left\langle aT;\ \lambda\left|\int_t^T (\mathrm{d}x)Q'(x)\right|bt;\ \lambda\right\rangle\langle\text{rest}\rangle \\
&+ \langle aT;\ \lambda\,|\,bt;\ \lambda\rangle\,\langle bt;\ \lambda|Q(x)|b't;\ \lambda\rangle \\
&\pm \mathrm{i}\left\langle b't;\ \lambda\left|\int_{T'}^t (\mathrm{d}x)Q'(x)\right|a'T';\ \lambda\right\rangle \\
&+ \langle aT;\ \lambda\,|\,bt;\ \lambda\rangle\,\frac{\delta}{\delta\lambda'}\langle bt;\ \lambda|Q(x)|b't;\ \lambda\rangle\,\langle b't;\ \lambda\,|\,a'T';\ \lambda\rangle \Bigg\}
\end{aligned}$$

$$\begin{aligned}
&-\mathrm{i}\frac{\delta}{\delta\lambda'}\langle aT;\ \lambda|Q(x)|a'T';\ \lambda\rangle \\
&\quad = \left\langle aT;\ \lambda\left|\int_t^T (\mathrm{d}x')Q'(x')Q(x) \pm Q(x)\int_{T'}^t (dy')Q'(y')\right|a'T';\ \lambda\right\rangle \\
&\quad + \sum_{b,b'}\langle aT;\ \lambda\,|\,bt;\ \lambda\rangle\,\frac{\delta}{\delta\lambda'}\underbrace{\langle bt;\ \lambda|Q(x)|b't;\ \lambda\rangle\,\langle b't;\ \lambda\,|\,a'T';\ \lambda\rangle}_{@}
\end{aligned} \tag{B.11}$$

where $@ = \left\langle b0;\ \lambda\left|\frac{\delta}{\delta\lambda'}Q(0,\mathbf{x})\right|b'0;\ \lambda\right\rangle$

Definition (Time Order Product)

$$(Q'(x')Q(x))_+ = Q'(x')Q(x)\Theta(x'^0 - x^0) \pm Q(x)Q(x')\Theta(x^0 - x'^0)$$

B.3 The Solution

Find the solution of vacuum-to-vacuum transition amplitude $\langle 0_+ | 0_- \rangle$

$$(-\mathrm{i})\frac{\partial}{\partial e}\langle 0_+ | 0_- \rangle = \int (\mathrm{d}x) \left\langle 0_+ \left| \left(\overline{\psi}\gamma^\mu A_\mu\right)_+ \right| 0_- \right\rangle$$

$$(-\mathrm{i})\frac{\partial}{\partial \overline{\eta}}\langle 0_+ | 0_- \rangle = \int (\mathrm{d}x) \langle 0_+ | \psi(x) | 0_- \rangle$$

$$(-\mathrm{i})\frac{\partial}{\partial \eta_\alpha(x)}(-\mathrm{i})\frac{\partial}{\partial \overline{\eta}_\beta(x)}\langle 0_+ | 0_- \rangle = \int (\mathrm{d}x) \left\langle 0_+ \left| \left(\overline{\psi}_\beta(x)\psi_\alpha(x)\right)_+ \right| 0_- \right\rangle$$

$$(-\mathrm{i})\frac{\partial}{\partial J^\mu}(-\mathrm{i})\frac{\partial}{\partial \eta_\alpha(x)}(-\mathrm{i})\frac{\partial}{\partial \overline{\eta}_\beta(x)}\langle 0_+ | 0_- \rangle = \int (\mathrm{d}x) \langle 0_+ | \left(\overline{\psi}_\beta(x)\psi_\alpha(x)A_\mu\right) | 0_- \rangle \,. \tag{B.12}$$

The functional derivative in Eq. (B.12) is defined with the independent fields and their conjugate momentum field. Hence

$$\langle 0_+ | 0_- \rangle_e = \exp\left[\mathrm{i}e \int (\mathrm{d}x)(-\mathrm{i})\frac{\partial}{\partial J^\mu}(-\mathrm{i})\frac{\partial}{\partial \eta_\alpha(x)}\gamma^\mu(-\mathrm{i})\frac{\partial}{\partial \overline{\eta}_\beta(x)}\right] \langle 0_+ | 0_- \rangle_0 \,, \tag{B.13}$$

where

$$\langle 0_+ | 0_- \rangle_0 = \exp \mathrm{i}\overline{\eta}S_+\eta \exp \frac{\mathrm{i}}{2} J^\mu D_G^{\mu\nu} J_\nu$$

$$\overline{\eta}S_+\eta = \int (\mathrm{d}x)(\mathrm{d}x')\overline{\eta}(x)S_+(x-x')\eta(x')$$

$$S_+(x-x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4}\mathrm{e}^{\mathrm{i}p(x-x')}\frac{(-\gamma p + m)}{p^2+m^2-\mathrm{i}\varepsilon},$$

for $D_G^{\mu\nu}(q) = \dfrac{1}{q^2-\mathrm{i}\varepsilon}\left[\delta^{\mu\nu} - \dfrac{q^\mu q^\nu}{q^2}\right] + q^\mu q^\nu G(q^2)$ is the photon propagator in any gauge.
Special case $G(q^2) = \dfrac{\xi_0}{q^2(q^2-\mathrm{i}\varepsilon)}$

1 $\xi_0 = 1$ (Feynman gauge)

2 $\xi_0 = 0$ (Landau gauge)

3 $\xi_0 = 3$ (Yennie gauge)

So we get $\langle 0_+ | 0_- \rangle_e$ up to e^2 in Eq. (B.13)as

$$\langle 0_+ | 0_- \rangle_e = \exp\left[a_0 + ea_1 + e^2 a_2 + e^3 a_3 + \ldots\right]. \tag{B.14}$$

Find a_0, a_1, a_2

$$\begin{aligned}\langle 0_+ | 0_- \rangle \big|_{e=0} \equiv \mathrm{e}^{a_0} = \langle 0_+ | 0_- \rangle_0 \\ = \exp[\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')\overline{\eta}(x) S_+(x,x')\eta(x')] \\ \times \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x') J^\mu(x) D_G^{\mu\nu}(x,x') J_\nu(x')\right].\end{aligned} \tag{B.15}$$

We obtain

$$\begin{aligned}a_0 = \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')\overline{\eta}(x) S_+(x,x')\eta(x') \\ + \frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x') J^\mu(x) D_G^{\mu\nu}(x,x') J_\nu(x')\end{aligned} \tag{B.16}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}e}\langle 0_+ | 0_- \rangle = [a_1 + 2ea_2 + \ldots]\exp[a_0 + ea_1 + e^2 a_2 + \ldots] \\ = [a_1 + 2ea_2 + \ldots]\langle 0_+ | 0_- \rangle \\ \frac{\mathrm{d}}{\mathrm{d}e}\langle 0_+ | 0_- \rangle \bigg|_{e=0} = a_1 \langle 0_+ | 0_- \rangle \big|_{e=0} = a_1 \langle 0_+ | 0_- \rangle_0\end{aligned} \tag{B.17}$$

$\therefore$ we obtain a_1 as

$$a_1 = \frac{1}{\langle 0_+ | 0_- \rangle_0}\frac{\mathrm{d}}{\mathrm{d}e}\langle 0_+ | 0_- \rangle \bigg|_{e=0} \tag{B.18}$$

from $\langle 0_+ | 0_- \rangle = \exp[\mathrm{i}e \int (\mathrm{d}x)(-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^\mu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(-\mathrm{i})\frac{\delta}{\delta J^\mu(x)}] \langle 0_+ | 0_- \rangle_0$

Choose

$$\langle 0_+ | 0_- \rangle \equiv \mathrm{e}^{\mathrm{i}\widehat{A}} \langle 0_+ | 0_- \rangle_0 \tag{B.19}$$

where

$$\widehat{A} \equiv \int (\mathrm{d}x)(-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^\mu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(-\mathrm{i})\frac{\delta}{\delta J^\mu(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}e} \langle 0_+ | 0_- \rangle = \mathrm{i}\widehat{A}\mathrm{e}^{\mathrm{i}\widehat{A}} \langle 0_+ | 0_- \rangle_0$$

$$\frac{\mathrm{d}}{\mathrm{d}e} \langle 0_+ | 0_- \rangle \bigg|_{e=0} = \mathrm{i}\widehat{A} \langle 0_+ | 0_- \rangle_0 \tag{B.20}$$

We can rewrite Eq. (B.18) as

$$a_1 = \frac{\mathrm{i}}{\langle 0_+ | 0_- \rangle_0} \widehat{A} \langle 0_+ | 0_- \rangle_0 \tag{B.21}$$

$$\begin{aligned} \langle 0_+ | 0_- \rangle_0 = {} & \exp[\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')\overline{\eta}(x)S_+(x,x')\eta(x')] \\ & \times \exp\left[\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}x')J^\mu(x)D_G^{\mu\nu}(x,x')J_\nu(x')\right] \end{aligned} \tag{B.22}$$

$$\begin{aligned} \widehat{A} \langle 0_+ | 0_- \rangle = {} & \int (\mathrm{d}x)(-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^\mu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(-\mathrm{i})\frac{\delta}{\delta J^\mu(x)} \\ & \times \exp \mathrm{i}\overline{\eta}S_+\eta \exp\frac{\mathrm{i}}{2}J^\mu D_G^{\mu\nu} J_\nu . \end{aligned} \tag{B.23}$$

We can write $JDJ \equiv J^\mu D_{\mu\nu} J^\nu \equiv J^\mu D_G^{\mu\nu} J_\nu$

$\therefore$

$$(-\mathrm{i})\frac{\delta}{\delta J^\mu(x)}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} = (-\mathrm{i})\frac{\delta}{\delta J^\mu(x)}\exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}y)(\mathrm{d}z)J^\alpha(y)D_{\alpha\beta}(y,z)J^\beta(z)\right]$$

$$= \frac{1}{2}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}\int(\mathrm{d}y)(\mathrm{d}z)\Big[\delta_\mu^\alpha\delta^4(y-x)D_{\alpha\beta}(y,z)J^\beta(z)$$

$$+ \delta_\mu^\beta\delta^4(z-x)J^\alpha(y)D_{\alpha\beta}(y,z)\Big]$$

$$= \frac{1}{2}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}\Big[\int(\mathrm{d}z)D_{\alpha\beta}(x,z)J^\beta(z) + \int(\mathrm{d}y)J^\alpha(y)D_{\alpha\beta}(y,x)\Big]$$

$$(-\mathrm{i})\frac{\delta}{\delta J^\mu(x)}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} = \left[\int(\mathrm{d}y)D_{\mu\nu}(x,y)J^\nu(y)\right]\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}, \tag{B.24}$$

and

$$(-\mathrm{i})\frac{\delta}{\delta J^\mu}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} \equiv [D_{\mu\nu}J^\nu]\,\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}. \tag{B.25}$$

For the η and $\overline{\eta}$ sources, we have

$$(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} = (-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\exp\left[\mathrm{i}\int(\mathrm{d}y)(\mathrm{d}z)\overline{\eta}(y)S_+(y,z)\eta(z)\right]$$

$$= \mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}\left[\int(\mathrm{d}y)(\mathrm{d}z)\delta^4(y-x)S_+(y,z)\eta(z)\right]$$

$$(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} = \left[\int(\mathrm{d}y)S_+(x,y)\eta(y)\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} \tag{B.26}$$

$$(-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^\mu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} = (-\mathrm{i})\frac{\delta}{\delta\eta(x)}\left[\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}\int(\mathrm{d}y)\gamma^\mu S_+(x,y)\eta(y)\right]$$

$$= \left[\int(\mathrm{d}y)\gamma^\mu S_+(x,y)\eta(y)\right]\left[\int(\mathrm{d}z)\overline{\eta}(z)S_+(z,x)\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}$$

$$
\begin{aligned}
&- \mathrm{i}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}\left[\int (\mathrm{d}y)\gamma^\mu S_+(x,y)\delta^4(y-x)\right] \\
&= \left[\int (\mathrm{d}y)\gamma^\mu S_+(x,y)\eta(y)\right]\left[\int (\mathrm{d}z)\overline{\eta}(z)S_+(z,x)\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} \\
&- \mathrm{i}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}\left[\gamma^\mu S_+(x,x)\right] \qquad \text{(B.27)}
\end{aligned}
$$

For $S_+(x,x) = 0$ (Fermion loop), we can rewrite Eq. (B.27) as

$$
(-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^\mu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} = \left[\int (\mathrm{d}y)(\mathrm{d}z)\overline{\eta}(z)S_+(z,x)\gamma^\mu S_+(x,y)\eta(y)\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} \qquad \text{(B.28)}
$$

From Eq. (B.24) and Eq. (B.28), we obtain

$$
\begin{aligned}
\widehat{A}\,\langle 0_+ | 0_- \rangle_0 = &\Big[\int (\mathrm{d}y)(\mathrm{d}z)\overline{\eta}(z)S_+(z,x)\gamma^\mu S_+(x,y)\eta(y) \\
&\times \int (\mathrm{d}x)(\mathrm{d}x')J^\mu(x)D_{\mu\nu}(x,x')J^\nu(x')\Big]\,\langle 0_+ | 0_- \rangle_0 \qquad \text{(B.29)}
\end{aligned}
$$

We can rewrite a_1 as

$$
\begin{aligned}
a_1 = \mathrm{i}&\Big[\int (\mathrm{d}y)(\mathrm{d}z)\overline{\eta}(z)S_+(z,x)\gamma^\mu S_+(x,y)\eta(y) \\
&\times \int (\mathrm{d}x)(\mathrm{d}x')J^\mu(x)D_{\mu\nu}(x,x')J^\nu(x')\Big] \qquad \text{(B.30)}
\end{aligned}
$$

For e in first order, we obtain up to e^2

$$
\begin{aligned}
\langle 0_+ | 0_- \rangle &= \exp[a_0 + ea_1 + e^2 a_2 + \ldots] \\
\frac{\mathrm{d}}{\mathrm{d}e}\,\langle 0_+ | 0_- \rangle &= \exp[a_0 + ea_1 + e^2 a_2 + \ldots]
\end{aligned}
$$

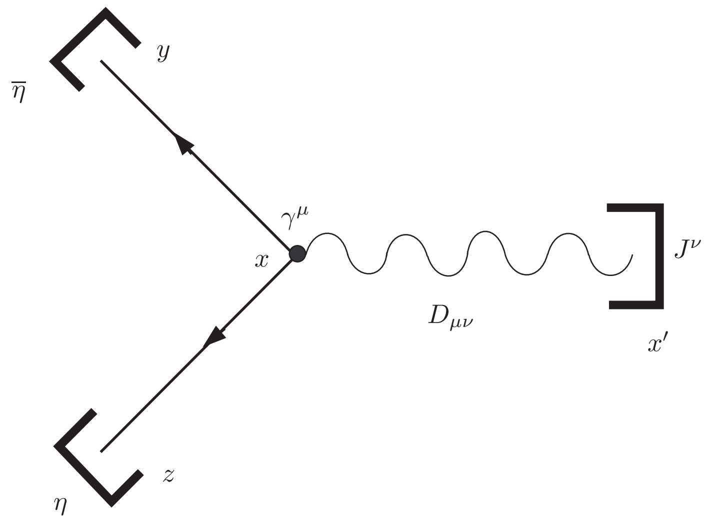

Figure B.1 Part of process of photon production.

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2}\langle 0_+ | 0_- \rangle \Big|_{e=0} = 2a_1 \langle 0_+ | 0_- \rangle + [a_1 + 2ea_2 + \ldots]^2 \langle 0_+ | 0_- \rangle$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2}\langle 0_+ | 0_- \rangle \Big|_{e=0} = (2a_2 + a_1^2) \langle 0_+ | 0_- \rangle_0$$

$$2a_2 + a_1^2 = \frac{1}{\langle 0_+ | 0_- \rangle_0} \frac{\mathrm{d}^2}{\mathrm{d}e^2}\langle 0_+ | 0_- \rangle \Big|_{e=0}$$

$$a_2 = -\frac{1}{2}a_1^2 + \frac{1}{2}\frac{1}{\langle 0_+ | 0_- \rangle_0} \frac{\mathrm{d}^2}{\mathrm{d}e^2}\langle 0_+ | 0_- \rangle \Big|_{e=0} \tag{B.31}$$

$$\because \langle 0_+ | 0_- \rangle_0 \equiv \mathrm{e}^{\mathrm{i}e\widehat{A}} \langle 0_+ | 0_- \rangle_0;\ \widehat{A} \equiv \int (\mathrm{d}x)(-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^\mu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(-\mathrm{i})\frac{\delta}{\delta J^\mu(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}e}\langle 0_+ | 0_- \rangle = \mathrm{i}\widehat{A}\mathrm{e}^{\mathrm{i}\widehat{A}} \langle 0_+ | 0_- \rangle_0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2}\langle 0_+ | 0_- \rangle = -(\widehat{A})^2\mathrm{e}^{\mathrm{i}\widehat{A}} \langle 0_+ | 0_- \rangle_0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2}\langle 0_+ | 0_- \rangle \Big|_{e=0} = -(\widehat{A})^2 \langle 0_+ | 0_- \rangle_0 \tag{B.32}$$

Find $(\widehat{A})^2$

$$(\widehat{A})^2 = \int (\mathrm{d}x)(\mathrm{d}y)(-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^{\mu}(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(-\mathrm{i})\frac{\delta}{\delta J^{\mu}(x)}$$

$$\times (-\mathrm{i})\frac{\delta}{\delta\eta(y)}\gamma^{\nu}(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(y)}(-\mathrm{i})\frac{\delta}{\delta J^{\nu}(y)}$$

From Eq. (B.24);

$$(-\mathrm{i})\frac{\delta}{\delta J^{\mu}(x)}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} = \left[\int (\mathrm{d}y)D_{\mu\nu}(x,y)J^{\nu}(y)\right]\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} \tag{B.33}$$

$$(-\mathrm{i})\frac{\delta}{\delta J^{\nu}(y)}(-\mathrm{i})\frac{\delta}{\delta J^{\mu}(x)}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}$$

$$= \Big[(-\mathrm{i})D_{\mu\nu}(x,y) + \times \int (\mathrm{d}y')(\mathrm{d}z')J^{\alpha}(y')D_{\alpha\mu}(y',x)D_{\nu\beta}(y,z')J^{\beta}(z')\Big]\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}. \tag{B.34}$$

Set

$$\odot^{\nu}(y) = (-\mathrm{i})\frac{\delta}{\delta\eta(y)}\gamma^{\mu}(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(y)}\mathrm{e}^{\mathrm{i}\overline{\eta}S_{+}\eta}$$

$$= \left[\int (\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')S_{+}(y',y)\gamma^{\nu}S_{+}(y,z')\eta(z')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_{+}\eta}, \tag{B.35}$$

and $[\ldots] \equiv \left[\int (\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')S_{+}(y',y)\gamma^{\nu}S_{+}(y,z')\eta(z')\right]$, so we obtain

$$(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\odot^{\nu}(y) = (-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\mathrm{e}^{\mathrm{i}\overline{\eta}S_{+}\eta}[\ldots] + (-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}[\ldots]\mathrm{e}^{\mathrm{i}\overline{\eta}S_{+}\eta}$$

$$= \left[\int (\mathrm{d}y'')S_{+}(x,y'')\eta(y'')\right][\ldots]\mathrm{e}^{\mathrm{i}\overline{\eta}S_{+}\eta}$$

$$- \mathrm{i}S_{+}(x,y)\left[\int (\mathrm{d}z')\gamma^{\nu}S_{+}(y,z')\eta(z')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_{+}\eta}$$

$$(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\odot^{\nu}(y) = \Big[\int(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')\overline{\eta}(y')S_+(y',y)\gamma^{\nu}S_+(y,z')\eta(z')$$
$$\times S_+(x,y'')\eta(y'')\Big]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}$$
$$-\,\mathrm{i}S_+(x,y)\left[\int(\mathrm{d}z')\gamma^{\nu}S_+(y,z')\eta(z')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}. \tag{B.36}$$

We define

$$\odot^{\mu\nu}(x,y) = (-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^{\mu}(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}\odot^{\nu}(y) \tag{B.37}$$

$$\odot^{\mu\nu}(x,y) = (-\mathrm{i})\frac{\delta}{\delta\eta(x)}\Bigg\{\left[\int(\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')\gamma^{\mu}S_+(y',y)\gamma^{\nu}S_+(y,z')\eta(z')\right]$$
$$\times\left[\int(\mathrm{d}y'')S_+(x,y'')\eta(y'')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}\Bigg\}$$
$$-\,\gamma^{\mu}S_+(x,y)\frac{\delta}{\delta\eta(x)}\left\{\left[\int(\mathrm{d}z')\gamma^{\nu}S_+(y,z')\eta(z')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}\right\}$$
$$= -\mathrm{i}\left[\int(\mathrm{d}y')\overline{\eta}(y')\gamma^{\mu}S_+(y',y)\gamma^{\nu}S_+(y,x)\right]$$
$$\times\left[\int(\mathrm{d}y'')S_+(x,y'')\eta(y'')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}$$
$$-\,\mathrm{i}\left[\int(\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')\gamma^{\mu}S_+(y',y)\gamma^{\nu}S_+(y,z')\eta(z')\right]\overbrace{S_+(x,x)}^{=0}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}$$
$$+\left[\int(\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y)\gamma^{\mu}S_+(y',y)\gamma^{\nu}S_+(y,z')\right]$$
$$\times\left[\int(\mathrm{d}y'')(\mathrm{d}z'')\overline{\eta}(z'')S_+(z'',x)S_+(x,y'')\eta(y'')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}$$
$$-\,\gamma^{\mu}S_+(x,y)\gamma^{\nu}S_+(y,x)\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}$$

$$- \mathrm{i}\gamma^\mu S_+(x,y)\left[\int (\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')\gamma^\mu S_+(y',x)\gamma^\nu S_+(y,z')\eta(z')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}$$

$$\begin{aligned}
\odot^\nu(x,y) = &-\mathrm{i}\left[\int (\mathrm{d}y')\overline{\eta}(y')\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,x)\right] \\
&\times\left[\int (\mathrm{d}y'')S_+(x,y'')\eta(y'')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} \\
&+\left[\int (\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y)\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,z')\right] \\
&\times\left[\int (\mathrm{d}y'')(\mathrm{d}z'')\overline{\eta}(z'')S_+(z'',x)S_+(x,y'')\eta(y'')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} \\
&-\gamma^\mu S_+(x,y)\gamma^\nu S_+(y,x)\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} \\
&-\mathrm{i}\gamma^\mu S_+(x,y)\left[\int (\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')\gamma^\mu S_+(y',x)\gamma^\nu S_+(y,z')\eta(z')\right]\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta}
\end{aligned} \tag{B.38}$$

where

$$\odot^{\mu\nu}(x,y) = (-\mathrm{i})\frac{\delta}{\delta\eta(x)}\gamma^\mu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(x)}(-\mathrm{i})\frac{\delta}{\delta\eta(y)}\gamma^\nu(-\mathrm{i})\frac{\delta}{\delta\overline{\eta}(y)}\mathrm{e}^{\mathrm{i}\overline{\eta}S_+\eta} \tag{B.39}$$

We obtain

$$\begin{aligned}
\odot(x,y) = \Bigg\{ &-\left[\int (\mathrm{d}y')(\mathrm{d}y'')\overline{\eta}(y')\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,x)D_{\mu\nu}(x,y)S_+(x,y'')\eta(y'')\right] \\
&-\mathrm{i}\left[\int (\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,z')\eta(z')\right]D_{\mu\nu}(x,y) \\
&\times\left[\int (\mathrm{d}y'')(\mathrm{d}z'')\overline{\eta}(z'')S_+(z'',x)S_+(x,y'')\eta(y'')\right] \\
&-D_{\mu\nu}(x,y)\gamma^\mu S_+(x,y)\left[\int (\mathrm{d}y')(\mathrm{d}z')\overline{\eta}(y')S_+(y',x)\gamma^\nu S_+(y,z')\eta(z')\right]
\end{aligned}$$

$$- \mathrm{i}\Big[\int(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')\overline{\eta}(y')\gamma^{\mu}S_{+}(y',y)\gamma^{\nu}S_{+}(y,x)\eta(y'')$$

$$\times J^{\alpha}(z')D_{\alpha\mu}(z',x)D_{\nu\beta}(y,z'')J^{\beta}(z'')\Big]$$

$$- \mathrm{i}\Big[\int(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}y''')(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}z''')\overline{\eta}(y')\gamma^{\mu}S_{+}(y',y)\gamma^{\nu}S_{+}(y,z')\eta(z')$$

$$\times \overline{\eta}(z'')S_{+}(z'',x)S_{+}(x,y'')\eta(y'')$$

$$\times J^{\alpha}(y''')D_{\alpha\mu}(y''',x)D_{\nu\beta}(y,z''')J^{\beta}(z''')\Big]$$

$$+ \mathrm{i}\gamma^{\mu}S_{+}(x,y)\gamma^{\nu}S_{+}(y,x)D_{\mu\nu}(x,y)$$

$$- \Big[\int(\mathrm{d}y')(\mathrm{d}z')J^{\alpha}(y')D_{\alpha\mu}(y',x)\gamma^{\mu}S_{+}(x,y)\gamma^{\nu}S_{+}(y,x)D_{\gamma\beta}(y,z')J^{\beta}(z')\Big]$$

$$- \mathrm{i}\gamma^{\mu}S_{+}(x,y)\Big[\int(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y'')\overline{\eta}(y')S_{+}(y',x)\gamma^{\nu}S_{+}(y,z')\eta(z')\Big]$$

$$\times J^{\alpha}(y'')D_{\alpha\mu}(y'',x)D_{\nu\beta}(y,z'')J^{\beta}(z'')\Big]\Big\}\langle 0_{+}|0_{-}\rangle_{0} \qquad \text{(B.40)}$$

$\because\ (\widehat{A})^{2}\langle 0_{+}|0_{-}\rangle_{0} = \int(\mathrm{d}x)(\mathrm{d}y)\odot(x,y)$

Let

$$\odot \equiv \frac{1}{\langle 0_{+}|0_{-}\rangle_{0}}(\widehat{A})^{2}\langle 0_{+}|0_{-}\rangle_{0} = \frac{1}{\langle 0_{+}|0_{-}\rangle_{0}}\int(\mathrm{d}x)(\mathrm{d}y)\odot(x,y) \qquad \text{(B.41)}$$

$\therefore\ \odot =$ I+II+III+IV+V+VI+VII

Consider for any process from $\odot$

$$\mathrm{I} \equiv -\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')\left[\overline{\eta}(y')\gamma^{\mu}S_{+}(y',y)\gamma^{\nu}S_{+}(y,x)D_{\mu\nu}(x,y)S_{+}(x,y'')\eta(y'')\right]$$

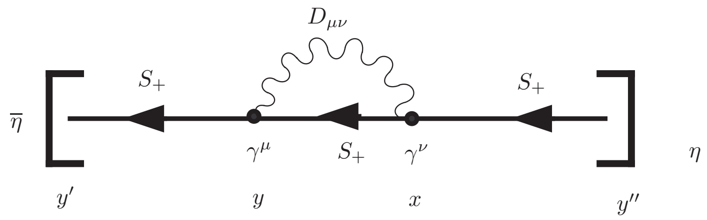

Figure B.2 Electron self energy.

$$\text{II} \equiv -\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')(\mathrm{d}z'')\,[\overline{\eta}(y')\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,z')\eta(z')]\,D_{\mu\nu}(x,y)$$

$$\times\,[\overline{\eta}(z'')S_+(z'',x)S_+(x,y'')\eta(y'')]$$

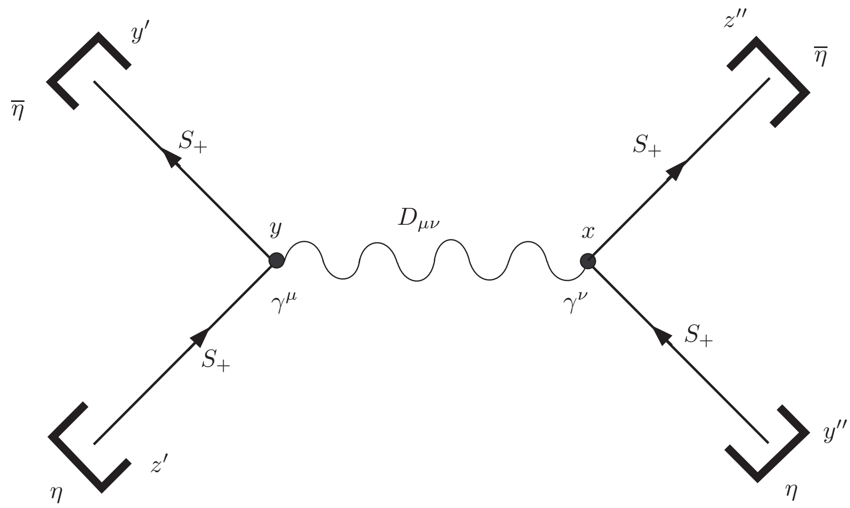

Figure B.3 Electron and a positron scattering processes.

$$\text{III} \equiv -\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')\,[D_{\mu\nu}(x,y)\gamma^\mu S_+(x,y)]$$

$$\times\,[\overline{\eta}(y')S_+(y',x)\gamma^\nu S_+(y,z')\eta(z')]$$

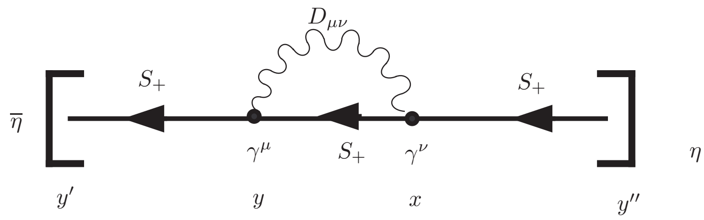

Figure B.4 Electron and a positron scattering processes.

$$\mathbf{IV} \equiv -\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')(\mathrm{d}z'')\,[\overline{\eta}(y')\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,x)S_+(x,y'')\eta(y'')]$$

$$\times\left[J^\alpha(z')D_{\alpha\mu}(z',x)D_{\mu\nu}(y,z'')J^\beta(z'')\right]$$

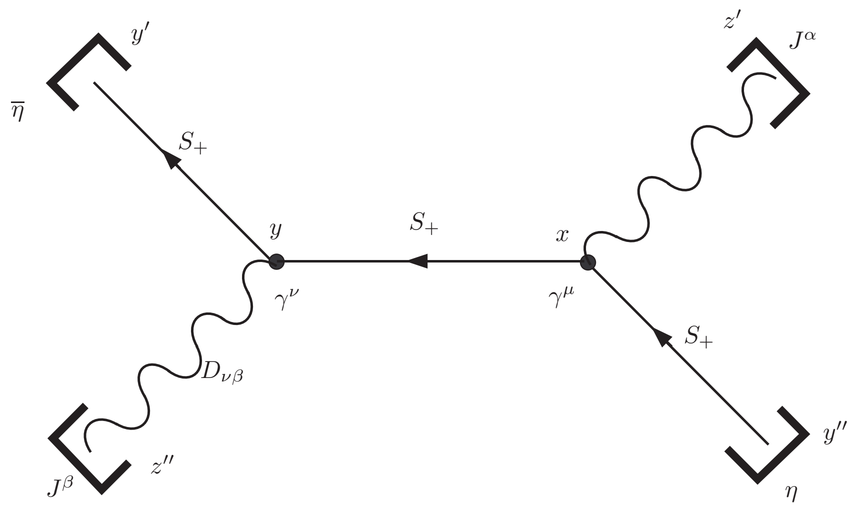

Figure B.5 Processes $e^-\gamma \to e^+\gamma$.

$$\mathbf{V} \equiv \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}y''')(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}z''')\Big[\overline{\eta}(y')\gamma^\mu S_+(y',y)\gamma^\nu S_+(y,z')\eta(z')$$

$$\times\,\overline{\eta}(z'')S_+(z'',x)S_+(x,y'')\eta(y'')$$

$$\times\, J^\alpha(y''')D_{\alpha\mu}(y''',x)D_{\nu\beta}(y,z''')J^\beta(z''')\Big]$$

Or

$$\mathrm{V} \equiv \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}y''')(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}z''')\Big[\overline{\eta}(y')\gamma^{\mu}S_{+}(x,y)S_{+}(y',x)\gamma^{\nu}$$

$$\times S_{+}(y,z')\eta(z')\Big]\Big[J^{\alpha}(y'')D_{\alpha\mu}(y'',x)D_{\nu\beta}(y,z'')J^{\beta}(z'')\Big]$$

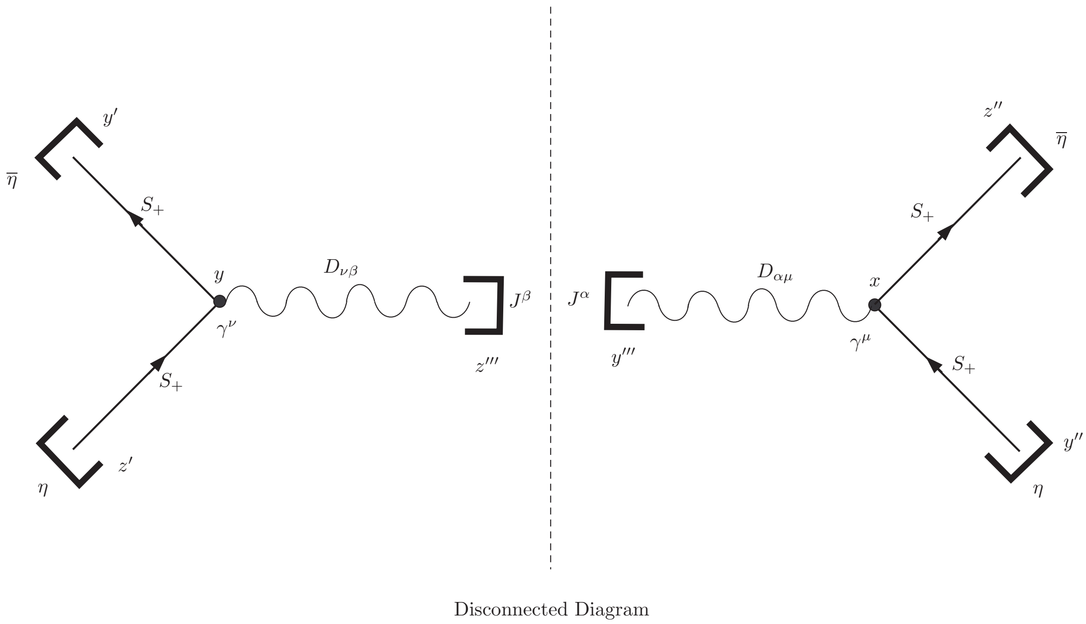

Figure B.6 Disconnected diagram.

$$\mathrm{VI} \equiv -\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')(\mathrm{d}z'')\Big[\overline{\eta}(y')\gamma^{\mu}S_{+}(x,y)S_{+}(y',x)\gamma^{\nu}S_{+}(y,z')\eta(z')\Big]$$

$$\times \Big[J^{\alpha}(y'')D_{\alpha\mu}(y'',x)D_{\nu\beta}(y,z'')J^{\beta}(z'')\Big]$$

$$\mathrm{VII} \equiv -\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')(\mathrm{d}z'')\Big[\overline{\eta}(y')\gamma^{\mu}S_{+}(x,y)S_{+}(y',x)\gamma^{\nu}S_{+}(y,z')\eta(z')\Big]$$

$$\times \Big[J^{\alpha}(y'')D_{\alpha\mu}(y'',x)D_{\nu\beta}(y,z'')J^{\beta}(z'')\Big]$$

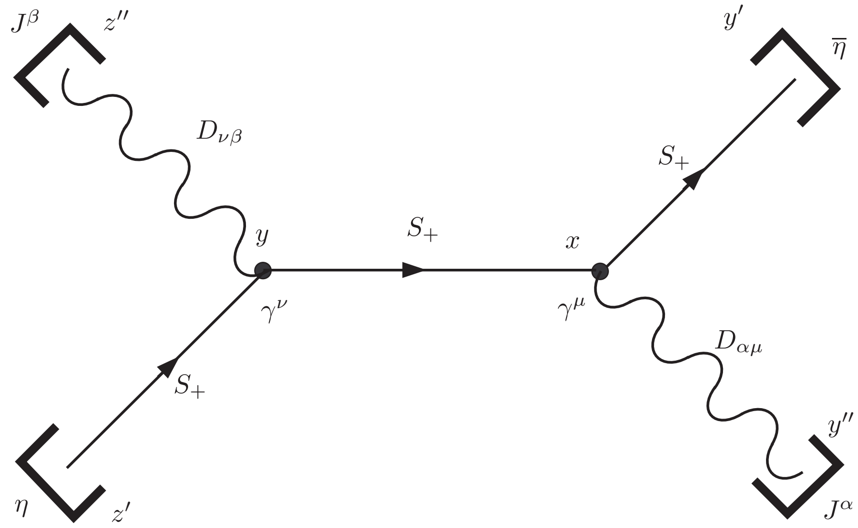

Figure B.7 A scattering process.

That is

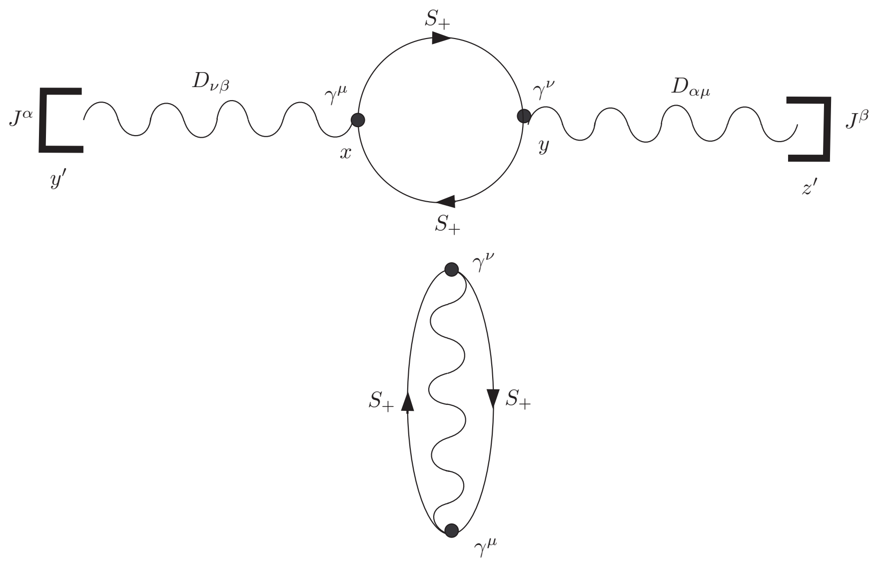

Figure B.8 Photon's self energy and a Bubble.

$$\text{I} = \text{III}$$

$$\text{IV} = \text{VI}$$

$$\therefore \quad \odot = 2\text{I} + \text{II} + 2\text{IV} + \text{V} + \text{VII} \tag{B.42}$$

From Eq. (B.31)

$$a_2 = -\frac{1}{2}a_1^2 - \frac{1}{2}\frac{1}{\langle 0_+ | 0_- \rangle_0}(\widehat{A})^2 \langle 0_+ | 0_- \rangle_0 \tag{B.43}$$

$\therefore$

$$a_2 = -\frac{1}{2}a_1^2 - \text{I} - \frac{1}{2}\text{II} - \text{IV} - \frac{1}{2}\text{V} - \frac{1}{2}\text{VII} \tag{B.44}$$

We have

$$a_1 = -\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)\left[\overline{\eta}(y)S_+(y,x)\gamma^\mu S_+(x,z)\eta(z)D_{\mu\nu}(x,x')J^\nu(x')\right]$$

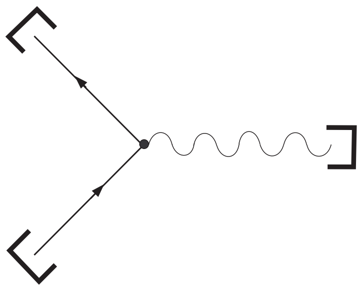

Figure B.9 Photon production scattering part.

$$a_1^2 = -\text{V} \text{ “Cancel disconnected part”}$$

$$a_2 = -\text{I} - \frac{1}{2}\text{II} - \text{IV} - \frac{1}{2}\text{VII}$$

where

$$A \equiv -\text{I} = \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')\left[\overline{\eta}(z)\gamma^\mu S_+(z,y)\gamma^\nu S_+(y,x))D_{\mu\nu}(x,y)S_+(x,z')\eta(z')\right]$$

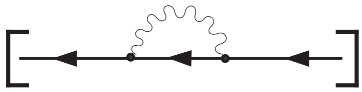

Figure B.10 Photon's self energy diagram.

$$B \equiv -\frac{1}{2}\mathbf{II} = \frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')(\mathrm{d}z'')\left[\overline{\eta}(y')\gamma^{\mu}S_{+}(y',y)\gamma^{\nu}S_{+}(y,z')\eta(z')\right]$$

$$\times\, D_{\mu\nu}(x,y)[\overline{\eta}(z'')S_{+}(z'',x)S_{+}(x,y'')\eta(y'')]$$

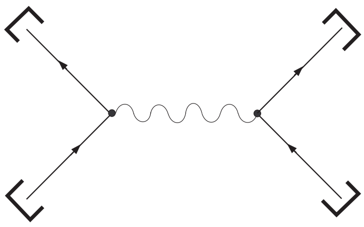

Figure B.11 Photon's self energy and a vacuum graph.

$$C \equiv -\mathbf{IV} = \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')(\mathrm{d}z'')\left[\overline{\eta}(y')\gamma^{\mu}S_{+}(y',y)\gamma^{\nu}S_{+}(y,x)S_{+}(x,y'')\eta(y'')\right]$$

$$\times\,[J^{\alpha}(z')D_{\alpha\mu}(z',x)D_{\nu\beta}(y,z'')J^{\beta}(z'')]$$

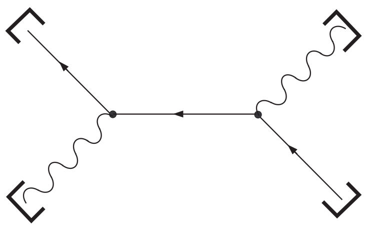

Figure B.12 A scattering process.

$$D \equiv -\frac{1}{2}\mathbf{VII} = -\frac{\mathrm{i}}{2}\int (\mathrm{d}x)(\mathrm{d}y)\Big[\gamma^\mu S_+(x,y)\gamma^\nu S_+(y,x)D_{\mu\nu}(x,y)\Big]$$

$$+\frac{1}{2}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')\big[J^\alpha(z')D_{\alpha\nu}(y',x)\gamma^\mu S_+(x,y)\gamma^\nu$$

$$\times S_+(y,x)D_{\nu\beta}(y,z')J^\beta(z')\big]$$

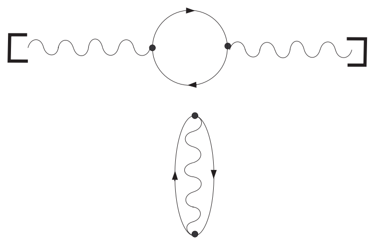

Figure B.13 A scattering process.

APPENDIX C

VACUUM-TO-VACUUM TRANSITION AMPLITUDE IN SCALAR ELECTRODYNAMICS

The vacuum-to-vacuum transition amplitude is derived to the leading order in scalar electrodynamics. Due to its complicated structure, it has never appeared in the literature to the best of my knowledge.

C.1 Vacuum-to-Vacuum Transition Amplitude

We start from the Lagrangian in Scalar Electrodynamics

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}^{\mathrm{EM}} + \mathscr{L}_{\mathrm{I}}, \tag{C.1}$$

where

$$\begin{aligned}
\mathscr{L}_0 &= -\frac{1}{2}(\partial^\mu \phi^\dagger \partial_\mu \phi + \partial^\mu \phi \partial_\mu \phi^\dagger) - \frac{1}{2} m_0 (\phi^\dagger \phi + \phi \phi^\dagger), \\
\mathscr{L}^{\mathrm{EM}} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \\
\mathscr{L}_{\mathrm{I}} &= e[\phi^\dagger \frac{\overleftrightarrow{\partial_\mu}}{\mathrm{i}} \phi] A^\mu - e^2 A_\mu A^\mu \phi \phi^\dagger,
\end{aligned}$$

and e is charge, J^μ are external source and A^μ is the vector potential.

To compute the vacuum-to-vacuum transition amplitude, we use the functional techniques, then interaction term in the Lagrangian is become the functional derivative operator. By replacing ϕ, ϕ', $A^\mu$and A_μ with the functional derivative operator

$$\phi(x) \to \frac{\delta}{\mathrm{i}\delta K^\dagger(x)},$$

$$\phi'(x) \to \frac{\delta}{\mathrm{i}\delta K(x)},$$

$$A^\mu \to \frac{\delta}{\mathrm{i}\delta J_\mu(x)},$$

$$A_\mu \to \frac{\delta}{\mathrm{i}\delta J^\mu(x)}.$$

Therefore, we obtain the interaction Lagrangian as:

$$\mathscr{L}_\mathrm{I} = e\frac{\delta}{\mathrm{i}\delta K(x)}\frac{\overleftrightarrow{\partial_\mu}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}$$

$$- e^2\frac{\delta}{\mathrm{i}\delta K(x)}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\frac{\delta}{\mathrm{i}\delta J^\mu(x)}.$$

C.2 Expression of The Vacuum-to-Vacuum Transition Amplitude in Scalar Electrodynamics

The vacuum-to-vacuum transition amplitude in term of the functional derivative operator, we write as

$$\exp\int(\mathrm{d}x)\left\{e\left[\frac{\delta}{\mathrm{i}\delta K(x)}\left(\frac{\partial_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\right)\frac{\delta}{\mathrm{i}\delta J_\mu(x)} - \left(\frac{\partial_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(x)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\right]\right.$$

$$\left.-e^2\frac{\delta}{\mathrm{i}\delta K(x)}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\frac{\delta}{\mathrm{i}\delta J^\mu(x)}\right\}\langle 0_+\,|\,0_-\rangle_0 \tag{C.2}$$

where

$$\langle 0_+\,|\,0_-\rangle_0 = \exp\mathrm{i}K^\dagger\varDelta_+K\exp\frac{\mathrm{i}}{2}J^\mu D_G^{\mu\nu}J_\nu,$$

$$K^\dagger \mathit{\Delta}_+ K = \int (\mathrm{d}x)(\mathrm{d}x') K^\dagger(x) \mathit{\Delta}_+(x-x') K(x'),$$

$$\mathit{\Delta}_+(x-x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2+m^2-\mathrm{i}\varepsilon},$$

We expand the vacuum-to-vacuum transition amplitude ($\langle 0_+ \,|\, 0_- \rangle_e$) into a power series according to

$$\langle 0_+ \,|\, 0_- \rangle_e = \exp \mathrm{i} \left[a_0 + e a_1 + e^2 a_2 + e^3 a_3 + \ldots \right]. \tag{C.3}$$

The first one, we find a_0 by setting $e = 0$. Therefore, Eq. (C.3) is

$$\begin{aligned} \langle 0_+ \,|\, 0_- \rangle \Big|_{e=0} = \mathrm{e}^{\mathrm{i}a_0} &= \langle 0_+ \,|\, 0_- \rangle_0 \\ &= \exp[\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x') K^\dagger(x) \mathit{\Delta}_+(x-x') K(x')] \\ &\times \exp \left[\frac{\mathrm{i}}{2} \int (\mathrm{d}x)(\mathrm{d}x') J_\mu(x) D_G^{\mu\nu}(x,x') J_\nu(x') \right]. \end{aligned} \tag{C.4}$$

We obtain

$$\begin{aligned} a_0 = &\int (\mathrm{d}x)(\mathrm{d}x') K^\dagger(x) \mathit{\Delta}_+(x-x') K(x') \\ &+ \frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}x') J_\mu(x) D_G^{\mu\nu}(x,x') J_\nu(x'). \end{aligned} \tag{C.5}$$

The second one, we find a_1 by using the first derivative $\langle 0_+ \,|\, 0_- \rangle$ with e and then give $e = 0$. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}e} \langle 0_+ \,|\, 0_- \rangle &= \mathrm{i}[a_1 + 2ea_2 + \ldots] \exp[a_0 + ea_1 + e^2 a_2 + \ldots] \\ &= \mathrm{i}[a_1 + 2ea_2 + \ldots] \langle 0_+ \,|\, 0_- \rangle, \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}e}\left\langle 0_{+}\,|\,0_{-}\right\rangle\bigg|_{e=0} = \mathrm{i}a_1\left\langle 0_{+}\,|\,0_{-}\right\rangle\big|_{e=0} = \mathrm{i}a_1\left\langle 0_{+}\,|\,0_{-}\right\rangle_0 , \tag{C.6}$$

therefore, we obtain a_1 as

$$a_1 = \frac{-\mathrm{i}}{\left\langle 0_{+}\,|\,0_{-}\right\rangle_0}\frac{\mathrm{d}}{\mathrm{d}e}\left\langle 0_{+}\,|\,0_{-}\right\rangle\bigg|_{e=0} . \tag{C.7}$$

To convenient in our computation of a_1, we rewrite Eq. (C.2) as

$$\left\langle 0_{+}\,|\,0_{-}\right\rangle \equiv \mathrm{e}^{\mathrm{i}[e\widehat{A}-e^2\widehat{B}]}\left\langle 0_{+}\,|\,0_{-}\right\rangle_0 , \tag{C.8}$$

where

$$\widehat{A} \equiv \int (\mathrm{d}x)\frac{\delta}{\mathrm{i}\delta K(x)}\left(\frac{\partial_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\right)\frac{\delta}{\mathrm{i}\delta J_\mu(x)} - \left(\frac{\partial_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(x)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}$$

$$\widehat{B} \equiv \int (\mathrm{d}x)\frac{\delta}{\mathrm{i}\delta K(x)}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\frac{\delta}{\mathrm{i}\delta J^\mu(x)} .$$

The derivative of Eq. (C.8) with respect to e, then set $e = 0$, given

$$\frac{\mathrm{d}}{\mathrm{d}e}\left\langle 0_{+}\,|\,0_{-}\right\rangle = \mathrm{i}(\widehat{A} - 2e\widehat{B})\mathrm{e}^{\mathrm{i}[e\widehat{A}-e^2\widehat{B}]}\left\langle 0_{+}\,|\,0_{-}\right\rangle_0 ,$$

$$\frac{\mathrm{d}}{\mathrm{d}e}\left\langle 0_{+}\,|\,0_{-}\right\rangle\bigg|_{e=0} = \mathrm{i}\widehat{A}\left\langle 0_{+}\,|\,0_{-}\right\rangle_0 . \tag{C.9}$$

Finally, we obtain a_1 in term of the functional derivative operator, given

$$a_1 = \frac{1}{\left\langle 0_{+}\,|\,0_{-}\right\rangle_0}\widehat{A}\left\langle 0_{+}\,|\,0_{-}\right\rangle_0 , \tag{C.10}$$

where

$$\left\langle 0_{+}\,|\,0_{-}\right\rangle_0 = \exp[\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}x')K^\dagger(x)\Delta_{+}(x,x')K(x')]$$

$$\times \exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')J_\mu(x)D_G^{\mu\nu}(x,x')J_\nu(x')\right]. \tag{C.11}$$

Next step, we calculate $\widehat{A}\,\langle 0_+\,|\,0_-\rangle_0$ in Eq. (C.10). We have

$$\begin{aligned}\widehat{A}\,\langle 0_+\,|\,0_-\rangle_0 = \int(\mathrm{d}x)\,&\left[\frac{\delta}{\mathrm{i}\delta K(x)}\left(\frac{\partial_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\right)\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\right.\\ &\left.-\left(\frac{\partial_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(x)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\right]\mathrm{e}^{\mathrm{i}K^\dagger\varDelta_+K}\mathrm{e}^{\mathrm{i}JDJ/2}.\end{aligned} \tag{C.12}$$

Here we note that $JDJ \equiv J^\mu D_{\mu\nu}J^\nu \equiv J_\mu D_G^{\mu\nu}J_\nu$. Then we calculate $\widehat{A}\,\langle 0_+\,|\,0_-\rangle_0$ with step by step. We start from

$$\begin{aligned}\frac{\delta}{\mathrm{i}\delta J^\mu(x)}\mathrm{e}^{\mathrm{i}JDJ/2} &= \frac{\delta}{\mathrm{i}\delta J^\mu(x)}\exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}y)(\mathrm{d}z)J^\alpha(y)D_{\alpha\beta}(y,z)J^\beta(z)\right]\\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}JDJ/2}\int(\mathrm{d}y)(\mathrm{d}z)\Big[\delta_\mu^\alpha\delta^4(y-x)D_{\alpha\beta}(y,z)J^\beta(z)\\ &\qquad + \delta_\mu^\beta\delta^4(z-x)J^\alpha(y)D_{\alpha\beta}(y,z)\Big]\\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}JDJ/2}\Big[\int(\mathrm{d}z)D_{\alpha\beta}(x,z)J^\beta(z)+\int(\mathrm{d}y)J^\alpha(y)D_{\alpha\beta}(y,x)\Big]\\ \frac{\delta}{\mathrm{i}\delta J^\mu(x)}\mathrm{e}^{\mathrm{i}JDJ/2} &= \left[\int(\mathrm{d}y)D_{\mu\nu}(x,y)J^\nu(y)\right]\mathrm{e}^{\mathrm{i}JDJ/2},\end{aligned} \tag{C.13}$$

by briefly

$$\frac{\delta}{\mathrm{i}\delta J^\mu}\mathrm{e}^{\mathrm{i}JDJ/2} \equiv [D_{\mu\nu}J^\nu]\,\mathrm{e}^{\mathrm{i}JDJ/2}. \tag{C.14}$$

For the derivative of $\langle 0_+\,|\,0_-\rangle_0$ with respect to K and $K^\dagger$ source, consider only $\mathrm{e}^{\mathrm{i}K^*\varDelta_+K}$ in $\langle 0_+\,|\,0_-\rangle_0$, we write as

$$\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\mathrm{e}^{\mathrm{i}K^*\varDelta_+K} = \frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\exp\left[\mathrm{i}\int(\mathrm{d}y)(\mathrm{d}z)K^\dagger(y)\varDelta_+(y,z)K(z)\right]$$

$$= \mathrm{e}^{\mathrm{i}K^\dagger \varDelta_+ K} \left[\int (\mathrm{d}y)(\mathrm{d}z)\delta^4(y-x)\varDelta_+(y,z)K(z) \right]. \tag{C.15}$$

Because of $\widehat{A}\,\langle 0_+ | 0_- \rangle_0$ has two term that it express as

$$\int (\mathrm{d}x) \left[\left(\frac{\partial^x_\mu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \right) \left(\frac{\partial^x_\mu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \right) \frac{\delta}{\mathrm{i}\delta J_\mu(x)} - \left(\frac{\partial^x_\mu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(x)} \right) \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \frac{\delta}{\mathrm{i}\delta J_\mu(x)} \right] \langle 0_+ | 0_- \rangle_0 \,,$$

then we separately consider above term. We have

$$\left(\frac{\partial^x_\mu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \right) \frac{\delta}{\mathrm{i}\delta J_\mu(x)} \langle 0_+ | 0_- \rangle_0 = \left[\int (\mathrm{d}y)(\mathrm{d}z) \left[\frac{1}{\mathrm{i}} \partial^x_\mu \delta^4(y-x) \right] \varDelta_+(y,z)K(z) \times \int (\mathrm{d}x') D_{\mu\beta}(x,x') J^\beta(x') \right] \langle 0_+ | 0_- \rangle_0 \,.$$

Then we operate above term with $\dfrac{\delta}{\mathrm{i}\delta K(x)}$, we obtain

$$\frac{\delta}{\mathrm{i}\delta K(x)} \left[\int (\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)\Delta_+(y,z)K(z) \left[\frac{1}{\mathrm{i}} \partial^x_\mu \delta^4(y-x) \right] D_{\mu\beta}(x,x') J^\beta(x') \right] \langle 0_+ | 0_- \rangle_0$$

$$= -\mathrm{i} \left[\int (\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)\varDelta_+(y,z)\delta^4(z-x) \left[\frac{1}{\mathrm{i}} \partial^x_\mu \delta^4(y-x) \right] D_{\mu\beta}(x,x') \times J^\beta(x') \right] \langle 0_+ | 0_- \rangle_0$$

$$+ \left[\int (\mathrm{d}x')(\mathrm{d}y)(\mathrm{d}z)\varDelta_+(y,z)K(z) \frac{1}{\mathrm{i}} [\partial^x_\mu \delta^4(y-x)] D_{\mu\beta}(x,x')] J^\beta(x') \times \int (\mathrm{d}y')(\mathrm{d}z') K^\dagger(y') \varDelta_+(y',z')\delta^4(z'-x) \right] \langle 0_+ | 0_- \rangle_0 \,,$$

therefore, the first term of Eq. (C.12) is written as

$$\int (\mathrm{d}x) \frac{\delta}{\mathrm{i}\delta K(x)} \left(\frac{\partial^x_\mu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \right) \frac{\delta}{\mathrm{i}\delta J_\mu(x)} \langle 0_+ | 0_- \rangle_0$$

$$= \Bigg\{ - \Bigg[\int (\mathrm{d}x')(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \delta^4(y-x)]\Delta_+(y,x)D_{\mu\beta}(x,x')J^\beta(x') \Bigg]$$

$$- \mathrm{i} \Bigg[\int (\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)K^\dagger(y')[\partial^x_\mu \delta^4(y-x)]\Delta_+(y',x)D_{\mu\beta}(x,x')$$

$$\times \Delta_+(y,z)K(z)J^\beta(x') \Bigg] \Bigg\} \langle 0_+ | 0_- \rangle_0 \,. \tag{C.16}$$

By using the property of $\partial_\mu(f(x)\delta^4(x-y))$, given

$$\partial_\mu(f(x)\delta^4(x-y)) = f(x)(\partial_\mu \delta^4(x-y)) + (\partial_\mu f(x))\delta^4(x-y),$$

where in this case, let $\partial_\mu(f(x)\delta^4(x-y)) = 0$. Hence we rewrite Eq. (C.16) as

$$= \Bigg\{ \int (\mathrm{d}x')(\mathrm{d}x)(\mathrm{d}y)\delta^4(y-x)[\partial^x_\mu \Delta_+(y,x)D_{\mu\beta}(x,x')]J^\beta(x')$$

$$+ \mathrm{i} \int (\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)K^\dagger(y')\delta^4(y-x)[\partial^x_\mu \Delta_+(y',x)D_{\mu\beta}(x,x')]$$

$$\times \Delta_+(y,z)K(z)J^\beta(x') \Bigg\} \langle 0_+ | 0_- \rangle_0$$

$$= \Bigg\{ \int (\mathrm{d}x')(\mathrm{d}x)(\mathrm{d}y)\delta^4(y-x)[\partial^x_\mu \Delta_+(y,x)]D_{\mu\beta}(x,x')J^\beta(x')$$

$$+ \int (\mathrm{d}x')(\mathrm{d}x)(\mathrm{d}y)\delta^4(y-x)\Delta_+(y,x)[\partial^x_\mu D_{\mu\beta}(x,x')]J^\beta(x')$$

$$+ \mathrm{i} \int (\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}z)K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')$$

$$+ \mathrm{i} \int (\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}z)K^\dagger(y')\Delta_+(y',x)\Delta_+(x,z)K(z)[\partial^x_\mu D_{\mu\beta}(x,x')]J^\beta(x')$$

$$\Bigg\} \langle 0_+ | 0_- \rangle_0 \,,$$

therefore, we obtain the explicit term of Eq. (C.16) as

$$= \Bigg\{ \int (\mathrm{d}x')(\mathrm{d}x)[\partial_\mu^x \Delta_+(y,x)]_{y=x} D_{\mu\beta}(x,x') J^\beta(x')$$

$$+ \int (\mathrm{d}x')(\mathrm{d}x) \Delta_+(x,x) [\partial_\mu^x D_{\mu\beta}(x,x')] J^\beta(x')$$

$$+ \mathrm{i} \int (\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}z) K^\dagger(y') [\partial_\mu^x \Delta_+(y',x)] \Delta_+(x,z) K(z) D_{\mu\beta}(x,x') J^\beta(x')$$

$$+ \mathrm{i} \int (\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}z) K^\dagger(y') \Delta_+(y',x) \Delta_+(x,z) K(z) [\partial_\mu^x D_{\mu\beta}(x,x')] J^\beta(x')$$

$$\Bigg\} \langle 0_+ | 0_- \rangle_0 \, .$$

And then we calculate the second term of Eq. (C.12), start with

$$\frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \mathrm{e}^{\mathrm{i}K^\dagger \Delta_+ K} = \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \exp\left[\mathrm{i} \int (\mathrm{d}y)(\mathrm{d}z) K^\dagger(y) \Delta_+(y,z) K(z)\right]$$

$$= \mathrm{e}^{\mathrm{i}K^\dagger \Delta_+ K} \left[\int (\mathrm{d}y)(\mathrm{d}z) \delta^4(y-x) \Delta_+(y,z) K(z)\right]$$

$$\frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \frac{\delta}{\mathrm{i}\delta J_\mu(x)} \langle 0_+ | 0_- \rangle_0 = \int (\mathrm{d}x')(\mathrm{d}z) \Delta_+(x,z) K(z) D_{\mu\beta}(x,x') J^\beta(x') \langle 0_+ | 0_- \rangle_0 \, . \tag{C.17}$$

Then we operate Eq. (C.17) with $\left(\frac{\partial_\mu^x}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(x)}\right)$, we obtain

$$\left(\frac{\partial_\mu^x}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(x)}\right) \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \frac{\delta}{\mathrm{i}\delta J_\mu(x)} \langle 0_+ | 0_- \rangle_0$$

$$= \frac{\partial_\mu^x}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(x)} \left[\int (\mathrm{d}x')(\mathrm{d}z) D_{\mu\beta}(x,x') J^\beta(x') \Delta_+(x,z) K(z) \langle 0_+ | 0_- \rangle_0\right]$$

$$= \Bigg\{ - \mathrm{i} \int (\mathrm{d}x')(\mathrm{d}z) \Delta_+(x,z) \left[\frac{1}{\mathrm{i}} \partial_\mu^x \delta^4(z-x)\right] D_{\mu\beta}(x,x') J^\beta(x')$$

$$+\int (\mathrm{d}x')(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')K^\dagger(y')$$

$$\times\Delta_+(y',z')\left[\frac{1}{\mathrm{i}}\partial^x_\mu\delta^4(z-x)\right]\Bigg\}\langle 0_+|0_-\rangle_0\,.$$

Using property in equation below Eq. (C.16), we have

$$\int(\mathrm{d}x)\left(\frac{\partial^x_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(x)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\langle 0_+|0_-\rangle_0$$

$$=\Bigg\{\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}z)\delta^4(z-x)[\partial^x_\mu\Delta_+(x,z)D_{\mu\beta}(x,x')]J^\beta(x')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')K(z)[\partial^x_\mu\Delta_+(x,z)D_{\mu\beta}(x,x')]$$

$$\times K^\dagger(y')\Delta_+(y',z')\delta^4(z'-x)J^\beta(x')\Bigg\}\langle 0_+|0_-\rangle_0\,,$$

therefore, the second term of Eq. (C.12) is written as

$$\int(\mathrm{d}x)\left(\frac{\partial^x_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(x)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\langle 0_+|0_-\rangle_0$$

$$=\Bigg\{\int(\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\beta}(x,x')J^\beta(x')$$

$$+\int(\mathrm{d}x)(\mathrm{d}x')\Delta_+(x,x)[\partial^x_\mu D_{\mu\beta}(x,x')]J^\beta(x')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)$$

$$\times\,D_{\mu\beta}(x,x')J^\beta(x')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\Delta_+(y',x)\Delta_+(x,z)K(z)$$

$$\times\,[\partial^x_\mu D_{\mu\beta}(x,x')]J^\beta(x')\Bigg\}\langle 0_+|0_-\rangle_0\,. \tag{C.18}$$

We can rewrite Eq. (C.12) as

$$\int(\mathrm{d}x)\Big[\frac{\delta}{\mathrm{i}\delta K(x)}\left(\frac{\partial^x_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\right)\frac{\delta}{\mathrm{i}\delta J_\mu(x)}-\left(\frac{\partial^x_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(x)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\Big]\langle 0_+|0_-\rangle_0$$

$$\begin{aligned}
=\Bigg\{&\int(\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu\Delta_+(y,x)]_{y=x}D_{\mu\beta}(x,x')J^\beta(x')\\
&+\int(\mathrm{d}x')(\mathrm{d}x)\Delta_+(x,x)[\partial^x_\mu D_{\mu\beta}(x,x')]J^\beta(x')\\
&+\mathrm{i}\int(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}z)K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')\\
&+\mathrm{i}\int(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x)(\mathrm{d}z)K^\dagger(y')\Delta_+(y',x)\Delta_+(x,z)K(z)[\partial^x_\mu D_{\mu\beta}(x,x')]J^\beta(x')\\
&-\int(\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\beta}(x,x')J^\beta(x')\\
&-\int(\mathrm{d}x)(\mathrm{d}x')\Delta_+(x,x)[\partial^x_\mu D_{\mu\beta}(x,x')J^\beta(x')\\
&-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\beta}(x,x')J^\beta(x')\\
&-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\Delta_+(y',x)\Delta_+(x,z)K(z)[\partial^x_\mu D_{\mu\beta}(x,x')]J^\beta(x')\\
&\Bigg\}\langle 0_+|0_-\rangle_0\,.
\end{aligned}\tag{C.19}$$

From the condition in Eq. (C.10), we write a_1 as

$$\begin{aligned}
a_1=&+\int(\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu\Delta_+(y,x)]_{y=x}D_{\mu\beta}(x,x')J^\beta(x')\\
&-\int(\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\beta}(x,x')J^\beta(x')\\
&+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')
\end{aligned}$$

$$- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z) K^\dagger(y') \mathit{\Delta}_+(y', x)[\partial_\mu^x \mathit{\Delta}_+(x, z)] K(z) D_{\mu\beta}(x, x') J^\beta(x'). \tag{C.20}$$

Next step, we will calculate a_2. So that we differential $\langle 0_+ | 0_- \rangle$ with respect to e, up to second order e^2. Giving

$$\langle 0_+ | 0_- \rangle = \exp \mathrm{i}[a_0 + e a_1 + e^2 a_2 + \ldots]$$

$$\frac{\mathrm{d}}{\mathrm{d}e} \langle 0_+ | 0_- \rangle = \mathrm{i}[a_1 + 2e a_2 + \ldots] \exp \mathrm{i}[a_0 + e a_1 + e^2 a_2 + \ldots]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2} \langle 0_+ | 0_- \rangle \Big|_{e=0} = 2\mathrm{i} a_1 \langle 0_+ | 0_- \rangle - [a_1 + 2e a_2 + \ldots]^2 \langle 0_+ | 0_- \rangle$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2} \langle 0_+ | 0_- \rangle \Big|_{e=0} = (2\mathrm{i} a_2 - a_1^2) \langle 0_+ | 0_- \rangle_0$$

$$2\mathrm{i} a_2 - a_1^2 = \frac{1}{\langle 0_+ | 0_- \rangle_0} \frac{\mathrm{d}^2}{\mathrm{d}e^2} \langle 0_+ | 0_- \rangle \Big|_{e=0}$$

$$a_2 = -\frac{\mathrm{i}}{2} a_1^2 - \frac{1}{2} \frac{\mathrm{i}}{\langle 0_+ | 0_- \rangle_0} \frac{\mathrm{d}^2}{\mathrm{d}e^2} \langle 0_+ | 0_- \rangle \Big|_{e=0}, \tag{C.21}$$

and we give $\langle 0_+ | 0_- \rangle \equiv \mathrm{e}^{\mathrm{i}[e\widehat{A} - e^2 \widehat{B}]} \langle 0_+ | 0_- \rangle_0$. Similarly above term, we have

$$\frac{\mathrm{d}}{\mathrm{d}e} \langle 0_+ | 0_- \rangle = \mathrm{i}[\widehat{A} - 2e\widehat{B}] \mathrm{e}^{\mathrm{i}[e\widehat{A} - e^2 \widehat{B}]} \langle 0_+ | 0_- \rangle_0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2} \langle 0_+ | 0_- \rangle = [-2\mathrm{i}\widehat{B} - [\widehat{A} - 2e\widehat{B}]^2] \mathrm{e}^{\mathrm{i}[e\widehat{A} - e^2 \widehat{B}]} \langle 0_+ | 0_- \rangle_0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}e^2} \langle 0_+ | 0_- \rangle \Big|_{e=0} = -2[\mathrm{i}\widehat{B} + \frac{1}{2}(\widehat{A})^2] \langle 0_+ | 0_- \rangle_0 . \tag{C.22}$$

We rewrite Eq. (C.21) as

$$a_2 = -\frac{\mathrm{i}}{2} a_1^2 + \frac{1}{\langle 0_+ | 0_- \rangle_0} [-\widehat{B} + \frac{\mathrm{i}}{2} (\widehat{A})^2] \langle 0_+ | 0_- \rangle_0 . \tag{C.23}$$

Find $(\widehat{A})^2 \langle 0_+ | 0_- \rangle_0$

$$= \int (\mathrm{d}x)(\mathrm{d}y) \left[\frac{\delta}{\mathrm{i}\delta K(y)} \left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(y)} \right) \frac{\delta}{\mathrm{i}\delta J_\nu(y)} - \left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)} \right) \frac{\delta}{\mathrm{i}\delta K^\dagger(y)} \frac{\delta}{\mathrm{i}\delta J_\nu(y)} \right]$$

$$\times \left[\frac{\delta}{\mathrm{i}\delta K(x)} \left(\frac{\partial^x_\mu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \right) \frac{\delta}{\mathrm{i}\delta J_\mu(x)} - \left(\frac{\partial^x_\mu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(x)} \right) \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \frac{\delta}{\mathrm{i}\delta J_\mu(x)} \right] \langle 0_+ | 0_- \rangle_0 \,.$$

From Eq. (C.20), we set

$$\begin{aligned} \odot^\mu(x) = & \int (\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu \Delta_+(y',x)]_{y'=x} D_{\mu\beta}(x,x') J^\beta(x') \\ & - \int (\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu \Delta_+(x,z)]_{z=x} D_{\mu\beta}(x,x') J^\beta(x') \\ & + \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z) K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)] \Delta_+(x,z) K(z) D_{\mu\beta}(x,x') J^\beta(x') \\ & - \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z) K^\dagger(y') \Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)] K(z) D_{\mu\beta}(x,x') J^\beta(x'). \end{aligned} \tag{C.24}$$

Then we operate Eq. (C.24) with $\dfrac{\delta}{\mathrm{i}\delta J_\nu(y)}$, given by

$$\frac{\delta}{\mathrm{i}\delta J_\nu(y)} [\odot^\mu(x) \langle 0_+ | 0_- \rangle_0] = \left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)} \odot^\mu (x) \right] \langle 0_+ | 0_- \rangle_0 + \odot^\mu(x) \left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)} \langle 0_+ | 0_- \rangle_0 \right]. \tag{C.25}$$

We define

$$\odot^{\mu\nu}(x,y) = \frac{\delta}{\mathrm{i}\delta J_\nu(y)} [\odot^\mu(x) \langle 0_+ | 0_- \rangle_0], \tag{C.26}$$

rewrite it as

$$\begin{aligned} \odot^{\mu\nu}(x,y) = \frac{\delta}{\mathrm{i}\delta J_\nu(y)} \Bigg\{ & \int (\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu \Delta_+(y',x)]_{y'=x} D_{\mu\beta}(x,x') J^\beta(x') \\ & - \int (\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu \Delta_+(x,z)]_{z=x} D_{\mu\beta}(x,x') J^\beta(x') \end{aligned}$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^{\dagger}(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^{\beta}(x')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^{\dagger}(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\beta}(x,x')J^{\beta}(x')$$

$$\Big\}\,\langle 0_+\,|\,0_-\rangle_0+\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right],$$

and we have

$$\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right]=\left[\int(\mathrm{d}x'')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')\right]\langle 0_+\,|\,0_-\rangle_0\,.$$

So that

$$\begin{aligned}\odot^{\mu\nu}(x,y)=\Big\{&-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu\Delta_+(y',x)]_{y'=x}D_{\mu\beta}(x,x')\delta^{\beta}_{\nu}\delta^4(x'-y)\\&+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\beta}(x,x')\delta^{\beta}_{\nu}\delta^4(x'-y)\\&+\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^{\dagger}(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)\\&\quad\times D_{\mu\beta}(x,x')\delta^{\beta}_{\nu}\delta^4(x'-y)\\&-\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z)K^{\dagger}(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)\\&\quad\times D_{\mu\beta}(x,x')\delta^{\beta}_{\nu}\delta^4(x'-y)\\&\Big\}\,\langle 0_+\,|\,0_-\rangle_0+\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right].\end{aligned}\tag{C.27}$$

We define

$$\odot^{\mu\nu}(x,y)=\Big\{-\mathrm{i}\int(\mathrm{d}x)[\partial^x_\mu\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)[\partial^x_\mu \Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\Big\}\,\langle 0_+\,|\,0_-\rangle_0 + \odot^\mu(x)\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right]. \tag{C.28}$$

We have

$$\begin{aligned}
&\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^{\mu\nu}(x,y)\\
&= \left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\langle 0_+\,|\,0_-\rangle_0\\
&+\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\,\langle 0_+\,|\,0_-\rangle_0\right]\\
&+\left[\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right]\\
&+\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right]\\
&+\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right]\\
&+\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right]\\
&+\left[\frac{\delta}{\mathrm{i}\delta K(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right]\\
&+\odot^\mu(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right].
\end{aligned} \tag{C.29}$$

To convenient, let

$$\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^{\mu\nu}(x,y) = (\mathrm{I.1}) + (\mathrm{I.2}) + (\mathrm{I.3}) + (\mathrm{I.4})$$

$$+ (\mathrm{I.5}) + (\mathrm{I.6}) + (\mathrm{I.7}) + (\mathrm{I.8}).$$

To express Eq. (C.29), we start from

$$\begin{aligned}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\left\langle 0_+ \,|\, 0_-\right\rangle_0 &= \frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\exp\left[\mathrm{i}\int(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y'')\varDelta_+(y'',z)K(z')\right]\mathrm{e}^{\mathrm{i}JDJ/2}\\ &= \mathrm{e}^{\mathrm{i}K^\dagger\varDelta_+K}\left[\int(\mathrm{d}y'')(\mathrm{d}z')\delta^4(y''-y)\varDelta_+(y'',z')K(z')\right]\mathrm{e}^{\mathrm{i}JDJ/2}\\ &= \left[\int(\mathrm{d}y'')(\mathrm{d}z')\delta^4(y''-y)\varDelta_+(y'',z')K(z')\right]\left\langle 0_+ \,|\, 0_-\right\rangle_0 . \qquad \text{(C.30)}\end{aligned}$$

Then we find each term in Eq. (C.29)

$$(\mathrm{I.1})\quad \left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left\langle 0_+ \,|\, 0_-\right\rangle_0 ,$$

we have

$$\begin{aligned}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x) = \Big\{ &-\mathrm{i}\int(\mathrm{d}x)[\partial^x_\mu\varDelta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\\ &+\mathrm{i}\int(\mathrm{d}x)[\partial^x_\mu\varDelta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\\ &+\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')[\partial^x_\mu\varDelta_+(y',x)]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)\\ &-\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu\varDelta_+(x,z)]K(z)D_{\mu\nu}(x,y)\\ &\Big\}\left\langle 0_+ \,|\, 0_-\right\rangle_0 + \odot^\mu(x)\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\left\langle 0_+ \,|\, 0_-\right\rangle_0\right],\end{aligned}$$

then operate above term with $\dfrac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}$, we obtain

$$\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)$$

$$=\Big\{-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)\delta^4(y'-y)[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)\delta^4(y'-y)\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)\Big\}\left\langle 0_+\,|\,0_-\right\rangle_0,$$

and then

$$\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)$$

$$=-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right][\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y).$$

Then operate above term with $\dfrac{\delta}{\mathrm{i}\delta K(y)}$, we obtain ((I.1))

$$\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\right]\left\langle 0_+\,|\,0_-\right\rangle_0$$

$$=\Big\{-\int(\mathrm{d}x)(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right][\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,y)D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,y)]D_{\mu\nu}(x,y)\Big\}\left\langle 0_+\,|\,0_-\right\rangle_0. \tag{C.31}$$

And

$$\text{(I.2)}\quad\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left\langle 0_+\,|\,0_-\right\rangle_0\right].$$

We have

$$\frac{\delta}{\mathrm{i}\delta K(y)}\,\langle 0_+\,|\,0_-\rangle_0 = \left[\int (\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',y)\right]\langle 0_+\,|\,0_-\rangle_0\,,$$

therefore, we obtain ((I.2))

$$\left[\left(\frac{\partial_\nu^y}{\mathrm{i}}\;\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\,\langle 0_+\,|\,0_-\rangle_0\right]$$

$$= \Big\{-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y'-y)\right][\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)$$

$$\times\, K(z)D_{\mu\nu}(x,y)K^\dagger(y'')\Delta_+(y'',y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y'-y)\right]\Delta_+(y',x)[\partial_\mu^x\Delta_+(x,z)]$$

$$\times\, K(z)D_{\mu\nu}(x,y)K^\dagger(y'')\Delta_+(y'',y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,. \qquad \text{(C.32)}$$

And

$$\text{(I.3)}\;\left[\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right].$$

We have

$$\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x) = \Big\{$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,y)D_{\mu\nu}(x,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial_\mu^x\Delta_+(x,y)]D_{\mu\nu}(x,y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,,$$

and

$$\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0 = \int(\mathrm{d}y'')(\mathrm{d}z')\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\varDelta_+(y'',z')K(z')\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,.$$

Therefore, we obtain ((I.3))

$$\begin{aligned}
&\left[\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right]\\
&=\Big\{-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y')[\partial_\mu^x\varDelta_+(y',x)]\varDelta_+(x,y)D_{\mu\nu}(x,y)\\
&\qquad\times\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\varDelta_+(y'',z')K(z')\\
&\qquad+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,y)]D_{\mu\nu}(x,y)\\
&\qquad\times\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\varDelta_+(y'',z')K(z')\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,.
\end{aligned}\tag{C.33}$$

And

$$\text{(I.4)}\;\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right].$$

We have

$$\begin{aligned}
&\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)=\Big\{-\mathrm{i}\int(\mathrm{d}x)[\partial_\mu^x\varDelta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\\
&+\mathrm{i}\int(\mathrm{d}x)[\partial_\mu^x\varDelta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\\
&+\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')[\partial_\mu^x\varDelta_+(y',x)]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)\\
&-\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,z)]K(z)D_{\mu\nu}(x,y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,,
\end{aligned}$$

and

$$\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0 = \Big\{-\,\mathrm{i}\int(\mathrm{d}y'')\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\Delta_+(y'',y)$$

$$+\int(\mathrm{d}y'')(\mathrm{d}y''')(\mathrm{d}z')\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\Delta_+(y'',z')K(z')$$

$$\times\, K^\dagger(y''')\Delta_+(y''',y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,.$$

Therefore, we obtain ((I.4))

$$\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right]$$

$$=\Big\{-\int(\mathrm{d}x)(\mathrm{d}y'')[\partial_\mu^x\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\Delta_+(y'',y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')[\partial_\mu^x\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]$$

$$\times\,\Delta_+(y'',z')K(z')K^\dagger(y''')\Delta_+(y''',y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y'')[\partial_\mu^x\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\Delta_+(y'',y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')[\partial_\mu^x\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]$$

$$\times\,\Delta_+(y'',z')K(z')K^\dagger(y''')\Delta_+(y''',y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\,D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\Delta_+(y'',y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\, D_{\mu\nu}(x,y)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y''-y)\right]\mathit{\Delta}_+(y'',z')K(z')K^\dagger(y''')\mathit{\Delta}_+(y''',y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')K^\dagger(y')\mathit{\Delta}_+(y',x)[\partial^x_\mu\mathit{\Delta}_+(x,z)]K(z)$$

$$\times\, D_{\mu\nu}(x,y)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y''-y)\right]\mathit{\Delta}_+(y'',y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}y''')K^\dagger(y')\mathit{\Delta}_+(y',x)[\partial^x_\mu\mathit{\Delta}_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y''-y)\right]\mathit{\Delta}_+(y'',z')K(z')K^\dagger(y''')\mathit{\Delta}_+(y''',y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,. \quad \text{(C.34)}$$

And

$$\text{(I.5)}\quad\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right].$$

We have

$$\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)=\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\left[\partial^x_\mu\mathit{\Delta}_+(y',x)\right]$$

$$\times\,\mathit{\Delta}_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')$$

$$-\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\mathit{\Delta}_+(y',x)\left[\partial^x_\mu\mathit{\Delta}_+(x,z)\right]K(z)$$

$$\times\, D_{\mu\beta}(x,x')J^\beta(x'),$$

and

$$\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)=-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]$$

$$\times\left[\partial^x_\mu\mathit{\Delta}_+(y',x)\right]\mathit{\Delta}_+(x,y)D_{\mu\beta}(x,x')J^\beta(x')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\varDelta_+(y',x)\left[\partial^x_\mu\varDelta_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x').$$

Therefore, we obtain ((I.5))

$$\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\left\langle 0_+\,|\,0_-\right\rangle_0\right]$$

$$=\Big\{-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\left[\partial^x_\mu\varDelta_+(y',x)\right]\varDelta_+(x,y)$$

$$\times\,D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\varDelta_+(y',x)\left[\partial^x_\mu\varDelta_+(x,y)\right]$$

$$\times\,D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')\Big\}\left\langle 0_+\,|\,0_-\right\rangle_0. \tag{C.35}$$

And

$$(\mathrm{I}.6)\ \left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\left\langle 0_+\,|\,0_-\right\rangle_0\right].$$

We have

$$\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)=\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\left[\partial^x_\mu\varDelta_+(y',x)\right]$$

$$\times\,\varDelta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')$$

$$-\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\varDelta_+(y',x)\left[\partial^x_\mu\varDelta_+(x,z)\right]$$

$$\times\,K(z)D_{\mu\beta}(x,x')J^\beta(x').$$

Therefore, we obtain ((I.6))

$$\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+\,|\,0_-\rangle_0\right]$$

$$= \Big\{\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\left[\partial^x_\mu\Delta_+(y',x)\right]$$

$$\times K^\dagger(y'')\Delta_+(y'',y)\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}y')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(y'-y)\right]\Delta_+(y',x)$$

$$\times K^\dagger(y'')\Delta_+(y'',y)\left[\partial^x_\mu\Delta_+(x,z)\right]K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$\Big\}\langle 0_+\,|\,0_-\rangle_0\,. \tag{C.36}$$

And

$$\text{(I.7)}\quad\left[\frac{\delta}{\mathrm{i}\delta K(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+\,|\,0_-\rangle_0\right].$$

We have

$$\frac{\delta}{\mathrm{i}\delta K(y)}\odot^\mu(x) = \int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')K^\dagger(y')\left[\partial^x_\mu\Delta_+(y',x)\right]$$

$$\times\Delta_+(x,y)D_{\mu\beta}(x,x')J^\beta(x')$$

$$-\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)\left[\partial^x_\mu\Delta_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x').$$

Therefore, we obtain ((I.7))

$$\left[\frac{\delta}{\mathrm{i}\delta K(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+\,|\,0_-\rangle_0\right]$$

$$= \Big\{\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y')\left[\partial^x_\mu\Delta_+(y',x)\right]\Delta_+(x,y)$$

$$\times \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \Delta_+(y'',z')K(z')D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y')\Delta_+(y',x)\left[\partial_\mu^x\Delta_+(x,y)\right]$$

$$\times \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \Delta_+(y'',z')K(z')D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$\Big\} \langle 0_+ \,|\, 0_-\rangle_0 \,. \tag{C.37}$$

So that, we rewrite Eq. (C.22) as

$$\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)} \odot^\mu (x) \langle 0_+ \,|\, 0_-\rangle_0$$

$$= \Big\{ -\int(\mathrm{d}x)(\mathrm{d}y')\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y'-y)\right][\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,y)D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y')\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y'-y)\right]\Delta_+(y',x)[\partial_\mu^x\Delta_+(x,y)]D_{\mu\nu}(x,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y'-y)\right][\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$\times \int(\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y'-y)\right]\Delta_+(y',x)[\partial_\mu^x\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times \int(\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,y)D_{\mu\nu}(x,y)$$

$$\times \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]\Delta_+(y'',z')K(z')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y')\Delta_+(y',x)[\partial_\mu^x\Delta_+(x,y)]D_{\mu\nu}(x,y)$$

$$\times \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \varDelta_+(y'',z')K(z')$$

$$- \int (\mathrm{d}x)(\mathrm{d}y'')[\partial_\mu^x \varDelta_+(y',x)]_{y'=x} D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \varDelta_+(y'',y)$$

$$- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')[\partial_\mu^x \varDelta_+(y',x)]_{y'=x} D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]$$

$$\times \varDelta_+(y'',z')K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y'')[\partial_\mu^x \varDelta_+(x,z)]_{z=x} D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \varDelta_+(y'',y)$$

$$+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')[\partial_\mu^x \varDelta_+(x,z)]_{z=x} D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right]$$

$$\times \varDelta_+(y'',z')K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')K^\dagger(y')[\partial_\mu^x \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \varDelta_+(y'',y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')K^\dagger(y')[\partial_\mu^x \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \varDelta_+(y'',z')K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x \varDelta_+(x,z)]K(z)$$

$$\times D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \varDelta_+(y'',y)$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}y''')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x \varDelta_+(x,z)]K(z)$$

$$\times D_{\mu\nu}(x,y) \left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(y''-y)\right] \varDelta_+(y'',z')K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$
\begin{aligned}
&- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y') \left[\frac{\partial_\nu^y}{\mathrm{i}} \delta^4(y'-y) \right] \left[\partial_\mu^x \Delta_+(y',x) \right] \Delta_+(x,y) \\
&\quad \times D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \\
&+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y') \left[\frac{\partial_\nu^y}{\mathrm{i}} \delta^4(y'-y) \right] \Delta_+(y',x) \left[\partial_\mu^x \Delta_+(x,y) \right] \\
&\quad \times D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \\
&+ \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}y') \left[\frac{\partial_\nu^y}{\mathrm{i}} \delta^4(y'-y) \right] \left[\partial_\mu^x \Delta_+(y',x) \right] \\
&\quad \times K^\dagger(y'') \Delta_+(y'',y) \Delta_+(x,z) K(z) D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \\
&- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}y') \left[\frac{\partial_\nu^y}{\mathrm{i}} \delta^4(y'-y) \right] \Delta_+(y',x) \\
&\quad \times K^\dagger(y'') \Delta_+(y'',y) \left[\partial_\mu^x \Delta_+(x,z) \right] K(z) D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \\
&+ \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')(\mathrm{d}y'') K^\dagger(y') \left[\partial_\mu^x \Delta_+(y',x) \right] \Delta_+(x,y) \\
&\quad \times \left[\frac{\partial_\nu^y}{\mathrm{i}} \delta^4(y''-y) \right] \Delta_+(y'',z') K(z') D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \\
&- \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')(\mathrm{d}y'') K^\dagger(y') \Delta_+(y',x) \left[\partial_\mu^x \Delta_+(x,y) \right] \\
&\quad \times \left[\frac{\partial_\nu^y}{\mathrm{i}} \delta^4(y''-y) \right] \Delta_+(y'',z') K(z') D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \\
&\Bigg\} \langle 0_+ | 0_- \rangle_0 + \odot^\mu(x) \left[\frac{\delta}{\mathrm{i}\delta K(y)} \left(\frac{\partial_\nu^y}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(y)} \right) \frac{\delta}{\mathrm{i}\delta J_\nu(y)} \langle 0_+ | 0_- \rangle_0 \right]. \qquad \text{(C.38)}
\end{aligned}
$$

Then we have

$$
\int (\mathrm{d}y) \frac{\delta}{\mathrm{i}\delta K(y)} \left(\frac{\partial_\nu^y}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^\dagger(y)} \right) \frac{\delta}{\mathrm{i}\delta J_\nu(y)} \odot^\mu(x) \langle 0_+ | 0_- \rangle_0
$$

$$= \Big\{ -\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')\delta^4(y'-y)[\partial^x_\mu \Delta_+(y',x)]\,[\partial^y_\nu \Delta_+(x,y)D_{\mu\nu}(x,y)]$$

$$+\,\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')\delta^4(y'-y)\Delta_+(y',x)\partial^y_\nu\left[[\partial^x_\mu \Delta_+(x,y)]D_{\mu\nu}(x,y)\right]$$

$$+\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z)(\mathrm{d}y'')\delta^4(y'-y)[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)K^\dagger(y'')$$

$$\times\,\partial^y_\nu\,[\Delta_+(y'',y)D_{\mu\nu}(x,y)]$$

$$-\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z)\delta^4(y'-y)\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)K^\dagger(y'')$$

$$\times\,\partial^y_\nu\,[\Delta_+(y'',y)D_{\mu\nu}(x,y)]$$

$$+\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\partial^y_\nu\,[\Delta_+(x,y)D_{\mu\nu}(x,y)]$$

$$\times\,\delta^4(y''-y)\Delta_+(y'',z')K(z')$$

$$-\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y')\Delta_+(y',x)\partial^y_\nu\left[[\partial^x_\mu \Delta_+(x,y)]D_{\mu\nu}(x,y)\right]$$

$$\times\,\delta^4(y''-y)\Delta_+(y'',z')K(z')$$

$$-\,\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')[\partial^x_\mu \Delta_+(y',x)]_{y'=x}\partial^y_\nu\,[D_{\mu\nu}(x,y)\Delta_+(y'',y)]\,\delta^4(y''-y)$$

$$+\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')[\partial^x_\mu \Delta_+(y',x)]_{y'=x}\partial^y_\nu\,[D_{\mu\nu}(x,y)\Delta_+(y''',y)]$$

$$\times\,\delta^4(y''-y)\Delta_+(y'',z')K(z')K^\dagger(y''')$$

$$+\,\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y'')[\partial^x_\mu \Delta_+(x,z)]_{z=x}\partial^y_\nu\,[D_{\mu\nu}(x,y)\Delta_+(y'',y)]\,\delta^4(y''-y)$$

$$-\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')[\partial^x_\mu \Delta_+(x,z)]_{z=x}\partial^y_\nu\,[D_{\mu\nu}(x,y)\Delta_+(y''',y)]$$

$$\times\,\delta^4(y''-y)\Delta_+(y'',z')K(z')K^\dagger(y''')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\,\partial^y_\nu\left[D_{\mu\nu}(x,y)\Delta_+(y'',y)\right]\delta^4(y''-y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\,\delta^4(y''-y)\Delta_+(y'',z')K(z')K^\dagger(y''')\partial^y_\nu\left[D_{\mu\nu}(x,y)\Delta_+(y''',y)\right]$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)$$

$$\times\,\delta^4(y''-y)\partial^y_\nu\left[D_{\mu\nu}(x,y)\Delta_+(y'',y)\right]$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y'')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)$$

$$\times\,\delta^4(y''-y)\Delta_+(y'',z')K(z')K^\dagger(y''')\partial^y_\nu\left[D_{\mu\nu}(x,y)\Delta_+(y''',y)\right]$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}x'')\delta^4(y'-y)\left[\partial^x_\mu\Delta_+(y',x)\right]\partial^y_\nu\left[D_{\nu\alpha}(y,x'')\Delta_+(x,y)\right]$$

$$\times\,D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y')\delta^4(y'-y)\Delta_+(y',x)\partial^y_\nu\left[D_{\nu\alpha}(y,x'')\left[\partial^x_\mu\Delta_+(x,y)\right]\right]$$

$$\times\,D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}y')\delta^4(y'-y)\left[\partial^x_\mu\Delta_+(y',x)\right]$$

$$\times\,K^\dagger(y'')\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')\partial^y_\nu\left[D_{\nu\alpha}(y,x'')\Delta_+(y'',y)\right]J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}y')\delta^4(y'-y)\Delta_+(y',x)$$

$$\times\,K^\dagger(y'')\left[\partial^x_\mu\Delta_+(x,z)\right]K(z)D_{\mu\beta}(x,x')J^\beta(x')\partial^y_\nu\left[D_{\nu\alpha}(y,x'')\Delta_+(y'',y)\right]J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y')\left[\partial_{\mu}^{x}\varDelta_{+}(y',x)\right]$$

$$\times\,\delta^{4}(y''-y)\partial_{\nu}^{y}\left[\varDelta_{+}(x,y)D_{\nu\alpha}(y,x'')\right]\varDelta_{+}(y'',z')K(z')D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y')\varDelta_{+}(y',x)\delta^{4}(y''-y)\varDelta_{+}(y'',z')$$

$$\times\,K(z')D_{\mu\beta}(x,x')J^{\beta}(x')\partial_{\nu}^{y}\left[\left[\partial_{\mu}^{x}\varDelta_{+}(x,y)\right]D_{\nu\alpha}(y,x'')\right]J^{\alpha}(x'')$$

$$\Bigg\}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}+\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right] \tag{C.39}$$

Simplify above term as

$$\int(\mathrm{d}y)\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$=\Big\{-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial_{\mu}^{x}\varDelta_{+}(y,x)]\,[\partial_{\nu}^{y}\varDelta_{+}(x,y)]\,D_{\mu\nu}(x,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial_{\mu}^{x}\varDelta_{+}(y,x)]\varDelta_{+}(x,y)\,[\partial_{\nu}^{y}D_{\mu\nu}(x,y)]$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)\varDelta_{+}(y,x)\left[\partial_{\nu}^{y}\partial_{\mu}^{x}\varDelta_{+}(x,y)\right]D_{\mu\nu}(x,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)\varDelta_{+}(y,x)[\partial_{\mu}^{x}\varDelta_{+}(x,y)]\,[\partial_{\nu}^{y}D_{\mu\nu}(x,y)]$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^{\dagger}(y'')\varDelta_{+}(y'',y)[\partial_{\mu}^{x}\varDelta_{+}(y,x)]\varDelta_{+}(x,z)K(z)\,[\partial_{\nu}^{y}D_{\mu\nu}(x,y)]$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^{\dagger}(y'')\,[\partial_{\nu}^{y}\varDelta_{+}(y'',y)]\,[\partial_{\mu}^{x}\varDelta_{+}(y,x)]\varDelta_{+}(x,z)K(z)D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^{\dagger}(y'')\,[\partial_{\nu}^{y}\varDelta_{+}(y'',y)]\,\varDelta_{+}(y,x)[\partial_{\mu}^{x}\varDelta_{+}(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^{\dagger}(y'')\varDelta_{+}(y'',y)\varDelta_{+}(y,x)[\partial_{\mu}^{x}\varDelta_{+}(x,z)]K(z)\,[\partial_{\nu}^{y}D_{\mu\nu}(x,y)]$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')[\partial^{x}_{\mu}\Delta_{+}(y',x)]\,[\partial^{y}_{\nu}\Delta_{+}(x,y)]\,\Delta_{+}(y,z')K(z')D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')[\partial^{x}_{\mu}\Delta_{+}(y',x)]\Delta_{+}(x,y)\Delta_{+}(y,z')K(z')\,[\partial^{y}_{\nu}D_{\mu\nu}(x,y)]$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')\Delta_{+}(y',x)\,\left[\partial^{y}_{\nu}\partial^{x}_{\mu}\Delta_{+}(x,y)\right]\,\Delta_{+}(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')\Delta_{+}(y',x)[\partial^{x}_{\mu}\Delta_{+}(x,y)]\Delta_{+}(y,z')K(z')\,[\partial^{y}_{\nu}D_{\mu\nu}(x,y)]$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\Delta_{+}(y',x)]_{y'=x}D_{\mu\nu}(x,y)\,[\partial^{y}_{\nu}\Delta_{+}(y'',y)]_{y''=y}$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\Delta_{+}(y',x)]_{y'=x}\,[\partial^{y}_{\nu}D_{\mu\nu}(x,y)]\,\Delta_{+}(y,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^{x}_{\mu}\Delta_{+}(y',x)]_{y'=x}\,[\partial^{y}_{\nu}D_{\mu\nu}(x,y)]\,K^{\dagger}(y''')\Delta_{+}(y''',y)$$

$$\times\,\Delta_{+}(y,z')K(z')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^{x}_{\mu}\Delta_{+}(y',x)]_{y'=x}D_{\mu\nu}(x,y)K^{\dagger}(y''')\,[\partial^{y}_{\nu}\Delta_{+}(y''',y)]$$

$$\times\,\Delta_{+}(y,z')K(z')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\Delta_{+}(x,z)]_{z=x}\,[\partial^{y}_{\nu}D_{\mu\nu}(x,y)]\,\Delta_{+}(y,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\Delta_{+}(x,z)]_{z=x}D_{\mu\nu}(x,y)\,[\partial^{y}_{\nu}\Delta_{+}(y'',y)]_{y''=y}$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^{x}_{\mu}\Delta_{+}(x,z)]_{z=x}\,[\partial^{y}_{\nu}D_{\mu\nu}(x,y)]\,K^{\dagger}(y''')\Delta_{+}(y''',y)$$

$$\times\,\Delta_{+}(y,z')K(z')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^{x}_{\mu}\Delta_{+}(x,z)]_{z=x}D_{\mu\nu}(x,y)\,\left[\partial^{y}_{\nu}K^{\dagger}(y''')\Delta_{+}(y''',y)\right]$$

$$\times\,\Delta_{+}(y,z')K(z')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial_\mu^x\varDelta_+(y',x)]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$\times\left[\partial_\nu^y\varDelta_+(y'',y)\right]_{y''=y}$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial_\mu^x\varDelta_+(y',x)]\varDelta_+(x,z)K(z)\left[\partial_\nu^y D_{\mu\nu}(x,y)\right]\varDelta_+(y,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial_\mu^x\varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times K^\dagger(y''')\varDelta_+(y''',y)\varDelta_+(y,z')K(z')\left[\partial_\nu^y D_{\mu\nu}(x,y)\right]$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial_\mu^x\varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times K^\dagger(y''')\left[\partial_\nu^y\varDelta_+(y''',y)\right]\varDelta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,z)]K(z)\left[\partial_\nu^y D_{\mu\nu}(x,y)\right]\varDelta_+(y,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\left[\partial_\nu^y\varDelta_+(y'',y)\right]_{y''=y}$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,z)]K(z)$$

$$\times K^\dagger(y''')\left[\partial_\nu^y\varDelta_+(y''',y)\right]\varDelta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,z)]K(z)$$

$$\times K^\dagger(y''')\varDelta_+(y''',y)\varDelta_+(y,z')K(z')\left[\partial_\nu^y D_{\mu\nu}(x,y)\right]$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial_\mu^x\varDelta_+(y,x)\right]\left[\partial_\nu^y\varDelta_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial_\mu^x \Delta_+(y,x)\right]\Delta_+(x,y)D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times\left[\partial_\nu^y D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\Delta_+(y,x)\left[\partial_\mu^x \Delta_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times\left[\partial_\nu^y D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\Delta_+(y,x)\left[\partial_\nu^y\partial_\mu^x \Delta_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\left[\partial_\nu^y \Delta_+(y'',y)\right]\left[\partial_\mu^x \Delta_+(y,x)\right]\Delta_+(x,z)$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',y)\left[\partial_\mu^x \Delta_+(y,x)\right]\Delta_+(x,z)$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')\left[\partial_\nu^y D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',y)\Delta_+(y,x)\left[\partial_\mu^x \Delta_+(x,z)\right]$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')\left[\partial_\nu^y D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\left[\partial_\nu^y \Delta_+(y'',y)\right]\Delta_+(y,x)\left[\partial_\mu^x \Delta_+(x,z)\right]$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial_\mu^x \Delta_+(y',x)\right]$$

$$\times\left[\partial_\nu^y \Delta_+(x,y)\right]D_{\nu\alpha}(y,x'')\Delta_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\left[\partial_{\mu}^{x}\varDelta_{+}(y',x)\right]$$

$$\times\,\varDelta_{+}(x,y)\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]\varDelta_{+}(y,z')K(z')D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\varDelta_{+}(y',x)\left[\partial_{\nu}^{y}\partial_{\mu}^{x}\varDelta_{+}(x,y)\right]$$

$$\times\,\varDelta_{+}(y,z')K(z')D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\varDelta_{+}(y',x)[\partial_{\mu}^{x}\varDelta_{+}(x,y)]$$

$$\times\,\varDelta_{+}(y,z')K(z')D_{\mu\beta}(x,x')J^{\beta}(x')\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]J^{\alpha}(x'')$$

$$\Bigg\}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}+\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]. \tag{C.40}$$

Find second term in Eq. (C.29), We have

$$\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\odot^{\mu\nu}(x,y)$$

$$=\left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\right]\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$+\left[\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\right]\left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]$$

$$+\left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]$$

$$+\left[\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\right]\left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]$$

$$+\left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\odot^{\mu}(x)\right]\left[\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]$$

$$+\left[\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\odot^{\mu}(x)\right]\left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]$$

$$+\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+|0_-\rangle_0\right]$$

$$+\odot^\mu(x)\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+|0_-\rangle_0\right]. \quad \text{(C.41)}$$

To convenient, let

$$\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\odot^{\mu\nu}(x,y) = (\mathrm{II}.1)+(\mathrm{II}.2)+(\mathrm{II}.3)+(\mathrm{II}.4)$$

$$+(\mathrm{II}.5)+(\mathrm{II}.6)+(\mathrm{II}.7)+(\mathrm{II}.8).$$

Then we find each term

$$(\mathrm{II}.1)\quad\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\langle 0_+|0_-\rangle_0 .$$

We have

$$\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x) = \Big\{-\mathrm{i}\int(\mathrm{d}x)[\partial^x_\mu\varDelta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)$$

$$+\mathrm{i}\int(\mathrm{d}x)[\partial^x_\mu\varDelta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')[\partial^x_\mu\varDelta_+(y',x)]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu\varDelta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\Big\}\langle 0_+|0_-\rangle_0+\odot^\mu(x)\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+|0_-\rangle_0\right],$$

then operate it with $\dfrac{\delta}{\mathrm{i}\delta K^\dagger(y)}$, we obtain

$$\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x) = \Big\{-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)[\partial^x_\mu\varDelta_+(y,x)]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)\mathit{\Delta}_+(y,x)[\partial^x_\mu\mathit{\Delta}_+(x,z)]K(z)D_{\mu\nu}(x,y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,.$$

Therefore, we obtain ((II.1))

$$\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\langle 0_+\,|\,0_-\rangle_0$$

$$=\Big\{-\int(\mathrm{d}x)(\mathrm{d}z)[\partial^x_\mu\mathit{\Delta}_+(y,x)]\mathit{\Delta}_+(x,z)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z-y)\right]D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}z)\mathit{\Delta}_+(y,x)[\partial^x_\mu\mathit{\Delta}_+(x,z)]\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z-y)\right]D_{\mu\nu}(x,y)\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,. \quad \text{(C.42)}$$

And

$$\text{(II.2)}\quad\left[\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right].$$

We have

$$\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\langle 0_+\,|\,0_-\rangle_0=\left[\int(\mathrm{d}y'')(\mathrm{d}z')K^\dagger(y'')\mathit{\Delta}_+(y'',z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]\right]\langle 0_+\,|\,0_-\rangle_0\,.$$

Therefore, we obtain ((II.2))

$$\left[\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\langle 0_+\,|\,0_-\rangle_0\right]$$

$$=\Big\{-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y'')[\partial^x_\mu\mathit{\Delta}_+(y,x)]\mathit{\Delta}_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$\times\,K^\dagger(y'')\mathit{\Delta}_+(y'',z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y'')\mathit{\Delta}_+(y,x)[\partial^x_\mu\mathit{\Delta}_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\,K^\dagger(y'')\mathit{\Delta}_+(y'',z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]\Big\}\,\langle 0_+\,|\,0_-\rangle_0\,. \quad \text{(C.43)}$$

And

$$\text{(II.3)}\quad \left[\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\left\langle 0_+\,|\,0_-\right\rangle_0\right].$$

We have

$$\begin{aligned}\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x) = \Big\{ -\mathrm{i}\int &(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial_\mu^x \varDelta_+(y',x)] \\ &\times \varDelta_+(x,z)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z-y)\right]D_{\mu\nu}(x,y) \\ &+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,z)] \\ &\times\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z-y)\right]D_{\mu\nu}(x,y)\Big\}\left\langle 0_+\,|\,0_-\right\rangle_0,\end{aligned}$$

and

$$\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\left\langle 0_+\,|\,0_-\right\rangle_0 = \int(\mathrm{d}z')\varDelta_+(y,z')K(z')\left\langle 0_+\,|\,0_-\right\rangle_0.$$

Therefore, we obtain ((II.3))

$$\begin{aligned}&\left[\left(\frac{\partial_\nu^y}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\right)\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\left\langle 0_+\,|\,0_-\right\rangle_0\right] \\ &= \Big\{ -\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial_\mu^x\varDelta_+(y',x)]\varDelta_+(x,z)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z-y)\right] \\ &\quad \times \varDelta_+(y,z')K(z')D_{\mu\nu}(x,y) \\ &\quad +\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\varDelta_+(y',x)[\partial_\mu^x\varDelta_+(x,z)]\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z-y)\right] \\ &\quad \times\varDelta_+(y,z')K(z')D_{\mu\nu}(x,y)\Big\}\left\langle 0_+\,|\,0_-\right\rangle_0. \qquad \text{(C.44)}\end{aligned}$$

And

$$\text{(II.4)}\quad \left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)} \odot^\mu (x)\right] \left[\left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \frac{\delta}{\mathrm{i}\delta K^\dagger(y)} \left\langle 0_+ \,|\, 0_- \right\rangle_0 \right].$$

We have

$$\begin{aligned}
\frac{\delta}{\mathrm{i}\delta J_\nu(y)} \odot^\mu (x) = \Big\{ & -\mathrm{i}\int (\mathrm{d}x)[\partial^x_\mu \Delta_+(y',x)]_{y'=x} D_{\mu\nu}(x,y) \\
& + \mathrm{i}\int (\mathrm{d}x)[\partial^x_\mu \Delta_+(x,z)]_{z=x} D_{\mu\nu}(x,y) \\
& + \int (\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z) K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y) \\
& - \int (\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z) K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)D_{\mu\nu}(x,y) \Big\} \left\langle 0_+ \,|\, 0_- \right\rangle_0 ,
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \frac{\delta}{\mathrm{i}\delta K^\dagger(y)} \left\langle 0_+ \,|\, 0_- \right\rangle_0 = \Big\{ & -\mathrm{i}\int (\mathrm{d}z')\Delta_+(y,z') \left[\frac{\partial^y_\nu}{\mathrm{i}} \delta^4(z'-y)\right] \\
& + \int (\mathrm{d}z')(\mathrm{d}y''')(\mathrm{d}z'')\Delta_+(y,z')K(z')K^\dagger(y''')\Delta_+(y''',z'') \left[\frac{\partial^y_\nu}{\mathrm{i}} \delta^4(z''-y)\right] \\
& \Big\} \left\langle 0_+ \,|\, 0_- \right\rangle_0 .
\end{aligned}$$

Therefore, we obtain ((II.4))

$$\begin{aligned}
& \left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)} \odot^\mu (x)\right] \left[\left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \frac{\delta}{\mathrm{i}\delta K^\dagger(y)} \left\langle 0_+ \,|\, 0_- \right\rangle_0 \right] \\
& = \Big\{ - \int (\mathrm{d}x)(\mathrm{d}z')[\partial^x_\mu \Delta_+(y',x)]_{y'=x} D_{\mu\nu}(x,y)\Delta_+(y,z') \left[\frac{\partial^y_\nu}{\mathrm{i}} \delta^4(z'-y)\right] \\
& -\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')[\partial^x_\mu \Delta_+(y,x)]_{y'=x} K^\dagger(y''')\Delta_+(y''',z'')\Delta_+(y,z')K(z')
\end{aligned}$$

$$\times \left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z''-y)\right] D_{\mu\nu}(x,y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}z')[\partial^x_\mu \varDelta_+(x,z)]_{z=x} D_{\mu\nu}(x,y)\varDelta_+(y,z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')[\partial^x_\mu \varDelta_+(x,z)]_{z=x} K^\dagger(y''')\varDelta_+(y''',z'')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z''-y)\right]$$

$$\times \varDelta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$- \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times D_{\mu\nu}(x,y)\varDelta_+(y,z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]$$

$$+ \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times D_{\mu\nu}(x,y)\varDelta_+(y,z')K(z')K^\dagger(y''')\varDelta_+(y''',z'')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z''-y)\right]$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,z)]K(z)$$

$$\times D_{\mu\nu}(x,y)\varDelta_+(y,z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}z'')(\mathrm{d}y''')K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,z)]K(z)$$

$$\times D_{\mu\nu}(x,y)\varDelta_+(y,z')K(z')K^\dagger(y''')\varDelta_+(y''',z'')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z''-y)\right]\Bigg\}\langle 0_+\,|\,0_-\rangle_0\,. \tag{C.45}$$

And

$$\text{(II.5)}\quad \left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\odot^\mu(x)\right]\left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+\,|\,0_-\rangle_0\right].$$

We have

$$\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \odot^{\mu}(x) = \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x') \left[\partial^x_\mu \mathit{\Delta}_+(y,x)\right] \mathit{\Delta}_+(x,z) K(z) D_{\mu\beta}(x,x') J^\beta(x')$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x') \mathit{\Delta}_+(y,x) \left[\partial^x_\mu \mathit{\Delta}_+(x,z)\right] K(z) D_{\mu\beta}(x,x') J^\beta(x'),$$

and

$$\left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \odot^{\mu}(x) = -\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}z) \left[\frac{\partial^y_\nu}{\mathrm{i}} \delta^4(z-y)\right]$$

$$\times \left[\partial^x_\mu \mathit{\Delta}_+(y,x)\right] \mathit{\Delta}_+(x,z) D_{\mu\beta}(x,x') J^\beta(x')$$

$$+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x') \left[\frac{\partial^y_\nu}{\mathrm{i}} \delta^4(z-y)\right] \mathit{\Delta}_+(y,x) \left[\partial^x_\mu \mathit{\Delta}_+(x,z)\right] D_{\mu\beta}(x,x') J^\beta(x').$$

Therefore, we obtain ((II.5))

$$\left[\left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \odot^{\mu}(x)\right] \left[\frac{\delta}{\mathrm{i}\delta J_\nu(y)} \langle 0_+ | 0_- \rangle_0\right]$$

$$= \Big\{ -\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}z) \left[\frac{\partial^y_\nu}{\mathrm{i}} \delta^4(z-y)\right] \left[\partial^x_\mu \mathit{\Delta}_+(y,x)\right] \mathit{\Delta}_+(x,z)$$

$$\times D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'')$$

$$+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}z) \left[\frac{\partial^y_\nu}{\mathrm{i}} \delta^4(z-y)\right] \mathit{\Delta}_+(y,x) \left[\partial^x_\mu \mathit{\Delta}_+(x,z)\right]$$

$$\times D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \Big\} \langle 0_+ | 0_- \rangle_0 . \tag{C.46}$$

And

$$\text{(II.6)} \; \left[\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \odot^{\mu}(x)\right] \left[\left(\frac{\partial^y_\nu}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \frac{\delta}{\mathrm{i}\delta J_\nu(y)} \langle 0_+ | 0_- \rangle_0\right].$$

We have

$$\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \odot^{\mu}(x) = \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x') \left[\partial_{\mu}^{x} \Delta_{+}(y,x)\right] \Delta_{+}(x,z) K(z) D_{\mu\beta}(x,x') J^{\beta}(x')$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x') \Delta_{+}(y,x) \left[\partial_{\mu}^{x} \Delta_{+}(x,z)\right] K(z) D_{\mu\beta}(x,x') J^{\beta}(x').$$

Therefore, we obtain ((II.6))

$$\left[\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \odot^{\mu}(x)\right] \left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \frac{\delta}{\mathrm{i}\delta J_{\nu}(y)} \langle 0_{+} \,|\, 0_{-}\rangle_{0}\right]$$

$$= \Bigg\{ \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}z')(\mathrm{d}x'')(\mathrm{d}y'') \left[\partial_{\mu}^{x} \Delta_{+}(y,x)\right] K^{\dagger}(y'') \Delta_{+}(y'',z')$$

$$\times \left[\frac{\partial_{\nu}^{y}}{\mathrm{i}} \delta^{4}(z'-y)\right] \Delta_{+}(x,z) K(z) D_{\mu\beta}(x,x') J^{\beta}(x') D_{\nu\alpha}(y,x'') J^{\alpha}(x'')$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}z') \Delta_{+}(y',x) K^{\dagger}(y'') \Delta_{+}(y'',z')$$

$$\times \left[\frac{\partial_{\nu}^{y}}{\mathrm{i}} \delta^{4}(z'-y)\right] \left[\partial_{\mu}^{x} \Delta_{+}(x,z)\right] K(z) D_{\mu\beta}(x,x') J^{\beta}(x') D_{\nu\alpha}(y,x'') J^{\alpha}(x'')$$

$$\Bigg\} \langle 0_{+} \,|\, 0_{-}\rangle_{0}. \tag{C.47}$$

And

$$\text{(II.7)} \quad \left[\left(\frac{\partial_{\nu}^{y}}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)}\right) \odot^{\mu}(x)\right] \left[\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \frac{\delta}{\mathrm{i}\delta J_{\nu}(y)} \langle 0_{+} \,|\, 0_{-}\rangle_{0}\right].$$

We have

$$\frac{\delta}{\mathrm{i}\delta K(y)} \odot^{\mu}(x) = \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y') K^{\dagger}(y') \left[\partial_{\mu}^{x} \Delta_{+}(y',x)\right]$$

$$\times \Delta_{+}(x,z) \left[\frac{\partial_{\nu}^{y}}{\mathrm{i}} \delta^{4}(z-y)\right] D_{\mu\beta}(x,x') J^{\beta}(x')$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')K^{\dagger}(y')\varDelta_{+}(y',x)\left[\partial^{x}_{\mu}\varDelta_{+}(x,z)\right]\left[\frac{\partial^{y}_{\nu}}{\mathrm{i}}\delta^{4}(z-y)\right]$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x').$$

Therefore, we obtain ((II.7))

$$\left[\left(\frac{\partial^{y}_{\nu}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\odot^{\mu}(x)\right]\left[\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\left\langle 0_{+}\,|\,0_{-}\right\rangle_{0}\right]$$

$$= \Big\{ \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\left[\partial^{x}_{\mu}\varDelta_{+}(y',x)\right]\varDelta_{+}(x,z)$$

$$\times \left[\frac{\partial^{y}_{\nu}}{\mathrm{i}}\delta^{4}(z-y)\right]\varDelta_{+}(y,z')K(z')D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\varDelta_{+}(y',x)\left[\partial^{x}_{\mu}\varDelta_{+}(x,z)\right]$$

$$\times \left[\frac{\partial^{y}_{\nu}}{\mathrm{i}}\delta^{4}(z-y)\right]\varDelta_{+}(y,z')K(z')D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$\Big\}\left\langle 0_{+}\,|\,0_{-}\right\rangle_{0}. \qquad \text{(C.48)}$$

So that, we obtain

$$\left(\frac{\partial^{y}_{\nu}}{\mathrm{i}}\;\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\odot^{\mu}(x)\left\langle 0_{+}\,|\,0_{-}\right\rangle_{0}$$

$$= \Big\{ - \int (\mathrm{d}x)(\mathrm{d}z)\left[\frac{\partial^{y}_{\nu}}{\mathrm{i}}\delta^{4}(z-y)\right][\partial^{x}_{\mu}\varDelta_{+}(y,x)]\varDelta_{+}(x,z)D_{\mu\nu}(x,y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}z)\left[\frac{\partial^{y}_{\nu}}{\mathrm{i}}\delta^{4}(z-y)\right]\varDelta_{+}(y,x)[\partial^{x}_{\mu}\varDelta_{+}(x,z)]D_{\mu\nu}(x,y)$$

$$- \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}z)\left[\frac{\partial^{y}_{\nu}}{\mathrm{i}}\delta^{4}(z'-y)\right][\partial^{x}_{\mu}\varDelta_{+}(y,x)]\varDelta_{+}(x,z)K(z)D_{\mu\nu}(x,y)$$

$$\times \int (\mathrm{d}y'')K^{\dagger}(y'')\varDelta_{+}(y'',z')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}z)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y,x)[\partial_\mu^x\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\int(\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',z')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y')(\mathrm{d}z)(\mathrm{d}z')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$\times\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y,z')K(z')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\Delta_+(y,x)[\partial_\mu^x\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y,z')K(z')$$

$$-\int(\mathrm{d}x)(\mathrm{d}z')[\partial_\mu^x\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y,z')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')[\partial_\mu^x\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z''-y)\right]$$

$$\times\,\Delta_+(y,z')K(z')K^\dagger(y''')\Delta_+(y''',z'')$$

$$+\int(\mathrm{d}x)(\mathrm{d}z')[\partial_\mu^x\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y'',z')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')[\partial_\mu^x\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z''-y)\right]$$

$$\times\,\Delta_+(y,z')K(z')K^\dagger(y''')\Delta_+(y''',z'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\,D_{\mu\nu}(x,y)\left[\frac{\partial_\nu^y}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y,z')$$

$$+\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\, D_{\mu\nu}(x,y)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z''-y)\right]\Delta_+(y,z')K(z')K^\dagger(y''')\Delta_+(y''',z'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)$$

$$\times\, D_{\mu\nu}(x,y)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y,z')$$

$$-\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z'')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)$$

$$\times\, D_{\mu\nu}(x,y)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z''-y)\right]\Delta_+(y,z')K(z')K^\dagger(y''')\Delta_+(y''',z'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}z)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z-y)\right]\left[\partial^x_\mu\Delta_+(y,x)\right]\Delta_+(x,z)$$

$$\times\, D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}z)\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z-y)\right]\Delta_+(y,x)\left[\partial^x_\mu\Delta_+(x,z)\right]$$

$$\times\, D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]\left[\partial^x_\mu\Delta_+(y,x)\right]$$

$$\times\, K^\dagger(y'')\Delta_+(y'',z')\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}z')\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z'-y)\right]\Delta_+(y,x)$$

$$\times\, K^\dagger(y'')\Delta_+(y'',z')\left[\partial^x_\mu\Delta_+(x,z)\right]K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\int(\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu\Delta_+(y',x)\right]\Delta_+(x,z)$$

$$\times\left[\frac{\partial^y_\nu}{\mathrm{i}}\delta^4(z-y)\right]\Delta_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$- \int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'') K^{\dagger}(y') \Delta_{+}(y', x) \left[\partial_{\mu}^{x} \Delta_{+}(x, z) \right]$$

$$\times \left[\frac{\partial_{\nu}^{y}}{\mathrm{i}} \delta^{4}(z - y) \right] \Delta_{+}(y, z') K(z') D_{\mu\beta}(x, x') J^{\beta}(x') D_{\nu\alpha}(y, x'') J^{\alpha}(x'')$$

$$\Bigg\} \langle 0_{+} | 0_{-} \rangle_{0} + \odot^{\mu}(x) \left[\frac{\delta}{\mathrm{i}\delta K(y)} \left(\frac{\partial_{\nu}^{y}}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \right) \frac{\delta}{\mathrm{i}\delta J_{\nu}(y)} \langle 0_{+} | 0_{-} \rangle_{0} \right]. \qquad \text{(C.49)}$$

Then we have

$$\int (\mathrm{d}y) \left(\frac{\partial_{\nu}^{y}}{\mathrm{i}} \frac{\delta}{\mathrm{i}\delta K(y)} \right) \frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)} \frac{\delta}{\mathrm{i}\delta J_{\nu}(y)} \odot^{\mu}(x) \langle 0_{+} | 0_{-} \rangle_{0}$$

$$= \Bigg\{ - \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z) \delta^{4}(z - y) \Delta_{+}(x, z) \partial_{\nu}^{y} \left[[\partial_{\mu}^{x} \Delta_{+}(y, x)] D_{\mu\nu}(x, y) \right]$$

$$+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z) \delta^{4}(z - y) [\partial_{\mu}^{x} \Delta_{+}(x, z)] \partial_{\nu}^{y} \left[\Delta_{+}(y, x) D_{\mu\nu}(x, y) \right]$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}z)(\mathrm{d}y'') \delta^{4}(z' - y) \Delta_{+}(y'', z') \Delta_{+}(x, z) K(z) K^{\dagger}(y'')$$

$$\times \partial_{\nu}^{y} \left[[\partial_{\mu}^{x} \Delta_{+}(y, x)] D_{\mu\nu}(x, y) \right]$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y'')(\mathrm{d}z) \delta^{4}(z' - y) [\partial_{\mu}^{x} \Delta_{+}(x, z)] K(z) K^{\dagger}(y'') \Delta_{+}(y'', z')$$

$$\times \partial_{\nu}^{y} \left[\Delta_{+}(y, x) D_{\mu\nu}(x, y) \right]$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z') K^{\dagger}(y') [\partial_{\mu}^{x} \Delta_{+}(y', x)] \partial_{\nu}^{y} \left[\Delta_{+}(y, z') D_{\mu\nu}(x, y) \right]$$

$$\times \delta^{4}(z - y) K(z') \Delta_{+}(x, z)$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z') K^{\dagger}(y') \Delta_{+}(y', x) \partial_{\nu}^{y} \left[\Delta_{+}(y, z') D_{\mu\nu}(x, y) \right]$$

$$\times \delta^{4}(z - y) [\partial_{\mu}^{x} \Delta_{+}(x, z)] K(z')$$

$$- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')[\partial^x_\mu \Delta_+(y',x)]_{y'=x} \partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right] \delta^4(z'-y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z'')(\mathrm{d}y''')[\partial^x_\mu \Delta_+(y',x)]_{y'=x} \partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right]$$

$$\times \delta^4(z''-y)K(z')K^\dagger(y''')\Delta_+(y''',z'')$$

$$+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')[\partial^x_\mu \Delta_+(x,z)]_{z=x} \partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right] \delta^4(z'-y)$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')[\partial^x_\mu \Delta_+(x,z)]_{z=x} \partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right]$$

$$\times \delta^4(z''-y)K(z')K^\dagger(y''')\Delta_+(y''',z'')$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times \partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right] \delta^4(z'-y)$$

$$+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}z'')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times \delta^4(z''-y)K(z')K^\dagger(y''')\Delta_+(y''',z'')\partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right]$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)$$

$$\times \delta^4(z'-y)\partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right]$$

$$- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)$$

$$\times \delta^4(z''-y)K(z')K^\dagger(y''')\Delta_+(y''',z'')\partial^y_\nu \left[D_{\mu\nu}(x,y)\Delta_+(y,z')\right]$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}z)(\mathrm{d}x'')\delta^4(z-y)\Delta_+(x,z)\partial^y_\nu \left[D_{\nu\alpha}(y,x'')\left[\partial^x_\mu \Delta_+(y,x)\right]\right]$$

$$\times D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$
-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}z)\delta^4(z-y)\left[\partial^x_\mu\varDelta_+(x,z)\right]\partial^y_\nu\left[D_{\nu\alpha}(y,x'')\varDelta_+(y,x)\right]
$$

$$
\times D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')
$$

$$
+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}z')\delta^4(z'-y)\varDelta_+(y'',z')
$$

$$
\times K^\dagger(y'')\varDelta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')\partial^y_\nu\left[D_{\nu\alpha}(y,x'')\left[\partial^x_\mu\varDelta_+(y,x)\right]\right]J^\alpha(x'')
$$

$$
-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')(\mathrm{d}z')\delta^4(z'-y)\varDelta_+(y'',z')
$$

$$
\times K^\dagger(y'')\left[\partial^x_\mu\varDelta_+(x,z)\right]K(z)D_{\mu\beta}(x,x')J^\beta(x')\partial^y_\nu\left[D_{\nu\alpha}(y,x'')\varDelta_+(y,x)\right]J^\alpha(x'')
$$

$$
+\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu\varDelta_+(y',x)\right]\varDelta_+(x,z)
$$

$$
\times\delta^4(z-y)\partial^y_\nu\left[\varDelta_+(y,z')D_{\nu\alpha}(y,x'')\right]K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')
$$

$$
-\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\varDelta_+(y',x)\left[\partial^x_\mu\varDelta_+(x,z)\right]
$$

$$
\times\delta^4(z-y)K(z')D_{\mu\beta}(x,x')J^\beta(x')\partial^y_\nu\left[\varDelta_+(y,z')D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')
$$

$$
\Bigg\}\langle 0_+\,|\,0_-\rangle_0+\odot^\mu(x)\left[\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\langle 0_+\,|\,0_-\rangle_0\right]. \tag{C.50}
$$

Simplify above term as

$$
\int(\mathrm{d}y)\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\odot^\mu(x)\,\langle 0_+\,|\,0_-\rangle_0
$$

$$
=\Big\{\int(\mathrm{d}x)(\mathrm{d}y)\varDelta_+(x,y)\left[\partial^y_\nu\partial^x_\mu\varDelta_+(y,x)\right]D_{\mu\nu}(x,y)
$$

$$
+\int(\mathrm{d}x)(\mathrm{d}y)\varDelta_+(x,y)[\partial^x_\mu\varDelta_+(y,x)]\left[\partial^y_\nu D_{\mu\nu}(x,y)\right]
$$

$$
-\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\varDelta_+(x,y)]\left[\partial^y_\nu\varDelta_+(y,x)\right]D_{\mu\nu}(x,y)
$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \Delta_+(x,y)]\Delta_+(y,x)\partial^y_\nu \left[D_{\mu\nu}(x,y)\right]$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')\Delta_+(y'',y)\Delta_+(x,z)K(z)K^\dagger(y'')$$

$$\times \left[\partial^y_\nu \partial^x_\mu \Delta_+(y,x)\right] D_{\mu\nu}(x,y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')\Delta_+(y'',y)\Delta_+(x,z)K(z)K^\dagger(y'')$$

$$\times [\partial^x_\mu \Delta_+(y,x)] \left[\partial^y_\nu D_{\mu\nu}(x,y)\right]$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)[\partial^x_\mu \Delta_+(x,z)]K(z)K^\dagger(y'')\Delta_+(y'',y)$$

$$\times \left[\partial^y_\nu \Delta_+(y,x)\right] D_{\mu\nu}(x,y)$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)[\partial^x_\mu \Delta_+(x,z)]K(z)K^\dagger(y'')\Delta_+(y'',y)$$

$$\times \Delta_+(y,x) \left[\partial^y_\nu D_{\mu\nu}(x,y)\right]$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)] \left[\partial^y_\nu \Delta_+(y,z')\right] D_{\mu\nu}(x,y)$$

$$\times K(z')\Delta_+(x,y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(y,z') \left[\partial^y_\nu D_{\mu\nu}(x,y)\right]$$

$$\times K(z')\Delta_+(x,y)$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\Delta_+(y',x) \left[\partial^y_\nu \Delta_+(y,z')\right] D_{\mu\nu}(x,y)$$

$$\times [\partial^x_\mu \Delta_+(x,y)]K(z')$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\Delta_+(y',x)\Delta_+(y,z') \left[\partial^y_\nu D_{\mu\nu}(x,y)\right]$$

$$\times\,[\partial^x_\mu \varDelta_+(x,y)]K(z')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(y',x)]_{y'=x} D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]_{z'=y}$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(y',x)]_{y'=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\varDelta_+(y,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y''')[\partial^x_\mu \varDelta_+(y',x)]_{y'=x} D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]$$

$$\times\,K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z'')(\mathrm{d}y''')[\partial^x_\mu \varDelta_+(y',x)]_{y'=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\varDelta_+(y,z')$$

$$\times\,K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(x,z)]_{z=x} D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]_{z'=y}$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(x,z)]_{z=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\varDelta_+(y,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu \varDelta_+(x,z)]_{z=x} D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]$$

$$\times\,K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu \varDelta_+(x,z)]_{z=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\varDelta_+(y,z')$$

$$\times\,K(z')K^\dagger(y''')\varDelta_+(y''',y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times\,D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]_{z'=y}$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times \, [\partial^y_\nu D_{\mu\nu}(x,y)] \, \Delta_+(y,y)$$

$$+ \, \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times \, K(z')K^\dagger(y''')\Delta_+(y''',y)D_{\mu\nu}(x,y) \, [\partial^y_\nu \Delta_+(y,z')]$$

$$+ \, \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times \, K(z')K^\dagger(y''')\Delta_+(y''',y) \, [\partial^y_\nu D_{\mu\nu}(x,y)] \, \Delta_+(y,z')$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)$$

$$\times \, D_{\mu\nu}(x,y) \, [\partial^y_\nu \Delta_+(y,z')]_{z'=y}$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)$$

$$\times \, [\partial^y_\nu D_{\mu\nu}(x,y)] \, \Delta_+(y,y)$$

$$- \, \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)$$

$$\times \, K(z')K^\dagger(y''')\Delta_+(y''',y)D_{\mu\nu}(x,y) \, [\partial^y_\nu \Delta_+(y,z')]$$

$$- \, \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)$$

$$\times \, K(z')K^\dagger(y''')\Delta_+(y''',y) \, [\partial^y_\nu D_{\mu\nu}(x,y)] \, \Delta_+(y,z')$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\Delta_+(x,y)D_{\nu\alpha}(y,x'') \left[\partial^y_\nu \partial^x_\mu \Delta_+(y,x)\right]$$

$$\times \, D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\Delta_+(x,y) \, [\partial^y_\nu D_{\nu\alpha}(y,x'')] \left[\partial^x_\mu \Delta_+(y,x)\right]$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial_{\mu}^{x}\Delta_{+}(x,y)\right]\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]\Delta_{+}(y,x)$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial_{\mu}^{x}\Delta_{+}(x,y)\right]D_{\nu\alpha}(y,x'')\left[\partial_{\nu}^{y}\Delta_{+}(y,x)\right]$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')\Delta_{+}(y'',y)$$

$$\times K^{\dagger}(y'')\Delta_{+}(x,z)K(z)D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')\left[\partial_{\nu}^{y}\partial_{\mu}^{x}\Delta_{+}(y,x)\right]J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')\Delta_{+}(y'',y)$$

$$\times K^{\dagger}(y'')\Delta_{+}(x,z)K(z)D_{\mu\beta}(x,x')J^{\beta}(x')\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]\left[\partial_{\mu}^{x}\Delta_{+}(y,x)\right]J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')\Delta_{+}(y'',y)$$

$$\times K^{\dagger}(y'')\left[\partial_{\mu}^{x}\Delta_{+}(x,z)\right]K(z)D_{\mu\beta}(x,x')J^{\beta}(x')\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]\Delta_{+}(y,x)J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')\Delta_{+}(y'',y)$$

$$\times K^{\dagger}(y'')\left[\partial_{\mu}^{x}\Delta_{+}(x,z)\right]K(z)D_{\mu\beta}(x,x')J^{\beta}(x')\left[\partial_{\nu}^{y}\Delta_{+}(y,x)\right]D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\left[\partial_{\mu}^{x}\Delta_{+}(y',x)\right]\Delta_{+}(x,y)$$

$$\times \Delta_{+}(y,z')\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]K(z')D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\left[\partial_{\mu}^{x}\Delta_{+}(y',x)\right]\Delta_{+}(x,y)$$

$$\times\ [\partial^y_\nu \Delta_+(y,z')]\, K(z')D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\Delta_+(y',x)\left[\partial^x_\mu \Delta_+(x,y)\right]$$

$$\times\, K(z')D_{\mu\beta}(x,x')J^\beta(x')\,[\partial^y_\nu \Delta_+(y,z')]\, D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\Delta_+(y',x)\left[\partial^x_\mu \Delta_+(x,y)\right]$$

$$\times\, K(z')D_{\mu\beta}(x,x')J^\beta(x')\Delta_+(y,z')\,[\partial^y_\nu D_{\nu\alpha}(y,x'')]\, J^\alpha(x'')$$

$$\Bigg\}\, \langle 0_+\,|\,0_-\rangle_0 + \odot^\mu(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\,\langle 0_+\,|\,0_-\rangle_0\right] \tag{C.51}$$

Find $\dfrac{\delta}{\mathrm{i}\delta K(y)}\dfrac{\delta}{\mathrm{i}\delta K^\dagger(y)}\dfrac{\delta}{\mathrm{i}\delta J_\nu(x)}\dfrac{\delta}{\mathrm{i}\delta J_\mu(x)}\,\langle 0_+\,|\,0_-\rangle_0$, we start from

$$\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} = \left[\int(\mathrm{d}y)D_{\mu\beta}(x,y)J^\beta(y)\right]\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}$$

$$\begin{aligned}\frac{\delta}{\mathrm{i}\delta J_\nu(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} &= \frac{\delta}{\mathrm{i}\delta J_\mu(x)}\left[\int(\mathrm{d}y)D_{\mu\beta}(x,y)J^\beta(y)\right]\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ}\\ &= -\mathrm{i}\left[\int(\mathrm{d}y)D_{\mu\beta}(x,y)\delta^\beta_\nu\delta^4(y-x)\right]\\ &\quad+\left[\int(\mathrm{d}y)D_{\mu\beta}(x,y)\right]\left[\int(\mathrm{d}y')D_{\nu\alpha}(x,y')J^\alpha(y')\right]\\ &= -\mathrm{i}D_{\mu\nu}(x,x)+\int(\mathrm{d}y)(\mathrm{d}y')D_{\mu\beta}(x,y)J^\beta(y)D_{\nu\alpha}(x,y')J^\alpha(y'),\end{aligned} \tag{C.52}$$

or

$$\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\mathrm{e}^{\frac{\mathrm{i}}{2}JDJ} = -\mathrm{i}D_{\mu\nu}(x,x)+\int(\mathrm{d}y)(\mathrm{d}x')D_{\mu\beta}(x,y)J^\beta(y)D_{\nu\alpha}(x,x')J^\alpha(x'). \tag{C.53}$$

And

$$\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\mathrm{e}^{\mathrm{i}K^\dagger \Delta_+ K} = \frac{\delta}{\mathrm{i}\delta K^\dagger(x)} \exp\left[\mathrm{i}\int (\mathrm{d}y)(\mathrm{d}z)K^\dagger(y)\Delta_+(y,z)K(z)\right]$$

$$= \mathrm{e}^{\mathrm{i}K^\dagger \Delta_+ K}\left[\int (\mathrm{d}y)(\mathrm{d}z)\delta^4(y-x)\Delta_+(y,z)K(z)\right]$$

$$= \mathrm{e}^{\mathrm{i}K^\dagger \Delta_+ K}\left[\int (\mathrm{d}z)\Delta_+(x,z)K(z)\right].$$

Then operate above term with $\frac{\delta}{\mathrm{i}\delta K(x)}$, we have

$$\frac{\delta}{\mathrm{i}\delta K(x)}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\mathrm{e}^{\mathrm{i}K^\dagger \Delta_+ K} = -\mathrm{i}\int (\mathrm{d}z)\Delta_+(x,z)K(z)$$

$$+\int (\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')\Delta_+(x,z)K(z)K^\dagger(y')\Delta_+(y',z')\delta^4(z'-x)$$

$$= -\mathrm{i}\int (\mathrm{d}z)\Delta_+(x,z)K(z)$$

$$+\int (\mathrm{d}z)(\mathrm{d}y')(K^\dagger(y')\Delta_+(y',x)\Delta_+(x,z)K(z). \tag{C.54}$$

Finally, we obtain

$$\frac{\delta}{\mathrm{i}\delta K(y)}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\langle 0_+|0_-\rangle_0 = -\int (\mathrm{d}x)(\mathrm{d}z)\Delta_+(x,z)K(z)D_{\mu\nu}(x,x)$$

$$-\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)\Delta_+(x,z)K(z)D_{\mu\nu}(x,x)$$

$$-\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')\Delta_+(x,z)K(z)D_{\mu\beta}(x,y)J^\beta(y)D_{\nu\alpha}(x,x')J^\alpha(x')$$

$$+\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)\Delta_+(x,z)K(z)D_{\mu\beta}(x,y)J^\beta(y)$$

$$\times D_{\nu\alpha}(x,x')J^\alpha(x'). \tag{C.55}$$

We have

$$a_2 = -\frac{\mathrm{i}}{2}a_1^2 + \frac{1}{\langle 0_+ \,|\, 0_- \rangle_0}[-\widehat{B} + \frac{\mathrm{i}}{2}(\widehat{A})^2] \langle 0_+ \,|\, 0_- \rangle_0 \,. \tag{C.56}$$

Then we find $(\widehat{A})^2 \langle 0_+ \,|\, 0_- \rangle_0$

$$= \int (\mathrm{d}x)(\mathrm{d}y)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\right)\frac{\delta}{\mathrm{i}\delta J_\nu(y)} - \left(\frac{\partial^y_\nu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(y)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(y)}\frac{\delta}{\mathrm{i}\delta J_\nu(y)}\right]$$

$$\times\left[\frac{\delta}{\mathrm{i}\delta K(x)}\left(\frac{\partial^x_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\right)\frac{\delta}{\mathrm{i}\delta J_\mu(x)} - \left(\frac{\partial^x_\mu}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K(x)}\right)\frac{\delta}{\mathrm{i}\delta K^\dagger(x)}\frac{\delta}{\mathrm{i}\delta J_\mu(x)}\right] \langle 0_+ \,|\, 0_- \rangle_0 \,.$$

Every term in above term, we have calculated it in Eq. (C.40) and Eq. (C.55). So that, we have

$$(\widehat{A})^2 \langle 0_+ \,|\, 0_- \rangle_0$$

$$= \Big\{ -\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(y,x)]\,[\partial^y_\nu \varDelta_+(x,y)]\, D_{\mu\nu}(x,y)$$

$$-\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(y,x)]\varDelta_+(x,y)\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)\varDelta_+(y,x)\,\left[\partial^y_\nu\partial^x_\mu \varDelta_+(x,y)\right]\, D_{\mu\nu}(x,y)$$

$$+\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)\varDelta_+(y,x)[\partial^x_\mu \varDelta_+(x,y)]\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^\dagger(y'')\varDelta_+(y'',y)[\partial^x_\mu \varDelta_+(y,x)]\varDelta_+(x,z)K(z)\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^\dagger(y'')\,[\partial^y_\nu \varDelta_+(y'',y)]\,[\partial^x_\mu \varDelta_+(y,x)]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$-\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^\dagger(y'')\,[\partial^y_\nu \varDelta_+(y'',y)]\,\varDelta_+(y,x)[\partial^x_\mu \varDelta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$-\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^\dagger(y'')\varDelta_+(y'',y)\varDelta_+(y,x)[\partial^x_\mu \varDelta_+(x,z)]K(z)\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\,[\partial^y_\nu \varDelta_+(x,y)]\,\varDelta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,y)\Delta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')\Delta_+(y',x)\left[\partial^y_\nu\partial^x_\mu\Delta_+(x,y)\right]\Delta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,y)]\Delta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y'',y)]_{y''=y}$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(y',x)]_{y'=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(y',x)]_{y'=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,K^{\dagger}(y''')\Delta_+(y''',y)$$

$$\times\,\Delta_+(y,z')K(z')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)K^{\dagger}(y''')\,[\partial^y_\nu\Delta_+(y''',y)]$$

$$\times\,\Delta_+(y,z')K(z')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(x,z)]_{z=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y'',y)]_{y''=y}$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(x,z)]_{z=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,K^{\dagger}(y''')\Delta_+(y''',y)$$

$$\times\,\Delta_+(y,z')K(z')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\left[\partial^y_\nu K^{\dagger}(y''')\Delta_+(y''',y)\right]$$

$$\times\,\Delta_+(y,z')K(z')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^{\dagger}(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)[\partial^y_\nu\Delta_+(y'',y)]_{y''=y}$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)K^\dagger(y''')$$

$$\times\,\Delta_+(y''',y)\Delta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\,K^\dagger(y''')\,[\partial^y_\nu\Delta_+(y''',y)]\,\Delta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\,\left[\partial^y_\nu\Delta_+(y'',y)\right]_{y''=y}$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)K^\dagger(y''')$$

$$\times\,[\partial^y_\nu\Delta_+(y''',y)]\,\Delta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)K^\dagger(y''')$$

$$\times\,\Delta_+(y''',y)\Delta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial^x_\mu\Delta_+(y,x)\right]\left[\partial^y_\nu\Delta_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times\,D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial^x_\mu\Delta_+(y,x)\right]\Delta_+(x,y)D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times\,\left[\partial^y_\nu D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\mathit{\Delta}_+(y,x)\left[\partial^x_\mu\mathit{\Delta}_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times\left[\partial^y_\nu D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\mathit{\Delta}_+(y,x)\left[\partial^y_\nu\partial^x_\mu\mathit{\Delta}_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\left[\partial^y_\nu\mathit{\Delta}_+(y'',y)\right]\left[\partial^x_\mu\mathit{\Delta}_+(y,x)\right]\mathit{\Delta}_+(x,z)$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\mathit{\Delta}_+(y'',y)\left[\partial^x_\mu\mathit{\Delta}_+(y,x)\right]\mathit{\Delta}_+(x,z)K(z)$$

$$\times D_{\mu\beta}(x,x')J^\beta(x')\left[\partial^y_\nu D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\mathit{\Delta}_+(y'',y)\mathit{\Delta}_+(y,x)\left[\partial^x_\mu\mathit{\Delta}_+(x,z)\right]$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')\left[\partial^y_\nu D_{\nu\alpha}(y,x'')\right]J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\left[\partial^y_\nu\mathit{\Delta}_+(y'',y)\right]\mathit{\Delta}_+(y,x)\left[\partial^x_\mu\mathit{\Delta}_+(x,z)\right]$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu\mathit{\Delta}_+(y',x)\right]\left[\partial^y_\nu\mathit{\Delta}_+(x,y)\right]D_{\nu\alpha}(y,x'')$$

$$\times\mathit{\Delta}_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu\mathit{\Delta}_+(y',x)\right]\mathit{\Delta}_+(x,y)\left[\partial^y_\nu D_{\nu\alpha}(y,x'')\right]$$

$$\times\mathit{\Delta}_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$
\begin{aligned}
&- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'') K^\dagger(y') \Delta_+(y',x) \left[\partial_\nu^y \partial_\mu^x \Delta_+(x,y)\right] \Delta_+(y,z') \\
&\quad \times K(z') D_{\mu\beta}(x,x') J^\beta(x') D_{\nu\alpha}(y,x'') J^\alpha(x'') \\
&- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'') K^\dagger(y') \Delta_+(y',x) [\partial_\mu^x \Delta_+(x,y)] \Delta_+(y,z') K(z') \\
&\quad \times D_{\mu\beta}(x,x') J^\beta(x') \left[\partial_\nu^y D_{\nu\alpha}(y,x'')\right] J^\alpha(x'') \\
&+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y) \Delta_+(x,y) \left[\partial_\nu^y \partial_\mu^x \Delta_+(y,x)\right] D_{\mu\nu}(x,y) \\
&+ \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y) \Delta_+(x,y) [\partial_\mu^x \Delta_+(y,x)] \left[\partial_\nu^y D_{\mu\nu}(x,y)\right] \\
&- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y) [\partial_\mu^x \Delta_+(x,y)] \left[\partial_\nu^y \Delta_+(y,x)\right] D_{\mu\nu}(x,y) \\
&- \mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y) [\partial_\mu^x \Delta_+(x,y)] \Delta_+(y,x) \partial_\nu^y \left[D_{\mu\nu}(x,y)\right] \\
&- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'') K^\dagger(y'') \Delta_+(y'',y) \left[\partial_\nu^y \partial_\mu^x \Delta_+(y,x)\right] \Delta_+(x,z) K(z) D_{\mu\nu}(x,y) \\
&- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'') K^\dagger(y'') \Delta_+(y'',y) [\partial_\mu^x \Delta_+(y,x)] \Delta_+(x,z) K(z) \left[\partial_\nu^y D_{\mu\nu}(x,y)\right] \\
&+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z) K^\dagger(y'') \Delta_+(y'',y) \left[\partial_\nu^y \Delta_+(y,x)\right] \left[\partial_\mu^x \Delta_+(x,z)\right] K(z) D_{\mu\nu}(x,y) \\
&+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z) K^\dagger(y'') \Delta_+(y'',y) \Delta_+(y,x) [\partial_\mu^x \Delta_+(x,z)] K(z) \left[\partial_\nu^y D_{\mu\nu}(x,y)\right] \\
&- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z') K^\dagger(y') [\partial_\mu^x \Delta_+(y',x)] \Delta_+(x,y) \left[\partial_\nu^y \Delta_+(y,z')\right] K(z') D_{\mu\nu}(x,y) \\
&- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z') K^\dagger(y') [\partial_\mu^x \Delta_+(y',x)] \Delta_+(x,y) \Delta_+(y,z') K(z') \left[\partial_\nu^y D_{\mu\nu}(x,y)\right] \\
&+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z') K^\dagger(y') \Delta_+(y',x) [\partial_\mu^x \Delta_+(x,y)] \left[\partial_\nu^y \Delta_+(y,z')\right] K(z') D_{\mu\nu}(x,y)
\end{aligned}
$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,y)]\Delta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y,z')]_{z'=y}$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(y',x)]_{y'=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y''')[\partial^x_\mu\Delta_+(y',x)]_{y'=x}K^\dagger(y''')\Delta_+(y''',y)\,[\partial^y_\nu\Delta_+(y,z')]\,K(z')$$

$$\times\,D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z'')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(y',x)]_{y'=x}K^\dagger(y''')\Delta_+(y''',y)\Delta_+(y,z')K(z')$$

$$\times\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y,z')]_{z'=y}$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(x,z)]_{z=x}\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(x,z)]_{z=x}K^\dagger(y''')\Delta_+(y''',y)\,[\partial^y_\nu\Delta_+(y,z')]\,K(z')$$

$$\times\,D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(x,z)]_{z=x}K^\dagger(y''')\Delta_+(y''',y)\Delta_+(y,z')K(z')$$

$$\times\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\,D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y,z')]_{z'=y}$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$- \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)K^\dagger(y''')$$

$$\times\, \Delta_+(y''',y)\,[\partial^y_\nu \Delta_+(y,z')]\,K(z')D_{\mu\nu}(x,y)$$

$$- \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \Delta_+(y',x)]\Delta_+(x,z)K(z)K^\dagger(y''')$$

$$\times\, \Delta_+(y''',y)\Delta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\, [\partial^y_\nu \Delta_+(y,z')]_{z'=y}$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)\,[\partial^y_\nu D_{\mu\nu}(x,y)]\,\Delta_+(y,y)$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)K^\dagger(y''')$$

$$\times\, \Delta_+(y''',y)\,[\partial^y_\nu \Delta_+(y,z')]\,K(z')D_{\mu\nu}(x,y)$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu \Delta_+(x,z)]K(z)K^\dagger(y''')$$

$$\times\, \Delta_+(y''',y)\Delta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\Delta_+(x,y)\left[\partial^y_\nu \partial^x_\mu \Delta_+(y,x)\right] D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$\times\, D_{\mu\beta}(x,x')J^\beta(x')$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\Delta_+(x,y)\left[\partial^x_\mu \Delta_+(y,x)\right][\partial^y_\nu D_{\nu\alpha}(y,x'')]\,J^\alpha(x'')$$

$$\times\, D_{\mu\beta}(x,x')J^\beta(x')$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial^x_\mu \Delta_+(x,y)\right]\Delta_+(y,x)\,[\partial^y_\nu D_{\nu\alpha}(y,x'')]\,J^\alpha(x'')$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x')$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial_{\mu}^{x}\Delta_{+}(x,y)\right]\left[\partial_{\nu}^{y}\Delta_{+}(y,x)\right]D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y'')\Delta_{+}(y'',y)\left[\partial_{\nu}^{y}\partial_{\mu}^{x}\Delta_{+}(y,x)\right]\Delta_{+}(x,z)$$

$$\times K(z)D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y'')\Delta_{+}(y'',y)\left[\partial_{\mu}^{x}\Delta_{+}(y,x)\right]\Delta_{+}(x,z)K(z)$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x')\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y'')\Delta_{+}(y'',y)\Delta_{+}(y,x)\left[\partial_{\mu}^{x}\Delta_{+}(x,z)\right]K(z)$$

$$\times D_{\mu\beta}(x,x')J^{\beta}(x')\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y'')\Delta_{+}(y'',y)\left[\partial_{\nu}^{y}\Delta_{+}(y,x)\right]\left[\partial_{\mu}^{x}\Delta_{+}(x,z)\right]$$

$$\times K(z)D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\left[\partial_{\mu}^{x}\Delta_{+}(y',x)\right]\Delta_{+}(x,y)\Delta_{+}(y,z')K(z')$$

$$\times\left[\partial_{\nu}^{y}D_{\nu\alpha}(y,x'')\right]D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\left[\partial_{\mu}^{x}\Delta_{+}(y',x)\right]\Delta_{+}(x,y)\left[\partial_{\nu}^{y}\Delta_{+}(y,z')\right]$$

$$\times K(z')D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\Delta_{+}(y',x)\left[\partial_{\mu}^{x}\Delta_{+}(x,y)\right]\left[\partial_{\nu}^{y}\Delta_{+}(y,z')\right]$$

$$\times\, K(z')D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\varDelta_{+}(y',x)\left[\partial^{x}_{\mu}\varDelta_{+}(x,y)\right]\varDelta_{+}(y,z')K(z')$$

$$\times\, D_{\mu\beta}(x,x')J^{\beta}(x')\left[\partial^{y}_{\nu}D_{\nu\alpha}(y,x'')\right]J^{\alpha}(x'')$$

$$\Bigg\}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}+\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^{y}_{\nu}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]$$

$$-\,\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^{y}_{\nu}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]. \tag{C.57}$$

Simplify above term

$$(\widehat{A})^{2}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}$$

$$=\Big\{-2\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\varDelta_{+}(y,x)]\,[\partial^{y}_{\nu}\varDelta_{+}(x,y)]\,D_{\mu\nu}(x,y)$$

$$+\,2\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)\varDelta_{+}(y,x)\left[\partial^{y}_{\nu}\partial^{x}_{\mu}\varDelta_{+}(x,y)\right]D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^{\dagger}(y'')\,[\partial^{y}_{\nu}\varDelta_{+}(y'',y)]\,[\partial^{x}_{\mu}\varDelta_{+}(y,x)]\varDelta_{+}(x,z)K(z)D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^{\dagger}(y'')\,[\partial^{y}_{\nu}\varDelta_{+}(y'',y)]\,\varDelta_{+}(y,x)[\partial^{x}_{\mu}\varDelta_{+}(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')[\partial^{x}_{\mu}\varDelta_{+}(y',x)]\,[\partial^{y}_{\nu}\varDelta_{+}(x,y)]\,\varDelta_{+}(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')\varDelta_{+}(y',x)\left[\partial^{y}_{\nu}\partial^{x}_{\mu}\varDelta_{+}(x,y)\right]\varDelta_{+}(y,z')K(z')D_{\mu\nu}(x,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\varDelta_{+}(y',x)]_{y'=x}D_{\mu\nu}(x,y)\,[\partial^{y}_{\nu}\varDelta_{+}(y'',y)]_{y''=y}$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\varDelta_{+}(x,z)]_{z=x}D_{\mu\nu}(x,y)\,[\partial^{y}_{\nu}\varDelta_{+}(y'',y)]_{y''=y}$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu \varDelta_+(x,z)]_{z=x} D_{\mu\nu}(x,y) K^\dagger(y''')\,[\partial^y_\nu \varDelta_+(y''',y)]$$

$$\times\, \varDelta_+(y,z')K(z')$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y') K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$\times\, [\partial^y_\nu \varDelta_+(y'',y)]_{y''=y}$$

$$+\,\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''') K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times\, K^\dagger(y''')\,[\partial^y_\nu \varDelta_+(y''',y)]\,\varDelta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y') K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\, [\partial^y_\nu \varDelta_+(y'',y)]_{y''=y}$$

$$-\,\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y''') K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,z)]K(z)$$

$$\times\, K^\dagger(y''')\,[\partial^y_\nu \varDelta_+(y''',y)]\,\varDelta_+(y,z')K(z')D_{\mu\nu}(x,y)$$

$$+\,2 \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\,\left[\partial^x_\mu \varDelta_+(y,x)\right]\,[\partial^y_\nu \varDelta_+(x,y)]\,D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times\, D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\,2 \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\varDelta_+(y,x)\,\left[\partial^y_\nu \partial^x_\mu \varDelta_+(x,y)\right]\,D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times\, D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\,\mathrm{i} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'') K^\dagger(y'')\,[\partial^y_\nu \varDelta_+(y'',y)]\,\left[\partial^x_\mu \varDelta_+(y,x)\right]$$

$$\times\, \varDelta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$- 2\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\,[\partial^y_\nu \varDelta_+(y'',y)]\,\varDelta_+(y,x)$$

$$\times\ \left[\partial^x_\mu \varDelta_+(x,z)\right] K(z) D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu \varDelta_+(y',x)\right]$$

$$\times\ [\partial^y_\nu \varDelta_+(x,y)]\,D_{\nu\alpha}(y,x'')\varDelta_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu \varDelta_+(y',x)\right]$$

$$\times\ \varDelta_+(x,y)\,[\partial^y_\nu D_{\nu\alpha}(y,x'')]\,\varDelta_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$- \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\varDelta_+(y',x)\left[\partial^y_\nu\partial^x_\mu \varDelta_+(x,y)\right]$$

$$\times\ \varDelta_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)\varDelta_+(x,y)\left[\partial^y_\nu\partial^x_\mu \varDelta_+(y,x)\right] D_{\mu\nu}(x,y)$$

$$- \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(x,y)]\,[\partial^y_\nu \varDelta_+(y,x)]\,D_{\mu\nu}(x,y)$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^\dagger(y'')\varDelta_+(y'',y)\left[\partial^y_\nu\partial^x_\mu \varDelta_+(y,x)\right]\varDelta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^\dagger(y'')\varDelta_+(y'',y)\,[\partial^y_\nu \varDelta_+(y,x)]\,[\partial^x_\mu \varDelta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$- \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]\,K(z')D_{\mu\nu}(x,y)$$

$$+ \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,y)]\,[\partial^y_\nu \varDelta_+(y,z')]\,K(z')D_{\mu\nu}(x,y)$$

$$+ \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]_{z'=y}$$

$$- \mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu \varDelta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]_{z'=y}$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu \varDelta_+(x,z)]_{z=x} K^\dagger(y''')\varDelta_+(y''',y)\,[\partial^y_\nu \varDelta_+(y,z')]\,K(z')$$

$$\times\, D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times\, D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]_{z'=y}$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times\,\, [\partial^y_\nu D_{\mu\nu}(x,y)]\,\varDelta_+(y,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times\, K^\dagger(y''')\varDelta_+(y''',y)\,[\partial^y_\nu \varDelta_+(y,z')]\,K(z')D_{\mu\nu}(x,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial^x_\mu \varDelta_+(y',x)]\varDelta_+(x,z)K(z)$$

$$\times\, K^\dagger(y''')\varDelta_+(y''',y)\varDelta_+(y,z')K(z')\,[\partial^y_\nu D_{\mu\nu}(x,y)]$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,z)]K(z)$$

$$\times\, D_{\mu\nu}(x,y)\,[\partial^y_\nu \varDelta_+(y,z')]_{z'=y}$$

$$+\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,z)]K(z)$$

$$\times\,\, [\partial^y_\nu D_{\mu\nu}(x,y)]\,\varDelta_+(y,y)$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\varDelta_+(y',x)[\partial^x_\mu \varDelta_+(x,z)]K(z)$$

$$\times\, K^\dagger(y''')\varDelta_+(y''',y)\,[\partial^y_\nu \varDelta_+(y,z')]\,K(z')D_{\mu\nu}(x,y)$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y'')\varDelta_{+}(y'',y)\left[\partial^{y}_{\nu}\partial^{x}_{\mu}\varDelta_{+}(y,x)\right]$$

$$\times\,\varDelta_{+}(x,z)K(z)D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^{\dagger}(y'')\varDelta_{+}(y'',y)\left[\partial^{y}_{\nu}\varDelta_{+}(y,x)\right]$$

$$\times\,\left[\partial^{x}_{\mu}\varDelta_{+}(x,z)\right]K(z)D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\left[\partial^{x}_{\mu}\varDelta_{+}(y',x)\right]\varDelta_{+}(x,y)$$

$$\times\,\varDelta_{+}(y,z')K(z')\left[\partial^{y}_{\nu}D_{\nu\alpha}(y,x'')\right]D_{\mu\beta}(x,x')J^{\beta}(x')J^{\alpha}(x'')$$

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^{\dagger}(y')\varDelta_{+}(y',x)\left[\partial^{x}_{\mu}\varDelta_{+}(x,y)\right]$$

$$\times\,\left[\partial^{y}_{\nu}\varDelta_{+}(y,z')\right]K(z')D_{\mu\beta}(x,x')J^{\beta}(x')D_{\nu\alpha}(y,x'')J^{\alpha}(x'')$$

$$\Bigg\}\langle 0_{+}\,|\,0_{-}\rangle_{0}+\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^{y}_{\nu}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right]$$

$$-\,\odot^{\mu}(x)\left[\frac{\delta}{\mathrm{i}\delta K(y)}\left(\frac{\partial^{y}_{\nu}}{\mathrm{i}}\frac{\delta}{\mathrm{i}\delta K^{\dagger}(y)}\right)\frac{\delta}{\mathrm{i}\delta J_{\nu}(y)}\,\langle 0_{+}\,|\,0_{-}\rangle_{0}\right].\tag{C.58}$$

Finally, we obtain a_2, simplify term of a_2. Each term in a_2 are separated with the properties of vary process in Scalar Electrodynamics. Let

$$\begin{aligned}a_2 &= \mathrm{P.1}+\mathrm{P.2}+\mathrm{P.3}+\mathrm{P.4}+\mathrm{P.5}\\ &\quad+\mathrm{P.6}+\mathrm{P.7}+\mathrm{P.8}+\mathrm{P.9}+\mathrm{P.10}\\ &\quad+\mathrm{P.11}\end{aligned}\tag{C.59}$$

where $\mathrm{P}.i$, $i=1,2,3,4,\ldots,11$ denote processes in Scalar Electrodynamics

The process P.1

$$\int (\mathrm{d}x)(\mathrm{d}z)\mathit{\Delta}_+(x,z)K(z)D_{\mu\nu}(x,x) \tag{C.60}$$

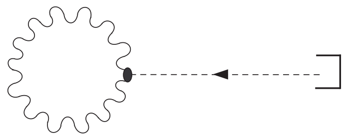

Figure C.1 Diagram corresponding to the process P.1.

The process P.2

$$\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\mathit{\Delta}_+(y',x)\mathit{\Delta}_+(x,z)K(z)D_{\mu\nu}(x,x) \tag{C.61}$$

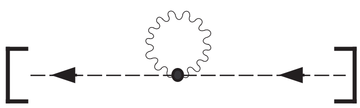

Figure C.2 Diagram corresponding to the process P.2.

The process P.3

$$\mathrm{i}\int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')\mathit{\Delta}_+(x,z)K(z)D_{\mu\beta}(x,y)J^\beta(y)D_{\nu\alpha}(x,x')J^\alpha(x') \tag{C.62}$$

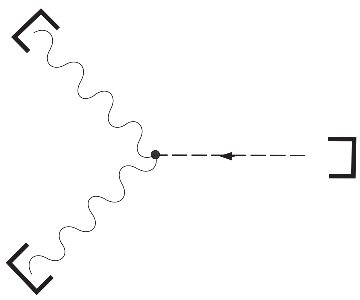

Figure C.3 Diagram corresponding to the process P.3.

The process P.4

$$-\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}y')K^{\dagger}(y')\Delta_{+}(y',x)\Delta_{+}(x,z)K(z)D_{\mu\beta}(x,y)J^{\beta}(y)$$

$$\times D_{\nu\alpha}(x,x')J^{\alpha}(x') \tag{C.63}$$

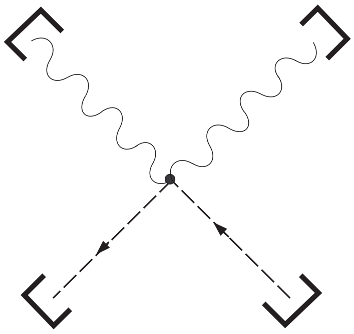

Figure C.4 A seagull diagram in scalar electrodynamics.

The process P.5

$$+\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\Delta_{+}(y,x)]\,[\partial^{y}_{\nu}\Delta_{+}(x,y)]\,D_{\mu\nu}(x,y)$$

$$-\int(\mathrm{d}x)(\mathrm{d}y)\Delta_{+}(y,x)\,\left[\partial^{y}_{\nu}\partial^{x}_{\mu}\Delta_{+}(x,y)\right]\,D_{\mu\nu}(x,y)$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)\Delta_{+}(x,y)\,\left[\partial^{y}_{\nu}\partial^{x}_{\mu}\Delta_{+}(y,x)\right]\,D_{\mu\nu}(x,y)$$

$$+\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^{x}_{\mu}\Delta_{+}(x,y)]\,[\partial^{y}_{\nu}\Delta_{+}(y,x)]\,D_{\mu\nu}(x,y) \tag{C.64}$$

The process P.6

$$+\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^{\dagger}(y'')[\partial^{y}_{\nu}\Delta_{+}(y'',y)][\partial^{x}_{\mu}\Delta_{+}(y,x)]\Delta_{+}(x,z)K(z)D_{\mu\nu}(x,y)$$

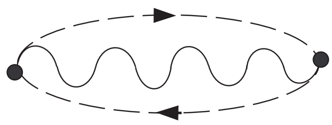

Figure C.5 Diagram corresponding to the process P.5.

$$
\begin{aligned}
&-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^{\dagger}(y'')[\partial^y_\nu\Delta_+(y'',y)]\Delta_+(y,x)[\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)\\
&+\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')[\partial^x_\mu\Delta_+(y',x)][\partial^y_\nu\Delta_+(x,y)]\Delta_+(y,z')K(z')D_{\mu\nu}(x,y)\\
&-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')\Delta_+(y',x)\left[\partial^y_\nu\partial^x_\mu\Delta_+(x,y)\right]\Delta_+(y,z')K(z')D_{\mu\nu}(x,y)\\
&-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y'')K^{\dagger}(y'')\Delta_+(y'',y)\left[\partial^y_\nu\partial^x_\mu\Delta_+(y,x)\right]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)\\
&+\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y'')(\mathrm{d}z)K^{\dagger}(y'')\Delta_+(y'',y)[\partial^y_\nu\Delta_+(y,x)][\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)\\
&-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,y)[\partial^y_\nu\Delta_+(y,z')]K(z')D_{\mu\nu}(x,y)\\
&+\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}y')(\mathrm{d}z')K^{\dagger}(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,y)][\partial^y_\nu\Delta_+(y,z')]K(z')D_{\mu\nu}(x,y)
\end{aligned}
\tag{C.65}
$$

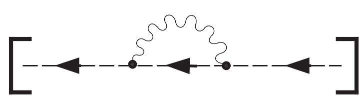

Figure C.6 spin 0 particle's self energy diagram in scalar electrodynamics.

The process P.7

$$
\begin{aligned}
&+\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y'',y)]_{y''=y}\\
&-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y'',y)]_{y''=y}
\end{aligned}
$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(y',x)]_{y'=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y,z')]_{z'=y}$$

$$+\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y,z')]_{z'=y} \qquad \text{(C.66)}$$

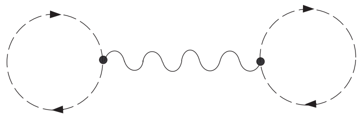

Figure C.7 Diagram corresponding to the process P.7.

The process P.8

$$-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(x,z)]_{z=x}D_{\mu\nu}(x,y)K^\dagger(y''')\,[\partial^y_\nu\Delta_+(y''',y)]$$

$$\times\,\Delta_+(y,z')K(z')$$

$$+\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)D_{\mu\nu}(x,y)$$

$$\times\,[\partial^y_\nu\Delta_+(y'',y)]_{y''=y}$$

$$-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')K^\dagger(y')\Delta_+(y',x)[\partial^x_\mu\Delta_+(x,z)]K(z)D_{\mu\nu}(x,y)$$

$$\times\,[\partial^y_\nu\Delta_+(y'',y)]_{y''=y}$$

$$+\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z')(\mathrm{d}y''')[\partial^x_\mu\Delta_+(x,z)]_{z=x}K^\dagger(y''')\Delta_+(y''',y)\,[\partial^y_\nu\Delta_+(y,z')]$$

$$\times\,K(z')D_{\mu\nu}(x,y)$$

$$-\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')K^\dagger(y')[\partial^x_\mu\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times\,D_{\mu\nu}(x,y)\,[\partial^y_\nu\Delta_+(y,z')]_{z'=y}$$

$$- \frac{\mathrm{i}}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y') K^\dagger(y') [\partial^x_\mu \Delta_+(y', x)] \Delta_+(x, z) K(z)$$

$$\times \, [\partial^y_\nu D_{\mu\nu}(x, y)] \, \Delta_+(y, y)$$

$$+ \frac{\mathrm{i}}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y') K^\dagger(y') \Delta_+(y', x) [\partial^x_\mu \Delta_+(x, z)] K(z)$$

$$\times \, D_{\mu\nu}(x, y) \, [\partial^y_\nu \Delta_+(y, z')]_{z'=y}$$

$$+ \frac{\mathrm{i}}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y') K^\dagger(y') \Delta_+(y', x) [\partial^x_\mu \Delta_+(x, z)] K(z)$$

$$\times \, [\partial^y_\nu D_{\mu\nu}(x, y)] \, \Delta_+(y, y) \tag{C.67}$$

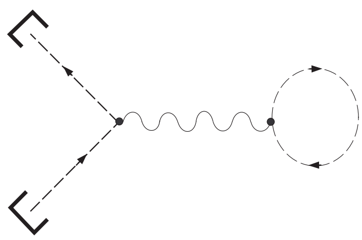

Figure C.8 Diagram corresponding to the process P.8.

The process P.9

$$- \frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''') K^\dagger(y') [\partial^x_\mu \Delta_+(y', x)] \Delta_+(x, z) K(z)$$

$$\times \, K^\dagger(y''') \, [\partial^y_\nu \Delta_+(y''', y)] \, \Delta_+(y, z') K(z') D_{\mu\nu}(x, y)$$

$$+ \frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}y''') K^\dagger(y') \Delta_+(y', x) [\partial^x_\mu \Delta_+(x, z)] K(z)$$

$$\times \, K^\dagger(y''') \, [\partial^y_\nu \Delta_+(y''', y)] \, \Delta_+(y, z') K(z') D_{\mu\nu}(x, y)$$

$$+ \frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''') K^\dagger(y') [\partial^x_\mu \Delta_+(y', x)] \Delta_+(x, z) K(z)$$

$$\times K^\dagger(y''')\Delta_+(y''',y)\,[\partial_\nu^y\Delta_+(y,z')]\,K(z')D_{\mu\nu}(x,y)$$

$$+\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}y''')K^\dagger(y')[\partial_\mu^x\Delta_+(y',x)]\Delta_+(x,z)K(z)$$

$$\times K^\dagger(y''')\Delta_+(y''',y)\Delta_+(y,z')K(z')\,[\partial_\nu^y D_{\mu\nu}(x,y)]$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}z')(\mathrm{d}y')(\mathrm{d}y''')K^\dagger(y')\Delta_+(y',x)[\partial_\mu^x\Delta_+(x,z)]K(z)$$

$$\times K^\dagger(y''')\Delta_+(y''',y)\,[\partial_\nu^y\Delta_+(y,z')]\,K(z')D_{\mu\nu}(x,y) \tag{C.68}$$

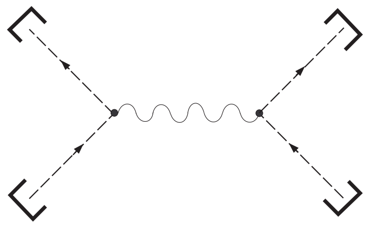

Figure C.9 A scattering diagram in scalar electrodynamics.

The process P.10

$$+\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\left[\partial_\mu^x\Delta_+(y,x)\right][\partial_\nu^y\Delta_+(x,y)]\,D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\,\mathrm{i}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}x'')\Delta_+(y,x)\left[\partial_\nu^y\partial_\mu^x\Delta_+(x,y)\right]D_{\mu\beta}(x,x')J^\beta(x')$$

$$\times D_{\nu\alpha}(y,x'')J^\alpha(x'') \tag{C.69}$$

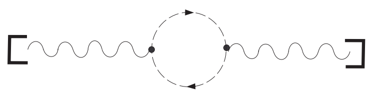

Figure C.10 Photon self energy diagram in scalar electrodynamics.

The process P.11

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\left[\partial^y_\nu\Delta_+(y'',y)\right]\left[\partial^x_\mu\Delta_+(y,x)\right]$$

$$\times\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\left[\partial^y_\nu\Delta_+(y'',y)\right]\Delta_+(y,x)\left[\partial^x_\mu\Delta_+(x,z)\right]$$

$$\times K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu\Delta_+(y',x)\right]\left[\partial^y_\nu\Delta_+(x,y)\right]D_{\nu\alpha}(y,x'')$$

$$\times\Delta_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\left[\partial^x_\mu\Delta_+(y',x)\right]\Delta_+(x,y)\left[\partial^y_\nu D_{\nu\alpha}(y,x'')\right]$$

$$\times\Delta_+(y,z')K(z')D_{\mu\beta}(x,x')J^\beta(x')J^\alpha(x'')$$

$$+\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'')K^\dagger(y')\Delta_+(y',x)\left[\partial^y_\nu\partial^x_\mu\Delta_+(x,y)\right]\Delta_+(y,z')$$

$$\times K(z')D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$+\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',y)\left[\partial^y_\nu\partial^x_\mu\Delta_+(y,x)\right]$$

$$\times\Delta_+(x,z)K(z)D_{\mu\beta}(x,x')J^\beta(x')D_{\nu\alpha}(y,x'')J^\alpha(x'')$$

$$-\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)(\mathrm{d}x')(\mathrm{d}x'')(\mathrm{d}y'')K^\dagger(y'')\Delta_+(y'',y)\left[\partial^y_\nu\Delta_+(y,x)\right]\left[\partial^x_\mu\Delta_+(x,z)\right]$$

$$\times K(z) D_{\mu\beta}(x,x') J^{\beta}(x') D_{\nu\alpha}(y,x'') J^{\alpha}(x'')$$

$$+ \frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'') K^{\dagger}(y') \left[\partial^x_\mu \Delta_+(y',x)\right] \Delta_+(x,y)$$

$$\times \Delta_+(y,z') K(z') \left[\partial^y_\nu D_{\nu\alpha}(y,x'')\right] D_{\mu\beta}(x,x') J^{\beta}(x') J^{\alpha}(x'')$$

$$- \frac{1}{2} \int (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}x')(\mathrm{d}y')(\mathrm{d}z')(\mathrm{d}x'') K^{\dagger}(y') \Delta_+(y',x) \left[\partial^x_\mu \Delta_+(x,y)\right] \left[\partial^y_\nu \Delta_+(y,z')\right]$$

$$\times K(z') D_{\mu\beta}(x,x') J^{\beta}(x') D_{\nu\alpha}(y,x'') J^{\alpha}(x'') \qquad \text{(C.70)}$$

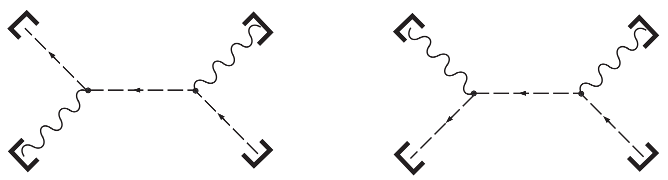

Figure C.11 Scattering diagrams in scalar electrodynamics.

APPENDIX D

COMPUTATIONS RELEVANT TO $e^-e^- \to e^-e^-$ SCATTERING

D.1 Polarizations Correlations: Initially Unpolarized Particles

The amplitude of process $e^-e^- \to e^-e^-$ may be written as

$$\mathcal{A} \propto \frac{\overline{u}(p_1')\gamma^\mu u(p_1)\overline{u}(p_2')\gamma_\mu u(p_2)}{(p_1'-p_1)^2} - \frac{\overline{u}(p_2')\gamma^\mu u(p_1)\overline{u}(p_1')\gamma_\mu u(p_2)}{(p_2'-p_1)^2}$$

$$= -\frac{\overline{u}(p_1')\gamma^0 u(p_1)\overline{u}(p_2')\gamma^0 u(p_2)}{(p_1'-p_1)^2} + \frac{\overline{u}(p_1')\gamma^j u(p_1)\overline{u}(p_2')\gamma^j u(p_2)}{(p_1'-p_1)^2}$$

$$+ \frac{\overline{u}(p_2')\gamma^0 u(p_1)\overline{u}(p_1')\gamma^0 u(p_2)}{(p_2'-p_1)^2} - \frac{\overline{u}(p_2')\gamma^j u(p_1)\overline{u}(p_1')\gamma^j u(p_2)}{(p_2'-p_1)^2}. \tag{D.1}$$

For the process $e^-e^- \to e^-e^-$, in the c.m., with initially unpolarized spins, with momenta $\mathbf{p}_1 = \gamma m\beta(0,1,0) = -\mathbf{p}_2$, we take for the momenta of the final electrons

$$\mathbf{p'}_1 = \gamma m\beta(1,0,0) = -\mathbf{p}_2', \tag{D.2}$$

and for the four-spinors

$$u(p_1') = \left(\frac{p^0+m}{2m}\right)^{1/2}\begin{pmatrix}\xi_1 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\xi_1\end{pmatrix}, \quad \xi_1 = \begin{pmatrix}-\mathrm{i}\cos\chi_1/2 \\ \sin\chi_1/2\end{pmatrix} \tag{D.3}$$

$$u(p_2') = \left(\frac{p^0+m}{2m}\right)^{1/2} \begin{pmatrix} \xi_2 \\ -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m}\xi_2 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -\mathrm{i}\cos\chi_2/2 \\ \sin\chi_2/2 \end{pmatrix} \tag{D.4}$$

$$\overline{u}(p_1') = \left(\frac{p^0+m}{2m}\right)^{1/2} \begin{pmatrix} \xi_1^\dagger & -\xi_1^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} -\mathrm{i}\cos\chi_1/2 \\ \sin\chi_1/2 \end{pmatrix} \tag{D.5}$$

$$\overline{u}(p_2') = \left(\frac{p^0+m}{2m}\right)^{1/2} \begin{pmatrix} \xi_2^\dagger & +\xi_2^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'}{p^0+m} \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -\mathrm{i}\cos\chi_2/2 \\ \sin\chi_2/2 \end{pmatrix}, \tag{D.6}$$

where

$$\xi_1 = \begin{pmatrix} -\mathrm{i}\cos\chi_1/2 \\ \sin\chi_1/2 \end{pmatrix} \tag{D.7}$$

$$\xi_2 = \begin{pmatrix} -\mathrm{i}\cos\chi_2/2 \\ \sin\chi_2/2 \end{pmatrix} \tag{D.8}$$

$$\xi_1^\dagger = \begin{pmatrix} \mathrm{i}\cos\chi_1/2 & \sin\chi_1/2 \end{pmatrix} \tag{D.9}$$

$$\xi_2^\dagger = \begin{pmatrix} \mathrm{i}\cos\chi_2/2 & \sin\chi_2/2 \end{pmatrix}. \tag{D.10}$$

A straightforward but tedious computation of the corresponding probability of occurrence with initially unpolarized electrons, (D.1) leads to

$$\begin{aligned}
\text{Prob} \propto\, & [\overline{u}(p_1')\gamma^\mu(-\gamma p_1+m)\gamma^\sigma u(p_1')]\,[\overline{u}(p_2')\gamma_\mu(-\gamma p_2+m)\gamma_\sigma u(p_2')] \\
& - [\overline{u}(p_1')\gamma^\mu(-\gamma p_1+m)\gamma^\sigma u(p_2')]\,[\overline{u}(p_2')\gamma_\mu(-\gamma p_2+m)\gamma_\sigma u(p_1')] \\
& - [\overline{u}(p_2')\gamma^\mu(-\gamma p_1+m)\gamma^\sigma u(p_1')]\,[\overline{u}(p_1')\gamma_\mu(-\gamma p_2+m)\gamma_\sigma u(p_2')] \\
& + [\overline{u}(p_2')\gamma^\mu(-\gamma p_1+m)\gamma^\sigma u(p_2')]\,[\overline{u}(p_1')\gamma_\mu(-\gamma p_2+m)\gamma_\sigma u(p_1')]\,.
\end{aligned} \tag{D.11}$$

By using program Mathematica 5 to calculate probability of this process, we obtain ($p \equiv |\mathbf{p}|$)

$$\text{Prob} \propto \frac{1}{2m^2(m+p^0)^2}\Big\{5m^2 - 15m^2p^2 + 31m^2p^4 + 3p^6 + 8m^2p^4\cos(\chi_1+\chi_2)$$

$$+ \left[-5m^6 - 25m^4p^2 + m^2p^4 - 3p^6\right]\cos(\chi_1-\chi_2) + 16mp^0p^4\cos(\chi_1+\chi_2)$$

$$- 4mp^0\left[-3m^4 + 6m^2p^2 - 15p^4\left[3m^4 + 14m^2p^2 - p^4\right]\cos(\chi_1-\chi_2)\right]$$

$$+ (p^0)^2\left[3m^4 + 30m^2p^2 + 31p^4\left[-3m^4 - 30m^2p^2 + p^4\right]\cos(\chi_1-\chi_2)\right]$$

$$+ 8(p^0)^2p^4\cos(\chi_1+\chi_2)$$

$$+ (p^0)^2 8m\left[-3m^2 + 9p^2 + \left[m^2+p^2\right]\cos(\chi_1-\chi_2)\right]$$

$$+ (p^0)^4\left[3[m^2+11p^2]\left[-3m^2+7p^2\right]\cos(\chi_1-\chi_2)\right]$$

$$+ \left[-12m(p^0)^5 - 5(p^0)^6\right]\left[-1+\cos(\chi_1-\chi_2)\right]\Big\}. \tag{D.12}$$

We define momenta of two initial electrons in term of speed (β) as

$$p = \frac{m\beta}{\sqrt{1-\beta^2}}, \tag{D.13a}$$

$$p^0 = \frac{m}{\sqrt{1-\beta^2}}. \tag{D.13b}$$

So that, we simplify Eq. D.12 in term of speed (β)

$$\text{Prob} \propto A(\beta)\left[1 + 2\beta^2 + 6\beta^4\left[-1 - 2\beta^2 + 3\beta^4\right]\cos(\chi_1-\chi_2) + \beta^4\cos(\chi_1+\chi_2)\right], \tag{D.14}$$

where

$$A(\beta) = -\frac{4m^2\left[-2\left(1-\sqrt{1-\beta^2}\right)+\beta^2\left(2+\sqrt{1-\beta^2}\right)\right]}{[1-\beta^2]^{5/2}\left(\sqrt{1-\beta^2}\right)^2}. \qquad \text{(D.15)}$$

from $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$, and we can neglect $A(\beta)$ because it disappear after the probability is normalized. After simplification and of collecting Eq. D.12 reduces to

$$\text{Prob} \propto (1-\beta^2)(1+3\beta^2)\sin^2\left(\frac{\chi_1-\chi_2}{2}\right)+\beta^4\cos^2\left(\frac{\chi_1+\chi_2}{2}\right)+4\beta^4. \qquad \text{(D.16)}$$

APPENDIX E

BELL'S THEOREM AND ENTANGLED STATES; THE C-H INEQUALITY

E.1 Bell's Theorem

The formulation of Bell's theorem started much later after Einstein, Podolsky and Rosen formulated the EPR paradox, in 1935, to argue that quantum mechanics is not complete as it is, in particular, at best probabilistic. They argue that the prediction of the measurement in quantum mechanics is inconsistent, and the theory should have some variables to make it complete. As a result, they suggested that variables, so-called "hidden variables" may be necessary for a more complete theory. We will describe a simplified version of EPR paradox. Consider two particles each with spin $1/2$ emerging from some source moving in opposite directions with total spin 0–the singlet. From the measurement of spin of one of the particles, call it particle 1, and found, say, to be along $\mathbf{n}$, one may conclude instantaneously, because of the correlation implied by the total spin 0-state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\mathbf{n}\rangle_1 \, |-\mathbf{n}\rangle_2 - |-\mathbf{n}\rangle_1 \, |\mathbf{n}\rangle_2 \right) \tag{E.1}$$

that the component of spin of the other particle, call it particle 2, is along $-\mathbf{n}$ without ever disturbing this latter particle. With no such disturbance, one may invoke locality, as a no-action at a distance, to infer that the value of the component of spin found indirectly for particle 2 must have existed prior to a measurement done on particle 1. Since $\mathbf{n}$ was arbitrary, one may also infer that all the components of spin of particle 2 were known to begin with. That is, all the components of spin of a particle are definite in clear contradiction with quantum mechanics of the non-commutativity of spins $[S_i, S_j] =$

$\mathrm{i}\hbar\epsilon_{ijk}S_k$, and the underlying theory of the latter is incomplete.

The above led to the belief that perhaps quantum mechanics is a limiting case of a more complete local theory which, involves, so-called, hidden variables. Such theories are referred to as *Local Hidden Variables* (LHV) theories.

In 1964, and in subsequent years, John Bell has put such theories to a test. Several tests have been also proposed in the literature by various authors. We refer to all such tests as Bell-like tests. We will discuss one originating from the work of Clauser, Horne, Shimoney and Holt (CHSH) [Clauser and Shimoney, 1978].

To the above end, and in view of applications to a system of two particles, as described below Eq. (E.1), and other similar processes, we consider the following in the light of LHV theories.

Let λ denote collectively the random variables expected to be relevant to the system under study with corresponding probability density or probability mass function $\mathrm{d}\rho(\lambda)$ normalized as

$$\int_{\Lambda} \mathrm{d}\rho(\lambda) = 1 \tag{E.2}$$

summed over the set Λ of all values that λ may take on.

One is interested in determining coincidence and single counts obtained in the measurements of the spins of the particles, after emerging from the process in question, making angles, say, χ_1, χ_2 with some given directions.

Suppose that the system is in a state specified by λ. We may introduce the following probabilities of counts:

$$p[\chi_1, \chi_2; \lambda], \quad p[\chi_1, -; \lambda], \quad p[-, \chi_2; \lambda] \tag{E.3}$$

correspondingly, respectively, to coincidence counts when measurements are made on both particles' spins, to a count when a measurement is made on only one particle (call it particle 1), and, finally, to a count when a measurement is made on particle 2 only.

In such a framework, one makes the key assumption that if the system is in any given state specified by λ, the probability count obtained from measurements performed on one particle is *independent* of the probability count corresponding to the other particle after they have emerged from the process. This is a key point. That is, the probability counts are necessarily factorable,

$$p[\chi_1, \chi_2; \lambda] = p[\chi_1, -; \lambda]p[-, \chi_2; \lambda] \tag{E.4}$$

for all λ in Λ, implying their *independence*, with all determined in the same state λ.

Now we use the fact that for any four numbers $0 \leqslant x_1, x_2, x_1', x_2' \leqslant 1$, we have the following elementary inequality

$$-1 \leqslant x_1 x_2 - x_1 x_2' + x_1' x_2 + x_1' x_2' - x_1' - x_2 \leqslant 0 \tag{E.5}$$

as established in Eq. (E.3).

Accordingly upon setting

$$\left.\begin{aligned} x_1 &= p[\chi_1, -; \lambda] \\ x_2 &= p[-, \chi_2; \lambda] \\ x_1' &= p[\chi_1', -; \lambda] \\ x_2' &= p[-, \chi_2'; \lambda] \end{aligned}\right\} \tag{E.6}$$

for four angles, χ_1, χ_2, χ_1', χ_2', and using the fact that probabilities, as in Eq. (E.3), necessarily must fall in the range $[0, 1]$, we have from Eq. (E.5) upon multiplying the latter by $\mathrm{d}\rho(\lambda)$ and summing (integrating) over λ:

$$-1 \leqslant p[\chi_1, \chi_2] - p[\chi_1, \chi_2'] + p[\chi_1', \chi_2] + p[\chi_1', \chi_2'] - p[\chi_1', -] - p[-, \chi_2] \leqslant 0 \tag{E.7}$$

where

$$p[\chi_1, \chi_2] = \int_\Lambda \mathrm{d}\rho(\lambda) p[\chi_1, -; \lambda] p[-, \chi_2; \lambda]$$

$$= \int_\Lambda \mathrm{d}\rho(\lambda) p[\chi_1, \chi_2; \lambda] \tag{E.8}$$

etc., where we have used, in particular, the factorization assumption in Eq. (E.5).

The inequality in Eq. (E.7) is expressed in terms of probability counts which may be determined experimentally putting LHV theories to a test. $p[\chi_1, \chi_2]$ denotes the joint probability count for measurements of both spins, while $p[\chi_1, -]$, $p[-, \chi_2]$ correspond to probability counts with measurements of only one of the spins.

In the bulk of this thesis, the probabilities computed from quantum theory corresponding to $p[\chi_1, \chi_2]$, $p[\chi_1, -]$, $p[-, \chi_2]$ will be denoted, respectively, by $P[\chi_1, \chi_2]$, $P[\chi_1, -]$, $P[-, \chi_2]$ with a capital "P".

In order to obtain a violation of the inequality in Eq. (E.7) experimentally, it is sufficient to choose any four angles χ_1, χ_2, χ_1', χ_2' that do the job since, according to the LHV reasoning, Eq. (E.7) must be true for all angles. Experiments show violation of the inequalities and are consistent with the quantum mechanical predictions. Experiments of optical nature have been performed and a classic one involving two photons with measurements made on photon polarization correlations is one due to Aspect *et. al.* [Aspect, Dalibard and Roger, 1982].

E.2 Entangled States

Consider two sets of independent vectors $|\alpha_i\rangle$, $|\beta_j\rangle$,

$$\langle \alpha_i | \alpha_j \rangle = \delta_{ij}, \quad \langle \beta_i | \beta_j \rangle = \delta_{ij} \tag{E.9}$$

then for any vector

$$|\psi\rangle = \sum_i c_i |\alpha_i\rangle |\beta_i\rangle \tag{E.10}$$

such that at least two of the coefficients c_i are non-zero, *cannot* be rewritten as a product

$$|\psi\rangle = |\psi_1\rangle |\psi_2\rangle \tag{E.11}$$

where

$$|\psi_1\rangle = \sum_i a_i |\alpha_i\rangle \tag{E.12}$$

$$|\psi_2\rangle = \sum_i b_i |\beta_i\rangle . \tag{E.13}$$

To show this, suppose, without any loss of generality that $c_1 \neq 0$ and $c_2 \neq 0$. Upon multiplying Eq. (E.10), in turn, by $\langle\alpha_1| \langle\beta_1|$, $\langle\alpha_2| \langle\beta_2|$ and using Eq. (E.3) we obtain

$$c_1 = a_1 b_1, \quad c_2 = a_2 b_2. \tag{E.14}$$

On the other hand by multiplying Eq. (E.10) in turn by $\langle\alpha_1| \langle\beta_2|$, $\langle\alpha_2| \langle\beta_1|$ and using (E.3) we obtain

$$0 = a_1 b_2, \quad 0 = a_2 b_1 \tag{E.15}$$

which upon comparison with Eq. (E.14) leads to the *contradiction* that at least one of c_1, c_2 is zero.

Definition: A state as defined in Eq. (E.10), with at least two of the coefficients c_i non-zero is called an *entangled state*.

E.3 The Clauser-Horne (C-H) Inequality

Consider four numbers $0 \leqslant x_1, x_2, x_1', x_2' \leqslant 1$, and set

$$U = x_1 x_2 - x_1 x_2' + x_1' x_2 + x_1' x_2' - x_1' - x_2. \tag{E.16}$$

We first derive the upper bound $U \leqslant 0$.

For $x_1 \geqslant x_1'$, we may rewrite

$$U = (x_1 - 1)x_2 + x_1'(x_2 - 1) + x_2'(x_1' - x_1)$$

$$\leqslant 0 \tag{E.17}$$

since every term is non-positive.

For $x_1 < x_1'$, we may rewrite

$$U = x_1(x_2 - x_2') + (x_1' - 1)x_2 - x_1'(1 - x_2')$$

$$\leqslant x_1(x_2 - x_2') + (x_1' - 1)x_2 - x_1(1 - x_2')$$

$$= x_1x_2 + (x_1' - 1)x_2 - x_1$$

$$= x_1(x_2 - 1) + (x_1' - 1)x_2 \leqslant 0. \tag{E.18}$$

We now derive the lower bound $-1 \leqslant U$.

For $x_1' \geqslant x_1$,

$$U + 1 = (1 - x_1')(1 - x_2) + x_1x_2 + x_2'(x_1' - x_1)$$

$$\geqslant 0 \tag{E.19}$$

since every term is non-negative.

For $x_1 > x_1'$,

$$U + 1 = (1 - x_1')(1 - x_2) - (x_1 - x_1')(x_2' - x_2) + x_1'x_2$$

$$\geqslant (x_1 - x_1')(1 - x_2) - (x_1 - x_1')(x_2' - x_2) + x_1'x_2$$

$$= (x_1 - x_1')(1 - x_2') + x_1' x_2 \geqslant 0. \tag{E.20}$$

All told, we have

$$-1 \leqslant U \leqslant 0. \tag{E.21}$$

APPENDIX F

THE ROTATION MATRIX

For a general orientation of any vector $\mathbf{r}$, we may rotate it, say, clockwise (c.w.) about vector $\mathbf{n}$ (see in figure **F.1**)

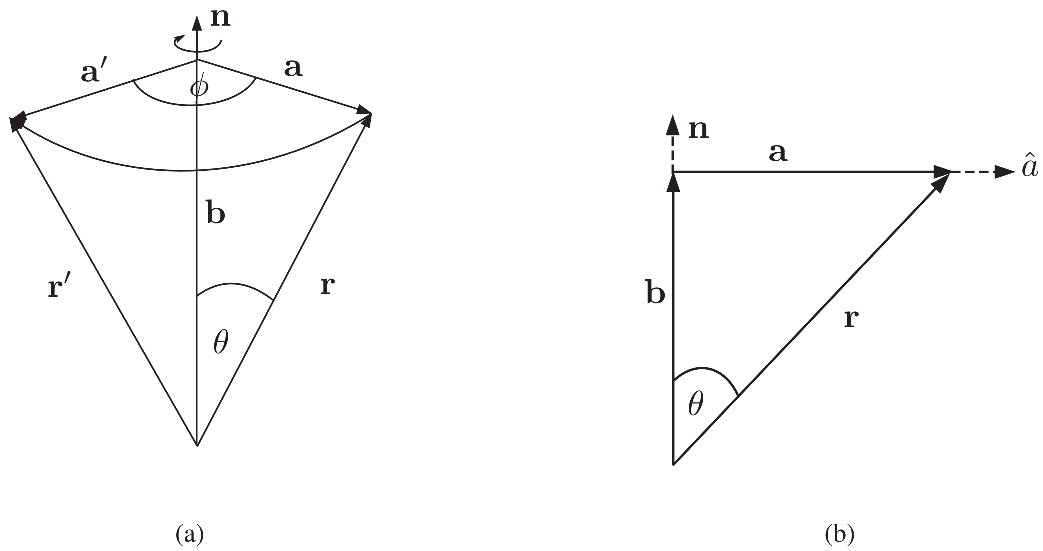

Figure F.1 (a) The figure shows the vector $\mathbf{r}$ rotated clockwise (c.w.) by an angle ϕ about a vector $\mathbf{n}$. (b) The figure depicts the $\mathbf{n}$-$\mathbf{r}$ plane, with $\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$ denoting a unit vector in the direction of $\mathbf{a}$.

By using the geometric properties, we have

$$\mathbf{a} = \mathbf{r} - \mathbf{b}$$

$$= \mathbf{r} - |\mathbf{r}| \cos\theta \mathbf{n}$$

$$\mathbf{a} = \mathbf{r} - (\mathbf{r} \cdot \mathbf{n})\mathbf{n}, \tag{F.1}$$

therefore, we get (from figure F.1(b))

$$|\mathbf{r}|\sin\theta\hat{a} = \mathbf{r} - (\mathbf{r}\cdot\mathbf{n})\mathbf{n}. \tag{F.2}$$

Let

$$\mathbf{c} = \mathbf{n}\times\mathbf{r} = |\mathbf{r}|\sin\theta\hat{c}. \tag{F.3}$$

From figures F.2(a) and F.2(b), we obtain

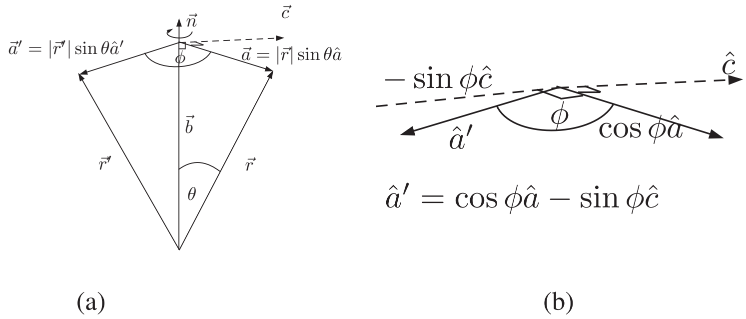

Figure F.2 (a) The figure depicts $\mathbf{n}\times\mathbf{r}$ in the direction of $\mathbf{c}$ (b) The figure depicts $\mathbf{a}' = \cos\phi\hat{a} - \sin\phi\hat{c}$.

$$\mathbf{a}' = |\mathbf{r}'|\sin\theta\hat{a}'$$

$$= |\mathbf{r}|\sin\theta(\cos\phi\hat{a} - \sin\phi\hat{c})$$

$$\mathbf{a}' = |\mathbf{r}|\sin\theta\cos\phi\hat{a} - |\mathbf{r}|\sin\theta\sin\phi\hat{c}, \tag{F.4}$$

giving

$$\mathbf{r}' - \mathbf{r} = \mathbf{a}' - \mathbf{a}'$$

$$= |\mathbf{r}|\sin\theta\cos\phi\hat{a} - |\mathbf{r}|\sin\theta\sin\phi\hat{c} - |\mathbf{r}|\sin\theta\hat{a}$$

$$\mathbf{a}' = |\mathbf{r}|\sin\theta(\cos\phi - 1)\hat{a} - |\mathbf{r}|\sin\theta\sin\phi\hat{c}. \tag{F.5}$$

From Eqs. (F.2) and (F.3), we can rewrite Eq. (F.5) as

$$\mathbf{r}' - \mathbf{r} = (\mathbf{r} - (\mathbf{r}\cdot\mathbf{n})\mathbf{n})(\cos\phi - 1) - \sin\phi(\mathbf{n}\times\mathbf{r})$$

$$\mathbf{r}' = \mathbf{r} - \sin\phi(\mathbf{n}\times\mathbf{r}) + (\mathbf{r} - (\mathbf{r}\cdot\mathbf{n})\mathbf{n})(\cos\phi - 1). \tag{F.6}$$

Use the properties of the tensor with

$$\epsilon^{ijk} = \begin{cases} 1 & \text{even permutation} \\ 0 & \text{(two or more)} \\ -1 & \text{odd permutation} \end{cases} \tag{F.7}$$

giving

$$(r')^i = r^i - \epsilon^{ijk} n^j r^k \sin\phi + (r^i - (r^k n^k) n^i \mathbf{n})(\cos\phi - 1)$$

$$= \delta^{ik} r^k - \epsilon^{ijk} n^j r^k \sin\phi + (\delta^{ik} r^k - (r^k n^k) n^i \mathbf{n})(\cos\phi - 1)$$

$$(r')^i = \left[\delta^{ik} - \epsilon^{ijk} n^j \sin\phi + (\delta^{ik} - n^k n^i \mathbf{n})(\cos\phi - 1)\right] r^k. \tag{F.8}$$

In the rotated form, we may write

$$(r')^i = R^{ik} r^k. \tag{F.9}$$

So that, we obtain for the rotation matrix the expression

$$R^{ik} = \delta^{ik} - \epsilon^{ijk} n^j \sin\phi + (\delta^{ik} - n^k n^i \mathbf{n})(\cos\phi - 1). \tag{F.10}$$

APPENDIX G

NUMERICAL RESULTS FOR THE STATISTIC S CORRESPONDING TO SPECIFIC ANGLES OF MEASUREMENTS

Table G.1 Angles used in calculating $S < -1$ in QED (Spin half) in Process 1

β	χ_1	χ_2	χ_3	χ_4	$S < -1$
0.00	0	23	45	67	-1.207
0.01	0	23	45	67	-1.207
0.02	0	23	45	67	-1.206
0.03	0	23	45	67	-1.204
0.04	0	23	45	67	-1.202
0.05	0	23	45	67	-1.200
0.06	0	23	45	67	-1.196
0.07	0	23	45	67	-1.193
0.08	0	23	45	67	-1.189
0.09	0	23	45	67	-1.185
0.10	0	23	45	67	-1.180
0.11	0	23	45	67	-1.174
0.12	0	23	45	67	-1.168
0.13	0	23	45	67	-1.161
0.14	0	23	45	67	-1.154
0.15	0	23	45	67	-1.147

β	χ_1	χ_2	χ_3	χ_4	$S < -1$
0.16	0	23	45	67	-1.139
0.17	0	23	45	67	-1.131
0.18	0	23	45	67	-1.123
0.19	0	23	45	67	-1.114
0.20	0	23	45	67	-1.105
0.21	0	23	45	67	-1.096
0.22	0	23	45	67	-1.086
0.23	0	23	45	67	-1.076
0.24	0	23	45	67	-1.066
0.25	0	23	45	67	-1.056
0.26	0	23	45	67	-1.046
0.27	0	23	45	67	-1.035
0.28	0	23	45	67	-1.025
0.29	0	23	45	67	-1.014
0.30	0	23	45	67	-1.003

Table G.2 Angles used in calculating $S > 0$ in QED (Spin half) in Process 1

β	χ_1	χ_2	χ_3	χ_4	$S > 0$
0.00	0	67	135	23	0.207
0.01	0	67	135	23	0.207
0.02	0	67	135	23	0.206
0.03	0	67	135	23	0.204
0.04	0	67	135	23	0.202
0.05	0	67	135	23	0.200
0.06	0	67	135	23	0.196
0.07	0	67	135	23	0.193
0.08	0	67	135	23	0.189
0.09	0	67	135	23	0.185
0.10	0	67	135	23	0.180
0.11	0	67	135	23	0.174
0.12	0	67	135	23	0.168
0.13	0	67	135	23	0.161
0.14	0	67	135	23	0.154
0.15	0	67	135	23	0.147
0.16	0	67	135	23	0.139
0.17	0	67	135	23	0.131
0.18	0	67	135	23	0.123
0.19	0	67	135	23	0.114
0.20	0	67	135	23	0.105
0.21	0	67	135	23	0.096
0.22	0	67	135	23	0.086
0.23	0	67	135	23	0.076
0.24	0	67	135	23	0.066
0.25	0	67	135	23	0.056
0.26	0	67	135	23	0.046
0.27	0	67	135	23	0.035
0.28	0	67	135	23	0.025
0.29	0	67	135	23	0.014
0.30	0	67	135	23	0.003

Table G.3 Angles used in calculating $S > 0$ in Scalar Electrodynamics (Spin 0) in Process 1

β	χ_1	χ_2	χ_3	χ_4	$S > 0$
0.00	0	23	45	67	0.207
0.01	0	23	45	67	0.207
0.02	0	23	45	67	0.206
0.03	0	23	45	67	0.204
0.04	0	23	45	67	0.202
0.05	0	23	45	67	0.200
0.06	0	23	45	67	0.196
0.07	0	23	45	67	0.193
0.08	0	23	45	67	0.189
0.09	0	23	45	67	0.185
0.10	0	23	45	67	0.180
0.11	0	23	45	67	0.174
0.12	0	23	45	67	0.168
0.13	0	23	45	67	0.161
0.14	0	23	45	67	0.154
0.15	0	23	45	67	0.147
0.16	0	23	45	67	0.139
0.17	0	23	45	67	0.131
0.18	0	23	45	67	0.123
0.19	0	23	45	67	0.114
0.20	0	23	45	67	0.105
0.21	0	23	45	67	0.096
0.22	0	23	45	67	0.086
0.23	0	23	45	67	0.076
0.24	0	23	45	67	0.066
0.25	0	23	45	67	0.056
0.26	0	23	45	67	0.046
0.27	0	23	45	67	0.035
0.28	0	23	45	67	0.025
0.29	0	23	45	67	0.014
0.30	0	23	45	67	0.003

Table G.4 Angles used in calculating $S < -1$ in Scalar Electrodynamics (Spin 0) in Process 1

β	χ_1	χ_2	χ_3	χ_4	$S < -1$
0.00	0	67	135	23	-1.207
0.01	0	67	135	23	-1.207
0.02	0	67	135	23	-1.206
0.03	0	67	135	23	-1.204
0.04	0	67	135	23	-1.202
0.05	0	67	135	23	-1.200
0.06	0	67	135	23	-1.196
0.07	0	67	135	23	-1.193
0.08	0	67	135	23	-1.189
0.09	0	67	135	23	-1.185
0.10	0	67	135	23	-1.180
0.11	0	67	135	23	-1.174
0.12	0	67	135	23	-1.168
0.13	0	67	135	23	-1.161
0.14	0	67	135	23	-1.154
0.15	0	67	135	23	-1.147

β	χ_1	χ_2	χ_3	χ_4	$S < -1$
0.16	0	67	135	23	-1.139
0.17	0	67	135	23	-1.131
0.18	0	67	135	23	-1.123
0.19	0	67	135	23	-1.114
0.20	0	67	135	23	-1.105
0.21	0	67	135	23	-1.096
0.22	0	67	135	23	-1.086
0.23	0	67	135	23	-1.076
0.24	0	67	135	23	-1.066
0.25	0	67	135	23	-1.056
0.26	0	67	135	23	-1.046
0.27	0	67	135	23	-1.035
0.28	0	67	135	23	-1.025
0.29	0	67	135	23	-1.014
0.30	0	67	135	23	-1.003

Table G.5 Angles used in calculating $S > 0$ for Spin half and Spin 0 in Process 2

β	χ_1	χ_2	χ_3	χ_4	$S > 0$
0.00	-	-	-	-	unviable
0.01	0	67	135	23	0.207
0.02	0	67	135	23	0.206
0.03	0	67	135	23	0.205
0.04	0	67	135	23	0.203
0.05	0	67	135	23	0.201
0.06	0	67	135	23	0.199
0.07	0	67	135	23	0.196
0.08	0	67	135	23	0.192
0.09	0	67	135	23	0.188
0.10	0	67	135	23	0.184
0.11	0	67	135	23	0.179
0.12	0	67	135	23	0.174
0.13	0	67	135	23	0.169
0.14	0	67	135	23	0.163
0.15	0	67	135	23	0.156

β	χ_1	χ_2	χ_3	χ_4	$S > 0$
0.16	0	67	135	23	0.150
0.17	0	67	135	23	0.143
0.18	0	67	135	23	0.136
0.19	0	67	135	23	0.128
0.20	0	67	135	23	0.120
0.21	0	67	135	23	0.112
0.22	0	67	135	23	0.104
0.23	0	67	135	23	0.095
0.24	0	67	135	23	0.087
0.25	0	67	135	23	0.078
0.26	0	67	135	23	0.068
0.27	0	67	135	23	0.059
0.28	0	67	135	23	0.049
0.29	0	67	135	23	0.039
0.30	0	67	135	23	0.029

Table G.6 Angles used in calculating $S < -1$ for Spin half and Spin 0 in Process 2

β	χ_1	χ_2	χ_3	χ_4	$S < -1$
0.00	-	-	-	-	unviable
0.01	0	23	45	67	-1.207
0.02	0	23	45	67	-1.206
0.03	0	23	45	67	-1.205
0.04	0	23	45	67	-1.203
0.05	0	23	45	67	-1.201
0.06	0	23	45	67	-1.199
0.07	0	23	45	67	-1.196
0.08	0	23	45	67	-1.192
0.09	0	23	45	67	-1.188
0.10	0	23	45	67	-1.184
0.11	0	23	45	67	-1.179
0.12	0	23	45	67	-1.174
0.13	0	23	45	67	-1.169
0.14	0	23	45	67	-1.163
0.15	0	23	45	67	-1.156

β	χ_1	χ_2	χ_3	χ_4	$S < -1$
0.16	0	23	45	67	-1.150
0.17	0	23	45	67	-1.143
0.18	0	23	45	67	-1.136
0.19	0	23	45	67	-1.128
0.20	0	23	45	67	-1.120
0.21	0	23	45	67	-1.112
0.22	0	23	45	67	-1.104
0.23	0	23	45	67	-1.095
0.24	0	23	45	67	-1.087
0.25	0	23	45	67	-1.078
0.26	0	23	45	67	-1.068
0.27	0	23	45	67	-1.059
0.28	0	23	45	67	-1.049
0.29	0	23	45	67	-1.039
0.30	0	23	45	67	-1.029

APPENDIX H

STRING THEORY ASPECTS USED IN THIS WORK

H.1 Introduction

We here consider some basic properties of relativistic strings needed in Chapter VI of this work. String theory is ambitious in that it promises to unify conventional particle theory with Einstein's theory of gravity. We will not, however, go into much details and deal with some general aspects only and we will eventually be dealing with those detail used in this work as studied in §H.2 and §H.3. In the following two sections we determine the electromagnetic current associated with a particular charged string, and the energy-momentum tensor associated with a Neutral Nambu string.

H.2 The Charged String

The dynamics of the string is described as follows. The trajectory of the string is described by a vector function $\mathbf{R}(\sigma, t)$, where σ is the parameter along the string. The equations of motion for the closed string is taken to be (see, eg. Kibble and Turok, 1982; Manoukian, 1991 and Sakellariadou, 1990)

$$\ddot{\mathbf{R}} - \mathbf{R}'' = 0 \tag{H.1}$$

where

$$\dot{\mathbf{R}} = \mathrm{d}\mathbf{R}/\mathrm{d}t \tag{H.2}$$

$$\mathbf{R}' = \partial\mathbf{R}/\partial\sigma \tag{H.3}$$

with the constraints equations

$$\dot{\mathbf{R}} \cdot \mathbf{R}' = 0 \tag{H.4}$$

$$\dot{\mathbf{R}}^2 + \mathbf{R}'^2 = 1 \tag{H.5}$$

$$\mathbf{R}(\sigma + (2\pi/m), t) = \mathbf{R}(\sigma, t) \tag{H.6}$$

where dots and primes stand for derivatives with respect to t and σ, respectively, and m, so far, is an arbitrary mass scale.

The general solution of Eq. (H.1) is

$$\mathbf{R} = \frac{1}{2}[\mathbf{A}(\sigma - t) + \mathbf{B}(\sigma + t)] \tag{H.7}$$

where $\mathbf{A}$, $\mathbf{B}$ satisfy, in particular, the normalization conditions that we obtain

$$\mathbf{A}'^2 = \mathbf{B}'^2 = 1 \tag{H.8}$$

we consider a solution of the form (Manoukian, 1991)

$$\mathbf{R} = \frac{1}{m}(\cos m\sigma, \sin m\sigma, 0)\sin mt \tag{H.9}$$

describing a radially oscillating circular string. The general expression for the electromagnetic current of the string is given by (Kibble and Turok, 1982; Albrecht and Turok, 1989; Manoukian, 1991; Sakellariadou, 1990)

$$\mathbf{J} = \frac{Qm}{2\pi}\int_0^{2\pi/m} \mathrm{d}\sigma \dot{\mathbf{R}} \delta^3(\mathbf{r} - \mathbf{R}(\sigma, t)) \tag{H.10a}$$

$$J^0 = \frac{Qm}{2\pi}\int_0^{2\pi/m} \mathrm{d}\sigma \delta^3(\mathbf{r} - \mathbf{R}(\sigma, t)) \tag{H.10b}$$

where Q denotes the total charge of string, $R^0 = t$, and $\mathbf{r}$ lies in the plane of string.

Because of the symmetry of the problem, we find it convenient to work in cylindrical coordinates. Thus, $\mathbf{r} = (r\cos\theta, r\sin\theta, z)$ and

$$\delta^3(\mathbf{r} - \mathbf{R}(\sigma,t)) = \frac{1}{r}\delta\left(r - \frac{|\sin mt|}{m}\right)\delta\left(\theta - m\sigma - \frac{\pi}{2}\{1 - \mathrm{sgn}(\sin mt)\}\right)\delta(z). \tag{H.11}$$

Therefore, the explicit evaluation of J^μ read as: when we have

$$\dot{\mathbf{R}} = (\cos m\sigma, \sin m\sigma, 0)\cos mt.$$

For $\mu = 1$, we read J^1 as

$$J^1 = \frac{Qm}{2\pi}\int_0^{2\pi/m} \mathrm{d}\sigma \cos m\sigma \cos mt\ \delta^3(\mathbf{r} - \mathbf{R}(\sigma,t)).$$

By using property in Eq. H.11, we rewrite J^1 as:

$$\begin{aligned}
&= \frac{Qm}{2\pi}\int_0^{2\pi/m}\mathrm{d}\sigma \cos m\sigma \cos mt \frac{1}{r}\ \delta\left(r - \frac{|\sin mt|}{m}\right)\delta\left(\theta - m\sigma - \frac{\pi}{2}\{1 - \mathrm{sgn}(\sin mt)\}\right)\delta(z) \\
&= \frac{Q}{2\pi}\cos\left(\theta - \frac{\pi}{2}\{1 - \mathrm{sgn}(\sin mt)\}\right)\cos mt \frac{1}{r}\delta\left(r - \frac{|\sin mt|}{m}\right)\delta(z) \\
&= \frac{Q}{2\pi}\cos\theta\ \mathrm{sgn}(\sin mt)\cos mt \frac{1}{r}\delta\left(r - \frac{|\sin mt|}{m}\right)\delta(z).
\end{aligned} \tag{H.12}$$

For $\mu = 2$, we read J^2 as

$$J^2 = \frac{Qm}{2\pi}\int_0^{2\pi/m} \mathrm{d}\sigma \sin m\sigma \cos mt\ \delta^3(\mathbf{r} - \mathbf{R}(\sigma,t)).$$

Similarly, by using property in Eq. H.11, we rewrite J^2 as:

$$\begin{aligned}
&= \frac{Qm}{2\pi}\int_0^{2\pi/m}\mathrm{d}\sigma \sin m\sigma \cos mt \frac{1}{r}\ \delta\left(r - \frac{|\sin mt|}{m}\right)\delta\left(\theta - m\sigma - \frac{\pi}{2}\{1 - \mathrm{sgn}(\sin mt)\}\right)\delta(z) \\
&= \frac{Q}{2\pi}\sin\left(\theta - \frac{\pi}{2}\{1 - \mathrm{sgn}(\sin mt)\}\right)\cos mt \frac{1}{r}\delta\left(r - \frac{|\sin mt|}{m}\right)\delta(z)
\end{aligned}$$

$$= \frac{Q}{2\pi}\sin\theta\,\mathrm{sgn}(\sin mt)\cos mt\frac{1}{r}\delta\left(r-\frac{|\sin mt|}{m}\right)\delta(z). \tag{H.13}$$

Finally, $\mu = 0$, we read J^0 as

$$J^0 = \frac{Qm}{2\pi}\int_0^{2\pi/m}\mathrm{d}\sigma\;\delta^3(\mathbf{r}-\mathbf{R}(\sigma,t))$$

$$= \frac{Qm}{2\pi}\int_0^{2\pi/m}\mathrm{d}\sigma\frac{1}{r}\delta\left(r-\frac{|\sin mt|}{m}\right)\delta\left(\theta-m\sigma-\frac{\pi}{2}\{1-\mathrm{sgn}(\sin mt)\}\right)\delta(z)$$

$$J^0 = \frac{Q}{2\pi}\frac{\delta\left(r-\frac{|\sin mt|}{m}\right)}{r}\delta(z). \tag{H.14}$$

So that, we obtain

$$\mathbf{J} = \frac{Q}{2\pi}(\cos\theta,\sin\theta,0)\frac{\delta\left(r-\frac{|\sin mt|}{m}\right)}{r}\delta(z)\cos mt\;\mathrm{sgn}(\sin mt) \tag{H.15a}$$

$$J^0 = \frac{Q}{2\pi}\frac{\delta\left(r-\frac{|\sin mt|}{m}\right)}{r}\delta(z) \tag{H.15b}$$

in cylindrical coordinates where $\mathbf{r}$ *now* lies in the plane of the string. Using the periodicity of J^μ in time we may write:

$$J^\mu(\mathbf{r},z,t) = \sum_{N=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}Nmt}\int\frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}q}{2\pi}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\mathrm{e}^{\mathrm{i}qz}B^\mu(\mathbf{p},N) \tag{H.16}$$

where

$$B^\mu(\mathbf{p},N) = \frac{m}{2\pi}\int_{-\pi/m}^{\pi/m}\mathrm{d}t\;\mathrm{e}^{\mathrm{i}Nmt}\int\mathrm{d}^2\mathbf{r}\int_{-\infty}^{\infty}\mathrm{d}z\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{i}qz}J^\mu(\mathbf{r},z,t) \tag{H.17}$$

Upon writing

$$\mathbf{p} = p(\cos\phi,\sin\phi,0) \tag{H.18}$$

$$\mathbf{p}\cdot\mathbf{r} = pr\cos(\phi' - \phi), \quad \mathrm{d}^2\mathbf{r} = r\mathrm{d}r\mathrm{d}\phi' \tag{H.19}$$

using the expansion (Table of Integral):

$$\exp\left[\mathrm{i}\rho\cos(\phi' - \phi)\right] = \sum_{n=-\infty}^{\infty} (\mathrm{i})^n \exp\left[\mathrm{i}n(\phi' - \phi)\right] J_n(\rho) \tag{H.20}$$

the integrals (Table of Integral):

$$\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}NT} J_M\left(a|\sin T|\right)\mathrm{d}T = 2\pi\cos\left(\frac{N\pi}{2}\right) J_{\frac{M-N}{2}}\left(\frac{a}{2}\right) J_{\frac{M+N}{2}}\left(\frac{a}{2}\right) \tag{H.21}$$

$$\int_{-\pi}^{\pi} \sin(NT) J_M\left(a\sin T\right)\mathrm{d}T = \pi\sin\left(\frac{N\pi}{2}\right) J_{\frac{M-N}{2}}\left(\frac{a}{2}\right) J_{\frac{M+N}{2}}\left(\frac{a}{2}\right) \tag{H.22}$$

or,

$$\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}NT}\mathrm{sgn}(\sin T) J_M\left(a|\sin T|\right)\mathrm{d}T = 2\pi\mathrm{i}\sin\left(\frac{N\pi}{2}\right) J_{\frac{M-N}{2}}\left(\frac{a}{2}\right) J_{\frac{M+N}{2}}\left(\frac{a}{2}\right) \tag{H.23}$$

the property,

$$\sum_{n=-\infty}^{\infty} (-\mathrm{i})^n \mathrm{e}^{\mathrm{i}n\phi} J_n(\rho) \int_0^{2\pi} \mathrm{d}\phi' \mathrm{e}^{-\mathrm{i}n\phi'}(\cos\phi', \sin\phi', 0)$$

$$= (-\mathrm{i})2\pi(\cos\phi, \sin\phi, 0) J_1(\rho) \tag{H.24}$$

and the elementary recurrence relation

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) \tag{H.25}$$

we obtain the following expressions for B^μ:

$$B^0(\mathbf{p}, N) = Q(-1)^{N/2}\cos\left(\frac{N\pi}{2}\right) J^2_{N/2}\left(\frac{p}{2m}\right) \tag{H.26}$$

$$\mathbf{B}(\mathbf{p},N) = \frac{Q_m}{p} N(-1)^{N/2} \cos\left(\frac{N\pi}{2}\right)(\cos\phi, \sin\phi, 0) J^2_{N/2}\left(\frac{p}{2m}\right) \tag{H.27}$$

and $J_n(x)$ is the Bessel function of order n.

The equations Eq. (H.16), Eq. (H.18), Eq. (H.25) and Eq. (H.26) may be written in a more convenient form as:

$$J^\mu(\mathbf{r},z,t) = \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi} \int \frac{\mathrm{d}P^0}{2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \mathrm{e}^{\mathrm{i}qz} \mathrm{e}^{-\mathrm{i}p^0 t} J^\mu(P) \tag{H.28}$$

$$J^\mu(P) = J^\mu(P^0,\mathbf{p}) = 2 \sum_{N=-\infty}^{\infty} \delta(P^0 - mN) B^\mu(\mathbf{p},N) \tag{H.29}$$

$$P = (P^0, \mathbf{p}, q), \tag{H.30}$$

$$B^0(\mathbf{p},N) = a_N J^2_{N/2}\left(\frac{p}{2m}\right), \tag{H.31}$$

$$\mathbf{B}(\mathbf{p},N) = \frac{mN}{p^2}\mathbf{p} B^0(\mathbf{p},N), \tag{H.32}$$

$$a_N = Q(-1)^{N/2} \cos\left(\frac{N\pi}{2}\right), \tag{H.33}$$

We note that there is no q dependence in Eq. (H.29), Eqs. (H.31)–(H.33).

H.3 The Neutral String

The general expression for the energy-momentum tensor of the string is given by (Kibble and Turok, 1982; Albrecht and Turok, 1989; Sakellariadou, 1990 and Manoukian, 1997)

$$T^{\mu\nu} = \frac{m^2}{2\pi} \int_0^{2\pi/m} \mathrm{d}\sigma \left(\partial_t R^\mu \partial_t R^\nu - \partial_\sigma R^\mu \partial_\sigma R^\nu\right) \delta^3(\mathbf{r} - \mathbf{R}(\sigma,t)). \tag{H.34}$$

More explicitly, it is given by

$$T^{\mu\nu}(\mathbf{r},z,t) = \int \frac{\mathrm{d}p^0}{2\pi} \int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \mathrm{e}^{\mathrm{i}qz} \mathrm{e}^{-\mathrm{i}p^0 t} T^{\mu\nu}(p^0,\mathbf{p},q) \tag{H.35}$$

$$T^{\mu\nu}(p^0,\mathbf{p},q) \equiv T^{\mu\nu}(p^0,\mathbf{p}) = 2\sum_{N=-\infty}^{\infty} \delta(p^0 - mN) B^{\mu}(\mathbf{p},N) \tag{H.36}$$

$$B^{00}(\mathbf{p},N) = \beta_N J^2_{N/2}(x)\,, \tag{H.37}$$

$$B^{0a}(\mathbf{p},N) = \beta_N \frac{p^0 p^a}{\mathbf{p}^2} J^2_{N/2}(x)\,; \quad a = 1,2, \tag{H.38}$$

$$B^{ab}(\mathbf{p},N) = \beta_N A_N \delta^{ab} + \beta_N E_N \frac{p^a p^b}{\mathbf{p}^2} J^2_{N/2}(x)\,; \quad a = 1,2, \tag{H.39}$$

$$B^{\mu 3}(\mathbf{p},N) = 0, \tag{H.40}$$

$$A_N = \frac{1}{4}\left[J^2_{\frac{N}{2}+1}(x) + J^2_{\frac{N}{2}-1}(x) - 2J_{\frac{N}{2}+1}(x) J_{\frac{N}{2}-1}(x) \right], \tag{H.41}$$

$$E_N = J_{\frac{N}{2}+1}(x) J_{\frac{N}{2}-1}(x), \tag{H.42}$$

$$\beta_N = m(-1)^{N/2} \cos\left(\frac{N\pi}{2}\right). \tag{H.43}$$

One readily verifies the conservation laws $\partial_\mu J^\mu = 0$, $\partial_\mu T^{\mu\nu} = 0$ directly by checking that

$$p_\mu B^\mu = 0, \quad p_\mu B^{\mu\nu} = 0 \tag{H.44}$$

where

$$p^0 = mN = -p_0 \tag{H.45}$$

is the total energy of the mono-energetic pair of the oppositely charged scalar particles each of mass m.

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