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ในบริบทที่เปลี่ยนแปลงไปอย่างสิ้นเชิง

นายนัฏพงษ์ ยงรัมย์

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สาขาวิชาฟิสิกส์

มหาวิทยาลัยเทคโนโลยีสุรนารี

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QUANTUM FIELD THEORY METHODS IN AN ENTIRELY DIFFERENT CONTEXT

Mr. Nattapong Yongram

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วิทยานิพนธ์ฉบับนี้เป็นงานแรกเริ่มในการใช้ประโยชน์ของระเบียบวิธีทางทฤษฎีสนาม-ควอนตัมกับปัญหาเชิงพลวัตแผนเดิม แท้ที่จริงในที่นี้คือประโยชน์ของระเบียบวิธีทางในบริบทใหม่ที่เปลี่ยนแปลงไปอย่างสิ้นเชิง เป็นการกระจายเพอร์เทอร์เบชันของการวิวัฒนาการตามเวลาของตัวแปรเชิงพลวัตในตัวแปรคู่ควบเป็นครั้งแรกในทฤษฎีสนามโดยการกระจายเพอร์เทอร์เบชันของ ชวิงเยอร์(Schwinger)-ฟายน์แมน(Feynman)-ดายสัน(Dyson) คล้ายกับผลคูณหน่วยเวลาตลอดช่วงเวลาของการแพร่กระจาย สองรูปนัยนิยมใหม่ได้ถูกพัฒนาขึ้นสำหรับความซับซ้อนของตัวดำเนินการวิวัฒนาการตามเวลาในการอธิบายพลศาสตร์ในปริภูมิเฟส วิธีหนึ่งเป็นวิธีแม่นตรงและอีกวิธีหนึ่งเป็นไปตามธรรมชาติเพอร์เทอร์เบชัน ที่สามารถขยายไปใช้ประโยชน์ยังลักษณะเชิงซ้อนสองมิติซึ่งเรียกว่าเรขาคณิตแบรีเฟสได้ สุดท้ายเป็นการหาปริพันธ์โดยตรงของสมการแฮมิลตัน เป็นพัฒนารูปนัยนิยมปริพันธ์ตามวิถีคล้ายกับการแจกแจงของเอกลักษณ์ คล้ายๆกับการแจกแจงของเอกลักษณ์ของตัวดำเนินการผูกพันในตัวในฟิสิกส์ควอนตัม

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Mr. NATTAPONG YONGRAM: QUANTUM FIELD THEORY METHODS IN AN ENTIRELY DIFFERENT CONTEXT
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This thesis is involved with a pioneering work on the application of quantum field theoretical methods to classical dynamical problems. This is indeed an application of methods in an entirely different context. A perturbation expansion of the time evolution of dynamical variables in the coupling parameter is derived for the first time in the spirit of the Schwinger-Feynman-Dyson perturbation expansion, in field theory, as multiple time integrals over the time interval of propagation. Two new formalisms are developed for the complexification of the time evolution operator suitable in describing the dynamics in phase space. One method is exact and the other is perturbation of nature. This is then extended to a two-dimensional complex setting with an application to the so-called geometrical Berry phase. Finally, by directly integrating Hamilton' s equations we develop a path integral formalism as a resolution of the identity very much in the spirit of the resolution of identity of a self-adjoint operator in quantum physics.

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Nattapong Yongram

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Chapter I

Introduction

With the rapid progress in quantum field theory as the only present available theory for elementary particle physics, it has become more urgent in recent years to develop unified field theories for the four basic interactions known in nature which are the electromagnetic, the weak, the strong and the gravitational interactions. The pioneering work in the method of quantum field theoretical methods is due to Schwinger (1958), Dyson (1949) and Feynman (1949, 1950). With the exception of the gravitational interaction all the interactions have met the severe test or renormalizability (Manoukian (1983)). Thanks to the developments of gauge theories (Weinberg (1980), Salam (1980), and Glashow (1980)), renormalizable field theories have been constructed for the electromagnetic, weak and strong interactions. New methods have been worked out in quantum field theory (see, e.g. Abers and Lee (1973)) for actual computations and the Schwinger functional formalism has been extented to the gauge theories of elementary particles in Manoukian (1986).

These recent developments have shown more and more clearly the emphasis put on the need of unification of computational as well as theoretical developments in describing nature. Although much work has been done in respect of the unification of quantum physical application methods, embracing the non-relativistic regime, and of quantum field theoretical application methods, embracing the relativistic high-energy regime of elementary particles (Feynman and Hibbs (1965)), very little has been done in promoting classical dynamical computational methods to the sophisticated level of quantum field theoretical methods. The present thesis addresses just this problem.

The research involved in this work is a pioneering work on the application of quantum field theory techniques in describing classical dynamics. Very little is avaliable in the literature about the problems addressed to in this work and some papers which are, however, only tangentially related appeared recently in the literature (Abrikosov (1993), Gozzi and Reuter (1989), Gozzi, Reuter and Thacker (1989), Schwartz (1976) and Wetterich (1997)). These authors were mainly interested in a path integral formulation for classical dynamics. The present work, however, aims in developing concrete techniques in solving dynamical problems as follow directly from Hamilton' s formulation of mechanics (Goldstein (1980)) working along the lines of developments in quantum field theory. This reflects the choice of the title of the thesis as the application of field theory methods in an entirely different context - that is, in a non-field theoretical context.

The basic and general idea used in this work is the following. If $q(0)$ denotes the position of a particle at time $t=0$, then $q(t)$ at time $t>0$ may be given by a general formula of the type

$$q(t) = \exp(tO)q(0) \tag{1.1}$$

where O is an operator function of $q(0)$ and $p(0)$, and depends on the details of the dynamics involved in a non-trivial manner. Here $p(0)$ denotes the so-called *generalized momentum* at time $t=0$. It is surprising how far one can go directly from

Eq. (1.1), together with the equivalent Hamilton' s equations, to describe the time evolution of $q(t)$, $p(t)$ or any function of them by the new methods proposed and derived in this thesis. First, a formal expansion of Eq. (1.1) in powers of t turns out to be not too useful as it involves highly corrected Poisson brackets expressions (Goldstein (1980), p 415, Schwartz (1976)) of limited practical value. We propose a new approach (Chapter III) directly in the spirit of field theory in the so-called *interaction representation to all orders in the coupling parameter involved*. This thesis is also involved in obtaining a formal path integral representation (Chapter VI) for classical dynamics as a resolution of the identity in the same spirit as of developing the resolution of the identity of a self-adjoint operator in quantum physics as a sum over projections on the quantum states of the operator in questions. Except in our case we have a sum over paths constrained, through the introduction of the Lagrange multipliers, at every given time in the history of the system, to coincide with the classical one.

The outline of the work carried out in this thesis is as follows: In Chapter II we give a brief review on what is called a *phase space* and develop the Poisson bracket formalism. This is followed by deriving the expression for the time evolution of any function $F[q(t), p(t)]$ of $q(t)$ and $p(t)$. In Chapter III, we develop a perturbation expansion in the coupling parameter involved and, most importantly, we develop rules of computations in the same spirit as done in field theory as multiple integrals in time as the dynamics develops from time $t = 0$ to any other time $t > 0$. The rules are quite general and illustrations are provided. Chapter IV deals with the complexification of the time evolution of dynamical systems. One method is exact of nature (Formalism I) and one is perturbative of nature (Formalism II). In Chapter V, we extend the work carried out in Chapter IV to a two-dimensional setting suitable for analyzing the concept of a geometrical phase (referred to, in general, as a Berry phase (Berry (1984), Shapere and Wilczek (1989)) as applied to the very intriguing problem of the Foucault pendulum. The physical nature of this phase is also discussed there in detail. Chapter VI deals in developing the path integral as a resolution of the identity. The departure of our analysis from previous attempts (Abrikosov (1993), Gozzi and Reuter (1989), Gozzi, Reuter and Thacker (1989), Schwartz (1976) and Wetterich (1997)) is that the restriction of the well known path integral (Feynman and Hibbs (1965)) just to the classical one does no amount to much as far as computations are concerned. The present expression follows directly by integrating in succession, over time, Hamilton' s equations as applied to the Poisson bracket formalism. In the concluding Chapter VII, we summarize all of our new findings with some additional comments. In writing this thesis we have found the classical mathematical techniques developed in Arfken and Weber (1995) quite useful. For basic field theoretical techniques the book by Greiner and Reinhardt (1996) turned to be also useful.

Chapter II

The time evolution operator

This Chapter is devoted exclusively to describe the time development of observables. Given any function of position and/or momentum at time $t=0$, we develop the formalism to obtain such a function at any time $t>0$. To do this, we construct the time evolution operator that describes such a dynamical process. As one considers, quite generally, observables which are functions of position and momentum, we first briefly develop the concept of phase space. This is followed by introducing the important definition of a Poisson bracket which finally leads, in the final section, to the time evolution operator.

2.1 Phase Space

With each degree of freedom of a mechanical system, there are two variables associated with it. These are the generalized coordinate and the generalized momentum. The generalized coordinates q_i can be used to define an n-dimensional configuration space with every point in it representing a certain state of the system, and the generalized momenta p_i define an n-dimensional momentum space with every point representing a certain condition of motion of the system. We can imagine the space of $2n$ dimensions spanned by the $2n$ variables $q_1, q_2, ..., q_n, p_1, p_2, ..., p_n$. Thus, every point in this space represents both the positions and momentum of the all particles in the system. Such a space is called the *phase space*. The time evolution of the system is given by the phase space trajectories $(q_1(t), q_2(t), \ldots, q_n(t), p_1(t),$ $p_2(t), \ldots, p_n(t))$ as functions of time t and is determined by the solution of Hamilton' s equations, which constitute a system of $2n$ ordinary differential equations of the first order. As initial conditions, we assume that the $2n$ values $q_1, q_2, \ldots, q_n, p_1, p_2$ $, \ldots, p_n$ are given for some $t = t_0$.

Through every point of phase space, there is exactly one trajectory, which is a solution of Hamilton' s equations. Thus, the state of a system of N particles with n degrees of freedom can be uniquely characterized by a corresponding single point in phase space. In configuration space, however, the position of all particles, at any time, can be represented, but not the velocities of the individual particles. As the time varies, the point $(q(t), p(t))$ in phase space, describing the state of system at any given time t, will move along a trajectory with an initial condition specified by the point $(q(0), p(0))$, say at $t = 0$. This is illustrated in figure 2.1

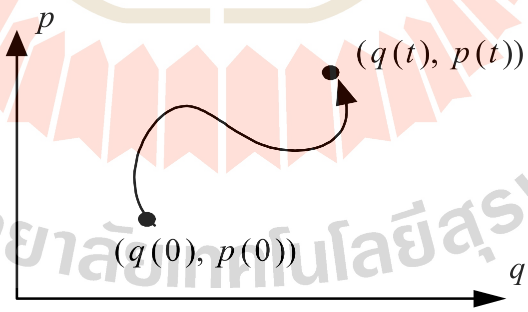

Figure 2.1. The point $(q(t), p(t))$ in phase space describe the state of system at any given time t falling on a trajectory with an initial condition corresponding to a given point $(q(0), p(0))$ say, at time $t = 0$.

2.2 The Poisson bracket

Before of considering the concept of a Poisson bracket, we will recall Hamilton's equations which are a set of first order differential equations, and are equivalent to the second-order Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{2.1}$$

The latter involve the $2n$ variables q_i and $\dot{q}_i$. On the other hand, Hamilton' s equations are $2n$ equations in the $2n$ variables q_i, p_i. Hamilton' s equations are in general functions of q_i, p_i and t. The transformation rule between $\mathcal{H}$ and $\mathcal{L}$, given that $\mathcal{L} = \mathcal{L}(q, \dot{q}, t)$ and $\mathcal{H} = \mathcal{H}(q, p, t)$, is known as a *Legendre transformation*. In the sequel, we suppress the index i in q_i, p_i for simplicity of the notation. By using the definition $\mathcal{L} = p\dot{q} - \mathcal{H}$, We have from elementary calculus

$$\begin{aligned} d\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial q} dq + \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q} + \frac{\partial \mathcal{L}}{\partial t} dt \\ &= d(p\dot{q} - \mathcal{H}) \\ &= \dot{q}dp + pd\dot{q} - \frac{\partial \mathcal{H}}{\partial q} dq - \frac{\partial \mathcal{H}}{\partial p} dp - \frac{\partial \mathcal{H}}{\partial t} dt \end{aligned} \tag{2.2}$$

The explicit quantities that $\mathcal{H}$ and $\mathcal{L}$ depend on specify the overall coefficients of dq, $d\dot{q}$, dp and dt. Upon the comparison of the first and last lines of Eq. (2.2) we arrive at the following equations

$$\frac{\partial L}{\partial \dot{q}} = p \tag{2.3}$$

$$\frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q} \tag{2.4}$$

$$\dot{q} = \frac{\partial H}{\partial p} \tag{2.5}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{2.6}$$

Eq. (2.3) is just the definition of the generalized momentum as given above. By invoking Lagrange' s equations, we then find from Eq. (2.4), since $\partial L/\partial q = dp/dt$, that

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{2.7}$$

By using Eqs. (2.5) and (2.7) in

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} \tag{2.8}$$

the following equation is obtained

$$\frac{\partial H}{\partial q} = -\dot{p} \tag{2.9}$$

and our earlier one

$$\frac{\partial H}{\partial p} = \dot{q}. \tag{2.10}$$

which together constitute as what are called Hamilton' s equations.

Hamilton' s equations of motion describe the time evolution of the position $q(t)$ and the momentum $p(t)$ in phase space. From these equations we can find the equations of motion for any arbitrary function of position $q(t)$ and momentum $p(t)$ by introducing the concept of the Poisson bracket. For Hamiltonian of a system $H[q(t), p(t)]$, we consider arbitrary function of position ($q(t)$) and momentum ($p(t)$) which we denote by $F[q(t), p(t)]$. The time derivative of $F[q(t), p(t)]$ is given by

$$\frac{dF}{dt} = \dot{q}\frac{\partial F}{\partial q} + \dot{p}\frac{\partial F}{\partial p} \tag{2.11}$$

From Eqs. (2.9) and (2.10).we may rewrite Eq. (2.11) as

$$\frac{dF}{dt} = \frac{\partial H}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial F}{\partial p} \tag{2.12}$$

Thus defining, in the process, the Poisson bracket:

$$\frac{dF}{dt} = \{\mathrm{H}, F\}_{P.B.} \tag{2.13}$$

The right-hand side of Eq. (2.13) is called *Poisson bracket* and is given explicitly by

$$\{\mathrm{H}, F\}_{P.B.} = \frac{\partial \mathrm{H}}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial \mathrm{H}}{\partial q}\frac{\partial F}{\partial p} \tag{2.14}$$

2.3 Derivation of the time evolution operator

Eq. (2.13) allows us to rewrite

$$\frac{dF}{dt} = \left(\frac{\partial \mathrm{H}}{\partial p}\frac{\partial}{\partial q} - \frac{\partial \mathrm{H}}{\partial q}\frac{\partial}{\partial p}\right)F \tag{2.15}$$

as

$$\frac{dF}{dt} = (O)F \tag{2.16}$$

where the operator O is a function of $q(t)$ and $p(t)$, and the first derivatives with respect to them. It is given symbolically by

$$O \equiv \left(\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \right)$$

we develop the first n derivatives of F with respect to t, to obtain in succession:

$$\left.\begin{aligned}
&\frac{dF}{dt} = (O)F \equiv F_1 \\
&\frac{d^2F}{dt^2} = \frac{dF_1}{dt} = (O)F_1 = (O)^2 F \equiv F_2 \\
&\frac{d^3F}{dt^3} = \frac{dF_2}{dt} = (O)F_2 = (O)^3 F \equiv F_3 \\
&\frac{d^4F}{dt^4} = (O)^4 F \equiv F_4 \\
&\vdots \\
&\frac{d^nF}{dt^n} = (O)^n F \equiv F_n
\end{aligned}\right\} \tag{2.17}$$

where $F_1, F_2, \ldots, F_n$ are function of $q(t)$ and $p(t)$.

We consider these equation for $t = 0$, thus obtaining in the process

$$F[q(t), p(t)]\big|_{t=0} = F(0)$$

$$\left.\frac{dF}{dt}\right|_{t=0} = (O(0))F(0)$$

$$\left.\frac{d^2F}{dt^2}\right|_{t=0} = (O(0))^2 F(0)$$

$$\left.\frac{d^3F}{dt^3}\right|_{t=0} = (O(0))^3 F(0)$$

$$\left.\frac{d^4F}{dt^4}\right|_{t=0} = (O(0))^4 F(0)$$

$$\vdots$$

$$\left.\frac{d^nF}{dt^n}\right|_{t=0} = (O(0))^n F(0) \tag{2.18}$$

We use the general expression of a Taylor expansion of a given function $f(t)$ about $t = a$,

$$f(t) = f(a) + (t-a)f'(a) + \frac{(t-a)^2}{2!} f''(a) + \ldots + \frac{(t-a)^n}{n!} f^n(a) + \ldots \tag{2.19}$$

to obtain for $F(t) \equiv F[q(t), p(t)]$ an expansion about $t = 0$

$$F(t) = F(0) + tF'(0) + \frac{t^2}{2!} F''(0) + \ldots + \frac{t^n}{n!} F^n(0) + \ldots \tag{2.20}$$

or equivalently the expansion

$$F(t) = F(0) + t(O(0))F(0) + \frac{t^2}{2!}(O(0))^2 F(0) + \ldots + \frac{t^n}{n!}(O(0))^n F(0) + \ldots \tag{2.21}$$

Upon the comparison of the latter with the expansion

$$e^{At} = 1 + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots + \frac{(At)^n}{n!} + \ldots \qquad (2.22)$$

for any operator A not acting on the time variable t, we obtain the final expression:

$$F(t) = [1 + (tO(0)) + \frac{(tO(0))^2}{2!} + \ldots + \frac{(tO(0))^n}{n!} + \ldots]F(0)$$

$$F(t) = \exp(tO(0))F(0) \qquad (2.23)$$

where

$$O(0) \equiv \left(\frac{\partial \mathcal{H}(0)}{\partial p(0)} \frac{\partial}{\partial q(0)} - \frac{\partial \mathcal{H}(0)}{\partial q(0)} \frac{\partial}{\partial p(0)} \right) \qquad (2.24)$$

and $\exp(tO(0))$ is called the *time evolution operator*. We rewrite Eq. (2.23) in the more explicit form

$$F[q(t), p(t)] = \exp\left[t \left(\frac{\partial \mathcal{H}(0)}{\partial p(0)} \frac{\partial}{\partial q(0)} - \frac{\partial \mathcal{H}(0)}{\partial q(0)} \frac{\partial}{\partial p(0)} \right) \right] F[q(0), p(0)] \qquad (2.25)$$

Here we have set

$$\mathcal{H}[q(0), p(0)] \equiv \mathcal{H}(0) \qquad (2.26)$$

Chapter III

The perturbation theory

In this Chapter, we use the time evolution operator derived in Chapter II to develop a general perturbation expansion, which describes the time evolution of any function $F[q(t), p(t)]$ as a function t. Our final result is summarized in Section 3.2

3.1 The fundamental lemma for the perturbation expansion

Before developing the perturbation expansion, we prove the following fundamental lemma that will be used in developing the perturbation theory we are seeking.

Lemma: For any two operators A and B that do not necessarily commute

$$e^{t(A+B)}e^{-tA} = 1 + \int_0^t dt' e^{t'(A+B)} B e^{-t'A} \tag{3.1}$$

Proof: We first use the following identity

$$\int_0^t dt' \frac{d}{dt'}(e^{t'(A+B)}e^{-t'A}) = e^{t(A+B)}e^{-tA} - 1$$

On the other hand

$$\frac{d}{dt'}(e^{t'(A+B)}e^{-t'A}) = e^{t'(A+B)}[A+B-A]e^{-t'A}$$

$$= e^{t'(A+B)}Be^{-t'A}$$

which lead immediately to Eq. (3.1)

For further analysis we set

$$B(t) = e^{tA}Be^{-tA} \tag{3.2}$$

and rewrite Eq. (3.1) in the equivalent form

$$e^{t(A+B)}e^{-tA} = 1 + \int_0^t dt' e^{t'(A+B)}e^{-t'A}B(t') \tag{3.3}$$

To zeroth order in B

$$e^{t(A+B)}e^{-tA} \approx 1 \tag{3.4}$$

Up to the first order in B, we have from

$$e^{t(A+B)}e^{-tA} = 1 + \int_0^t dt_1 e^{t_1(A+B)}Be^{-t_1A}$$

$$= 1 + \int_0^t dt_1 e^{t_1(A+B)}e^{-t_1A}e^{t_1A}Be^{-t_1A}$$

$$= 1 + \int_0^t dt_1 e^{t_1(A+B)} e^{-t_1 A} B(t_1)$$

the expression

$$e^{t(A+B)} e^{-tA} \approx 1 + \int_0^t dt_1 B(t_1) \tag{3.5}$$

Up to the second order in B

$$e^{t(A+B)} e^{-tA} = 1 + \int_0^t dt_1 e^{t_1(A+B)} e^{-t_1 A} B(t_1)$$

$$= 1 + \int_0^t dt_1 [1 + \int_0^{t_1} dt_2 e^{t_2(A+B)} B e^{-t_2 A}] B(t_1)$$

gives

$$e^{t(A+B)} e^{-tA} \approx 1 + \int_0^t dt_1 B(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 B(t_2) B(t_1) \tag{3.6}$$

$$\vdots$$

Up to the n^{th} order in B, we finally have

$$e^{t(A+B)} e^{-tA} \approx 1 + \int_0^t dt_1 B(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 B(t_2) B(t_1) + \cdots + \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n B(t_n) \cdots B(t_1) \tag{3.7}$$

leading to the explicit expansion

$$e^{t(A+B)} e^{-tA} = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n B(t_n) \cdots B(t_2) B(t_1) \tag{3.8}$$

Eqs. (3.7) and (3.8) are referred to as *perturbation expansions*.

3.2 The perturbation expansion and the rules for computations

If $q(0)$ and $p(0)$ denotes the position and the momentum of a particle at time $t=0$. Then $q(t)$ and $p(t)$ at time $t>0$ is given by a general formula in Eq. (2.23), and for an arbitrary function $F[q(t),p(t)]$ of $q(t)$ and $p(t)$, we have

$$F[q(t),p(t)]=\exp(tO(0))F[q(0),p(0)] \tag{3.9}$$

or more explicitly by

$$F[q(t),p(t)]=\exp\left[t\left(\frac{\partial \mathrm{H}(0)}{\partial p(0)}\frac{\partial}{\partial q(0)}-\frac{\partial \mathrm{H}(0)}{\partial q(0)}\frac{\partial}{\partial p(0)}\right)\right]F[q(0),p(0)] \tag{3.10}$$

If the Hamiltonian of the system of the particle is denoted by

$$\mathrm{H}=\mathrm{H}_1+\lambda \mathrm{H}_2 \tag{3.11}$$

where

$$\mathrm{H}_1=\frac{p^2}{2m}$$

describing the kinetic energy, and

$$\mathrm{H}_2 = V(q)$$

with the latter describing the potential energy, then

$$\frac{\partial \mathrm{H}}{\partial p} = \frac{p}{m} \quad \text{and} \quad \frac{\partial \mathrm{H}}{\partial q} = \lambda V'(q) \tag{3.12}$$

Here λ is a coupling parameter useful for bookkeeping purposes. Eq. (3.10) then explicitly reads

$$F[q(t), p(t)] = \exp\left[t\left(\frac{\partial \mathrm{H}(0)}{\partial p(0)}\frac{\partial}{\partial q(0)} - \frac{\partial \mathrm{H}(0)}{\partial q(0)}\frac{\partial}{\partial p(0)}\right)\right] F[q(0), p(0)] \tag{3.13}$$

We set

$$A = \frac{p}{m}\frac{\partial}{\partial q} \quad \text{and} \quad B = -\lambda V'(q)\frac{\partial}{\partial p} \tag{3.14}$$

And hence

$$\exp t\left(\frac{p}{m}\frac{\partial}{\partial q} - \lambda V'(q)\frac{\partial}{\partial p}\right) \equiv \exp t(A+B) \tag{3.15}$$

These definitions allow us to write

$$F[q(t),p(t)] = e^{t(A+B)}F[q(0),p(0)]$$

$$= e^{t(A+B)}e^{-tA}e^{tA}F[q(0),p(0)]$$

We use the expansion developed in section 3.1 to write symbolically

$$F[q(t),p(t)] = (\sum perturbation)\ \ e^{tA}F[q(0),p(0)] \tag{3.16}$$

We first provide some examples which show how to find $B(t)$.

Example 1: If $A = \frac{p}{m}\frac{\partial}{\partial q}$ and $B = \frac{\partial}{\partial p}$

then

$$B(t) = e^{tA}Be^{-tA}$$

gives

$$B(t) = e^{\frac{tp}{m}\frac{\partial}{\partial q}}\left[\left(\frac{\partial}{\partial p}\right)e^{-\frac{tp}{m}\frac{\partial}{\partial q}}\right]$$

$$= e^{\frac{tp}{m}\frac{\partial}{\partial q}}e^{-\frac{tp}{m}\frac{\partial}{\partial q}}[-\frac{t}{m}\frac{\partial}{\partial q}+\frac{\partial}{\partial p}]$$

$$B(t) = \frac{\partial}{\partial p} - \frac{t}{m}\frac{\partial}{\partial q}$$

Example 2: If $A = \frac{p}{m}\frac{\partial}{\partial q}$ and $B = \widetilde{F}(q)\frac{\partial}{\partial p}$, $\widetilde{F}(q) \equiv -V'(q)$

then

$$B(t) = e^{tA} B e^{-tA}$$

$$B(t) = e^{\frac{tp}{m}\frac{\partial}{\partial q}} \left(\widetilde{F}(q) \frac{\partial}{\partial p} \right) B e^{-\frac{tp}{m}\frac{\partial}{\partial q}}$$

$$= \widetilde{F}(q + \frac{tp}{m}) e^{\frac{tp}{m}\frac{\partial}{\partial q}} e^{-\frac{tp}{m}\frac{\partial}{\partial q}} [-\frac{t}{m}\frac{\partial}{\partial q} + \frac{\partial}{\partial p}]$$

where we have used the fact that $e^{\frac{tp}{m}\frac{\partial}{\partial q}}$ is a translation operator of any function of q to $(q + \frac{tp}{m})$. That is,

$$B(t) = \widetilde{F}(q + \frac{tp}{m})[\frac{\partial}{\partial p} - \frac{t}{m}\frac{\partial}{\partial q}]$$

Finally, we provide the following important example, which will be used below.

Example 3: We wish to find $F[q(t), p(t)]$, if $F[q(0), p(0)] \equiv q$

From Example 1, $B(t) = \frac{\partial}{\partial p} - \frac{t}{m}\frac{\partial}{\partial q}$ and $F[q(t), p(t)] \equiv q(t)$

From Eq. (3.16), we then have

$$q(t) = (\sum perturbation) e^{\frac{tp}{m}\frac{\partial}{\partial q}} q$$

Since $e^{\frac{tp}{m}\frac{\partial}{\partial q}} q = [q + \frac{tp}{m}]$, we have the explicit expression

$$q(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n B(t_n) \cdots B(t_2) B(t_1) [q + \frac{tp}{m}] \tag{3.17}$$

3.2.1 The general formula for the perturbation expansion

If $q(0)$ denotes the position of a particle at time $t = 0$, then $q(t)$ at time $t > 0$ is given by the perturbation expansion in Eq. (3.17).

Consider a Hamiltonian H given by

$$\mathrm{H} = \frac{p^2}{2m} + \lambda V(q)$$

Hence we may set

$$A = \frac{p}{m}\frac{\partial}{\partial q}$$

and

$$B = \tilde{F}(q)\frac{\partial}{\partial p} \quad , \tilde{F}(q) \equiv -V'(q)$$

$$B(t) = \tilde{F}(q + \frac{tp}{m})[\frac{\partial}{\partial p} - \frac{t}{m}\frac{\partial}{\partial q}] \tag{3.18}$$

to obtain

$$q(t) = \sum_{n=0}^{\infty} (\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n B(t_n) \cdots B(t_2) B(t_1) [q + \frac{tp}{m}] \tag{3.19}$$

The presence of the coupling parameter λ is explicit and it conveniently specifies the order of the expansion in the interaction. This is exactly in the spirit of the perturbation expansion in quantum field theory. We consider term by term on the right-hand side of the expansion in Eq. (3.19). To first order in B or equivalently in λ, we are led to evaluate:

$$B(t_1)[q+\frac{tp}{m}] = \widetilde{F}[q+\frac{t_1 p}{m}]\left(\frac{\partial}{\partial p} - \frac{t_1}{m}\frac{\partial}{\partial q}\right)[q+\frac{tp}{m}]$$

which is equal to

$$= \widetilde{F}[q+\frac{t_1 p}{m}]\left(\frac{\partial}{\partial p}[q+\frac{tp}{m}] - \frac{t_1}{m}\frac{\partial}{\partial q}[q+\frac{tp}{m}]\right)$$

$$= \widetilde{F}[q+\frac{t_1 p}{m}]\left(\frac{t}{m} - \frac{t_1}{m}\right)$$

$$B(t_1)[q+\frac{tp}{m}] = \frac{1}{m}\widetilde{F}[q+\frac{t_1 p}{m}](t-t_1) \tag{3.20}$$

Similarly,

$$B(t_2)B(t_1)[q+\frac{tp}{m}] = \widetilde{F}[q+\frac{t_2 p}{m}]\left(\frac{\partial}{\partial p} - \frac{t_2}{m}\frac{\partial}{\partial q}\right)\left\{\frac{1}{m}\widetilde{F}[q+\frac{t_1 p}{m}](t-t_1)\right\}$$

$$= \frac{1}{m}(t-t_1)\widetilde{F}[q+\frac{t_2 p}{m}]\left(\frac{\partial}{\partial p}\widetilde{F}[q+\frac{t_1 p}{m}] - \frac{t_2}{m}\frac{\partial}{\partial q}\widetilde{F}[q+\frac{t_1 p}{m}]\right)$$

$$= \frac{1}{m}(t-t_1)\tilde{F}[q+\frac{t_2 p}{m}]\tilde{F}'[q+\frac{t_1 p}{m}]\left(\frac{t_1}{m}-\frac{t_2}{m}\right)$$

$$B(t_2)B(t_1)[q+\frac{tp}{m}] = \frac{1}{m^2}(t-t_1)(t_1-t_2)\tilde{F}[q+\frac{t_2 p}{m}]\tilde{F}'[q+\frac{t_1 p}{m}] \tag{3.21}$$

Also in the same manner

$$B(t_3)B(t_2)B(t_1)[q+\frac{tp}{m}]$$

$$= \tilde{F}[q+\frac{t_3 p}{m}]\left(\frac{\partial}{\partial p}-\frac{t_3}{m}\frac{\partial}{\partial q}\right)\left\{\frac{1}{m^2}(t-t_1)(t_1-t_2)\tilde{F}[q+\frac{t_2 p}{m}]\tilde{F}'[q+\frac{t_1 p}{m}]\right\}$$

$$= \frac{1}{m^2}(t-t_1)(t_1-t_2)\tilde{F}[q+\frac{t_3 p}{m}]$$

$$\times\left(\frac{\partial}{\partial p}\tilde{F}[q+\frac{t_2 p}{m}]\tilde{F}'[q+\frac{t_1 p}{m}]-\frac{t_3}{m}\frac{\partial}{\partial q}\tilde{F}[q+\frac{t_2 p}{m}]\tilde{F}'[q+\frac{t_1 p}{m}]\right)$$

$$= \frac{1}{m^2}(t-t_1)(t_1-t_2)\tilde{F}[q+\frac{t_3 p}{m}]\left\{\tilde{F}'[q+\frac{t_2 p}{m}]\tilde{F}'[q+\frac{t_1 p}{m}]\left(\frac{t_2}{m}-\frac{t_3}{m}\right)\right.$$

$$\left.+\tilde{F}[q+\frac{t_2 p}{m}]\tilde{F}''[q+\frac{t_1 p}{m}]\left(\frac{t_1}{m}-\frac{t_3}{m}\right)\right\}$$

or

$$B(t_3)B(t_2)B(t_1)[q+\frac{tp}{m}] = \frac{1}{m^3}(t-t_1)(t_1-t_2)\tilde{F}[q+\frac{t_3 p}{m}]\left\{\tilde{F}'[q+\frac{t_2 p}{m}]\tilde{F}'[q+\frac{t_1 p}{m}](t_2-t_3)\right.$$

$$\left.+\tilde{F}[q+\frac{t_2 p}{m}]\tilde{F}''[q+\frac{t_1 p}{m}](t_1-t_3)\right\} \tag{3.22}$$

To fourth order we may write the expression

$$B(t_4)B(t_3)B(t_2)B(t_1)[q+\frac{tp}{m}]$$

$$=\frac{1}{m^4}(t-t_1)\widetilde{F}[u_4]\times\sum\Big\{(t_1-t_2)^{\delta(k_1,2)}(t_1-t_3)^{\delta(k_1,3)}(t_1-t_4)^{\delta(k_1,4)}\widetilde{F}^{(k_1)}[u_1]$$
$$\times(t_2-t_3)^{\delta(k_2,3)}(t_2-t_4)^{\delta(k_2,4)}\widetilde{F}^{(k_2)}[u_2] \qquad (3.23)$$
$$\times(t_3-t_4)^{\delta(k_3,4)}\widetilde{F}^{(k_3)}[u_3]\Big\}$$

<u>and the summation is over:</u>

$$\left.\begin{array}{l} k_1=1,2,3 \\ k_2=0,1,2 \\ k_3=0,1 \end{array}\right\} \text{such that : } k_1+k_2+k_3=3$$

$$\left.\begin{array}{l} \delta(k_1,2),\delta(k_1,3),\delta(k_1,4) \\ \delta(k_2,3),\delta(k_2,4) \\ \delta(k_3,4) \end{array}\right\} \text{which are either 0 or 1}$$

such that

$$\delta(k_1,2)+\delta(k_1,3)+\delta(k_1,4)=k_1$$
$$\delta(k_2,3)+\delta(k_2,4)=k_2$$
$$\delta(k_3,4)=k_3$$

and $\delta(k_1,j)+\delta(k_2,j)+\delta(k_3,j)=1$ (3.24)

for $j=2,3,4$

where in Eq. (3.24) we set

$\delta(k_1, j) \equiv 0$ for $j = 1$

$\delta(k_2, j) \equiv 0$ for $j = 1,2$

$\delta(k_3, j) \equiv 0$ for $j = 1,2,3$

That is t_j appears only once in

$$[\ (t_1 - t_j)^{\cdot\cdot}(t_2 - t_j)^{\cdot\cdot}\ldots]$$

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Finally, for the n^{th} order we have

$$B(t_n)B(t_{n-1})\Lambda\ B(t_2)B(t_1)[q + \frac{tp}{m}]$$

$$= \frac{1}{m^n}(t - t_1)\widetilde{F}[u_n] \times \sum \left\{ \left(\prod_{j=2}^{n} (t_1 - t_j)^{\delta(k_1,j)} \right) \widetilde{F}^{(k_1)}[u_1] \right.$$

$$\times \left(\prod_{j=3}^{n} (t_2 - t_j)^{\delta(k_2,j)} \right) \widetilde{F}^{(k_2)}[u_2]$$

$$\times \left(\prod_{j=4}^{n} (t_2 - t_j)^{\delta(k_3,j)} \right) \widetilde{F}^{(k_3)}[u_3]$$

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$$\times \left(\prod_{j=n-2}^{n} (t_{n-3} - t_j)^{\delta(k_{n-3},j)} \right) \widetilde{F}^{(k_{n-3})}[u_{n-3}]$$

$$\times \left(\prod_{j=n-1}^{n} (t_{n-2} - t_j)^{\delta(k_{n-2},j)} \right) \widetilde{F}^{(k_{n-2})}[u_{n-2}] \qquad (3.25)$$

$$\left. \times (t_{n-1} - t_n)^{\delta(k_{n-1},n)}\ \widetilde{F}^{(k_{n-1})}[u_{n-1}]\ \right\}$$

<u>and the summation is over:</u>

$$\left.\begin{array}{l} k_1 = 1,2,3,\ldots,n-1 \\ k_2 = 0,1,2,\ldots,n-2 \\ k_3 = 0,1,2,\ldots,n-3 \\ \vdots \\ k_{n-2} = 0,1,2 \\ k_{n-1} = 0,1 \end{array}\right\} \text{ such that : } k_1 + k_2 + \ldots + k_{n-1} = n-1$$

$$\left.\begin{array}{l} \delta(k_1, j), \quad j = 2,3,\ldots,n-1 \\ \delta(k_2, j), \quad j = 3,4,\ldots,n-1 \\ \delta(k_3, j), \quad j = 4,5,\ldots,n-3 \\ \vdots \\ \delta(k_{n-2}, j), \quad j = (n-1),n \\ \delta(k_{n-1}, n), \end{array}\right\} \text{ which are either 0 or 1}$$

such that

$$\left\{\begin{array}{l} \sum_{j=2}^{n} \delta(k_1, j) = k_1 \\ \sum_{j=3}^{n} \delta(k_2, j) = k_2 \\ \vdots \\ \delta(k_{n-1}, n) = k_{n-1} \end{array}\right.$$

and $$\sum_{i=1}^{n-1} \delta(k_i, j) = 1 \qquad (3.26)$$

where we see from Eq. (3.26) that we have to set

$$\delta(k_i, j) \equiv 0 \quad \text{For} \quad j = 1,2,3,\ldots,i$$

That is, for a fixed i, t_j appears only once in

$$\left.\begin{array}{l} \prod_{j=2}^{n}(t_1 - t_j)^{\delta(k_1,j)} \\ \prod_{j=3}^{n}(t_2 - t_j)^{\delta(k_2,j)} \\ \qquad\vdots \end{array}\right\}$$

Hence, from Eq. (3.19), we can rewrite the general form of the perturbation expansion as

$$q(t) = \sum_{n=0}^{\infty}(\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\times \frac{1}{m^n}(t - t_1)\tilde{F}[u_n] \times \sum \left\{ \left(\prod_{j=2}^{n}(t_1 - t_j)^{\delta(k_1,j)} \right) \tilde{F}^{(k_1)}[u_1] \right.$$

$$\times \left(\prod_{j=3}^{n}(t_2 - t_j)^{\delta(k_2,j)} \right) \tilde{F}^{(k_2)}[u_2]$$

$$\times \left(\prod_{j=4}^{n}(t_2 - t_j)^{\delta(k_3,j)} \right) \tilde{F}^{(k_3)}[u_3]$$

$$\vdots$$

$$\times \left(\prod_{j=n-2}^{n}(t_{n-3} - t_j)^{\delta(k_{n-3},j)} \right) \tilde{F}^{(k_{n-3})}[u_{n-3}]$$

$$\times\left(\prod_{j=n-1}^{n}\left(t_{n-2}-t_j\right)^{\delta(k_{n-2},j)}\right)\widetilde{F}^{(k_{n-2})}[u_{n-2}]$$
$$\times\left(t_{n-1}-t_n\right)^{\delta(k_{n-1},n)}\widetilde{F}^{(k_{n-1})}[u_{n-1}]\Big\} \qquad (3.27)$$

with $u_n \equiv \left[q+\frac{t_n p}{m}\right]$

3.3 Illustrative applications of the new formalism

Example 1: For the harmonic oscillator with the addition of a linear coupling to q we set

$$\mathrm{H}=\frac{p^2}{2m}+\frac{m\omega^2 q^2}{2}+\lambda q \qquad (3.28)$$

We will find $q(t)$ for any time $t>0$ according to the perturbation expansion in Eq. (3.27). From $\widetilde{F}[q]=-\lambda V'(q)$ and Eq. (3.27), we obtain

$$\widetilde{F}[q]=-(m\omega^2 q+\lambda)$$
$$\widetilde{F}'[q]=-m\omega^2$$
$$\widetilde{F}''[q]=0$$
$$\vdots \qquad (3.29)$$

Upon replacing q by $\left(q+\frac{tp}{m}\right)$, as shown in Eq. (3.19) we obtain

$$\tilde{F}[q+\frac{tp}{m}] = -(m\omega^2\left(q+\frac{tp}{m}\right)+\lambda)$$

$$\tilde{F}'[q+\frac{tp}{m}] = -m\omega^2$$

$$\tilde{F}''[q+\frac{tp}{m}] = 0$$

$$\vdots \qquad (3.30)$$

Upon substituting the latter in Eq. (3.27), we obtain

$$q(t)=\left(q+\frac{tp}{m}\right)+\int_0^t dt_1\frac{1}{m}(t-t_1)(-m\omega^2\left(q+\frac{t_1 p}{m}\right)-\lambda)+\int_0^t dt_1\int_0^{t_1} dt_2\frac{1}{m^2}(t-t_1)(t_1-t_2)$$

$$\times\left\{m^2\omega^4\left(q+\frac{t_2 p}{m}\right)+\lambda m\omega^2\right\}+\cdots \qquad (3.31)$$

The time integrations are explicitly carried out to give

$$q(t)=\left(q+\frac{tp}{m}\right)-\frac{\omega^2 t^2}{2}\left(q+\frac{tp}{3m}\right)-\frac{\lambda t^2}{2m}+\frac{\omega^4 t^4}{24}\left(q+\frac{tp}{5m}\right)+\frac{\lambda\omega t^4}{24m}+\cdots$$

$$=\left(q-\frac{\omega^2 t^2 q}{2!}+\frac{\omega^4 t^4 q}{4!}+\cdots\right)+\left(\frac{tp}{m}-\frac{\omega^2 t^3 p}{3!m}+\frac{\omega^4 t^5 p}{5!m}-\cdots\right)+\left(-\frac{\lambda t^2}{2!m}+\frac{\lambda\omega^4 t^4}{4!m}-\cdots\right)$$

$$=q\left(1-\frac{\omega^2 t^2}{2!}+\frac{\omega^4 t^4}{4!}+\cdots\right)+\frac{p}{\omega m}\left(\omega t-\frac{\omega^3 t^3}{3!}+\frac{\omega^5 t^5}{5!}-\cdots\right)+\frac{\lambda}{\omega^2 m}\left(1-\frac{t^2}{2!}+\frac{\omega^4 t^4}{4!}-\cdots\right)$$

$$-\frac{\lambda}{\omega^2 m}$$

which lead, upon recognizing the sine and cosine functions expansions, to

$$q(t) = q\cos\omega t + \frac{p}{\omega m}\sin\omega t + \frac{\lambda}{\omega^2 m}\cos\omega t - \frac{\lambda}{\omega^2 m}$$

$$= \left(q + \frac{\lambda}{\omega^2 m}\right)\cos\omega t + \frac{p}{\omega m}\sin\omega t - \frac{\lambda}{\omega^2 m}$$

$$= \sqrt{\left(q + \frac{\lambda}{\omega^2 m}\right)^2 + \left(\frac{p}{\omega m}\right)^2}\cos(\omega t - \delta) - \frac{\lambda}{\omega^2 m}$$

or finally to

$$q(t) = A\cos(\omega t - \delta) - \frac{\lambda}{\omega^2 m} \tag{3.32}$$

where

$$A \equiv \sqrt{\left(q + \frac{\lambda}{\omega^2 m}\right)^2 + \left(\frac{p}{\omega m}\right)^2} \quad \text{and} \quad \delta = \tan^{-1}\left(\frac{p/\omega m}{\left(q + \frac{p}{\omega^2 m}\right)}\right)$$

here (q, p) represents the initial condition at time $t = 0$.

Example 2: We consider the perturbation expansion of the Hamiltonian

$$\mathrm{H} = \frac{p^2}{2m} + \lambda e^{-\alpha q} \tag{3.33}$$

To first order in λ but up to all order in α.

We will find $q(t)$ according to the perturbation expansion in Eq. (3.27).

From $\widetilde{F}[q] = -\lambda V'(q)$ and Eq. (3.27), we obtain

$$\widetilde{F}[q] = \lambda \alpha e^{-\alpha q}$$

$$\widetilde{F}'[q] = -\lambda \alpha^2 e^{-\alpha q}$$

$$\widetilde{F}''[q] = \lambda \alpha^3 e^{-\alpha q}$$

$$\vdots$$

$$\widetilde{F}^{(n)}[q] = (-1)^n \lambda \alpha^{n+1} e^{-\alpha q} \tag{3.34}$$

Upon replacing q by $\left(q + \frac{tp}{m}\right)$, we obtain

$$\widetilde{F}[q + \frac{tp}{m}] = \lambda \alpha e^{-\alpha[q + \frac{tp}{m}]}$$

$$\widetilde{F}'[q + \frac{tp}{m}] = -\lambda \alpha^2 e^{-\alpha[q + \frac{tp}{m}]}$$

$$\widetilde{F}''[q + \frac{tp}{m}] = \lambda \alpha^3 e^{-\alpha[q + \frac{tp}{m}]}$$

$\vdots$

$$\widetilde{F}^{(n)}[q+\frac{tp}{m}] = (-1)^n \lambda \alpha^{n+1} e^{-\alpha[q+\frac{tp}{m}]} \tag{3.35}$$

Upon substituting the latter in Eq. (3.27), we obtain

$$q(t) = \left(q+\frac{tp}{m}\right) + \int_0^t dt_1 \frac{1}{m}(t-t_1)\lambda\alpha e^{-\alpha[q+\frac{t_1 p}{m}]} + \int_0^t dt_1 \int_0^{t_1} dt_2 \frac{1}{m^2}(t-t_1)(t_1-t_2)$$

$$\times\left\{-\lambda^2\alpha^3 e^{-\alpha[q+\frac{t_1 p}{m}]} e^{-\alpha[q+\frac{t_2 p}{m}]}\right\} + \ldots \tag{3.36}$$

The time integration are explicitly carried out to yield, to first order in λ but up to all order in α, the expression

$$q(t) = \left(q+\frac{tp}{m}\right) + \frac{\lambda\alpha}{m} e^{-\alpha q}\left[t\int_0^t dt_1 e^{-\alpha\frac{t_1 p}{m}} - \int_0^t dt_1 (t_1) e^{-\alpha\frac{t_1 p}{m}}\right] + \ldots$$

$$= \left(q+\frac{tp}{m}\right) + \frac{\lambda\alpha}{m} e^{-\alpha q}\left[t\left(-\frac{m}{\alpha p}\right)e^{-\alpha\frac{t_1 p}{m}}\Big|_0^t - t_1\left(-\frac{m}{\alpha p}\right)e^{-\alpha\frac{t_1 p}{m}}\Big|_0^t + \left(-\frac{m^2}{\alpha^2 p^2}\right)e^{-\alpha\frac{t_1 p}{m}}\Big|_0^t\right] + \ldots$$

$$q(t) = (q+\frac{tp}{m}) + \frac{\lambda e^{-\alpha q}}{p}\left\{t - \frac{m}{\alpha p}(e^{-t\alpha\frac{p}{m}} - 1)\right\} + \ldots \tag{3.37}$$

Here (q, p) represents the initial condition at time $t = 0$.

Chapter IV

Complexification of the time evolution

The purpose of this Chapter is to recast the time evolution of dynamical variables developed in the previous two Chapters in term of the complex dynamical variable defined by

$$z(t) = aq(t) + ibp(t) \tag{4.1}$$

where a and b are some conveniently chosen real numbers appropriate for the problem at hand. The complex representation of the dynamical variables ($q(t), p(t)$) in phase space turns out to be a useful alternative in dealing separately with the dynamical variables $q(t)$ and $p(t)$.

We develop two formalisms for dealing with the time evolution of the complex dynamical variable $z(t)$ in Eq. (4.1). The first one deals directly with the time evolution expressed completely in term of the complex variables $z(0)$ and its complex conjugate and is written in its exact form. The second formalism follows directly from the perturbation expansion derived in Chapter III and follows directly from it as now applied to the variable $z(t)$.

4.1 Formalism I

Eq. (2.23) in Chapter II reads

$$F[q(t), p(t)] = \exp(tO(0))F[q(0), p(0)] \tag{4.2}$$

where

$$O(0) = \left(\frac{\partial \mathcal{H}(0)}{\partial p(0)} \frac{\partial}{\partial q(0)} - \frac{\partial \mathcal{H}(0)}{\partial q(0)} \frac{\partial}{\partial p(0)} \right) \tag{4.3}$$

For the complex dynamical variable in Eq. (4.1) we have for its complex conjugate

$$z^*(t) = aq(t) - ibp(t) \tag{4.4}$$

At $t = 0$, we will write

$$z(0) = aq(0) + ibp(0) \tag{4.5}$$

and

$$z^*(0) = aq(0) - ibp(0) \tag{4.6}$$

Therefore,

$$q(t) = \frac{1}{2a}(z(t) + z^*(t)) \tag{4.7}$$

$$p(t) = \frac{1}{2ib}(z(t) - z^*(t)) \tag{4.8}$$

and at $t = 0$

$$q(0) = \frac{1}{2a}(z(0) + z^*(0)) \tag{4.9}$$

$$p(0) = \frac{1}{2ib}(z(0) - z^*(0)) \tag{4.10}$$

In reference to the operator $O(0)$, we deduce that

$$\frac{\partial \mathcal{H}(0)}{\partial q(0)} = a\left(\frac{\partial \mathcal{H}(0)}{\partial z(0)} + \frac{\partial \mathcal{H}(0)}{\partial z^*(0)}\right) \tag{4.11}$$

$$\frac{\partial \mathcal{H}(0)}{\partial p(0)} = ib\left(\frac{\partial \mathcal{H}(0)}{\partial z(0)} - \frac{\partial \mathcal{H}(0)}{\partial z^*(0)}\right) \tag{4.12}$$

where we have used the differentiation identities

$$\frac{\partial}{\partial q(0)} = a\left(\frac{\partial}{\partial z(0)} + \frac{\partial}{\partial z^*(0)}\right) \tag{4.13}$$

$$\frac{\partial}{\partial p(0)} = ib\left(\frac{\partial}{\partial z(0)} - \frac{\partial}{\partial z^*(0)}\right) \tag{4.14}$$

substituting these in Eq. (4.3) that for the operator $O(0)$, one may equivalently write

$$O(0) = iab\left(\left(\frac{\partial \mathcal{H}(0)}{\partial z(0)} - \frac{\partial \mathcal{H}(0)}{\partial z^*(0)}\right)\left(\frac{\partial}{\partial z(0)} + \frac{\partial}{\partial z^*(0)}\right) - \left(\frac{\partial \mathcal{H}(0)}{\partial z(0)} + \frac{\partial \mathcal{H}(0)}{\partial z^*(0)}\right)\left(\frac{\partial}{\partial z(0)} - \frac{\partial}{\partial z^*(0)}\right)\right)$$

The present formalism for the time development of the complex dynamical variable $z(t)$ then becomes

$$z(t) = \exp(tO(0))z(0) \tag{4.15}$$

where

$$O(0) = iab\left(\left(\frac{\partial\mathcal{H}(0)}{\partial z(0)} - \frac{\partial\mathcal{H}(0)}{\partial z^*(0)}\right)\left(\frac{\partial}{\partial z(0)} + \frac{\partial}{\partial z^*(0)}\right) - \left(\frac{\partial\mathcal{H}(0)}{\partial z(0)} + \frac{\partial\mathcal{H}(0)}{\partial z^*(0)}\right)\left(\frac{\partial}{\partial z(0)} - \frac{\partial}{\partial z^*(0)}\right)\right)$$

Eq. (4.15) provides the exact complex representation of the time evolution of complex dynamical variable $z(t)$ expressed completely in term of the complex variables $z(0)$ and $z^*(0)$.

4.2 Formalism II

The second formalism is a recast of the perturbation expansion derived in Chapter III for the complex dynamical variable $z(t)$ and directly follows from the detailed treatment given in the just mentioned Chapter.

At any time t, we have shown that

$$q(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$\times \frac{1}{m^n}(t - t_1)\widetilde{F}[u_n] \times \sum \left\{ \left(\prod_{j=2}^{n} (t_1 - t_j)^{\delta(k_1, j)} \right) \widetilde{F}^{(k_1)}[u_1] \right.$$

$$\times\left(\prod_{j=3}^{n}(t_2-t_j)^{\delta(k_2,j)}\right)\widetilde{F}^{(k_2)}[u_2]$$

$$\times\left(\prod_{j=4}^{n}(t_2-t_j)^{\delta(k_3,j)}\right)\widetilde{F}^{(k_3)}[u_3]$$

$$\vdots$$

$$\times\left(\prod_{j=n-2}^{n}(t_{n-3}-t_j)^{\delta(k_{n-3},j)}\right)\widetilde{F}^{(k_{n-3})}[u_{n-3}]$$

$$\times\left(\prod_{j=n-1}^{n}(t_{n-2}-t_j)^{\delta(k_{n-2},j)}\right)\widetilde{F}^{(k_{n-2})}[u_{n-2}]$$

$$\times(t_{n-1}-t_n)^{\delta(k_{n-1},n)}\widetilde{F}^{(k_{n-1})}[u_{n-1}]\Bigg\} \tag{4.16}$$

Similarly, for the momentum, we may write

$$p(t)=(\sum perturbation)e^{\frac{ip}{m}\frac{\partial}{\partial q}}p \tag{4.17}$$

where $p \equiv p(0)$, to obtain

$$p(t)=\sum_{n=0}^{\infty}\int_0^t dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n B(t_n)\ldots B(t_2)B(t_1)p \tag{4.18}$$

where we have used the fact that

$$e^{\frac{ip}{m}\frac{\partial}{\partial q}}p=p \tag{4.19}$$

In details we have

$$B(t_1)p = \widetilde{F}[q + \frac{t_1 p}{m}]\left(\frac{\partial}{\partial p} - \frac{t_1}{m}\frac{\partial}{\partial q}\right)p$$

$$= \widetilde{F}[q + \frac{t_1 p}{m}]$$

$$B(t_1)p = \widetilde{F}[q + \frac{t_1 p}{m}] \tag{4.20}$$

$$B(t_2)B(t_1)p = \widetilde{F}[q + \frac{t_2 p}{m}]\left(\frac{\partial}{\partial p} - \frac{t_2}{m}\frac{\partial}{\partial q}\right)\left\{\widetilde{F}[q + \frac{t_1 p}{m}]\right\}$$

$$= \widetilde{F}[q + \frac{t_2 p}{m}]\left(\frac{\partial}{\partial p}\widetilde{F}[q + \frac{t_1 p}{m}] - \frac{t_2}{m}\frac{\partial}{\partial q}\widetilde{F}[q + \frac{t_1 p}{m}]\right)$$

$$= \widetilde{F}[q + \frac{t_2 p}{m}]\widetilde{F}'[q + \frac{t_1 p}{m}]\left(\frac{t_1}{m} - \frac{t_2}{m}\right)$$

$$B(t_2)B(t_1)p = \frac{1}{m}(t_1 - t_2)\widetilde{F}[q + \frac{t_2 p}{m}]\widetilde{F}'[q + \frac{t_1 p}{m}] \tag{4.21}$$

$$B(t_3)B(t_2)B(t_1)p$$

$$= \widetilde{F}[q + \frac{t_3 p}{m}]\left(\frac{\partial}{\partial p} - \frac{t_3}{m}\frac{\partial}{\partial q}\right)\left\{\frac{1}{m}(t_1 - t_2)\widetilde{F}[q + \frac{t_2 p}{m}]\widetilde{F}'[q + \frac{t_1 p}{m}]\right\}$$

$$= \frac{1}{m}(t_1 - t_2)\widetilde{F}[q + \frac{t_3 p}{m}]$$
$$\times\left(\frac{\partial}{\partial p}\widetilde{F}[q + \frac{t_2 p}{m}]\widetilde{F}'[q + \frac{t_1 p}{m}] - \frac{t_3}{m}\frac{\partial}{\partial q}\widetilde{F}[q + \frac{t_2 p}{m}]\widetilde{F}'[q + \frac{t_1 p}{m}]\right)$$

$$= \frac{1}{m}(t_1 - t_2)\widetilde{F}[q + \frac{t_3 p}{m}]\left\{ \widetilde{F}'[q + \frac{t_2 p}{m}]\widetilde{F}'[q + \frac{t_1 p}{m}]\left(\frac{t_2}{m} - \frac{t_3}{m}\right) + \widetilde{F}[q + \frac{t_2 p}{m}]\widetilde{F}''[q + \frac{t_1 p}{m}]\left(\frac{t_1}{m} - \frac{t_3}{m}\right)\right\}$$

$$B(t_3)B(t_2)B(t_1)[q + \frac{tp}{m}] = \frac{1}{m^2}(t_1 - t_2)\widetilde{F}[q + \frac{t_3 p}{m}]\left\{ \widetilde{F}'[q + \frac{t_2 p}{m}]\widetilde{F}'[q + \frac{t_1 p}{m}](t_2 - t_3) + \widetilde{F}[q + \frac{t_2 p}{m}]\widetilde{F}''[q + \frac{t_1 p}{m}](t_1 - t_3)\right\} \tag{4.22}$$

$$B(t_n)B(t_{n-1})\cdots B(t_2)B(t_1)p$$

$$= \frac{1}{m^{n-1}}\widetilde{F}[u_n] \times \sum \left\{ \left(\prod_{j=2}^{n}(t_1 - t_j)^{\delta(k_1,j)}\right)\widetilde{F}^{(k_1)}[u_1] \right.$$

$$\times \left(\prod_{j=3}^{n}(t_2 - t_j)^{\delta(k_2,j)}\right)\widetilde{F}^{(k_2)}[u_2]$$

$$\times \left(\prod_{j=4}^{n}(t_2 - t_j)^{\delta(k_3,j)}\right)\widetilde{F}^{(k_3)}[u_3]$$

$$\vdots$$

$$\times \left(\prod_{j=n-2}^{n}(t_{n-3} - t_j)^{\delta(k_{n-3},j)}\right)\widetilde{F}^{(k_{n-3})}[u_{n-3}]$$

$$\times \left(\prod_{j=n-1}^{n}(t_{n-2} - t_j)^{\delta(k_{n-2},j)}\right)\widetilde{F}^{(k_{n-2})}[u_{n-2}]$$

$$\left. \times (t_{n-1} - t_n)^{\delta(k_{n-1},n)}\widetilde{F}^{(k_{n-1})}[u_{n-1}] \right\} \tag{4.23}$$

where $u_n \equiv \left[q + \frac{t_n p}{m} \right]$

where the summation is over:

$$\left.\begin{array}{l} k_1 = 1,2,3,\ldots,n-1 \\ k_2 = 0,1,2,\ldots,n-2 \\ k_3 = 0,1,2,\ldots,n-3 \\ \quad\vdots \\ k_{n-2} = 0,1,2 \\ k_{n-1} = 0,1 \end{array}\right\} \quad \text{such that : } k_1 + k_2 + \ldots + k_{n-1} = n-1$$

$$\left.\begin{array}{l} \delta(k_1, j), \quad j = 2,3,\ldots,n-1 \\ \delta(k_2, j), \quad j = 3,4,\ldots,n-1 \\ \delta(k_3, j), \quad j = 4,5,\ldots,n-3 \\ \quad\vdots \\ \delta(k_{n-2}, j), \quad j = (n-1), n \\ \delta(k_{n-1}, n), \end{array}\right\} \quad \text{which are either 0 or 1}$$

such that

$$\left\{\begin{array}{l} \sum_{j=2}^{n} \delta(k_1, j) = k_1 \\ \sum_{j=3}^{n} \delta(k_2, j) = k_2 \\ \qquad\vdots \\ \delta(k_{n-1}, n) = k_{n-1} \end{array}\right.$$

and $$\sum_{i=1}^{n-1}\delta(k_i,j)=1 \tag{4.24}$$

where we see in Eq. (4.24) we set

$\delta(k_i,j)\equiv 0$ For $j=1,2,3,\ldots,i$

That is, for a fixed i, t_j appears only once in

$$\left.\begin{array}{l}\prod_{j=2}^{n}(t_1-t_j)^{\delta(k_1,j)}\\ \prod_{j=3}^{n}(t_2-t_j)^{\delta(k_2,j)}\\ \quad\vdots\end{array}\right\}$$

From Eq. (4.17), then we may rewrite in general

$$\begin{aligned}p(t)=\sum_{n=0}^{\infty}\int_0^t dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n\\ \times\frac{1}{m^{n-1}}\widetilde{F}[u_n]\times\sum\Bigg\{&\left(\prod_{j=2}^{n}(t_1-t_j)^{\delta(k_1,j)}\right)\widetilde{F}^{(k_1)}[u_1]\\ &\times\left(\prod_{j=3}^{n}(t_2-t_j)^{\delta(k_2,j)}\right)\widetilde{F}^{(k_2)}[u_2]\\ &\times\left(\prod_{j=4}^{n}(t_2-t_j)^{\delta(k_3,j)}\right)\widetilde{F}^{(k_3)}[u_3]\\ &\vdots\\ &\times\left(\prod_{j=n-2}^{n}(t_{n-3}-t_j)^{\delta(k_{n-3},j)}\right)\widetilde{F}^{(k_{n-3})}[u_{n-3}]\end{aligned}$$

$$\times\left(\prod_{j=n-1}^{n}\left(t_{n-2}-t_j\right)^{\delta(k_{n-2},j)}\right)\widetilde{F}^{(k_{n-2})}[u_{n-2}]$$

$$\times\left(t_{n-1}-t_n\right)^{\delta(k_{n-1},n)}\widetilde{F}^{(k_{n-1})}[u_{n-1}]\Bigg\} \qquad (4.25)$$

For the complex dynamical variable $z(t) = aq(t) + ibp(t)$, we then obtain

$$z(t)=\sum_{n=0}^{\infty}\int_0^t dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{n-1}}dt_n$$

$$\times\left(\frac{a(t-t_1)}{m^n}+\frac{ib}{m^{n-1}}\right)\widetilde{F}[u_n]\times\sum\left\{\left(\prod_{j=2}^{n}\left(t_1-t_j\right)^{\delta(k_1,j)}\right)\widetilde{F}^{(k_1)}[u_1]\right.$$

$$\times\left(\prod_{j=3}^{n}\left(t_2-t_j\right)^{\delta(k_2,j)}\right)\widetilde{F}^{(k_2)}[u_2]$$

$$\times\left(\prod_{j=4}^{n}\left(t_2-t_j\right)^{\delta(k_3,j)}\right)\widetilde{F}^{(k_3)}[u_3]$$

$$\vdots$$

$$\times\left(\prod_{j=n-2}^{n}\left(t_{n-3}-t_j\right)^{\delta(k_{n-3},j)}\right)\widetilde{F}^{(k_{n-3})}[u_{n-3}]$$

$$\times\left(\prod_{j=n-1}^{n}\left(t_{n-2}-t_j\right)^{\delta(k_{n-2},j)}\right)\widetilde{F}^{(k_{n-2})}[u_{n-2}]$$

$$\left.\times\left(t_{n-1}-t_n\right)^{\delta(k_{n-1},n)}\widetilde{F}^{(k_{n-1})}[u_{n-1}]\right\} \qquad (4.26)$$

where now the variables $u_1, u_2, \ldots, u_{n-1}$ are define by the general expression

$$u_n = \frac{(z(0)+z^*(0))}{2a} + \frac{t_n(z(0)-z^*(0))}{2ibm}$$

4.3 Applications

As applications of the above formalisms we consider the following examples.

Example 1:

For the presence of an arbitrary potential energy $\lambda V(q)$ in addition to the harmonic oscillator potential we may write

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} + \lambda V(q) \tag{4.27}$$

From the expressions $z = aq + ibp$ and $z^* = aq - ibp$, we obtain

$$z + z^* = 2aq$$

$$z - z^* = 2ibp$$

$$zz^* = a^2q^2 + b^2p^2 \tag{4.28}$$

From Eqs. (4.27) and (4.28), it is convenient to choose

$$a = \omega\sqrt{\frac{m}{2}} \qquad \text{and} \qquad b = \frac{1}{\sqrt{2m}}$$

so that

$$q = \frac{z + z^*}{\omega\sqrt{2m}}$$

$$z = \omega\sqrt{\frac{m}{2}}q + i\frac{1}{\sqrt{2m}}p$$

$$z^* = \omega\sqrt{\frac{m}{2}}q - i\frac{1}{\sqrt{2m}}p$$

The Hamiltonian in the terms of z and z^* then takes the particularly simple form

$$\mathcal{H}[z, z^*] = zz^* + \lambda V\left(\frac{z + z^*}{\omega\sqrt{2m}}\right) \tag{4.29}$$

In reference to formalism I, we have

$$\frac{\partial\mathcal{H}}{\partial q} = \omega\sqrt{\frac{m}{2}}\left((z + z^*) + \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}V'\left(\frac{z + z^*}{\omega\sqrt{2m}}\right)\right) \tag{4.30}$$

$$\frac{\partial\mathcal{H}}{\partial p} = \frac{i}{\sqrt{2m}}\left(z^* - z\right) \tag{4.31}$$

$$\frac{\partial}{\partial q} = \omega\sqrt{\frac{m}{2}}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right) \tag{4.32}$$

$$\frac{\partial}{\partial p} = \frac{i}{\sqrt{2m}}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right) \tag{4.33}$$

So, the time evolution operator becomes

$$\exp t\left(\frac{i\omega}{2}\left(z^{*}-z\right)\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z^{*}}\right)+\frac{i\omega}{2}\left\{\left(z^{*}+z\right)+\frac{\lambda}{\omega}\sqrt{\frac{2}{m}}V'\left(\frac{z+z^{*}}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^{*}}-\frac{\partial}{\partial z}\right)\right\}\right) \tag{4.34}$$

We set

$$-V'\left(\frac{z+z^{*}}{\omega\sqrt{2m}}\right)\equiv F\left(\frac{z+z^{*}}{\omega\sqrt{2m}}\right) \tag{4.35}$$

to rewrite Eq. (4.33) as

$$\exp t\left(\frac{i\omega}{2}\left(z^{*}-z\right)\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z^{*}}\right)+\frac{i\omega}{2}\left\{\left(z^{*}+z\right)-\frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^{*}}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^{*}}-\frac{\partial}{\partial z}\right)\right\}\right) \tag{4.36}$$

or

$$\exp\frac{i\omega t}{2}\left(2\left(z^{*}\frac{\partial}{\partial z^{*}}-z\frac{\partial}{\partial z}\right)-\frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^{*}}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^{*}}-\frac{\partial}{\partial z}\right)\right) \tag{4.37}$$

To finally obtain the exact expression

$$z(t)=\exp\frac{i\omega t}{2}\left(2\left(z^{*}\frac{\partial}{\partial z^{*}}-z\frac{\partial}{\partial z}\right)-\frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^{*}}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^{*}}-\frac{\partial}{\partial z}\right)\right)z \tag{4.38}$$

Upon the expansion of the exponential in powers of t, we have

$$z(t) = 1 + \frac{i\omega t}{2}\left(2\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right) - \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}\right)\right)$$

$$+\frac{1}{2!}\left(\frac{i\omega t}{2}\right)^2\left\{2\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right) - \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}\right)\right.$$

$$\left.\times 2\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right) - \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}\right)\right\} + \ldots$$

that is,

$$z(t) = [1 + \frac{i\omega t}{2}\left(2\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right) - \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}\right)\right)$$

$$+\frac{1}{2!}\left(\frac{i\omega t}{2}\right)^2\left\{2\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right) - \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}\right)\right.$$

$$\left.\times 2\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right) - \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\left(\frac{\partial}{\partial z^*} - \frac{\partial}{\partial z}\right)\right\} + \ldots]z$$

$$z(t) = z + \frac{i\omega t}{2}\left(-2z - \frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\right) + \frac{(i\omega t)^2}{4\times 2!}\left\{4z + 2\frac{\lambda}{\omega}\sqrt{\frac{2}{m}}\right.$$

$$\left.\times F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)(1-z) - 2\frac{\lambda}{\omega}\sqrt{\frac{2}{m}}F\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)F'\left(\frac{z+z^*}{\omega\sqrt{2m}}\right)\frac{1}{\omega\sqrt{2m}}(z^*-z)\right\} + \ldots$$

(4.39)

In particular for $\lambda = 0$, we obtain directly from Eq. (4.38) the exact expression

$$z(t) = \exp\left(-i\omega t z \frac{\partial}{\partial z}\right) z$$

or

$$z(t) = \exp(-i\omega t) z \tag{4.40}$$

since

$$z(t) = \omega\sqrt{\frac{m}{2}} q(t) + i\frac{1}{\sqrt{2m}} p(t)$$

$$z = \omega\sqrt{\frac{m}{2}} q + i\frac{1}{\sqrt{2m}} p$$

we may rewrite Eq.(4.36) as

$$\omega\sqrt{\frac{m}{2}} q(t) + i\frac{1}{\sqrt{2m}} p(t) = \exp(-i\omega t)(\omega\sqrt{\frac{m}{2}} q + i\frac{1}{\sqrt{2m}} p)$$

$$\omega\sqrt{\frac{m}{2}} (q(t) + i\frac{1}{m\omega} p(t)) = \exp(-i\omega t)(\omega\sqrt{\frac{m}{2}} (q + i\frac{1}{m\omega} p))$$

$$q(t) + i\frac{1}{m\omega} p(t) = \exp(-i\omega t)(q + i\frac{1}{m\omega} p)$$

$$= (q + i\frac{1}{m\omega} p)(\cos\omega t + i\sin\omega t)$$

$$= q\cos\omega t - iq\sin\omega t + i\frac{1}{m\omega} p\cos\omega t + \frac{p}{m\omega}\sin\omega t)$$

$$q(t) + i\frac{1}{m\omega} p(t) = \left(q\cos\omega t + \frac{p}{m\omega}\sin\omega t\right) + i\left(-q\sin\omega t + \frac{p}{m\omega}\cos\omega t\right)$$

leading to the familiar solution

$$q(t) = q\cos\omega t + \frac{p}{m\omega}\sin\omega t$$

$$p(t) = -q\sin\omega t + \frac{p}{m\omega}\cos\omega t$$

Example 2:

Here we choose

$$a = 1\,,\; b = \frac{1}{m\omega}\,,\; V(q) = \frac{1}{2}m\omega^2 q^2$$

and apply directly formalism II for direct comparison.

From the definition

$$\widetilde{F}[q] = -V'(q)$$

we obtain

$$\widetilde{F}[q] = -m\omega^2 q$$

$$\widetilde{F}'[q] = -m\omega^2$$

$$\widetilde{F}''[q] = 0$$

$$\vdots$$

In the light of formalism II, we have to make the replacement $\widetilde{F}[q] \rightarrow \widetilde{F}[q + \frac{tp}{m}]$ thus obtaining

$$\tilde{F}[q+\frac{tp}{m}] = -m\omega^2(q+\frac{tp}{m})$$

$$\tilde{F}'[q+\frac{tp}{m}] = -m\omega^2$$

$$\tilde{F}''[q+\frac{tp}{m}] = 0$$

$$\vdots$$

From Eq. (4.26) we then have

$$\begin{aligned}
z(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \\
\times\left(\frac{(t-t_1)}{m^n}+\frac{i}{\omega m^n}\right)\tilde{F}[u_n]\times\sum\Bigg\{\left(\prod_{j=2}^{n}(t_1-t_j)^{\delta(k_1,j)}\right)\tilde{F}^{(k_1)}[u_1] \\
\times\left(\prod_{j=3}^{n}(t_2-t_j)^{\delta(k_2,j)}\right)\tilde{F}^{(k_2)}[u_2] \\
\vdots \\
\times\left(\prod_{j=n-2}^{n}(t_{n-3}-t_j)^{\delta(k_{n-3},j)}\right)\tilde{F}^{(k_{n-3})}[u_{n-3}] \\
\times\left(\prod_{j=n-1}^{n}(t_{n-2}-t_j)^{\delta(k_{n-2},j)}\right)\tilde{F}^{(k_{n-2})}[u_{n-2}] \\
\times(t_{n-1}-t_n)^{\delta(k_{n-1},n)}\tilde{F}^{(k_{n-1})}[u_{n-1}]\Bigg\}
\end{aligned} \qquad (4.41)$$

Or equivalently we obtain

$$z(t)=\left(q+\left(\frac{t}{m}+\frac{i}{m\omega}\right)p\right)+\int_0^t dt_1\widetilde{F}[q+\frac{t_1 p}{m}]((t-t_1)+\frac{i}{\omega})\frac{1}{m}+\int_0^t dt_1\int_0^{t_1} dt_2\widetilde{F}[q+\frac{t_2 p}{m}]$$

$$\times((t-t_1)+\frac{i}{\omega})\frac{1}{m^2}\widetilde{F}'[q+\frac{t_1 p}{m}](t_1-t_2)+\ldots \tag{4.42}$$

The explicit time integrals are:

$$\int_0^t dt_1\widetilde{F}[q+\frac{t_1 p}{m}]((t-t_1)+\frac{i}{\omega})\frac{1}{m}$$

$$=\int_0^t dt_1 -m\omega^2(q+\frac{t_1 p}{m})((t-t_1)+\frac{i}{\omega})\frac{1}{m}$$

$$=-\omega^2\int_0^t\left\{q\left\{(t-t_1)+\frac{i}{\omega}\right\}+\frac{p}{m}\left\{(tt_1-t_1^2)+\frac{it_1}{\omega}\right\}\right\}dt_1$$

$$=-\omega^2\left[q\left\{(tt_1-\frac{t_1^2}{2})+\frac{it_1}{\omega}\right\}+\frac{p}{m}\left\{(\frac{tt_1^2}{2}-\frac{t_1^3}{3})+\frac{it_1^2}{2\omega}\right\}\right]_{t_1=0}^{t_1=t}$$

$$=-\omega^2\left(q\frac{t^2}{2}+iq\frac{t}{\omega}+\frac{p}{m}\frac{t^3}{3}+\frac{ip}{m}\frac{t^2}{2\omega}\right) \tag{4.43}$$

Similarly, we may carry out the following time integrations in

$$\int_0^t dt_1\int_0^{t_1} dt_2\widetilde{F}[q+\frac{t_2 p}{m}]((t-t_1)+\frac{i}{\omega})\frac{1}{m^2}\widetilde{F}'[q+\frac{t_1 p}{m}](t_1-t_2)$$

$$= \int_0^t dt_1 \int_0^{t_1} dt_2 ((t-t_1)+\frac{i}{\omega})\frac{1}{m^2}(t_1-t_2)(-m\omega^2[q+\frac{t_2 p}{m}])(-m\omega^2)$$

$$= \omega^4 \int_0^t dt_1 \int_0^{t_1} dt_2 ((t-t_1)+\frac{i}{\omega})(t_1-t_2)[q+\frac{t_2 p}{m}]$$

$$= \omega^4 \int_0^t \left\{(t-t_1)+\frac{i}{\omega}\right\} dt_1 \int_0^{t_1} (t_1-t_2)[q+\frac{t_2 p}{m}] dt_2$$

$$= \omega^4 \int_0^t \left\{(t-t_1)+\frac{i}{\omega}\right\} dt_1 \left[\int_0^{t_1} \left\{q(t_1-t_2)+\frac{p}{m}(t_1 t_2 - t_2^2)\right\} dt_2\right]$$

$$= \omega^4 \int_0^t \left\{(t-t_1)+\frac{i}{\omega}\right\} dt_1 \left[q(t_1 t_2 - \frac{t_2^2}{2}) + \frac{p}{m}(\frac{t_1 t_2^2}{2} - \frac{t_2^3}{3})\right]_{t_2=0}^{t_2=t_1}$$

$$= \omega^4 \int_0^t \left\{(t-t_1)+\frac{i}{\omega}\right\} dt_1 \left[q\frac{t_1^2}{2} + \frac{p t_1^3}{6m}\right]$$

$$= \omega^4 \int_0^t \left(q\frac{t_1^2}{2}(t-t_1) + \frac{iqt_1^2}{2\omega} + \frac{pt_1^3}{6m}(t-t_1) + \frac{ipt_1^3}{6m\omega}\right) dt_1$$

$$= \omega^4 \left[\frac{q}{2}(\frac{tt_1^3}{3} - \frac{t_1^4}{4}) + \frac{iqt_1^3}{6\omega} + \frac{pt_1^3}{6m}(\frac{tt_1^4}{4} - \frac{t_1^5}{5}) + \frac{ipt_1^4}{24m\omega}\right]_{t_1=0}^{t_1=t}$$

$$= \omega^4 \left(\frac{qt^4}{24} + \frac{iqt^3}{6\omega} + \frac{pt^5}{120m} + \frac{ipt^4}{24m\omega}\right) \tag{4.44}$$

Upon the substitution of Eq. (4.22) and Eq. (4.44) in Eq. (4.42), we obtain

$$z(t) = \left(q + \left(\frac{t}{m} + \frac{i}{m\omega}\right)p\right) + -\omega^2\left(q\frac{t^2}{2} + iq\frac{t}{\omega} + \frac{p}{m}\frac{t^3}{3} + \frac{ip}{m}\frac{t^2}{2\omega}\right)$$

$$+\,\omega^4\left(\frac{qt^4}{24}+\frac{iqt^3}{6\omega}+\frac{pt^5}{120m}+\frac{ipt^4}{24m\omega}\right)-\ldots$$

$$=\left(q-\frac{q\omega^2t^2}{2}+\frac{q\omega^4t^4}{24}-\ldots\right)+\left(-\frac{iq\omega^2t}{\omega}+\frac{iq\omega^4t^3}{6\omega}-\ldots\right)$$

$$+\left(\frac{pt}{m}-\frac{p\omega^2t^3}{6m}+\frac{p\omega^4t^5}{120m}-\ldots\right)+\left(\frac{ip}{m\omega}-\frac{ipt^2\omega^2}{2m\omega}+\frac{ipt^4\omega^4}{24m\omega}-\ldots\right)$$

$$=q\left(1-\frac{\omega^2t^2}{2!}+\frac{\omega^4t^4}{4!}-\ldots\right)+(-iq)\left(\omega t-\frac{\omega^3t^3}{3!}+\ldots\right)$$

$$+\left(\frac{p}{m\omega}\right)\left(\omega t-\frac{\omega^3t^3}{3!}+\frac{\omega^5t^5}{5!}-\ldots\right)+\left(\frac{ip}{m\omega}\right)\left(1-\frac{\omega^2t^2}{2!}+\frac{\omega^4t^4}{3!}-\ldots\right)$$

$$=\left(q+\frac{ip}{m\omega}\right)\left(1-\frac{\omega^2t^2}{2!}+\frac{\omega^4t^4}{4!}-\ldots\right)-i\left(q+\frac{ip}{m\omega}\right)\left(\omega t-\frac{\omega^3t^3}{3!}+\frac{\omega^5t^5}{5!}-\ldots\right)$$

$$=\left(q+\frac{ip}{m\omega}\right)\left(1+(-i\omega t)+\frac{(-i)^2\omega^2t^2}{2!}+\frac{(-i)^3\omega^3t^3}{3!}+\frac{(-i)^4\omega^4t^4}{4!}+\frac{(-i)^5\omega^5t^5}{5!}+\ldots\right)$$

leading to

$$z(t)=\exp(-i\omega t)z(0) \tag{4.45}$$

Chapter V

Complexification and the geometrical Berry phase

In this Chapter, we will extend the complex representation of formalism I that was developed in Chapter IV. This is extended to a two-dimensional representation, in a non-trivial way, in view of an application to the classical Foucault pendulum, which provides a modified motion to the standard linear pendulum due to the rotation of the earth. Our exact complex representation developed earlier is quite convenient as it provides, in closed form, two phases one is the so-called *dynamical phase* and the other is a *geometrical phase*. The dynamical phase is, by definition, a function of the coupling parameter (the gravitational coupling constant g) and the earth angular velocity of rotation ω. The geometrical phase, on the other hand, is independent of the coupling parameter and the rotational parameter ω and we will dwell further on why such a geometrical phase arises in this intriguing dynamical problem. The geometrical phase is often referred to as the Berry phase (Berry (1984), Shapere and Wilczek (1989)) for his pioneering work of the nature of geometrical phases as arising in quantum physics. As our problem is purely classical of origin we will not dwell on the quantum counterpart.

Before considering our detailed treatment, we would like to make some additional remarks. A phase, for our purposes, is not a state of matter but a complex number of unit modulus ("phase" is used with the meaning just as an "angle" for whatever possible argument of $\sin(\cdot)$ or $\cos(\cdot)$ or $\exp i(\cdot)$).

The phase we shall be interested in are often associated with cyclic evolution of a physical system. More specifically, we shall find that the cyclic variation of external parameters often leads to a net evolution involving a phase depending only on the geometry of the path traversed in parameter space. In other words, this phase is independent of how fast the various parts of the path are traversed and of the dynamical coupling parameter. For non-cyclic evolution, the extra phase will depend on the end points of the path. The phase is non-integrable; it can not be written as a function just of end points, because it depends on the geometry of the path connecting them as well. Examples of geometric phases abound in many areas of physics. Many familiar problems that we do not ordinarily associate with geometric phases may be phrased in terms of them. Often, the result is a clearer understanding of the structure of the problem and provides an elegant expression for its solution.

5.1 Two-dimensional complexfication and the Foucault pendulum

Here we extend our formalism I of the complex representation derided in Chapter IV to a two-dimensional setting in view of application to the Foucault pendulum. The Hamiltonian for the Foucault pendulum is defined by

$$\mathrm{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{mg}{2L}(x^2 + y^2) + \left(p_x y - p_y x\right)\omega_z \tag{5.1}$$

or

$$\mathrm{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega_0^2}{2}(x^2 + y^2) + \left(p_x y - p_y x\right)\omega_z \tag{5.2}$$

where

$$\omega_z = \omega \sin\lambda \tag{5.3}$$

and λ is the latitude (see figure 5.2)

$$\omega_0 \equiv \sqrt{\frac{g}{L}} \tag{5.4}$$

Here ω is the angular velocity of rotation of the earth, and ω_0 is the angular velocity of the so-called simple pendulum.

Our purpose is to develop the two-dimensional complex representation of the time evolution of the dynamical variable

$$z(t) = x(t) + iy(t) \tag{5.5}$$

We are particularly interested with the problem when

$$t = \frac{2\pi}{\omega} \tag{5.6}$$

That is, when the earth makes a complete rotation about its axis. As initial conditions we take

$$x(0) = x_0 \text{ and } y(0) = 0 \tag{5.7}$$

$$p_x(0) = 0,\; p_y(0) = x_0 m\omega_z \tag{5.8}$$

where g is the gravitational acceleration and L is the length of pendulum. Physically, the initial conditions in Eqs. (5.5) and (5.6) mean that the blob of the pendulum is initially displaced by a distance x_0 and released from rest ($\dot{x}(0)=0$, $\dot{y}(0)=0$).

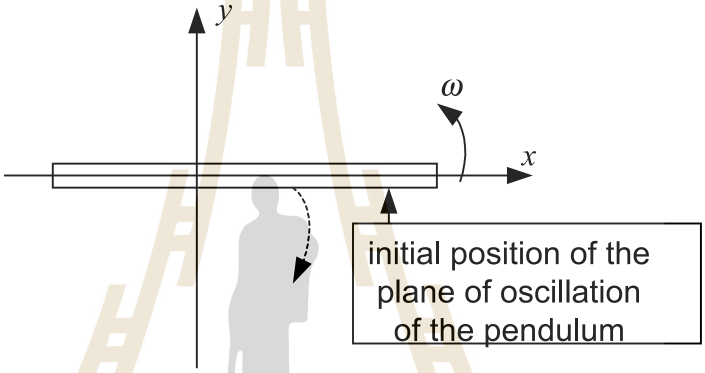

Figure 5.1. The figure shows the initial position of plane of oscillation of the pendulum and the direction of the rotation of the earth (solid line) which is in the opposite direction of the rotation of pendulum (dotted line). (The direction of the z-axis is perpendicular to the paper).

From elementary mechanics, the initial condition $x(0)=x_0$, $y(0)=0$ means that the plane of oscillations of the pendulum is along the x-axis. Since this plane rotates in opposite direction of the earth, the initial position of the plane is 2π rather than zero. That is, we should formally write

$$x(0)=e^{i2\pi}x_0 \tag{5.9}$$

emphasizing the physical aspect that the phase of the plane will reduce from 2π to smaller values as the earth rotates. At this stage it is advisable not to replace $e^{i2\pi}$ by one.

Suppressing the time dependence, we introduce the following variables. We define

$$z = x + iy$$

$$z_1 = x + \frac{ip_x}{m\omega_0}$$

$$z_2 = y + \frac{ip_y}{m\omega_0}$$

The complex conjugates are given by

$$z^* = x - iy$$

$$z_1^* = x - \frac{ip_x}{m\omega_0}$$

$$z_2^* = y - \frac{ip_y}{m\omega_0}$$

so that

$$x = \frac{(z_1 + z_1^*)}{2}$$

$$y = \frac{(z_2 + z_2^*)}{2}$$

$$p_x = \frac{m\omega_0}{2i}(z_1 - z_1^*)$$

$$p_y = \frac{m\omega_0}{2}(z_2 - z_2^*)$$

We may then rewrite Eq. (5.2) as

$$\mathrm{H} = \frac{m\omega_0^2}{2}(z_1 z_1^* + z_2 z_2^*) + \frac{(m\omega_0)}{4i}\omega_z \left\{(z_1 - z_1^*)(z_2 + z_2^*) - (z_1 + z_1^*)(z_2 - z_2^*)\right\}$$

$$= \frac{m\omega_0^2}{2}(z_1 z_1^* + z_2 z_2^*) + \frac{(m\omega_0)}{4i}\omega_z \left\{ z_1 z_2 - z_2 z_1^* + z_1 z_2^* + z_1^* z_2^* - \left\{z_1 z_2 + z_2 z_1^* - z_1 z_2^* - z_1^* z_2^*\right\} \right\}$$

$$= \frac{m\omega_0^2}{2}(z_1 z_1^* + z_2 z_2^*) + \frac{m\omega_z\omega_0}{4i}\{2 z_1 z_2^* - 2 z_2 z_1^*\}$$

$$= \frac{m\omega_0^2}{2}(z_1 z_1^* + z_2 z_2^*) + \frac{m\omega_z\omega_0}{2i}\{z_1 z_2^* - z_2 z_1^*\}$$

$$\mathrm{H} = \frac{m\omega_0^2}{2}(z_1 z_1^* + z_2 z_2^*) + \frac{im\omega_z\omega_0}{2}\{z_2 z_1^* - z_1 z_2^*\} \qquad 5.10)$$

Form Eq. (3.9) in Chapter III, we recall the expression

$$F[q(t), p(t)] = \exp\left[t\left(\frac{\partial \mathrm{H}(0)}{\partial p(0)}\frac{\partial}{\partial q(0)} - \frac{\partial \mathrm{H}(0)}{\partial q(0)}\frac{\partial}{\partial p(0)}\right)\right]F[q(0), p(0)] \qquad (5.11)$$

for further investigations in reference to Eq (5.10), we write

$$\left.\begin{array}{l} \mathrm{H} \equiv \mathrm{H}(0) \\ p \equiv p(0) \\ q \equiv q(0) \\ z \equiv z(0) \\ z_1 \equiv z_1(0) \\ z_1^* \equiv z_1^*(0) \\ z_2 \equiv z_2^*(0) \\ z_2^* \equiv z_2^*(0) \end{array}\right\} \qquad (5.12)$$

In particular,

$$\frac{\partial \mathrm{H}}{\partial p_x} = \frac{\partial \mathrm{H}}{\partial z_1}\frac{\partial z_1}{\partial p_x} + \frac{\partial \mathrm{H}}{\partial z_1^*}\frac{\partial z_1^*}{\partial p_x}$$

$$= \left(\frac{m\omega_0^2}{2} z_1^* - \frac{im\omega_0\omega_z}{2} z_2^*\right)\left(\frac{i}{m\omega_0}\right) + \left(\frac{m\omega_0^2}{2} z_1 + \frac{im\omega_0\omega_z}{2} z_2\right)\left(-\frac{i}{m\omega_0}\right)$$

$$= \frac{i\omega_0}{2}(z_1^* - z_1) + \frac{\omega_z}{2}(z_2^* + z_2) \qquad (5.13)$$

$$\frac{\partial}{\partial x} = \frac{\partial z_1}{\partial x}\frac{\partial}{\partial z_1} + \frac{\partial z_1^*}{\partial x}\frac{\partial}{\partial z_1^*}$$

$$= \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_1^*} \qquad (5.14)$$

$$\frac{\partial \mathrm{H}}{\partial x} = \frac{\partial \mathrm{H}}{\partial z_1}\frac{\partial z_1}{\partial x} + \frac{\partial \mathrm{H}}{\partial z_1^*}\frac{\partial z_1^*}{\partial x}$$

$$= \left(\frac{m\omega_0^2}{2} z_1^* - \frac{im\omega_0\omega_z}{2} z_2^*\right)(1) + \left(\frac{m\omega_0^2}{2} z_1 + \frac{im\omega_0\omega_z}{2} z_2\right)(1)$$

$$= \frac{im\omega_0^2}{2}(z_1^* + z_1) + \frac{im\omega_0\omega_z}{2}(z_2^* - z_2) \tag{5.15}$$

$$\frac{\partial}{\partial p_x} = \frac{\partial z_1}{\partial p_x}\frac{\partial}{\partial z_1} + \frac{\partial z_1^*}{\partial p_x}\frac{\partial}{\partial z_1^*}$$

$$= \left(\frac{i}{m\omega_0}\right)\frac{\partial}{\partial z_1} - \left(\frac{i}{m\omega_0}\right)\frac{\partial}{\partial z_1^*}$$

$$= \frac{i}{m\omega_0}\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_1^*}\right) \tag{5.16}$$

From Eqs. (5.13)-(5.16), we rewrite the first part of the exponential in Eq. (5.11) as

$$\frac{\partial \mathrm{H}}{\partial p_x}\frac{\partial}{\partial x} - \frac{\partial \mathrm{H}}{\partial x}\frac{\partial}{\partial p_x} = \left\{\frac{i\omega_0}{2}(z_1^* - z_1) + \frac{\omega_z}{2}(z_2^* + z_2)\right\}\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_1^*}\right)$$

$$- \left\{\frac{m\omega_0^2}{2}(z_1^* + z_1) + \frac{im\omega_0\omega_z}{2}(z_2 - z_2^*)\right\}\left(\frac{i}{m\omega_0}\right)\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_1^*}\right)$$

$$= \left\{\frac{i\omega_0}{2}(z_1^* - z_1) + \frac{\omega_z}{2}(z_2^* + z_2)\right\}\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_1^*}\right)$$

$$- \left\{\frac{i\omega_0}{2}(z_1^* + z_1) - \frac{\omega_z}{2}(z_2 - z_2^*)\right\}\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_1^*}\right)$$

$$= (\omega_z z_2 - i\omega_0 z_1)\frac{\partial}{\partial z_1} + (\omega_z z_2^* + i\omega_0 z_1^*)\frac{\partial}{\partial z_1^*} \tag{5.17}$$

Similarly, we obtain

$$\frac{\partial \mathrm{H}}{\partial p_y} = \frac{\partial \mathrm{H}}{\partial z_2}\frac{\partial z_2}{\partial p_y} + \frac{\partial \mathrm{H}}{\partial z_2^*}\frac{\partial z_2^*}{\partial p_y}$$

$$= \left(\frac{m\omega_0^2}{2} z_2^* + \frac{im\omega_0\omega_z}{2} z_1^*\right)\left(\frac{i}{m\omega_0}\right) + \left(\frac{m\omega_0^2}{2} z_2 - \frac{im\omega_0\omega_z}{2} z_1\right)\left(-\frac{i}{m\omega_0}\right)$$

$$= \frac{i\omega_0}{2}(z_2^* - z_2) - \frac{\omega_z}{2}(z_1^* + z_1) \tag{5.18}$$

$$\frac{\partial}{\partial y} = \frac{\partial z_2}{\partial y}\frac{\partial}{\partial z_2} + \frac{\partial z_2^*}{\partial y}\frac{\partial}{\partial z_2^*}$$

$$= \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2^*} \tag{5.19}$$

$$\frac{\partial \mathrm{H}}{\partial y} = \frac{\partial \mathrm{H}}{\partial z_2}\frac{\partial z_2}{\partial y} + \frac{\partial \mathrm{H}}{\partial z_2^*}\frac{\partial z_2^*}{\partial y}$$

$$= \left(\frac{m\omega_0^2}{2} z_2^* + \frac{im\omega_0\omega_z}{2} z_1^*\right)(1) + \left(\frac{m\omega_0^2}{2} z_2 - \frac{im\omega_0\omega_z}{2} z_1\right)(1)$$

$$= \frac{im\omega_0^2}{2}(z_2^* + z_2) + \frac{m\omega_0\omega_z}{2}(z_1^* - z_1) \tag{5.20}$$

$$\frac{\partial}{\partial p_y} = \frac{\partial z_2}{\partial p_y}\frac{\partial}{\partial z_2} + \frac{\partial z_2^*}{\partial p_y}\frac{\partial}{\partial z_2^*}$$

$$= \left(\frac{i}{m\omega_0} \right) \frac{\partial}{\partial z_2} - \left(\frac{i}{m\omega_0} \right) \frac{\partial}{\partial z_2^*}$$

$$= \frac{i}{m\omega_0} \left(\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_2^*} \right) \tag{5.21}$$

From Eqs. (5.13)-(5.16), we rewrite the second part of the exponential in Eq. (5.11) as

$$\begin{aligned}
\frac{\partial \mathrm{H}}{\partial p_y}\frac{\partial}{\partial y} - \frac{\partial \mathrm{H}}{\partial y}\frac{\partial}{\partial p_y} &= \left\{ \frac{i\omega_0}{2}(z_2^* - z_2) - \frac{\omega_z}{2}(z_1^* + z_1) \right\} \left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2^*} \right) \\
&\quad - \left\{ \frac{m\omega_0^2}{2}(z_2^* + z_2) + \frac{im\omega_0\omega_z}{2}(z_1^* - z_1) \right\} \left(\frac{i}{m\omega_0} \right) \left(\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_2^*} \right) \\
&= \left\{ \frac{i\omega_0}{2}(z_2^* - z_2) - \frac{\omega_z}{2}(z_1^* + z_1) \right\} \left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2^*} \right) \\
&\quad - \left\{ \frac{i\omega_0}{2}(z_2^* + z_2) - \frac{\omega_z}{2}(z_1^* - z_1) \right\} \left(\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_2^*} \right) \\
&= \left(i\omega_0 z_2^* - \omega_z z_1^* \right) \frac{\partial}{\partial z_2^*} - \left(i\omega_0 z_2 + \omega_z z_1 \right) \frac{\partial}{\partial z_2}
\end{aligned} \tag{5.22}$$

Upon combining Eq. (5.17) and Eq. (5.22), we obtain

$$\left(\frac{\partial \mathrm{H}}{\partial p_x}\frac{\partial}{\partial x} - \frac{\partial \mathrm{H}}{\partial x}\frac{\partial}{\partial p_x} \right) + \left(\frac{\partial \mathrm{H}}{\partial p_y}\frac{\partial}{\partial y} - \frac{\partial \mathrm{H}}{\partial y}\frac{\partial}{\partial p_y} \right) = (\omega_z z_2 - i\omega_0 z_1) \frac{\partial}{\partial z_1} + \left(\omega_z z_2^* + i\omega_0 z_1^* \right) \frac{\partial}{\partial z_1^*}$$

$$+\left(i\omega_0 z_2^* - \omega_z z_1^*\right)\frac{\partial}{\partial z_2^*} - \left(i\omega_0 z_2 + \omega_z z_1\right)\frac{\partial}{\partial z_2}$$

(5.23)

Now it is convenient to introduce the following new variables: U, U^*, V and V^*, defined as follows:

$$U = z_1 + iz_2$$

$$V = z_1 - iz_2$$

$$U^* = z_1^* - iz_2^*$$

$$V^* = z_1^* + iz_2^*$$

Or the old variables we may conveniently write

$$z_1 = \frac{U+V}{2}$$

$$z_2 = \frac{U-V}{2i}$$

$$z_1^* = \frac{U^* + V^*}{2}$$

$$z_2^* = \frac{V^* - U^*}{2i}$$

$$z = x + iy = \frac{U + V^*}{2}$$

$$\frac{\partial}{\partial z_1} = \frac{\partial U}{\partial z_1}\frac{\partial}{\partial U} + \frac{\partial V}{\partial z_1}\frac{\partial}{\partial V} = \frac{\partial}{\partial U} + \frac{\partial}{\partial V}$$

$$\frac{\partial}{\partial z_2}=\frac{\partial U}{\partial z_2}\frac{\partial}{\partial U}+\frac{\partial V}{\partial z_2}\frac{\partial}{\partial V}=i\left(\frac{\partial}{\partial U}-\frac{\partial}{\partial V}\right)$$

$$\frac{\partial}{\partial z_1^*}=\frac{\partial U^*}{\partial z_1^*}\frac{\partial}{\partial U^*}+\frac{\partial V^*}{\partial z_1^*}\frac{\partial}{\partial V^*}=\frac{\partial}{\partial U^*}+\frac{\partial}{\partial V^*}$$

$$\frac{\partial}{\partial z_2^*}=\frac{\partial U^*}{\partial z_2^*}\frac{\partial}{\partial U^*}+\frac{\partial V^*}{\partial z_2^*}\frac{\partial}{\partial V^*}=i\left(\frac{\partial}{\partial V^*}-\frac{\partial}{\partial U^*}\right)$$

In terms of the variables U, U^*, V and V^*, we rewrite Eq. (5.23) as

$$\left(\frac{\partial \mathrm{H}}{\partial p_x}\frac{\partial}{\partial x}-\frac{\partial \mathrm{H}}{\partial x}\frac{\partial}{\partial p_x}\right)+\left(\frac{\partial \mathrm{H}}{\partial p_y}\frac{\partial}{\partial y}-\frac{\partial \mathrm{H}}{\partial y}\frac{\partial}{\partial p_y}\right)$$

$$=\left(\omega_z\left(\frac{U-V}{2i}\right)-i\omega_0\left(\frac{U+V}{2}\right)\right)\left(\frac{\partial}{\partial U}+\frac{\partial}{\partial V}\right)$$

$$+\left(\omega_z\left(\frac{V^*-U^*}{2i}\right)-i\omega_0\left(\frac{U^*+V^*}{2}\right)\right)\left(\frac{\partial}{\partial U^*}+\frac{\partial}{\partial V^*}\right)$$

$$+\left(i\omega_0\left(\frac{V^*-U^*}{2i}\right)-\omega_z\left(\frac{U^*+V^*}{2}\right)\right)(i)\left(\frac{\partial}{\partial V^*}-\frac{\partial}{\partial U^*}\right)$$

$$-\left(i\omega_0\left(\frac{U-V}{2i}\right)+\omega_z\left(\frac{U+V}{2}\right)\right)(i)\left(\frac{\partial}{\partial U}-\frac{\partial}{\partial V}\right)$$

$$=\left(-\frac{i}{2}\right)\left(\omega_z(U-V)+\omega_0(U+V)\right)\left(\frac{\partial}{\partial U}+\frac{\partial}{\partial V}\right)$$

$$+\left(\frac{i}{2}\right)\left(-\omega_z\left(V^*-U^*\right)+\omega_0\left(U^*+V^*\right)\right)\left(\frac{\partial}{\partial U^*}+\frac{\partial}{\partial V^*}\right)$$

$$+\left(\frac{i}{2}\right)\left(\omega_0\left(V^* - U^*\right) - \omega_z\left(U^* + V^*\right)\right)\left(\frac{\partial}{\partial V^*} - \frac{\partial}{\partial U^*}\right)$$

$$-\left(\frac{i}{2}\right)\left(\omega_0(U - V) + \omega_z(U + V)\right)\left(\frac{\partial}{\partial U} - \frac{\partial}{\partial V}\right)$$

$$= -\frac{i}{2}\omega_z U\left(\frac{\partial}{\partial U} + \frac{\partial}{\partial V}\right) + \frac{i}{2}\omega_z V\left(\frac{\partial}{\partial U} + \frac{\partial}{\partial V}\right) - \frac{i}{2}\omega_0 U\left(\frac{\partial}{\partial U} + \frac{\partial}{\partial V}\right) - \frac{i}{2}\omega_0 V\left(\frac{\partial}{\partial U} + \frac{\partial}{\partial V}\right)$$

$$-\frac{i}{2}\omega_0 U\left(\frac{\partial}{\partial U} - \frac{\partial}{\partial V}\right) + \frac{i}{2}\omega_0 V\left(\frac{\partial}{\partial U} - \frac{\partial}{\partial V}\right) - \frac{i}{2}\omega_z U\left(\frac{\partial}{\partial U} - \frac{\partial}{\partial V}\right) - \frac{i}{2}\omega_z V\left(\frac{\partial}{\partial U} - \frac{\partial}{\partial V}\right)$$

$$+\frac{i}{2}\omega_0 U^*\left(\frac{\partial}{\partial U^*} + \frac{\partial}{\partial V^*}\right) + \frac{i}{2}\omega_0 V^*\left(\frac{\partial}{\partial U^*} + \frac{\partial}{\partial V^*}\right) + \frac{i}{2}\omega_z U^*\left(\frac{\partial}{\partial U^*} + \frac{\partial}{\partial V^*}\right)$$

$$-\frac{i}{2}\omega_z V^*\left(\frac{\partial}{\partial U^*} + \frac{\partial}{\partial V^*}\right) - \frac{i}{2}\omega_0 U^*\left(\frac{\partial}{\partial V^*} - \frac{\partial}{\partial U^*}\right) + \frac{i}{2}\omega_0 V^*\left(\frac{\partial}{\partial V^*} - \frac{\partial}{\partial U^*}\right)$$

$$-\frac{i}{2}\omega_z U^*\left(\frac{\partial}{\partial V^*} - \frac{\partial}{\partial U^*}\right) - \frac{i}{2}\omega_z V^*\left(\frac{\partial}{\partial V^*} - \frac{\partial}{\partial U^*}\right)$$

$$= -i\omega_z U\frac{\partial}{\partial U} + i\omega_z V\frac{\partial}{\partial V} + i\omega_z U^*\frac{\partial}{\partial U^*} - i\omega_z V^*\frac{\partial}{\partial V^*} - i\omega_0 U\frac{\partial}{\partial U} - i\omega_0 V\frac{\partial}{\partial V}$$

$$+ i\omega_0 U^*\frac{\partial}{\partial U^*} + i\omega_0 V^*\frac{\partial}{\partial V^*}$$

$$= (-i)(\omega_0 + \omega_z)U\frac{\partial}{\partial U} + (-i)(\omega_0 - \omega_z)V\frac{\partial}{\partial V} + (i)(\omega_0 + \omega_z)U^*\frac{\partial}{\partial U^*}$$

$$+ (i)(\omega_0 - \omega_z)V^*\frac{\partial}{\partial V^*} \tag{5.24}$$

The exponential expression in Eq. (5.10) then reads

$$\exp t\left\{(-i)(\omega_0+\omega_z)U\frac{\partial}{\partial U}+(-i)(\omega_0-\omega_z)V\frac{\partial}{\partial V}+(i)(\omega_0+\omega_z)U^*\frac{\partial}{\partial U^*}\right.$$
$$\left.+(i)(\omega_0-\omega_z)V^*\frac{\partial}{\partial V^*}\right\} \qquad (5.25)$$

We also rewrite the variable $z=x+iy$ in terms of U, U^*, V and V^*:

$$z=\left(\frac{z_1+z_1^*}{2}\right)+i\left(\frac{z_2+z_2^*}{2}\right)$$
$$=\frac{1}{2}\left\{\left(\frac{U+V}{2}\right)+\left(\frac{U^*+V^*}{2}\right)\right\}+i\frac{1}{2}\left\{\left(\frac{U-V}{2i}\right)+\left(\frac{V^*-U^*}{2i}\right)\right\}$$
$$=\frac{1}{2}(U+V^*) \qquad (5.26)$$

To emphasize that we are dealing with the actual non-static earth, that is, in rotation, we will denote our dynamical variable by

$$z[\omega,t] \text{ and } z[\omega,0]=z(\omega) \qquad (5.27)$$

At any time t, we have from Eq. (5.10)

$$z[\omega,t]=\exp t\left\{(-i)(\omega_0+\omega_z)U\frac{\partial}{\partial U}+(-i)(\omega_0-\omega_z)V\frac{\partial}{\partial V}+(i)(\omega_0+\omega_z)U^*\frac{\partial}{\partial U^*}\right.$$
$$\left.+(i)(\omega_0-\omega_z)V^*\frac{\partial}{\partial V^*}\right\}z(\omega) \tag{5.28}$$

From Eq . (5.27), we see, that $z(\omega)$ is independent on U^* and V, but depends on U and V^*. That is, Eq. (5.28) simplifies to

$$z[\omega,t]=\exp t\left\{(-i)(\omega_0+\omega_z)U\frac{\partial}{\partial U}+(i)(\omega_0-\omega_z)V^*\frac{\partial}{\partial V^*}\right\}z(\omega) \tag{5.29}$$

where

$$z(\omega)=\frac{U+V^*}{2} \tag{5.30}$$

We use the identity

$$e^{\beta x\frac{d}{dx}}x=xe^{\beta} \tag{5.31}$$

where β is an arbitrary constant, to rewrite Eq. (5.29) as

$$z[\omega,t]=\exp t\left\{(-i)(\omega_0+\omega_z)U\frac{\partial}{\partial U}+(i)(\omega_0-\omega_z)V^*\frac{\partial}{\partial V^*}\right\}\left(\frac{U+V^*}{2}\right)$$

$$= \frac{1}{2}\left\{ e^{(-i)t(\omega_0+\omega_z)U\frac{\partial}{\partial U}} U + e^{(i)t(\omega_0-\omega_z)V^*\frac{\partial}{\partial V^*}} V^* \right\}$$

$$= e^{-it\omega_z}\frac{1}{2}\left\{ Ue^{-it\omega_0} + V^* e^{it\omega_0} \right\}$$

From the initial conditions given in Eqs. (5.7)-(5.8), we have since $\omega << \omega_0$ ($\omega/\omega_0 \approx 10^{-5}$ for Foucault pendulum)

$$U\Big|_{t=0} = e^{i2\pi} x_0 \tag{5.32}$$

$$V^*\Big|_{t=0} = e^{i2\pi} x_0 \tag{5.33}$$

leading to the final solution at any time t is given by

$$z[\omega, t] = e^{-it\omega_z} e^{i2\pi} x_0 \cos(\omega_0 t) \tag{5.34}$$

The phase factor $\omega_0 t$, with $\omega_0 = \sqrt{g/L}$, is the familiar dynamical phase of the simple pendulum. The other phase factor $e^{-it\omega_z}$, with $\omega_z = \omega \sin\lambda$, is discussed next.

5.2 Derivation of the geometrical phase and its significance

In reference to Section 5.1, we have the solution of the two-dimensional of Foucault pendulum when the earth makes a full rotation, that is, when t is equal to $2\pi/\omega$. From Eq. (5.34) this is given by

$$z[\omega, \frac{2\pi}{\omega}] = e^{i2\pi(1-\sin\lambda)} x_0 \cos(2\pi \frac{\omega_0}{\omega}) \tag{5.35}$$

where

$$z[\omega, t] = x(t) + iy(t)$$

$$z[\omega, 0] = x(0) + iy(0) \equiv x(0)$$

To obtain insight into the phase $e^{i2\pi(1-\sin\lambda)}$, we note that the solid angle subtended at the center of the earth when the earth make a full rotation is given by

$$\Omega = \int_0^\theta d\Omega \tag{5.36}$$

where

$$d\Omega = 2\pi \sin\theta' d\theta'$$

$$\theta = \left(\frac{\pi}{2} - \lambda\right)$$

Upon integration of Eq. (5.36), we obtain

$$\Omega = 2\pi \int_0^\theta \sin\theta' d\theta'$$

$$= 2\pi\left(1 - \cos\left(\frac{\pi}{2} - \lambda\right)\right)$$

$$\therefore \quad = 2\pi(1 - \sin\lambda) \tag{5.37}$$

The solid angle is depicted in figure 5.2 and is given by Eq. (5.37).

That is, we may write

$$z[\omega, \frac{2\pi}{\omega}] = e^{i\Omega} x_0 \cos(2\pi \frac{\omega_0}{\omega}) \tag{5.38}$$

where

$$\Omega = 2\pi(1 - \sin\lambda)$$

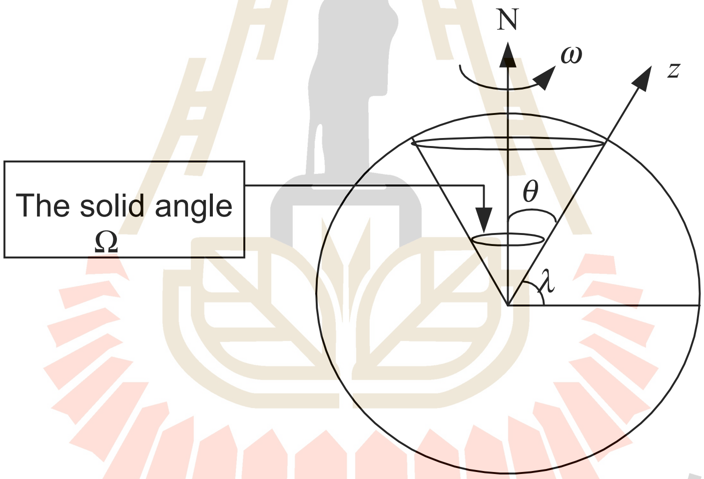

Figure 5.2. The solid angle subtended at the center of the earth and the definition of Ω.

The phase factor $\exp i2\pi(1-\sin\lambda) = \exp i\Omega$ is independent of g and ω. That is, it is independent on how fast the earth is rotating and how strong the gravitational strength is. It is thus a geometrical phase as opposed to the phase factor $2\pi\,\omega_0/\omega \equiv 2\pi\sqrt{g/L}/\omega$ which depends on both g and ω, and is termed as a dynamical phase. At the North Pole $\sin\lambda = 1$. That is, at the North Pole, the initial phase goes from 2π to 0. The physical reason why such a general phase occurs is simple and is easily explained. The plane of oscillations, in which the pendulum swings, lags behind the earth rotation thus not necessarily making a complete revolution to catch up with the earth rotation. At the North Pole, the earth, at that point, is stationary, while the plane of oscillations of the pendulum makes a complete revolution going from a phase of 2π to 0. At the equator, the plane of oscillations of the pendulum does not rotate and the initial phase of the plane stays always at a 2π angle with the x-axis. The interesting situation arises for latitudes $0 < \lambda < \pi/2$. It is of great physical interest that such a geometrical phase is measurable since $x(2\pi/\omega)$, $y(2\pi/\omega)$ may be measured and hence $z(2\pi/\omega)$ is measurable. The determined phase gives us the latitude at which the experiment was performed. The geometrical phase is of a discontinuous in nature for $\omega \equiv 0$ corresponding to a “static” earth. In such a case the plane of oscillations of the pendulum does not rotate and hence here there is no question of a geometrical phase.

Chapter VI

A path integral formulation

One of the most powerful tools that theoretical physicists use today is the path integral formulation of the quantum mechanics developed by Feynman. The Feynman path integral method is a reformulation of quantum mechanics in an alternate way to the earlier methods developed by Schroedinger and Heisenberg about three decades earlier. The Feynman path integral gives rise to the expression for the so-called *Green function* or for the *propagator*, which propagates a wave function in time and hence, as in the conventional formulation, is solves the dynamical problem. The Feynman path integral, in its simplest form may be written as

$$\langle q(T)|q(0)\rangle = \sum_{paths} \exp\left(\frac{i}{\hbar}\int_0^T \mathcal{L}\,(q(t),\dot{q}(t))dt\right) \qquad (6.1)$$

where $\mathcal{L}\,(q(t),\dot{q}(t))$ is the Lagrangian of the theory, T denotes the time of propagation and the sum is over all paths, including the classical one, from end point $q(0)$ to end point $q(T)$. For $\hbar \to 0$, the classical path dominates in Eq. (6.1) in conformity with one's intuition that quantum physics goes over to classical physics in this limit. Clearly, to develop the corresponding expression for classical dynamics using Eq. (6.1) as it stands is not very useful as one would have to restrict the sum in Eq. (6.1) just to one term corresponding to the classical path. Accordingly, the so-called action integral

$$A = \int_0^T L\,(q(t), \dot{q}(t))dt \tag{6.2}$$

becomes just replaced by the classical one

$$A_C = \int_0^T L\,(q_C(t), \dot{q}_C(t))dt \tag{6.3}$$

where $(q(t)_C, \dot{q}(t)_C)$ corresponds to the classical trajectory and Eq. (6.1) as it stands would not be of practical value in classical dynamics.

The above then shows that an alternative method is necessary to be developed for a path integral formulation of classical dynamical problems, which we now propose.

For a self-adjoint operator A in quantum physics with a spectrum say λ_1, λ_2, … and eigenvectors $|\lambda_1\rangle, |\lambda_2\rangle, \ldots$, one may define the unitary operator

$$e^{itA} = \sum_k e^{it\lambda_k} |\lambda_k\rangle\langle\lambda_k| \tag{6.4}$$

where t in an arbitrary parameter. In particular for $t = 0$, the left-hand side of Eq. (6.4) becomes the unit operator ($\mathbf{1}$), and the equation reduces to

$$\mathbf{1} = \sum_k |\lambda_k\rangle\langle\lambda_k| \tag{6.5}$$

For any vector $|\Psi\rangle$ in the Hilbert space we may then write

$$\mathbf{1}|\Psi\rangle = |\Psi\rangle = \sum_k |\lambda_k\rangle\langle\lambda_k|\Psi\rangle \tag{6.6}$$

The decomposition in Eq. (6.5) in terms of the projection operators $|\lambda_k\rangle\langle\lambda_k|$ on physical states is referred to as the resolution of the identity. We will essentially use this idea of developing a resolution of the identity, as directly obtained from Hamilton' s equations, q and p with projections on the "paths" as dictated from these equations. It is almost obvious that we will need additional variables to q and p which are referred to as Lagrange multipliers variables which restrict, at every given time t, the pair of variables $(q(t), p(t))$ to fall on the correct classical path in phase space.

The path integral formulation, as a solution of the identity, is directly obtained by integrating Hamilton' s equation. This we do next. To simplify the notation, we suppress the indices referring to the various degrees of freedom. Our results are valid for any arbitrary number of degrees of freedom. Thus $(q(t), p(t))$ stands for ($q_1(t), q_2(t), q_3(t), \ldots, p_1(t), p_2(t), p_3(t), \ldots$).

6.1 A path integral as a resolution of the identity

We begin with Hamilton' s equation

$$\frac{\partial \mathrm{H}(t)}{\partial q} = -\dot{p}(t) \tag{6.7}$$

and

$$\frac{\partial \mathrm{H}(t)}{\partial p} = \dot{q}(t) \tag{6.8}$$

To find $q(t)$, we integrate Eq (6.7) over *t*, to obtain from

$$\frac{dq(t')}{dt'} = \frac{\partial \mathrm{H}(t')}{\partial p}$$

the following expressions

$$\int_{q(0)}^{q(t)} dq(t') = \int_0^t \frac{\partial \mathrm{H}(t')}{\partial p} dt'$$

$$q(t) - q(0) = \int_0^t \frac{\partial \mathrm{H}(t')}{\partial p} dt'$$

$$q(t) = q(0) + \int_0^t \frac{\partial \mathrm{H}(t')}{\partial p} dt' \tag{6.9}$$

where $q(0)$ denote represents the initial condition.

Similarly, to find $p(t)$ we integrate Eq. (6.8) over t with an initial condition $p(0)$ at $t = 0$, to obtain from

$$\frac{dp(t')}{dt'} = -\frac{\partial \mathrm{H}(t)}{\partial q}$$

the following expressions

$$\int_{p(0)}^{p(t)} dp(t') = -\int_0^t \frac{\partial \mathrm{H}(t')}{\partial q} dt'$$

$$p(t) - p(0) = -\int_0^t \frac{\partial \mathrm{H}(t')}{\partial q} dt'$$

$$p(t) = p(0) - \int_0^t \frac{\partial \mathrm{H}(t')}{\partial q} dt' \qquad (6.10)$$

We are interested in motion in phase space with a given initial condition ($q(0)$, $p(0)$). Let $F[q(t), p(t)]$ be any arbitrary functions of $q(t)$ and $p(t)$. From Eq. (6.9) and Eq. (6.4) we may rewrite $F[q(t), p(t)]$ as

$$F[q(t), p(t)] = F[q(0) + \int_0^t \frac{\partial \mathrm{H}(t')}{\partial p} dt', p(0) - \int_0^t \frac{\partial \mathrm{H}(t')}{\partial q} dt'] \qquad (6.11)$$

To analyze Eq. (6.4) we have to consider the meaning of the corresponding ordinary Riemann integral. For a very fine partition of the t-axis, under the curve, we have the situation shown in figure 6.1 below.

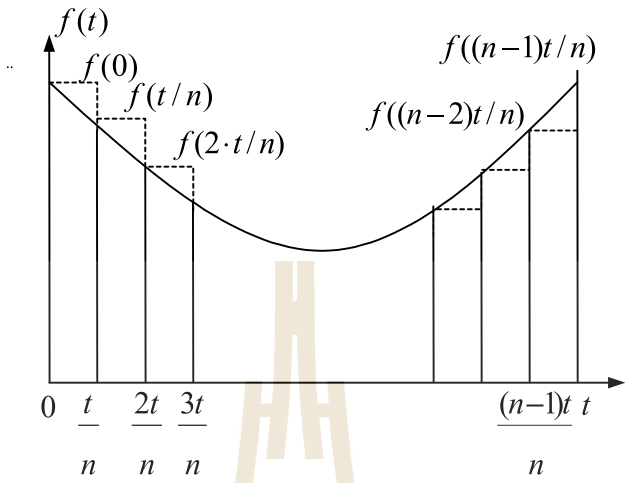

Figure 6.1. The definition of the ordinary Riemann integral. A set of ordinates is drawn from the abscissa to the curve. The ordinates are spaced a distance Δt apart. The integral (area between the curve and the abscissa) is approximated by Δt times the sum of the ordinates. This approximation approaches the correct value of the integral as Δt approaches zero.

In reference to Figure.6.1, we have

$$\Delta t = \frac{t}{n}$$

The area under the curve is then approximately given by

$$A \approx \Delta t f(0) + \Delta t f(\frac{t}{n}) + \Delta t f(\frac{2t}{n}) + \text{K} + \Delta t f(\frac{(n-1)t}{n})$$

$$= \frac{t}{n}\sum_{m=0}^{n-1} f(\frac{mt}{n}) \tag{6.12}$$

We have divided the interval from 0 to t into n subintervals each of length t/n. Evaluating each of the summands at the left-end points of these subintervals, we rewrite the right-hand side of Eq. (6.11) rigorously as

$$\lim_{n\to\infty} F[q_0 + \frac{t}{n}\sum_{m=0}^{n-1} \frac{\partial \mathrm{H}}{\partial p}(\frac{mt}{n}), p_0 - \frac{t}{n}\sum_{m=0}^{n-1} \frac{\partial \mathrm{H}}{\partial q}(\frac{mt}{n})] \tag{6.13}$$

where

$$q_0 = q(0) \text{ and } p_0 = p(0)$$

To evaluate Eq. (6.6), we use in the process, the definition of the Dirac delta function δ which, in particular, satisfies the properties

$$\int_a^b f(x)\delta(x-x_0)dx = f(x_0), \qquad a < x_0 < b \tag{6.14}$$

$$\delta(x) = \delta(-x) \tag{6.15}$$

The expression in Eq. (6.13), to which the limit $n \to \infty$ is to be taken, may be equivalently rewritten as

$$F[q_0 + \frac{t}{n}\sum_{m=0}^{n-1}\frac{\partial H}{\partial p}(\frac{mt}{n}), p_0 - \frac{t}{n}\sum_{m=0}^{n-1}\frac{\partial H}{\partial q}(\frac{mt}{n})]$$

$$= \int dq_n dp_n \delta((q_0 + \frac{t}{n}\sum_{m=0}^{n-1}\frac{\partial H}{\partial p}(\frac{mt}{n})) - q_n)\delta((p_0 - \frac{t}{n}\sum_{m=0}^{n-1}\frac{\partial H}{\partial q}(\frac{mt}{n})) - p_n)F[q_n, p_n]$$

(6.16)

As is easily checked for our purposes, the later may be also rewritten in the more convenient form:

$$= \left[\int dq_1 dp_1 \delta((q_0 + \frac{t}{n}\frac{\partial H}{\partial p}(0)) - q_1)\delta((p_0 - \frac{t}{n}\frac{\partial H}{\partial q}(0)) - p_1)\right.$$

$$\times \int dq_2 dp_2 \delta((q_0 + \frac{t}{n}(\frac{\partial H}{\partial p}(0) + \frac{\partial H}{\partial p}(\frac{1t}{n}))) - q_2)\delta((p_0 - \frac{t}{n}(\frac{\partial H}{\partial q}(0) + \frac{\partial H}{\partial q}(\frac{1t}{n}))) - p_2)$$

$$\vdots$$

$$\times \int dq_n dp_n \delta((q_0 + \frac{t}{n}\left\{\frac{\partial H}{\partial p}(0) + \frac{\partial H}{\partial p}(\frac{1t}{n}) + \ldots + \frac{\partial H}{\partial p}(\frac{(n-1)t}{n})\right\}) - q_n)\delta((p_0 - \frac{t}{n}\left\{\frac{\partial H}{\partial q}(0)\right.$$

$$\left.\left. \times \frac{\partial H}{\partial q}(\frac{1t}{n}) + \ldots + \frac{\partial H}{\partial q}(\frac{(n-1)t}{n})\right\}) - p_n)\right] \times F[q_n, p_n] \qquad (6.17)$$

In the above equation, we have used the facts that:

$$q_0 \equiv q(0)$$

$$q(\frac{t}{n}) = q_0 + \frac{t}{n}\frac{\partial H}{\partial p}(0)$$

$$q(\frac{2t}{n}) = q_1 + \frac{t}{n}\frac{\partial H}{\partial p}(\frac{1t}{n})$$

$$\vdots$$

$$q(\frac{(m-1)t}{n}) = q_{m-2} + \frac{t}{n}\frac{\partial H}{\partial p}(\frac{(m-2)t}{n})$$

$$q(\frac{mt}{n}) = q_{m-1} + \frac{t}{n}\frac{\partial H}{\partial p}(\frac{(m-1)t}{n}) \qquad (6.18)$$

Similarly, for the momentum values, we have

$$p_0 \equiv p(0)$$

$$p(\frac{t}{n}) = p_0 + \frac{t}{n}\frac{\partial H}{\partial q}(0)$$

$$p(\frac{2t}{n}) = p_1 + \frac{t}{n}\frac{\partial H}{\partial q}(\frac{1t}{n})$$

$$\vdots$$

$$p(\frac{(m-1)t}{n}) = p_{m-2} + \frac{t}{n}\frac{\partial H}{\partial q}(\frac{(m-2)t}{n})$$

$$p(\frac{mt}{n}) = p_{m-1} + \frac{t}{n}\frac{\partial H}{\partial q}(\frac{(m-1)t}{n}) \qquad (6.19)$$

We may then rewrite the Eq. (17) in the form of a product as follows:

$$= \Big[\int dq_1 dp_1 \delta((q_0 + \frac{t}{n}\frac{\partial H}{\partial p}(0)) - q_1)\delta((p_0 - \frac{t}{n}\frac{\partial H}{\partial q}(0)) - p_1)$$

$$\times\int dq_2dp_2\delta((q_1+\frac{t}{n}\frac{\partial H}{\partial p}(\frac{1t}{n}))-q_2)\delta((p_1-\frac{t}{n}\frac{\partial H}{\partial q}(\frac{1t}{n}))-p_2)$$

M

$$\times\int dq_ndp_n\delta((q_{n-1}+\frac{t}{n}\frac{\partial H}{\partial p}(\frac{(n-1)t}{n}))-q_n)\delta((p_{n-1}-\frac{t}{n}\frac{\partial H}{\partial q}(\frac{(n-1)t}{n}))-p_n)\Bigg]$$

$$\times F[q_n,p_n] \tag{6.20}$$

$$=\int[\prod_{m=1}^{n} dq_mdp_m\delta((q_{m-1}+\frac{t}{n}\frac{\partial H}{\partial p}(\frac{(m-1)t}{n}))-q_m)\delta((p_{m-1}-\frac{t}{n}\frac{\partial H}{\partial q}(\frac{(m-1)t}{n}))-p_m)]$$

$$\times F[q_n,p_n] \tag{6.21}$$

For simplicity of the notation, we use the notation $f(k)$ for $f(\frac{kt}{n})$ to rewrite the latter as

$$=\int[\prod_{k=1}^{n} dq_kdp_k\delta((q_{k-1}+\frac{t}{n}\frac{\partial H}{\partial p}(k-1))-q_k)\delta((p_{k-1}-\frac{t}{n}\frac{\partial H}{\partial q}(k-1))-p_k)]$$
$$\times F[q_n,p_n]$$
$$=\int[\prod_{k=1}^{n} dq_kdp_k\delta(\frac{t}{n}\frac{\partial H}{\partial p}(k-1)-(q_k-q_{k-1}))\delta(-\frac{t}{n}\frac{\partial H}{\partial q}(k-1)-(p_k-p_{k-1}))]$$
$$\times F[q_n,p_n] \tag{6.22}$$

or equivalently as

$$= \int [\prod_{k=1}^{n} dq_k dp_k \delta(\frac{t}{n}(\frac{\partial H}{\partial p}(k-1) - (\frac{q_k - q_{k-1}}{t/n})))\delta(\frac{t}{n}(\frac{\partial H}{\partial q}(k-1) + (\frac{p_k - p_{k-1}}{t/n})))]$$

$$\times F[q_n, p_n] \tag{6.23}$$

From Eq. (6.23) we shall develop a path integral expression. To this end, we recall the formal Dirac delta expression:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} \tag{6.24}$$

This allows us to write

$$\delta(\frac{t}{n}(\frac{\partial H}{\partial p}(k-1) - (\frac{q_k - q_{k-1}}{t/n}))) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial p}(k-1) - (\frac{q_k - q_{k-1}}{t/n}))\lambda_k} d\lambda_k$$

and

$$\delta(\frac{t}{n}(\frac{\partial H}{\partial q}(k-1) + (\frac{p_k - p_{k-1}}{t/n}))) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial q}(k-1) + (\frac{p_k - p_{k-1}}{t/n}))\eta_k} d\eta_k$$

and hence obtain for the expression in Eq. (6.23) the following very convenient one:

$$= \int [\prod_{k=1}^{n} dq_k dp_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial p}(k-1) - (\frac{q_k - q_{k-1}}{t/n}))\lambda_k} d\lambda_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial q}(k-1) + (\frac{p_k - p_{k-1}}{t/n}))\eta_k} d\eta_k] F[q_n, p_n] \tag{6.25}$$

In detail, the above is given by the following multiple integral

$$=\Bigg[\int dq_1 dp_1 dq_2 dp_2 \ldots dq_n dp_n \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial p}(0)-(\frac{q_1-q_0}{t/n}))\lambda_1} d\lambda_1 \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial q}(0)+(\frac{p_1-p_0}{t/n}))\eta_1} d\eta_1$$

$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial p}(1)-(\frac{q_2-q_1}{t/n}))\lambda_2} d\lambda_2 \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial q}(1)+(\frac{p_2-p_1}{t/n}))\eta_2} d\eta_2$$

$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial p}(2)-(\frac{q_3-q_2}{t/n}))\lambda_3} d\lambda_3 \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial q}(2)+(\frac{p_3-p_2}{t/n}))\eta_k} d\eta_3$$

$$\vdots$$

$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial p}(n-1)-(\frac{q_n-q_{n-1}}{t/n}))\lambda_n} d\lambda_n \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{it}{n}(\frac{\partial H}{\partial q}(n-1)+(\frac{p_n-p_{n-1}}{t/n}))\eta_k} d\eta_n\Bigg]$$

$$\times F[q_n, p_n] \tag{6.26}$$

Upon combining the exponential, we are lead to the expressions

$$e^{\sum_{k=1}^{n}\frac{it}{n}(\frac{\partial H}{\partial p}(k-1)-(\frac{q_k-q_{k-1}}{t/n}))\lambda_k} \text{ and } e^{\sum_{k=1}^{n}\frac{it}{n}(\frac{\partial H}{\partial q}(k-1)+(\frac{p_k-p_{k-1}}{t/n}))\eta_k}. \tag{6.27}$$

in Eq. (6.26) to simplify the latter to

$$=\int[\prod_{k=1}^{n} dq_k dp_k \left(\frac{d\lambda_k}{2\pi}\right)\left(\frac{d\eta_k}{2\pi}\right)] e^{\sum_{k=1}^{n}\frac{it}{n}(\frac{\partial H}{\partial p}(k-1)-(\frac{q_k-q_{k-1}}{t/n}))\lambda_k} e^{\sum_{k=1}^{n}\frac{it}{n}(\frac{\partial H}{\partial q}(k-1)+(\frac{p_k-p_{k-1}}{t/n}))\eta_k} F[q_n, p_n]$$

(6.28)

since rigorously

$$F[q(t),p(t)] = \lim_{n\to\infty} F\left[q_0 + \frac{t}{n}\sum_{m=0}^{n-1}\frac{\partial \mathrm{H}}{\partial p}\left(\frac{mt}{n}\right),\, p_0 - \frac{t}{n}\sum_{m=0}^{n-1}\frac{\partial \mathrm{H}}{\partial q}\left(\frac{mt}{n}\right)\right]$$

We now have to consider the limit

$$= \lim_{n\to\infty}\int\left[\prod_{k=1}^{n} dq_k\, dp_k \left(\frac{d\lambda_k}{2\pi}\right)\left(\frac{d\eta_k}{2\pi}\right)\right] e^{\sum_{k=1}^{n}\frac{it}{n}\left(\frac{\partial \mathrm{H}}{\partial p}(k-1)-\left(\frac{q_k-q_{k-1}}{t/n}\right)\right)\lambda_k}\, e^{\sum_{k=1}^{n}\frac{it}{n}\left(\frac{\partial \mathrm{H}}{\partial q}(k-1)+\left(\frac{p_k-p_{k-1}}{t/n}\right)\right)\eta_k}\, F[q_n,p_n]$$

(6.29)

Using the formal identities

$$\int_0^t d\tau f(\tau) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{t}{n} f\left(\frac{kt}{n}\right)$$

$$\lim_{n\to\infty}\left(\frac{q_k-q_{k-1}}{t/n}\right) = \dot{q}_k$$

$$\lim_{n\to\infty}\left(\frac{p_k-p_{k-1}}{t/n}\right) = \dot{p}_k$$

We obtain the final expression of the path integral:

$$F[q(t),p(t)] = \left[\int \mathcal{D}(q)\mathcal{D}(p)\mathcal{D}(\lambda)\mathcal{D}(\eta)\, e^{i\int_0^t d\tau\left(\frac{\partial \mathrm{H}}{\partial p}(\tau)-\dot{q}(\tau)\right)\lambda(\tau)}\, e^{i\int_0^t d\tau\left(\frac{\partial \mathrm{H}}{\partial q}(\tau)+\dot{p}(\tau)\right)\eta(\tau)}\right]\times F[q(t),p(t)]$$

(6.30)

where the measure of integration is formally defined by

$$\int \mathcal{D}(q)\mathcal{D}(p)\mathcal{D}(\lambda)\mathcal{D}(\eta) = \lim_{n\to\infty}\int\prod_{k=1}^{n} dq_k dp_k \left(\frac{d\lambda_k}{2\pi}\right)\left(\frac{d\eta_k}{2\pi}\right)$$

Eq. (6.29) provides the path integral as a resolution of the identity. The additional variables λ, η are referred to as Lagrange multiplier variables which restrict, at every given time in the interval $(0,t)$, to the correct classical path. Since time is a continuous variable we obviously need an uncountable infinite number of them as the path integral shows. In quantum field theory such variables are often called ghost fields.

6.2 Application of the formalism

As an application we consider the dynamics of an electron in external uniform magnetic field.

We will show how the new formalism applies to the dynamics of an electron (e^-) in a uniform magnetic field $\vec{B}$. The direction of the magnetic field is chosen along the z-axis ($\vec{B} = (0,0,B)$). Since this a two-dimensional problem we may write for the vector potential

$$\vec{A} = \frac{(-q_2, q_1)}{2} B \tag{6.31}$$

and for the Hamiltonian

$$\mathrm{H} = \frac{\left(p_1 + q_2 \frac{m\omega}{2}\right)^2 + \left(p_2 - q_1 \frac{m\omega}{2}\right)^2}{2m}$$

Here we have the equivalent convenient notations:

$$\vec{x} \equiv (q_1, q_2) \equiv (x^1, x^2) \equiv (x, y)$$

where the parameter ω, the so-called angular frequency is given by

$$\omega \equiv \frac{eB}{mc}$$

where e is the charge, and c is the velocity of light.

First we check that the expression for the vector potential $\vec{A}$ in Eq. (6.31) gives the correct magnetic field $\vec{B}$. That is, we have to check that:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{6.32}$$

we write

$$\vec{B} = B\hat{k}$$

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

And obtain

$$\therefore \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\hat{i} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\hat{j} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\hat{k} \tag{6.33}$$

From Eqs. (6.32) and (6.33), we have

$$\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} = 0 \tag{6.34}$$

$$\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} = 0 \tag{6.35}$$

$$\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} = B \tag{6.36}$$

Since $a_1 = -\frac{y}{2}B$, $a_2 = \frac{x}{2}B$, $a_3 = 0$, Eqs. (6.34)-(6.36) are verified.

The free Hamiltonian for a free particle in two dimensions is given by

$$H_0 = \frac{p_1^{\,2}}{2m} + \frac{p_2^{\,2}}{2m} \tag{6.37}$$

In the presence of an external electromagnetic field we need the replacements:

$$\vec{p} \to \left(\vec{p} - \frac{e\vec{A}}{c} \right)$$

In detail, we have

$$p_1 \to p_1 - \frac{e(-q_2)B}{2c} = p_1 + q_2 \frac{m\omega}{2}$$

and

$$p_2 \to p_2 - \frac{e(q_1)B}{2c} = p_2 - q_1 \frac{m\omega}{2}$$

These lead to replacing H_0 by

$$\mathrm{H} = \frac{\left(p_1 + q_2 \frac{m\omega}{2} \right)^2 + \left(p_2 - q_1 \frac{m\omega}{2} \right)^2}{2m} \tag{6.38}$$

For the interacting system we now consider the exponential terms in the path integral of Eq. (6.30). We first consider the expression

$$\exp(i\int_0^t [\frac{\partial \mathrm{H}}{\partial p}(\tau) - \dot{q}(\tau)]\lambda(\tau)d\tau)$$

which in two dimensions is rewritten as

$$\exp(i\int_0^t [(\frac{\partial \mathrm{H}}{\partial p_1} - \dot{q}_1)\lambda_1 + (\frac{\partial \mathrm{H}}{\partial p_2} - \dot{q}_2)\lambda_2]d\tau) \tag{6.39}$$

Since

$$\frac{\partial \mathrm{H}}{\partial p_1} = \frac{1}{2}\left(p_1 + q_2\frac{m\omega}{2}\right)$$

$$\frac{\partial \mathrm{H}}{\partial p_2} = \frac{1}{2}\left(p_2 - q_1\frac{m\omega}{2}\right)$$

we obtain for Eq. (6.39)

$$\exp(i\int_0^t [(\frac{1}{2}\left(p_1 + q_2\frac{m\omega}{2}\right) - \dot{q}_1)\lambda_1 + (\frac{1}{2}\left(p_2 - q_1\frac{m\omega}{2}\right) - \dot{q}_2)\lambda_2]d\tau) \tag{6.40}$$

Similarly, for

$$\exp(i\int_0^t [\frac{\partial \mathrm{H}}{\partial q}(\tau) + \dot{p}(\tau)]\eta(\tau)d\tau)$$

we have to write in two dimensions

$$\exp(i\int_0^t [(\frac{\partial \mathrm{H}}{\partial q_1} + \dot{p}_1)\eta_1 + (\frac{\partial \mathrm{H}}{\partial q_2} + \dot{p}_2)\eta_2]d\tau) \qquad (6.41)$$

Using

$$\frac{\partial \mathrm{H}}{\partial q_1} = \left(-\frac{\omega}{2}\right)\left(p_2 - q_1\frac{m\omega}{2}\right)$$

and

$$\frac{\partial \mathrm{H}}{\partial q_2} = \left(\frac{\omega}{2}\right)\left(p_1 + q_2\frac{m\omega}{2}\right)$$

Eq. (6.41) reduces to

$$\exp(i\int_0^t [(-\frac{\omega}{2}\left(p_2 - q_1\frac{m\omega}{2}\right) + \dot{p}_1)\eta_1 + (\frac{\omega}{2}\left(p_1 + q_2\frac{m\omega}{2}\right) + \dot{p}_2)\eta_2]d\tau) \qquad (6.42)$$

We define the following new variables

$$U = \left(q_1\frac{m\omega}{2} - p_2\right) + i\left(p_1 + q_2\frac{m\omega}{2}\right)$$

$$V = \left(q_1\frac{m\omega}{2} + p_2\right) + i\left(p_1 - q_2\frac{m\omega}{2}\right)$$

$$U^* = \left(q_1 \frac{m\omega}{2} - p_2\right) - i\left(p_1 + q_2 \frac{m\omega}{2}\right)$$

$$V^* = \left(q_1 \frac{m\omega}{2} + p_2\right) - i\left(p_1 - q_2 \frac{m\omega}{2}\right) \qquad (6.43)$$

We find that when q_1, q_2, p_1 and p_2 are spelled out in terms of U, U^*, V and V^*, they are given by

$$p_1 = \frac{(U - U^*) + (V - V^*)}{4i}$$

$$p_2 = \frac{(V + V^*) - (U + U^*)}{4}$$

$$q_1 = \left(\frac{2}{m\omega}\right)\left(\frac{(U + U^*) + (V + V^*)}{4}\right)$$

$$q_2 = \left(\frac{2}{m\omega}\right)\left(\frac{(U - U^*) - (V - V^*)}{4i}\right)$$

Also

$$\dot{p}_1 = \frac{d}{dt}\left(\frac{(U - U^*) + (V - V^*)}{4i}\right)$$

$$\dot{p}_2 = \frac{d}{dt}\left(\frac{(V + V^*) - (U + U^*)}{4}\right)$$

$$\dot{q}_1 = \left(\frac{2}{m\omega}\right)\frac{d}{dt}\left(\frac{(U + U^*) + (V + V^*)}{4}\right)$$

$$\dot{q}_2 = \left(\frac{2}{m\omega}\right)\frac{d}{dt}\left(\frac{(U-U^*)-(V-V^*)}{4i}\right)$$

For the time-integrand in the exponential in Eqs. (6.40) and (6.42), without the i factor, we have

$$\equiv \left\{\frac{1}{m}\left(\frac{(U-U^*)}{2i}\right)-\left(\frac{2}{m\omega}\right)\frac{d}{dt}\left(\frac{(U+U^*)+(V+V^*)}{4}\right)\right\}\lambda_1$$

$$+\left\{\frac{1}{m}\left(-\frac{(U+U^*)}{2}\right)-\left(\frac{2}{m\omega}\right)\frac{d}{dt}\left(\frac{(U-U^*)-(V-V^*)}{4i}\right)\right\}\lambda_2$$

$$+\left\{\frac{\omega}{2}\left(\frac{(U+U^*)}{2}\right)+\frac{d}{dt}\left(\frac{(U-U^*)+(V-V^*)}{4i}\right)\right\}\eta_1$$

$$+\left\{\frac{\omega}{2}\left(\frac{(U-U^*)}{2i}\right)+\frac{d}{dt}\left(\frac{(V+V^*)-(U+U^*)}{4}\right)\right\}\eta_2 \qquad (6.44)$$

By rearranging the terms, the terms, the above is equivalent to the expression

$$\equiv -\left(\frac{\lambda_1}{2m\omega}\right)\left\{\left(\frac{d}{dt}+i\omega\right)U+\left(\frac{d}{dt}-i\omega\right)U^*+\frac{d}{dt}(V+V^*)\right\}$$

$$-\left(\frac{\lambda_2}{2m\omega}\right)\left\{\frac{1}{i}\left(\frac{d}{dt}+i\omega\right)U-\frac{1}{i}\left(\frac{d}{dt}-i\omega\right)U^*-\frac{1}{i}\frac{d}{dt}(V-V^*)\right\}$$

$$+\left(\frac{\eta_1}{4}\right)\left\{\frac{1}{i}\left(\frac{d}{dt}+i\omega\right)U-\frac{1}{i}\left(\frac{d}{dt}-i\omega\right)U^*+\frac{1}{i}\frac{d}{dt}(V-V^*)\right\}$$

$$-\left(\frac{\eta_2}{4}\right)\left\{\left(\frac{d}{dt}+i\omega\right)U+\left(\frac{d}{dt}-i\omega\right)U^*-\frac{d}{dt}\left(V+V^*\right)\right\} \tag{6.45}$$

The coefficients of $\lambda_1, \lambda_2, \eta_1, \eta_2$ on the right-hand side of Eq. (6.45) are all reals. Upon integration over $\lambda_1, \lambda_2, \eta_1, \eta_2$ we learn from the path integral expression Eq. (6.30) that the real and imaginary parts of $\left(\frac{d}{dt}+i\omega\right)U \pm \frac{dV}{dt}$ separately must vanish. That is,

$$\left(\frac{d}{dt}+i\omega\right)U \pm \frac{dV}{dt} = 0 \tag{6.46}$$

$$\left(\frac{d}{dt}+i\omega\right)U = 0 \quad \text{and} \quad \frac{dV}{dt} = 0$$

$$\therefore \quad \frac{dU}{dt} = -i\omega$$

$$\int_{U(0)}^{U(t)} \frac{1}{U}\,dU = -\int_0^t i\omega dt$$

$$\ln\frac{U(t)}{U(0)} = -i\omega t$$

$$\therefore \quad U(t) = \exp(-i\omega t)U(0) \tag{6.47}$$

$$V(t) = V(0) \tag{6.48}$$

On the other hand from Eq. (6.43) at $t = 0$, we obtain

$$U(0)=\left(q_1(0)\frac{m\omega}{2}-p_2(0)\right)+i\left(p_1(0)+q_2(0)\frac{m\omega}{2}\right)$$

$$V(0)=\left(q_1(0)\frac{m\omega}{2}+p_2(0)\right)+i\left(p_1(0)-q_2(0)\frac{m\omega}{2}\right)$$

$$U^*(0)=\left(q_1(0)\frac{m\omega}{2}-p_2(0)\right)-i\left(p_1(0)+q_2(0)\frac{m\omega}{2}\right)$$

$$V^*(0)=\left(q_1(0)\frac{m\omega}{2}+p_2(0)\right)-i\left(p_1(0)-q_2(0)\frac{m\omega}{2}\right) \tag{6.49}$$

Eqs. (6.47)-(6.49) then immediately lead to

$$U(t)=U(0)(\cos\omega t-i\sin\omega t) \tag{6.50}$$

$$U(t)=\left\{\left(q_1(0)\frac{m\omega}{2}-p_2(0)\right)+i\left(p_1(0)+q_2(0)\frac{m\omega}{2}\right)\right\}(\cos\omega t-i\sin\omega t)$$

$$=\left\{\left(q_1(0)\frac{m\omega}{2}-p_2(0)\right)\cos\omega t+\left(p_1(0)+q_2(0)\frac{m\omega}{2}\right)\sin\omega t\right\}$$

$$+i\left\{-\left(q_1(0)\frac{m\omega}{2}-p_2(0)\right)\sin\omega t+\left(p_1(0)+q_2(0)\frac{m\omega}{2}\right)\cos\omega t\right\} \tag{6.51}$$

$$U^*(t)=\exp(i\omega t)U^*(0)$$

$$U^*(t)=U^*(0)(\cos\omega t+i\sin\omega t) \tag{6.52}$$

$$U^*(t)=\left\{\left(q_1(0)\frac{m\omega}{2}-p_2(0)\right)-i\left(p_1(0)+q_2(0)\frac{m\omega}{2}\right)\right\}(\cos\omega t+i\sin\omega t)$$

$$= \left\{ \left[\left(q_1(0)\frac{m\omega}{2} - p_2(0) \right) \cos\omega t + \left(p_1(0) + q_2(0)\frac{m\omega}{2} \right) \sin\omega t \right] \right.$$

$$\left. + i \left[\left(q_1(0)\frac{m\omega}{2} - p_2(0) \right) \sin\omega t - \left(p_1(0) + q_2(0)\frac{m\omega}{2} \right) \cos\omega t \right] \right\} \quad (6.53)$$

$$V(t) = V(0) = \left(q_1(0)\frac{m\omega}{2} + p_2(0) \right) + i\left(p_1(0) - q_2(0)\frac{m\omega}{2} \right) \quad (6.54)$$

$$V^*(t) = V^*(0) = \left(q_1(0)\frac{m\omega}{2} + p_2(0) \right) - i\left(p_1(0) - q_2(0)\frac{m\omega}{2} \right) \quad (6.55)$$

Upon equating the real and imaginary parts of each of the latter two equations we finally obtain our solutions. By using the identity

$$q_1 = \left(\frac{2}{m\omega} \right) \left(\frac{(U + U^*) + (V + V^*)}{4} \right)$$

$\therefore$ And comparing the real parts of Eqs. (6.47)-(6.55), we obtain

$$q_1 = \left(\frac{1}{2m\omega} \right) \left\{ \left[\left(q_1(0)\frac{m\omega}{2} - p_2(0) \right) \cos\omega t + \left(p_1(0) + q_2(0)\frac{m\omega}{2} \right) \sin\omega t \right. \right.$$

$$+ \left(q_1(0)\frac{m\omega}{2} - p_2(0) \right) \cos\omega t + \left(p_1(0) + q_2(0)\frac{m\omega}{2} \right) \sin\omega t$$

$$\left. \left. + \left(q_1(0)\frac{m\omega}{2} + p_2(0) \right) + \left(q_1(0)\frac{m\omega}{2} + p_2(0) \right) \right] \right\}$$

$$= \left(\frac{1}{2m\omega}\right)\left\{2\left(q_1(0)\frac{m\omega}{2} - p_2(0)\right)\cos\omega t + 2\left(p_1(0) + q_2(0)\frac{m\omega}{2}\right)\sin\omega t\right.$$

$$\left. + 2\left(q_1(0)\frac{m\omega}{2} + p_2(0)\right)\right\}$$

$$q_1 = \left(\frac{q_1(0)}{2} + \frac{p_2(0)}{m\omega}\right) + \left(\frac{q_1(0)}{2} - \frac{p_2(0)}{m\omega}\right)\cos\omega t + \left(\frac{p_1(0)}{m\omega} + \frac{q_2(0)}{2}\right)\sin\omega t \qquad (6.56)$$

Similarly, from the identity

$$iq_2 = \left(\frac{2}{m\omega}\right)\left(\frac{(U - U^*) - (V - V^*)}{4}\right)$$

$\therefore$ And upon comparing the imaginary parts of Eqs. (6.47)-(6.55), we finally obtain

$$q_2 = \left(\frac{1}{2m\omega}\right)\left\{-\left(q_1(0)\frac{m\omega}{2} - p_2(0)\right)\sin\omega t + \left(p_1(0) + q_2(0)\frac{m\omega}{2}\right)\cos\omega t\right.$$

$$-\left(q_1(0)\frac{m\omega}{2} - p_2(0)\right)\sin\omega t + \left(p_1(0) + q_2(0)\frac{m\omega}{2}\right)\cos\omega t$$

$$\left. -\left(p_1(0) - q_2(0)\frac{m\omega}{2}\right) - \left(p_1(0) - q_2(0)\frac{m\omega}{2}\right)\right\}$$

$$= \left(\frac{1}{2m\omega}\right)\left\{-2\left(q_1(0)\frac{m\omega}{2} - p_2(0)\right)\sin\omega t + 2\left(p_1(0) + q_2(0)\frac{m\omega}{2}\right)\cos\omega t\right.$$

$$-2\left(p_1(0) - q_2(0)\frac{m\omega}{2}\right)\Bigg]\Bigg\}$$

$$q_2 = \left(\frac{q_2(0)}{2} + \frac{p_1(0)}{m\omega}\right)\cos\omega t - \left(\frac{q_1(0)}{2} - \frac{p_2(0)}{m\omega}\right)\sin\omega t - \left(\frac{p_1(0)}{m\omega} - \frac{q_2(0)}{2}\right) \qquad (6.57)$$

6.3 From the path integral to the Poisson bracket

To check the consistency of the path integral derived, we explicitly show that it leads to a derivation of the Poisson bracket expression for the time evolution of dynamical variable.

To the above end, we consider the limit in

$$F[q(t), p(t)]$$

$$= \lim_{n\to\infty}\int [\prod_{k=1}^{n} dq_k dp_k \delta(\frac{t}{n}\frac{\partial \mathrm{H}}{\partial p}(k-1) - (q_k - q_{k-1}))\delta(\frac{t}{n}\frac{\partial \mathrm{H}}{\partial q}(k-1) + (p_k - p_{k-1}))]$$

$$\times F[q_n, p_n] \qquad (6.58)$$

In particular, in reference to the first equality on the right-hand side of Eq. (6.26), $\exp\{(t/n)(\partial \mathrm{H}(k-1)/\partial p)(\partial/\partial q_{k-1})\}$ is not quite a translation operator for a function of q_{k-1} since $(\partial \mathrm{H}(k-1)/\partial p)$ may, in general, depend on q_{k-1} as well. However, in view of the fact that the limit $n \to \infty$ is to be taken, this operator may be indeed taken to have such a property for the accuracy needed. A similar comment applies to operator for translation of a function of p_{k-1} via the operator $\exp\{(-t/n)(\partial H(k-1)/\partial q)(\partial/\partial p_{k-1})\}$. Since

$$e^{a\frac{d}{dx}} f(x) = f(x+a) \tag{6.59}$$

we have

$$\delta\left(\frac{t}{n}\frac{\partial H}{\partial p}(k-1) - (q_k - q_{k-1})\right) = e^{\frac{t}{n}\frac{\partial H}{\partial p}(k-1)\frac{\partial}{\partial q_k}} \delta(q_{k-1} - q_k) \tag{6.60}$$

and

$$\delta\left(\frac{t}{n}\frac{\partial H}{\partial q}(k-1) + (p_k - p_{k-1})\right) = e^{-\frac{t}{n}\frac{\partial H}{\partial q}(k-1)\frac{\partial}{\partial p_k}} \delta(p_k - p_{k-1}) \tag{6.61}$$

Also using the property $\delta(x) = \delta(-x)$, we rewrite Eq. (6.61) as

$$\delta\left(\frac{t}{n}\frac{\partial H}{\partial q}(k-1) + (p_k - p_{k-1})\right) = e^{-\frac{t}{n}\frac{\partial H}{\partial q}(k-1)\frac{\partial}{\partial p_k}} \delta(p_{k-1} - p_k) \tag{6.62}$$

Replacing Eqs. (6.60) and (6.61) in Eq. (6.58), we obtain

$$F[q(t),p(t)] = \lim_{n\to\infty}\int[\prod_{k=1}^{n} dq_k dp_k e^{\frac{t}{n}\frac{\partial H}{\partial p}(k-1)\frac{\partial}{\partial q_k}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(k-1)\frac{\partial}{\partial p_k}} \delta(q_{k-1}-q_k)\delta(p_{k-1}-p_k)]$$
$$\times F[q_n,p_n]$$

(6.63)

In detail, this equation reads

$$F[q(t),p(t) = \lim_{n\to\infty}\Bigg[\int dq_1 dp_1 e^{\frac{t}{n}\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_1}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_1}} \delta(q_0-q_1)\delta(p_0-p_1)$$
$$\times\int dq_2 dp_2 e^{\frac{t}{n}\frac{\partial H}{\partial p}(1)\frac{\partial}{\partial q_2}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(1)\frac{\partial}{\partial p_2}} \delta(q_1-q_2)\delta(p_1-p_2)$$
$$\times\int dq_3 dp_3 e^{\frac{t}{n}\frac{\partial H}{\partial p}(2)\frac{\partial}{\partial q_3}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(2)\frac{\partial}{\partial p_3}} \delta(q_2-q_3)\delta(p_2-p_3)$$
$$\vdots$$
$$\times\int dq_n dp_n e^{\frac{t}{n}\frac{\partial H}{\partial p}(n-1)\frac{\partial}{\partial q_{n-1}}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(n-1)\frac{\partial}{\partial p_{n-1}}} \delta(q_{n-1}-q_n)\delta(p_{n-1}-p_n)\Bigg]F[q_n,p_n]$$

(6.64)

In reference to the right-hand side of Eq. (6.64), we may integrate over q_k, p_k, $k = 1,2,\ldots,n-1$ with the aid of the delta functions. These integrations eventually picks up the q_0, p_0 values for the former integrals, thus obtaining

$$\int dq_n dp_n e^{\frac{t}{n}\frac{\partial H}{\partial p}(n-1)\frac{\partial}{\partial q_{n-1}}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(n-1)\frac{\partial}{\partial p_{n-1}}} \delta(q_{n-1}-q_n)\delta(p_{n-1}-p_n)F[q_n,p_n]$$
$$= e^{\frac{t}{n}\frac{\partial H}{\partial p}(n-1)\frac{\partial}{\partial q_n}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(n-1)\frac{\partial}{\partial p_n}} F[q_{n-1},p_{n-1})]$$

$$\int dq_n dp_n e^{\frac{t}{n}\frac{\partial H}{\partial p}(n-1)\frac{\partial}{\partial q_n}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(n-1)\frac{\partial}{\partial p_n}} \delta(q_{n-1}-q_n)\delta(p_{n-1}-p_n)F[q_n,p_n]$$

$$\times\int dq_{n-1} dp_{n-1} e^{\frac{t}{n}\frac{\partial H}{\partial p}(n-2)\frac{\partial}{\partial q_{n-2}}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(n-2)\frac{\partial}{\partial p_{n-2}}} \delta(q_{n-2}-q_{n-1})\delta(p_{n-2}-p_{n-1})$$

$$=\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(n-2)\frac{\partial}{\partial q_{n-2}}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(n-2)\frac{\partial}{\partial p_{n-2}}}\right)^2 F[q_{n-2},p_{n-2}]$$

N

$$\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(2)\frac{\partial}{\partial q_2}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(2)\frac{\partial}{\partial p_2}}\right)\int dq_3 dp_3 \left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(3)\frac{\partial}{\partial q_3}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(3)\frac{\partial}{\partial p_3}}\right)^{n-3} \delta(q_2-q_3)\delta(p_2-p_3)F[q_3,p_3]$$

$$=\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(2)\frac{\partial}{\partial q_2}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(2)\frac{\partial}{\partial p_2}}\right)^{n-2} F[q_2,p_2]$$

$$\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(1)\frac{\partial}{\partial q_1}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(1)\frac{\partial}{\partial p_1}}\right)\int dq_2 dp_2 \left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(2)\frac{\partial}{\partial q_2}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(2)\frac{\partial}{\partial p_2}}\right)^{n-2} \delta(q_1-q_2)\delta(p_1-p_2)F[q_2,p_2]$$

$$=\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(1)\frac{\partial}{\partial q_{n-2}}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(1)\frac{\partial}{\partial p_{n-2}}}\right)^{n-1} F[q_1,p_1]$$

The final integral then gives

$$\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}}\right)\int dq_1 dp_1 \left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(1)\frac{\partial}{\partial q_1}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(1)\frac{\partial}{\partial p_1}}\right)^{n-1} \delta(q_0-q_1)\delta(p_0-p_1)F[q_1,p_1]$$

$$=\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}}\right)^{n} F[q_0,p_0]$$

So, we may rewrite Eq. (6.64) as

$$F[q(t),p(t)] = \lim_{n\to\infty}\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}} \right)^n F[q_0,p_0] \tag{6.65}$$

Finally we use the identity

$$e^A e^B = e^{(A+B)} e^{\left\{\frac{1}{2}[A,B]+\ldots\right\}}$$

For any two operators A and B, the terms in the curly brackets { } above will turn out to be unimportant as we now show for our problem. To this end, for $n\to\infty$

$$\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}} \right)^n = \left(e^{\frac{t}{n}\left(\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0}-\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}\right)} e^{\left\{\frac{1}{2}\left(\frac{t}{n}\right)^2\left[\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0},-\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}\right]+\ldots\right\}} \right)^n$$

$$\left(e^{\frac{t}{n}\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0}} e^{-\frac{t}{n}\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}} \right)^n = \left(e^{\frac{t}{n}\left(\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0}-\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}\right)} \right)^n \left(e^{\left\{\frac{1}{2}\left(\frac{t}{n}\right)^2\left[\frac{\partial H}{\partial p}(0)\frac{\partial}{\partial q_0},-\frac{\partial H}{\partial q}(0)\frac{\partial}{\partial p_0}\right]+\ldots\right\}} \right)^n \tag{6.66}$$

The commutators in the exponential on the right-hand sides of these equations are or the order $O(1/n^2)$ or smaller and hence with the overall power n, the corresponding exponential, just mentioned, goes over to 1 for $n\to\infty$. All told, we have for $n\to\infty$

$$\lim_{n->\infty}\left(e^{\frac{t}{n}\frac{\partial \mathrm{H}}{\partial p}(0)\frac{\partial}{\partial q_0}}e^{-\frac{t}{n}\frac{\partial \mathrm{H}}{\partial q}(0)\frac{\partial}{\partial p_0}}\right)^n = e^{t\left(\frac{\partial \mathrm{H}}{\partial p}(0)\frac{\partial}{\partial q_0}-\frac{\partial \mathrm{H}}{\partial q}(0)\frac{\partial}{\partial p_0}\right)} \tag{6.67}$$

which leads to the desired result

$$F[q(t),p(t)] = \exp\left[t\left(\frac{\partial \mathrm{H}}{\partial p}(0)\frac{\partial}{\partial q_0}-\frac{\partial \mathrm{H}}{\partial q}(0)\frac{\partial}{\partial p_0}\right)\right]F[q(0),p(0)] \tag{6.68}$$

thus checking the correctness of the path integral formulation and its consistency.

Chapter VII

Conclusion

In this Chapter, we summarize all of our new findings and make some additional comments. By the application of the fundamental lemma established in Chapter III and given in Eq. (3.1), we have developed the following perturbation expansion of the position $q(t)$ of a particle for any time $t > 0$, for which the Hamiltonian □ H is given by $\mathrm{H} = (p^2/2m) + \lambda V(q)$, with λ denoting a coupling parameter introduced to specify the order of the perturbation expansion. Our perturbation expansion is given by:

$$
\begin{aligned}
q(t) = \sum_{n=0}^{\infty} (\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\
\times \frac{1}{m^n} (t - t_1) \widetilde{F}[u_n] \times \sum \Bigg\{ \left(\prod_{j=2}^{n} (t_1 - t_j)^{\delta(k_1, j)} \right) \widetilde{F}^{(k_1)}[u_1] \\
\times \left(\prod_{j=3}^{n} (t_2 - t_j)^{\delta(k_2, j)} \right) \widetilde{F}^{(k_2)}[u_2] \\
\times \left(\prod_{j=4}^{n} (t_2 - t_j)^{\delta(k_3, j)} \right) \widetilde{F}^{(k_3)}[u_3] \\
\vdots \\
\times \left(\prod_{j=n-2}^{n} (t_{n-3} - t_j)^{\delta(k_{n-3}, j)} \right) \widetilde{F}^{(k_{n-3})}[u_{n-3}]
\end{aligned}
$$

$$\times\left(\prod_{j=n-1}^{n}\left(t_{n-2}-t_j\right)^{\delta(k_{n-2},j)}\right)\widetilde{F}^{(k_{n-2})}[u_{n-2}]$$

$$\left.\times\left(t_{n-1}-t_n\right)^{\delta(k_{n-1},n)}\,\widetilde{F}^{(k_{n-1})}[u_{n-1}]\right\} \tag{7.1}$$

with $u_n \equiv \left[q+\dfrac{t_n p}{m}\right]$

$$\widetilde{F}[q+\frac{t_n p}{m}] \equiv -V(q+\frac{t_n p}{m})$$

<u>and the summation is over:</u>

$$\left.\begin{array}{l} k_1 = 1,2,3,\ldots,n-1 \\ k_2 = 0,1,2,\ldots,n-2 \\ k_3 = 0,1,2,\ldots,n-3 \\ \quad\vdots \\ k_{n-2} = 0,1,2 \\ k_{n-1} = 0,1 \end{array}\right\} \text{such that : } k_1+k_2+\ldots+k_{n-1} = n-1$$

$$\left.\begin{array}{l} \delta(k_1,j), \quad j = 2,3,\ldots,n-1 \\ \delta(k_2,j), \quad j = 3,4,\ldots,n-1 \\ \delta(k_3,j), \quad j = 4,5,\ldots,n-3 \\ \quad\vdots \\ \delta(k_{n-2},j), \quad j = (n-1),n \\ \delta(k_{n-1},n), \end{array}\right\} \text{which are either 0 or 1}$$

such that

$$\begin{cases} \sum_{j=2}^{n} \delta(k_1, j) = k_1 \\ \sum_{j=3}^{n} \delta(k_2, j) = k_2 \\ \qquad \vdots \\ \quad \delta(k_{n-1}, n) = k_{n-1} \end{cases}.$$

and $\sum_{i=1}^{n-1} \delta(k_i, j) = 1$ (7.2)

where we see from Eq. (7.2) that we have to set

$\delta(k_i, j) \equiv 0$ For $j = 1,2,3,\ldots,i$

<u>That is, for a fixed</u> i, t_j <u>appears only once in</u>

$$\left.\begin{array}{l} \prod_{j=2}^{n} (t_1 - t_j)^{\delta(k_1, j)} \\ \prod_{j=3}^{n} (t_2 - t_j)^{\delta(k_2, j)} \\ \qquad \vdots \end{array}\right\}$$

Two formalisms were developed for the complexification of the time evolution operator suitable for studying the dynamics in phase space. The formalism I is given by

$$z(t) = \exp(tO(0))z(0) \tag{7.3}$$

where

$$O(0) = iab\left(\left(\frac{\partial \mathcal{H}(0)}{\partial z(0)} - \frac{\partial \mathcal{H}(0)}{\partial z^*(0)}\right)\left(\frac{\partial}{\partial z(0)} + \frac{\partial}{\partial z^*(0)}\right) - \left(\frac{\partial \mathcal{H}(0)}{\partial z(0)} + \frac{\partial \mathcal{H}(0)}{\partial z^*(0)}\right)\left(\frac{\partial}{\partial z(0)} - \frac{\partial}{\partial z^*(0)}\right)\right)$$

$z(t)$ is the complex dynamical variable at any time $t > 0$ and $z(0)$ is the complex dynamical variable at any time $t = 0$. $z^*(0)$ is the complex conjugate. Formalism I is exact of nature. The formalism II is given through the perturbation expansion

$$\begin{aligned}
z(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \\
\times\left(\frac{a(t-t_1)}{m^n} + \frac{ib}{m^{n-1}}\right)\widetilde{F}[u_n] \times \sum \Bigg\{ \left(\prod_{j=2}^{n}(t_1 - t_j)^{\delta(k_1,j)}\right)\widetilde{F}^{(k_1)}[u_1] \\
\times\left(\prod_{j=3}^{n}(t_2 - t_j)^{\delta(k_2,j)}\right)\widetilde{F}^{(k_2)}[u_2] \\
\times\left(\prod_{j=4}^{n}(t_2 - t_j)^{\delta(k_3,j)}\right)\widetilde{F}^{(k_3)}[u_3] \\
\vdots \\
\times\left(\prod_{j=n-2}^{n}(t_{n-3} - t_j)^{\delta(k_{n-3},j)}\right)\widetilde{F}^{(k_{n-3})}[u_{n-3}] \\
\times\left(\prod_{j=n-1}^{n}(t_{n-2} - t_j)^{\delta(k_{n-2},j)}\right)\widetilde{F}^{(k_{n-2})}[u_{n-2}] \\
\times(t_{n-1} - t_n)^{\delta(k_{n-1},n)}\, \widetilde{F}^{(k_{n-1})}[u_{n-1}] \Bigg\}
\end{aligned} \qquad (7.4)$$

where now the variables $u_1, u_2, \ldots, u_{n-1}$ are defined by the general expression

$$u_n = \frac{(z(0)+z^*(0))}{2a} + \frac{t_n(z(0)-z^*(0))}{2ibm}$$

The complexification was then extended to a two-dimensional setting suitable in studying the so-called geometrical Berry phase as arising in the famous Foucault pendulum problem on our rotating earth. The Hamiltonian for the latter is defined by

$$\mathrm{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{mg}{2L}(x^2+y^2) + \left(p_x y - p_y x\right)\omega_z \qquad (7.5)$$

or

$$\mathrm{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega_0^2}{2}(x^2+y^2) + \left(p_x y - p_y x\right)\omega_z \qquad (7.6)$$

where

$$\omega_z = \omega \sin\lambda \qquad (7.7)$$

and λ is the latitude (see figure 5.2)

$$\omega_0 \equiv \sqrt{\frac{g}{L}} \qquad (7.8)$$

With the definitions

$$z = x + iy$$

$$z_1 = x + \frac{ip_x}{m\omega_0}$$

$$z_2 = y + \frac{ip_y}{m\omega_0}$$

$$z^* = x - iy$$

$$z_1^* = x - \frac{ip_x}{m\omega_0}$$

$$z_2^* = y - \frac{ip_y}{m\omega_0}$$

$$U = z_1 + iz_2$$

$$V = z_1 - iz_2$$

$$U^* = z_1^* - iz_2^*$$

$$V^* = z_1^* + iz_2^*$$

$$z[\omega,0] = z(\omega) = \frac{1}{2}(U + V^*)$$

we obtain the exact solution

$$z(\omega,t) = \exp t\left\{(-i)(\omega_0 + \omega_z)U\frac{\partial}{\partial U} + (i)(\omega_0 - \omega_z)V^*\frac{\partial}{\partial V^*}\right\}\left(\frac{U + V^*}{2}\right) \quad (7.9)$$

leading to

$$z[\omega,t] = e^{-it\omega_z} e^{i2\pi} x_0 \cos(\omega_0 t) \quad (7.10)$$

For $t = 2\pi/\omega$, as the earth makes a complete revolution about its axis, the plane of oscillations of the pendulum does not come back to its initial position for $0 < \lambda < \pi/2$, where λ is the latitude at which the experiment is carried out. For $t = 2\pi/\omega$, we obtain

$$z[\omega, \frac{2\pi}{\omega}] = e^{i\Omega} x_0 \cos(2\pi \frac{\omega_0}{\omega}) \tag{7.11}$$

where

$$\Omega = 2\pi(1 - \sin\lambda)$$

and the solid angle Ω is depicted in figure 5.2, with $e^{i\Omega}$ denoting the geometrical phase. Since $x(t)$ and $y(t)$ for $t = 2\pi/\omega$ may be determined experimentally and hence also $z(t)$, this phase factor is measurable and of direct physical context. It provides the value of the latitude angle λ.

Finally we have developed a path integral expression by directly integrating Hamilton' s equation by dissecting the time interval into infinitestimals in a limiting scene. Lagrange multipliers were automatically introduced which restrict, at every given time, the paths in the path integral to the classical path. The final expression is given by

$$F[q(t), p(t)] = [\int \mathcal{D}(q)\mathcal{D}(p)\mathcal{D}(\lambda)\mathcal{D}(\eta) e^{i\int_0^t d\tau (\frac{\partial H}{\partial p}(\tau) - \dot{q}(\tau))\lambda(\tau)} e^{i\int_0^t d\tau (\frac{\partial H}{\partial q}(\tau) + \dot{p}(\tau))\eta(\tau)}] \times F[q(t), p(t)]$$

(7.12)

where the measure of integration is formally defined by

$$\int \mathcal{D}(q)\mathcal{D}(p)\mathcal{D}(\lambda)\mathcal{D}(\eta) = \lim_{n\to\infty}\int\prod_{k=1}^{n} dq_k dp_k \left(\frac{d\lambda_k}{2\pi}\right)\left(\frac{d\eta_k}{2\pi}\right)$$

and λ, η denote the Lagrange multipliers.

The results obtained in this work are numerous. Needless to say that these investigations open the way to analyze all sorts of classical dynamical questions in the light of the present formalisms. We hope that this work will attract enough attention of several practitioners in investigating classical dynamical problems through these newly developed methods. It is gratifying to see that in such a well established branch of physics one may offer something new as shown in the bulk of this thesis.

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Curriculum Vitae

Nattapong Yongram was born on May 19th, 1975 in Surin, NorthEast Thailand. He graduated with a High School Diploma from Surawittayakarn in 1993. After that he went to study in the Department of Physics, Faculty of Science, Khonkhaen University and graduated with a B.Sc. degree in 1998. After that, he decided to study for his Master's degree in the School of Physics, Institute of Science of the Suranaree University of Technology.