การหาผลเฉลยยืนยงและผลเฉลยยืนยงเป็นบางส่วนของสมการนาเวียร์-สโตกส์ที่สัมพันธ์กับกลุ่มการหมุน

นายอภิชัย เหมะธุลิน

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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INVARIANT AND PARTIALLY INVARIANT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS RELATED WITH THE GROUP OF ROTATIONS

MR. APICHAI HEMATULIN

A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in Applied Mathematics

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อภิชัย เหมะธุลิน : การหาผลเฉลยยืนยงและยืนยงเป็นบางส่วนของสมการนาเวียร์-สโตกส์ ที่สัมพันธ์กับกลุ่มการหมุน (INVARIANT AND PARTIALLY INVARIANT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS RELATED WITH THE GROUP OF ROTATIONS)
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วิทยานิพนธ์ฉบับนี้ศึกษาอย่างเป็นระบบถึงการหาผลเฉลยของสมการที่อธิบายการเคลื่อนที่ของของไหลชนิดมีความหนืดแบบนิวโตเนียนซึ่งสมการเหล่านี้จะสัมพันธ์กับกลุ่มการหมุนในปริภูมิสามมิติ

ในตอนแรกจะเริ่มด้วยการศึกษาผลเฉลยเป็นบางส่วนที่สัมพันธ์กับกลุ่มการหมุนก่อนซึ่งส่วนหลักๆ ที่ยุ่งยากในการวิเคราะห์หาผลเฉลยยืนยงเป็นบางส่วน คือการศึกษาว่าเมื่อใดระบบสมการเชิงอนุพันธ์ย่อยจะมีผลเฉลย ซึ่งในวิทยานิพนธ์ฉบับนี้จะแสดงว่าการมีผลเฉลยจะมีภายใต้เงื่อนไขอะไร และผลเฉลยเหล่านี้จะถูกลดรูปให้เป็นผลเฉลยยืนยงแบบเอกฐาน ซึ่งก็คือ การสมมาตรแบบทรงกลม ในตอนต่อมาของงานวิจัยนี้ได้ศึกษาถึงการจัดจำพวกกลุ่มของสมการที่สมมาตรแบบทรงกลม โดยเทียบกับสัมประสิทธิ์ความหนืด สัมประสิทธิ์การนำความร้อน และฟังก์ชันสถานะ ซึ่งฟังก์ชันนี้จะทำให้มีการขยายส่วนกลางของกลุ่มที่แอดมิต ได้ 3 กรณี แล้วตัวแทนของผลเฉลยยืนยงจะถูกสร้างขึ้นจากทั้ง 3 กรณีนี้

การศึกษาผลเฉลยเหล่านี้จะสัมพันธ์กับกลุ่มการหมุนซึ่งจำเป็นต้องใช้ระบบพิกัดทรงกลม การได้มาของสมการในระบบดังกล่าวและการวิเคราะห์หาเงื่อนไขของการมีผลเฉลยเป็นสิ่งที่ยุ่งยากซับซ้อน ดังนั้นเพื่อเอาชนะอุปสรรคเหล่านี้ จึงจำเป็นต้องใช้การคำนวณเชิงสัญลักษณ์เข้าช่วย

สาขาวิชาคณิตศาสตร์ ____________

นักศึกษา

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VISCOUS FLUID DYNAMICS EQUATIONS / GROUP ANALYSIS / INVARIANT AND PARTIALLY INVARIANT SOLUTIONS /ANALYSIS OF COMPATIBILITY / SPHERICALLY SYMMETRIC FLOWS/

The thesis systematically studies solutions of the equations governing a motion of Newtonian viscous fluid, which is related with the group of rotations in three-dimensional space.

At first, partially invariant solutions with respect to the group of rotations are studied. The main difficulty in analyzing partially invariant solutions consists of study a compatibility of the system of partial differential equations. In the thesis, the compatibility conditions for these solutions were established. It is proven that they are reduced to singular invariant solutions, which are spherically symmetric. The next part of the research is devoted to the group classification of the spherically symmetric equations with respect to viscosity heat conductivity coefficient and state functions. It is shown that the extension of the kernel of admitted groups occurs in three cases. For all these cases, representations of invariant solutions were constructed.

A study of solutions related with the group of rotations requires using the spherical coordinate system. The derivation of the equations in spherical coordinates and an analysis of compatibility are very cumbersome. Symbolic calculations were used to overcoming these difficulties.

School of Mathematics ____________ Student

Academic Year 2001 ____________ Advisor

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Apichai Hematulin

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Chapter I

Introduction

Almost all important governing equations in physics take the form of nonlinear partial differential equations, usually as systems of equations. Nonlinearity and the presence of a large number of variables in the initial equations are sources of significant mathematical difficulties in the analysis of the solutions of these equations. Frequently, it is virtually impossible to give explicit solutions, and while a multitude of numerical methods has been developed to obtain approximate solutions, there remains intense interest in finding exact solutions. Each solution has value, first, as the exact description of the real process in the framework of a given model; secondly, as a model to compare various numerical methods; thirdly, as a basis to improve the models used. One of the methods for constructing exact solutions is group analysis of differential equations. At present, numerous differential equations are being investigated by this method. A historical review of a group analysis development can be found in Ibragimov, (1999). Many results of the group analysis are collected in CRC Handbooks edited by Ibragimov (ed.), (1994), (1995), (1996). Historically, many solutions of partial differential equations were constructed by educated guesses of the representations of solutions. Besides producing new solutions, group analysis provides a more systematic method for constructing such representations.

This research is devoted to an application of group analysis to the equations which govern the motion of Newtonian viscous fluid. These equations make it possible to obtain full information about the structure of flows under usual temperature and pressure, and they play a central role in a variety of research within the fields of applied mathematics, physics and engineering. We called these equations viscous gas dynamics equations[1]. In compact form, the viscous gas dynamics equations can be written as follows:

$$\begin{aligned}
&\frac{d\mathbf{v}}{dt} + \tau\nabla p = \tau[(\lambda+\mu)\nabla(div\mathbf{v}) + (div\mathbf{v})\nabla\lambda + 2\mathbf{D} < \nabla\mu > + \mu\Delta\mathbf{v}]\ , \\
&\frac{d\tau}{dt} - \tau div\mathbf{v} = 0\ , \qquad (1.1) \\
&\frac{dp}{dt} + A(p,\tau)(div\mathbf{v}) = B(p,\tau)(\lambda(div\mathbf{v})^2 + 2\mu\mathbf{D}:\mathbf{D} + (\nabla\kappa)(\nabla T) + \kappa\triangle T).
\end{aligned}$$

[1] In the literature viscous gas dynamics equations are often called the Navier-Stokes equations

Where

$$A(p,\tau) = \frac{\tau(p+U_\tau)}{U_p}, \quad B(p,\tau) = \frac{\tau}{U_p}$$

and $\mathbf{v} = (u_1, u_2, u_3)$ is the velocity vector, p is pressure, $\tau = \frac{1}{\rho}$ is specific volume, ρ is density, U is the internal energy, T is the temperature, λ and μ are first and the second coefficients of viscosity, κ is a coefficient of a heat conductivity, t is time, ∇ and Δ are the gradient and the Laplacian with respect to the space variables $\mathbf{x} = (x_1, x_2, x_3)$, respectively.

The equations for incompressible fluid (τ is constant) are obtained from the viscous gas dynamics equations by chooing τ and μ constant. In this case, the viscous gas dynamics equations consist of two parts, and the first part is

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= -\tau(\nabla p + \mu\Delta\mathbf{v}) \\ div\mathbf{v} &= 0. \end{aligned} \tag{1.2}$$

These equations are called the Navier-Stokes equations.

Many of the invariant solutions of the Navier-Stokes equations have been known for a long time; however, their systematic analysis became possible only with the development of the modern methods of *group analysis* of differential equations (cf. Ovsiannikov (1978)). The first group classification of the Navier-Stokes equations in the three-dimensional case was done in Bytev (1972). It was shown that the Lie group admitted by the Navier-Stokes equations is infinite-dimensional. Its Lie algebra can be presented in the form of the direct sum $L^\infty \oplus L^{11}$, where the infinite-dimensional ideal L^∞ is generated by the operators

$$\begin{aligned} \zeta_{\phi_i} \cdot \partial &= \phi_i(t)\partial_{x_i} + \phi_i'(t)\partial_{u_i} - \phi_i''(t)x_i\partial_p \\ \zeta_\psi \cdot \partial &= \psi(t)\partial_p \end{aligned}$$

with arbitrary functions $\phi_i(t) : R \to R, (i = 1, 2, 3)$, and $\psi(t) : R \to R$.

The *subalgebra* L^{11} has the following basis:

$$\begin{aligned} \xi_0 \cdot \partial &= \partial_t, \xi_i \cdot \partial = \partial_{x_i}, \eta_i \cdot \partial = t\partial_{x_i} + \partial_{u_i} (i = 1, 2, 3), \\ \zeta_{ik} \cdot \partial &= x_i\partial_{x_k} - x_k\partial_{x_i} + u_i\partial_{u_k} - u_k\partial_{u_i}, (i < k), \\ \tau \cdot \partial &= 2t\partial_t + x_i\partial_{x_i} - u_i\partial_{u_i} - 2p\partial_p. \end{aligned} \tag{1.3}$$

If the dilatation operator $\tau \cdot \partial$ is neglected, then the remaining operators of (1.3) generate a basis of the algebra L^{10}, the well-known *Lie algebra of the Galilean group* G^{10}, whose transformations operate on the seven-dimensional space $(t, \mathbf{x}, \mathbf{u})$. Its subalgebra generated by the three operators $\zeta_{ik} \cdot \partial$ composes the *Lie algebra* $SO(3)$.

Note that there is still no classification of the full group admitted by the Navier-Stokes equations. In spite of this several papers (cf. Puchnachov (1974), Lloyd (1981), Boisvert and Ames (1983), Grauel and Steeb (1985), Fushchich and Popovych (1994), Ibragimov (1994) and Popovych (1995)) are devoted to invariant solutions of the Navier-Stokes equations. Reviews devoted to invariant solutions of the Navier-Stokes equations can be found in Puchanchov (1974), Fushchich and Popovych (1994) and Ludlow, Clarkson and Bassom (1999).

Partially invariant solutions of the Navier–Stokes equations have been less studied (cf. Puchnachov (1974), Meleshko and Puchnachov (1999)). At the same time there has been progress in studying such classes of solutions of inviscid gas dynamics equations (cf. Ovsiannikov (1978), Sidorov, Shapeev and Yanenko (1984) and Meleshko (1991)). Recently, Ovsiannikov (1995) found one class of partially invariant solutions, called a special vortex. These solutions are based on the group of rotations $O(3)$. An ideal fluid and an inviscid gas have the same class of solutions. Therefore, it is natural to investigate the existence of special vortex type solutions for the Navier-Stokes equations and viscous gas dynamics equations.

One part of our study is devoted to answering this question. Along with the same partially invariant solutions were studied for the viscous gas dynamics equations.

It is well-known that the main difficulty in the study of partially invariant solutions is the analysis of the compatibility (cf. Finikov (1948) and Kuranishi (1967)) of the appearing overdetermined systems. The analysis of compatibility can be reduced to the consecutive performance of algebraic operations of symbolic nature. These operations are related with a prolongation of the system, substitution of composite expressions (transition onto manifold), and finding ranks of matrices. Typically, the compatibility study of systems of partial differential equations requires a large amount of analytical calculations, and it is necessary to use a computer system for these calculations.

A brief review of computer systems can be found, for example, in Ibragimov (1994), (1995), (1996), (1999). In our calculations the system REDUCE (cf. Hearn (1999)) was used.

The study of partially invariant solutions based on the group of rotations depends on the value of H which is modulus of velocity vectors V, W in the spherical coordinates. If the value of $H = 0$, then the partially invariant solution is reduced to a singular invariant solution, which is spherically symmetric flow.

Another part of the study in this thesis is devoted to the group classification of spherically symmetric viscous gas dynamics equations. The group classification problem consists of searching for groups of transformations admitted by the system for all arbitrary elements and all specifications of arbitrary elements. By a special choice of the arbitrary elements, one can extend the admitted group.

After finding the admitted group, one can try to construct exact solutions: every subgroup of the admitted group can be a source of invariant or partially invariant solutions. There is an infinite number of subgroups, even in cases where the admitted groups are finite-dimensional. But if two subgroups are similar, i.e., they are related with each other by a symmetry transformation, then their corresponding invariant solutions are connected with each other by the same transformation. Since the set of subgroups can be divided into classes of similar subgroups, therefore, it is sufficient to find only one representative solution from each similar class of subgroups. A set of representatives of equivalent subgroup classes is called an optimal system of subgroups. Because there is a one-to-one correspondence between groups and Lie algebras one can study the Lie algebra of the admitted group. In this thesis, representations of all invariant solutions with respect to subgroups of two-dimensional admitted groups of spherically symmetric viscous gas dynamics equations are presented.

We should also note here that, as for the Navier-Stokes equations, many of the invariant solutions of the viscous gas dynamics equations have been obtained by other methods (cf. Williams (1967), Shennikov (1969), Shidlovskii (1972), Aristov (1990), (1995) and Byrkin (1969), (1970), (1976), (1996)). The group classification of the viscous gas equations (in case when the first λ and the second μ coefficients of viscosity are related by the equation $\lambda = -2\mu/3$) was done in Bublik (2000). For some models of viscous gas dynamics equations, group analysis was used in Bublik (1996) and Meleshko (1998). There also exist other similar approaches for constructing exact solutions of the Navier–Stokes equations. We note here two of them: nonclassical symmetry reductions (cf. Ludlow and others (1998), (1999)) and linear profile of velocity (cf. Sidorov (1989)).

The thesis is organized as follows. Chapter II mainly introduces notations of group analysis and provides references to well known facts on the application of group analysis for constructing exact solutions of partial differential equations. Chapter III deals with obtaining the viscous gas dynamics equations. In this chapter, we establish the derivation of the viscous gas dynamics equations in the spherical coordinates by using symbolic calculations. The necessity of this is stipulated by the fact that all textbooks have some misprints in writing the Navier-Stokes equations in curvilinear coordinates. Chapter IV is devoted to an analysis of compatibility of partially invariant solutions of the Navier-Stokes and viscous gas dynamics equations related with the group of rotations. It is proved that all such partially invariant solutions are reduced to spherically symmetric solutions, which are singular invariant. The spherically symmetric viscous gas dynamics equations are studied in the next chapter V. Equivalence and admitted group for viscous gas dynamics is given in this chapter. Also representations of all invariant solutions of spherically symmetric viscous gas equations are considered there.

Chapter II

Group Analysis

2.1 Lie Groups

In this chapter, we will discuss the group analysis method for constructing solutions of partial differential equations. Discussions of the Lie group analysis can be found in the textbooks (cf. Ovsiannikov (1978), Ibragimov (1994), (1995), (1996)).

We consider invertible point transformations

$$\begin{aligned} &x_i^{'} = f^i(x,u;a), u^{'j} = \varphi^j(x,u;a), \\ &(i = 1,2,...,n; \quad j = 1,2,...,m). \end{aligned} \tag{2.1}$$

with $(x,u) \in V \subset Z = R^n(x) \times R^m(u)$ and a a group parameter, $a \in \Delta$, where $x = (x_1, x_2, ..., x_n)$ and $u = (u_1, u_2, ..., u_m)$ are independent variables and dependent variables respectively. The set V is an open set in Z, Δ is a symmetric interval of R^1 with respect to zero.

Definition 1 *A set of transformations (2.1) is called a local one-parameter group G^1 if it has the following properties:*

1. $f(x,u;0) = x$, $\varphi(x,u;0) = u$ for all $(x,u) \in V$.
2. $f(f(x,u;a), \varphi(x,u;a);b) = f(x,u;a+b)$ and $\varphi(f(x,u;a), \varphi(x,u;a);b) = \varphi(x,u;a+b)$ for all $a, b, a+b \in \Delta, (x,u) \in V$.
3. If $a \in \Delta$ and $f(x,u;a) = x, \varphi(x,u;a) = u$ for all $(x,u) \in V$,then $a = 0$.
4. $f, \varphi \in C_\infty(V \times \Delta)$.

The functions f^i and φ^j can be presented via their Taylor series expansion with respect to the parameter a in the neighborhood of $a = 0$ and written as the infinitesimal transformation (2.1)

$$\begin{aligned} x_i^{'} &\approx x_i + \xi^i(x,u)a \\ u^{'j} &\approx u^j + \zeta^j(x,u)a \end{aligned}$$

where

$$\xi^i(x,u) = \frac{\partial f^i(x,u;a)}{\partial a}\Big|_{a=0}, \quad \zeta^j(x,u) = \frac{\partial \varphi^j(x,u;a)}{\partial a}\Big|_{a=0} \tag{2.2}$$

The vector (ξ, ζ) given by (2.2) where $\xi = (\xi^1, ..., \xi^n), \zeta = (\zeta^1, ..., \zeta^m)$ is a tangent vector (at the point (x,u)) to the curve determined by transformation (2.1).

This vector is called a tangent vector field of the group G^1. A tangent vector field can be written in terms of the first-order differential operator

$$X = \xi^i(x,u)\partial_{x_i} + \zeta^j(x,u)\partial_{u^j} \tag{2.3}$$

The operator X is transformed as a scalar under a change of variables. There is a one to one correspondence between group of transformation (2.1) and infinitesimal generators (2.3).

Theorem 1 *A local Lie group of transformations (2.1) can be completely determined by the solution of the Cauchy problem of ordinary differential equations*

$$\frac{dx_i^{'}}{da} = \xi^i(x^{'},u^{'}), \quad \frac{du^{'j}}{da} = \zeta^j(x^{'},u^{'}) \tag{2.4}$$

with the initial data

$$f^i(x,u,0) = x, \quad \varphi^j(x,u,0) = 0.$$

Here equations (2.4) are called *Lie equations.*

Assume that there is a given function $u = u_0(x)$. Let us show how to construct a transformation of this function by the Lie group(2.1). First we substitute $u_0(x)$ into the expression for the transformation of the independent variables (2.1)

$$x^{'} = f(x, u_0(x); a). \tag{2.5}$$

because $f(x,u_0(x),0) = x$,then by using the inverse function theorem, in a neighborhood of $a = 0$ we can get

$$x = g(x^{'}, a) \tag{2.6}$$

Note that after substitution of $x = g(x^{'},a)$ into the first equation of (2.1) we obtain is the identity

$$x^{'} = f(g(x^{'},a), u_0(g(x^{'},a)); a). \tag{2.7}$$

The transformed function is obtained after substitution of (2.6) into the second part of(2.1)

$$u_a(x^{'}) = \varphi(g(x^{'},a), u_0(g(x^{'},a)); a) \tag{2.8}$$

We arrive at the identity

$$u_a(f(x,u_0(x),a)) = \varphi(x, u_0(x); a). \tag{2.9}$$

For applying Lie group to differential equations, one needs to know the transformation of derivatives. Let us study a transformation of the derivatives of the given function $u_0(x)$. This is called a prolongation of group (2.1) to the first derivatives.

For the sake of simplicity we study this idea in the case $n = 1, m = 1$.

A derivative of the transformed function $u_a(x^{'})$ is found by differentiating (2.8)

$$\frac{\partial u_a(x^{'})}{\partial x^{'}} = \frac{\partial \varphi}{\partial x} \cdot \frac{\partial g}{\partial x^{'}} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x} \cdot \frac{\partial g}{\partial x^{'}} =$$

$$= \frac{\partial g}{\partial x^{'}}(x^{'}, a)[\frac{\partial \varphi}{\partial x}(g(x^{'}, a), u_0(g(x^{'}, a); a) + \frac{\partial \varphi}{\partial u}(g(x^{'}, a), u_0(g(x^{'}, a); a) \cdot \frac{\partial u_0}{\partial x}(g(x^{'}, a))]$$

Here the derivative $\frac{\partial g}{\partial x^{'}}$ can be found by differentiating the left and the right sides of (2.7) with respect to $x^{'}$:

$$1 \quad = \quad (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x})(\frac{\partial g}{\partial x^{'}})$$

Because

$$\frac{\partial f}{\partial x}(g(x^{'}, 0), u_0(g(x^{'}, 0); 0) = 1$$
$$\frac{\partial f}{\partial u}(g(x^{'}, 0), u_0(g(x^{'}, 0); 0) = 1$$

then the value $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x} \neq 0$ in the neighborhood of $a = 0$.

Hence,

$$\frac{\partial g}{\partial x^{'}} = \frac{1}{(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x})} \tag{2.10}$$

Therefore

$$\frac{\partial u_a(x^{'})}{\partial x^{'}} = \frac{\frac{\partial \varphi(x, u_0(x); a)}{\partial x} + \frac{\partial \varphi(x, u_0(x); a)}{\partial u}\frac{\partial u_0(x)}{\partial x}}{\frac{\partial f(x, u_0(x); a)}{\partial x} + \frac{\partial f(x, u_0(x); a)}{\partial u}\frac{\partial u_0(x)}{\partial x}}$$

$$= F(x, u_0(x), p(x); a) \tag{2.11}$$

where $p_0 = \dfrac{\partial u_0}{\partial x}$ and we have the transformation for the derivative

$$p_a = Q(x, u_0, p; a)$$

We have obtained the first prolongation $\underset{1}{G}$ of the group G acting in the space (x, u_0, p). By using the same way, one can find the second prolongation $\underset{2}{G}$, etc. The infinitesimal generator of the $\underset{1}{G}$ is

$$\underset{1}{X} = \xi(x, u)\partial_x + \eta(x, u)\partial_u + \zeta(x, u, y^{'})\partial_{y^{'}}$$

where $\zeta(x,u,p)\partial_p = \frac{\partial Q(x,u,p;a)}{\partial a}|_{a=0}$ and where we have defined y' as the derivative of y. Direct calculations of $\frac{\partial F}{\partial a}$ give that $\zeta = D_x\eta - y' D_x\eta$ where $D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + ...$

Here we present the formulas for the case when the number of independent and dependent variables is greater than one. In this case, the prolonged operator is obtained by a similar way as explained above. For the derivatives we use the notations

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x_i}, u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j}, \ldots .$$

and $u_{(k)}$ is a collection of all derivatives of the k-th order $u_{(k)} = \{u^\alpha_{j_1 .. j_k}\}$. The first prolongation of the generator (2.3) is given by

$$\underset{1}{X} = X + \zeta_i^\alpha \partial_{u_i^\alpha} \ , (\alpha = 1, ..., m)$$

where

$$\begin{array}{c} \zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), (i = 1, ..., n) \\ D_i = \partial_{x_i} + \sum\limits_\alpha u_i^\alpha \partial_{u^\alpha} + \sum\limits_{\alpha,\beta} u_{i\beta}^\alpha \partial_{u_\beta^\alpha} + ... \end{array}$$

The second prolongation of the generator X is

$$\underset{2}{X} = \underset{1}{X} + \zeta_{i_1 i_2}^\alpha \partial_{u_{i_1 i_2}^\alpha}$$

where

$$\zeta_{i_1 i_2}^\alpha = D_{i_2}(\zeta_{i_1}^\alpha) - u_{j i_1}^\alpha D_{i_2}(\xi^j), (i_1, i_2 = 1, ..., n)$$

In a general case, a k-th prolongation of the generator X is

$$\underset{k}{X} = \underset{k-1}{X} + \zeta_{i_1 \ldots i_k}^\alpha \partial_{u_{i_1 \ldots i_k}^\alpha}$$

where

$$\zeta_{i_1 \ldots i_s}^\alpha = D_{i_s}(\zeta_{i_1 \ldots i_{s-1}}) - u_{j i_1 \ldots i_{s-1}}^\alpha D_{i_s}(\xi^j) \ , (i_1, i_2, ..., i_s = 1, ..., n.)$$

Now we link Lie groups and differential equations. Let us consider the k-th order system of partial differential equations.

$$F_\alpha(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) = 0, (\alpha = 1, 2, ..., s) \tag{2.12}$$

Definition 2 *A one parameter group (2.1) is called group admitted by partial differential equations (2.12) if*

$$\underset{(k)}{X} F_\alpha \left(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}\right)\Big|_{(F)} = 0. \tag{2.13}$$

or in a short form as follows:

$$\underset{(k)}{X} F_\alpha \left(x, u, p\right)|_{(F)} = 0$$

.

where the sign $\big|_{(F)}$ means that equations (2.13) are considered on the manifold described by equations (2.12). The process of obtaining equations (2.13) consists of the following steps.

The first step is to get the k-th prolongation of the generator X . The second step is acting of the k-th prolongation $\underset{k}{X}$ on the equations $F_\alpha = 0, (\alpha = 1, 2, ..., s)$. The next step is a transition onto the manifold described by equation (2.13). These equations are called determining equations. The solutions of these equations compose a Lie algebra of generators. The Lie group corresponding to this Lie algebra is called an admitted Lie group and the Lie algebra is called an admitted Lie algebra.

2.2 Lie Algebras

Let

$$X_1 = \xi_1^i(x)\partial_{x_i}$$
$$X_2 = \xi_2^i(x)\partial_{x_i}$$

be two infinitesimal operators. Their commutator (Lie bracket) is defined by

$$[X_1, X_2] = X_1X_2 - X_2X_1 \tag{2.14}$$

or

$$[X_1, X_2] = (X_1(\xi_2^i) - X_2(\xi_1^i))\partial_{x_i}$$

A vector space L of infinitesimal operators is called a *Lie algebra* L of operators if it satisfies the following properties:

$$[aX_1 + bX_2, X_3] = a\,[X_1, X_3] + b\,[X_2, X_3]$$

$$[X_1, aX_2 + bX_3] = a\,[X_1, X_2] + b\,[X_1, X_3]$$

$$[X_1, X_2] = -\,[X_2, X_1]$$

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0$$

Here X_1, X_2, X_3 are arbitrary elements in L; a and b are arbitrary real numbers.

A vector subspace $H \subseteq L$ is said to be a *subalgebra* of the Lie algebra L if it is closed under commutator, i.e. $[H, H] \subset H$.

Let L and K be two Lie algebras of operators. A linear mapping $f : L \to K$ of L onto K is called *an homomorphism* if it satisfies the relation

$$f([X_1, X_2]_L) = [f(X_1), f(X_2)]_K \tag{2.15}$$

for any $X_1, X_2 \in L$.

A one-to-one homomorphism is called an isomorphism.

Let $X_1, X_2, ..., X_r$ be a basis of vector space L_r. If L_r is the Lie algebra, then

$$[X_i, X_j] = c_{ij}^{\alpha}X_\alpha\ ; i, j = 1, 2, ...r.$$

The constants c_{ij}^{α} are called structure constants. If two finite-dimensional Lie algebras L and K are isomorphic, then there exist bases $\{X_1, ..., X_r\} \in L$ and $\{Y_1, ..., Y_r\} \in K$ so that both have the same structure constants. An isomorphism of L onto itself is called an *automorphism.*

Assume that $X_1, ..., X_r$ is a basis of a r-dimensional Lie algebra L_r. Any $X \in L$ is written as $X = x^{\alpha} X_{\alpha}$.

Let L_r^A be a Lie algebra spanned by the following operators:

$$E_i = c_{ij}^{\alpha} x^j \partial_{x^{\alpha}} \ ; i = 1, ..., r$$

The algebra L_r^A generates the group G^A of transformations of the space of $\{X_1, ..., X_n\}$. These transformations determine automorphisms of the Lie algebra L_r, known as inner automorphisms.

Two subalgebras are similar if there exists an inner automorphism which takes one subalgebra into the other. The similarity relation divides the set of all subalgebras of L_r into disjoint classes of similar subalgebras. In this partition, we take the classes of subalgebras of the same dimension s, and choose a representative for each of the classes. The set of pairwise nonconjugate subalgebras is known as an optimal system of s-dimensional subalgebras of the Lie algebra L_r.

2.3 Equivalence Groups

Let a system of differential equations

$$F_{\alpha}(x, u, p, \theta) = 0; (\alpha = 1, 2, ..., s) \tag{2.16}$$

be given where $\theta = \theta(u, x)$ are arbitrary elements of the system (2.16).

A nondegenerated transformation that changes the independent variables x, dependent variables u and arbitrary elements θ of system (2.16) to a system of equations of the same differential structure, is called an equivalence transformation.

The problem is how to find the transformations which preserve the equations without changing the differential structure.

Assume that the transformations

$$\begin{aligned} x^{'} &= f^x(x, u, \theta; a) \\ u^{'} &= f^u(x, u, \theta; a) \\ \theta^{'} &= f^{\theta}(x, u, \theta; a) \end{aligned} \tag{2.17}$$

compose a Lie group of such transformations. This group can be found by a similar way as an admitted Lie group.

Let

$$X^e = \xi^{x_i} \partial_{x_i} + \zeta^{u^{\alpha}} \partial_{u^{\alpha}} + \zeta^{\theta^j} \partial_{\theta^j}$$

be an infinitesimal generator of the Lie group (2.17) with the coefficients:

$$\begin{aligned} \xi^{x_i} &= \frac{\partial f^x(x,u,\theta;a)}{\partial a}\,|_{a=0} \\ \zeta^{u^\alpha} &= \frac{\partial f^u(x,u,\theta;a)}{\partial a}\,|_{a=0} \\ \zeta^{\theta^j} &= \frac{\partial f^\theta(x,u,\theta;a)}{\partial a}\,|_{a=0} \end{aligned}$$

and

$$\underset{1}{\overset{e}{X}} = X^e + \zeta^{u^\alpha_{x_i}}\partial_{u^\alpha_{x_i}} + \zeta^{\theta^j_{x_\alpha}}\partial_{\theta^j_{x_\alpha}} + \zeta^{\theta^j_{u^\beta}}\partial_{\theta^j_{u^\beta}} + ...$$

be a prolonged generator. Here the coefficients $\zeta^{u^\alpha_{x_i}}, \zeta^{\theta^j_{x_\alpha}}, \zeta^{\theta^j_{u^\beta}}, ...$ are expressed by the following :

$$\begin{aligned} \zeta^{u^\alpha_{x_i}} &= D^e_{x_i}\zeta^{u^\alpha} - u^\alpha_{x_\beta}D^e_{x_i}\xi^{x_\beta} \\ \zeta^{u^\alpha_{x_i x_j}} &= D^e_{x_i}\zeta^{u^\alpha_{x_i}} - u^\alpha_{x_i x_\beta}D^e_{x_j}\xi^{x_\beta} \\ \zeta^{\theta^j_{x_i}} &= \widetilde{D}^e_{x_i}\zeta^{\theta^j} - \theta^j_{x_\alpha}\widetilde{D}^e_{x_i}\xi^{x_\alpha} - \theta^j_{u^\beta}\widetilde{D}^e_{x_i}\zeta^{u^\beta} \\ \zeta^{\theta^j_{u^k}} &= \widetilde{D}^e_{u^k}\zeta^{\theta^j} - \theta^j_{x_\alpha}\widetilde{D}^e_{u^k}\xi^{x_\alpha} - \theta^j_{u^\beta}\widetilde{D}^e_{u^k}\zeta^{u^\beta} \end{aligned}$$

where

$$\begin{aligned} D^e_{x_i} &= \partial_{x_i} + u^\alpha_{x_i}\partial_{u^\alpha} + (\theta^j_{x_i} + \theta^j_{u^\alpha}u^\alpha_{x_i})\partial_{\theta^j} + ... \\ \widetilde{D}^e_{x_i} &= \partial_{x_i} + \theta^j_{x_i}\partial_{\theta^j} + ... \\ \widetilde{D}^e_{u^k} &= \partial_{u^k} + \theta^j_{u^k}\partial_{\theta^j} + ... \end{aligned}$$

The determining equations for the Lie group (2.17) are

$$\widetilde{X}^e F_\alpha\,|_{F=0} = 0; \quad \alpha = 1,2,...,s.$$

The solution of the determining equations gives us coefficients of the infinitesimal generator. By solving the Lie equations, one can obtain the equivalence group of transformations (2.17).

We use the main feature of the Lie group that any solution $u_0(x)$ of the systems (2.16) with the functions $\theta(x,u)$ is transformed by (2.17) into the solution $u = u_a(x')$ of the system (2.16), but with another (transformed) functions $\theta_a(x,u)$ which are defined in the following way. By solving the relations for (x,u)

$$x' = f^x(x,u,\theta(x,u);a), \quad u' = f^u(x,u,\theta(x,u);a)$$

we obtain

$$x = g^x(x',u';a), \quad u = g^u(x',u';a). \tag{2.18}$$

Then a transformed function is

$$\theta_a(x',u') = f^\theta(x,u,\theta(x,u);a),$$

where, instead of (x,u) we have to substitute their expressions (2.18). The transformed solution $T_a(u) = u_a(x)$ is obtained by solving the relations

$$x' = f^x(x, u_0(x), \theta(x, u_0(x)); a)$$

for $x = \psi^x(x'; a)$ and substituting these solutions into

$$u_a(x') = f^u(x, u_0(x), \theta_a(x, u_0(x)); a).$$

Lemma. The transformations $T_a(u)$ constructed in this way form a group.

Because the transformed function $u_a(x')$ is a solution of the same system with transformed arbitrary elements $\theta_a(x', u')$, then the equations

$$F_\alpha(x', u_a(x'), p_a'(x'), \theta_a(x', u_a(x'))) = 0, (\alpha = 1, 2, \dots, s)$$

are valid for arbitrary x'. But by virtue of one–to–one correspondence between x and x' we have

$$F_\alpha(f^x(z(x), a), f^u(z(x), a), f^p(z_p(x), a), f^\theta(z(x)))) = 0, (\alpha = 1, 2, \dots, s)$$

where
$z(x) = (x, u_0(x), \theta(x, u_0(x))), z_p(x) = (x, u_0(x), \theta(x, u_0(x)), p(x), \theta(x, u_0(x)), \dots).$

After differentiating this equations with respect to the group parameter a we get the usual algorithm of a finding an admissible group of continuous transformations (equivalence transformations) (2.17). The differences are only in the prolongation of infinitesimal generator X^e.

2.4 Invariant and Partially Invariant Solutions

Let G^r be a Lie group admitted by equations (2.12) . Assume that $X_1, ..., X_r$ is a basis of the Lie algebra L^r, which corresponds to the Lie group G^r.

Definition 3 *A function $\phi(x,u)$ is an invariant of the group G^r if it satisfies the conditions*

$$X_i \phi(x,u) = 0, (i = 1, 2, ..., r). \tag{2.19}$$

In order to find an invariant, one needs to solve the overdetermined system of linear equations (2.19). Note that this system is a complete system. A set of functionally independent invariants

$$J = \left(J^1(x,u), J^2(x,u), ..., J^{m+n-r}(x,u)\right)$$

such that any invariant ϕ can be expressed through this set

$$\phi = \Psi(J^1, J^2, ..., J^{m+n-r})$$

is called a universal invariant.

Note that if the rank of the Jacobi matrix $\dfrac{\partial(J^1,...,J^{m+n-r})}{\partial(u_1,...,u_m)}$ is equal to k , then without loss of generality, one can choose first k invariants $J^1,...,J^k$ dependent on x and u with the rank of Jacobi matrix k, $k = rank\dfrac{\partial(J^1,...,J^k)}{\partial(u_1,...,u_m)}$. The remaining invariants J^{k+1}, $J^{k+2},...,$ J^{m+n-r} only depend on the independent variables x.

In order to construct a representation of invariant or partially invariant solutions, one needs to separate the universal invariant into two parts. The group analysis method requires that the first part be a function of the second part.

For the invariant solutions $k = m$: the Jacobian of the first part with respect to the dependent variables $u_1,...,u_m$ is not equal to zero. Therefore, the invariant solution has the representation

$$J^i = \Psi^i(J^{m+1},...,J^{m+n-r}), (i = 1,2,...,m). \tag{2.20}$$

From the first m invariants $J^i, (i = 1,2,...,m)$ one can define the dependent variables as

$$u^i = \Phi^i(J^1,...,J^m,x), (i = 1,...,m) \tag{2.21}$$

These expressions of u^i after substituting (2.20) into them are called a *representation of the invariant solution.*

Let us consider the construction of a representation of partially invariant solutions.

Assume that rank $\dfrac{\partial(J^1,...,J^{m+n-r})}{\partial(u_1,...,u_m)} = k \le m$. Let $1 \le l \le k \le m$. For partially invariant solution group analysis requires that the first l invariants $J^1,...,J^l$ are functions of the remained invariants $J^{l+1},...,J^{m+n-r}$:

$$J^i = \Psi^i(J^{l+1},...,J^k,J^{k+1},...,J^{m+n-r}), (i = 1,2,...,l). \tag{2.22}$$

From the first l invariants $J^1,...,J^l$ one can find l functions u^i. Without loss of generality, we consider that

$$u_i = \phi^i(J^1,...,J^l,u_{l+1},...,u_m,x), (i = 1,...,l).$$

These expressions of u_i after substituting (2.22) into them are called a representation of partially invariant solution.

The next step in constructing an invariant or partially invariant solution consists of substituting the representation of the solutions into the original system of equations. The obtained system on functions Ψ^i is called a reduced system.

For invariant solutions, this system is a system of equations for the functions Ψ^i with $n - r$ independent variables.

For partially invariant solutions, the reduced system is an overdetermined system of m equations for $m - l$ functions $u_{l+1}, u_{l+2},...,u_m$. Compatibility conditions of this overdetermined system are equations on function Ψ^i. The analysis of partially invariant solutions requires a more difficult analysis of compatibility.

Chapter III

Viscous Fluid Dynamics Equations

3.1 Coordinateless Form of Viscous Fluid Dynamics Equations

We study unsteady viscous fluid dynamics equations. These equations govern a three-dimensional motion of a thermal conductive, Newtonian viscous flow

$$\frac{d\mathbf{v}}{dt} = \tau div(\mathbf{P}) \tag{3.1}$$

$$\frac{d\tau}{dt} - \tau div\mathbf{v} = 0 \tag{3.2}$$

$$\frac{dU}{dt} = \tau\mathbf{P} : \mathbf{D} + \tau div(\kappa\nabla T). \tag{3.3}$$

Here $\tau = 1/\rho$ is the specific volume, ρ is the density, $\mathbf{v}$ is the velocity, $\mathbf{P}$ is the stress tensor, $\mathbf{D} = \frac{1}{2}\left(\frac{\partial\mathbf{v}}{\partial\mathbf{x}} + (\frac{\partial\mathbf{v}}{\partial\mathbf{x}})^*\right)$ is the rate-of-strain tensor, U is internal energy, T is temperature, κ is the coefficient of heat conductivity, and $\mathbf{P} : \mathbf{D}$ is a contraction of tensors $\mathbf{P}$ and $\mathbf{D}$. The Stokes axioms for a viscous fluid give

$$\mathbf{P} = (-p + \lambda div\mathbf{v})\mathbf{I} + 2\mu\mathbf{D} \tag{3.4}$$

where p is pressure, λ and μ are the first and the second coefficients of viscosity, respectively. Let us exclude the tensor $\mathbf{P}$ from equations (3.1-3.3). After substituting the stress tensor $\mathbf{P}$ from (3.4), into $\mathbf{P}$:$\mathbf{D}$, we get

$$\begin{aligned} \mathbf{P} : \mathbf{D} &= ((-p + \lambda div\mathbf{v})\mathbf{I} + 2\mu\mathbf{D}) : \mathbf{D} \\ &= (-p + \lambda div\mathbf{v})\mathbf{I} : \mathbf{D} + 2\mu\mathbf{D} : \mathbf{D} \\ &= -p\mathbf{I} : \mathbf{D} + (\lambda div\mathbf{v})\mathbf{I} : \mathbf{D} + 2\mu\mathbf{D} : \mathbf{D} \end{aligned}$$

where $\mathbf{I} : \mathbf{D} = div\mathbf{v}$. Hence,

$$\mathbf{P} : \mathbf{D} = -\, p div\mathbf{v} + \lambda(div\mathbf{v})^2 + 2\mu\mathbf{D} : \mathbf{D} \tag{3.5}$$

and the energy equation becomes

$$\frac{dU}{dt} = -\tau p div\mathbf{v} + \tau\lambda(div\mathbf{v})^2 + 2\tau\mu\mathbf{D} : \mathbf{D} + \tau div(\kappa\nabla T) \tag{3.6}$$

For the divergence $div\mathbf{P}$ in equation (3.1), we have

$$div\mathbf{P} = div[(-p + \lambda div\mathbf{v})\mathbf{I} + 2\mu\mathbf{D}]$$

and

$$div\mathbf{P} = -\nabla p + \nabla(\lambda div\mathbf{v}) + div(2\mu\mathbf{D}) \tag{3.7}$$

Substituting equation (3.7) into equation (3.1), we get

$$\frac{d\mathbf{v}}{dt} = \tau(-\nabla p + \nabla(\lambda div\mathbf{v}) + div(2\mu\mathbf{D})). \tag{3.8}$$

A viscous fluid is a two parametric medium. As the main thermodynamic variables, we choose the pressure p and specific volume τ. The entropy η, the internal energy U and the temperature T are functions of the pressure and specific volume

$$\eta = \eta(p,\tau),\ U = U(p,\tau),\ T = T(p,\tau),$$

which are related by the main thermodynamic identity

$$Td\eta = dU + pd\tau. \tag{3.9}$$

The rate form of this equation is

$$T\frac{d\eta}{dt} = \frac{dU}{dt} + p\frac{d\tau}{dt}. \tag{3.10}$$

In fluid mechanics, it is that assumed $U_p \neq 0$. The thermal identity relates the functions $\eta(p,\tau)$, $U(p,\tau)$, $T(p,\tau)$ by the formulas

$$U_p = \frac{\tau}{B},\ U_\tau = \frac{A}{B} - p,\ \eta_p = \frac{\tau}{BT},\ \eta_\tau = \frac{A}{BT},$$

where

$$A(p,\tau) = \frac{\tau(p + U_\tau)}{U_p}, B(p,\tau) = \frac{\tau}{U_p} \tag{3.11}$$

or $A = B(p + U_\tau)$. Using the relations $U_{p\tau} = U_{\tau p}$ and $\eta_{p\tau} = \eta_{\tau p}$ we have

$$(\frac{\tau}{B})_\tau = (\frac{A}{B} - p)_p, (\frac{A}{BT})_p = (\frac{\tau}{BT})_\tau$$

or

$$B - \tau B_\tau = BA_p - AB_p - B^2 \tag{3.12}$$

and

$$T(BA_p - AB_p) = ABT_p + T(B - \tau B_\tau) - \tau BT_\tau \tag{3.13}$$

after substituting equation (3.12) into equation (3.13), we obtain

$$\begin{aligned} B(AT_p - \tau T_\tau) = TB^2 \\ \tau T_\tau = AT_p - TB. \end{aligned} \tag{3.14}$$

In the case of an ideal gas, $T = R^{-1}p\tau$ where R is the gas constant. Hence, the equation $\eta_{p\tau} = \eta_{\tau p}$, gives

$$(\frac{U_\tau + p}{R^{-1}p\tau})_p = (\frac{U_p}{R^{-1}p\tau})_\tau$$

or

$$\tau U_\tau - pU_p = 0.$$

The general solution of this equation is

$$U = g(\tau p)$$

where g is an arbitrary function of one independent variable.

Substituting $U_\tau = pg^{'}, U_p = \tau g^{'}$ into (3.11), we have

$$A = p(1 + \frac{1}{g^{'}}), B = \frac{1}{g^{'}}$$

or we can rewrite A, B as the following

$$\begin{aligned} A &= p(1 + B), \\ B &= B(\tau p). \end{aligned}$$

For a polytropic gas $U = \dfrac{\tau p}{\gamma - 1}, where$ γ is the polytropic exponent. Hence,

$$\begin{aligned} U_\tau &= \frac{p}{\gamma - 1}, \\ U_p &= \frac{\tau}{\gamma - 1} \end{aligned}$$

Substituting these expressions into A and B in (3.11), we obtain

$$\begin{aligned} A &= p\gamma, \\ B &= \gamma - 1. \end{aligned}$$

For the next study, let us rewrite the energy equation in a suitable form. Since $U = U(p, \tau)$, thus

$$\frac{dU}{dt} = U_p \frac{dp}{dt} + U_\tau \frac{d\tau}{dt} \tag{3.15}$$

After substituting $\frac{dU}{dt}$ from equation (3.15) into equation(3.3) and using equation (3.2), we can express the energy equation (3.3) as the following

$$U_p \frac{dp}{dt} + (\tau U_\tau + \tau p)(div\mathbf{v}) = \tau\lambda(div\mathbf{v})^2 + 2\tau\mu\mathbf{D} : \mathbf{D} + \tau div(\kappa\nabla T).$$

Because $U_p \neq 0$, then this equation can be rewritten

$$\frac{dp}{dt} + \frac{\tau(U_\tau + p)}{U_p}(div\mathbf{v}) = \frac{\tau}{U_p}[\lambda(div\mathbf{v})^2 + 2\mu\mathbf{D} : \mathbf{D} + div(\kappa\nabla T)].$$

or

$$\frac{dp}{dt} + A(p,\tau)(div\mathbf{v}) = B(p,\tau)[\lambda(div\mathbf{v})^2 + 2\mu\mathbf{D} : \mathbf{D} + div(\kappa\nabla T)]. \tag{3.16}$$

Because

$$\begin{aligned} div(\kappa\nabla T) &= (\nabla\kappa)(\nabla T) + \kappa div(\nabla T) \\ &= (\nabla\kappa)(\nabla T) + \kappa\triangle T), \end{aligned} \tag{3.17}$$

then

$$\frac{dp}{dt} + A(p,\tau)(div\mathbf{v}) = B(p,\tau)[\lambda(div\mathbf{v})^2 + 2\mu\mathbf{D} : \mathbf{D} + (\nabla\kappa)(\nabla T) + \kappa\triangle T) \tag{3.18}$$

For the term $\nabla(\lambda div\mathbf{v})$ in equation (3.8), we have

$$\nabla(\lambda div\mathbf{v}) = (div\mathbf{v})\nabla\lambda + \lambda\nabla(div\mathbf{v}). \tag{3.19}$$

By the definition of divergence of the tensor $2\mu\mathbf{D}$ there is

$$\mathbf{a}div(2\mu\mathbf{D}) = 2tr\frac{\partial}{\partial x}(\mu\mathbf{D}^* < \mathbf{a} >). \tag{3.20}$$

By using Leibnitz rule, we get

$$\begin{aligned} 2tr\frac{\partial}{\partial x}(\mu\mathbf{D}^* < \mathbf{a} >) &= 2\mu tr\frac{\partial}{\partial x}(\mathbf{D}^* < \mathbf{a} >) + 2\mathbf{D}^* < \mathbf{a} > \nabla\mu \\ &= (2\mu div\mathbf{D} + 2\mathbf{D} < \nabla\mu >)\mathbf{a} \\ &= \mathbf{a}(\mu div\frac{\partial\mathbf{v}}{\partial x} + \mu div(\frac{\partial\mathbf{v}}{\partial x})^* + 2\mathbf{D} < \nabla\mu >) \end{aligned} \tag{3.21}$$

Let us consider $div(\frac{\partial\mathbf{v}}{\partial x})$ and $div(\frac{\partial\mathbf{v}}{\partial x})^*$:

$$\begin{aligned} \mathbf{a}\cdot div(\frac{\partial\mathbf{v}}{\partial x})^* &= tr(\frac{\partial}{\partial x}(\frac{\partial v}{\partial x} < \mathbf{a} >)) = \frac{\partial}{\partial x_\alpha}(\frac{\partial v_\alpha}{\partial x_\beta}a_\beta) = a_\beta\frac{\partial^2 v_\alpha}{\partial x_\alpha\partial x_\beta} \\ &= a_\beta\frac{\partial}{\partial x_\beta}(\frac{\partial v_\alpha}{\partial x_\alpha}) = a_\beta(\frac{\partial}{\partial x_\beta}div\mathbf{v}) \\ &= a\cdot\nabla(div\mathbf{v}) \end{aligned}$$

and

$$\mathbf{a}\cdot div(\frac{\partial \mathbf{v}}{\partial x}) = tr(\frac{\partial}{\partial x}((\frac{\partial v}{\partial x})^* < \mathbf{a} >)) = tr(\frac{\partial}{\partial x}(\frac{\partial a_\alpha v_\alpha}{\partial x}))$$
$$= \sum_\beta \frac{\partial^2}{\partial x_\beta^2}(v_\alpha a_\alpha) = a_\alpha \sum_\beta \partial^2 v_\alpha \partial x_\beta^2$$
$$= \mathbf{a}\cdot\Delta\mathbf{v}$$

Hence, we can rewrite the equation (3.20) as

$$div(2\mu\mathbf{D}) = 2\mathbf{D} < \nabla\mu > +\mu\nabla(div\mathbf{v}) + \mu\Delta\mathbf{v} \tag{3.22}$$

Substituting equation (3.19) and equation (3.22) into equation (3.8), we obtain

$$\frac{d\mathbf{v}}{dt} + \tau\nabla p = \tau[(div\mathbf{v})\nabla\lambda + \lambda\nabla(div\mathbf{v}) + 2\mathbf{D} < \nabla\mu > +\mu\nabla(div\mathbf{v}) + \mu\Delta\mathbf{v}]$$
$$= \tau[(\tau+\mu)\nabla(div\mathbf{v}) + (div\mathbf{v})\nabla\lambda + 2\mathbf{D} < \nabla\mu > +\mu\Delta\mathbf{v}] \tag{3.23}$$

Therefore, the viscous gas dynamics equations which we study are

$$\frac{d\mathbf{v}}{dt} + \tau\nabla p = \tau[(\lambda+\mu)\nabla(div\mathbf{v}) + (div\mathbf{v})\nabla\lambda + 2\mathbf{D} < \nabla\mu > +\mu\Delta\mathbf{v}],$$
$$\frac{d\tau}{dt} - \tau div\mathbf{v} = 0, \tag{3.24}$$
$$\frac{dp}{dt} + A(p,\tau)(div\mathbf{v}) = B(p,\tau)(\lambda(div\mathbf{v})^2 + 2\mu\mathbf{D}:\mathbf{D} + (\nabla\kappa)(\nabla T) + \kappa\triangle T).$$

3.2 Thermodynamics of Gas and Fluid

The second law of thermodynamic requires that the total entropy production is not negative:

$$R = \frac{dS}{dt} + \int_{\partial\omega}(\frac{q}{T})\mathbf{n}d\sigma \geq o \tag{3.25}$$

where $S = \int_\omega \rho\eta d\omega$, q is heat flux given by the Fourier heat conduction law $q = -\kappa\nabla T$, and $\kappa \geq 0$, is the coefficient of heat conductivity.

We can rewrite equation (3.25) as the following

$$\frac{d}{dt}\int_\omega \rho\eta d\omega + \int_{\partial\omega}(\frac{q}{T})\mathbf{n}d\sigma \geq o.$$

By using the Gauss-Ostrogradskii theorem, we obtain

$$\int_\omega[\rho\frac{d\eta}{dt} + div(\frac{q}{T})]d\omega \geq 0$$

For continuous motions, one obtains

$$\rho\frac{d\eta}{dt} + div(\frac{q}{T}) \geq 0$$

or

$$T\frac{d\eta}{dt} + \tau div(q) - \frac{\tau}{T}(q \cdot \nabla T) \geq 0 \quad (3.26)$$

After substituting $T\frac{d\eta}{dt}$ found from, the rate form of the main thermodynamics identity (3.10), into equation (3.26), we have

$$\frac{dU}{dt} + p\frac{d\tau}{dt} + \tau div(q) - \frac{\tau}{T}(q \cdot \nabla T) \geq 0 \quad (3.27)$$

By using the energy equation (3.6), we obtain

$$-\tau p div\mathbf{v} + \tau\lambda(div\mathbf{v})^2 + 2\tau\mu\mathbf{D} : \mathbf{D} + \tau div(\kappa\nabla T) + p\frac{d\tau}{dt} + \tau div(q) - \frac{\tau}{T}(q \cdot \nabla T) \geq 0$$

Because

$$\frac{d\tau}{dt} = \tau div\mathbf{v}$$

and

$$q = -\kappa\nabla T$$

we get

$$\tau\lambda(div\mathbf{v})^2 + 2\tau\mu\mathbf{D} : \mathbf{D} + \frac{\tau}{T}(\kappa\nabla T \cdot \nabla T) \geq 0$$

or

$$\lambda(div\mathbf{v})^2 + 2\mu\mathbf{D} : \mathbf{D} \geq 0.$$

Assume

$$\Phi = \lambda(div\mathbf{v})^2 + 2\mu\mathbf{D} : \mathbf{D} \geq 0$$

Let us consider condition under which Φ is always nonnegative, that is $\Phi \geq 0$. In order to find these conditions, we first of all write values of the function Φ with reference to the principal axes of rate-of-strain tensor:

$$D = O \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} O^*,$$

where O is an orthogonal transformation. In matrix representation we have

$$\Phi = \lambda(J_1(D))^2 + 2\mu O_{ik}d_kO_{lk}O_{i\beta}d_\beta O_{l\beta} = \lambda(J_1(D))^2 + 2\mu\delta_{k\beta}d_k\delta_{k\beta}d_\beta$$

then

$$\lambda(d_1 + d_2 + d_3)^2 + 2\mu(d_1^2 + d_2^2 + d_3^2) \geq 0$$

or

$$\frac{1}{3}[(3\lambda + 2\mu)(d_1 + d_2 + d_3)^2 + 2\mu((d_1 - d_2)^2 + (d_2 - d_3)^2 + (d_3 - d_1)^2)] \geq 0$$

then we get

$$3\lambda + 2\mu \geq 0, \mu \geq 0.$$

3.3 The Navier-Stokes Equations

The Navier-Stokes equations are obtained from the viscous gas dynamics equations by setting τ and μ constant. In this case, the viscous gas dynamics equations consist of two parts. The first part is

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= -\tau(\nabla p + \mu\Delta\mathbf{v}) \\ div\mathbf{v} &= 0\end{aligned} \tag{3.28}$$

These equations are called the Navier-Stokes equations. The remaining equation is the energy equation, which can be solved afterwards. For the Cartesian coordinates, the Navier-Stokes equations can be written as the following equations

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) \tag{3.29}$$

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.30}$$

The dependent variables u, v, w, p are functions of the space variables x, y, z and time t.

3.3.1 Dimensionless Analysis of the Navier-Stokes Equations

It is useful to write the Navier -Stokes equations in dimensionless form. Let us consider the new variables $\mathbf{v}^*, p^*, \mathbf{x}^*, t^*$ by giving $\mathbf{v}^* = \frac{\mathbf{v}}{V}, p^* = \frac{p}{p_0}$,$\mathbf{x}^* = \frac{\mathbf{x}}{L}$, $t^* = \frac{t}{T_0}$, where V, p_0, L and T_o are velocity, pressure, length and time units, respectively.

We can rewrite these variables as $\mathbf{v} = V\mathbf{v}^*, p = p_0 p^*, \mathbf{x} = L\mathbf{x}^*, t = T_0 t^*$.

Differentiating these functions with respect to independent variables, we have

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{V}{L}\frac{\partial \mathbf{v}^*}{\partial \mathbf{x}^*}, \frac{\partial \mathbf{v}}{\partial t} = \frac{V}{T_0}\frac{\partial \mathbf{v}^*}{\partial t^*}, \frac{\partial p}{\partial \mathbf{x}} = \frac{p_0}{L}\frac{\partial p^*}{\partial \mathbf{x}^*} \tag{3.31}$$

and similarly

$$\frac{\partial^2 \mathbf{v}}{\partial \mathbf{x}^2} = \frac{V}{L^2}\frac{\partial^2 \mathbf{v}^*}{\partial (\mathbf{x}^*)^2} \tag{3.32}$$

Substituting equations (3.31) and (3.32) into equation (3.29) and equation (3.30), we obtain

$$(\frac{\rho V}{T}_0)\frac{\partial u^*}{\partial t^*} + (\frac{\rho V^2}{L})(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}) = -\frac{p_0}{L}\frac{\partial p^*}{\partial x^*} + \frac{\mu V}{L^2}(\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2} + \frac{\partial^2 u^*}{\partial (z^*)^2})$$
$$(\frac{\rho V}{T}_0)\frac{\partial v^*}{\partial t^*} + (\frac{\rho V^2}{L})(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}) = -\frac{p_0}{L}\frac{\partial p^*}{\partial y^*} + \frac{\mu V}{L^2}(\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 v^*}{\partial (y^*)^2} + \frac{\partial^2 v^*}{\partial (z^*)^2})$$
$$(\frac{\rho V}{T}_0)\frac{\partial z^*}{\partial t^*} + (\frac{\rho V^2}{L})(u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}) = -\frac{p_0}{L}\frac{\partial p^*}{\partial z^*} + \frac{\mu V}{L^2}(\frac{\partial^2 w^*}{\partial (x^*)^2} + \frac{\partial^2 w^*}{\partial (y^*)^2} + \frac{\partial^2 w^*}{\partial (z^*)^2}) \tag{3.33}$$

$$\frac{V}{L}\frac{\partial u^*}{\partial x^*} + \frac{V}{L}\frac{\partial v^*}{\partial y^*} + \frac{V}{L}\frac{\partial w^*}{\partial z^*} = 0 \tag{3.34}$$

Multiplying equation (3.33) by $\frac{L}{\rho V^2}$ and multiplying equation (3.34) by $\frac{L}{V}$, we get

$$(\frac{L}{VT_0})\frac{\partial u^*}{\partial t^*} + (u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}) = -\frac{p_0}{L}\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho VL}(\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2} + \frac{\partial^2 u^*}{\partial (z^*)^2})$$
$$(\frac{L}{VT_0})\frac{\partial v^*}{\partial t^*} + (u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}) = -\frac{p_0}{\rho V^2}\frac{\partial p^*}{\partial y^*} + \frac{\mu}{\rho VL}(\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 v^*}{\partial (y^*)^2} + \frac{\partial^2 v^*}{\partial (z^*)^2})$$
$$(\frac{L}{VT_0})\frac{\partial z^*}{\partial t^*} + (u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}) = -\frac{p_0}{\rho V^2}\frac{\partial p^*}{\partial z^*} + \frac{\mu}{\rho VL}(\frac{\partial^2 w^*}{\partial (x^*)^2} + \frac{\partial^2 w^*}{\partial (y^*)^2} + \frac{\partial^2 w^*}{\partial (z^*)^2}) \tag{3.35}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0. \tag{3.36}$$

The dimensionless constants of the Navier-Stokes equations $St = \frac{L}{VT}, Eu = \frac{p_0}{\rho V^2}, Re = \frac{\rho VL}{\mu}$ are called the Strouhal Number, Euler Number and Reynolds Number, respectively. So by choosing the units L, V, T_o, p_0 such that

$$\frac{L}{VT} = 1, \frac{p_0}{\rho V^2} = 1 \tag{3.37}$$

one obtains

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*} = \frac{\partial p^*}{\partial x^*} + \frac{1}{Re}(\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2} + \frac{\partial^2 u^*}{\partial (z^*)^2})$$

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*} = \frac{\partial p^*}{\partial y^*} + \frac{1}{Re}(\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 v^*}{\partial (y^*)^2} + \frac{\partial^2 v^*}{\partial (z^*)^2})$$

$$\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*} = \frac{\partial p^*}{\partial z^*} + \frac{1}{Re}(\frac{\partial^2 w^*}{\partial (x^*)^2} + \frac{\partial^2 w^*}{\partial (y^*)^2} + \frac{\partial^2 w^*}{\partial (z^*)^2})$$

After omitting (*), we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{Re}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\frac{1}{Re}(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{Re}(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

For the sake of simplicity of the mathematical study in addition to (3.37), one can choose units such that $Re = 1$. Then the system of equations become

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}) \tag{3.38}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}) \tag{3.39}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}) \tag{3.40}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{3.41}$$

From equations (3.38) to (3.41), we can write the system in the short form as follows:

$$\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla p+\Delta\mathbf{v}$$

$$div\mathbf{v}=0$$

where ∇ and Δ are gradient and Laplacian with respect to the space variables $\mathbf{x}=(x,y,z)$, respectively.
Alternatively, we can write in the form

$$\frac{d\mathbf{v}}{dt}+\nabla p=\Delta\mathbf{v}, \tag{3.42}$$

$$div\mathbf{v}=0 \tag{3.43}$$

where $\frac{d\mathbf{v}}{dt}=\mathbf{v}_t+v^i\frac{\partial\mathbf{v}}{\partial x^i}$

3.4 Spherical Coordinates

Here all coordinateless operators of the viscous gas dynamics equations are given in spherical coordinates.

Let $\Omega \subset R^3(\mathbf{x})$ be open set. A one-to-one continuous differential mapping $K : \Omega \to R^3$ is called a coordinate system. We define the mapping by the formulas

$$\mathbf{x} \to K(\mathbf{x}) = (K^1(\mathbf{x}), K^2(\mathbf{x}), K^3(\mathbf{x}))$$

Values of the functions $K^i(\mathbf{x})$ are called curvilinear coordinates of the point $\mathbf{x}$. For the spherical coordinates :

$$K^1 = r = \sqrt{x^2 + y^2 + z^2};\ K^2 = \theta = arctg\,\tfrac{\sqrt{x^2+y^2}}{z};$$
$$K^3 = \varphi = arctg\,\tfrac{y}{x}.$$

The inverse mapping K^{-1} is

$$x = r\ \sin\theta\cos\varphi;\ y = r\ \sin\theta\sin\varphi;\ z = r\ \cos\theta;$$
$$(0 \le \varphi \le 2\pi),\ (0 \le \theta \le \pi)$$

The vectors $e_i = \frac{\partial\mathbf{x}}{\partial K^i}, (i = 1, 2, 3)$ compose a basis and they are called basis vectors.

The vectors $e^i = \frac{\partial K^i}{\partial\mathbf{x}} = \nabla K^i = (\frac{\partial K^i}{\partial x}, \frac{\partial K^i}{\partial y}, \frac{\partial K^i}{\partial z}), (i = 1, 2, 3)$ are called cobasis vectors.

For the spherical coordinates system, we get

$$\begin{aligned}
e_1 &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = e^1 \\
e_2 &= r(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), e^2 = \frac{1}{r^2}e_2 \\
e_3 &= r\sin\theta(-\sin\varphi, \cos\varphi, 0) = e^3 = \frac{1}{r^2\sin^2\theta}e_3
\end{aligned}$$

Definition 4 *The tensor $< a, b >$ is called a fundamental tensor. The coordinates of the fundamental tensor are*

$$(g_{ij}) = e_i \cdot e_j, \quad (g^{ij}) = e^i \cdot e^j$$

For the spherical coordinate system, the coordinates of the fundamental tensor are

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix};\ (g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} \qquad (3.44)$$

where $|g| = \det(g_{ij})$ then $|g| = r^4\sin^2\theta$.

Definition 5 *The values*

$$\Gamma^l_{ij} = \frac{1}{2}g^{ls}\left(\frac{\partial g_{is}}{\partial K^j} + \frac{\partial g_{js}}{\partial K^i} - \frac{\partial g_{ij}}{\partial K^s}\right) \qquad (3.45)$$

are called the Christoffel symbols. For the spherical coordinate system of coordinates, the Christoffel symbols

$$\begin{array}{c} \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\ \ \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}\ \ \Gamma^1_{22} = -r, \\ \Gamma^3_{23} = \Gamma^3_{32} = \cot\,\theta,\ \ \Gamma^1_{33} = -r\,\sin^2\,\theta\ \ \Gamma^2_{33} = -\sin\,\theta\,\cos\,\theta \end{array} \tag{3.46}$$

and the others Christoffel symbols vanish.

Definition 6 *The numerical values of the tensor components divided by the length of the corresponding basis or cobasis vectors are called physical components of the tensor.*

Let $\mathbf{v} = (v_r, v_\theta, v_\varphi)$ be the physical components of the vector $\mathbf{v}$. The tensor components of the vector $\mathbf{v}$ are $(v^1, v^2, v^3) = (v_r, \frac{v_\theta}{r}, \frac{v_\varphi}{r\sin\theta})$, $(v_1, v_2, v_3) = (v_r, rv_\theta, r\sin\theta v_\varphi)$.

Here we use the gradient of a function F as:

$$\begin{aligned} (\nabla F)^1 &= \frac{\partial F}{\partial r}, \\ (\nabla F)^2 &= \frac{1}{r^2}\frac{\partial F}{\partial \theta}, \\ (\nabla F)^3 &= \frac{1}{(r sin\theta)^2}\frac{\partial F}{\partial \varphi} \end{aligned} \tag{3.47}$$

The divergence of vector the $\mathbf{v}$ is expressed in the form

$$div\,\mathbf{v} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial K^i}\left(\sqrt{|g|}v^i\right)$$

For the spherical coordinates system

$$\begin{aligned} div\mathbf{v} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(r^2\sin\theta v_r) + \frac{\partial}{\partial\theta}(r^2\sin\theta)\frac{v_\theta}{r} + \frac{\partial}{\partial\varphi}(r^2\sin\theta\frac{v_\varphi}{r\sin\theta}\right) \\ &= \frac{1}{r^2}\frac{\partial(r^2v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} = 0 \end{aligned} \tag{3.48}$$

The contravariant components of the Laplace operator of a vector $\mathbf{v}$ are defined by the formulas

$$\begin{aligned} (\triangle\mathbf{v})^l &= g^{ij}v^l{}_{,ij} \\ &= \triangle(v^l) + 2g^{ij}\,\Gamma^l_{is}\frac{\partial v^s}{\partial K^j} + \\ & \quad g^{ij}\left(\frac{\partial\Gamma^l_{ip}}{\partial K^j} + \Gamma^s_{ip}\Gamma^l_{js} - \Gamma^s_{ij}\Gamma^l_{ps}\right)v^p, (l = 1, 2, 3) \end{aligned}$$

Let $l = 1$, then

$$(\triangle \mathbf{v})^1 = \triangle(v_r) - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_r}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial v_\varphi}{\partial \varphi} - \frac{2\cot\theta v_\theta}{r^2}$$

and the calculations show that for the spherical coordinates

$$\begin{aligned}
(\triangle \mathbf{v})^2 &= \triangle(\frac{v_\theta}{r}) + \frac{2}{r^2}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^3}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^3\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_\theta}{r^3\sin^2\theta} \\
(\triangle \mathbf{v})^3 &= \triangle(\frac{v_\varphi}{r\sin\theta}) + \frac{2}{r\sin\theta}\frac{\partial}{\partial r}(\frac{v_\varphi}{r}) + \frac{2\cot\theta}{r^3}\frac{\partial}{\partial \theta}(\frac{v_\varphi}{\sin\theta}) + \\
&\quad \frac{2}{r^3\sin^2\theta}\frac{\partial v_r}{\partial \varphi} + \frac{2\cot\theta}{r^3\sin^2\theta}\frac{\partial v_\theta}{\partial \varphi}
\end{aligned}$$

Here we use the Laplace operator of a scalar function F

$$\triangle F = div\ (\nabla F)$$

In terms of covariant components, it is

$$\triangle F = g^{ij}F_{,ij} = \left(g^{is}\frac{\partial F}{\partial K^s}\right)_{,i} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial K^i}\left(\sqrt{|g|}g^{is}\frac{\partial F}{\partial K^s}\right). \tag{3.49}$$

For the spherical coordinates system

$$\triangle F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial F}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial F}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial \varphi^2}.$$

Therefore,

$$\begin{aligned}
\triangle(v_r) &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial v_r}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial v_r}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \varphi^2} \\
\triangle(\frac{v_\theta}{r}) &= \frac{1}{r}\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^3}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^3}\frac{\cos\theta}{\sin\theta}\frac{\partial v_\theta}{\partial \theta}) + \frac{1}{r^3\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \varphi^2} \\
\triangle(\frac{v_\varphi}{r\sin\theta}) &= \frac{1}{r^2\sin\theta}(r\frac{\partial^2 v_\varphi}{\partial r^2}) + \frac{1}{r^3\sin\theta}(v_\varphi\csc^2\theta - \cot\theta\frac{\partial v_\varphi}{\partial \theta} + \frac{\partial^2 v_\varphi}{\partial \theta^2}) + \\
&\quad \frac{1}{r^3\sin^3\theta}\frac{\partial^2 v_\varphi}{\partial \varphi^2}
\end{aligned}$$

The coordinates of the acceleration of vector $\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \overline{x}}\langle \mathbf{v}\rangle$ are defined by the formulas:

$$(\frac{d\mathbf{v}}{dt})^i = \frac{\partial v^i}{\partial t} + v^s v^i_{.,s} = \frac{\partial v^i}{\partial t} + v^s\frac{\partial v^i}{\partial K^s} + \Gamma^i_{js}v^j v^s, (i, j = 1, 2, 3),$$

which in spherical coordinates system are

$$(\frac{d\mathbf{v}}{dt})^1 = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial v_\varphi} - \frac{1}{r}(v_\theta^2 + v_\varphi^2) \tag{3.50}$$

$$(\frac{d\mathbf{v}}{dt})^2 = \frac{1}{r}(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial \varphi}) + \frac{1}{r^2}(v_r v_\theta - \cot\theta v_\varphi^2) \quad (3.51)$$

$$(\frac{d\mathbf{v}}{dt})^3 = \frac{1}{r\sin\theta}(\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}) + \frac{1}{r^2\sin\theta}(v_r v_\varphi + v_\theta v_\varphi \cot\theta) \quad (3.52)$$

For example, the Navier-Stokes equations in the spherical coordinates are

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial v_\varphi} - \frac{1}{r}(v_\theta^2 + v_\varphi^2) + \frac{\partial p}{\partial r} \\
&\quad = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial v_r}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial v_r}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\varphi^2} \\
&\qquad - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2v_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{2\cot\theta v_\theta}{r^2}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{r}(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial \varphi}) + \frac{1}{r^2}(v_r v_\theta - \cot\theta v_\varphi^2) + \frac{\partial p}{r\partial\theta} \\
&\quad = \frac{1}{r}\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^3}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r^3}\frac{\cos\theta}{\sin\theta}\frac{\partial v_\theta}{\partial\theta}) \\
&\qquad + \frac{1}{r^3\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\varphi^2} + \frac{2}{r^2}\frac{\partial v_\theta}{\partial r} \\
&\qquad + \frac{2}{r^3}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^3\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\theta}{r^3\sin^2\theta}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{r\sin\theta}(\frac{\partial v_\varphi}{\partial t} + v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}) + \frac{1}{r^2\sin\theta}(v_r v_\varphi + v_\theta v_\varphi\cot\theta) + \frac{\partial p}{r\sin\theta\partial\varphi} \\
&\quad = \frac{1}{r\sin\theta}\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{v_\varphi}{r^3\sin^3\theta} - \frac{\cos\theta}{r^3\sin^2\theta}\frac{\partial v_\varphi}{\partial\theta} + \frac{1}{r^3\sin\theta}\frac{\partial^2 v_\varphi}{\partial\theta^2} \\
&\qquad + \frac{1}{r^3\sin^3\theta}\frac{\partial^2 v_\varphi}{\partial\varphi^2} - \frac{2v_\varphi}{r^3\sin\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_\varphi}{\partial r} - \frac{2\cos^2\theta}{r^3\sin^3\theta}v_\varphi \\
&\qquad + \frac{2\cos\theta}{r^3\sin^2\theta}\frac{\partial v_\varphi}{\partial\theta} + \frac{2}{r^3\sin^2\theta}\frac{\partial v_r}{\partial\varphi} + \frac{2\cot\theta}{r^3\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi}
\end{aligned}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} = 0$$

The viscous gas dynamics equations in spherical coordinates are very cumbersome. For their expressions, we use the REDUCE-program, which is described in the next section.

3.5 Application of Computer Algebra for Constructing the Viscous Gas Dynamics Equations in Spherical Coordinates

A study of compatibility conditions and rewriting of the gas dynamics equations in spherical coordinates requires cumbersome symbolic calculations.

The advent of algebraic computing and the use of computer languages for symbolic computations such as REDUCE, MAPLE, MATHEMATICA etc., have made possible the use of computers for our goals. In this study, we use the REDUCE-program (cf. Hearn (1999)).

The main goal of our research is to study partially invariant solutions with respect to all rotations in three-dimensional space. For rotations, it is convenient to work in the spherical coordinates.

The procedure of an application of computor algebra can be described as follows:

1. We write the Navier-Stokes equations and the viscous gas dynamics equations in the spherical coordinates. Cartesian coordinates of point $X(x, y, z)$, velocity vector $\mathbf{v}(u, v, w)$ are related to the spherical coordinates by the formulas

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\theta \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} U &= u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta, \\ V &= u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta, \\ W &= -u\sin\varphi + v\cos.\varphi \end{aligned} \tag{3.54}$$

From these equations, one can find u, v, w :

$$\begin{aligned} u &= U\sin\theta\cos\varphi + V\cos\theta\cos\varphi - W\sin\varphi, \\ v &= U\sin\theta\sin\varphi + V\cos\theta\sin\varphi + W\cos\varphi, \\ w &= -U\cos\theta + v\sin\theta \end{aligned} \tag{3.55}$$

Here $U = v_r, V = v_\theta, W = v_\varphi$ are physical coordinates of the velocity vector. Dependent variables are v_r, v_θ, v_φ and p which can be rewritten in computer program as vr, vte, vfi and p respectively. Independent variables are r, θ, φ and t, in REDUCE-program we use r, te, fi and t. The full REDUCE-program for obtaining the Navier-Stokes equations, the viscous gas dynamics equations are enclosed in Appendix A.

2. We write the group of rotations in the spherical coordinates. This group corresponds to the algebra, whose generators in Cartesian coordinates are

$$\begin{aligned} X_7 &= y\partial_z - z\partial_y + v\partial_w - w\partial_v \\ X_8 &= x\partial_z - z\partial_x + u\partial_w - w\partial_u \\ X_9 &= x\partial_y - y\partial_x + u\partial_v - v\partial_u \end{aligned}$$

In order to transform these generators into the spherical coordinates, we have to make use of an arbitary function f which depends on $x, y, z, u, v, w, U, V, W, r, \theta, \varphi$ as the initial step in using the REDUCE-program. Then we define variables u, v, w depending on x, y, z and use M, L, K for the variables U, V, W which depend on r, θ, φ.

The result of these calculations is the representation of the generators X_7, X_8, X_9 in spherical coordinates

$$X_7 = -\sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi + \cos\varphi(\sin\theta)^{-1}(V\partial_W - W\partial_V) \tag{3.56}$$

$$X_8 = -\cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi + \sin\varphi(\sin\theta)^{-1}(V\partial_W - W\partial_V) \tag{3.57}$$

$$X_9 = \partial_\varphi \tag{3.58}$$

We further simplify operators (3.56)–(3.57) by introducing cylindrical coordinates (H, ω) into the two-dimensional space of vectors (V, W)

$$V = H\cos\omega, \quad W = H\sin\omega \tag{3.59}$$

or

$$\omega = \omega(V, W), \quad H = H(V, W) \tag{3.60}$$

In this case

$$\begin{aligned} \partial_v &= \cos\omega\partial_H - \frac{\sin\omega}{H}\partial_\omega \\ \partial_w &= \sin\omega\partial_H + \frac{\cos\omega}{H}\partial_\omega \end{aligned}$$

and the operator

$$V\partial_W - W\partial_V = \partial\omega$$

Hence, the generators (3.56)–(3.58) in the spherical coordinates are rewritten as

$$X_7 = -\sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi + \cos\varphi(\sin\theta)^{-1}\partial_\omega \tag{3.61}$$

$$X_8 = -\cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi + \sin\varphi(\sin\theta)^{-1}\partial_\omega \tag{3.62}$$

$$X_9 = \partial_\varphi \tag{3.63}$$

The full REDUCE-program for the generators of the group of rotations in spherical coordinates is in Appendix B.

Chapter IV

Analysis of Compatibility

4.1 Representation of Partially Invariant Solution

In this section we construct a representation of partially invariant solution of the viscous gas dynamics equations with respect to the Lie group with the generators (3.61, 3.62, 3.63). The first step in finding the representation is a construction of the universal invariant, which is a set of functionally independent solutions of the equations

$$X_7 F = -\sin\varphi F_\theta - \cos\varphi \cot\theta F_\varphi + \cos\varphi (\sin\theta)^{-1} F_\omega \tag{4.1}$$

$$X_8 F = -\cos\varphi F_\theta - \sin\varphi \cot\theta F_\varphi + \sin\varphi (\sin\theta)^{-1} F_\omega \tag{4.2}$$

$$X_9 F = F_\varphi \tag{4.3}$$

Here we use the space of the variables $t, r, \theta, \varphi, U, H, \omega, \tau, \rho$.

System (4.1)–(4.3) is a overdetermined homogeneous linear system of the first order equations. For solving these equations, one needs to use Poisson brackets. Because the generators X_7, X_8, X_9 compose a Lie algebra, then this system is a complete system. For a complete systems the method of their solving consists of sequential solving of the equations of the system. For example, from the equation $X_9 F = 0$, we have

$$F = F(t, r, \theta, U, H, \omega, \tau, \rho)$$

By taking linear combinations of the equations $X_7 F = 0$ and $X_8 F = 0$, these equations can be rewritten as $F_\theta = 0, F_\omega = 0$. Therefore, the universal invariant is $J = J(t, r, U, H, \tau, \rho)$. The rank of the Jacobi matrix of the universal invariant with respect to the dependent functions is

$$rank\ (\frac{\partial(t, r, U, H, \rho, \tau)}{\partial(U, H, \rho, \omega, \tau)}) = 4$$

Hence, according to Ovsiannikov, (1978), there are no nonsingular invariant solutions, only partially invariant solutions are possible. A minimal possible defect of the partially invariant solution is equal to one. In this case, a representation of the partially invariant solution is

$$U = U(t,r),\ H = H(t,r),\ p = p(t,r),\ \tau = \tau(t,r), \omega = \omega(t,r,\theta,\varphi) \qquad (4.4)$$

The function $\omega(t,r,\theta,\varphi)$ is "superfluous": it depends on all independent variables. If $H = 0$, then the tangent component of the velocity vector is equal to zero. This corresponds to the spherically symmetric flows. In our first study, we assume $H \neq 0$.

Theorem 2 *The class of solutions that is partially invariant with respect to all rotations is confined to spherically symmetric solutions.*

A description of the REDUCE-program of study compatibility conditions is given in the next section. The result of the calculations is the following:

4.2 Analysis of Compatibility of Partially Invariant Solution ($H \neq 0$)

For the sake of simplicity, we present here the analysis of compatibility of partially invariant solution for the Navier-Stokes equations, i.e., when τ and μ are constants.

The analysis of compatibility for the viscous gas dynamics equations is similar, but it needs more cumbersome symbolic calculations. After substituting the representation of the partially invariant solution (4.4) into the Navier-Stokes equations, [1] some combinations of the second and the third equations the initial system can be split on two subsystems: the invariant system

$$D_0 U + p_r - [r^{-1}H^2 + (U_{rr} + 4r^{-1}U_r + 2r^{-2}U)] = 0 \qquad (4.5)$$

with the operator $D_0 = \partial_t + U\partial_r$ and the supplementary system

$$\begin{aligned} D_0(rH) - (rH)_{rr} + (r\sin^2\theta)^{-1}H + rH(\omega_r^2 + r^{-2}\omega_\theta^2 + \\ (r\sin\theta)^{-2}\omega_\varphi^2 + 2(r^2\sin\theta)^{-1}\cot\theta\omega_\varphi) = 0 \end{aligned} \qquad (4.6)$$

$$\begin{aligned} D_0\omega + (r\sin\theta)^{-1}H(\sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi + \cos\theta\sin\omega) - \\ \omega_{rr} - 2(rH)^{-1}(rH)_r\omega_r - r^{-2}\omega_{\theta\theta} - r^{-2}\cot\theta\omega_\theta - (r\sin\theta)^{-2}\omega_{\varphi\varphi} = 0 \end{aligned} \qquad (4.7)$$

$$\sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi - \cos\theta\cos\omega - \sin\theta(rH)^{-1}(r^2U)_r = 0 \qquad (4.8)$$

For the analysis of compatibility of system (4.5) and (4.6) it is convenient to use an implicit representation for the function $\omega = \omega(t,r,\theta,\varphi)$:

$$F(\omega,t,r,\theta,\varphi) = 0, \quad (F_\omega \neq 0). \qquad (4.9)$$

[1] Here we use the dimensionless representation of the Navier-Stokes equations.

In this case, the derivatives are

$$\begin{aligned} F_r &= F_\omega \omega_r + F_r \\ F_\theta &= F_\omega \omega_\theta + F_\theta \\ F_{rr} &= F_\omega \omega_{rr} + \omega_r^2 F_{\omega\omega} + F_{r\omega}\omega_r \\ F_{\theta\theta} &= F_\omega \omega_{\theta\theta} + \omega_\theta^2 F_{\omega\omega} + F_{\theta\omega}\omega_\theta \end{aligned}$$

All derivatives of the function $\omega(t, r, \theta, \varphi)$ can be calculated through the derivatives of the function $F(\omega, t, r, \theta, \varphi)$:

$$\begin{gathered} \omega_t = -F_t/F_\omega,\ \omega_r = -F_r/F_\omega,\ \omega_\theta = -F_\theta/F_\omega,\ \omega_\varphi = -F_\varphi/F_\omega \\ \omega_{rr} = -\tfrac{1}{F_\omega}(F_{\omega\omega}\omega_r^2 + 2F_{\omega r}\omega_r + F_{rr}) \\ \omega_{\theta\theta} = -\tfrac{1}{F_\omega}(F_{\omega\omega}\omega_\theta^2 + 2F_{\omega\theta}\omega_\theta + F_{\theta\theta}) \\ \omega_{\varphi\varphi} = -\tfrac{1}{F_\omega}(F_{\omega\omega}\omega_\varphi^2 + 2F_{\omega\varphi}\omega_\varphi + F_{\varphi\varphi}) \end{gathered} \tag{4.10}$$

Then equation (4.8) becomes

$$\sin\theta \sin\omega F_\theta - \cos\omega F_\varphi + F_\omega(\cos\theta\cos\omega + k\sin\theta) = 0$$

where the function $k = (rH)^{-1}(r^2U)_r$ only depends on t and r. Note that for the viscous gas dynamics equations, there is the same equation with the function $k(t, r) = (Hr\tau)^{-1}(-rD_0\tau + \tau(r^2U)_r$. The general solution of the last equation is

$$F = \Phi\left(\varphi + arctan(\frac{\sin\omega}{k\sin\theta + \cos\theta\cos\omega}), \sin\theta\cos\omega - k\cos\theta, t, r\right).$$

Here the function $\Phi = \Phi(y_1, y_2, t, r)$ is an arbitrary function of the arguments t, r and

$$y_1 = \varphi + arctan(\frac{\sin\omega}{k\sin\theta + \cos\theta\cos\omega}),\ y_2 = \sin\theta\cos\omega - k\cos\theta.$$

All further intermediate calculations in studying the compatibility of the overdetermined system (4.5)-(4.8) were made on computer in the system REDUCE (Hearn (1999)). Here we give the way of computing and the final results.

Note that the Jacobian $\dfrac{\partial(y_1, y_2, \theta, t, r)}{\partial(\omega, \theta, \varphi, t, r)} \neq 0$; therefore, one can choose (y_1, y_2, θ, t, r) as the new independent variables instead of $\omega, \theta, \varphi, t, r$. All derivatives of the function $F(t, r, \theta, \varphi)$ can be written through the derivatives of the function $\Phi(y_1, y_2, t, r)$. After that, the equation (4.7) takes the form

$$\sin\omega G_1(y_1, y_2, t, r, \theta) + G_2(y_1, y_2, t, r, \theta) = 0 \tag{4.11}$$

where the functions $G_1(y_1, y_2, t, r, \theta)$ and $G_2(y_1, y_2, t, r, \theta)$ do not include ω and its derivatives. In the last equation, $\sin\omega$ can be excluded by using the trigonometry identity:

$$G_1^2(1 - (y_2 + k\cos\theta)^2) - G_2^2(1 - \cos^2\theta) = 0 \tag{4.12}$$

where the equality $\cos\omega = \sin^{-1}\theta(y_2 + k\cos\theta)$ is found from the representation of y_2.

Further calculations show that the last equation depends on θ as the polynomial of the degree 8 with respect to $\cos\theta$:

$$P_8 = \sum_{k=0}^{8} a_k \cos^k\theta = 0. \tag{4.13}$$

The coefficients a_k, $(k = 0, 1, \dots, 8)$ only depend on y_1, y_2, t, r and do not depend on θ. This allows splitting the equation with respect to $\cos\theta$: $a_k = 0$, $(k = 1, 2, \dots, 8)$.

The equality $a_8 = 0$ gives

$$D_0 h = h_{rr} + h(k^2+1)^{-1}h_r, \tag{4.14}$$

where $h = rH$. Substituting h_t found from (4.14) into $a_6 = 0$, we obtain

$$k_r\left((k^2+1)\Phi_r + kk_r y_2 \Phi_{y_2}\right) = 0. \tag{4.15}$$

If $(k^2+1)\Phi_r + kk_r y_2\Phi_{y_2} = 0$, then the equation $a_4 = 0$ gives the equation

$$y_2^2 - (k^2+1) = 0$$

or

$$(\sin\theta\cos\omega - k\cos\theta)^2 = k^2 + 1.$$

Note that substituting the representation of the function $\omega(t, r, \theta, \varphi)$ found from this equation into (4.5) to (4.8) and splitting them with respect to $\cos\theta$ gives the expression $H = 0$ that contradicts the assumption about H.

For the second case in (4.15), when $k_r = 0$ we will obtain a contradiction with the help of the equation of (4.6). In reality, the same study of the equation of (4.6) as for the equation (4.7) leads to a polynomial of the degree 10 with respect to $\cos\theta$:

$$P_{10} = \sum_{k=0}^{10} b_k \cos^k\theta = 0,$$

where the coefficients b_k, $(k = 0, 1, ..., 10)$ only depend on y_1, y_2, t, r. The equality $b_{10} = 0$ gives

$$k_t = r^{-2}h(k^2+1). \tag{4.16}$$

By virtue of $k_r = 0$, (4.16) and the definition of $k = (r^2U)_r/h$ one can obtain that

$$h(t, r) = 3c(t)r^2, \quad r^2U(t, r) = k(t)c(t)r^3 + \lambda(t),$$

where $c(t) = (k^2(t)+1)^{-1}k'(t)/3$. Substitution of this representation into (4.14) and splitting it with respect to r gives $c(t) = 0$ which contradicts the assumption $H \neq 0$. Similar calculations have been done for the viscous gas dynamics equations. The analysis that done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal incompressible inviscid fluid dynamics equations, are only spherically symmetric solutions.

In order to use REDUCE-program for analysis of compatibility, we summarize the main steps as follows:

1. Substituting the representation of partially invariant solution into the Navier-Stokes equations.
2. Using the implicit representation for the function $\omega = \omega(t, r, \theta, \varphi)$ described in equations (4.5) and (4.6).
3. Substituting the general solution of equation (4.8) into equations (4.5)–(4.7).
4. Changing the independent variables from $\omega, \theta, \varphi, t, r$ into y_1, y_2, θ, t, r.
5. Substituting all derivatives of the function $F(t, r, \theta, \varphi)$ through the derivative of the function $\Phi(y_1, y_2, t, r)$ into the system of equations in the previous step.
6. Splitting modified equation (4.7) with respect to $cos\theta$.
7. Analyzing the equations obtained.

For the final result of analysis of the equations, it is found that $H = 0$ which contradicts to the initial assumption. This means that there are no partially invariant solutions in this case.

The full REDUCE-program for analysis of compatibility is in Appendix C.

4.3 Spherically Symmetric Flows of the Navier-Stokes Equations

Here we study the case $H = 0$, which corresponds to a spherically symmetric flow of the Navier-Stokes equations. After substituting the representation $U = U(t, r), V = 0, W = 0, P = P(t, r)$ into the Navier-Stokes equations we have

$$D_0 U + p_r - (U_{rr} + 4r^{-1}U_r + 2r^{-2}U) = 0 \tag{4.17}$$

$$(r^2 U)_r = 0 \tag{4.18}$$

where the operator $D_0 = \partial_t + U\partial_r$. From equation (4.18) we obtain

$$r^2 U = f(t)$$

or

$$U = r^{-2} f(t)$$

where $f = f(t)$ is an arbitrary function of time t. After substituting the function U into equation (4.17). we have

$$p_r = -\frac{1}{r^2} f' + 2\frac{f^2}{r^5}.$$

Therefore, the spherically symmetric solution of the Navier-Stokes equations is

$$p = q(t) + \frac{f'(t)}{r} - \frac{1}{2}\frac{f^2(t)}{r^4}, U = \frac{f(t)}{r^2}, V = 0, W = 0 \tag{4.19}$$

with arbitrary functions $q = q(t), f = f(t)$.

Chapter V

Spherically Symmetric Flows

5.1 Spherically Symmetric Flows of a Viscous Gas

The case $H = 0$ corresponds to a spherically symmetric flow of a viscous gas. According to the definitions of group analysis, the case $H = 0$ corresponds to a singular invariant solution with respect to the group of rotations $O(3)$. The viscous gas dynamics equations in this case are

$$\begin{aligned} D_0\tau - \tau(U_r + 2r^{-1}U) = 0, \\ D_0U + \tau p_r = \tau(\lambda + 2\mu)(U_{rr} + 2r^{-1}U_r - 2r^{-2}U) + 6\tau(\mu_\tau\tau_r + \mu_p p_r) + \\ +\tau(U_r + 2r^{-1}U)(\lambda_\tau\tau_r + \lambda_p p_r), \\ D_0p + A(U_r + 2r^{-1}U) = B[\lambda(U_r + 2r^{-1}U)^2 + 2\mu(U_r^2 + 2r^{-2}U^2) + \\ +\kappa(T_{\tau\tau}\tau_r^2 + 2T_{\tau p}\tau_r p_r + T_{pp}p_r^2 + T_\tau(\tau_{rr} + 2r^{-1}\tau_r) + \\ +T_p(p_{rr} + 2r^{-1}p_r) + (\kappa_p p_r + \kappa_\tau\tau_r)(T_\tau\tau_r + T_p p_r)], \end{aligned} \tag{5.1}$$

where $D_0 = \partial_t + U\partial_r$. In this section we study a group classification of equations (5.1) with respect to the arbitrary elements A , B, λ, μ, κ, T. Henceforth, we shall use the letter ε for the internal energy because we use U for the velocity.

5.2 Equivalence Transformations

The first stage of group classification requires determining a group of equivalence transformations of equations (5.1). An equivalence transformation is a nondegenerated change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to a system of equations of the same class. It allows using the simplest representation of given equations. Here we give a construction of the group of equivalence transformations without restrictions on the representation of equivalence transformations (Ovsiannikov, 1982). We follow the approach for the calculation of equivalence transformations developed in Meleshko (1995).

Since arbitrary elements satisfy restrictions (3.14) and $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$, then for calculating an equivalence group of transformations we have to append the equations

$$\begin{aligned} A_r &= 0, A_t = 0, A_U = 0, B_r = 0, B_t = 0, B_U = 0, \\ \lambda_r &= 0, \lambda_t = 0, \lambda_U = 0, \mu_r = 0, \mu_t = 0, \mu_U = 0, \\ \kappa_r &= 0, \kappa_t = 0, \kappa_U = 0, T_r = 0, T_t = 0, T_U = 0 \end{aligned}$$

to equations (5.1). All coefficients of the infinitesimal generator of the equivalence group

$$X^e = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^U\partial_U + \zeta^\tau\partial_\tau + \zeta^p\partial_p + \zeta^A\partial_A + \zeta^B\partial_B + \zeta^\lambda\partial_\lambda + \zeta^\mu\partial_\mu + \zeta^\kappa\partial_\kappa + \zeta^T\partial_T$$

are dependent on all independent, dependent variables and arbitrary elements

$$r,\ t,\ U,\ \tau,\ p,\ A,\ B,\ \lambda,\ \mu,\ \kappa,\ T.$$

With the following notation:

$$u^1 = U,\ u^2 = \tau,\ u^3 = p,\ a^1 = A,\ a^2 = B,\ a^3 = \lambda,\ a^4 = \mu,\ a^5 = \kappa,\ a^6 = T$$

and

$$z^1 = r,\ z^2 = t,\ z^3 = U,\ z^4 = \tau,\ z^5 = p, a^k_\beta = \frac{\partial a^k}{\partial z^\beta}, a^k_{j\beta} = \frac{\partial^2 a^k}{\partial z^j \partial z^\beta},$$

the coefficients of the prolonged operator

$$\bar{X}^e = X^e + \sum_i (\zeta^{u^i_r}\partial_{u^i_r} + \zeta^{u^i_t}\partial_{u^i_t}) + \sum_{k,j} \zeta^{a^k_{z^j}}\partial_{a^k_{z^j}} + \ldots$$

can be constructed with the prolongation formulas:

$$\zeta^{u^i_r} = D_r\zeta^{u^i} - u^i_r D_r\zeta^r - u^i_t D_r\zeta^t,\ \zeta^{u^i_t} = D_t\zeta^{u^i} - u^i_r D_t\zeta^r - u^i_t D_t\zeta^t,$$

$$\zeta^{u^i_{rr}} = D_r\zeta^{u^i_r} - u^i_{rr} D_r\zeta^r - u^i_{rt} D_r\zeta^t.$$

$$\zeta^{a^k_\beta} = D^e_{z^\beta}\zeta^{a^k} - \sum_{\alpha=1}^{5} a^k_\alpha D^e_{z^\beta}\zeta^{z^\alpha},\ \zeta^{a^k_{j\beta}} = D^e_{z^\beta}\zeta^{a^k_j} - \sum_{\alpha=1}^{5} a^k_{j\alpha} D^e_{z^\beta}\zeta^{z^\alpha}.$$

Here the operators $D_r,\ D_t$ denote the total derivative operators with respect to r and t, respectively. For example,

$$D_r = \partial_r + \sum_\alpha u^\alpha_r \partial_{u^\alpha} + \sum_i (a^i_r + \sum_j a^i_{u^j} u^j_r)\partial_{a^i} + \ldots$$

When we use the operator $D^e_{z^j}$, we consider $z^1, \ldots, z^5$ as independent variables and $a^1, ..., a^6$ as dependent variables. As a result, we obtain:

$$D^e_{z^j} = \partial_{z^j} + \sum_i a^i_{z^j}\partial_{a^i} + \ldots .$$

All necessary calculations here as in the previous sections were carried out on a computer using the symbolic manipulation program REDUCE (Hearn, 1999).

The calculations show that the group of equivalence transformations of equations (5.1) corresponds to Lie algebra with generators

$$X_1^e = \partial_t,\ X_2^e = \partial_p,\ X_3^e = r\partial_r + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa,$$

$$X_4^e = r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa,\ X_5^e = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

Remark. If instead of the functions $A(p,\tau), B(p,\tau)$, one considers the internal energy $\varepsilon(p,\tau)$, then the operators X_2^e, X_4^e, and X_5^e are changed to

$$\begin{aligned} X_2^e &= \partial_p - \tau\partial_\varepsilon, \\ X_4^e &= r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa + 2\varepsilon\partial_\varepsilon, \\ X_5^e &= -\tau\partial_\tau + p\partial_p + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa. \end{aligned}$$

and there is one more generator $X_6^e = \partial_\varepsilon$.

Remark. By direct checking, one can obtain that in the general case[1] (equations (3.24)) the equivalence group includes the transformations with the generators

$$\begin{aligned} X_1^e &= \partial_t, \\ X_2^e &= \partial_p, \\ X_3^e &= \mathbf{x}\partial_{\mathbf{x}} + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa, \\ X_4^e &= \mathbf{x}\partial_{\mathbf{x}} + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa, \\ X_5^e &= -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa. \end{aligned}$$

There are also other generators, for example, that correspond to the Galilei transformations and to the rotations in the three-dimensional case.

5.3 Admitted Group

Finding an admitted group consists of seeking solutions of the determining equations (Ovsiannikov, 1982). We are looking for the generator

$$X = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^U\partial_U + \zeta^\tau\partial_\tau + \zeta^p\partial_p$$

with the coefficients depending on r, t, U, τ, p. Calculations lead to the following result.

The kernel of the fundamental Lie algebra is made up of the generator

$$X = \partial_t.$$

[1] Group classification of three-dimensional viscous gas dynamics equations with $\lambda = -2\mu/3$ was studied in Bublik (2000).

Extension of the kernel of the main Lie algebra occurs by specializing the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. Note that the functions $A = A(p,\tau), B = B(p,\tau), T = T(p,\tau)$ have to satisfy equations (3.14). There are three types of the generators admitted by system (5.1). Further α, β and δ are arbitrary constants.

Type (a). If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{array}{l} \alpha\tau A_\tau + A_p = 0,\ \alpha\tau B_\tau + B_p = 0, \\ \alpha\tau\mu_\tau + \mu_p = \beta\mu,\ \alpha\tau\lambda_\tau + \lambda_p = \beta\lambda, \\ \alpha\tau T_\tau + T_p = \delta T,\ \alpha\tau\kappa_\tau + \kappa_p = (-\delta + \alpha + \beta)\kappa, \end{array} \tag{5.2}$$

then there is one more admitted generator:

$$Y_a = \alpha U\partial_U + 2\alpha\tau\partial_\tau + 2\partial_p + (\alpha + 2\beta)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (5.2) is

$$\begin{array}{c} A = A(\tau e^{-\alpha p}),\ B = B(\tau e^{-\alpha p}),\ \mu = e^{\beta p}M(\tau e^{-\alpha p}),\ \lambda = e^{\beta p}\Lambda(\tau e^{-\alpha p}), \\ T = e^{\delta p}\Theta(\tau e^{-\alpha p}),\ \kappa = e^{(-\delta+\alpha+\beta)p}K(\tau e^{-\alpha p}), \end{array}$$

where the functions $A(z), B(z)$ and $\Theta(z)$ satisfy the equations $(z \equiv \tau e^{-\alpha p})$

$$-\alpha zBA' + zB'(1 + \alpha A) = B^2 + B, \quad (1 + \alpha A)z\Theta' = (\delta A - B)\Theta. \tag{5.3}$$

The internal energy is represented by the formula

$$\varepsilon = e^{\alpha p}(\varphi(z) - zp) + \psi(p),\ \psi'(p) = Ce^{\alpha p},$$

where the function $\varphi(z)$ and the constant C can be accounted as arbitrary and they are related with the functions $A(z)$ and $B(z)$ by the formulas

$$\varphi'(z) = \frac{A(z)}{B(z)}, \quad C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - \alpha\varphi(z).$$

In this case, the function $\Theta(z)$ has to satisfy the equation

$$(C - z + \alpha\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z).$$

Type (b). If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{array}{c} \alpha\tau A_\tau + pA_p = A,\ \alpha\tau B_\tau + pB_p = 0, \\ \alpha\tau\mu_\tau + p\mu_p = (\beta + 1)\mu,\ \alpha\tau\lambda_\tau + p\lambda_p = (\beta + 1)\lambda, \\ \alpha\tau T_\tau + pT_p = \delta T,\ \alpha\tau\kappa_\tau + pT_p = (-\delta + 2 + \alpha + \beta)\kappa, \end{array} \tag{5.4}$$

then there is an extension by the generator

$$Y_b = (1 + \alpha)U\partial_U + 2\alpha\tau\partial_\tau + 2p\partial_p + (\alpha + 2\beta + 1)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (5.4) is

$$A = p\widehat{A}(\tau p^{-\alpha}),\ B = B(\tau p^{-\alpha}),\ \mu = p^{\beta+1}M(\tau p^{-\alpha}),\ \lambda = p^{\beta+1}\Lambda(\tau p^{-\alpha}),$$
$$T = p^{\delta}\Theta(\tau p^{-\alpha}),\ \kappa = p^{-\delta+\alpha+\beta+2}K(\tau p^{-\alpha}),$$

where the functions $\widehat{A}(z), B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau p^{-\alpha}$)

$$-\alpha z B\widehat{A}' + zB'(1+\alpha\widehat{A}) = B^2 + B - B\widehat{A}, \quad (1+\alpha\widehat{A})z\Theta' = (\delta\widehat{A} - B)\Theta. \quad (5.5)$$

The internal energy is represented by the formula

$$\varepsilon = p^{(\alpha+1)}(\varphi(z) - z) + \psi(p),\ \psi'(p) = Cp^{\alpha},$$

where the function $\varphi(z)$ and the constant C are arbitrary and they are related with the functions $\widehat{A}(z)$ and $B(z)$ by the formulas

$$\varphi'(z) = \frac{\widehat{A}(z)}{B(z)},\ C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - (\alpha+1)\varphi(z)$$

The function $\Theta(z)$ is represented through the function $\varphi(z)$ by the formula

$$(C - z + (\alpha+1)\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z)$$

Note that an ideal gas belongs to this type in case of $\delta = \alpha + 1$ and the function $\varphi(z)$ satisfies the equation

$$\delta(z\varphi' - \varphi) = C.$$

Type (c). If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$A_\tau = 0,\ B_\tau = 0,\ \tau\mu_\tau = \beta\mu,\ \tau\lambda_\tau = \beta\lambda,$$
$$\tau T_\tau = \delta T,\ \tau\kappa_\tau = (-\delta + 1 + \beta)\kappa, \quad (5.6)$$

then there is one more admitted generator:

$$Y_c = U\partial_U + 2\tau\partial_\tau + (1+2\beta)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (5.6) is

$$A = A(p),\ B = B(p),\ \mu = \tau^{\beta}M(p),\ \lambda = \tau^{\beta}\Lambda(p),$$
$$T = \tau^{\delta}\Theta(p),\ \kappa = \tau^{-\delta+\beta+1}K(p),$$

where the functions $A(p), B(p)$ and $\Theta(p)$ satisfy the equations

$$BA' - AB' = B^2 + B, \quad A\Theta' = (\delta + B)\Theta. \quad (5.7)$$

The internal energy is represented by the formula

$$\varepsilon = \tau\varphi(p) - \tau p,$$

where the function $\varphi(p)$ is an arbitrary function and is related with the functions $A(p)$ and $B(p)$ by the formula

$$\varphi(p) = \frac{A(p)}{B(p)}.$$

In this case the function $\Theta(p)$ is related with the function $\varphi(p)$ by the formula

$$\varphi(p)\Theta'(p) = (1 - \delta + \delta\varphi'(p))\Theta(p).$$

Note that if $\delta = 1$ and $\varphi = Cp$, then the gas is ideal.

The final results of the group classification are presented in Table 6.1. In this table the first column means the type of the extension of the algebra $\{X\}$: the types a, b, or c, respectively. The last column means conditions for the state functions.

Therefore, there are three kinds of admitted by equations (5.1) groups, which depend on the specifications of the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. These groups are one-dimensional, two-dimensional and three-dimensional.

The two-dimensional admitted groups are groups with the generators either $\{X, Y_a\}$ or $\{X, Y_b\}$ or $\{X, Y_c\}$. The three-dimensional admitted groups are the groups with the generators either $\{X, Y_a, Y_b\}$ or $\{X, Y_a, Y_c\}$ or $\{X, Y_b, Y_c\}$.

The group with the generators $\{X, Y_a, Y_b\}$ is admitted by equations (5.1) if

$$A = A_0\tau^\alpha,\ B = -1,\ \mu = \mu_0\tau^{\beta+\alpha},\ \lambda = \lambda_0\tau^{\beta+\alpha},\ \kappa = \kappa_0\tau^{\beta+2\alpha},\ T = T_0\tau,\ \alpha \neq 0.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0\int \tau^\alpha\, d\tau)$. Instead, the operators Y_a and Y_b, one can use their linear combinations:

$$\widehat{Y}_a = \partial_p,\ \widehat{Y}_b = (1+\alpha)U\partial_U + 2\tau\partial_\tau + (\alpha + 2\beta + 1)r\partial_r + 2\beta t\partial_t.$$

The algebra of the type $\{X, Y_a, Y_c\}$ is admitted by equations (5.1) if

$$A = A_0,\ B = -1,\ \mu = \mu_0\tau^\beta e^{\alpha p},\ \lambda = \lambda_0\tau^\beta e^{\alpha p},$$

$$\kappa = \kappa_0\tau^{\beta - A_0\sigma}e^{(\alpha-\sigma)p},\ \ T = T_0\tau^{1+A_0\sigma}e^{\sigma p}.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0\tau)$ and by taking linear combinations of the operators Y_a and Y_c one obtains another basis of the generators:

$$\widehat{Y}_a = \partial_p + \alpha(r\partial_r + t\partial_t),\ \widehat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta+1)r\partial_r + 2\beta t\partial_t.$$

The third type of the algebras $\{X, Y_b, Y_c\}$ is admitted by (5.1) if

$$A = \gamma p,\ B = \gamma - 1,\ \mu = \mu_0\tau^\beta p^{1+\alpha},\ \lambda = \lambda_0\tau^\beta p^{1+\alpha},$$
$$\kappa = \kappa_0\tau^{\gamma(1-\alpha)+\beta}p^{\alpha-\delta+2},\ T = T_0\tau^{\gamma(\delta-1)+1}p^\delta,\ \gamma \neq 1.$$

The internal the energy in this case is

$$\varepsilon = \frac{\tau p}{\gamma - 1}$$

and linear combinations of the operators Y_b and Y_c are:

$$\widehat{Y}_b = U\partial_U + 2p\partial_p + (2\alpha+1)r\partial_r + 2\alpha t\partial_t,\ \widehat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta+1)r\partial_r + 2\beta t\partial_t$$

Note that a polytropic gas belongs to the last case of gases, where γ is a polytropic exponent.

In the formulas above $A_0, \mu_0, \lambda_0, \kappa_0, T_0, \alpha, \beta, \gamma, \delta, \sigma$ are arbitrary constants; the commutators

$$[\widehat{Y}_a, \widehat{Y}_b] = 0,\ [\widehat{Y}_a, \widehat{Y}_c] = 0,\ [\widehat{Y}_b, \widehat{Y}_c] = 0.$$

5.4 Optimal Systems of Subalgebras

Here we study subalgebras of the two-dimensional admitted algebras $\{X, Y_a\}$, $\{X, Y_b\}$, $\{X, Y_c\}$.

The commutator $[X, Y]$ of the generators X and Y is

$$[X, Y] = zX.$$

Here either $Y = Y_a$ or $Y = Y_b$ or $Y = Y_c$ and $z = 2\beta$. Automorphisms are recovered by the table of commutators and consists of the automorphisms

$$A_1:\ x' = x + zya_1,\ y' = y,$$
$$A_2:\ x' = e^{-za_2}x,\ y' = y,$$

where x and y are coordinates of the operator $Z = xX + yY$, x' and y' are coordinates of the operator Z' after actions of the automorphisms, and $a_1,\ a_2$ are parameters of the automorphisms. There is also one involution

$$E:\ x' = -x,\ y' = y,$$

which corresponds to the change of the variables $t \to -t$ and $U \to -U$ without changes of equations (5.1). Note that if $z = 0$, then the automorphisms are identity transformations. This leads to two optimal systems of subalgebras.

If $z = 0$ (or $\beta = 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\},\ \{Y + hX\},\ \{X, Y\},$$

where h is an arbitrary positive constant.

If $z \neq 0$ (or $\beta \neq 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\},\ \{Y\},\ \{X, Y\}.$$

Therefore, one can summarize: optimal systems of subalgebras for the two-dimensional algebras are described by the following system of subalgebras

$$\{X\},\ \{Y + hX\},\ \{X, Y\},\ \beta h = 0. \qquad (5.8)$$

5.5 Representations of Invariant Solutions

The next step in the construction of representations of invariant solutions consists of finding universal invariants. Note that invariant solutions corresponding to the case of the subalgebra $\{X\}$ are the well-known stationary solutions. The universal invariants for the other subalgebras of the optimal system (5.8) of the algebras $\{X, Y_a\}$, $\{X, Y_b\}$ and $\{X, Y_c\}$ are presented in Tables 5.2, 5.3 and 5.4, respectively.

According to the theory of the group analysis (Ovsiannikov, 1982), in the next step in constructing of invariant solutions, one needs to separate the universal invariant into two parts: one part has to be solvable with respect to the dependent variables U, τ, p. After that, the representations of invariant solutions are obtained by supposing that the first part of the universal invariant depends on the second part. Because of this requirement, there are no invariant solutions in the cases: $a.1$ if $h = 0$, $a.4$, $b.1$, $b.5$ and $c.3$. The cases $a.5$, $b.6$ and $c.4$ correspond to the special cases of stationary solutions, which we also exclude from our consideration[2].

All possible representations of invariant solutions of equations (5.1) are presented in Table 5.5, where the functions f^u, f^τ, f^p are functions of one independent variable presented in the last column. These functions must satisfy ordinary differential equations, which are obtained after substituting the representation of solution into system (5.1).

Remark. Invariant solutions $a.3$, $b.2$, $b.4$, $c.2$ are self-similar solutions.

Remark. One of the well-known solutions of the Boltzmann equation (the BKW-solution[3]) has the representation (Bobylev,1976; Krook,1977).

$$f = \phi(|u|e^{ct}),$$

where f is a distribution function, $|u|$ is a modulus of the velocity. The invariant solution of the viscous gas equations, which corresponds to the case $b.3$ gives

$$|u|e^{-t(\alpha+1)/h} = qf^u(q),$$

with $q = re^{-t(\alpha+1)/h}$. Therefore this solution can correspond to the BKW-solution and generalize it on molecules with exponent intermolecular potentials. For the molecules with exponent intermolecular potentials the coefficients of viscosity and conductivity are in Bird(1994).

$$\mu = \mu_0 T^k, \kappa = \kappa_0 T^k,$$

where $T = T_0 p\tau$, $k = (n-1)/m + 1/2$, n is a dimension of the problem, m is the exponent of intermolecular potentials. In this case $\alpha = 1/k - 1 = (m + 2n - 2)^{-1}(m - 2n + 2)$. For the Maxwell molecules, for which the BKW-solution was constructed, the exponent of intermolecular potentials is $m = 4$, and hence, in the three-dimensional case $\alpha = 0$ and $k = 1$.

[2]If an universal invariant is three-dimensional (consists of three invariants), such as in the cases of $a.5$, $b.6$, $c.4$, then the representation of the invariant solution is obtained by assuming that all invariants of the universal invariant are constants.

[3]This solution is constructed for the Maxwell molecules.

Table 5.1: Group classification.

	λ	μ	T	κ	A	B	z	Cond.
a	$e^{\beta p}\Lambda(z)$	$e^{\beta p}M(z)$	$e^{\delta p}\Theta(z)$	$e^{(-\delta+\alpha+\beta)p}K(z)$	$A(z)$	$B(z)$	$\tau e^{-\alpha p}$	(5.3)
b	$p^{\beta+1}\Lambda(z)$	$p^{\beta+1}M(z)$	$p^{\delta}\Theta(z)$	$p^{-\delta+\alpha+\beta+2}K(z)$	$p\widehat{A}(z)$	$B(z)$	$\tau p^{-\alpha}$	(5.5)
c	$\tau^{\beta}\Lambda(p)$	$\tau^{\beta}M(p)$	$\tau^{\delta}\Theta(p)$	$\tau^{-\delta+\beta+1}K(p)$	$A(p)$	$B(p)$	p	(5.7)

Table 5.2: Universial invariants of subalgebras of the algebra $\{X, Y_a\}$.

N	Subalgebra	consts	Universal invariant
$a.1$	$Y_a + hX$	$\beta = 0, \alpha = 0$	$U, \tau, t - hp/2, r$
$a.2$	$\beta h = 0$	$\beta = 0, \alpha \neq 0$	$Ur^{-1}, \tau r^{-2}, p - 2\alpha^{-1}\ln r, t - h\alpha^{-1}\ln r$
$a.3$		$\beta \neq 0$	$Ut^{-(\alpha)/(2\beta)}, \tau t^{-\alpha/\beta}, p - ln(t^{1/\beta}), rt^{-(\alpha+2\beta)/(2\beta)}$
$a.4$	X, Y_a	$\alpha + 2\beta = 0$	$Ue^{-\alpha p/2}, \tau e^{-\alpha p}, r$
$a.5$		$\alpha + 2\beta \neq 0$	$Ur^{-\alpha/(\alpha+2\beta)}, \tau r^{-2\alpha/(\alpha+2\beta)}, p - 2(\alpha+2\beta)^{-1}\ln r$

Table 5.3: Universial invariants of subalgebras of the algebra $\{X, Y_b\}$ $(k \equiv \alpha + 2\beta + 1)$.

N	Subalgebra	consts	Universal invariant
$b.1$	$Y_b + hX$	$\beta = 0, \alpha = -1, h = 0$	$U, p/\tau, r, t$
$b.2$	$\beta h = 0$	$\beta = 0, \alpha \neq -1, h = 0$	$Ur^{-1}, \tau r^{-2\alpha/(\alpha+1)}, pr^{-2/(\alpha+1)}, t$
$b.3$		$\beta = 0, h \neq 0$	$Ue^{-t(\alpha+1)/h}, \tau e^{-2t\alpha/h}, pe^{-2t/h}, re^{-t(\alpha+1)/h}$
$b.4$		$\beta \neq 0$	$Ut^{-(\alpha+1)/(2\beta)}, \tau t^{-\alpha/\beta}, pt^{-1/\beta}, rt^{-k/(2\beta)}$
$b.5$	X, Y_b	$k = 0$	$Up^{-(\alpha+1)/2}, \tau p^{-\alpha}, r$
$b.6$		$k \neq 0$	$Ur^{-(\alpha+1)/k}, \tau r^{-2\alpha/k}, pr^{-2/k}$

Table 5.4: Universial invariants of subalgebras of the algebra $\{X, Y_c\}$.

N	Subalgebra	consts	Universal invariant
$c.1$	$Y_c + hX$	$\beta = 0$	$Ur^{-1}, \tau r^{-2}, p, t - h\ln r$
$c.2$	$\beta h = 0$	$\beta \neq 0$	$Ut^{-1/(2\beta)}, \tau t^{-1/\beta}, p, rt^{-(2\beta+1)/(2\beta)}$
$c.3$	X, Y_c	$2\beta + 1 = 0$	$U\tau^{-1/2}, p, r$
$c.4$		$2\beta + 1 \neq 0$	$Ur^{-1/(2\beta+1)}, \tau r^{-2/(2\beta+1)}, p$

Table 5.5: Representations of invariant solutions.

N	Representation of invariant solution	Ind. variable	Model
1	$U = f^u, \tau = f^\tau, p = 2th^{-1} + f^p$	r	$a.1$
2	$U = rf^u, \tau = r^2 f^\tau, p = 2\alpha^{-1}\ln r + f^p$	$t - h\alpha^{-1}\ln r$	$a.2$
3	$U = t^{(\alpha)/(2\beta)}f^u, \tau = t^{\alpha/\beta}f^\tau, p = ln(t^{1/\beta}) + f^p$	$rt^{-(\alpha+2\beta)/(2\beta)}$	$a.3$
4	$U = rf^u, \tau = r^{2\alpha/(\alpha+1)}f^\tau, p = r^{2/(\alpha+1)}f^p$	t	$b.2$
5	$U = e^{t(\alpha+1)/h}f^u, \tau = e^{2t\alpha/h}f^\tau, p = e^{2t/h}f^p,$	$re^{-t(\alpha+1)/h}$	$b.3$
6	$U = t^{(\alpha+1)/(2\beta)}f^u, \tau = t^{\alpha/\beta}f^\tau, p = t^{1/\beta}f^p$	$rt^{-(\alpha+2\beta+1)/(2\beta)}$	$b.4$
7	$U = rf^u, \tau = r^2 f^\tau, p = f^p$	$t - h\ln r$	$c.1$
8	$U = t^{1/(2\beta)}f^u, \tau = t^{1/\beta}f^\tau, p = f^p$	$rt^{-(2\beta+1)/(2\beta)}$	$c.2$

Chapter VI

Conclusions

6.1 Thesis Summary

In this thesis, partially invariant and invariant solutions of the Navier-Stokes and viscous gas dynamics equations related with the group of rotations in three-dimensional space were studied.

6.1.1 Problems

The thesis is devoted to an application of group analysis to the equations governing a motion of Newtonian viscous fluid. In the compact form, these equations can be written as follows [1]

$$\frac{d\mathbf{v}}{dt} + \tau\nabla p = \tau[(\lambda+\mu)\nabla(div\mathbf{v}) + (div\mathbf{v})\nabla\lambda + 2\mathbf{D} < \nabla\mu > + \mu\Delta\mathbf{v}],$$

$$\frac{d\tau}{dt} - \tau div\mathbf{v} = 0, \quad (6.1)$$

$$\frac{dp}{dt} + A(p,\tau)(div\mathbf{v}) = B(p,\tau)(\lambda(div\mathbf{v})^2 + 2\mu\mathbf{D}:\mathbf{D} + (\nabla\kappa)(\nabla T) + \kappa\triangle T),$$

where

$$A(p,\tau) = \frac{\tau(p+U_\tau)}{U_p}, \quad B(p,\tau) = \frac{\tau}{U_p}$$

and $\mathbf{v} = (u_1, u_2, u_3)$ is the velocity vector, p is pressure, $\tau = \dfrac{1}{\rho}$ is specific volume, ρ is density, U is the internal energy, T is the temperature, λ and μ are first and the second coefficient of viscosity, κ is a coefficient of a heat conductivity, t is time, ∇ and Δ are the gradient and the Laplacian with respect to the space variables $\mathbf{x} = (x_1, x_2, x_3)$, respectively.

In the case of incompressible fluid, τ and μ are assumed to be constants and the equations are split on two parts: the momentum equations and the continuity equation

$$\frac{d\mathbf{v}}{dt} = -\tau(\nabla p + \mu\Delta\mathbf{v}), \quad div\mathbf{v} = 0.$$

[1] In the literature, these equations are often called the Navier-Stokes equations.

Recently, Ovsiannikov (1995) found one class of partially invariant solutions of inviscid gas dynamics equations ($\mu = \lambda = \kappa = 0$), which is called a special vortex. This solution is based on the group of rotations in three-dimensional space. Therefore, it is natural to investigate the solutions of the Navier-Stokes and viscous gas dynamics equations, which are related with this group.

The group of rotations in three-dimensional space can be described by the generators as:

$$\zeta_{ik} \cdot \partial = x_i \partial_{x_k} - x_k \partial_{x_i} + u_i \partial_{u_k} - u_k \partial_{u_i}, (i < k)$$

A study of solutions related with the group of rotations requires using the spherical coordinate system. The derivation of equations (6.1) in the spherical coordinate system is done by a computer program in REDUCE. Using symbolic calculations prevents us from errors in doing analytical studies.

The thesis has considered the following problems.

1. Partially invariant solutions with respect to group of rotations have the representation

$$U = U(t, r),\ H = H(t, r),\ p = p(t, r),\ \tau = \tau(t, r), \omega = \omega(t, r, \theta, \varphi)$$

where (U, V, W) is a velocity vector in spherical coordinate system, $V = Hcos\omega$, $W = Hsin\omega$. Here $\omega = \omega(t, r, \theta, \varphi)$ is a "superfluous" function. The first study is devoted to establishing a compatibility of essentially $(H \neq 0)$ partially invariant solutions of the Navier-Stokes and viscous gas dynamics equations.

2. The case $H = 0$ corresponds to a singular invariant solution. The problem is to study the Navier-Stokes and viscous gas dynamics equations by group analysis. The first step in applying the group analysis is a group classification. The group classification problem consists of searching for groups of transformations admitted by the system for all arbitrary elements and all specifications of arbitrary elements. By the special choice of the arbitrary elements, one can extend the admitted group.

After finding the admitted group, one can try to construct exact solutions: every subgroup of the admitted group can be a source of invariant or partially invariant solutions. There is an infinite number of subgroups, even in cases where the admitted groups are finite-dimensional. But if two subgroups are similar, i.e., they are connected with each other by a symmetry transformation, then their corresponding invariant solutions are connected with each other by the same transformation. Since the set of subgroups can be divided into classes of similar subgroups, it is sufficient to find only one representative solution from each class of subgroups. A set of representatives of equivalent subgroup classes is called an optimal system of subgroups.

6.1.2 Results

1. It is proven that partially invariant solutions of the singular vortex type of the Navier-Stokes and viscous gas dynamics equations are reduced to singular

invariant solutions (H=0).

2. The group classification of the spherically symmetric viscous gas dynamics equations has been done. The kernel of the fundamental Lie algebra is made up of the generator

$$X = \partial_t$$

Extension of the kernel of the main Lie algebra occurs by specializing the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. There are three types of generators admitted by system (5.1).

3. An optimal system of subalgebras of the algebra $\{X, Y\}$ was presented.

4. All representations of invariant solutions of the spherically symmetric viscous gas dynamics equations were constructed.

6.2 Applications

Expected benefits of this research include the following. Exact solutions are good tests for the comparisons of various numerical methods. By comparison of the solutions of different simplifications with the solutions of the complete Navier-Stokes equations can be useful for a determination of scopes of simplifications of the complete equations.

The immediate beneficiaries of this research will be to those who use the Navier-Stokes equations for modeling the processes in different kinds of technology: superconductor, aerodynamics and geodynamics. In the longer term, the results of this research will be used by researchers doing theoretical investigations of the Navier-Stokes equations and other scientific studies.

6.3 Limitations

In the thesis, the group analysis was applied to the Navier-Stokes and viscous gas dynamics equations. Solutions related with the group of rotations were studied.

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Appendix

Appendix A

Deriving Equations in Spherical Coordinates

A.1 The full program for deriving the Navier-Stokes equations in spherical coordinates

The first part of the program is devoted to deriving the Navier-Stokes Equations in spherical coordinates by using REDUCE. Here we explain the identificators and main commands.

The commands

```
s1:=dv1dt+grp1-lapv1;
s2:=dv2dt+grp2-lapv2;
s3:=dv3dt+grp3-lapv3;
s4:=divv;
```

correspond to equations (3.42) and (3.43), where

$$\texttt{dv1dt} = (\frac{d\mathbf{v}}{dt})^1, \quad \texttt{dv2dt} = (\frac{d\mathbf{v}}{dt})^2, \quad \texttt{dv3dt} = (\frac{d\mathbf{v}}{dt})^3$$
$$\texttt{grp1} = (\nabla p)^1, \quad \texttt{grp2} = (\nabla p)^2, \quad \texttt{grp3} = (\nabla p)^3$$
$$\texttt{lapv1} = (\Delta \mathbf{v})^1, \quad \texttt{lapv2} = (\Delta \mathbf{v})^2, \quad \texttt{lapv3} = (\Delta \mathbf{v})^3$$

The commands

```
depend vr,r,te,fi,t;
depend vte,r,te,fi,t;fi,t;
depend vfi,r,te,fi,t;
depend p,r,te,fi,t;
depend f,r,te,fi,t;
```

mean that `vr, vte, vfi, p` and `f` are functions of the dependent variables `r,te,fi,t`. Here `vr=U, vte=V, vfi=W` are physical coordinates of velocity **v**.

The command

```
lap:=df(r**2*df(f,r),r)/r**2+df(sin(te)*df(f,te),te)/(r**2*sin(te))
    + df(f,fi,2)/(r**2*sin(te)**2);
```

defines the Laplace operator of the function `f`.

The command

```
dd:=df(f,t)+vr*df(f,r)+vte*df(f,te)/r+vfi*df(f,fi)/(r*sin(te));
```

defines the operator $\dfrac{Df}{Dt}$

The identificators

```
k(1):=r;
k(2):=te;
k(3):=fi;
```

define spherical coordinates r, θ and φ, respectively.

The operators

```
vs(1):=vr;
vs(2):=vte/r;
vs(3):=vfi/(r*sin(te));
```

define contravariant coordinates of the velocity vector $\mathbf{v}$.

The following formulas define contravariant (g^{ij})

```
g(1,1):=1;g(1,2):=0;g(1,3):=0;
g(2,1):=0;g(2,2):=1/r**2;g(2,3):=0;
g(3,1):=0;g(3,2):=0;g(3,3):=1/(r**2*sin(te)**2);
```

and covariant coordinates (g_{ij})

```
gd(1,1):=1; gd(1,2):=0; gd(1,3):=0;
gd(2,1):=0; gd(2,2):=r**2; gd(2,3):=0;
gd(3,1):=0; gd(3,2):=0; gd(3,3):=r**2*sin(te)**2;
```

of the fundamental tensor. The Christoffel symbols are `ga(l,i,j)`$=\Gamma^l_{ij}$:

```
for l:=1:3 do for i:=1:3 do for j:=1:3 do
ga(l,i,j):= for s:=1:3 sum
g(l,s)*( df(gd(i,s),k(j))+df(gd(j,s),k(i))-df(gd(i,j),k(s)) )/2;
```

The loop

```
for l:=1:3 do
for i:=1:3 do
for j:=1:3 do
write(''ga('',l,'','',i,'','',j,'') = '',ga(l,i,j));
```

output the Christoffel symbols.

The following loop of commands corresponds to equations (3.23):

```
for l:=1:3 do begin
ss1:=sub(f=vs(l),lap);
ss2:=for i:=1:3 sum for j:=1:3 sum for s:=1:3 sum
g(i,j)*ga(l,i,s)*df(vs(s),k(j));
ss3:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
vs(pk)*g(i,j)*df(ga(l,i,pk),k(j));
ss4:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum for s:=1:3 sum
vs(pk)*g(i,j)*ga(s,i,pk)*ga(l,j,s);
ss5:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum for s:=1:3 sum
vs(pk)*g(i,j)*ga(s,i,j)*ga(l,pk,s);
qq(l):=ss1+2*ss2+ss3+ss4-ss5;
end;
```

Here

$$\texttt{lapv1} = \texttt{qq}(1) = (\Delta v)^1$$
$$\texttt{lapv2} = \texttt{qq}(2) = (\Delta v)^2$$
$$\texttt{lapv3} = \texttt{qq}(3) = (\Delta v)^3$$

For the components of the acceleration vector $(\dfrac{d\mathbf{v}}{dt})^l$ (3.50)-(3.52)

```
dv1dt:=sub(f=vr,dd)-(vte**2+vfi**2)/r;
dv2dt:=sub(f=vte,dd)/r+(vr*vte-(cos(te)/sin(te))*vfi**2)/r**2;
dv3dt:=sub(f=vfi,dd)/(r*sin(te))
+(vr*vfi+(cos(te)/sin(te))*vfi*vte)/(r**2*sin(te));
```

The coordinates of the gradient of p are defined by equations (3.47)

```
grp1:=df(p,r);
grp2:=df(p,te)/r**2;
grp3:=df(p,fi)/(r*sin(te))**2;
```

The divergence of the vector $\mathbf{v}$ is

```
divv:=df(r**2*vr,r)/(r**2)+df(sin(te)*vte,te)/(r*sin(te))
  +df(vfi,fi)/(r*sin(te));
```

The full program can be written as follows:

```
s1:=dv1dt+grp1-nu*lapv1;
s2:=dv2dt+grp2-nu*lapv2;
s3:=dv3dt+grp3-nu*lapv3;
s4:=divv;
factor nu;
depend vr,r,te,fi,t;
depend vte,r,te,fi,t;
depend vfi,r,te,fi,t;
depend p,r,te,fi,t;
depend f,r,te,fi,t;
lap:=df(r**2*df(f,r),r)/r**2+df(sin(te)*df(f,te),te)
    /(r**2*sin(te))+df(f,fi,2)/(r**2*sin(te)**2);
dd:=df(f,t)+vr*df(f,r)+vte*df(f,te)/r+vfi*df(f,fi)/(r*sin(te));
operator g,gd,ga,k,vs,qq,qqq;
k(1):=r;
k(2):=te;
k(3):=fi;
vs(1):=vr;
vs(2):=vte/r;
vs(3):=vfi/(r*sin(te));
g(1,1):=1;g(1,2):=0;g(1,3):=0;
g(2,1):=0;g(2,2):=1/r**2;g(2,3):=0;
g(3,1):=0;g(3,2):=0;g(3,3):=1/(r**2*sin(te)**2);
gd(1,1):=1;gd(1,2):=0;gd(1,3):=0;
gd(2,1):=0;gd(2,2):=r**2;gd(2,3):=0;
gd(3,1):=0;gd(3,2):=0;gd(3,3):=r**2*sin(te)**2;
for l:=1:3 do for i:=1:3 do for j:=1:3 do
ga(l,i,j):= for s:=1:3 sum g(l,s)*(df(gd(i,s),k(j))+
df(gd(j,s),k(i))-df(gd(i,j),k(s)))/2;
for l:=1:3 do for i:=1:3 do for j:=1:3 do write
("ga(",l,",",i,",",j,") = ",ga(l,i,j));
for l:=1:3 do begin
ss1:=sub(f=vs(l),lap);
ss2:=for i:=1:3 sum for j:=1:3 sum for s:=1:3 sum
    g(i,j)*ga(l,i,s)*df(vs(s),k(j));
ss3:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
    vs(pk)*g(i,j)*df(ga(l,i,pk),k(j));
ss4:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
    for s:=1:3 sum vs(pk)*g(i,j)*ga(s,i,pk)*ga(l,j,s);
```

```
ss5:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
    for s:=1:3 sum vs(pk)*g(i,j)*ga(s,i,j)*ga(l,pk,s);
    qq(l):=ss1+2*ss2+ss3+ss4-ss5;
end;
for l:=1:3 do begin
qqq(l):=sub(f=vs(l),lap)+
    2*(for i:=1:3 sum for j:=1:3 sum for s:=1:3 sum
    g(i,j)*ga(l,i,s)*df(vs(s),k(j)))+
    for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
    vs(pk)*g(i,j)*(df(ga(l,i,pk),k(j))+
    for s:=1:3 sum
    ga(s,i,pk)*ga(l,j,s)-
    for s:=1:3 sum
    ga(s,i,j)*ga(l,pk,s));
end;
qq(1);
qq(2);
qq(3);
dv1dt:=sub(f=vr,dd)-(vte**2+vfi**2)/r;
lapv1:=sub(f=vr,lap)-2*df(vte,te)/(r**2)-2*df(vfi,fi)/((r**2)*
    sin(te))-2*vr/(r**2)-2*(cos(te)/sin(te))*vte/(r**2);
lapv2:=sub(f=vte/r,lap)+2*df(vte,r)/r**2+2*df(vr,te)/r**3-
    2*(cos(te)/sin(te))*df(vfi,fi)/(r**3*sin(te))-vte/(r**3*sin(te)**2);
%lapv2:=sub(f=vte/r,lap)+2*df(vte,r)/(r**2)+2*df(vr,te)/(r**3)-
% 2*(cos(te)/sin(te))*df(vfi,fi)/((r**3)*sin(te)**2)-
% vte/((r**3)*(sin(te)**2));
lapv3:= sub(f=vfi/(r*sin(te)),lap)+2*df(vfi/r,r)/(r*sin(te))+
    (2*(cos(te)/sin(te))*df(vfi/(sin(te)),te)/(r**3))+
    2*df(vr,fi)/((r**3)*(sin(te)**2))+
    (2*(cos(te)/sin(te))*df(vte,fi)/((r**3)*(sin(te)**2)));
let sin(te)**2=1-cos(te)**2;
ss1:=qq(1)-lapv1;
ss2:=qq(2)-lapv2;
ss3:=qq(3)-lapv3;
pause;
clear sin(te)**2;
lapv1:=qq(1);
lapv2:=qq(2);
lapv3:=qq(3);
grp1:=df(p,r);
grp2:=df(p,te)/r**2;
grp3:=df(p,fi)/(r*sin(te))**2;
on div;

dv2dt:=sub(f=vte,dd)/r+(vr*vte-(cos(te)/sin(te))*vfi**2)/r**2;
dv3dt:=sub(f=vfi,dd)/(r*sin(te))+(vr*vfi+(cos(te)/sin(te))*vfi*vte)/
    (r**2*sin(te));
divv:=df(r**2*vr,r)/(r**2)+df(sin(te)*vte,te)/(r*sin(te))+
df(vfi,fi)/(r*sin(te));
s1:=s1;
s2:=r*s2;
s3:=r*sin(te)*s3;
s4:=s4;
ss1:=df(vr,t)+vr*df(vr,r)+vte*df(vr,te)/r+vfi*df(vr,fi)/(r*sin(te))-
    (vte**2+vfi**2)/r+df(p,r)-nu*(df(r**2*df(vr,r),r)/r**2+
    df(sin(te)*df(vr,te),te)/(r**2*sin(te))+df(vr,fi,2)/(r**2*sin(te)**2)
    -2*vr/r**2-2*df(vte,te)/r**2-2*vte*cos(te)/(r**2*sin(te))-
    2*df(vfi,fi)/(r**2*sin(te)));
ss2:=df(vte,t)+vr*df(vte,r)+vte*df(vte,te)/r+vfi*df(vte,fi)/(r*sin(te))+
    vte*vr/r-vfi**2*cos(te)/(r*sin(te))+df(p,te)/r-nu*
    (df(r**2*df(vte,r),r)/r**2+df(sin(te)*df(vte,te),te)/(r**2*sin(te))+
    df(vte,fi,2)/(r**2*sin(te)**2)+2*df(vr,te)/r**2-
    (vte+2*cos(te)*df(vfi,fi))/(r**2*sin(te)**2));
ss3:=df(vfi,t)+vr*df(vfi,r)+vte*df(vfi,te)/r+vfi*df(vfi,fi)/(r*sin(te))+
    vfi*vr/r+vte*vfi*cos(te)/(r*sin(te))+df(p,fi)/(r*sin(te))-nu*
    (df(r**2*df(vfi,r),r)/r**2+df(sin(te)*df(vfi,te),te)/(r**2*sin(te))+
    df(vfi,fi,2)/(r**2*sin(te)**2)-vfi/(r**2*sin(te)**2)+
    2*df(vr,fi)/(r**2*sin(te))+2*cos(te)*df(vte,fi)/(r**2*sin(te)**2));
```

```
ssl1:=df(vr,t)+vr*df(vr,r)+vte*df(vr,te)/r+vfi*df(vr,fi)/(r*sin(te))-
    (vte**2+vfi**2)/r+df(p,r)-nu*(df(r**2*df(vr,r),r)/r**2+
    df(sin(te)*df(vr,te),te)/(r**2*sin(te))+df(vr,fi,2)/(r**2*sin(te)**2)
    -2*vr/r**2-2*df(sin(te)*vte,te)/(r**2*sin(te))-
    2*df(vfi,fi)/(r**2*sin(te)));

ssl2:=df(vte,t)+vr*df(vte,r)+vte*df(vte,te)/r+vfi*df(vte,fi)/(r*sin(te))+
    vte*vr/r-vfi**2*cos(te)/(r*sin(te))+df(p,te)/r-nu*
    (df(r**2*df(vte,r),r)/r**2+df(sin(te)*df(vte,te),te)/(r**2*sin(te))+
    df(vte,fi,2)/(r**2*sin(te)**2)+2*df(vr,te)/r**2-vte/(r*sin(te))**2-
    2*cos(te)*df(vfi,fi)/(r**2*sin(te)**2));

ssl3:=df(vfi,t)+vr*df(vfi,r)+vte*df(vfi,te)/r+vfi*df(vfi,fi)/(r*sin(te))+
    vfi*vr/r+vte*vfi*cos(te)/(r*sin(te))+df(p,fi)/(r*sin(te))-nu*
    (df(r**2*df(vfi,r),r)/r**2+df(sin(te)*df(vfi,te),te)/(r**2*sin(te))+
    df(vfi,fi,2)/(r**2*sin(te)**2)-vfi/(r**2*sin(te)**2)+
    2*df(vr,fi)/(r**2*sin(te))+2*cos(te)*df(vte,fi)/(r**2*sin(te)**2));

nodepend vr,te,fi;
depend H,t,r;
nodepend p,te,fi;
depend om,t,r,te,fi;
depend co,t,r,te,fi;
depend si,t,r,te,fi;

let sin(te)**2=1-cos(te)**2;
let sin(fi)**2=1-cos(fi)**2;
s2:=s2*r;
s3:=s3*sin(te)*r;

%co:=cos(om);
vfi:=W=H sin(om);
vte:=V=H cos(om);
vte:=H*co;
vfi:=H*si;
df(co,t):= -si*df(om,t);
df(co,r):= -si*df(om,r);
df(co,te):= -si*df(om,te);
df(co,fi):= -si*df(om,fi);
df(si,t):= co*df(om,t);
df(si,r):= co*df(om,r);
df(si,te):= co*df(om,te);
df(si,fi):= co*df(om,fi);
```

```
let si**2=1-co**2;

factor df(om,t),df(om,r),df(om,te),df(om,fi),
    df(om,t,2),df(om,r,t),df(om,t,te),df(om,fi,t),
    df(om,r,2),df(om,r,te),df(om,fi,r),
    df(om,te,2),df(om,fi,te),
    df(om,fi,2),
    df(vr,t),df(vr,r),df(vr,r,t),df(vr,r,2),df(vr,t,2),
    df(h,t),df(h,r),df(h,r,t),df(h,r,2),df(h,t,2),
    df(p,t),df(p,r),df(p,r,t),df(p,r,2),df(p,t,2);
    j:=df(s2,df(om,t));
    omt:=df(om,t):=df(om,t)-s2/j;
s2;
clear sin(te)**2;
let cos(te)**2=1-sin(te)**2;
off nat;
s3:=s3*si;
    j:=df(s4,df(om,te));
    omte:=df(om,te):=df(om,te)-s4/j;
s4;
s1:=s1;
s3;
clear df(om,te),df(om,t);
s2:=df(om,t)-omt;
s4:=s4;
s3:=s3/r/sin(te);
ss1 := - 4*nu*df(vr,r)*r**(-1) - nu*df(vr,r,2) - 2*nu*r**(-2)*vr +
    df(vr,t) + df(vr,r)*vr + df(p,r) - h**2*r**(-1);

ss2 := - nu*df(om,r)**2*co*si**(-1) - 2*nu*df(om,r)*df(h,r)*h**(-1) -
    2*nu*df(om,r)*r**(-1) - nu*df(om,te)**2*co*r**(-2)*si**(-1) -
    nu*df(om,te)*cos(te)*sin(te)**(-1)*r**(-2) - nu*df(om,fi)**2*
    sin(te)**(-2)*co*r**(-2)*si**(-1) - 2*nu*df(om,fi)*cos(te)*
    sin(te)**(-2)*co*r**(-2)*si**(-1) - nu*df(om,r,2) -
    nu*df(om,te,2)*r**(-2) - nu*df(om,fi,2)*sin(te)**(-2)*r**(-2) +
    2*nu*df(h,r)*co*h**(-1)*r**(-1)*si**(-1) +
    nu*df(h,r,2)*co*h**(-1)*si**(-1) - nu*sin(te)**(-2)*co*r**(-2)*si**(-1) +
    df(om,t) + df(om,r)*vr + df(om,te)*co*h*r**(-1) + df(om,fi)*
    sin(te)**(-1)*h*r**(-1)*si**(-1)*( - co**2 + 1) -
    df(h,t)*co*h**(-1)*si**(-1) -df(h,r)*co*h**(-1)*si**(-1)*vr +
    r**(-1)*si**(-1)*( - cos(te)*sin(te)**(-1)*co**2*h +
    cos(te)*sin(te)**(-1)*h - co*vr);

ss4 := - df(om,te)*h*r**(-1)*si + df(om,fi)*sin(te)**(-1)*co*h*r**(-1)
    + df(vr,r) + r**(-1)*(cos(te)*sin(te)**(-1)*co*h + 2*vr);
```

```
ss3 := nu*df(om,r)**2*h + nu*df(om,te)**2*h*r**(-2) + nu*df(om,fi)**2*
    sin(te)**(-2)*h*r**(-2) + 2*nu*df(om,fi)*cos(te)*sin(te)**(-2)*
    h*r**(-2) - 2*nu*df(h,r)*r**(-1) - nu*df(h,r,2) +
    nu*sin(te)**(-2)*h*r**(-2) +df(h,t) + df(h,r)*vr + h*r**(-1)*vr;

ss1-s1;
ss2-s2;
ss3-s3;
ss4-s4;

end;
```

A.2 The full program of viscous gas dynamics equations in spherical coordinates

The last part of the programs is devoted to obtaining equations for compatibility analysis after substitution of the representation of partially invariat solution.

```
operator Dgrmu,vu,gu,grmup,Dud,grmuup,grtemd,grkaup,gd,ga,k,qq,qqq; factor apta,bpta;
s1:=dv1dt+tau*grp1-tau*((la+mu)*grdivv1+mu*lapv1+grala1*divv+Dgrmu(1));
s2:=dv2dt+tau*grp2-tau*((la+mu)*grdivv2+mu*lapv2+grala2*divv+Dgrmu(2));
s3:=dv3dt+tau*grp3-tau*((la+mu)*grdivv3+mu*lapv3+grala3*divv+Dgrmu(3));
s4:=dtaudt-tau*divv;
s5:=dpdt+Apta*divv-Bpta*
(la*divv**2+mu*contrd+grkapgrtem+kap*laptem);
depend vr,r,te,fi,t; depend vte,r,te,fi,t;
depend vfi,r,te,fi,t; depend tau,r,te,fi,t; depend p,r,te,fi,t;
depend tem,r,te,fi,t; depend kap,r,te,fi,t;
depend la,r,te,fi,t; depend mu,r,te,fi,t; depend f,r,te,fi,t;
lap:=df(r**2*df(f,r),r)/r**2+df(sin(te)*df(f,te),te)/(r**2*sin(te))+
df(f,fi,2)/(r**2*sin(te)**2);
dd:=df(f,t)+vr*df(f,r)+vte*df(f,te)/r+vfi*df(f,fi)/(r*sin(te));
divv:=df(r**2*vr,r)/(r**2)+df(sin(te)*vte,te)/(r*sin(te))+
df(vfi,fi)/(r*sin(te));
grdivv1:=df(divv,r); grdivv2:=df(divv,te)/r**2;
grdivv3:=df(divv,fi)/(r*sin(te))**2;
k(1):=r; k(2):=te; k(3):=fi;
vu(1):=vr; vu(2):=vte/r; vu(3):=vfi/(r*sin(te));
gu(1,1):=1; gu(1,2):=0; gu(1,3):=0;
gu(2,1):=0; gu(2,2):=1/r**2; gu(2,3):=0;
gu(3,1):=0; gu(3,2):=0; gu(3,3):=1/(r**2*sin(te)**2);
gd(1,1):=1; gd(1,2):=0; gd(1,3):=0;
gd(2,1):=0; gd(2,2):=r**2; gd(2,3):=0;
gd(3,1):=0; gd(3,2):=0; gd(3,3):=r**2*sin(te)**2;
for l:=1:3 do for i:=1:3 do for j:=1:3 do ga(l,i,j):= for s:=1:3
sum gu(l,s)*(df(gd(i,s),k(j))+df(gd(j,s),k(i))-df(gd(i,j),k(s)))/2;
for l:=1:3 do for i:=1:3 do for j:=1:3 do if not
(ga(l,i,j)=0) then write ("ga(",l,",",i,",",j,") = ",ga(l,i,j));
for l:=1:3 do begin
ss1:=sub(f=vu(l),lap);
ss2:=for i:=1:3 sum for j:=1:3 sum for s:=1:3 sum
gu(i,j)*ga(l,i,s)*df(vu(s),k(j));
ss3:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
vu(pk)*gu(i,j)*df(ga(l,i,pk),k(j));
ss4:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
for s:=1:3 sum vu(pk)*gu(i,j)*ga(s,i,pk)*ga(l,j,s);
ss5:= for i:=1:3 sum for j:=1:3 sum for pk:=1:3 sum
for s:=1:3 sum vu(pk)*gu(i,j)*ga(s,i,j)*ga(l,pk,s);
```

```
qq(l):=ss1+2*ss2+ss3+ss4-ss5;
end; lapv1:=qq(1); lapv2:=qq(2); lapv3:=qq(3);
operator vud;
for j:=1:3 do for l:=1:3 do vud(j,l):=df(vu(j),k(l))+for s:=1:3
sum (ga(j,l,s)*vu(s));
for kk:=1:3 do for j:=1:3 do write (" vud(",kk,",",j,") = ",vud(kk,j));
divv-for kk:=1:3 sum vud(kk,kk);
for j:=1:3 do for kk:=1:3 do Dud(j,kk):=vud(j,kk)+for be:=1:3
sum for al:=1:3 sum (gu(j,be)*gd(kk,al)*vud(al,be));
for j:=1:3 do Dgrmu(j):=for al:=1:3 sum (Dud(j,al)*grmuup(al));
contrd:=(for j:=1:3 sum for kk:=1:3 sum (dud(j,kk)*dud(kk,j)))/2;
hh:=for j:=1:3 sum for kk:=1:3 sum vud(kk,j)*
(vud(j,kk)+for al:=1:3 sum for be:=1:3 sum
vud(al,be)*gu(j,be)*gd(kk,al));
hh-contrd; grdivv1:=df(divv,r); grdivv2:=df(divv,te)/r**2;
grdivv3:=df(divv,fi)/(r*sin(te))**2;
grmuup(1):=df(mu,r); grmuup(2):=df(mu,te)/r**2;
grmuup(3):=df(mu,fi)/(r*sin(te))**2;
grtemd(1):=df(tem,r); grtemd(2):=df(tem,te);
grtemd(3):=df(tem,fi);
grkaup(1):=df(kap,r); grkaup(2):=df(kap,te)/r**2;
grkaup(3):=df(kap,fi)/(r*sin(te))**2;
grkapgrtem:=for j:=1:3 sum grtemd(j)*grkaup(j);
laptem:=sub(f=tem,lap);
dv1dt:=sub(f=vr,dd)-(vte**2+vfi**2)/r;
dv2dt:=sub(f=vte,dd)/r+(vr*vte-(cos(te)/sin(te))*vfi**2)/r**2;
dv3dt:=sub(f=vfi,dd)/(r*sin(te))+(vr*vfi+(cos(te)/sin(te))*vfi*vte)/
(r**2*sin(te));
dtaudt:=sub(f=tau,dd); dpdt:=sub(f=p,dd);
grala1:=df(la,r); grala2:=df(la,te)/r**2;
grala3:=df(la,fi)/(r*sin(te))**2;
grp1:=df(p,r); grp2:=df(p,te)/r**2;
grp3:=df(p,fi)/(r*sin(te))**2;
factor df(tau,t),df(tau,r),df(tau,te),df(tau,fi);
factor df(p,t),df(p,r),df(p,te),df(p,fi);
factor df(vr,t),df(vr,r,2),df(vr,r,te),df(vr,r,fi),
df(vr,te,2),df(vr,te,fi),df(vr,fi,2);
factor df(vte,t),df(vte,r,2),df(vte,r,te),df(vte,r,fi),
df(vte,te,2),df(vte,te,fi),df(vte,fi,2);
factor df(vfi,t),df(vfi,r,2),df(vfi,r,te),df(vfi,r,fi),
df(vfi,te,2),df(vfi,te,fi),df(vfi,fi,2);
s1:=s1; s2:=r*s2; s3:=r*sin(te)*s3; s4:=s4; s5:=s5;
%let sin(te)**2=1-cos(te)**2;
on div; s1:=s1; s2:=s2; s3:=s3; s4:=s4; s5:=s5;
off div;
%******************** depend tem,p,tau; depend la,p,tau; depend mu,p,tau; depend kap,p,tau;
df(tem,r) :=df(tem,p)*df(p,r) +df(tem,tau)*df(tau,r);
df(tem,te):=df(tem,p)*df(p,te)+df(tem,tau)*df(tau,te);
df(tem,fi):=df(tem,p)*df(p,fi)+df(tem,tau)*df(tau,fi);
df(la,r) :=df(la,p)*df(p,r)+ df(la,tau)*df(tau,r);
df(la,te):=df(la,p)*df(p,te)+df(la,tau)*df(tau,te);
df(la,fi):=df(la,p)*df(p,fi)+df(la,tau)*df(tau,fi);
df(mu,r) :=df(mu,p)*df(p,r)+ df(mu,tau)*df(tau,r);
df(mu,te):=df(mu,p)*df(p,te)+df(mu,tau)*df(tau,te);
df(mu,fi):=df(mu,p)*df(p,fi)+df(mu,tau)*df(tau,fi);
df(kap,r) :=df(kap,p)*df(p,r)+ df(kap,tau)*df(tau,r);
df(kap,te):=df(kap,p)*df(p,te)+df(kap,tau)*df(tau,te);
df(kap,fi):=df(kap,p)*df(p,fi)+df(kap,tau)*df(tau,fi);
s1:=s1; s2:=s2; s3:=s3; s4:=s4; s5:=s5;
%********************
nodepend vr,te,fi; depend H,t,r; nodepend tau,te,fi;
nodepend p,te,fi; depend om,t,r,te,fi; depend co,t,r,te,fi;
depend si,t,r,te,fi;
let sin(te)**2=1-cos(te)**2; let sin(fi)**2=1-cos(fi)**2;
s1:=s1; s2:=s2; s3:=s3; s4:=s4; s5:=s5;
% co:=cos(om);
vfi:=W=H sin(om);
vte:=V=H cos(om);
```

```
vte:=H*co; vfi:=H*si; df(co,t):= -si*df(om,t); df(co,r):=-si*df(om,r);
df(co,te):= -si*df(om,te); df(co,fi):= -si*df(om,fi);
df(si,t):= co*df(om,t); df(si,r):= co*df(om,r);
df(si,te):= co*df(om,te); df(si,fi):= co*df(om,fi);

let si**2=1-co**2;
factor df(om,t),df(om,r),df(om,te),df(om,fi),
df(om,t,2),df(om,r,t),df(om,t,te),df(om,fi,t),
df(om,r,2),df(om,r,te),df(om,fi,r),
df(om,te,2),df(om,fi,te),
df(om,fi,2),
df(vr,t),df(vr,r),df(vr,r,t),df(vr,r,2),df(vr,t,2),
df(h,t),df(h,r),df(h,r,t),df(h,r,2),df(h,t,2),
df(p,t),df(p,r),df(p,r,t),df(p,r,2),df(p,t,2);
j:=df(s2,df(om,t)); omt:=df(om,t):=df(om,t)-s2/j; s2; clear
sin(te)**2; let cos(te)**2=1-sin(te)**2; off nat; s3:=s3*si;
j:=df(s4,df(om,te)); omte:=df(om,te):=df(om,te)-s4/j; s4;
s1:=s1; s5:=s5; s3; clear df(om,te),df(om,t);
s2:=df(om,t)-omt; s4:=s4; s5:=s5; s3:=s3/r/sin(te);
end;
```

Appendix B

Generators of the Group of Rotations in Spherical Coordinates

Here we rewrite the generators of rotations in spherical coordinates by using REDUCE. Let us define these generators as

```
EQ1 :  = v*Fw-w*Fv+y*Fz-z*Fy ;
EQ2 :  = u*Fw-w*Fu+x*Fz-z*Fx ;
EQ3 :  = u*Fv-v*Fu+x*Fy-y*Fx ;
```

The command

```
depend f,x,y,z,u,v,w,M,L,K,r,th,fi;
```

means that `f` depends on `x, y, z, u, v, w, M, L, K, r, th, fi`.

The commands

```
depend u, x, y, z ;
depend v, x, y, z ;
depend w, x, y, z ;
depend M, r, th, fi ;
depend L, r, th, fi ;
depend K, r, th, fi ;
```

describe that `u, v, w` depend on `x, y, z` and `M, L, K` depend on `r, th, fi`. Here `M`=U, `L`=V, `K`=W and they are defined by equations (3.54):

```
MM:= u*sin(th)*cos(fi)+v*sin(th)*sin(fi)+w*cos(th);
LL:= u*cos(th)*cos(fi)+v*cos(th)*sin(fi)-w*sin(th);
KK:= -u*sin(fi)+v*cos(fi);
```

For the spherical coordinates, we use

```
rr:=SQRT(x^2+y^2+z^2);
tht:=ATAN(SQRT(x^2+y^2+z^2);
fif:=ATAN(y/x);
```

For the calculations we use

```
a1:=df(rr,x); a2:=df(tht,x); a3:=df(fif,x);
a4:=df(MM,th)*a2+df(MM,fi)*a3;
a5:=df(LL,th)*a2+df(LL,fi)*a3;
a6:=df(KK,fi)*a3;
```

The command

```
Fx:=a1*df(f,r)+a2*df(f,th)+a3*df(f,fi)+a4*df(f,M)+
a5*df(f,L)+a6*df(f,K);
```

describes the application of the chain rule to the total derivative $D_x f$:

$$D_x f = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial f}{\partial M}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial L}\frac{\partial V}{\partial x} + \frac{\partial f}{\partial K}\frac{\partial W}{\partial x}.$$

By the same way we construct the derivatives $D_y f$ and $D_z f$:

```
Fy:=b1*df(f,r)+b2*df(f,th)+b3*df(f,fi)+b4*df(f,M)+
b5*df(f,L)+b6*df(f,K);
Fz:=c1*df(f,r)+c2*df(f,th)+c3*df(f,fi)+c4*df(f,M)+c5*df(f,L)+
c6*df(f,K);
```

Here

```
b1:=df(rr,y); b2:=df(tht,y); b3:=df(fif,y);
b4:=df(MM,th)*b2+df(MM,fi)*b3;
b5:=df(LL,th)*b2+df(LL,fi)*b3;
b6:=df(KK,fi)*b3;
c1:=df(rr,z); c2:=df(tht,z); c3:=df(fif,z);
c4:=df(MM,th)*c2+df(MM,fi)*c3;
c5:=df(LL,th)*c2+df(LL,fi)*c3;
c6:=df(KK,fi)*c3;
```

Similarly for the total derivatives $D_u f, D_r f$ and $D_w f$:

```
Fu:=d1*df(f,r)+d2*df(f,th)+d3*df(f,fi)+d4*df(f,M)+
d5*df(f,L)+d6*df(f,K);
Fv:=e1*df(f,r)+e2*df(f,th)+e3*df(f,fi)+e4*df(f,M)+
e5*df(f,L)+e6*df(f,K);
Fw:=f1*df(f,r)+f2*df(f,th)+f3*df(f,fi)+f4*df(f,M)+
f5*df(f,L)+f6*df(f,K);
```

where the coefficients `di`, `ei` and `fi`, (i=1,2...,6) are

```
d1:=df(rr,u);d2:=df(tht,u);d3:=df(fif,u);
d4:=df(MM,u); d5:=df(LL,u);d6:=df(KK,u);
e1:=df(rr,v);e2:=df(tht,v);e3:=df(fif,v);
e4:=df(MM,v); e5:=df(LL,v);e6:=df(KK,v);
f1:=df(rr,w);f2:=df(tht,w);f3:=df(fif,w);
f4:=df(MM,w); f5:=df(LL,w);f6:=df(KK,w);
```

The full program can be written as follows:

```
EQ1:=v*Fw-w*Fv+y*Fz-z*Fy;

EQ2:=u*Fw-w*Fu+x*Fz-z*Fx;

EQ3:=u*Fv-v*Fu+x*Fy-y*Fx;

depend f,x,y,z,u,v,w,M,L,K,r,th,fi;

ssm:=MM;

ssL:=LL;

ssK:=KK;

MM:= u*sin(th)*cos(fi)+v*sin(th)*sin(fi)+w*cos(th);

LL:= u*cos(th)*cos(fi)+v*cos(th)*sin(fi)-w*sin(th);
```

```
KK:= -u*sin(fi)+v*cos(fi);
rr:= SQRT (x**2+y**2+z**2);
tht:=ATAN (SQRT(x**2+y**2)/z);
fif:=ATAN (y/x);
a1:=df(rr,x);
a2:=df(tht,x);
a3:=df(fif,x);
a4:=df(MM,th)*a2+df(MM,fi)*a3;
a5:=df(LL,th)*a2+df(LL,fi)*a3;
a6:=df(KK,fi)*a3;
b1:=df(rr,y);
b2:=df(tht,y);
b3:=df(fif,y);
b4:=df(MM,th)*b2+df(MM,fi)*b3;
b5:=df(LL,th)*b2+df(LL,fi)*b3;
b6:=df(KK,fi)*b3;
c1:=df(rr,z);
c2:=df(tht,z);
c3:=df(fif,z);
c4:=df(MM,th)*c2+df(MM,fi)*c3;
c5:=df(LL,th)*c2+df(LL,fi)*c3;
c6:=df(KK,fi)*c3;
Fx:=a1*df(f,r)+a2*df(f,th)+a3*df(f,fi)+a4*df(f,M)+a5*df(f,L)+a6*df(f,K);
Fy:=b1*df(f,r)+b2*df(f,th)+b3*df(f,fi)+b4*df(f,M)+b5*df(f,L)+b6*df(f,K);
Fz:=c1*df(f,r)+c2*df(f,th)+c3*df(f,fi)+c4*df(f,M)+c5*df(f,L)+c6*df(f,K);
d1:=df(rr,u);
d2:=df(tht,u);
d3:=df(fif,u);
d4:=df(MM,u);
d5:=df(LL,u);
d6:=df(KK,u);
e1:=df(rr,v);
e2:=df(tht,v);
e3:=df(fif,v);
e4:=df(MM,v);
e5:=df(LL,v);
e6:=df(KK,v);
f1:=df(rr,w);
f2:=df(tht,w);
f3:=df(fif,w);
f4:=df(MM,w);
f5:=df(LL,w);
f6:=df(KK,w);
Fu:=d1*df(f,r)+d2*df(f,th)+d3*df(f,fi)+d4*df(f,M)+d5*df(f,L)+d6*df(f,K);
Fv:=e1*df(f,r)+e2*df(f,th)+e3*df(f,fi)+e4*df(f,M)+e5*df(f,L)+e6*df(f,K);
```

```
Fw:=f1*df(f,r)+f2*df(f,th)+f3*df(f,fi)+f4*df(f,M)+f5*df(f,L)+f6*df(f,K);

factor df(f,r),df(f,fi),df(f,th),df(f,M),df(f,L),df(f,K);

EQ1:=v*Fw-w*Fv+y*Fz-z*Fy;
EQ2:=u*Fw-w*Fu+x*Fz-z*Fx;
EQ3:=u*Fv-v*Fu+x*Fy-y*Fx;
x:=r*sin(th)*cos(fi);
y:=r*sin(th)*sin(fi);
z:=r*cos(th);
u:=M*sin(th)*cos(fi)+L*cos(th)*cos(fi)-K*sin(fi);
v:=M*sin(th)*sin(fi)+L*cos(th)*sin(fi)+K*cos(fi);
w:=M*cos(th)-L*sin(th);
EQ1:=v*Fw-w*Fv+y*Fz-z*Fy;
EQ2:=u*Fw-w*Fu+x*Fz-z*Fx;
EQ3:=u*Fv-v*Fu+x*Fy-y*Fx;
let sin(fi)**2=1-cos(fi)**2, sin(th)**2=1-cos(th)**2;
EQ1:=EQ1;
EQ2:=EQ2;
EQ3:=EQ3;

clear sin(th)**2,sin(fi)**2;

let cos(fi)**2=1-sin(fi)**2, cos(th)**2=1-sin(th)**2;
EQ1:=EQ1;
EQ2:=EQ2;
EQ3:=EQ3;
ssm:=ssm;
ssl:=ssl;
ssk:=ssk;
end;
```

Appendix C

Program for Analysis of Compatibility

Here we describe the program for compatibility analysis of partially invariant solutions by using REDUCE.

The commands

```
depend vr,t,r;
depend H,t,r;
depend p,t,r;
depend om,t,r,te,fi;
depend co,t,r,te,fi;
depend si,t,r,te,fi;
```

mean that `vr`, `H`, `p`, depend on `t`, and `r`, the functions `om`, `co` and `si` depend on `t`,`r`,`te` and `fi` where we use identificators `om`= ω, `cos`= $\cos\omega$, `si`= $\sin\omega$ and `vr`= U.

The commands

```
df(co,t):= -si*df(om,t);
df(co,r):= -si*df(om,r);
df(co,te):= -si*df(om,te);
df(co,fi):= -si*df(om,fi);
df(si,t):= co*df(om,t);
df(si,r):= co*df(om,r);
df(si,te):= co*df(om,te);
df(si,fi):= co*df(om,fi);
```

describe the derivatives of the functions $\cos\omega$ and $\sin\omega$ with respect to `t`,`r`,`te` and fi.

The expressions

```
   s1 := - 4*nu*df(vr,r)*r**(-1) - nu*df(vr,r,2) -2*nu*r**(-2)*vr+
df(vr,t) + df(vr,r)*vr + df(p,r) - h**2*r**(-1);

s2 := - nu*df(om,r)**2*co*si**(-1) -2*nu*df(om,r)*df(h,r)*h**(-1)
- 2 *nu*df(om,r)*r**(-1) -nu*df(om,te)**2*co*r**(-2)*si**(-1) -
nu*df(om,te)*cos(te)*sin(te)**(-1)*r**(-2) - nu*df(om,fi)**2*
sin(te)**(-2)* co*r**(-2)*si**(-1) -2*nu*df(om,fi)*cos(te)*
sin(te)**(-2)*co*r**(-2)* si**(-1) -nu*df(om,r,2) - nu*
```

```
df(om,te,2)*r**(-2) - nu*df(om,fi,2)*sin(te)**(-2)*r**(-2)+
2*nu*df(h,r)*co*h**(-1)*r**(-1)*si**(-1) + nu*df(h,r,2)*co*h**(-1)*
si**(-1) - nu*sin(te)**(-2)*co*r**(-2)*si**(-1)+ df(om,t) +
df(om,r)*vr +df(om,te)*co*h*r**(-1) + df(om,fi)*sin(te)**(-1)*
h*r**(-1)*si**(-1)*( - co**2 + 1) - df(h,t)*co*h**(-1)*si**(-1) -
df(h,r)*co*h**(-1)*si**(-1)*vr +r**(-1)*si**(-1)*(-cos(te)*
sin(te)** (-1)*co**2*h +cos(te)*sin(te)**(-1)*h - co*vr);

s3 := nu*df(om,r)**2*h + nu*df(om,te)**2*h*r**(-2) + nu*
df(om,fi)**2*sin(te)**(-2)*h*r**(-2) + 2*nu*df(om,fi)*
cos(te)*sin(te)**(-2)*h*r**(-2) - 2*nu*df(h,r)*r**(-1) -
nu*df(h,r,2) + nu*sin(te)**(-2)*h*r**(-2) +df(h,t) + df(h,r)*vr
+ h*r**(-1)*vr;
s4 := - df(om,te)*h*r**(-1)*si +df(om,fi)*sin(te)**(-1)*
co*h*r**(-1) + df(vr,r) +r**(-1)*(cos(te)*sin(te)**(-1)*co*h +
2*vr);
```

represent the left sides of equations (4.5), (4.6), (4.7) and (4.8), which are obtained after substituting the representation of solution (4.4) into the Navier-Stokes equations.

Let us explain an example of a computational trick for solving the linear equation. The equation $S3 = 0$ is linear with respect to the derivative H_t`df(h,t)`. On the first step, one can find the coefficient

```
j:=df(s3,df(h,t));
```

On the second step the value of the derivative is defined

```
df(h,t):=df(h,t) - s3/j.
```

For the sake of simplicity, we use the function $h1 = rH$ instead of $H(t,r)$. It is introduced by the commands

```
depend h1,t,r;
h:=h1/r;
h1t:=df(h1,t):=r*ht;
```

Substitutions of the derivatives (4.10) have to be done in two steps.

The second derivatives $\omega_{rr}, \omega_{\theta\theta}, \omega_{\varphi\varphi}$:

```
df(om,r,2):=-(df(F,r,2)+2*df(F,o,r)*df(om,r)+df(F,o,2)*
df(om,r)**2)/df(F,o);
df(om,te,2):=-(df(F,te,2)+2*df(F,o,te)*df(om,te)+df(F,o,2)*
df(om,te)**2)/df(F,o);
df(om,fi,2):=-(df(F,fi,2)+2*df(F,o,fi)*df(om,fi)+df(F,o,2)*
df(om,fi)**2)/df(F,o);
```

have to be substituted in the expressions `s1, s2, s4` before the first derivatives, because the recursive algorithm is used in REDUCE. After that, the expressions `s1, s2, s4` have to be recalculated. On the next step, the first derivatives $\omega_t, \omega_r, \omega_\theta, \omega_\varphi$:

```
df(om,t):=-df(F,t)/df(F,o);
df(om,r):=-df(F,r)/df(F,o); df(om,te):=-df(F,te)/df(F,o);
df(om,fi):=-df(F,fi)/df(F,o);
```

are substituted in expressions for `s1, s2, s4`.
For the suitable calculation, we define $a = \dfrac{rU_r + 2U}{h}$ by using the commands

```
depend a,t,r;
df(vr,r):=(a*h-2*vr)/r;
```

For the functions ϕ in the calculation, we use the identificator `ps`.

```
depend ps,t,r,be,la;
```

Expressions of y_1 and y_2 are defined by the identificators `bbe` and `lla`

```
lla:=cos(o)*sin(te)-a*cos(te);
```

correspondingly. Since there is no explicit expression of the `bbl`, we define only the derivatives

```
depend bbe,t,r,te,fi,o; df(bbe,t):=df(arbe,t)/(1+arbe**2);
df(bbe,r):=df(arbe,r)/(1+arbe**2); df(bbe,t)/df(bbe,r);
df(bbe,te):=df(arbe,te)/(1+arbe**2); df(bbe,fi):=1;
df(bbe,o):=df(arbe,o)/(1+arbe**2);
```

where for the sake of simplicity, we use the idetificator

```
arbe:=sin(o)/(a*sin(te)+cos(o)*cos(te));
```

The second derivatives of the function F expressed through the function ϕ are defined by the commands.

```
   df(f,r,2):= df(bbe,r)*(df(ps,be,2)*df(bbe,r)+df(ps,be,la)*
   df(lla,r)+df(ps,be,r))+df(ps,be)*df(bbe,r,2)+df(lla,r)*
   (df(ps,be,la)*df(bbe,r)+df(ps,la,2)*df(lla,r)+df(ps,la,r))+
   df(ps,la)*df(lla,r,2)+df(ps,be,r)*df(bbe,r)+df(ps,la,r)*
   df(lla,r)+df(ps,r,2);

df(f,te,2):= df(bbe,te)*(df(ps,be,2)*df(bbe,te)+df(ps,be,la)*
   df(lla,te))+df(ps,be)*df(bbe,te,2)+df(lla,te)*
   (df(ps,be,la)*df(bbe,te)+df(ps,la,2)*df(lla,te))+
   df(ps,la)*df(lla,te,2);

df(f,fi,2):= df(bbe,fi)*(df(ps,be,2)*df(bbe,fi)+df(ps,be,la)*
   df(lla,fi))+ df(ps,be)*df(bbe,fi,2)+df(lla,fi)*
   (df(ps,be,la)*df(bbe,fi)+df(ps,la,2)*df(lla,fi))+
   df(ps,la)*df(lla,fi,2);

df(f,o,2):= df(bbe,o)*(df(ps,be,2)*df(bbe,o)+df(ps,be,la)*
   df(lla,o))+df(ps,be)*df(bbe,o,2)+df(lla,o)*(df(ps,be,la)*
   df(bbe,o)+df(ps,la,2)*df(lla,o))+ df(ps,la)*df(lla,o,2);

df(f,o,r):= df(bbe,o)*(df(ps,be,2)*df(bbe,r)+df(ps,be,la)*
```

```
   df(lla,r)+df(ps,be,r))+df(ps,be)*df(bbe,o,r)+df(lla,o)*
   (df(ps,be,la)*df(bbe,r)+df(ps,la,2)*df(lla,r)+df(ps,la,r))+
   df(ps,la)*df(lla,o,r);

df(f,o,te):= df(bbe,o)*(df(ps,be,2)*df(bbe,te)+df(ps,be,la)*
   df(lla,te))+df(ps,be)*df(bbe,o,te)+df(lla,o)*(df(ps,be,la)*
   df(bbe,te)+df(ps,la,2)*df(lla,te))+df(ps,la)*df(lla,o,te);

df(f,o,fi):= df(bbe,o)*(df(ps,be,2)*df(bbe,fi)+df(ps,be,la)*
   df(lla,fi))+df(ps,be)*df(bbe,fi,o)+df(lla,o)*(df(ps,be,la)*
   df(bbe,fi)+df(ps,la,2)*df(lla,fi))+df(ps,la)*df(lla,fi,o);
```

According to the REDUCE in order to prevent mistakes, the substitution of the second derivatives has to be done before the substitution of the first derivatives, which are given by the commands

```
df(f,t):=df(ps,be)*df(bbe,t)+df(ps,la)*df(lla,t)+df(ps,t);
df(f,r):=df(ps,be)*df(bbe,r)+df(ps,la)*df(lla,r)+df(ps,r);
df(f,te):=df(ps,be)*df(bbe,te)+df(ps,la)*df(lla,te);
df(f,fi):=df(ps,be)*df(bbe,fi)+df(ps,la)*df(lla,fi);
df(f,o):=df(ps,be)*df(bbe,o)+df(ps,la)*df(lla,o);
```

After the substitutions of the second and first order derivatives in the equation `s3=0`, this equation becomes linear with respect to the $\sin\omega$. The coefficients G_1 and G_2 in equations (4.11) are defined by the commands

```
ko1:=df(s3,sin(o));
ko2:=s3-ko1*sin(o);
```

Then by using trigonometry, we have equation (4.12) which is represented by the command

```
s3:=ko1**2*(1-cos(o)**2)-ko2**2;
```

Here the identificator $S3$ corresponds to the polynomial P_8 (4.13). The coefficient of this polynomial are defined by the commands

```
ss8:=df(s3,cos(te),8)/(8*7*6*5*4*3*2);
s3:=s3-ss8*cos(te)**8;
ss7:=df(s3,cos(te),7)/(7*6*5*4*3*2);
s3:=s3-ss7*cos(te)**7;
ss6:=df(s3,cos(te),6)/(6*5*4*3*2);
s3:=s3-ss6*cos(te)**6;
ss5:=df(s3,cos(te),5)/(5*4*3*2);
s3:=s3-ss5*cos(te)**5;
ss4:=df(s3,cos(te),4)/(4*3*2);
s3:=s3-ss4*cos(te)**4;
ss3:=df(s3,cos(te),3)/(3*2);
s3:=s3-ss3*cos(te)**3;
ss2:=df(s3,cos(te),2)/2;
s3:=s3-ss2*cos(te)**2;
ss1:=df(s3,cos(te));
ss0:=s3-ss1*cos(te);
```

The result of this calculations is the equation (4.16)

$$k_r = 0.$$

Substituting this into the first equation ($s2$ in the program) and splitting it with respect to $\cos(\theta)$ is given by the following:

```
off factor;
s2:=num s2;
ko1:=df(s2,sin(o));
ko2:=s2-ko1*sin(o);
s2:=ko1**2*(1-cos(o)**2)-ko2**2;
ss:=df(s2,sin(te));
ss10:=df(s2,cos(te),10)/(10*9*8*7*6*5*4*3*2);
s2:=s2-ss10*cos(te)**10;
ss9:=df(s2,cos(te),9)/(9*8*7*6*5*4*3*2);
s2:=s2-ss9*cos(te)**9;
ss8:=df(s2,cos(te),8)/(8*7*6*5*4*3*2);
s2:=s2-ss8*cos(te)**8;
ss7:=df(s2,cos(te),7)/(7*6*5*4*3*2);
s2:=s2-ss7*cos(te)**7;
ss6:=df(s2,cos(te),6)/(6*5*4*3*2);
s2:=s2-ss6*cos(te)**6;
ss5:=df(s2,cos(te),5)/(5*4*3*2);
s2:=s2-ss5*cos(te)**5;
ss4:=df(s2,cos(te),4)/(4*3*2);
s2:=s2-ss4*cos(te)**4;
ss2:=df(s2,cos(te),3)/(3*2);
s2:=s2-ss2*cos(te)**3;
ss2:=df(s2,cos(te),2)/2;
s2:=s2-ss2*cos(te)**2;
ss1:=df(s2,cos(te));
ss0:=s2-ss1*cos(te);
ss10;
```

The coefficient $b10 = 0$ ($ss10$ in the program) gives

$$k_t = r^{-2}h(k^2 + 1).$$

For checking it, we use

```
df(a,r):=(a**2+1)*h1/r**2;
ss10;
```

The full program can be written as the following:

```
depend vr,t,r;
depend H,t,r;
depend p,t,r;
depend om,t,r,te,fi;
depend co,t,r,te,fi;
depend si,t,r,te,fi;
df(co,t):= -si*df(om,t);
```

```
df(co,r):= -si*df(om,r);
df(co,te):= -si*df(om,te);
df(co,fi):= -si*df(om,fi);
df(si,t):= co*df(om,t);
df(si,r):= co*df(om,r);
df(si,te):= co*df(om,te);
df(si,fi):= co*df(om,fi);

let si**2=1-co**2;

factor df(om,t),df(om,r),df(om,te),df(om,fi),
    df(om,t,2),df(om,r,t),df(om,t,te),df(om,fi,t),
    df(om,r,2),df(om,r,te),df(om,fi,r),
    df(om,te,2),df(om,fi,te),
    df(om,fi,2),
    df(vr,t),df(vr,r),df(vr,r,t),df(vr,r,2),df(vr,t,2),
    df(h,t),df(h,r),df(h,r,t),df(h,r,2),df(h,t,2),
    df(p,t),df(p,r),df(p,r,t),df(p,r,2),df(p,t,2);

s1 := - 4*nu*df(vr,r)*r**(-1) - nu*df(vr,r,2) - 2*nu*r**(-2)*vr +
    df(vr,t) + df(vr,r)*vr + df(p,r) - h**2*r**(-1);

s2 := - nu*df(om,r)**2*co*si**(-1) - 2*nu*df(om,r)*df(h,r)*h**(-1) - 2
    *nu*df(om,r)*r**(-1) - nu*df(om,te)**2*co*r**(-2)*si**(-1) -
    nu*df(om,te)*cos(te)*sin(te)**(-1)*r**(-2) - nu*df(om,fi)**2*
    sin(te)**(-2)*co*r**(-2)*si**(-1) - 2*nu*df(om,fi)*
    cos(te)*sin(te)**(-2)*co*r**(-2)*si**(-1) - nu*df(om,r,2) -
    nu*df(om,te,2)*r**(-2) - nu*df(om,fi,2)*sin(te)**(-2)*
    r**(-2) + 2*nu*df(h,r)*co*h**(-1)*r**(-1)*si**(-1) +
    nu*df(h,r,2)*co*h**(-1)*si**(-1) - nu*sin(te)**(-2)*
    co*r**(-2)*si**(-1) +df(om,t) + df(om,r)*vr + df(om,te)*
    co*h*r**(-1) + df(om,fi)*sin(te)**(-1)*h*r**(-1)*si**(-1)*
    (- co**2 + 1) - df(h,t)*co*h**(-1)*si**(-1) -df(h,r)*
    co*h**(-1)*si**(-1)*vr + r**(-1)*si**(-1)*( - cos(te)*
    sin(te)**(-1)*co**2*h + cos(te)*sin(te)**(-1)*h - co*vr);

s4 := - df(om,te)*h*r**(-1)*si + df(om,fi)*sin(te)**(-1)*co*h*r**(-1)
    + df(vr,r) + r**(-1)*(cos(te)*sin(te)**(-1)*co*h + 2*vr);

s3 := nu*df(om,r)**2*h + nu*df(om,te)**2*h*r**(-2) + nu*df(om,fi)**2*
    sin(te)**(-2)*h*r**(-2) + 2*nu*df(om,fi)*cos(te)*sin(te)**(-2)*
    h*r**(-2) - 2*nu*df(h,r)*r**(-1) - nu*df(h,r,2) + nu*sin(te)**(-2)*
    h*r**(-2) +df(h,t) + df(h,r)*vr + h*r**(-1)*vr;

j:=df(s3,df(h,t));
ht:=df(h,t):=df(h,t)-s3/j;
on div;
depend h1,t,r;
h:=h1/r;
h1t:=df(h1,t):=r*ht;
clear si**2;
let co**2=1-si**2;
s2:=s2;
s4:=s4;
clear df(h1,t);
s3:=s3/h1;
s4:=r**2*s4/h1;
depend F,o,t,r,te,fi;
df(om,t,2):=-(df(F,t,2)+2*df(F,o,t)*df(om,t)+df(F,o,2)*df(om,t)**2)/df(F,o);
df(om,r,2):=-(df(F,r,2)+2*df(F,o,r)*df(om,r)+df(F,o,2)*df(om,r)**2)/df(F,o);
df(om,te,2):=-(df(F,te,2)+2*df(F,o,te)*df(om,te)+df(F,o,2)*df(om,te)**2)/ df(F,o);
df(om,fi,2):=-(df(F,fi,2)+2*df(F,o,fi)*df(om,fi)+df(F,o,2)*df(om,fi)**2)/ df(F,o);
s1:=s1;
s2:=s2;
s3:=s3;
s4:=s4;
df(om,t):=-df(F,t)/df(F,o);
df(om,r):=-df(F,r)/df(F,o);
```

```
df(om,te):=-df(F,te)/df(F,o);
df(om,fi):=-df(F,fi)/df(F,o);
si:=sin(o);
co:=cos(o);
s1:=s1;
depend a,t,r;
df(vr,r):=(a*h-2*vr)/r;
s1;
s2:=s2*df(F,o)**3;
s3:=s3*df(F,o)**2;
s4:=s4*df(F,o);

    lla:=cos(o)*sin(te)-a*cos(te);
arbe:=sin(o)/(a*sin(te)+cos(o)*cos(te));
depend bbe,t,r,te,fi,o;
df(bbe,t):=df(arbe,t)/(1+arbe**2);
df(bbe,r):=df(arbe,r)/(1+arbe**2);
df(bbe,t)/df(bbe,r);
df(bbe,te):=df(arbe,te)/(1+arbe**2);
df(bbe,fi):=1;
df(bbe,o):=df(arbe,o)/(1+arbe**2);

depend ps,t,r,be,la;
df(f,r,2):= df(bbe,r)*(df(ps,be,2)*df(bbe,r)+
    df(ps,be,la)*df(lla,r)+df(ps,be,r))+
    df(ps,be)*df(bbe,r,2)+df(lla,r)*(df(ps,be,la)*df(bbe,r)+
    df(ps,la,2)*df(lla,r)+df(ps,la,r))+ df(ps,la)*df(lla,r,2)+
    df(ps,be,r)*df(bbe,r)+df(ps,la,r)*df(lla,r)+df(ps,r,2);
df(f,te,2):= df(bbe,te)*(df(ps,be,2)*df(bbe,te)+df(ps,be,la)*df(lla,te))+
    df(ps,be)*df(bbe,te,2)+ df(lla,te)*(df(ps,be,la)*df(bbe,te)+
    df(ps,la,2)*df(lla,te))+ df(ps,la)*df(lla,te,2);
df(f,fi,2):= df(bbe,fi)*(df(ps,be,2)*df(bbe,fi)+df(ps,be,la)*df(lla,fi))+
    df(ps,be)*df(bbe,fi,2)+ df(lla,fi)*(df(ps,be,la)*df(bbe,fi)+
    df(ps,la,2)*df(lla,fi))+ df(ps,la)*df(lla,fi,2);
    df(f,o,2):= df(bbe,o)*(df(ps,be,2)*df(bbe,o)+df(ps,be,la)*df(lla,o))+
    df(ps,be)*df(bbe,o,2)+ df(lla,o)*(df(ps,be,la)*df(bbe,o)+
    df(ps,la,2)*df(lla,o))+ df(ps,la)*df(lla,o,2);
df(f,o,r):=
df(bbe,o)*(df(ps,be,2)*df(bbe,r)+df(ps,be,la)*df(lla,r)+df(ps,be,r))+
    df(ps,be)*df(bbe,o,r)+
df(lla,o)*(df(ps,be,la)*df(bbe,r)+df(ps,la,2)*df(lla,r)+df(ps,la,r))+
    df(ps,la)*df(lla,o,r);
df(f,o,te):=
df(bbe,o)*(df(ps,be,2)*df(bbe,te)+df(ps,be,la)*df(lla,te))+
    df(ps,be)*df(bbe,o,te)+
df(lla,o)*(df(ps,be,la)*df(bbe,te)+df(ps,la,2)*df(lla,te))+
    df(ps,la)*df(lla,o,te);
df(f,o,fi):=
df(bbe,o)*(df(ps,be,2)*df(bbe,fi)+df(ps,be,la)*df(lla,fi))+
    df(ps,be)*df(bbe,fi,o)+
df(lla,o)*(df(ps,be,la)*df(bbe,fi)+df(ps,la,2)*df(lla,fi))+
    df(ps,la)*df(lla,fi,o);

s1:=s1;
s2:=s2;
s3:=s3;
s4:=s4;
df(f,t):=df(ps,be)*df(bbe,t)+df(ps,la)*df(lla,t)+df(ps,t);
df(f,r):=df(ps,be)*df(bbe,r)+df(ps,la)*df(lla,r)+df(ps,r);
df(f,te):=df(ps,be)*df(bbe,te)+df(ps,la)*df(lla,te);
df(f,fi):=df(ps,be)*df(bbe,fi)+df(ps,la)*df(lla,fi);
df(f,o):=df(ps,be)*df(bbe,o)+df(ps,la)*df(lla,o);


    s1:=s1;

s2:=s2;

s3:=s3;
```

```
s4:=s4;

let sin(o)**2=1-cos(o)**2;
let sin(te)**2=1-cos(te)**2;

s1:=s1;
s2:=num s2;
s3:=num s3;
s4:=s4;

cos(o):=(la+a*cos(te))/sin(te);
factor cos(te);
s2:=num s2;
s3:=num s3;
ko1:=df(s3,sin(o));
ko2:=s3-ko1*sin(o);
s3:=ko1**2*(1-cos(o)**2)-ko2**2;

ss:=df(s3,sin(te));
ss8:=df(s3,cos(te),8)/(8*7*6*5*4*3*2);
s3:=s3-ss8*cos(te)**8;
ss7:=df(s3,cos(te),7)/(7*6*5*4*3*2);
s3:=s3-ss7*cos(te)**7;
ss6:=df(s3,cos(te),6)/(6*5*4*3*2);
s3:=s3-ss6*cos(te)**6;
ss5:=df(s3,cos(te),5)/(5*4*3*2);
s3:=s3-ss5*cos(te)**5;
ss4:=df(s3,cos(te),4)/(4*3*2);
s3:=s3-ss4*cos(te)**4;
ss3:=df(s3,cos(te),3)/(3*2);
s3:=s3-ss3*cos(te)**3;
ss2:=df(s3,cos(te),2)/2;
s3:=s3-ss2*cos(te)**2;
ss1:=df(s3,cos(te));
ss0:=s3-ss1*cos(te);

on factor;
df(f,o);
depend qq,t,r,be,la;
let df(ps,be)**2=-( df(ps,la)**2*a**4-2*df(ps,la)**2*a**2*la**2-2*df(ps,la)**2*la**2+
    2*df(ps,la)**2*a**2+df(ps,la)**2*la**4+df(ps,la)**2)/(a**2+1)+qq;
ss8:=ss8/qq;
ss7:=ss7/qq;
ss6:=ss6/qq;
ss5:=ss5/qq;
```

```
ss4:=ss4/qq;
ss3:=ss3/qq;
ss2:=ss2/qq;
ss1:=ss1/qq;
ss0:=ss0/qq;
ss8;
df(h1,t):=-vr*df(h1,r)+nu*(df(h1,r,2)+h1*df(a,r)**2/(a**2+1));
ss8;
ss7;
ss6:=ss6;
% first case df(a,r) neq 0
df(ps,r):=-df(ps,la)*a*la*df(a,r)/(a**2+1);
ss6;
ss5;
ss4;
% it's contradiction:  a**2+1=0
clear df(ps,r);
clear df(ps,be)**2;
qq:= df(ps,be)**2+(df(ps,la)**2*a**4-2*df(ps,la)**2*a**2*la**2-
    2*df(ps,la)**2*la**2+2*df(ps,la)**2*a**2+
    df(ps,la)**2*la**4+df(ps,la)**2)/(a**2+1);

df(a,r):=0;
ss6;
ss5;
ss4;
off factor;
s2:=num s2;
ko1:=df(s2,sin(o));
ko2:=s2-ko1*sin(o);
s2:=ko1**2*(1-cos(o)**2)-ko2**2;

ss:=df(s2,sin(te));
ss10:=df(s2,cos(te),10)/(10*9*8*7*6*5*4*3*2);
s2:=s2-ss10*cos(te)**10;
ss9:=df(s2,cos(te),9)/(9*8*7*6*5*4*3*2);
s2:=s2-ss9*cos(te)**9;
ss8:=df(s2,cos(te),8)/(8*7*6*5*4*3*2);
s2:=s2-ss8*cos(te)**8;
ss7:=df(s2,cos(te),7)/(7*6*5*4*3*2);
s2:=s2-ss7*cos(te)**7;
ss6:=df(s2,cos(te),6)/(6*5*4*3*2);
s2:=s2-ss6*cos(te)**6;
ss5:=df(s2,cos(te),5)/(5*4*3*2);
s2:=s2-ss5*cos(te)**5;
```

```
ss4:=df(s2,cos(te),4)/(4*3*2);
s2:=s2-ss4*cos(te)**4;
ss2:=df(s2,cos(te),3)/(3*2);
s2:=s2-ss2*cos(te)**3;
ss2:=df(s2,cos(te),2)/2;
s2:=s2-ss2*cos(te)**2;
ss1:=df(s2,cos(te));
ss0:=s2-ss1*cos(te);
on factor;
ss10;
df(a,t) := (a**2 + 1)*h1/r**2;
depend cc,t;
h1:=3*cc*r**2;
ss10;
end;
```

Appendix D

Program of Deriving Determining Equations

```
nx:=2; nu:=3; nua:=0; off echo;
operator s,sk,u,ua,uu,uua,x,rru,ffu,ffua,fu,fua,zu,zua,kx;
algebraic procedure gkk(m,l,q);
begin h:=1; if m=l and l=q then h:=6 else if l=q or m=l or m=q
then h:=2; return h; end;algebraic procedure gkkk(k,m,l,q);
begin h:=1; if k=m then begin
    if gkk(m,l,q)=6 then h:=24 else
    if gkk(m,l,q)=2 then <<if m=l then h:=6 else
h:=4; >>else h:=2; end else h:=gkk(m,l,q); return h; end;
for ju:=1:nu do for ku:=1:nu do depend zu(ju),u(ku);
for ju:=1:nu do for kxn:=1:nx do depend zu(ju),x(kxn);
for jx:=1:nx do for ku:=1:nu do depend kx(jx),u(ku);
for jx:=1:nx do for kxn:=1:nx do depend kx(jx),x(kxn);
for ju:=1:nu do for ku:=1:nu do factor df(zu(ju),u(ku));
for ju:=1:nu do for kxn:=1:nx do factor df(zu(ju),x(kxn));
for jx:=1:nx do for ku:=1:nu do factor df(kx(jx),u(ku));
for jx:=1:nx do for kxn:=1:nx do factor df(kx(jx),x(kxn));
for ju:=1:nu do factor zu(ju); for jx:=1:nx do factor kx(jx);

if not(nua=0) then begin
for ju:=1:nu do for kua:=1:nua do depend zu(ju),ua(kua);
for jua:=1:nua do for ku:=1:nu do depend zua(jua),u(ku);
for jua:=1:nua do for kxn:=1:nx do depend zua(jua),x(kxn);
for jua:=1:nua do for kua:=1:nua do depend zua(jua),ua(kua);
for jx:=1:nx do for kua:=1:nua do depend kx(jx),ua(kua);
for ju:=1:nu do for kua:=1:nua do factor df(zu(ju),ua(kua));
for jua:=1:nua do for ku:=1:nu do factor df(zua(jua),u(ku));
for jua:=1:nua do for kxn:=1:nx do factor df(zua(jua),x(kxn));
for jua:=1:nua do for kua:=1:nua do factor df(zua(jua),ua(kua));
for jx:=1:nx do for kua:=1:nua do factor df(kx(jx),ua(kua));
for jua:=1:nua do factor zua(jua);
end;

    if nua = 0 then in "groeq2.new" else in "equieq2.new" ;
for k:=1:nu do for j:=1:nx do for l:=(j+1):nx do u(k,l,j):=u(k,j,l);
if not (nua=0) then for k:=1:nua do for j:=1:(nx+nu) do for
l:=(j+1):(nx+nu) do ua(k,l,j):=ua(k,j,l);

koutlet:=-3;

for each nomu in 1,2,3 do
begin
    write ("nomu = ",nomu);

% act on the equation by prolonged operator
jj1:= for l:=1:nx sum for m:=1:nu sum
df(s(nomu),u(m,l))*ffu(m,l);
if nomu = koutlet then write ("JJ1 = ",jj1);
jj2:= for m:=1:nu sum
df(s(nomu),u(m))*zu(m);
```

```
if nomu = koutlet then write ("JJ2 = ",jj2);
jj3:= for l:=1:nx sum kx(l)*df(s(nomu),x(l)) ;
if nomu = koutlet then write ("JJ3 = ",jj3);
jj4 := for k := 1:nu sum
for i:= 1:nx sum
for j:= i:nx sum
ffu(k,i,j)*df(s(nomu),u(k,i,j)) ;
if nomu = koutlet then write ("JJ4 = ",jj4);
if not(nua=0) then
<<jja0:= for m:=1:nua sum
    df(s(nomu),ua(m))*zua(m) >>
else jja0:=0; if nomu = koutlet then
    write ("JJa0 = ",jja0);

if not(nua=0) then
<<jja1:= for l:=1:nx+nu sum for m:=1:nua sum
    df(s(nomu),ua(m,l))*ffua(m,l) >>
else jja1:=0; if nomu = koutlet then
    write ("JJa1 = ",jja1);

rru:=jj1+jj2+jj3+jj4+jja0+jja1;
if nomu = koutlet then write ("rru = ",rru);
clear jj1,jj2,jj3,jj4,jja0,jja1;

% coeff.  at the df(derivatives) in the generator
for i:=1:nu do begin
fu(i):=zu(i) - for l:=1:nx sum kx(l)*u(i,l);
if nomu = koutlet then write "fu(",i,")=",fu(i);
end ;
if not(nua=0) then begin
% for fua()
for i:=1:nua do begin
fua(i):=zua(i) - ( for l:=1:nx sum kx(l)*ua(i,l) )
    - ( for j:=1:nu sum zu(j)*ua(i,j+nx) );
if nomu = koutlet then write "fua(",i,")=",fua(i);
end;
end;
% for fua()
for i:=1:nu do for j:=1:nx do begin
    ffu(i,j):= df(fu(i),x(j)) +
for m:=1:nu sum u(m,j)*df(fu(i),u(m)) ;
ffu(i,j):=ffu(i,j)+for k:=1:nua sum
df(fu(i),ua(k))*( ua(k,j) + for l:=1:nu sum ua(k,nx+l)*u(l,j) );
if nomu = koutlet then write "ffu(",i,",",j,")=",ffu(i,j)
end;
if not(nua=0) then begin % for ffua()
for i:=1:nua do begin
    for j:=1:nx do begin
    ffua(i,j):= df(fua(i),x(j)) + for m:=1:nua sum
    ua(m,j)*df(fua(i),ua(m));
if nomu = koutlet then write "ffua(",i,",",j,")=",ffua(i,j);
end;
for j:=1:nu do begin ffua(i,j+nx):= df(fua(i),u(j)) +
for m:=1:nua sum ua(m,j+nx)*df(fua(i),ua(m));
if nomu = koutlet then write "ffua(",i,",",j+nx,")=",ffua(i,j+nx);
end;
end;
end;
% for ffua()
for i := 1:nu do
    for j := 1:nx do
    for k:= j:nx do
begin
ss:=-for l:=1:nx sum
u(i,j,l)*(df(kx(l),x(k))+ for kl:=1:nu sum u(kl,k)*df(kx(l),u(kl)) );
if not(nua=0) then
% for df(kx(l),ua()
ss:=ss-for ka:=1:nua sum
```

```
df(kx(l),ua(ka))*( ua(ka,k) + for la:=1:nu sum ua(ka,nx+la)*u(la,k) );
ss:=ss+ df(ffu(i,j), x(k)) + for m:= 1:nu sum u(m,k)*df(ffu(i,j),u(m));
ss:=ss + for m:= 1:nu sum for kl:=1:nx sum u(m,k,kl)*df(ffu(i,j),u(m,kl));
if not(nua=0) then begin
% for df(ffu(i,j),ua()
ss:=ss + for ka:= 1:nua sum df(ffu(i,j),ua(ka))*( ua(ka,k) +
    for la:=1:nu sum ua(ka,nx+la)*u(la,k) );
ss:=ss + for ka:= 1:nua sum for kl:=1:nx+nu sum
    df(ffu(i,j),ua(ka,kl))*( ua(ka,kl,k) +
    for la:=1:nu sum ua(ka,kl,nx+la)*u(la,k) );
end;
% for df(ffu(i,j),ua()
ffu(i,j,k) :=ss;
clear ss;

if nomu = koutlet then write "ffu(",i,",",j,",",k,")=",ffu(i,j,k);
end;
rru:=rru;
% second order stuff above
for i:=1:nu do clear fu(i); off nat;

    write ("nomu = ",nomu);
if nua=0 then in "groma2.new" else in "equima2.new";

    for m:=1:ms do for l:=m:ms do for q:=l:ms do begin h:=gkk(m,l,q);
ss:=df(df(df(rru,sk(m)),sk(l)),sk(q))/h;
rru:=rru-ss*sk(m)*sk(l)*sk(q);
if not (ss=0) then write "fu(",nomu,",",m,",",l,",",q,") := ",num ss;
%clear fu(nomu,m,l,q);
end;
for m:=1:ms do for l:=m:ms do begin if m=l then h:=2 else h:=1;
ss:=df(df(rru,sk(m)),sk(l))/h; rru:=rru-ss*sk(m)*sk(l);
if not (ss=0) then write "fu(",nomu,",",m,",",l,") := ",num ss;
%clear fu(nomu,m,l);
end;
for m:=1:ms do begin ss:=df(rru,sk(m)); rru:=rru-ss*sk(m);
if not (ss=0) then write "fu(",nomu,",",m,") := ",num ss;
clear
ss;
end;
write "result rru(",nomu,") := ",rru;
write ("end of splitting");
end;
end;
```

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