INVARIANTS OF THE EQUIVALENCE GROUP OF A SYSTEM OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Mr. Sakka Sookmee

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วิทยานิพนธ์นี้ศึกษาระบบสมการของสมการเชิงอนุพันธ์สามัญอันดับสอง 2 สมการ โดยปัญหาที่นำมาศึกษาเกี่ยวข้องกับการหาการแปลงที่ผกผันได้ของตัวแปรไม่อิสระและตัวแปรอิสระ ทำให้ระบบสมการดังกล่าวเปลี่ยนรูปเป็นระบบสมการเชิงเส้น เรียกปัญหานี้ว่าปัญหาการทำให้เป็นเชิงเส้น การศึกษานี้พบว่าผลยืนยงสัมพัทธ์ของกลุ่มสมมูลช่วยในการวิเคราะห์เงื่อนไขที่จำเป็นสำหรับปัญหาการทำให้เป็นเชิงเส้น ในวิทยานิพนธ์นี้ได้แสดงผลยืนยงสัมพัทธ์อันดับหนึ่งและอันดับสอง ซึ่งผลยืนยงดังกล่าวคำนวณได้จากระบบสมการของสมการเชิงอนุพันธ์ย่อยเอกพันธุ์ ซึ่งมีจำนวนสมการมากกว่าจำนวนตัวแปรไม่อิสระ

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This thesis is devoted to the study of a system of two second-order ordinary differential equations. The problem considered is related to the linearization problem, which is to find a transformation consisting of an invertible change of the independent and dependent variables, transforming the given system into a linear system. Necessary conditions for the linearization problem can be found through relative invariants of the equivalence group. In this thesis relative invariants of first and second orders are found. For finding these invariants, an overdetermined system of linear homogeneous partial differential equations was solved.

School of Mathematics | Student's Signature Sakka Sookmee

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Sakka Sookmee

CONTENTS

CHAPTER I

INTRODUCTION

1.1 Background and History

Many physical phenomena are described by differential equations. Ordinary differential equations play a significant role in the theory of differential equations. In the 19th century, one of the most important problems in analysis was the problem of classification of ordinary differential equations.

One type of classification problem is the equivalence problem: a differential equation is said to be equivalent to another equation, if there exists an invertible transformation of the independent and dependent variables (point transformation) which transforms one equation into the another. The linearization problem is a particular case of the equivalence problem, where one of the equations is a linear equation.

In mathematical history, Sophus Lie (1883) was the first who studied the linearization problem of second-order ordinary differential equations. He gave the criteria for a second-order ordinary differential equation to be linearizable. He showed that every linearizable second-order ordinary differential equation has the form

$$y'' + a(x,y)y'^3 + b(x,y)y'^2 + c(x,y)y' + d(x,y) = 0, \tag{1.1}$$

where

$$y' = \frac{dy}{dx}\ , \quad y'' = \frac{d^2y}{dx^2}\ .$$

To see this, notice that the general form of a linear second-order ordinary differential equation is

$$u'' + e(t)u' + f(t)u + g(t) = 0, \tag{1.2}$$

where

$$u' = \frac{du}{dt}\ , \quad u'' = \frac{d^2u}{dt^2}\ .$$

Equation (1.2) can be also reduced to the equation

$$u'' = 0. \tag{1.3}$$

Assume that equation (1.1) is obtained from (1.3) by an invertible transformation of the independent and dependent variables,

$$t = \varphi(x, y),\ u = \psi(x, y). \tag{1.4}$$

It is assumed that $D_x\varphi = \varphi_x + y'\varphi_y \neq 0$ and $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$. The derivatives are transformed by formulae

$$\begin{aligned} u' &= g(x, y, y') = \frac{D_x\psi}{D_x\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y}\ , \\ u'' &= p(x, y, y', y'') = \frac{D_x g}{D_x\varphi} = \frac{g_x + y'g_y + y''g_{y'}}{\varphi_x + y'\varphi_y} \\ &= \frac{(\varphi_x + y'\varphi_y)^{-3}}{\Delta}[y'' + a(x, y)y'^3 + b(x, y)y'^2 + c(x, y)y' + d(x, y)], \end{aligned}$$

where

$$\begin{aligned} a &= \Delta^{-1}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y), \\ b &= \Delta^{-1}(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)), \\ c &= \Delta^{-1}(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)), \\ d &= \Delta^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x). \end{aligned} \tag{1.5}$$

Substituting u' and u'' into (1.3), one obtains (1.1). Thus, if a second-order ordinary differential equation is linearizable, then it has the form (1.1). Notice that not every equation of form (1.1) is linearizable. For finding linearizing transformation (1.4), one has to solve the overdetermined system (1.5) with respect to the functions $\varphi(x, y)$ and $\psi(x, y)$. Hence, the linearization problem is to obtain conditions which guarantee the existence of functions (1.4) satisfying system (1.5). Lie found that equation (1.1) can be transformed into equation (1.3) by transformation (1.4) if and only if the coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$ and $d(x, y)$ satisfy the conditions

$$\begin{aligned} H &= 3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0, \\ K &= b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y a - 3d_x a + 3d_y b = 0. \end{aligned} \tag{1.6}$$

The function H and K are called relative invariants. Notice that the functions H and K are relative invariants with respect to invertible transformation of independent and dependent variables.

There are other approaches for solving the linearization problem, for example, one was developed by E.Cartan (1924). He used differential geometry for solving this problem.

These two approaches (Lie's approach and Cartan's approach) were also applied to third-order ordinary differential equations, for examples, by Chern (1943), G.Grebot (1996), M.Petitot and S.Neut (2002), N.Ibragimov and S.Meleshko (2004, 2005).

1.2 Statement of the Problem

This thesis is devoted to the study a system of two second-order ordinary differential equations

$$y_1'' = f_1(x, y_1, y_2, y_1', y_2')\ ,\ y_2'' = f_2(x, y_1, y_2, y_1', y_2'). \tag{1.7}$$

The problem considered is part of the linerization problem. For system (1.7) the linearization problem is to find an invertible transformation of independent and dependent variables

$$t = \varphi(x, y_1, y_2),\ u_1 = \psi_1(x, y_1, y_2),\ u_2 = \psi_2(x, y_1, y_2), \tag{1.8}$$

which transforms system of equations (1.7) into a linear system of equations

$$u'' + B(t)u = 0, \tag{1.9}$$

where

$$u = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}.$$

Notice that any system of linear equations

$$u'' + C(t)u' + D(t)u + E(t) = 0,$$

where

$$C(t) = \begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix}, \quad D(t) = \begin{pmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & d_{22}(t) \end{pmatrix},$$

$$u = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad E(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix},$$

can be reduced to the form (1.9).

Similar to Lie's case of a single second-order ordinary differential equation, in order to obtain necessary conditions, we assume that system (1.7) is obtained from the linear system of differential equations (1.9) by an invertible transformation (1.8). Since the derivatives are changed by formulae

$$u_1' = g_1(x, y_1, y_2, y_1', y_2') = \frac{D_x\psi_1}{D_x\varphi} \ ,$$
$$u_1'' = p_1(x, y_1, y_2, y_1', y_2', y_1'', y_2'') = \frac{D_x g_1}{D_x\varphi} \ ,$$
$$u_2' = g_2(x, y_1, y_2, y_1', y_2') = \frac{D_x\psi_2}{D_x\varphi} \ ,$$
$$u_2'' = p_2(x, y_1, y_2, y_1', y_2', y_1'', y_2'') = \frac{D_x g_2}{D_x\varphi},$$

where $D_x = \frac{\partial}{\partial x} + y_1'\frac{\partial}{\partial y_1} + y_2'\frac{\partial}{\partial y_2} + y_1''\frac{\partial}{\partial y_1'} + y_2''\frac{\partial}{\partial y_2'}$, replacing u_1', u_1'' ,u_2', and u_2'' in system (1.9), it becomes

$$\begin{aligned} F_1 = y_1'' + a_{11}{y_1'}^3 + a_{12}{y_1'}^2y_2' + a_{13}y_1'{y_2'}^2 + a_{14}{y_1'}^2 \\ +a_{15}y_1'y_2' + a_{16}{y_2'}^2 + a_{17}y_1' + a_{18}y_2' + a_{19} = 0, \\ F_2 = y_2'' + a_{13}{y_2'}^3 + a_{12}{y_2'}^2y_1' + a_{11}y_2'{y_1'}^2 + a_{24}{y_1'}^2 \\ +a_{25}y_2'y_1' + a_{26}{y_2'}^2 + a_{27}y_1' + a_{28}y_2' + a_{29} = 0, \end{aligned} \tag{1.10}$$

where the coefficients a_{ij} are expressed through the functions φ, ψ_1, ψ_2, their partial derivatives and functions b_{ij}. Thus, if system (1.7) is linearizable, then it has the form (1.10).

The computational calculations show that form (1.10) is not changed by any transformation (1.8). The main goal of this thesis is to find other invariants and to obtain some necessary conditions for the linearization problem.

For solving the problem of the thesis, Lie's approach was used. This approach contains of the following steps.

1. Find the equivalence group of transformations for system (1.10).
2. Obtain equations for invariants.
3. Solve the equations defining invariants.

Since each step needs a huge amount of analytical calculations, it is necessary to use a computer for these calculations. A brief review of computer systems of symbolic manipulations can be found, for example, in Davenport (1993). In our calculations the system REDUCE (cf. Hearn (1999)) was used.

This thesis is provided as follows. Chapter II introduces some background knowledge of elementary Lie group analysis, which is necessary for our study. Chapter III presents the equivalence group of transformations for system (1.10). Chapter IV presents relative invariants. The source code of the Reduce computer program can be obtained by contacting the author by e-mail at the address shown in the Curriculum Vitae.

CHAPTER II

GROUP ANALYSIS

Group analysis is a powerful method for analyzing differential equations. A part of the group analysis method is devoted to equivalence transformations. A Lie group of equivalence transformations can be applied for finding invariants of equations. These invariants are not changed during any change of the independent and dependent variables. An introduction to this method can be found in textbooks (cf. Ovsiannikov (1978), Olver (1984), Ibragimov (1999). Many results obtained by this method are collected in the Handbook of Lie group analysis (1994), (1995), (1996)).

2.1 Local Lie Group

In this section, we review some background knowledge of elementary Lie group analysis, which is necessary for our study.

We consider invertible point transformations

$$\bar{z}^i = g^i(z; a), \tag{2.1}$$

where $i = 1, 2, ..., N, z \in V \subset R^N$ and a is parameter, $a \in \Delta$. The set V is an open set in R^N, and Δ is an symmetrical interval in R^1 w.r.t. zero.

For differential equations the variable z is separated into two parts, $z = (x, u) \in V \subset R^n \times R^m, N = n + m$. Here $x = (x_1, x_2, ..., x_n) \in R^n$ is considered as the independent variable, $u = (u^1, u^2, ..., u^m) \in R^m$ is considered as the dependent variable. For the transformations we use

$$\bar{x}_i = \varphi^i(x, u; a), \;\; \bar{u}^j = \psi^j(x, u; a), \tag{2.2}$$

where $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, $(x, u) \in V \subset R^n \times R^m$, and the set V is open in $R^n \times R^m$.

2.1.1 One-Parameter Lie Group of Transformations

Definition 1. *A set of transformation (2.1) is called a local one-parameter Lie group if it has the following properties*

1. $g(z; 0) = z$ *for all* $z \in V$*.*

2. $g(g(z; a), b) = g(z; a + b)$ *for all* $a, b, a + b \in \Delta, z \in V$*.*

3. If for $a \in \Delta$ *we have* $g(z; a) = z$ *for all* $z \in V$*, then* $a = 0$*.*

4. $g \in C^\infty(V, \Delta)$*.*

On definition of Lie group is a local, one because we only require that V is an open neighborhood of some z_0 and Δ is a small symmetrical interval around zero.

Define

$$\xi^i(x, u) = \left.\frac{\partial \varphi^i(x, u; a)}{\partial a}\right|_{a=\;0}, \eta^j(x, u) = \left.\frac{\partial \psi^j(x, u; a)}{\partial a}\right|_{a=\;0},$$

and,

$$X = \xi^i(x, u)\partial_{x_i} + \eta^j(x, u)\partial_{u^j}. \tag{2.3}$$

The operator X is called an infinitesimal generator or a generator of the Lie group of transformations (2.2). The coefficients ξ^i, η^j are called the coefficients of the generator.

A local Lie group of transformations (2.2) can be completely determined by the solution of the Cauchy problem of ordinary differential equations, which are called Lie equations:

$$\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}a} = \xi^i(\bar{x}, \bar{u}), \quad \frac{\mathrm{d}\bar{u}^j}{\mathrm{d}a} = \eta^j(\bar{x}, \bar{u}) \tag{2.4}$$

with the initial data

$$\bar{x}_i\big|_{a=\ 0} = x_i, \ \ \bar{u}^j\big|_{a=\ 0} = u^j. \tag{2.5}$$

Theorem 1 (Lie). *Given a vector field* $\zeta = (\xi, \eta) : V \to R^N$ *of class* $C^\infty(V)$ *with* $\zeta(z_0) \neq 0$ *for some* $z_0 \in V$. *Then the solution of the Cauchy problem (2.4), (2.5) generates a local Lie group with the infinitesimal generator* $X = \xi^i(x, u)\partial_{x_i} + \eta^j(x, u)\partial_{u^j}$. *Conversely, let functions* $\varphi^i(x, u; a),\ i = 1, ..., n$ *and* $\psi^j(x, u; a),\ j = 1, ..., m$ *satisfy the properties of a Lie group and have the expansion*

$$\bar{x}_i = \varphi^i(x, u; a) \approx x_i + \xi^i(x, u)a,$$

$$\bar{u}^j = \psi^j(x, u; a) \approx u^j + \eta^j(x, u)a$$

where

$$\xi^i(x, u) = \frac{\partial\varphi^i(x, u; a)}{\partial a}\bigg|_{a=\ 0}, \eta^j(x, u) = \frac{\partial\psi^j(x, u; a)}{\partial a}\bigg|_{a=\ 0},$$

then the functions $\varphi^i(x, u; a),\ \psi^j(x, u; a)$ *solve the Cauchy problem (2.4), (2.5).*

Precisely, the Lie's theorem establishes a one-to-one correspondence between Lie group of transformations and infinitesimal generator.

2.1.2 Prolongation of a Lie Group

Given $Z = R^n \times R^m$, consider when space Z is prolonged by introducing the variables $p = (p^k_\alpha)$. Here $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index. For a multi-index the notations $|\alpha| \equiv \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\alpha, i \equiv (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_n)$ are used. The variable p^k_α plays a role of the derivative,

$$p^k_\alpha = \frac{\partial^{|\alpha|}u^k}{\partial x^\alpha} = \frac{\partial^{|\alpha|}u^k}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}\ .$$

The space J^l of the variables

$$x = (x_i), u = (u_k), p = (p^k_\alpha)$$

$$(i = 1, 2, ..., n;\ k = 1, 2, ..., m;\ |\alpha| \leq l)$$

is called the l-th prolongation of the space Z. This space can be provided with a manifold structure. For convenience we agree that $J^0 \equiv Z$.

Definition 2. *The generator*

$$X^l = X + \sum_{j,\alpha} \eta^j_\alpha \partial_{p^j_\alpha}, \qquad (j = 1, ..., m,\ |\alpha| \leq l),$$

with the coefficients

$$\eta^j_{\widetilde{\alpha},k} = D_k \eta^j_{\widetilde{\alpha}} - \sum_i p^j_{\widetilde{\alpha},i} D_k \xi^i, \qquad (|\widetilde{\alpha}| \leq l-1), \tag{2.6}$$

is called the l-th prolongation of the generator X.

Here the operators

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha} p^j_{\alpha,k} \frac{\partial}{\partial p^j_\alpha}\ , \qquad (k = 1, 2, ..., n),$$

are operators of the total derivatives with respect to x_k and $\eta^j_0 = \eta^j$, where ξ^i, η^j are defined as in (2.3).

For a simple illustration of using the prolongation formulae (2.6), let us study the first prolongation of the generator X with $n = m = 1$. In this case, the generator X^1 induces a local Lie group of transformations in the space J^1:

$$\bar{x} = \varphi(x, u; a),\ \bar{u} = \psi(x, u; a),\ \bar{p} = f(x, u, p; a), \tag{2.7}$$

with the generator

$$X^1 = \xi^x(x, u)\partial_x + \eta^u(x, u)\partial_u + \zeta^p(x, u, p)\partial_p\ , \tag{2.8}$$

where

$$\zeta^p = D_x(\eta^u) - pD_x(\xi^x),\ p = \frac{du}{dx}\ ,\ \bar{p} = \frac{d\bar{u}}{d\bar{x}}\ .$$

Notice that the coefficients ξ^x, η^u are defined as in (2.3). Let us show in the following why coefficient ζ^p must be of this form. Let a function $u_0(x)$ be given. Substituting it into the first equation of (2.7), one obtains

$$\bar{x} = \varphi(x, u_0(x); a).$$

Since $\varphi(x, u_0(x); 0) = x$, the Jacobian at $a = 0$ is

$$\left.\frac{\partial \bar{x}}{\partial x}\right|_{a=\ 0} = \left.\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x}\right)\right|_{a=\ 0} = 1.$$

Thus, by virtue of the inverse function theorem, in some neighborhood of $a = 0$ one can express x as a function of $\bar{x}$ and a,

$$x = \phi(\bar{x}, a). \tag{2.9}$$

Note that after substituting (2.9) into the first equation (2.7), one has the identity

$$\bar{x} = \varphi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \tag{2.10}$$

Substituting (2.9) into the second equation of (2.7), one obtains the transformed function

$$u_a(\bar{x}) = \psi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \tag{2.11}$$

Differentiating the function $u_a(\bar{x})$ with respect to $\bar{x}$, one gets

$$\bar{u}_{\bar{x}} = \frac{\partial u_a(\bar{x})}{\partial \bar{x}} = \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \psi}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}u_0'\right)\frac{\partial \phi}{\partial \bar{x}},$$

where the derivative $\frac{\partial \phi}{\partial \bar{x}}$ can be found by differentiating equation (2.10) with respect to $\bar{x}$,

$$1 = \frac{\partial \varphi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\right)\frac{\partial \phi}{\partial \bar{x}}.$$

Since

$$\frac{\partial \varphi}{\partial x}(\phi(\bar{x},0),u_0(\phi(\bar{x},0));0)=1,\ \frac{\partial \varphi}{\partial u}(\phi(\bar{x},0),u_0(\phi(\bar{x},0));0)=0, \tag{2.12}$$

one has $\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}u_0'\neq 0$ in some neighborhood of $a=0$. Thus,

$$\frac{\partial \phi}{\partial \bar{x}}=\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}u_0'\right)^{-1},$$

and

$$\bar{u}_{\bar{x}}=\left(\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial u}u_0'\right)\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}u_0'\right)^{-1}=:g(x,u_0,u_0';a). \tag{2.13}$$

Transformation (2.7) together with

$$\bar{u}_{\bar{x}}=g(x,u,u_x;a)$$

is called the prolongation of (2.7). Now, we define the coefficient ζ^p as follows:

$$\zeta^p(x,u,p)=\left.\frac{\partial g(x,u,p;a)}{\partial a}\right|_{a=0},\ \ g|_{a=0}=p. \tag{2.14}$$

Equation (2.13) can be rewritten

$$g(x,u,p;a)\left(\frac{\partial \varphi(x,u;a)}{\partial x}+p\frac{\partial \varphi(x,u;a)}{\partial u}\right)=\left(\frac{\partial \psi(x,u;a)}{\partial x}+p\frac{\partial \psi(x,u;a)}{\partial u}\right).$$

Differentiating this equation with respect to the group parameter a and substituting $a=0$, one finds

$$\left.\left(\frac{\partial g}{\partial a}\left(\frac{\partial \varphi}{\partial x}+p\frac{\partial \varphi}{\partial u}\right)+g\left(\frac{\partial^2 \varphi}{\partial x\partial a}+p\frac{\partial^2 \varphi}{\partial u\partial a}\right)\right)\right|_{a=0}=\left.\left(\frac{\partial^2 \psi}{\partial x\partial a}+p\frac{\partial^2 \psi}{\partial u\partial a}\right)\right|_{a=0}$$

or

$$\begin{aligned}
\zeta^p(x,u,p) &= \left.\frac{\partial g}{\partial a}\right|_{a=0}\left.\left(\frac{\partial \varphi}{\partial x}+p\frac{\partial \varphi}{\partial u}\right)\right|_{a=0},\ \text{since by (2.12)}\left.\left(\frac{\partial \varphi}{\partial x}+p\frac{\partial \varphi}{\partial u}\right)\right|_{a=0}=1\\
&= \left.\left(\frac{\partial^2 \psi}{\partial x\partial a}+p\frac{\partial^2 \psi}{\partial u\partial a}\right)\right|_{a=0}-g|_{a=0}\left.\left(\frac{\partial^2 \varphi}{\partial x\partial a}+p\frac{\partial^2 \varphi}{\partial u\partial a}\right)\right|_{a=0}\\
&= \left(\frac{\partial \eta^u}{\partial x}+p\frac{\partial \eta^u}{\partial u}\right)-p\left(\frac{\partial \xi^x}{\partial x}+p\frac{\partial \xi^x}{\partial u}\right)\\
&= D_x(\eta^u)-pD_x(\xi^x)
\end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + p_x\frac{\partial}{\partial p} + ... \ , \ \ \xi^x = \left.\frac{\partial \varphi}{\partial a}\right|_{a=0} , \ \ \eta^u = \left.\frac{\partial \psi}{\partial a}\right|_{a=0} , \ \ \zeta^p = \left.\frac{\partial g}{\partial a}\right|_{a=0} .$$

Thus, the first prolongation of the generator (2.3) is given by

$$X^{(1)} = X + \zeta^p(x, u, p)\partial_p.$$

Similarly one can obtain prolongation formulae for any order prolongation of an infinitesimal generator.

Admitted Lie groups of transformations are related with differential equations by the following.

2.1.3 Lie groups admitted by differential equations

Consider a manifold M which is defined by a system of partial differential equations

$$F^k(x, u, p) = 0, \quad (k = 1, 2, ..., s). \tag{2.15}$$

Hence

$$M = \{(x, u, p) \mid \quad F^k(x, u, p) = 0, \quad (k = 1, ..., s)\}.$$

Here x is the independent variable, u is the dependent variable and p are arbitrary partial derivatives of u with respect x. The manifold M is assumed to be regular, i.e.

$$rank \left(\frac{\partial(F)}{\partial\ (u, p)}\right) = s.$$

Definition 3. *A manifold M is said to be invariant with respect to the group of transformations (2.2), if these transformations carry every point of the manifold M along this manifold, i.e.*

$$F^k(\bar{x}, \bar{u}, \bar{p}) = 0, \quad (k = 1, 2, ..., s).$$

Accordingly, it is said that equations (2.15) are not changed under the Lie group of transformations or, in the other word, the Lie group of transformations (2.2) is admitted by equations (2.15).

In order to find an infinitesimal generator of a Lie group admitted by differential equations (2.15) one can use the following theorem.

Theorem 2. *A system of equations (2.15) is not changed with respect to the Lie group of transformations (2.2) with the infinitesimal generator*

$$X = \xi^i \partial_{x_i} + \eta^j \partial_{u^j} \ .$$

if and only if

$$X^{(p)} F^k \,|_{M=0} = 0, \ \ (k = 1, ..., s). \tag{2.16}$$

Equations (2.16) are called determining equations.

Definition 4. *A function $J(x, u)$ is called an invariant of a Lie group if*

$$J(\bar{x}, \bar{u}) = J(x, u).$$

Theorem 3. *A function $J(x, u)$ is an invariant of the Lie group with the generator X if and only if,*

$$XJ(x, u) = 0. \tag{2.17}$$

Definition 5. *A vector function $\vec{J}(x, u)$ defines a relative invariant if the manifold defined by the equation $\vec{J}(x, u) = 0$ is an invariant manifold.*

Using theorem 2, one obtains the following theorem.

Theorem 4. *The functions $J^k(x, u)$ are relative invariant of Lie group with the generator X if and only if,*

$$XJ^k \,|_{\vec{J}=0} = 0,$$

where J^k are all components of $\vec{J}$.

2.2 Equivalence Group

Consider a system of differential equations

$$F^k(x, u, p, \theta) = 0, \quad (k = 1, 2, ..., s). \tag{2.18}$$

Here $\theta = \theta(x, u)$ are arbitrary elements of system (2.18), $(x, u) \in V \subset R^{n+m}$, and $\theta : V \to R^t$.

A nondegenerate change of dependent and independent variables, which transforms a system of differential equations (2.18) to a system of equations of the same class or same differential structure is called an equivalence transformation.

The problem of finding a Lie group of equivalent transformations consists of the construction a transformation of the space $R^{n+m+t}(x, u, \theta)$ that preserves the equations, only changing their representative $\theta = \theta(x, u)$. For this purpose a one parameter Lie group of transformations of the space R^{n+m+t} with the group parameter a is used. Assume that the transformations

$$\begin{aligned} x' &= f^x(x, u, \theta; a), \\ u' &= f^u(x, u, \theta; a), \\ \theta' &= f^\theta(x, u, \theta; a), \end{aligned} \tag{2.19}$$

compose a Lie group of equivalence transformations. So the infinitesimal generator of this group (2.19) has the form:

$$X^e = \xi^{x_i}\partial_{x_i} + \zeta^{u^j}\partial_{u^j} + \zeta^{\theta^k}\partial_{\theta^k},$$

with the coefficients:

$$\begin{aligned} \xi^{x_i} &= \frac{\partial f^{x_i}(x, u, \theta; a)}{\partial a}\Big|_{a=0}, \\ \zeta^{u^j} &= \frac{\partial f^{u^j}(x, u, \theta; a)}{\partial a}\Big|_{a=0}, \\ \zeta^{\theta^k} &= \frac{\partial f^{\theta^k}(x, u, \theta; a)}{\partial a}\Big|_{a=0}, \end{aligned}$$

where $(i = 1, ..., n; j = 1, ..., m; k = 1, ..., t)$.

We use the main requirement for the Lie group of equivalence transformations that any solution $u_0(x)$ of the system (2.18) with the functions $\theta(x, u)$ is transformed by (2.19) into the solution $u = u_a(x')$ of the system (2.18) of the same equations F^k, but with another (transformed) functions $\theta_a(x, u)$. The functions $\theta_a(x, u)$ are defined as follows. Solving the relations

$$x' = f^x(x, u, \theta(x, u); a), \quad u' = f^u(x, u, \theta(x, u); a),$$

for (x, u), one obtains

$$x = g^x(x', u'; a), \quad u = g^u(x', u'; a). \tag{2.20}$$

The transformed function is

$$\theta_a(x', u') = f^\theta(x, u, \theta(x, u); a),$$

where, instead of (x, u) we have to substitute their expressions (2.20). Because of the definition of the function $\theta(x, u)$, there is the following identity with respect to x and u:

$$(\theta \circ (f^x, f^u))(x, u, \theta(x, u); a) = f^\theta(x, u, \theta(x, u); a).$$

The transformed solution $T_a(u) = u_a(x)$ is obtained by solving the relations

$$x' = f^x(x, u_0(x), \theta(x, u_0(x)); a),$$

with respect to x, then obtaining $x = \psi^x(x'; a)$. Substituting $x = \psi^x(x'; a)$ into the second equation in (2.19), one obtains the transformed function

$$u_a(x') = f^u(x, u_0(x), \theta_a(x, u_0(x)); a).$$

Notice that, there is the identity with respect to x:

$$(u_a \circ f^x)((x, u_0(x), \theta_a(x, u_0(x)); a)) = f^u(x, u_0(x), \theta_a(x, u_0(x)); a). \tag{2.21}$$

Formulae for transformations of partial derivatives are obtained by differentiating (2.21) with respect to x'.

Lemma 1. *The transformations $T_a(u)$ constructed in this way form a Lie group.*

Because the transformed function $u_a(x')$ is a solution of system (2.18) with transformed arbitrary elements $\theta_a(x', u')$, then the equations

$$F^k(x', u_a(x'), p'_a(x'), \theta_a(x', u_a(x'))) = 0, (k = 1, 2, \dots, s)$$

must be satisfied for an arbitrary x'. Because of a one–to–one correspondence between x and x' one has

$$F^k(f^x(z(x), a), f^u(z(x), a), f^p(z_p(x), a), f^\theta(z(x)))) = 0, (k = 1, 2, \dots, s) \quad (2.22)$$

where $z(x) = (x, u_0(x), \theta(x, u_0(x)))$, $z_p(x) = (x, u_0(x), \theta(x, u_0(x)), p_0(x), \dots)$.

After differentiating equations (2.22) with respect to the group parameter a, we obtain an algorithm for finding equivalence transformations (2.19). The differences in the algorithms for obtaining an admitted Lie group and equivalence group are only in the prolongation formulae of the infinitesimal generator.

In agreement with the construction, after differentiating equations (2.22) with respect to the group parameter a, one obtains the determining equations

$$\widetilde{X}^e F^k(x, u, \theta) \,|_{F=0} = 0, \quad k = 1, 2, ..., s, \quad (2.23)$$

with the prolonged operator $\widetilde{X}^e$,

$$\widetilde{X}^e = X^e + \zeta^{u^j_{x_i}} \partial_{u^j_{x_i}} + \zeta^{\theta^k_{x_i}} \partial_{\theta^k_{x_i}} + \zeta^{\theta^k_{u^j}} \partial_{\theta^k_{u^j}} + \dots .$$

Here the coefficients $\zeta^{u^j_{x_i}}, \zeta^{\theta^k_{x_i}}, \zeta^{\theta^k_{u^j}}, \dots$ are expressed by the following :

$$\begin{aligned}
\zeta^{u^j_{x_i}} &= D^e_{x_i} \zeta^{u^j} - u^j_{x_\beta} D^e_{x_i} \xi^{x_\beta} \\
\zeta^{\theta^k_{x_i}} &= \widetilde{D}^e_{x_i} \zeta^{\theta^k} - \theta^k_{x_\beta} \widetilde{D}^e_{x_i} \xi^{x_\beta} - \theta^k_{u^j} \widetilde{D}^e_{x_i} \zeta^{u^j} \\
\zeta^{\theta^k_{u^j}} &= \widetilde{D}^e_{u^j} \zeta^{\theta^k} - \theta^k_{x_i} \widetilde{D}^e_{u^j} \xi^{x_i} - \theta^k_{u^\beta} \widetilde{D}^e_{u^j} \zeta^{u^\beta}
\end{aligned}$$

where

$$\begin{aligned}
D^e_{x_i} &= \partial_{x_i} + u^j_{x_i}\partial_{u^j} + (\theta^k_{x_i} + \theta^k_{u^j}u^j_{x_i})\partial_{\theta^k} + ... \\
\widetilde{D}^e_{x_i} &= \partial_{x_i} + \theta^k_{x_i}\partial_{\theta^k} + ... \\
\widetilde{D}^e_{u^j} &= \partial_{u^j} + \theta^k_{u^j}\partial_{\theta^k} + ...
\end{aligned}$$

The solution of determining equations (2.23) gives us the coefficients of an infinitesimal generator. By solving the Lie equations, one can obtain the equivalence group of transformations (2.19).

CHAPTER III

COMPUTATIONAL PROCEDURE

We consider a system of two second-order ordinary differential equations in the following form,

$$\begin{aligned} F_1 = y_1'' + a_{11}{y_1'}^3 + a_{12}{y_1'}^2 y_2' + a_{13} y_1' {y_2'}^2 + a_{14}{y_1'}^2 \\ +a_{15} y_1' y_2' + a_{16}{y_2'}^2 + a_{17} y_1' + a_{18} y_2' + a_{19} = 0, \\ F_2 = y_2'' + a_{13}{y_2'}^3 + a_{12}{y_2'}^2 y_1' + a_{11} y_2' {y_1'}^2 + a_{24}{y_1'}^2 \\ +a_{25} y_2' y_1' + a_{26}{y_2'}^2 + a_{27} y_1' + a_{28} y_2' + a_{29} = 0, \end{aligned} \tag{3.1}$$

where the coefficients a_{ij} are expressed through the functions x, y_1, y_2. We have known from the details given in Chapter I, that the form of system(3.1) is not changed by any transformation

$$t = \varphi(x, y_1, y_2),\ u_1 = \psi_1(x, y_1, y_2),\ u_2 = \psi_2(x, y_1, y_2). \tag{3.2}$$

This thesis is devoted to find other invariants of transformation (3.2).

3.1 Method of Solving

For seeking invariants, Lie's approach is used. This approach contains the following steps.

First, we have to find an equivalence group of transformations for system (3.1), that is to find a Lie group of transformations, which transforms system (3.1) into a new system with the same differential structure. For constructing this Lie group one has to find an infinitesimal generator

$$X^e = \xi^x \partial_x + \zeta^{y_1} \partial_{y_1} + \zeta^{y_2} \partial_{y_2} + \zeta^{\theta_j} \partial_{\theta_j}$$

with the coefficients

$$\xi^x = \xi^x(x, y_1, y_2, \theta), \zeta^{y_1} = \zeta^{y_1}(x, y_1, y_2, \theta), \zeta^{y_2} = \zeta^{y_2}(x, y_1, y_2, \theta), \zeta^{z_j} = \zeta^{z_j}(x, y_1, y_2, \theta),$$

by solving the equations

$$\widetilde{X}^e F_{i|(3.1)} = 0, \quad (i = 1, 2). \tag{3.3}$$

Here $\theta_1 = a_{11},\ \theta_2 = a_{12},\ \theta_3 = a_{13},\ \theta_4 = a_{14},\ \theta_5 = a_{15},\ \theta_6 = a_{16},\ \theta_7 = a_{17},\ \theta_8 = a_{18},\ \theta_9 = a_{19},\ \theta_{10} = a_{24},\ \theta_{11} = a_{25},\ \theta_{12} = a_{26},\ \theta_{13} = a_{27},\ \theta_{14} = a_{28},\ \theta_{15} = a_{29}$, the coefficients $\xi^x, \zeta^{y_1}, \zeta^{y_2}, \zeta^{\theta_j},$ (j = 1, 2,..., 15) depend on x, y_1, y_2, θ and $\theta = (\theta_1, \theta_2, ..., \theta_{15})$. Here we also used the agreement that there is a summation with respect to a repeat index j.

The next step consists of solving equations which define differential invariants of the equivalence group

$$\widetilde{X}^e J = 0. \tag{3.4}$$

Here J depends on the coefficients a_{ij} and their derivatives, $\widetilde{X}$ is the prolongation of the generator X up to the maximal order of derivatives involved in the invariant J.

Notice that for relative invariants $J^k,\ (k = 1, 2, ..., w)$, one has to solve the equations

$$\widetilde{X}^e J^k_{|(\mathrm{S})} = 0, \quad (i = 1, 2, ..., s), \tag{3.5}$$

where the manifold (S) is defined by the equations $J^k = 0,\ (k = 1, 2, ..., w)$.

3.2 Equivalence Group of System (3.1)

The prolonged operator is

$$\widetilde{X}^e = X^e + \zeta^{y_1'}\partial_{y_1'} + \zeta^{y_2'}\partial_{y_2'} + \zeta^{y_1''}\partial_{y_1''} + \zeta^{y_2''}\partial_{y_2''} + \zeta^{\theta_{j_x}}\partial_{\theta_{j_x}} + \zeta^{\theta_{j_{y_1}}}\partial_{\theta_{j_{y_1}}} + \zeta^{\theta_{j_{y_2}}}\partial_{\theta_{j_{y_2}}}$$
$$+\zeta^{\theta_{j_{xy_1}}}\partial_{\theta_{j_{xy_1}}} + \zeta^{\theta_{j_{xy_2}}}\partial_{\theta_{j_{xy_2}}} + \zeta^{\theta_{j_{y_1y_2}}}\partial_{\theta_{j_{y_1y_2}}} + \zeta^{\theta_{j_{xx}}}\partial_{\theta_{j_{xx}}} + \zeta^{\theta_{j_{y_1y_1}}}\partial_{\theta_{j_{y_1y_1}}}$$
$$+\zeta^{\theta_{j_{y_2y_2}}}\partial_{\theta_{j_{y_2y_2}}}.$$

The coefficients of the prolonged operators are show as the follows

$$\zeta^{y_1'} = D_x^e\zeta^{y_1} - y_1' D_x^e\xi^x.$$

$$\zeta^{y_2'} = D_x^e\zeta^{y_2} - y_2' D_x^e\xi^x.$$

$$\zeta^{y_1''} = D_x^e\zeta^{y_1'} - y_1'' D_x^e\xi^x.$$

$$\zeta^{y_2''} = D_x^e\zeta^{y_2'} - y_2'' D_x^e\xi^x.$$

$$\zeta^{\theta_{j_x}} = \widetilde{D}_x^e\zeta^{\theta_j} - \theta_{j_x}\widetilde{D}_x^e\xi^x - \theta_{j_{y_1}}\widetilde{D}_x^e\xi^{y_1} - \theta_{j_{y_2}}\widetilde{D}_x^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{y_1}}} = \widetilde{D}_{y_1}^e\zeta^{\theta_j} - \theta_{j_x}\widetilde{D}_{y_1}^e\xi^x - \theta_{j_{y_1}}\widetilde{D}_{y_1}^e\xi^{y_1} - \theta_{j_{y_2}}\widetilde{D}_{y_1}^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{y_2}}} = \widetilde{D}_{y_2}^e\zeta^{\theta_j} - \theta_{j_x}\widetilde{D}_{y_2}^e\xi^x - \theta_{j_{y_1}}\widetilde{D}_{y_2}^e\xi^{y_1} - \theta_{j_{y_2}}\widetilde{D}_{y_2}^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{y_1x}}} = \widetilde{D}_x^e\zeta^{\theta_{j_{y_1}}} - \theta_{j_{y_1x}}\widetilde{D}_x^e\xi^x - \theta_{j_{y_1y_1}}\widetilde{D}_x^e\xi^{y_1} - \theta_{j_{y_1y_2}}\widetilde{D}_x^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{y_2x}}} = \widetilde{D}_x^e\zeta^{\theta_{j_{y_2}}} - \theta_{j_{y_2x}}\widetilde{D}_x^e\xi^x - \theta_{j_{y_2y_1}}\widetilde{D}_x^e\xi^{y_1} - \theta_{j_{y_2y_2}}\widetilde{D}_x^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{y_2y_1}}} = \widetilde{D}_{y_1}^e\zeta^{\theta_{j_{y_2}}} - \theta_{j_{y_2x}}\widetilde{D}_{y_1}^e\xi^x - \theta_{j_{y_2y_1}}\widetilde{D}_{y_1}^e\xi^{y_1} - \theta_{j_{y_2y_2}}\widetilde{D}_{y_1}^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{xx}}} = \widetilde{D}_x^e\zeta^{\theta_{j_x}} - \theta_{j_{xx}}\widetilde{D}_x^e\xi^x - \theta_{j_{xy_1}}\widetilde{D}_x^e\xi^{y_1} - \theta_{j_{xy_2}}\widetilde{D}_x^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{y_1y_1}}} = \widetilde{D}_{y_1}^e\zeta^{\theta_{j_{y_1}}} - \theta_{j_{y_1x}}\widetilde{D}_{y_1}^e\xi^x - \theta_{j_{y_1y_1}}\widetilde{D}_{y_1}^e\xi^{y_1} - \theta_{j_{y_1y_2}}\widetilde{D}_{y_1}^e\xi^{y_2}.$$

$$\zeta^{\theta_{j_{y_2y_2}}} = \widetilde{D}_{y_2}^e\zeta^{\theta_{j_{y_2}}} - \theta_{j_{y_2x}}\widetilde{D}_{y_2}^e\xi^x - \theta_{j_{y_2y_1}}\widetilde{D}_{y_2}^e\xi^{y_1} - \theta_{j_{y_2y_2}}\widetilde{D}_{y_2}^e\xi^{y_2}.$$

Here the operators are

$$\begin{aligned} D^e_x = \; & \partial_x + y'_1\partial y_1 + y'_2\partial y_2 + y''_1\partial y'_1 + y''_2\partial y'_2 + (\theta_{j_x} + \theta_{j_{y_1}} y'_1 + \theta_{j_{y_2}} y'_2)\partial_{\theta_j} \\ & +(\theta_{j_{y_1 x}} + \theta_{j_{y_1 y_2}} y'_2 + \theta_{j_{y_1 y_1}} y'_1)\partial_{\theta_{j_{y_1}}} + (\theta_{j_{xx}} + \theta_{j_{xy_2}} y'_2 + \theta_{j_{xy_1}} y'_1)\partial_{\theta_{j_x}} \\ & +(\theta_{j_{y_2 x}} + \theta_{j_{y_2 y_2}} y'_2 + \theta_{j_{y_2 y_1}} y'_1)\partial_{\theta_{j_{y_2}}}. \\ \widetilde{D}^e_x = \; & \partial_x + \theta_{j_x}\partial\theta_j. \\ \widetilde{D}^e_{y_1} = \; & \partial_{y_1} + \theta_{j_{y_1}}\partial\theta_j. \\ \widetilde{D}^e_{y_2} = \; & \partial_{y_2} + \theta_{j_{y_2}}\partial\theta_j. \end{aligned}$$

Substituting the coefficients of the prolonged operator $\widetilde{X}^e$ into the determining equations (3.3), where the manifold defined by equations (3.1) is described by the equations

$$\begin{aligned} y''_1 &= -(a_{11}{y'_1}^3 + a_{12}{y'_1}^2 y'_2 + a_{13}y'_1{y'_2}^2 + a_{14}{y'_1}^2 \\ &\quad + a_{15}y'_1y'_2 + a_{16}{y'_2}^2 + a_{17}y'_1 + a_{18}y'_2 + a_{19}), \\ y''_2 &= -(a_{13}{y'_2}^3 + a_{12}{y'_2}^2 y'_1 + a_{11}y'_2{y'_1}^2 + a_{24}{y'_1}^2 \\ &\quad + a_{25}y'_2y'_1 + a_{26}{y'_2}^2 + a_{27}y'_1 + a_{28}y'_2 + a_{29}), \end{aligned} \tag{3.6}$$

one gets equations which can be split with respect to the variables $y'_1,\ y'_2,\ \theta_{j_x},\ \theta_{j_{y_1}},\ \theta_{j_{y_2}},\ \theta_{j_{xy_1}},\ \theta_{j_{xy_2}},\ \theta_{j_{y_1y_2}},\ \theta_{j_{xx}},\ \theta_{j_{y_1y_1}},\ \theta_{j_{y_2y_2}}$. All calculations were done on computer by using the system of symbolic calculations REDUCE (cf. Hearn (1999)). The method of calculating consists of sequential calculations, after analysis of the result of calculations, we add the new relations which were obtained into the program and repeat the calculations. As the result of calculations we obtained the generator of an equivalence group

$\xi^x = f(x, y_1, y_2),\ \zeta^{y_1} = g(x, y_1, y_2),\ \zeta^{y_2} = h(x, y_1, y_2),$

$\zeta^{a_{11}} = -a_{24}\frac{\partial f}{\partial y_2} - a_{14}\frac{\partial f}{\partial y_1} + a_{11}\frac{\partial f}{\partial x} - a_{12}\frac{\partial h}{\partial y_1} - 2a_{11}\frac{\partial g}{\partial y_1} + \frac{\partial f}{\partial y_1^2}\ ,$

$\zeta^{a_{12}} = -a_{25}\frac{\partial f}{\partial y_2} - a_{15}\frac{\partial f}{\partial y_1} + a_{12}\frac{\partial f}{\partial x} - a_{12}\frac{\partial h}{\partial y_2} - 2a_{13}\frac{\partial h}{\partial y_1} - 2a_{11}\frac{\partial g}{\partial y_2} - a_{12}\frac{\partial g}{\partial y_1} + 2\frac{\partial f}{\partial y_2 \partial y_1}\ ,$

$\zeta^{a_{13}} = -a_{26}\frac{\partial f}{\partial y_2} - a_{16}\frac{\partial f}{\partial y_1} + a_{13}\frac{\partial f}{\partial x} - 2a_{13}\frac{\partial h}{\partial y_2} - a_{12}\frac{\partial g}{\partial y_2} + \frac{\partial f}{\partial y_2^2}\ ,$

$\zeta^{a_{14}} = -a_{27}\frac{\partial f}{\partial y_2} - 2a_{17}\frac{\partial f}{\partial y_1} - a_{15}\frac{\partial h}{\partial y_1} - a_{12}\frac{\partial h}{\partial x} + a_{24}\frac{\partial g}{\partial y_2} - a_{14}\frac{\partial g}{\partial y_1} - 3a_{11}\frac{\partial g}{\partial x} + 2\frac{\partial f}{\partial y_1 x} - \frac{\partial g}{\partial y_1^2}\ ,$

$\zeta^{a_{15}} = -(a_{28} + a_{17})\frac{\partial f}{\partial y_2} - 2a_{18}\frac{\partial f}{\partial y_1} - a_{15}\frac{\partial h}{\partial y_2} - 2a_{16}\frac{\partial h}{\partial y_1} - 2a_{13}\frac{\partial h}{\partial x} + (a_{25} - 2a_{14})\frac{\partial g}{\partial y_2}$
$\quad -2a_{12}\frac{\partial g}{\partial x} + 2(\frac{\partial f}{\partial y_2 \partial x} - \frac{\partial g}{\partial y_2 \partial y_1})\ ,$

$\zeta^{a_{16}} = -a_{18}\frac{\partial f}{\partial y_2} - 2a_{16}\frac{\partial h}{\partial y_2} + (a_{26} - a_{15})\frac{\partial g}{\partial y_2} + a_{16}\frac{\partial g}{\partial y_1} - a_{13}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y_2^2}\ ,$

$\zeta^{a_{17}} = -a_{29}\frac{\partial f}{\partial y_2} - 3a_{19}\frac{\partial f}{\partial y_1} - a_{17}\frac{\partial f}{\partial x} - a_{18}\frac{\partial h}{\partial y_1} - a_{15}\frac{\partial h}{\partial x} + a_{27}\frac{\partial g}{\partial y_2} - 2a_{14}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x^2}$
$\quad -2\frac{\partial g}{\partial y_1 \partial x}\ ,$

$\zeta^{a_{18}} = -2a_{19}\frac{\partial f}{\partial y_2} - a_{18}\frac{\partial f}{\partial x} - a_{18}\frac{\partial h}{\partial y_2} - 2a_{16}\frac{\partial h}{\partial x} + (a_{28} - a_{17})\frac{\partial g}{\partial y_2} + a_{18}\frac{\partial g}{\partial y_1} - a_{15}\frac{\partial g}{\partial x}$
$\quad -2\frac{\partial g}{\partial y_2 \partial x}\ ,$

$\zeta^{a_{19}} = -2a_{19}\frac{\partial f}{\partial x} - a_{18}\frac{\partial h}{\partial x} + a_{29}\frac{\partial g}{\partial y_2} + a_{19}\frac{\partial g}{\partial y_1} - a_{17}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial x^2}\ ,$

$\zeta^{a_{24}} = -a_{27}\frac{\partial f}{\partial y_1} + a_{24}\frac{\partial h}{\partial y_2} + (a_{14} - a_{25})\frac{\partial h}{\partial y_1} - a_{11}\frac{\partial h}{\partial x} - 2a_{24}\frac{\partial g}{\partial y_1} - \frac{\partial h}{\partial y_1^2}\ ,$

$\zeta^{a_{25}} = -2a_{27}\frac{\partial f}{\partial y_2} - (a_{28} + a_{17})\frac{\partial f}{\partial y_1} + (a_{15} - 2a_{26})\frac{\partial h}{\partial y_1} - 2a_{12}\frac{\partial h}{\partial x} - 2a_{24}\frac{\partial g}{\partial y_2} - a_{25}\frac{\partial g}{\partial y_1}$
$\quad -2a_{11}\frac{\partial g}{\partial x} + 2(\frac{\partial g}{\partial y_1 \partial x} - \frac{\partial h}{\partial y_2 \partial y_1})\ ,$

$\zeta^{a_{26}} = -2a_{28}\frac{\partial f}{\partial y_2} - a_{18}\frac{\partial f}{\partial y_1} - a_{26}\frac{\partial h}{\partial y_2} + a_{16}\frac{\partial h}{\partial y_1} - 3a_{13}\frac{\partial h}{\partial x} - a_{25}\frac{\partial g}{\partial y_2} - a_{12}\frac{\partial g}{\partial x} + 2\frac{\partial f}{\partial y_2 \partial x}$
$\quad -\frac{\partial h}{\partial y_2^2}\ ,$

$\zeta^{a_{27}} = -2a_{29}\frac{\partial f}{\partial y_1} - a_{27}\frac{\partial f}{\partial x} + a_{27}\frac{\partial h}{\partial y_2} + (a_{17} - a_{28})\frac{\partial h}{\partial y_1} - a_{25}\frac{\partial h}{\partial x} - a_{27}\frac{\partial g}{\partial y_1} - 2a_{24}\frac{\partial g}{\partial x}$
$\quad -2\frac{\partial h}{\partial y_1 \partial x}\ ,$

$\zeta^{a_{28}} = -3a_{29}\frac{\partial f}{\partial y_2} - a_{19}\frac{\partial f}{\partial y_1} - a_{28}\frac{\partial f}{\partial x} + a_{18}\frac{\partial h}{\partial y_1} - 2a_{26}\frac{\partial h}{\partial x} - a_{27}\frac{\partial g}{\partial y_2} - a_{25}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x^2}$
$\quad -2\frac{\partial h}{\partial y_2 \partial x}\ ,$

$\zeta^{a_{29}} = -2a_{29}\frac{\partial f}{\partial x} + a_{29}\frac{\partial h}{\partial y_2} + a_{19}\frac{\partial h}{\partial y_1} - a_{28}\frac{\partial h}{\partial x} - a_{27}\frac{\partial g}{\partial x} - \frac{\partial h}{\partial x^2}\ .$

Here the functions $f(x, y_1, y_2)$, $g(x, y_1, y_2)$ and $h(x, y_1, y_2)$ are arbitrary functions of their arguments.

CHAPTER IV

INVESTIGATION OF INVARIANTS

In this research, the invariants that we were looking for are the functions of the coefficients appearing in system (3.1) and their derivatives up to first and second orders. In order to find these invariants, we have to apply the operator $\widetilde{X}^e$ to an invariant as in equation (3.4). One obtains an equation which is a polynomial with respect to the functions f, g, h and their derivatives. Since these functions are arbitrary, this equation can be split with respect to them. Splitting gives us a system of linear homogeneous first-order partial differential equations. This system includes 60 equations with 60 independent variables for an invariant of first-order. For second-order invariants this system consists of 105 equations with 150 independent variables.

4.1 First-Order Relative Invariants

Analysis of the system of linear homogeneous first-order partial differential equations shows that most of these equations are of one of the following types:

$$\frac{\partial F}{\partial x} + a_1\frac{\partial F}{\partial y_1} + a_2\frac{\partial F}{\partial y_2} + ... + a_n\frac{\partial F}{\partial y_n} = 0, \tag{4.1}$$

$$x\frac{\partial F}{\partial x} + \alpha_1 y_1\frac{\partial F}{\partial y_1} + \alpha_2 y_2\frac{\partial F}{\partial y_2} + ... + \alpha_n y_n\frac{\partial F}{\partial y_n} = 0, \tag{4.2}$$

$$\begin{aligned} &x\frac{\partial F}{\partial x} + \alpha_1 y_1\frac{\partial F}{\partial y_1} + \alpha_2 y_2\frac{\partial F}{\partial y_2} + ... + \alpha_n y_n\frac{\partial F}{\partial y_n} \\ &+\alpha_0(\frac{\partial F}{\partial x} + a_1\frac{\partial F}{\partial y_1} + a_2\frac{\partial F}{\partial y_2} + ... + a_n\frac{\partial F}{\partial y_n}) = 0, \end{aligned} \tag{4.3}$$

where $a_i = \beta_{i,1}(x)y_1 + \beta_{i,2}(x)y_2 + ... + \beta_{i,i-1}(x)y_{i-1} + \gamma_i(x)$, $\alpha_i =$ constant, $F = F(x, y_1, y_2, ..., y_n)$. The General solutions of these equations can be found analytically as follows.

The characteristic system for equation (4.1) is

$$\frac{dx}{1} = \frac{dy_1}{\gamma_1(x)} = \frac{dy_2}{\beta_{2,1}(x)y_1 + \gamma_2(x)} = \frac{dy_3}{\beta_{3,1}(x)y_1 + \beta_{3,2}(x)y_2 + \gamma_3(x)} = ... \,.$$

Integrating recurrently, one finds

$$y_1 = C_1 + F_1(x),\ y_2 = C_2 + F_2(x, C_1), ...,\ y_n = C_n + F_n(x, C_1, C_2, ..., C_{n-1}),$$

where

$$F_i(x, C_1, C_2, ..., C_{i-1}) = \int a_i(x, y_1(x), y_2(x), ..., y_{i-1}(x))dx.$$

The variables C_i in this step are considered to be constant. The general solution of equation (4.1) is $F = \phi(C_1, C_2, ..., C_n)$ where ϕ is an arbitrary function and $C_1, C_2, ..., C_n$ are defined by the formulae

$$C_1 = y_1 - F_1,\ C_2 = y_2 - F_2, ...,\ C_n = y_n - F_n.$$

The characteristic system for equation (4.2) is

$$\frac{dx}{x} = \frac{dy_1}{\alpha_1 y_1} = \frac{dy_2}{\alpha_2 y_2} = \frac{dy_3}{\alpha_3 y_3} = ... \,.$$

The general solution of equation (4.2) is $F = \phi(C_1, C_2, ..., C_n)$, where ϕ is an arbitrary function and $C_i = \frac{y_i}{x^{\alpha_i}}, i = 1, ..., n$.

The characteristic system for equation (4.3) is

$$\frac{dx}{x + \alpha_0} = \frac{dy_1}{\alpha_1 y_1 + \alpha_0 \gamma_1(x)} = \frac{dy_2}{\alpha_2 y_2 + \alpha_0 a_2(x, y_1)} = \frac{dy_3}{\alpha_3 y_3 + \alpha_0 a_3(x, y_1, y_2)} = ... \,.$$

Considering the first term and the second term, one has

$$\frac{dy_1}{dx} = \frac{\alpha_1 y_1}{x + \alpha_0} + \frac{\alpha_0 \gamma_1(x)}{x + \alpha_0} \,.$$

This is a first-order linear ordinary differential equation. The general solution of this equation is

$$y_1 = (x+\alpha_0)^{\alpha_1}(C_1 + \int \alpha_0\gamma_1(x)(x+\alpha_0)^{-(\alpha_1+1)}dx),$$

or

$$C_1 = y_1(x+\alpha_0)^{-\alpha_1} - \int \alpha_0\gamma_1(x)(x+\alpha_0)^{-(\alpha_1+1)}dx.$$

Considering the first term and the third term, after substituting $y_1(x)$, one also obtains a first-order linear ordinary differential equation

$$\frac{dy_2}{dx} = \frac{\alpha_2 y_2}{x+\alpha_0} + \frac{\alpha_0 a_2(x, y_1(x))}{x+\alpha_0}\ .$$

As in the above, the general solution of this equation is

$$y_2 = (x+\alpha_0)^{\alpha_2}(C_2 + \int \alpha_0 a_2(x, y_1(x))(x+\alpha_0)^{-(\alpha_2+1)}dx),$$

or

$$C_2 = y_2(x+\alpha_0)^{-\alpha_2} - \int \alpha_0 a_2(x, y_1(x))(x+\alpha_0)^{-(\alpha_2+1)}dx.$$

Recurrently, one finds all $C_i\ ,\ i = 1, 2, ..., n$. The general solution of equation (4.3) is $F = \phi(C_1, C_2, ..., C_n)$, where ϕ is an arbitrary function.

For solving the system of linear homogeneous first-order partial differential equations, we used the method of successive solving one equation, and substituting the solution into the other equations. Notice that after the substitutions the equations obtained can be split with respect to the independent variable x.

After repeatedly applying the previous step, one obtains the following first-order relative invariants

$$J^1 = -2a_{26}a_{12} + 2a_{25}a_{13} - 4a_{16}a_{11} + a_{15}a_{12} - 4a_{13y_1} + 2a_{12y_2} = 0.$$

$$J^2 = a_{28}a_{12} - 4a_{27}a_{13} + a_{25}a_{15} - 4a_{24}a_{16} + 4a_{18}a_{11} - 3a_{17}a_{12} + 2a_{15y_1} - 4a_{14y_2}$$
$$+2a_{12x} = 0.$$

$$J^3 = -3a_{28}a_{12} + 4a_{27}a_{13} - 4a_{26y_1} + 2a_{25y_2} + a_{25}a_{15} - 4a_{24}a_{16} - 4a_{18}a_{11} + a_{17}a_{12}$$
$$+2a_{12x} = 0.$$

$$J^4 = 2a_{28}a_{16} - 2a_{26}a_{18} + 4a_{19}a_{13} + 2a_{18y_2} + a_{18}a_{15} - 2a_{17}a_{16} - 4a_{16x} = 0.$$

$$J^5 = 2a_{28}a_{13} - 2a_{26}a_{15} + 2a_{25}a_{16} + 2a_{17}a_{13} - 4a_{16y_1} - 4a_{16}a_{14} + 2a_{15y_2} + a_{15}^2$$
$$-4a_{13x} = 0.$$

$$J^6 = 12a_{29}a_{13} + 2a_{28x} - 3a_{28}a_{15} + 4a_{27}a_{16} - 4a_{26x} + a_{25}a_{18} + 4a_{19}a_{12} - 6a_{18y_1}$$
$$+6a_{18}a_{14} + 6a_{17y_2} + 3a_{17}a_{15} = 0.$$

$$J^7 = -4a_{29}a_{13} + a_{28}a_{15} - 4a_{27}a_{16} + a_{25}a_{18} - 4a_{19}a_{12} + 2a_{18y_1} + 2a_{18}a_{14} - 4a_{17y_2}$$
$$-a_{17}a_{15} + 2a_{15x} = 0.$$

$$J^8 = -4a_{29}a_{16} + a_{28}a_{18} - 4a_{19y_2} - 2a_{19}a_{15} + 2a_{18x} + a_{18}a_{17} = 0.$$

$$J^9 = 4a_{29}a_{12} + 6a_{28y_1} + 3a_{28}a_{25} - 6a_{27y_2} - 6a_{27}a_{26} + a_{27}a_{15} - 3a_{25}a_{17} + 4a_{24}a_{18}$$
$$+12a_{19}a_{11} + 2a_{17y_1} - 4a_{14x} = 0.$$

$$J^{10} = -4a_{29}a_{12} - 4a_{28y_1} - a_{28}a_{25} + 2a_{27y_2} + 2a_{27}a_{26} + a_{27}a_{15} + 2a_{25x} + a_{25}a_{17}$$
$$-4a_{24}a_{18} - 4a_{19}a_{11} = 0.$$

$$J^{11} = 2a_{28}a_{11} - 4a_{26}a_{24} + 2a_{25y_1} + a_{25}^2 - 2a_{25}a_{14} - 4a_{24y_2} + 2a_{24}a_{15} + 2a_{17}a_{11}$$
$$-4a_{11x} = 0.$$

$$J^{12} = -4a_{29y_2} - 4a_{29}a_{26} + 2a_{29}a_{15} + 2a_{28x} + a_{28}^2 - 2a_{25}a_{19} + 4a_{19y_1} + 4a_{19}a_{14}$$
$$-2a_{17x} - a_{17}^2 = 0.$$

$$J^{13} = -4a_{29y_1} - 2a_{29}a_{25} + a_{28}a_{27} + 2a_{27x} + a_{27}a_{17} - 4a_{24}a_{19} = 0.$$

$$J^{14} = 4a_{29}a_{11} - 2a_{28}a_{24} + 2a_{27y_1} + a_{27}a_{25} - 2a_{27}a_{14} - 4a_{24x} + 2a_{24}a_{17} = 0.$$

$$J^{15} = a_{25}a_{12} - 4a_{24}a_{13} + 2a_{15}a_{11} - 2a_{14}a_{12} + 2a_{12y_1} - 4a_{11y_2} = 0.$$

4.2 Second-Order Relative Invariants

Finding second-order relative invariants is similar to the case of first-order relative invariants. The difference is only that in equation (3.4), we substitute J which depends on coefficients a_{ij} and their derivatives up to second-order instead of first-order. The set second-order relative invariants are shown as the following:

$$
\begin{aligned}
J^1 = &-a_{28}a_{13}a_{12} + 2a_{27}a_{13}^2 - 2a_{26_{y_1}}a_{13} - a_{26_{y_2}}a_{12} + 2a_{26}^2a_{12} - 2a_{26}a_{25}a_{13} \\
&+4a_{26}a_{16}a_{11} - a_{26}a_{15}a_{12} + 4a_{26}a_{13_{y_1}} - 3a_{26}a_{12_{y_2}} + 2a_{25_{y_2}}a_{13} + a_{25}a_{16}a_{12} \\
&+a_{25}a_{13_{y_2}} - 4a_{24}a_{16}a_{13} - 2a_{18}a_{13}a_{11} + a_{17}a_{13}a_{12} - a_{16_{y_1}}a_{12} - 2a_{16_{y_2}}a_{11} \\
&+2a_{16}a_{15}a_{11} - 2a_{16}a_{14}a_{12} + a_{16}a_{12_{y_1}} - 4a_{16}a_{11_{y_2}} + a_{15_{y_2}}a_{12} \\
&+a_{15}a_{13_{y_1}} - a_{13_x}a_{12} - 2a_{13_{y_1y_2}} + a_{13}a_{12_x} + a_{12_{y_2y_2}} = 0.
\end{aligned}
$$

$$
J^2 = -2a_{26}a_{12} + 2a_{25}a_{13} - 4a_{16}a_{11} + a_{15}a_{12} - 4a_{13_{y_1}} + 2a_{12_{y_2}} = 0.
$$

$$
\begin{aligned}
J^3 = &\, a_{28}a_{12}a_{11} - 2a_{27}a_{13}a_{11} - 2a_{26}a_{24}a_{12} + a_{25_{y_1}}a_{12} + 2a_{25}a_{24}a_{13} - a_{25}a_{14}a_{12} \\
&+a_{25}a_{11_{y_2}} - 2a_{24_{y_1}}a_{13} - a_{24_{y_2}}a_{12} - 4a_{24}a_{16}a_{11} + a_{24}a_{15}a_{12} + 4a_{24}a_{14}a_{13} \\
&-4a_{24}a_{13_{y_1}} + a_{24}a_{12_{y_2}} + 2a_{18}a_{11}^2 - a_{17}a_{12}a_{11} + 2a_{15_{y_1}}a_{11} - 2a_{15}a_{14}a_{11} \\
&+a_{15}a_{11_{y_1}} - a_{14_{y_1}}a_{12} - 2a_{14_{y_2}}a_{11} + 2a_{14}^2a_{12} - 3a_{14}a_{12_{y_1}} + 4a_{14}a_{11_{y_2}} + a_{12_x}a_{11} \\
&+a_{12_{y_1y_1}} - a_{12}a_{11_x} - 2a_{11_{y_1y_2}} = 0.
\end{aligned}
$$

$$
J^4 = a_{25}a_{12} - 4a_{24}a_{13} + 2a_{15}a_{11} - 2a_{14}a_{12} + 2a_{12_{y_1}} - 4a_{11_{y_2}} = 0.
$$

$$
J^5 = 2a_{28}a_{16} - 2a_{26}a_{18} + 4a_{19}a_{13} + 2a_{18_{y_2}} + a_{18}a_{15} - 2a_{17}a_{16} - 4a_{16_x} = 0.
$$

$$
J^6 = a_{28_{y_2}} - 4a_{27}a_{16} - 2a_{26_x} + 2a_{25}a_{18} - 4a_{19}a_{12} - 3a_{17_{y_2}} + 3a_{15_x} = 0.
$$

$$
J^7 = -4a_{29}a_{16} + a_{28}a_{18} - 4a_{19_{y_2}} - 2a_{19}a_{15} + 2a_{18_x} + a_{18}a_{17} = 0.
$$

$$
\begin{aligned}
J^8 = &-4a_{29}a_{13} + a_{28}a_{15} - 4a_{27}a_{16} + a_{25}a_{18} - 4a_{19}a_{12} + 2a_{18_{y_1}} + 2a_{18}a_{14} - 4a_{17_{y_2}} \\
&-a_{17}a_{15} + 2a_{15_x} = 0.
\end{aligned}
$$

$$
\begin{aligned}
J^9 = &\, 4a_{29}a_{16}a_{13} + 2a_{28_{y_2}}a_{16} - 2a_{28}a_{26}a_{16} - a_{28}a_{18}a_{13} + a_{28}a_{16_{y_2}} - 2a_{27}a_{16}^2 \\
&-2a_{26_x}a_{16} - a_{26_{y_2}}a_{18} + 2a_{26}^2a_{18} - 4a_{26}a_{19}a_{13} - 3a_{26}a_{18_{y_2}} - a_{26}a_{18}a_{15}
\end{aligned}
$$

$$+2a_{26}a_{17}a_{16}+4a_{26}a_{16x}+a_{25}a_{18}a_{16}+4a_{19y_2}a_{13}-2a_{19}a_{16}a_{12}+2a_{19}a_{15}a_{13}$$
$$+2a_{19}a_{13y_2}-a_{18x}a_{13}-a_{18y_1}a_{16}+a_{18y_2y_2}+a_{18y_2}a_{15}-a_{18}a_{17}a_{13}+a_{18}a_{16y_1}$$
$$+a_{18}a_{13x}-2a_{17y_2}a_{16}-a_{17}a_{16y_2}-2a_{16xy_2}-a_{16x}a_{15}+2a_{16}a_{15x}=0.$$

$$J^{10}=2a_{28}a_{13}-2a_{26}a_{15}+2a_{25}a_{16}+2a_{17}a_{13}-4a_{16y_1}-4a_{16}a_{14}+2a_{15y_2}+a_{15}^2$$
$$-4a_{13x}=0.$$

$$J^{11}=-3a_{28}a_{12}+4a_{27}a_{13}-4a_{26y_1}+2a_{25y_2}+a_{25}a_{15}-4a_{24}a_{16}-4a_{18}a_{11}+a_{17}a_{12}$$
$$+2a_{12x}=0.$$

$$J^{12}=a_{28}a_{12}-4a_{27}a_{13}+a_{25}a_{15}-4a_{24}a_{16}+4a_{18}a_{11}-3a_{17}a_{12}+2a_{15y_1}-4a_{14y_2}$$
$$+2a_{12x}=0.$$

$$J^{13}=4a_{29}a_{13}^2+2a_{28y_2}a_{13}-2a_{28}a_{26}a_{13}-a_{28}a_{15}a_{13}+a_{28}a_{13y_2}-2a_{26x}a_{13}$$
$$-2a_{26y_1}a_{16}-a_{26y_2}a_{15}+2a_{26}^2a_{15}-2a_{26}a_{25}a_{16}-2a_{26}a_{17}a_{13}+4a_{26}a_{16y_1}$$
$$+4a_{26}a_{16}a_{14}-3a_{26}a_{15y_2}-a_{26}a_{15}^2+4a_{26}a_{13x}+2a_{25y_2}a_{16}+a_{25}a_{16y_2}$$
$$+a_{25}a_{16}a_{15}-4a_{24}a_{16}^2+2a_{19}a_{13}a_{12}-2a_{18y_1}a_{13}+a_{18y_2}a_{12}+2a_{18}a_{16}a_{11}$$
$$-2a_{18}a_{14}a_{13}+2a_{18}a_{13y_1}-a_{18}a_{12y_2}+2a_{17y_2}a_{13}-2a_{17}a_{16}a_{12}+a_{17}a_{15}a_{13}$$
$$+a_{17}a_{13y_2}-2a_{16x}a_{12}-2a_{16y_1y_2}-2a_{16y_2}a_{14}+a_{16}a_{15y_1}-4a_{16}a_{14y_2}+2a_{16}a_{12x}$$
$$+a_{15x}a_{13}+a_{15y_2y_2}+a_{15y_2}a_{15}-2a_{13xy_2}=0.$$

$$J^{14}=-2a_{29x}a_{16}-a_{29y_2}a_{18}+2a_{29}a_{28}a_{16}-2a_{29}a_{26}a_{18}+4a_{29}a_{19}a_{13}+a_{29}a_{18y_2}$$
$$+a_{29}a_{18}a_{15}-2a_{29}a_{17}a_{16}-4a_{29}a_{16x}+a_{28x}a_{18}+a_{28}a_{19y_2}+2a_{27}a_{19}a_{16}$$
$$-a_{25}a_{19}a_{18}-2a_{19xy_2}-a_{19x}a_{15}+a_{19y_1}a_{18}-a_{19y_2}a_{17}+2a_{19}^2a_{12}-a_{19}a_{18y_1}$$
$$+2a_{19}a_{17y_2}-2a_{19}a_{15x}+a_{18xx}+a_{18x}a_{17}=0.$$

$$J^{15}=2a_{29xy_2}+2a_{29x}a_{26}-a_{29x}a_{15}-2a_{29y_1}a_{18}-4a_{29}^2a_{13}-a_{29}a_{28y_2}+a_{29}a_{28}a_{15}$$
$$+2a_{29}a_{27}a_{16}+4a_{29}a_{26x}-2a_{29}a_{25}a_{18}+2a_{29}a_{18y_1}+2a_{29}a_{18}a_{14}-a_{29}a_{17y_2}$$
$$-a_{29}a_{17}a_{15}-2a_{29}a_{15x}-a_{28xx}-a_{28x}a_{28}+a_{28y_1}a_{19}+a_{28}a_{25}a_{19}+a_{27x}a_{18}$$
$$-2a_{27y_2}a_{19}-2a_{27}a_{26}a_{19}+2a_{27}a_{19y_2}+2a_{27}a_{19}a_{15}-a_{27}a_{18x}+2a_{25x}a_{19}$$
$$+a_{25}a_{19x}-a_{25}a_{19}a_{17}-2a_{24}a_{19}a_{18}-2a_{19xy_1}-2a_{19x}a_{14}+4a_{19}^2a_{11}+a_{19}a_{17y_1}$$
$$-4a_{19}a_{14x}+a_{17xx}+a_{17x}a_{17}=0.$$

$$J^{16} = -4a_{29_{y_1}} - 2a_{29}a_{25} + a_{28}a_{27} + 2a_{27_x} + a_{27}a_{17} - 4a_{24}a_{19} = 0.$$

$$J^{17} = -2a_{29_x}a_{11} - 2a_{29_{y_1y_1}} - 2a_{29_{y_1}}a_{25} + 2a_{29_{y_1}}a_{14} + 2a_{29_{y_2}}a_{24} + 2a_{29}a_{27}a_{12}$$
$$-2a_{29}a_{24_{y_2}} + 4a_{29}a_{17}a_{11} - 4a_{29}a_{11_x} + 2a_{28_{y_1}}a_{27} + 2a_{28}a_{24_x} - 2a_{28}a_{24}a_{17}$$
$$+2a_{27_{y_1}}a_{17} - a_{27}^2a_{15} - 2a_{27}a_{25_x} + a_{27}a_{25}a_{17} + 2a_{27}a_{24}a_{18} - 2a_{27}a_{17}a_{14}$$
$$+2a_{27}a_{14_x} + 2a_{24_{xx}} - 4a_{24_x}a_{17} - 2a_{24_{y_1}}a_{19} - 4a_{24}a_{19_{y_1}} + 2a_{24}a_{17}^2 = 0.$$

$$J^{18} = 4a_{29}a_{11} - 2a_{28}a_{24} + 2a_{27_{y_1}} + a_{27}a_{25} - 2a_{27}a_{14} - 4a_{24_x} + 2a_{24}a_{17} = 0.$$

$$J^{19} = -4a_{29}a_{12} - 4a_{28_{y_1}} - a_{28}a_{25} + 2a_{27_{y_2}} + 2a_{27}a_{26} + a_{27}a_{15} + 2a_{25_x} + a_{25}a_{17}$$
$$-4a_{24}a_{18} - 4a_{19}a_{11} = 0.$$

$$J^{20} = -2a_{29_{xy_1}} - a_{29_x}a_{25} - a_{29_{y_1}}a_{28} + a_{29_{y_1}}a_{17} + a_{29_{y_2}}a_{27} + 2a_{29}^2a_{12} + 2a_{29}a_{28_{y_1}}$$
$$-a_{29}a_{27_{y_2}} - a_{29}a_{27}a_{15} - 2a_{29}a_{25_x} + 2a_{29}a_{24}a_{18} + 4a_{29}a_{19}a_{11} + a_{28}a_{27_x}$$
$$-2a_{28}a_{24}a_{19} + a_{27_{xx}} + a_{27_{y_1}}a_{19} + a_{27}a_{25}a_{19} - a_{27}a_{19_{y_1}} - 2a_{27}a_{19}a_{14} + a_{27}a_{17_x}$$
$$-4a_{24_x}a_{19} - 2a_{24}a_{19_x} + 2a_{24}a_{19}a_{17} = 0.$$

$$J^{21} = 4a_{29_{y_2}} + 4a_{29}a_{26} - 2a_{29}a_{15} - 2a_{28_x} - a_{28}^2 + 2a_{25}a_{19} - 4a_{19_{y_1}} - 4a_{19}a_{14}$$
$$+2a_{17_x} + a_{17}^2 = 0.$$

$$J^{22} = 4a_{29}a_{12} + 6a_{28_{y_1}} + 3a_{28}a_{25} - 6a_{27_{y_2}} - 6a_{27}a_{26} + a_{27}a_{15} - 3a_{25}a_{17} + 4a_{24}a_{18}$$
$$+12a_{19}a_{11} + 2a_{17_{y_1}} - 4a_{14_x} = 0.$$

$$J^{23} = 4a_{29_{y_2}}a_{16} + 2a_{29}a_{16_{y_2}} - 2a_{28}^2a_{16} + 2a_{28}a_{26}a_{18} - 4a_{28}a_{19}a_{13} - 2a_{28}a_{18_{y_2}}$$
$$-a_{28}a_{18}a_{15} + 2a_{28}a_{17}a_{16} + 4a_{28}a_{16_x} - 2a_{27}a_{18}a_{16} - 2a_{26_x}a_{18} - 2a_{26}a_{19_{y_2}}$$
$$+a_{25}a_{18}^2 + 2a_{19_x}a_{13} - 2a_{19_{y_1}}a_{16} + 2a_{19_{y_2y_2}} + 2a_{19_{y_2}}a_{15} - 2a_{19}a_{18}a_{12}$$
$$+2a_{19}a_{16_{y_1}} + 4a_{19}a_{13_x} - 2a_{18}a_{17_{y_2}} + 2a_{18}a_{15_x} - 2a_{17}a_{16_x} - 2a_{16_{xx}} = 0.$$

$$J^{24} = 4a_{29_{y_2}}a_{13} + 4a_{29}a_{16}a_{12} + 2a_{29}a_{13_{y_2}} - 2a_{28}^2a_{13} + 2a_{28}a_{26}a_{15} - 4a_{28}a_{25}a_{16}$$
$$-a_{28}a_{18}a_{12} - 2a_{28}a_{17}a_{13} + 4a_{28}a_{16_{y_1}} + 4a_{28}a_{16}a_{14} - 2a_{28}a_{15_{y_2}} - a_{28}a_{15}^2$$
$$+4a_{28}a_{13_x} + 4a_{27_{y_2}}a_{16} - 2a_{27}a_{18}a_{13} + 2a_{27}a_{16_{y_2}} - 2a_{27}a_{16}a_{15} - 2a_{26_x}a_{15}$$
$$-2a_{26_{y_1}}a_{18} + 2a_{26}a_{25}a_{18} - 2a_{26}a_{17_{y_2}} - 4a_{25}a_{19}a_{13} - 2a_{25}a_{18_{y_2}} + a_{25}a_{18}a_{15}$$
$$+2a_{25}a_{17}a_{16} + 4a_{25}a_{16_x} - 4a_{24}a_{18}a_{16} + 6a_{19_{y_2}}a_{12} + 6a_{19}a_{13_{y_1}} - 2a_{18_x}a_{12}$$

$$+2a_{18}^2a_{11} - 3a_{18}a_{17}a_{12} + 2a_{18}a_{15_{y_1}} - 4a_{18}a_{14_{y_2}} + 2a_{18}a_{12_x} + 2a_{17_x}a_{13}$$
$$-2a_{17_{y_1}}a_{16} + 2a_{17_{y_2y_2}} + 2a_{17}a_{13_x} - 4a_{16_{xy_1}} - 4a_{16_x}a_{14} + 2a_{15_x}a_{15}$$
$$-2a_{13_{xx}} = 0.$$

$$J^{25} = -4a_{29}a_{13}a_{12} - 2a_{28}a_{26}a_{12} + 4a_{28}a_{25}a_{13} - 4a_{28}a_{16}a_{11} + 2a_{28}a_{15}a_{12} - 4a_{28}a_{13_{y_1}}$$
$$+2a_{28}a_{12_{y_2}} - 4a_{27_{y_2}}a_{13} - 2a_{27}a_{16}a_{12} + 2a_{27}a_{15}a_{13} - 2a_{27}a_{13_{y_2}} + 2a_{26_x}a_{12}$$
$$+2a_{26_{y_1}}a_{15} - 2a_{26}a_{25}a_{15} + 2a_{26}a_{14_{y_2}} + 2a_{25}^2a_{16} - a_{25}a_{18}a_{12} + 2a_{25}a_{17}a_{13}$$
$$-4a_{25}a_{16_{y_1}} - 4a_{25}a_{16}a_{14} + 2a_{25}a_{15_{y_2}} - 4a_{25}a_{13_x} - 4a_{24_{y_2}}a_{16} + 4a_{24}a_{18}a_{13}$$
$$-2a_{24}a_{16_{y_2}} + 4a_{24}a_{16}a_{15} - 2a_{19}a_{12}^2 + 2a_{18_{y_1}}a_{12} - 4a_{18}a_{15}a_{11} + 4a_{18}a_{14}a_{12}$$
$$-2a_{18}a_{12_{y_1}} + 6a_{18}a_{11_{y_2}} - 4a_{17_{y_2}}a_{12} + a_{17}a_{15}a_{12} - 4a_{17}a_{13_{y_1}} + 6a_{16_x}a_{11}$$
$$+2a_{16_{y_1y_1}} + 2a_{16_{y_1}}a_{14} + 2a_{16}a_{14_{y_1}} - 2a_{15_{y_1}}a_{15} + 2a_{15}a_{14_{y_2}} - 2a_{15}a_{12_x}$$
$$-2a_{14_x}a_{13} - 2a_{14_{y_2y_2}} + 4a_{13_{xy_1}} = 0.$$

$$J^{26} = 6a_{29_{y_1}}a_{12} - 4a_{29}a_{15}a_{11} + 6a_{29}a_{11_{y_2}} + 2a_{28_x}a_{11} + 2a_{28_{y_1y_1}} - 2a_{28_{y_1}}a_{14}$$
$$-2a_{28_{y_2}}a_{24} - 3a_{28}a_{27}a_{12} + 2a_{28}a_{24}a_{15} - 2a_{28}a_{17}a_{11} + 2a_{28}a_{11_x} - 2a_{27_x}a_{12}$$
$$-2a_{27_{y_1}}a_{15} + 2a_{27}^2a_{13} - 4a_{27}a_{26_{y_1}} + 2a_{27}a_{25_{y_2}} + a_{27}a_{25}a_{15} - 4a_{27}a_{24}a_{16}$$
$$-2a_{27}a_{18}a_{11} - a_{27}a_{17}a_{12} + 2a_{27}a_{15}a_{14} - 2a_{27}a_{14_{y_2}} + 2a_{27}a_{12_x} - 4a_{26}a_{24_x}$$
$$+4a_{26}a_{24}a_{17} + 2a_{25_x}a_{25} - 2a_{25_{y_1}}a_{17} - a_{25}^2a_{17} - 2a_{25}a_{24}a_{18} + 2a_{25}a_{17}a_{14}$$
$$-2a_{25}a_{14_x} - 4a_{24_{xy_2}} + 4a_{24_x}a_{15} + 2a_{24_{y_1}}a_{18} + 4a_{24_{y_2}}a_{17} + 4a_{24}a_{19}a_{12}$$
$$+4a_{24}a_{18_{y_1}} - 4a_{24}a_{17}a_{15} + 4a_{19_{y_1}}a_{11} + 2a_{19}a_{11_{y_1}} - 2a_{17}^2a_{11} + 4a_{17}a_{11_x}$$
$$-2a_{11_{xx}} = 0.$$

$$J^{27} = 4a_{29_{y_1}}a_{11} + 2a_{29}a_{25}a_{11} - 2a_{29}a_{24}a_{12} - 4a_{29}a_{14}a_{11} + 2a_{29}a_{11_{y_1}} - 2a_{28_{y_1}}a_{24}$$
$$-a_{28}a_{27}a_{11} - a_{28}a_{24_{y_1}} + 2a_{28}a_{24}a_{14} - a_{27_x}a_{11} + a_{27_{y_1y_1}} + a_{27_{y_1}}a_{25} - 3a_{27_{y_1}}a_{14}$$
$$-a_{27_{y_2}}a_{24} - a_{27}a_{25}a_{14} + a_{27}a_{24_{y_2}} + a_{27}a_{24}a_{15} - a_{27}a_{17}a_{11} - a_{27}a_{14_{y_1}} + 2a_{27}a_{14}^2$$
$$+a_{27}a_{11_x} + 2a_{25_x}a_{24} - a_{25}a_{24_x} - 2a_{24_{xy_1}} + 4a_{24_x}a_{14} + a_{24_{y_1}}a_{17} - 2a_{24}^2a_{18}$$
$$+4a_{24}a_{19}a_{11} + 2a_{24}a_{17_{y_1}} - 2a_{24}a_{17}a_{14} - 2a_{24}a_{14_x} = 0.$$

$$J^{28} = 2a_{28}a_{11} - 4a_{26}a_{24} + 2a_{25_{y_1}} + a_{25}^2 - 2a_{25}a_{14} - 4a_{24_{y_2}} + 2a_{24}a_{15} + 2a_{17}a_{11}$$
$$-4a_{11_x} = 0.$$

$$J^{29} = 2a_{29}a_{12}^2 + 4a_{28y_1}a_{12} - a_{28}a_{25}a_{12} - 2a_{28}a_{15}a_{11} + 4a_{28}a_{11y_2} - 2a_{27y_2}a_{12}$$
$$-4a_{27}a_{26}a_{12} + 4a_{27}a_{25}a_{13} - 4a_{27}a_{16}a_{11} + a_{27}a_{15}a_{12} - 6a_{27}a_{13y_1} + 2a_{27}a_{12y_2}$$
$$+2a_{26x}a_{11} + 2a_{26y_1y_1} - 2a_{26y_1}a_{25} - 2a_{26y_1}a_{14} - 2a_{26y_2}a_{24} - 2a_{26}a_{24y_2}$$
$$+4a_{26}a_{24}a_{15} - 2a_{25y_1}a_{15} + 2a_{25y_2}a_{25} - 4a_{25}a_{24}a_{16} - 2a_{25}a_{18}a_{11} - 2a_{25}a_{17}a_{12}$$
$$+2a_{25}a_{15}a_{14} - 2a_{25}a_{14y_2} + 2a_{25}a_{12x} - 6a_{24x}a_{13} + 2a_{24y_1}a_{16} - 2a_{24y_2y_2}$$
$$+4a_{24y_2}a_{15} + 2a_{24}a_{18}a_{12} + 4a_{24}a_{17}a_{13} + 4a_{24}a_{16y_1} - 2a_{24}a_{15}^2 + 4a_{19}a_{12}a_{11}$$
$$+4a_{18y_1}a_{11} + 2a_{18}a_{11y_1} - 4a_{17}a_{15}a_{11} + 2a_{17}a_{14}a_{12} - 2a_{17}a_{12y_1} + 4a_{17}a_{11y_2}$$
$$+4a_{15}a_{11x} - 2a_{14x}a_{12} - 4a_{11xy_2} = 0.$$

$$J^{30} = a_{28}a_{12}^2 - 2a_{27}a_{13}a_{12} + 2a_{26y_1}a_{12} - 2a_{26}a_{25}a_{12} + 2a_{26}a_{11y_2} + 2a_{25}^2a_{13}$$
$$-4a_{25}a_{16}a_{11} - 4a_{25}a_{13y_1} + 2a_{25}a_{12y_2} - 4a_{24y_2}a_{13} + 4a_{24}a_{15}a_{13} - 2a_{24}a_{13y_2}$$
$$+2a_{18}a_{12}a_{11} - a_{17}a_{12}^2 + 4a_{16y_1}a_{11} + 2a_{16}a_{11y_1} - 2a_{15}^2a_{11} + 2a_{15}a_{14}a_{12}$$
$$-2a_{15}a_{12y_1} + 4a_{15}a_{11y_2} - 2a_{14y_2}a_{12} - 2a_{14}a_{13y_1} + 2a_{13x}a_{11} + 2a_{13y_1y_1}$$
$$-2a_{13}a_{11x} - 2a_{11y_2y_2} = 0.$$

$$J^{31} = 2a_{29}a_{12}a_{11} + 2a_{28y_1}a_{11} + a_{28}a_{25}a_{11} - 2a_{28}a_{24}a_{12} - 2a_{28}a_{14}a_{11} + a_{28}a_{11y_1}$$
$$+a_{27y_1}a_{12} - 2a_{27y_2}a_{11} - 2a_{27}a_{26}a_{11} + 2a_{27}a_{24}a_{13} - a_{27}a_{12y_1} + 2a_{27}a_{11y_2}$$
$$-4a_{26y_1}a_{24} - 2a_{26}a_{24y_1} + 4a_{26}a_{24}a_{14} + a_{25x}a_{11} + a_{25y_1y_1} + a_{25y_1}a_{25}$$
$$-3a_{25y_1}a_{14} + a_{25y_2}a_{24} - a_{25}^2a_{14} + a_{25}a_{24}a_{15} - a_{25}a_{17}a_{11} - a_{25}a_{14y_1} + 2a_{25}a_{14}^2$$
$$-2a_{24x}a_{12} - 2a_{24y_1y_2} + a_{24y_1}a_{15} + 4a_{24y_2}a_{14} - 4a_{24}^2a_{16} + 2a_{24}a_{15y_1}$$
$$-2a_{24}a_{15}a_{14} - 2a_{24}a_{14y_2} + 2a_{24}a_{12x} + 4a_{19}a_{11}^2 + 2a_{17y_1}a_{11} - 2a_{17}a_{14}a_{11}$$
$$+a_{17}a_{11y_1} - 2a_{14x}a_{11} + 4a_{14}a_{11x} - 2a_{11xy_1} = 0.$$

$$J^{32} = -12a_{29y_2}a_{13} + 8a_{29}a_{16}a_{12} + 4a_{29}a_{15}a_{13} - 6a_{29}a_{13y_2} + 2a_{28x}a_{13} + 6a_{28y_1}a_{16}$$
$$-2a_{28y_2y_2} + 2a_{28y_2}a_{26} + 2a_{28y_2}a_{15} + 4a_{28}^2a_{13} - 4a_{28}a_{26}a_{15} - 2a_{28}a_{25}a_{16}$$
$$-a_{28}a_{18}a_{12} + 2a_{28}a_{17}a_{13} - 4a_{28}a_{16y_1} - 4a_{28}a_{16}a_{14} + 4a_{28}a_{15y_2} + a_{28}a_{15}^2$$
$$-6a_{28}a_{13x} + 4a_{27y_2}a_{16} + 2a_{27}a_{18}a_{13} + 2a_{27}a_{16y_2} - 6a_{27}a_{16}a_{15} + 4a_{26xy_2}$$
$$-4a_{26x}a_{26} - 2a_{26y_1}a_{18} + 6a_{26}a_{25}a_{18} - 4a_{26}a_{19}a_{12} - 4a_{26}a_{18y_1} - 4a_{26}a_{18}a_{14}$$
$$+2a_{26}a_{17}a_{15} + 4a_{26}a_{15x} - 8a_{25x}a_{16} - 2a_{25y_2}a_{18} - 8a_{25}a_{19}a_{13} - 6a_{25}a_{18y_2}$$

$$-a_{25}a_{18}a_{15}+4a_{25}a_{17}a_{16}+6a_{25}a_{16x}+8a_{24}a_{18}a_{16}+8a_{19y_1}a_{13}+2a_{19y_2}a_{12}$$
$$-8a_{19}a_{16}a_{11}+8a_{19}a_{14}a_{13}-2a_{19}a_{13y_1}+4a_{19}a_{12y_2}+4a_{18y_1y_2}+4a_{18y_2}a_{14}$$
$$-2a_{18}^2a_{11}+a_{18}a_{17}a_{12}+4a_{18}a_{14y_2}-4a_{17x}a_{13}-4a_{17y_1}a_{16}-2a_{17y_2}a_{15}$$
$$-2a_{17}^2a_{13}-2a_{17}a_{15y_2}+8a_{16}a_{14x}-4a_{15xy_2}+2a_{13xx}=0.$$

$$J^{33}=-2a_{29}a_{13}a_{12}-4a_{28y_1}a_{13}-a_{28y_2}a_{12}+4a_{28}a_{26}a_{12}-3a_{28}a_{25}a_{13}+6a_{28}a_{16}a_{11}$$
$$-a_{28}a_{15}a_{12}+5a_{28}a_{13y_1}-4a_{28}a_{12y_2}+4a_{27y_2}a_{13}+2a_{27}a_{13y_2}-2a_{26x}a_{12}$$
$$-2a_{26y_1y_2}+2a_{26y_1}a_{26}+a_{26y_1}a_{15}-a_{26}a_{25y_2}-a_{26}a_{25}a_{15}+2a_{26}a_{18}a_{11}$$
$$-2a_{26}a_{12x}+3a_{25x}a_{13}+a_{25y_1}a_{16}+a_{25y_2y_2}+a_{25}^2a_{16}+a_{25}a_{18}a_{12}+a_{25}a_{17}a_{13}$$
$$-a_{25}a_{16y_1}-2a_{25}a_{16}a_{14}+a_{25}a_{15y_2}-4a_{24y_2}a_{16}-6a_{24}a_{18}a_{13}-2a_{24}a_{16y_2}$$
$$+2a_{24}a_{16}a_{15}-4a_{19}a_{13}a_{11}-a_{18y_1}a_{12}-2a_{18y_2}a_{11}+2a_{18}a_{15}a_{11}-2a_{18}a_{14}a_{12}$$
$$+a_{18}a_{12y_1}-4a_{18}a_{11y_2}+a_{17y_2}a_{12}+2a_{17}a_{16}a_{11}+a_{17}a_{13y_1}-2a_{16x}a_{11}$$
$$-4a_{16}a_{11x}+a_{15x}a_{12}-2a_{13xy_1}+2a_{12xy_2}=0.$$

$$J^{34}=2a_{29x}a_{13}+2a_{29y_1}a_{16}+a_{29y_2}a_{15}-2a_{29}a_{28}a_{13}+2a_{29}a_{26}a_{15}-2a_{29}a_{25}a_{16}$$
$$-2a_{29}a_{17}a_{13}+4a_{29}a_{16y_1}+4a_{29}a_{16}a_{14}-a_{29}a_{15y_2}-a_{29}a_{15}^2+4a_{29}a_{13x}$$
$$-a_{28x}a_{15}-a_{28y_1}a_{18}-2a_{28}a_{27}a_{16}-a_{28}a_{17y_2}+2a_{27x}a_{16}+a_{27y_2}a_{18}$$
$$+2a_{27}a_{26}a_{18}-6a_{27}a_{19}a_{13}-a_{27}a_{18y_2}-a_{27}a_{18}a_{15}+4a_{27}a_{16x}-a_{25x}a_{18}$$
$$-a_{25}a_{19y_2}+a_{25}a_{19}a_{15}+a_{25}a_{18}a_{17}-4a_{24}a_{19}a_{16}+2a_{19x}a_{12}+2a_{19y_1y_2}$$
$$+2a_{19y_2}a_{14}+2a_{19}a_{18}a_{11}-4a_{19}a_{17}a_{12}+3a_{19}a_{15y_1}-4a_{19}a_{14y_2}+4a_{19}a_{12x}$$
$$-2a_{18xy_1}-2a_{18x}a_{14}-a_{18}a_{17y_1}+2a_{17xy_2}+a_{17x}a_{15}-a_{17y_2}a_{17}+a_{17}a_{15x}$$
$$-a_{15xx}=0.$$

$$J^{35}=2a_{29x}a_{12}+4a_{29y_1y_2}+4a_{29y_1}a_{26}-4a_{29y_1}a_{15}-2a_{29}a_{28}a_{12}-4a_{29}a_{27}a_{13}$$
$$+2a_{29}a_{25y_2}-4a_{29}a_{18}a_{11}-2a_{29}a_{17}a_{12}-2a_{29}a_{14y_2}+4a_{29}a_{12x}-2a_{28y_1}a_{28}$$
$$-2a_{28y_2}a_{27}+2a_{28}a_{27}a_{15}+a_{28}a_{25}a_{17}-2a_{27y_1}a_{18}-2a_{27y_2}a_{17}+2a_{27}^2a_{16}$$
$$+4a_{27}a_{26x}-2a_{27}a_{26}a_{17}-3a_{27}a_{25}a_{18}+2a_{27}a_{19}a_{12}+2a_{27}a_{18y_1}+4a_{27}a_{18}a_{14}$$
$$+a_{27}a_{17}a_{15}-4a_{27}a_{15x}-2a_{25xx}+2a_{25x}a_{17}+2a_{25y_1}a_{19}+2a_{25}a_{19y_1}-a_{25}a_{17}^2$$
$$+6a_{24x}a_{18}+6a_{24}a_{19y_2}-4a_{24}a_{18}a_{17}-2a_{19x}a_{11}-2a_{19y_1y_1}-2a_{19y_1}a_{14}$$

$$+4a_{19}a_{17}a_{11} - 2a_{19}a_{14y_1} - 4a_{19}a_{11x} + 2a_{17y_1}a_{17} - 2a_{17}a_{14x} + 2a_{14xx} = 0.$$

$$J^{36} = 2a_{29x}a_{12} + 2a_{29y_1y_2} + 2a_{29y_1}a_{26} - a_{29y_1}a_{15} - 4a_{29}a_{28}a_{12} + 2a_{29}a_{27}a_{13}$$
$$-4a_{29}a_{26y_1} + 3a_{29}a_{25y_2} + a_{29}a_{25}a_{15} - 4a_{29}a_{24}a_{16} - 6a_{29}a_{18}a_{11} + 4a_{29}a_{12x}$$
$$+2a_{28xy_1} + a_{28x}a_{25} - a_{28y_1}a_{28} - a_{28y_1}a_{17} - a_{28y_2}a_{27} + a_{28}a_{27}a_{15} + a_{28}a_{25x}$$
$$-2a_{28}a_{19}a_{11} - 2a_{27xy_2} - 2a_{27x}a_{26} - a_{27y_1}a_{18} - a_{27}a_{25}a_{18} + a_{27}a_{18y_1}$$
$$+2a_{27}a_{18}a_{14} - a_{27}a_{17y_2} - a_{27}a_{15x} + 4a_{26}a_{24}a_{19} - a_{25xx} - a_{25y_1}a_{19} - a_{25}^2a_{19}$$
$$+a_{25}a_{19y_1} + 2a_{25}a_{19}a_{14} - a_{25}a_{17x} + 4a_{24x}a_{18} + 4a_{24y_2}a_{19} + 2a_{24}a_{19y_2}$$
$$-2a_{24}a_{19}a_{15} + 2a_{24}a_{18x} - 2a_{24}a_{18}a_{17} + 2a_{19x}a_{11} - 2a_{19}a_{17}a_{11} + 4a_{19}a_{11x} = 0.$$

$$J^{37} = 2a_{29x}a_{13} - 6a_{29y_1}a_{16} + 2a_{29y_2y_2} + 2a_{29y_2}a_{26} - 2a_{29y_2}a_{15} - 4a_{29}a_{28}a_{13}$$
$$+2a_{29}a_{26y_2} - 2a_{29}a_{18}a_{12} - 2a_{29}a_{15y_2} + 4a_{29}a_{13x} - 2a_{28y_2}a_{28} + a_{28}^2a_{15}$$
$$+4a_{28}a_{27}a_{16} + 2a_{28}a_{26x} - a_{28}a_{25}a_{18} + 2a_{28}a_{19}a_{12} + 2a_{28}a_{18y_1} + 2a_{28}a_{18}a_{14}$$
$$-a_{28}a_{17}a_{15} - 2a_{28}a_{15x} - 2a_{27y_2}a_{18} - 4a_{27}a_{26}a_{18} + 4a_{27}a_{19}a_{13} + 2a_{27}a_{18y_2}$$
$$+3a_{27}a_{18}a_{15} - 6a_{27}a_{16x} - 2a_{26xx} + 2a_{26y_1}a_{19} + 4a_{25x}a_{18} + 4a_{25}a_{19y_2}$$
$$-2a_{25}a_{18}a_{17} - 2a_{24}a_{18}^2 - 2a_{19x}a_{12} - 4a_{19y_1y_2} - 4a_{19y_2}a_{14} + 4a_{19}a_{18}a_{11}$$
$$+2a_{19}a_{17}a_{12} - 2a_{19}a_{15y_1} - 4a_{19}a_{12x} + 2a_{18}a_{17y_1} - 4a_{18}a_{14x} + 2a_{17y_2}a_{17}$$
$$+2a_{15xx} = 0.$$

$$J^{38} = -4a_{29y_1}a_{13} + 2a_{29y_2}a_{12} + 4a_{29}a_{26}a_{12} - 4a_{29}a_{25}a_{13} - 8a_{29}a_{16}a_{11} - 4a_{29}a_{15}a_{12}$$
$$+4a_{29}a_{13y_1} - 2a_{29}a_{12y_2} - 2a_{28x}a_{12} - 2a_{28y_2}a_{25} + a_{28}^2a_{12} + 2a_{28}a_{26y_1}$$
$$-2a_{28}a_{25y_2} + a_{28}a_{25}a_{15} + 8a_{28}a_{24}a_{16} + 6a_{28}a_{18}a_{11} - 2a_{28}a_{12x} + 2a_{27x}a_{13}$$
$$-6a_{27y_1}a_{16} + 2a_{27y_2y_2} + 2a_{27y_2}a_{26} - 4a_{27y_2}a_{15} + 2a_{27}a_{26y_2} - 4a_{27}a_{26}a_{15}$$
$$+4a_{27}a_{25}a_{16} + 4a_{27}a_{17}a_{13} - 6a_{27}a_{16y_1} + 3a_{27}a_{15}^2 - 2a_{27}a_{13x} - 4a_{26xy_1}$$
$$+2a_{26x}a_{25} + 2a_{26y_1}a_{17} - 8a_{26}a_{24}a_{18} + 4a_{25x}a_{15} + 4a_{25y_1}a_{18} - a_{25}^2a_{18}$$
$$+4a_{25}a_{19}a_{12} + 2a_{25}a_{18y_1} - 2a_{25}a_{18}a_{14} + 4a_{25}a_{17y_2} - 3a_{25}a_{17}a_{15} - 2a_{25}a_{15x}$$
$$-4a_{24y_2}a_{18} + 8a_{24}a_{19}a_{13} + 4a_{24}a_{18y_2} + 2a_{24}a_{18}a_{15} - 12a_{24}a_{16x} - 6a_{19y_1}a_{12}$$
$$-12a_{19y_2}a_{11} - 6a_{19}a_{12y_1} + 4a_{18x}a_{11} + 2a_{18}a_{17}a_{11} - 4a_{18}a_{11x} - 4a_{17y_1y_2}$$
$$+2a_{17y_1}a_{15} + 3a_{17}^2a_{12} - 2a_{17}a_{15y_1} + 4a_{17}a_{14y_2} - 4a_{17}a_{12x} + 4a_{15xy_1} - 4a_{15}a_{14x}$$

$$+2a_{12xx} = 0.$$

$$J^{39} = -12a_{29_{y_1}}a_{13} - 6a_{29_{y_2}}a_{12} + 8a_{29}a_{16}a_{11} + 4a_{29}a_{15}a_{12} - 6a_{29}a_{12_{y_2}} - 4a_{28_{y_1y_2}}$$
$$+4a_{28_{y_1}}a_{15} + 2a_{28_{y_2}}a_{25} + 3a_{28}^2a_{12} + 2a_{28}a_{27}a_{13} + 4a_{28}a_{26_{y_1}} - 2a_{28}a_{25_{y_2}}$$
$$-3a_{28}a_{25}a_{15} + 4a_{28}a_{18}a_{11} + 2a_{28}a_{14_{y_2}} - 4a_{28}a_{12x} + 4a_{27x}a_{13} + 4a_{27_{y_1}}a_{16}$$
$$+2a_{27_{y_2}}a_{15} - 2a_{27}a_{26}a_{15} + 2a_{27}a_{25}a_{16} + 6a_{27}a_{17}a_{13} - 4a_{27}a_{16_{y_1}} - 8a_{27}a_{16}a_{14}$$
$$+4a_{27}a_{15_{y_2}} - a_{27}a_{15}^2 - 4a_{27}a_{13x} - 4a_{26x}a_{25} + 4a_{25xy_2} - 2a_{25x}a_{15} + 3a_{25}^2a_{18}$$
$$-4a_{25}a_{19}a_{12} - 4a_{25}a_{18_{y_1}} - 4a_{25}a_{18}a_{14} + a_{25}a_{17}a_{15} + 4a_{25}a_{15x} - 12a_{24x}a_{16}$$
$$-6a_{24_{y_2}}a_{18} - 8a_{24}a_{19}a_{13} - 6a_{24}a_{18_{y_2}} + 4a_{24}a_{18}a_{15} + 8a_{24}a_{17}a_{16} + 2a_{19_{y_1}}a_{12}$$
$$-4a_{19_{y_2}}a_{11} - 4a_{19}a_{15}a_{11} + 4a_{19}a_{14}a_{12} - 2a_{19}a_{12_{y_1}} + 4a_{19}a_{11_{y_2}} + 2a_{18x}a_{11}$$
$$+2a_{18_{y_1y_1}} + 2a_{18_{y_1}}a_{14} + 2a_{18}a_{14_{y_1}} - 2a_{18}a_{11x} - 2a_{17x}a_{12} - 2a_{17_{y_1}}a_{15} + a_{17}^2a_{12}$$
$$-2a_{17}a_{15_{y_1}} + 2a_{17}a_{14_{y_2}} - 2a_{17}a_{12x} + 2a_{15}a_{14x} - 4a_{14xy_2} + 2a_{12xx} = 0.$$

$$J^{40} = 4a_{29_{y_2}}a_{12} + 8a_{29}a_{26}a_{12} - 8a_{29}a_{25}a_{13} + 8a_{29}a_{16}a_{11} - 6a_{29}a_{15}a_{12} + 12a_{29}a_{13_{y_1}}$$
$$-4a_{29}a_{12_{y_2}} - 4a_{28x}a_{12} - 2a_{28_{y_1}}a_{15} - 2a_{28_{y_2}}a_{25} + a_{28}^2a_{12} - 4a_{28}a_{27}a_{13}$$
$$+2a_{28}a_{26_{y_1}} - 2a_{28}a_{25_{y_2}} + a_{28}a_{25}a_{15} + 4a_{28}a_{24}a_{16} + 4a_{28}a_{18}a_{11} - 2a_{28}a_{14_{y_2}}$$
$$-2a_{28}a_{12x} + 6a_{27x}a_{13} - 2a_{27_{y_1}}a_{16} + 2a_{27_{y_2y_2}} + 2a_{27_{y_2}}a_{26} - 2a_{27_{y_2}}a_{15}$$
$$+2a_{27}a_{26_{y_2}} - 4a_{27}a_{17}a_{13} + 2a_{27}a_{16_{y_1}} + 4a_{27}a_{16}a_{14} - 2a_{27}a_{15_{y_2}} + a_{27}a_{15}^2$$
$$+6a_{27}a_{13x} - 4a_{26xy_1} + 2a_{26x}a_{25} + 2a_{26_{y_1}}a_{17} - 4a_{26}a_{24}a_{18} + 2a_{25x}a_{15}$$
$$+2a_{25_{y_1}}a_{18} - a_{25}^2a_{18} + 6a_{25}a_{19}a_{12} + 2a_{25}a_{18_{y_1}} + 2a_{25}a_{17_{y_2}} - a_{25}a_{17}a_{15}$$
$$-2a_{25}a_{15x} + 4a_{24x}a_{16} - 2a_{24_{y_2}}a_{18} - 8a_{24}a_{19}a_{13} + 2a_{24}a_{18_{y_2}} - 4a_{24}a_{17}a_{16}$$
$$-4a_{24}a_{16x} - 4a_{19_{y_1}}a_{12} + 8a_{19}a_{15}a_{11} - 8a_{19}a_{14}a_{12} + 4a_{19}a_{12_{y_1}} - 12a_{19}a_{11_{y_2}}$$
$$-6a_{18x}a_{11} - 2a_{18_{y_1y_1}} - 2a_{18_{y_1}}a_{14} + 4a_{18}a_{17}a_{11} - 2a_{18}a_{14_{y_1}} - 6a_{18}a_{11x}$$
$$+4a_{17x}a_{12} + 2a_{17_{y_1}}a_{15} - a_{17}^2a_{12} + 2a_{17}a_{15_{y_1}} - 2a_{17}a_{14_{y_2}} + 2a_{17}a_{12x} - 2a_{15}a_{14x}$$
$$+4a_{14xy_2} = 0.$$

$$J^{41} = 2a_{29_{y_1}}a_{12} + 8a_{29_{y_2}}a_{11} + 8a_{29}a_{26}a_{11} - 8a_{29}a_{24}a_{13} - 8a_{29}a_{15}a_{11} - 4a_{29}a_{14}a_{12}$$
$$+4a_{29}a_{12_{y_1}} - 2a_{29}a_{11_{y_2}} - 4a_{28x}a_{11} - 2a_{28_{y_1}}a_{25} - 4a_{28_{y_2}}a_{24} - 2a_{28}^2a_{11}$$
$$+a_{28}a_{27}a_{12} - 2a_{28}a_{25_{y_1}} + 2a_{28}a_{25}a_{14} + 4a_{28}a_{24}a_{15} + 2a_{28}a_{17}a_{11} + 4a_{27_{y_1y_2}}$$

$$+4a_{27y_1}a_{26}-6a_{27y_1}a_{15}-4a_{27y_2}a_{14}-2a_{27}^2a_{13}+4a_{27}a_{26y_1}-4a_{27}a_{26}a_{14}$$
$$-a_{27}a_{25}a_{15}+8a_{27}a_{24}a_{16}+2a_{27}a_{18}a_{11}-a_{27}a_{17}a_{12}-2a_{27}a_{15y_1}+6a_{27}a_{15}a_{14}$$
$$-2a_{27}a_{14y_2}+8a_{26x}a_{24}-4a_{26}a_{24}a_{17}-4a_{25xy_1}+4a_{25x}a_{14}+4a_{25y_1}a_{17}+a_{25}^2a_{17}$$
$$-6a_{25}a_{24}a_{18}+4a_{25}a_{19}a_{11}+2a_{25}a_{17y_1}-4a_{25}a_{17}a_{14}+6a_{24x}a_{15}+2a_{24y_1}a_{18}$$
$$-4a_{24y_2}a_{17}+8a_{24}a_{19}a_{12}+4a_{24}a_{18y_1}+6a_{24}a_{17y_2}-2a_{24}a_{17}a_{15}-8a_{24}a_{15x}$$
$$-12a_{19y_1}a_{11}-6a_{19}a_{11y_1}+2a_{17x}a_{11}-2a_{17y_1y_1}+2a_{17y_1}a_{14}+4a_{17}^2a_{11}$$
$$-6a_{17}a_{11x}+4a_{14xy_1}-4a_{14x}a_{14}+2a_{11xx}=0.$$
$$J^{42}=-4a_{29}a_{13}a_{11}+a_{28y_1}a_{12}+2a_{28}a_{24}a_{13}+a_{28}a_{15}a_{11}+a_{28}a_{11y_2}-2a_{27y_1}a_{13}$$
$$-a_{27y_2}a_{12}-2a_{27}a_{26}a_{12}+2a_{27}a_{25}a_{13}-6a_{27}a_{16}a_{11}+a_{27}a_{15}a_{12}+2a_{27}a_{14}a_{13}$$
$$-4a_{27}a_{13y_1}+a_{27}a_{12y_2}-2a_{26}a_{24}a_{15}+a_{25x}a_{12}+a_{25y_1}a_{15}+2a_{25}a_{24}a_{16}$$
$$-a_{25}a_{17}a_{12}-a_{25}a_{15}a_{14}+a_{25}a_{14y_2}-2a_{24x}a_{13}-2a_{24y_1}a_{16}-a_{24y_2}a_{15}$$
$$+6a_{24}a_{17}a_{13}-4a_{24}a_{16y_1}+a_{24}a_{15y_2}+a_{24}a_{15}^2-4a_{24}a_{13x}-2a_{19}a_{12}a_{11}$$
$$+4a_{18y_1}a_{11}+2a_{18}a_{11y_1}-a_{17y_1}a_{12}-4a_{17y_2}a_{11}-3a_{17}a_{15}a_{11}+4a_{17}a_{14}a_{12}$$
$$-4a_{17}a_{12y_1}+5a_{17}a_{11y_2}+3a_{15x}a_{11}+a_{15y_1y_1}-a_{15y_1}a_{14}-2a_{14x}a_{12}-2a_{14y_1y_2}$$
$$+2a_{14y_2}a_{14}-2a_{14}a_{12x}+2a_{12xy_1}-2a_{11xy_2}=0.$$

Notice that for verification, these two sets (set of first-order relative invariants and set of second-order relative invariants) are relative invariants, if they satisfy equation (3.5).

For the linearization problem, analysis of these relative invariants gives us that the equations $J^i=0,\ (i=1,2,...,15)$ and $J^m=0,\ (m=1,2,...,42)$ are necessary conditions for system of equations (3.1) to be equivalent to the trivial system:

$$\bar{y}_1''=0,\ \bar{y}_2''=0.$$

4.3 Analytical Testing of the Results

Example 1. Let us consider the system

$$\begin{aligned} &y_1'' + a_{11}(y_1')^3 + a_{14}(y_1')^2 + a_{17}y_1' + a_{19} = 0, \\ &y_2'' + a_{11}y_2'(y_1')^2 = 0. \end{aligned} \tag{4.4}$$

In this case

$$\begin{aligned} &J^1 = J^3 = J^4 = J^5 = J^{13} = J^{14} = 0, \\ &J^{15} = a_{11y_2}, \\ &J^2 = a_{14y_2}, \\ &J^6 = J^7 = a_{17y_2}, \\ &J^8 = a_{19y_2} = 0, \\ &J^9 = 6a_{19}a_{11} + a_{17y_1} - 2a_{14x}, \\ &J^{10} = -2a_{19}a_{11}, \\ &J^{11} = a_{17}a_{11} - 2a_{11x}, \\ &J^{12} = (4a_{19y_1} + 4a_{19}a_{14} - 2a_{17x} - a_{17}^2)/2\ . \end{aligned}$$

Assume that the relative invariants vanish. Because $J^2 = 0, J^6 = 0, J^8 = 0, J^{15} = 0$, the first equation of system (4.4) becomes an equation in only one dependent function, namely y_1. For this equation,

$$\begin{aligned} H = {} & 3a_{11xx} - 2a_{14xy_1} + a_{17y_1y_1} - 3a_{11x}a_{17} + 3a_{11y_1}a_{19} + 2a_{14x}a_{14} - 3a_{17x}a_{11} \\ & -a_{17y_1}a_{14} + 6a_{19y_1}a_{11}, \\ K = {} & a_{14xx} - 2a_{17xy_1} + 3a_{19y_1y_1} - 6a_{11x}a_{19} + a_{14x}a_{17} + 3a_{14y_1}a_{19} - 2a_{17y_1}a_{11} \\ & -3a_{19x}a_{11} + 3a_{19y_1}a_{14}. \end{aligned}$$

By virtue of $J^9 = 0, J^{10} = 0, J^{11} = 0, J^{12} = 0$, one finds that $H = 0,\ K = 0$. Hence, according to Lie's test, the vanishing of the relative invariants guarantee that the first equation is linearizable.

Example 2. Let us consider the system

$$
\begin{aligned}
&y_1'' + a_{13}y_1'(y_2')^2 = 0,\\
&y_2'' + a_{13}(y_2')^3 + a_{26}(y_2')^2 + a_{28}y_2' + a_{29} = 0.
\end{aligned}
\tag{4.5}
$$

In this case

$$
\begin{aligned}
&J^2 = J^4 = J^8 = J^{11} = J^{14} = J^{15} = 0,\\
&J^1 = a_{13y_1},\\
&J^3 = a_{26y_1},\\
&J^9 = J^{10} = a_{28y_1},\\
&J^{13} = a_{29y_1},\\
&J^6 = 6a_{29}a_{13} + a_{28y_2} - 2a_{26x},\\
&J^7 = -2a_{29}a_{13},\\
&J^5 = a_{28}a_{13} - 2a_{13x},\\
&J^{12} = (4a_{29y_2} + 4a_{29}a_{26} - 2a_{28x} - a_{28}^2)/2\ .
\end{aligned}
$$

Assume that the relative invariants vanish. Because of $J^1 = 0, J^3 = 0, J^9 = 0, J^{13} = 0$ the second equation of system (4.5) becomes an equation in only one dependent function, namely y_2. For this equation

$$
\begin{aligned}
H = {} & 3a_{13xx} - 2a_{26xy_2} + a_{28y_2y_2} - 3a_{13x}a_{28} + 3a_{13y_2}a_{29} + 2a_{26x}a_{26} - 3a_{28x}a_{13}\\
& -a_{28y_2}a_{26} + 6a_{29y_2}a_{13},\\
K = {} & a_{26xx} - 2a_{28xy_2} + 3a_{29y_2y_2} - 6a_{13x}a_{29} + a_{26x}a_{28} + 3a_{26y_2}a_{29} - 2a_{28y_2}a_{13}\\
& -3a_{29x}a_{13} + 3a_{29y_2}a_{26}.
\end{aligned}
$$

By virtue of $J^6 = 0, J^7 = 0, J^5 = 0, J^{12} = 0$, one finds that $H = 0,\ K = 0$. Hence, according to Lie's test, the vanishing of the relative invariants guarantee that the second equation is linearizable.

CHAPTER V

CONCLUSION

5.1 Thesis Summary

This thesis is devoted to the study a system of two second-order ordinary differential equations,

$$y_1'' = f_1(x, y_1, y_2, y_1', y_2') \ , \ y_2'' = f_2(x, y_1, y_2, y_1', y_2'). \tag{5.1}$$

by group analysis.

5.1.1 Problems

The problem considered in the thesis is related to the linerization problem. For system (5.1), the linearization problem is to find an invertible transformation of independent and dependent variables,

$$t = \varphi(x, y_1, y_2), \ u_1 = \psi_1(x, y_1, y_2), \ u_2 = \psi_2(x, y_1, y_2), \tag{5.2}$$

which transforms the system of equations (5.1) into a system of linear equations

$$u'' + B(t)u = 0, \tag{5.3}$$

Similar to a single second-order ordinary differential equation, in order to obtain necessary conditions, we assume that system (5.1) is obtained from a system of linear differential equations (5.3) by invertible transformation (5.2). Replacing the derivatives u_1', u_1'' ,u_2', and u_2'' in system (5.3), it becomes

$$\begin{aligned} & y_1'' + a_{11}{y_1'}^3 + a_{12}{y_1'}^2 y_2' + a_{13}y_1'{y_2'}^2 + a_{14}{y_1'}^2 \\ & +a_{15}y_1'y_2' + a_{16}{y_2'}^2 + a_{17}y_1' + a_{18}y_2' + a_{19} = 0, \\ & y_2'' + a_{13}{y_2'}^3 + a_{12}{y_2'}^2 y_1' + a_{11}y_2'{y_1'}^2 + a_{24}{y_1'}^2 \\ & +a_{25}y_2'y_1' + a_{26}{y_2'}^2 + a_{27}y_1' + a_{28}y_2' + a_{29} = 0. \end{aligned} \tag{5.4}$$

Thus, if system (5.1) is linearizable, then it must be of the form (5.4).

In this thesis it is shown that form (5.4) is not changed with respect to any transformation (5.2). The main goal of this thesis was to find invariants of transformation (5.2). These invariants constitute necessary conditions for the linearization problem.

For solving the problem of the thesis, Lie's approach was used. This approach contains the following steps:

1. Finding the equivalence group of transformations for system (5.4).

2. Obtaining equations for the invariants.

3. Solving the equations defining invariants.

5.1.2 Results

1. The equivalence group of transformations for system (5.4) is shown by its generator on page 23.

2. All obtained equations which define invariants compose a system of linear homogeneous first-order partial differential equations. For first-order invariants, this system includes 60 equations with 60 independent variables. For second-order invariants, this system consists of 105 equations with 150 independent variables.

3. In case of first-order relative invariants, we found 15 relative invariants as presented on page 27. In case of second-order relative invariants , we found

42 relative invariants as presented on pages 28-36. All these invariants help us to analyze necessary conditions for linearizing system (5.4).

5.1.3 Limitations

In this thesis, group analysis was applied to find invariants of the equivalence group of a system of two second-order ordinary differential equations in the shape of a system (5.4). We only considered first and second orders relative invariants. In further research work, one may look for absolute invariants as in definition 4.

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CURRICULUM VITAE

NAME: Sakka Sookmee.

SEX: Male.

NATIONALITY: Thai.

DATE OF BIRTH: August 24, 1978.

MARITAL STATUS: Single.

PHONE NUMBER AND E-MAIL: 06-8675277, s_sakka2001us@yahoo.com.

EDUCATIONAL BACKGROUND:

- Bachelor of Engineering in Civil Engineering, Walailak University, Nakhon Si Thammarat, Thailand, 2002.

WORK EXPERIENCE:

- Teaching Assistant, Department of Mathematics, Suranaree University of Technology, Nakhon Ratchasima, Thailand. 1/2004.