การหาผลเฉลยยืนยงเป็นบางส่วนของสมการนาเวียร์-สโตคส์ ที่มีค่าดีเฟกท์เท่ากับหนึ่ง

นางกัลทิมา ไทยเลิศ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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ON PARTIALLY INVARIANT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS WITH DEFECT $\delta = 1$

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วิทยานิพนธ์ฉบับนี้ เป็นการนำเอาประโยชน์ของกลุ่มวิเคราะห์ ไปใช้กับสมการนาเวียร์-สโตคส์ โดยมีการศึกษาผลเฉลยยืนยงเป็นบางส่วนแบบปรกติของสมการนาเวียร์-สโตคส์ที่มีค่าดีเฟกท์และค่าลำดับชั้นเท่ากับหนึ่ง ผลที่ได้นี้เป็นการพิสูจน์ว่าสามารถขยายวิธีการสร้างผลเฉลยยืนยงเป็นบางส่วน นั่นคือ มีผลเฉลยยืนยงเป็นบางส่วนของกลุ่มลีย์ (Lie group) ที่ไม่ยอมรับกับสมการนาเวียร์-สโตคส์ ส่วนหนึ่งของวิทยานิพนธ์ได้ศึกษากลุ่มลีย์ของการแปลงแบบแบคกลันด์ (Bäcklund transformations) กลุ่มลีย์นี้ยอมรับกับระบบสมการอนุพันธ์ย่อยซึ่งเกิดจากการศึกษาผลเฉลยยืนยงเป็นบางส่วนของสมการนาเวียร์-สโตคส์ ผลลัพธ์นี้เป็นการพิสูจน์การมีอยู่จริงของกลุ่มลีย์สำหรับการแปลงแบบแบคกลันด์ของภาวะสัมผัสอันดับจำกัด บางผลเฉลยของผลเฉลยยืนยงเป็นบางส่วน สามารถลดรูปสมการนาเวียร์-สโตคส์ไปยังสมการความร้อน โดยมีการจำแนกพีชคณิตย่อยที่ยอมรับกับสมการความร้อน และมีการหาผลเฉลยยืนยงของสมการความร้อนนี้ด้วย

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KANTHIMA THAILERT : ON PARTIALLY INVARIANT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS WITH DEFECT $\delta = 1$. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D. 113 PP.

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This thesis deals with an application of group analysis to the Navier-Stokes equations. All regular partially invariant solutions of the Navier-Stokes equations with defect one and rank one are studied. It is proven that the area of applications of the algorithm for constructing partially invariant solutions can be extended. There exist partially invariant solutions with respect to Lie groups which are not admitted by the Navier-Stokes equations. A part of the thesis is devoted to Lie groups of Bäcklund transformations. These Lie groups are admitted by the system of partial differential equations which arise from the study of partially invariant solutions of the Navier-Stokes equations. The existence of Lie groups of Bäcklund transformations of finite order tangency is proven. Some partially invariant solutions are reduced to the heat equation. Classification of subalgebras admitted by this equation and its invariant solutions are obtained.

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Kanthima Thailert

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Introduction

Mathematical modelling is a basis for analyzing physical phenomena by partial differential equations. Almost all fundamental equations of physics are nonlinear, and in general, are very difficult to solve explicitly. Numerical methods are often used with much success for obtaining approximate, not exact solutions. Hence, there is interest in obtaining exact solutions of nonlinear equations. Each solution has value, firstly, as an exact description of a real process in the framework of a given model; secondly, as a model to compare various numerical methods; thirdly, as a basis to improve the models used.

Group analysis is one of the methods for constructing particular exact solutions of partial differential equations. A survey of this method can be found in Ovsiannikov (1978) and Ibragimov (1994-1996). This method makes use of symmetry properties of differential equations. Symmetry means that any solution of a given system of partial differential equations is transformed by a Lie group of transformations to a solution of the same system. Moreover a symmetry allows finding new solutions of the system. There are two types of solutions which can be obtained by group analysis: invariant and partially invariant solutions.

In this thesis we study the unsteady Navier-Stokes equations. These equations are fundamental partial differential equations that describe the flow of an incompressible viscous fluid with usual temperature and pressure. In compact form, the Navier-Stokes equations are

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\rho^{-1}\nabla p + \nu \triangle \mathbf{u} + f, \quad \nabla \cdot \mathbf{u} = 0, \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, t is time, $p(\mathbf{x}, t)$ is the fluid pressure, $\nu = \mu/\rho$ is the kinematic viscosity, μ is viscosity, ρ is density, and f is the external body force. The operators ∇ and Δ are the gradient and the Laplacian, which in the Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ are $\nabla \equiv (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\Delta \equiv (\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3})$ where $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $(i = 1, 2, 3)$.

Many invariant solutions of the Navier-Stokes equations have been known for a long time; however their systematic analysis became possible only with the development of the modern methods of group analysis of differential equations (Ovsiannikov, 1978). The first group classification of the Navier-Stokes equations in the three-dimensional case was done in Bytev (1972). It was shown that the Lie group admitted by the Navier-Stokes equations is infinite-dimensional. There is still no classification of this group. Several papers (Puchnachov (1974), Lloyd (1981), Boisvert, Ames and Srivastava (1983), Grauel and Steeb (1985), Fushchich and Popovych (1994), Ibragimov (1994) and Popovych (1995)) are devoted to invariant solutions of the Navier-Stokes equations. Partially invariant solutions of the Navier–Stokes equations have been less studied (Puchnachov (1974), Meleshko and Puchnachov (1999), Hematulin (2001)). At the same time there has been progress in studying such classes of solutions of inviscid gas dynamics equations (Ovsiannikov (1978), Sidorov, Shapeev and Yanenko (1984), Meleshko (1991) and Ovsiannikov (1995)). Therefore, it is natural to investigate partially invariant solutions for the Navier-Stokes equations.

The construction of partially invariant solutions consists of a sequence of steps: choosing a subgroup, finding a representation of a solution, substituting the representation into the studied system of equations, and studying compatibility of the obtained (reduced) system of equations. This thesis deals with partially invariant solutions of the Navier-Stokes equations with defect $\delta = 1$ and rank

$\sigma = 1$. The subgroups studied are taken from part of the optimal system of subalgebras for the gas dynamics equations considered in Ovsiannikov and Chupakhin (1996). It should be noted that the notion of compatibility plays the key role in constructing partially invariant solutions. At the same time, when constructing a representation of a partially invariant solution, the property that the group is admitted is not used. These facts give rise to the assumption that one can construct partially invariant solutions with respect to a Lie group which is not necessary admitted. In the next sections examples of such partially invariant solutions are presented for the Navier-Stokes equations.

It is well-known that the main difficulty in the study of partially invariant solutions is the analysis of the compatibility (cf. Finikov (1948) and Kuranishi (1967)) of the appearing overdetermined systems. The analysis of compatibility can be reduced to the consecutive performance of algebraic operations of symbolic nature. The compatibility study of systems of partial differential equations requires a large amount of analytical calculations, and it is necessary to use a computer system for these calculations. A brief review of computer systems can be found, for example, in Ibragimov (1994), (1995), (1996), (1999). In our calculations the system REDUCE (cf. Hearn (1999)) and the MAPLE 8 (cf. Schwartz (2003)) program were used.

The partially invariant solutions of some subalgebras lead to the heat equation. For completeness of the study, we consider group classification of the admitted Lie algebra of the heat equation. Invariant solutions corresponds to these subalgebras give partially invariant solution of the Navier-Stokes equations.

Admitted Lie groups of the heat equation are obtained. This group is called G^{10}. The admitted subgroup G^9 allows dividing all exact solutions of the heat equation into classes of essentially different solutions with respect to G^9, where two solutions u_1, u_2 are nonessentially different if one is transformable into the other by a transformation belonging to the group G^9. Essentially different solutions are obtained with respect to different classes of similar subgroups. The set of all representatives (one from each class) is called an optimal system of subgroups. For each subgroup, one can try to find invariant or partially invariant solutions. For this, one has to find a universal invariant, a representation of a solution, substitute it into the given system and study the compatibility of the resulting system.

Part of this thesis is devoted to constructing a Lie group of tangent transformations for a system of partial differential equations. The Bäcklund theorem states that in the general case there are no nontrivial tangent transformations of finite order except contact transformations. This theorem is proven under the assumption that all derivatives involved in the transformations are free: they only satisfy the tangent conditions. On the other hand, if the derivatives appearing in a system of partial differential equations satisfy additional relations other than the tangent conditions, then there may exist nontrivial tangent transformations of finite order. These transformations are called Bäcklund transformations (Ibragimov, 1983). In this thesis, the existence of Bäcklund transformations for systems which arise from the study of partially invariant solutions of the Navier-Stokes equations is proven.

This thesis is organized as follows. Chapter II introduces notations of group analysis and provides references to known facts on application of group analysis to the construction of exact solutions of partial differential equations. Chapter III is devoted to the Navier-Stokes equations and admitted groups of the Navier-Stokes equations. Coordinate systems are also considered in this chapter. These coordinate systems are used for convenience of writing representations of partially invariant solutions. Chapter IV deals with the study of regular partially invariant solutions of the Navier-Stokes equations with respect to the subalgebras presented in Table 4.1. Subgroups for study are taken from the optimal system of subalgebras considered for the gas dynamic equations (Ovsiannikov and Chupakhin (1996)). Analysis of compatibility of these partially invariant solutions related to those subgroups is given in this chapter. Final results are presented in tables, which collect the results according to the type of a coordinate system. Chapter V considers two-dimensional subalgebras of the optimal system of the Lie algebra admitted by the heat equation. Invariant solutions of the heat equation are also studied. The result of representative calculations is presented in Table 5.2. Chapter VI is devoted to Lie groups of Bäcklund transformations. These Lie groups are admitted by a system of partial differential equations which arises from the study of partially invariant solutions of the Navier-Stokes equations.

Chapter II

Group Analysis Method

In this chapter, the group analysis method is discussed. An introduction to this method can be found in various textbooks (cf. Ovsiannikov (1978), Handbook of Lie group analysis (1994), (1995), (1996)).

2.1 Lie Groups

Consider a set of invertible point transformations

$$\bar{z}^i = \varphi^i(z; a), \ a \in \Delta, \ z \in V, \tag{2.1}$$

where $i = 1, 2, ..., N$, a is a parameter, and Δ is a symmetric interval in R^1. The set V is an open set in R^N.

If $z = (x, u)$, then one uses the notation $\varphi = (f, g)$. Here $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the vector of the independent variables, and $u = (u^1, u^2, ..., u^m) \in \mathbb{R}^m$ is the vector of the dependent variables. The transformation of the independent variables x, and the dependent variables u has the form

$$\bar{x}_i = f^i(x, u; a), \ \bar{u}^j = g^j(x, u; a), \tag{2.2}$$

where $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, $(x, u) \in V \subset R^n \times R^m$, and the set V is open in $R^n \times R^m$.

2.1.1 One-Parameter Lie-Group of Transformations

Definition 1. *A set of transformations (2.1) is called a local one-parameter Lie group if it has the following properties*

1. $\varphi(z;0) = z$ *for all* $z \in V$.

2. $\varphi(\varphi(z;a), b) = \varphi(z; a+b)$ *for all* $a, b, a+b \in \Delta, z \in V$.

3. If for $a \in \Delta$ *one has* $\varphi(z;a) = z$ *for all* $z \in V$*, then* $a = 0$.

4. $\varphi \in C^{\infty}(V, \Delta)$.

Transformations (2.2) are called a one-parameter Lie group of point transformations. For Lie groups of point transformations, the functions f^i and g^j can be written by Taylor series expansion with respect to the parameter a in the neighborhood of $a = 0$

$$\begin{aligned} \bar{x}_i &= x_i + a\left.\frac{\partial f^i}{\partial a}\right|_{a=0} + O(a^2), \\ \bar{u}^j &= u^j + a\left.\frac{\partial g^j}{\partial a}\right|_{a=0} + O(a^2). \end{aligned} \tag{2.3}$$

The transformations $x_i + a\xi^{x_i}(x,u)$ and $u^j + a\zeta^{u^j}(x,u)$ are called infinitesimal transformations of the Lie group of transformation (2.2), where

$$\xi^{x_i}(x,u) = \left.\frac{\partial f^i(x,u;a)}{\partial a}\right|_{a=0}, \zeta^{u^j}(x,u) = \left.\frac{\partial g^j(x,u;a)}{\partial a}\right|_{a=0}.$$

The components $\xi = (\xi^{x_1}, \xi^{x_2}, ..., \xi^{x_n})$, $\zeta = (\zeta^{u^1}, \zeta^{u^2}, ..., \zeta^{u^m})$ are called the infinitesimal of (2.2). This can be written in terms of the first-order differential operator

$$X = \xi^{x_i}(x,u)\partial_{x_i} + \zeta^{u^j}(x,u)\partial_{u^j}. \tag{2.4}$$

This operator X is called an infinitesimal generator.

There is a theorem, which relates a one-parameter Lie group G with its infinitesimal generator.

Theorem 1 (Lie). *Let functions* $f^i(x,u;a)$, $i = 1, ..., n$ *and* $g^j(x,u;a)$, $j = 1, ..., m$ *satisfy the group properties and have the expansion*

$$\bar{x}_i = f^i(x,u;a) \approx x_i + \xi^{x_i}(x,u)a,$$

$$\bar{u}^j = g^j(x,u;a) \approx u^j + \zeta^{u^j}(x,u)a$$

where

$$\xi^{x_i}(x,u) = \left.\frac{\partial f^i(x,u;a)}{\partial a}\right|_{a=0}, \zeta^{u^j}(x,u) = \left.\frac{\partial g^j(x,u;a)}{\partial a}\right|_{a=0}.$$

Then it solves the Cauchy problem

$$\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}a} = \xi^{x_i}(\bar{x},\bar{u}), \ \frac{\mathrm{d}\bar{u}^j}{\mathrm{d}a} = \zeta^{u^j}(\bar{x},\bar{u}) \tag{2.5}$$

with the initial data

$$\bar{x}_i|_{a=0} = x_i, \ \bar{u}^j\big|_{a=0} = u^j. \tag{2.6}$$

Conversely, given $\xi^{x_i}(x,u)$ and $\zeta^{u^j}(x,u)$, the solution of the Cauchy problem (2.5),(2.6) forms a Lie group.

Equations (2.5) are called Lie equations.

To apply a Lie group of transformations (2.2) for studying differential equations one needs to know how this group acts on the functions $u^j(x)$ and their derivatives. For the sake of simplicity, let us explain the basic idea for the case $n = 1$ and $m = 1$. Assume that $u_0(x)$ is a given known function, and the transformation is

$$\begin{aligned} \bar{x} &= f(x,u;a) \approx x + a\xi^x(x,u) \\ \bar{u} &= g(x,u;a) \approx u + a\zeta^u(x,u). \end{aligned} \tag{2.7}$$

Substituting $u_0(x)$ into the first equation (2.7), one obtains

$$\bar{x} = f(x,u_0(x);a).$$

Since $f(x,u_0(x);0) = x$, the Jacobian at $a = 0$ is

$$\left.\frac{\partial \bar{x}}{\partial x}\right|_{a=0} = \left.\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x}\right)\right|_{a=0} = 1.$$

Thus, by virtue of the inverse function theorem, in some neighborhood of $a = 0$ one can express x as a function of $\bar{x}$ and a,

$$x = \theta(\bar{x},a). \tag{2.8}$$

Note that after substituting (2.8) into the first equation (2.7), one has the identity

$$\bar{x} = f(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{2.9}$$

Substituting (2.8) into the second equation (2.7), one obtains the transformed function

$$u_a(\bar{x}) = g(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{2.10}$$

Differentiating equation (2.10) with respect to $\bar{x}$, one gets

$$\bar{u}_{\bar{x}} = \frac{\partial u_a(\bar{x})}{\partial \bar{x}} = \frac{\partial g}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial g}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}u_0'(x)\right)\frac{\partial \theta}{\partial \bar{x}},$$

where the derivative $\frac{\partial \theta}{\partial \bar{x}}$ can be found by differentiating equation (2.9) with respect to $\bar{x}$,

$$1 = \frac{\partial f}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right)\frac{\partial \theta}{\partial \bar{x}}.$$

Since

$$\frac{\partial f}{\partial x}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 1, \ \frac{\partial f}{\partial u}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 0, \tag{2.11}$$

one has $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x) \neq 0$ in some neighborhood of $a = 0$. Thus,

$$\frac{\partial \theta}{\partial \bar{x}} = \frac{1}{\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right)},$$

and

$$\bar{u}_{\bar{x}} = \frac{\frac{\partial g(x,u_0;a)}{\partial x} + \frac{\partial g(x,u_0;a)}{\partial u}u_0'(x)}{\frac{\partial f(x,u_0;a)}{\partial x} + \frac{\partial f(x,u_0;a)}{\partial u}u_0'(x)} = h(x, u_0(x), u_0'(x); a).$$

Transformation (2.2) together with

$$\bar{u}_{\bar{x}} = h(x, u, u_x; a) \tag{2.12}$$

is called the prolongation of (2.2).

As before, the function h can be written by Taylor series expansion with respect to the parameter a in the neighborhood of the point $a = 0$:

$$\bar{u}_{\bar{x}} = h(x, u, u_x; a) \approx u_x + a\zeta^{u_x}(x, u, u_x), \tag{2.13}$$

where

$$\zeta^{u_x}(x,u,u_x) = \left.\frac{\partial h(x,u,u_x;a)}{\partial a}\right|_{a=0}, \quad h|_{a=0} = u_x.$$

Equation (2.12) can be rewritten

$$h(x,u,u_x;a)\left(\frac{\partial f(x,u;a)}{\partial x} + u_x\frac{\partial f(x,u;a)}{\partial u}\right) = \left(\frac{\partial g(x,u;a)}{\partial x} + u_x\frac{\partial g(x,u;a)}{\partial u}\right).$$

Differentiating this equation with respect to the group parameter a and substituting $a = 0$, one finds

$$\left.\left(\frac{\partial h}{\partial a}\left(\frac{\partial f}{\partial x} + u_x\frac{\partial f}{\partial u}\right) + h\left(\frac{\partial^2 f}{\partial x\partial a} + u_x\frac{\partial^2 f}{\partial u\partial a}\right)\right)\right|_{a=0} = \left.\left(\frac{\partial^2 g}{\partial x\partial a} + u_x\frac{\partial^2 g}{\partial u\partial a}\right)\right|_{a=0}$$

or

$$\begin{aligned}
\zeta^{u_x}(x,u,u_x) &= \left.\frac{\partial h}{\partial a}\right|_{a=0}\left.\left(\frac{\partial f}{\partial x} + u_x\frac{\partial f}{\partial u}\right)\right|_{a=0} \\
&= \left.\left(\frac{\partial^2 g}{\partial x\partial a} + u_x\frac{\partial^2 g}{\partial u\partial a}\right)\right|_{a=0} - h|_{a=0}\left.\left(\frac{\partial^2 f}{\partial x\partial a} + u_x\frac{\partial^2 f}{\partial u\partial a}\right)\right|_{a=0} \\
&= \left(\frac{\partial\zeta^u}{\partial x} + u_x\frac{\partial\zeta^u}{\partial u}\right) - u_x\left(\frac{\partial\xi^x}{\partial x} + u_x\frac{\partial\xi^x}{\partial u}\right) \\
&= D_x(\zeta^u) - u_x D_x(\xi^x)
\end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \ldots, \quad \xi^x = \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad \zeta^u = \left.\frac{\partial g}{\partial a}\right|_{a=0}, \quad \zeta^{u_x} = \left.\frac{\partial h}{\partial a}\right|_{a=0}.$$

The first prolongation of the generator (2.4) is given by

$$X^{(1)} = X + \zeta^{u_x}(x,u,u_x)\partial_{u_x}.$$

In the same way, one obtains the infinitesimal transformation of the second derivative

$$\bar{u}_{\bar{x}\bar{x}} \approx u_{xx} + a\zeta^{u_{xx}}(x,u,u_x,u_{xx}),$$

where $\zeta^{u_{xx}} = D_x(\zeta^{u_x}) - u_{xx}D_x(\xi^x)$, and the second prolongation of the generator (2.4) is

$$X^{(2)} = X^{(1)} + \zeta^{u_{xx}}(x,u,u_x,u_{xx})\partial_{u_{xx}}.$$

For constructing prolongations of an infinitesimal generator in case $n, m \geq 2$ one proceeds similarly.

Let $x = \{x_i\}$ be the set of independent variables and $u = \{u^j\}$ the set of dependent variables. The derivatives of the dependent variables are given by the sets $u_{(1)} = \left\{u_i^j\right\}$, $u_{(2)} = \left\{u_{is}^j\right\}, \ldots$, where $j = 1, \ldots, m$ and $i, s = 1, \ldots, n$. The derivatives of the differentiable functions u^j can be written in terms of the total differentiation D_i operator given below,

$$\begin{aligned} u_i^j &= D_i(u^j), \\ u_{is}^j &= D_s(u_i^j), \end{aligned}$$

where

$$D_i = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + u_{is}^j \frac{\partial}{\partial u_s^j} + \ldots, \quad (i, s = 1, 2, \ldots, n; \; j = 1, 2, \ldots, m). \quad (2.14)$$

The formula of the first prolongation of the generator $X = \xi^{x_i}(x, u)\partial_{x_i} + \zeta^{u^j}(x, u)\partial_{u^j}$ is

$$X^{(1)} = X + \zeta^{u_i^j}(x, u, u_{(1)})\partial_{u_i^j},$$

where

$$\zeta^{u_i^j} = D_i\left(\zeta^{u^j}\right) - u_s^j D_i\left(\xi^{x_s}\right) \quad ; \; i, s = 1, ..., n \;\; ; \; j = 1, ..., m.$$

The second prolongation of the generator X is

$$X^{(2)} = X^{(1)} + \zeta^{u_{i_1,i_2}^j}(x, u, u_{(1)}, u_{(2)})\partial_{u_{i_1,i_2}^j},$$

where

$$\zeta^{u_{i_1,i_2}^j} = D_{i_2}\left(\zeta^{u_{i_1}^j}\right) - u_{i_1,s}^j D_{i_2}\left(\xi^{x_s}\right) \quad ; \; i_1, i_2, s = 1, ..., n \;\; ; \; j = 1, ..., m. \quad (2.15)$$

In the general case, the k-th prolongation of the generator X is

$$X^{(k)} = X^{(k-1)} + \zeta^{u_{i_1,\ldots,i_k}^j}(x, u, u_{(1)}, ..., u_{(k)})\partial_{u_{i_1,\ldots,i_k}^j}$$

where

$$\zeta^{u^j_{i_1,\dots,i_k}} = D_{i_k}\left(\zeta^{u^j_{i_1,\dots,i_{k-1}}}\right) - u^j_{i_1,\dots,i_{k-1},s} D_{i_k}\left(\xi^{x_s}\right);\ i_1,...,i_k,s = 1,...,n;\ j = 1,...,m.$$

Lie groups of transformations are related with differential equations by the following.

Definition 2. *Given a partial differential equation, a Lie group of transformations, which transforms a solution $u_0(x)$ to a solution $u_a(x)$ of the same equation is called an admitted Lie group of transformations.*

Let $\mathcal{F} = (F^1, ..., F^k),\ k = 1, ..., N$ be differential functions of order p. The equations

$$F^k\left(x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}\right) = 0,\ k = 1, ..., N \tag{2.16}$$

compose a manifold $[\mathcal{F} = 0]$ in the space of the variables $x, u, u_{(1)}, u_{(2)}, ..., u_{(p)}$.

After applying an admitted Lie group of transformations to a solution $u(x)$, one has

$$F^k\left(\bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, ..., \bar{u}_{(p)}\right) = 0,\ (k = 1, ..., N). \tag{2.17}$$

Differentiating these equations with respect to the group parameter a, and substituting $a = 0$, one finds

$$\left.\left(\frac{\partial F^k}{\partial x_i}\frac{\partial \bar{x}_i}{\partial a} + \frac{\partial F^k}{\partial u^j}\frac{\partial \bar{u}^j}{\partial a} + \frac{\partial F^k}{\partial u^j_{i_1}}\frac{\partial \bar{u}^j_{i_1}}{\partial a} + ... + \frac{\partial F^k}{\partial u^j_{i_1,i_2,\dots,i_p}}\frac{\partial \bar{u}^j_{i_1,i_2,\dots,i_p}}{\partial a}\right)\right|_{a=0} = 0$$

or

$$\xi^{x_i}\frac{\partial F^k}{\partial x_i} + \zeta^{u^j}\frac{\partial F^k}{\partial u^j} + \zeta^{u^j_{i_1}}\frac{\partial F^k}{\partial u^j_{i_1}} + \zeta^{u^j_{i_1,i_2}}\frac{\partial F^k}{\partial u^j_{i_1,i_2}} + ... + \zeta^{u^j_{i_1,i_2,\dots,i_p}}\frac{\partial F^k}{\partial u^j_{i_1,i_2,\dots,i_p}} = 0,$$

where

$$\xi^{x_i} = \left.\frac{\partial \bar{x}_i}{\partial a}\right|_{a=0},\ \zeta^{u^j} = \left.\frac{\partial \bar{u}^j}{\partial a}\right|_{a=0},\ \zeta^{u^j_{i_1}} = \left.\frac{\partial \bar{u}^j_{i_1}}{\partial a}\right|_{a=0}, \dots,\ \zeta^{u^j_{i_1,\dots,i_p}} = \left.\frac{\partial \bar{u}^j_{i_1,\dots,i_p}}{\partial a}\right|_{a=0}.$$

The last equation can be expressed as an action of the prolonged infinitesimal generator

$$X^{(p)}F^k\Big|_{[\mathcal{F}=0]} = 0, \ (k = 1, ..., N), \tag{2.18}$$

where

$$X^{(p)} = \xi^{x_i}\frac{\partial}{\partial x_i} + \zeta^{u^j}\frac{\partial}{\partial u^j} + \zeta^{u^j_{i_1}}\frac{\partial}{\partial u^j_{i_1}} + \zeta^{u^j_{i_1,i_2}}\frac{\partial}{\partial u^j_{i_1,i_2}} + ... + \zeta^{u^j_{i_1,i_2,...,i_p}}\frac{\partial}{\partial u^j_{i_1,i_2,...,i_p}}.$$

Hence, in order to find the infinitesimal generator of the Lie group admitted by differential equations (2.16) one can use the following theorem.

Theorem 2. *The differential equation (2.16) admits the group G with the generator X, if and only if, the following equations hold:*

$$X^{(p)}F^k\Big|_{[\mathcal{F}=0]} = 0, \ (k = 1, ..., N). \tag{2.19}$$

Equations (2.19) are called the determining equations.

2.1.2 Multi-Parameter Lie-Group of Transformations

Let O be a ball in the space R^r with center at the origin. Assume that ψ is a mapping, $\psi : O \times O \longrightarrow R^r$. The pair (O, ψ) is called a local multi-parameter Lie group with the multiplication law ψ if it has the following properties:

1. $\psi(a, 0) = \psi(0, a) = a$ for all $a \in O$.
2. $\psi(\psi(a, b), c) = \psi(a, \psi(b, c))$ for all $a, b, c \in O$ for which $\psi(a, b), \psi(b, c) \in O$.
3. $\psi \in C^\infty(O, O)$.

Let V be an open set in Z. Consider transformations

$$\bar{z}^i = \varphi^i(z; a), \tag{2.20}$$

where $i = 1, 2, ..., N$, $z \in V \subset Z = R^N$, and the vector-parameter $a \in O$.

Definition 3. *The set of transformations (2.20) is called a local r-parameter Lie group G^r if it has the following properties:*

1. $\varphi(z,0) = z$ *for all* $z \in V$.

2. $\varphi(\varphi(z,a),b) = \varphi(z,\psi(a,b))$ *for all* $a, b, \psi(a,b) \in O,\ z \in V$.

3. If for $a \in O$ *one has* $\varphi(z,a) = z$ *for all* $z \in V$, *then* $a = 0$.

Note that if one fixes all parameters except one, for example a_k, then the multi-parameter Lie group of transformations (2.20) composes a one-parameter Lie group. Conversely, in group analysis it is proven that any r-parameter group is a union of one-parameter subgroups belonging to it.

Let G^r be a Lie group admitted by the system of partial differential equations

$$F^k(x,u,p) = 0,\ k = 1, ..., s.$$

Assume that $\{X_1, X_2, ..., X_r\}$ is a basis of the Lie algebra L^r, which corresponds to the Lie group G^r.

Definition 4. *A function* $\Phi(x,u)$ *is called an invariant of a Lie group* G^r *if*

$$\Phi(\bar{x},\bar{u}) = \Phi(x,u).$$

Theorem 3. *A function* $\Phi(x,u)$ *is an invariant of the group* G^r *with the generators* $X_i,\ (i = 1, ..., r)$ *if and only if,*

$$X_i\Phi(x,u) = 0,\ (i = 1, ..., r). \tag{2.21}$$

In order to find an invariant, one needs to solve the overdetermined system of linear equations (2.21). Any invariant Φ can be expressed through this set

$$\Phi = \phi\left(\ J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u)\ \right).$$

where n, m is the numbers of independent and dependent variables, respectively and r_* is the total rank of the matrix composed by the coefficients of the generators $X_i,\ (i = 1, 2, ..., r)$. A set of functionally independent invariants

$$J = (\ J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u)\ )$$

is called an universal invariant.

Definition 5. *A set M is said to be invariant with respect to the group G^r, if the transformation* (2.20) *carries every point z of M to a point of M.*

Definition 6. *Let V be an open subset of R^N, and $\Psi : V \longrightarrow R^t,\ t \leq N$ a mapping belonging to the class $C^1(V)$. The system of equations $\Psi(z) = 0$ is called regular, if for any point $z \in V$:*

$$rank\ \left(\frac{\partial(\ \psi^1, ..., \psi^t)}{\partial\ (\ z_1, ..., z_N\ )} \right) = t$$

where $\Psi = (\psi^1, ..., \psi^t)$.

If a system $\Psi(z) = 0$ is regular, then for each $z_0 \in V$ with $\Psi(z_0) = 0$ there exists a neighborhood U of z_0 in V such that

$$M = \{\ z \in U\ :\ \Psi(z) = 0\ \}$$

is a manifold. Such a manifold is called a regularly assigned manifold.

Theorem 4. *A regularly assigned manifold M is an invariant manifold with respect to a Lie group G^r with the generator $X_i,\ (i = 1, ..., r)$, if*

$$X_i \psi^k(z)\big|_M = 0,\ (i = 1, ..., r),\ k = 1, ..., t.$$

2.2 Lie algebra

Before giving the definition of a Lie algebra, one needs to introduce the commutator. Let $X_1 = \xi_1 \partial_x + \zeta_1 \partial_u,\ X_2 = \xi_2 \partial_x + \zeta_2 \partial_u$ be two generators. Let us define a new generator X, denoted by $[X_1, X_2]$, by the following formula

$$X = [X_1, X_2] = (X_1\xi_2 - X_2\xi_1)\,\partial_x + (X_1\zeta_2 - X_2\zeta_1)\,\partial_u.$$

The generator X is called the commutator of the generators X_1, X_2.

Definition 7. *A vector space L over the field of real numbers with the operation of commutation $[\,\cdot\,,\,\cdot\,]$ is called a Lie algebra if $[X_1, X_2] \in L$ for any $X_1,\ X_2\ \in L$, and if the operation $[\,\cdot\,,\,\cdot\,]$ satisfies the axioms:*

a.1 (bilinearity) : for any $X_1,\ X_2,\ X_3\ \in L$ and $a,\ b\ \in R$

$$\begin{aligned}[aX_1 + bX_2, X_3] &= a\,[X_1, X_3] + b\,[X_2, X_3]\\ [X_1, aX_2 + bX_3] &= a\,[X_1, X_2] + b\,[X_1, X_3]\end{aligned}$$

a.2 (antisymmetry) : for any $X_1,\ X_2\ \in L$

$$[X_1, X_2] = -\,[X_2, X_1]$$

a.3 (the Jacobi identity) : for any $X_1,\ X_2,\ X_3\ \in L$

$$[[X_1, X_2]\,, X_3] + [[X_2, X_3]\,, X_1] + [[X_3, X_1]\,, X_2] = 0$$

Let L^r be an r-dimensional Lie algebra with basis $X_1, X_2, \ldots, X_r$: i.e., any vector $X \in L^r$ can be decomposed as

$$X = \sum_{k=1}^{r} x_k X_k$$

where x_k are the coordinates of the vector X in the basis $\{X_1, \ldots, X_r\}$. Then

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k; \quad i,\, j = 1, 2, \ldots, r$$

with real constants c_{ij}^k.

The numbers c_{ij}^k are called the structural constants of the Lie algebra L^r for the basis $\{X_1, \ldots, X_r\}$.

Definition 8. *A vector space $H \subset L$ is called a subalgebra of the Lie algebra L, if $[H_1, H_2] \in H$ for any $H_1, H_2 \in H$.*

Definition 9. *A subalgebra $I \subset L$ is called an ideal of the Lie algebra L if for any $X \in L$, $Y \in I$ it is also true that $[X, Y] \in I$.*

Definition 10. *An element $Y \in L$ is called central, if $[X,Y] = 0$ for any $X \in L$. The set of all central elements is called the center of the Lie algebra L.*

2.3 Classification of subalgebras

One of the main aims of group analysis is to construct exact solutions of differential equations. The set of all solutions can be divided into equivalence classes of solutions:

Definition 11. *Two solutions u_1 and u_2 of a differential equation are said to be equivalent with respect to a Lie group G, if one of the solutions can be transformed into the other by a transformation belonging to the group G.*

The problem of classification of exact solutions is equivalent to the classification of subgroups (or subalgebras) of the group G (or the subalgebra L). Because there is a one-to-one correspondence between Lie groups and Lie algebras let us explain here the classification of subalgebras. For this purpose, one needs the following definitions.

Definition 12. *Let L and $\overline{L}$ be Lie algebras. A linear one-to-one map f of L onto $\overline{L}$ is called an isomorphism if it satisfies the equation*

$$f\left([X_1, X_2]_L\right) = [f(X_1), f(X_2)]_{\overline{L}}, \ \forall \ X_1, \ X_2 \ \in L$$

where the indices L and $\overline{L}$ denote the commutators in the corresponding algebras. An isomorphism of L onto itself is called an automorphism of the Lie algebra L. This mapping will be denoted by the symbol $A : L \to L$.

In the finite-dimensional case, isomorphic Lie algebras have the same dimensions. The criterion for two Lie algebras to be equivalent can be stated in terms of their structural constants. If two of Lie algebra L and $\overline{L}$ are isomorphic, then there exist bases for each of them in which their structural constants are equal.

Let L be a Lie algebra with basis $\{X_1, X_2, \ldots, X_n\}$. Then one has

$$[X_i, X_j] = \sum_{\alpha=1}^{n} c_{ij}^{\alpha} X_{\alpha}; \quad (i, j = 1, 2, \ldots, n),$$

where c_{ij}^{α} are the structural constants. One constructs a one-parameter family of automorphism, A_i, $(i = 1, ..., n)$ on L,

$$A_i : \sum_{i=1}^{n} x_i X_i \to \sum_{i=1}^{n} \bar{x}_i X_i$$

where $\bar{x}_i = \bar{x}_i(a)$, as follows. Consider the system

$$\frac{\mathrm{d}\overline{x}_j}{\mathrm{d}a} = \sum_{\beta=1}^{n} c_{\beta i}^{j} \overline{x}_{\beta}, \quad (j = 1, 2, \ldots, n)\,. \tag{2.22}$$

Initial values for this system are $\overline{x}_j = x_j$ at $a = 0$. The set of solutions of these equations determines the set of automorphisms $\{A_i\}$.

The set of all subalgebras is divided into equivalence classes with respect to these automorphisms. A list of representatives, where each element of this list is one representative from every class, is called an optimal system of subalgebras.

Because of the difficulties in constructing the optimal system of subalgebras for Lie algebras of large dimension, there is a two-step algorithm (Ovsiannikov, 1994), which reduces this problem to the problem for constructing an optimal system of algebras of lower dimensions. In brief, let us consider an algebra L^r with basis $\{X_1, X_2, \ldots, X_r\}$. According to the algorithm, the algebra L^r is decomposed as $I_1 \oplus N_1$, where I_1 is an ideal of L^r and N_1 is a subalgebra of the algebra L^r. In the same way, the subalgebra N_1 can also be decomposed as $N_1 = I_2 \oplus N_2$. Repeats the same process $(\alpha - 1)$ times one ends up with an algebra N_α, for which an optimal system of subalgebras can be easily constructed. By gluing the ideals I_l and subalgebras N_l starting from $l = \alpha$ to $l = 1$, together one constructs the optimal system of subalgebras for the algebra L^r. Note that for every subalgebra N_l one needs to check the subalgebra conditions and use the automorphisms to simplify the coefficients of these systems. Therefore, the problem for constructing an optimal system of subalgebras of the algebra L^r by this method is reduced to the problem of classification of algebras of lower dimensions.

After constructing the optimal system, one can start seeking invariant and partially invariant solutions of subalgebras from the optimal system.

2.4 Invariant and partially invariant solutions

The notion of invariant solution was introduced by Sophus Lie (1895). The notion of a partially invariant solution was introduced by Ovsiannikov (1958). This notion of partially invariant solutions generalizes the notion of an invariant solution, and extends the scope of applications of group analysis for constructing exact solutions of partial differential equations. The algorithm of finding invariant and partially invariant solutions consists of the following steps.

Let L^r be a Lie algebra with the basis $X_1, ..., X_r$. The universal invariant J consists of $s = m + n - r_*$ functionally independent invariants

$$J = \left(\; J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u) \; \right),$$

where n, m is the numbers of independent and dependent variables, respectively and r_* is the total rank of the matrix composed by the coefficients of the generators X_i, $(i = 1, 2, ..., r)$. If the rank of the Jacobi matrix $\dfrac{\partial(J^1, ..., J^{m+n-r_*})}{\partial(u^1, ..., u^m)}$ is equal to q, then one can choose the first $q \leq m$ invariants $J^1, ..., J^q$ such that the rank of the Jacobi matrix $\dfrac{\partial(J^1, ..., J^q)}{\partial(u_1, ..., u_m)}$ is equal to q. A partially invariant solution is characterized by two integers: $\sigma \geq 0$ and $\delta \geq 0$. These solutions are also called $H(\sigma, \delta)$-solutions. The number σ is called the rank of a partially invariant solution. This number gives the number of the independent variables in the representation of the partially invariant solution. The number δ is called the defect of a partially invariant solution. The defect is the number of the dependent functions which can not be found from the representation of partially invariant solution. The rank σ and the defect δ must satisfy the conditions

$$\sigma = \delta + n - r_* \geq 0, \ \delta \geq 0,$$

$$\rho \leq \sigma < n, \ \max\{r_* - n, m - q, 0\} \leq \delta \leq \min\{r_* - 1, m - 1\},$$

where ρ is the maximum number of invariants which depend on the independent variables only. Note that for invariant solutions, $\delta = 0$ and $q = m$.

For constructing a representation of a $H(\sigma, \delta)$-solution one needs to choose $l = m - \delta$ invariants and separate the universal invariant in two parts:

$$\overline{J} = (J^1, ..., J^l), \ \overline{\overline{J}} = (J^{l+1}, J^{l+2}, ..., J^{m+n-r_*}).$$

The number l satisfies the inequality $1 \leq l \leq q \leq m$. The representation of the $H(\sigma, \delta)$-solution is obtained by assuming that the first l coordinates $\overline{J}$ of the universal invariant are functions of the invariants $\overline{\overline{J}}$:

$$\overline{J} = W(\overline{\overline{J}}). \tag{2.23}$$

Equation (2.23) form the invariant part of the representation of a solution. The next assumption about a partially invariant solution is that equation (2.23) can be solved for the first l dependent functions, for example,

$$u^i = \phi^i(u^{l+1}, u^{l+2}, ..., u^m, x), \ (i = 1, ..., l). \tag{2.24}$$

It is important to note that the functions W^i, $(i = 1, ..., l)$ are involved in the expressions for the functions ϕ^i, $(i = 1, ..., l)$. The functions $u^{l+1}, u^{l+2}, ..., u^m$ are called superfluous. The rank and the defect of the $H(\sigma, \delta)$-solution are $\delta = m - l$ and $\sigma = m + n - r_* - l = \delta + n - r_*$, respectively.

Note that if $\delta = 0$, the above algorithm is the algorithm for finding a representation of an invariant solution. If $\delta \neq 0$, then equations (2.24) do not define all dependent functions. Since a partially invariant solution satisfies the restrictions (2.23), this algorithm cuts out some particular solutions from the set of all solutions.

After constructing the representation of an invariant or partially invariant solution (2.24), it has to be substituted into the original system of equations. The system of equations obtained for the functions W and superfluous functions u^k, $(k = l + 1, 2, ..., m)$ is called the reduced system. This system is overdetermined and requires an analysis of compatibility. Compatibility analysis for invariant solutions is easier than for partially invariant solutions. Another case of partially invariant solutions which is easier than the general case occurs when $\overline{\overline{J}}$ only depends on the independent variables

$$J^{l+1} = J^{l+1}(x), J^{l+2} = J^{l+2}(x), ..., J^{m+n-r_*} = J^{m+n-r_*}(x).$$

In this case, a partially invariant solution is called regular, otherwise it is irregular (Ovsiannikov, 1995). The number $\sigma - \rho$ is called the measure of irregularity.

The process of studying compatibility consists of reducing the overdetermined system of partial differential equations to an involutive system (cf.

Meleshko (2001)). During this process different subclasses of $H(\sigma,\delta)$ partially invariant solutions can be obtained. Some of these subclasses can be $H_1(\sigma_1,\delta_1)$-solutions with subalgebra $H_1 \subset H$. In this case $\sigma_1 \geq \sigma,\ \delta_1 \leq \delta$ (Ovsiannikov, 1978). The study of compatibility of partially invariant solutions with the same rank $\sigma_1 = \sigma$, but with smaller defect $\delta_1 < \delta$ is simpler than the study of compatibility for $H(\sigma,\delta)$-solutions. In many applications there is a reduction of a $H(\sigma,\delta)$-solution to a $H_1(\sigma,0)$ solution. In this case the $H(\sigma,\delta)$-solution is called reducible to an invariant solution. The problem of reduction to an invariant solution is important since invariant solutions are usually studied first.

Chapter III

Navier-Stokes Equations

3.1 Navier-Stokes Equations

The Navier-Stokes equations are fundamental partial differential equations that describe flows of incompressible fluids.

In order to derive these equations one starts from the conservation laws of mass, linear momentum and energy.

The conservation law of mass (or continuity equation) is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\texttt{div}(\mathbf{u}) = 0. \tag{3.1}$$

The conservation law of momentum (or motion equation) is

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \texttt{div}(P) + \rho f. \tag{3.2}$$

The conservation law of energy (or energy equation) is

$$\rho\frac{\mathrm{d}U}{\mathrm{d}t} = P : D + \texttt{div}(k\nabla\theta), \tag{3.3}$$

where ρ is the density, $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$ is the velocity, t is time, U is the internal energy, θ is the absolute temperature, P is the stress tensor, f is the external body force, $D = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial x} + \left(\frac{\partial \mathbf{u}}{\partial x}\right)^*\right)$ is the rate-of-strain tensor, k is the coefficient of a heat conductivity, $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$ stands for the total derivative with respect to time, ∇ is the gradient, $P : D$ is the contraction of the tensors P and D. By virtue of the Stokes axioms one gets that the stress tensor is

$P = (-p + \lambda \mathtt{div}(\mathbf{u}))I + 2\mu D$. Here p is the pressure, λ and μ are the first and the second coefficients of viscosity, respectively.

After substituting the stress tensor P into equations (3.2) and (3.3) one rewrites the motion equation as

$$\begin{aligned} \rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} &= \mathtt{div}\,[(-p + \lambda \mathtt{div}(\mathbf{u}))I + 2\mu D] + \rho f \\ &= -\nabla p + \nabla(\lambda \mathtt{div}(\mathbf{u})) + \mathtt{div}(2\mu D) + \rho f \end{aligned} \tag{3.4}$$

and the energy equation as

$$\begin{aligned} \rho \frac{\mathrm{d}U}{\mathrm{d}t} &= [(-p + \lambda \mathtt{div}(\mathbf{u}))I + 2\mu D] : D + \mathtt{div}(k\nabla\theta) \\ &= [-pI : D + (\lambda \mathtt{div}(\mathbf{u}))I : D + 2\mu D : D] + \mathtt{div}(k\nabla\theta) \\ &= -p\mathtt{div}(\mathbf{u}) + \lambda(\mathtt{div}(\mathbf{u}))^2 + 2\mu D : D + \mathtt{div}(k\nabla\theta) \\ &= -p\mathtt{div}(\mathbf{u}) + \phi + \mathtt{div}(k\nabla\theta) \end{aligned} \tag{3.5}$$

where $\phi = \lambda(\mathtt{div}(\mathbf{u}))^2 + 2\mu D : D$. The function ϕ is called the dissipation function.

According to the state axiom, fluids are two-parameter media which satisfy the main thermodynamic identity

$$\theta \mathrm{d}\eta = \mathrm{d}U + p\mathrm{d}\tau, \tag{3.6}$$

where η is the entropy and $\tau = \frac{1}{\rho}$ is the specific volume.

Thus, the rate form of equation (3.6) is

$$\theta \frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}t} + p\frac{\mathrm{d}\tau}{\mathrm{d}t}.$$

Since $\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\frac{1}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}$, then

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\frac{1}{\rho^2}(-\rho \mathtt{div}(\mathbf{u})) = \frac{1}{\rho}\mathtt{div}(\mathbf{u})$$

where $\frac{\mathrm{d}\rho}{\mathrm{d}t}$ is found from the continuity equation.

Hence,

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \theta\frac{\mathrm{d}\eta}{\mathrm{d}t} - \frac{p}{\rho}\,\mathtt{div}(\mathbf{u}). \tag{3.7}$$

Substituting (3.7) into (3.5), one obtains the equation

$$\rho\theta\frac{\mathrm{d}\eta}{\mathrm{d}t} = \mathtt{div}(k\nabla\theta) + \phi.$$

The second law of thermodynamics gives

$$\phi + \frac{k}{\theta}\left(\nabla\theta\right)^2 \geq 0.$$

In summary, the conservation laws of mass, momentum and energy are reduced to the following equations

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\mathtt{div}(\mathbf{u}) = 0, \\ \rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \nabla(\lambda\mathtt{div}(\mathbf{u})) + \mathtt{div}(2\mu D) + \rho f, \\ \rho\theta\frac{\mathrm{d}\eta}{\mathrm{d}t} = \mathtt{div}(k\nabla\theta) + \phi,\ \theta = \frac{\partial U}{\partial\eta},\ p = \rho^2\frac{\partial U}{\partial\rho}. \end{aligned} \tag{3.8}$$

One calls these equations the viscous gas dynamics equations.

If one sets ρ and μ to constant, then the first two equations of (3.8) can be rewritten as follows

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \mu \triangle \mathbf{u} + \rho f,\ \mathtt{div}(\mathbf{u}) = 0, \tag{3.9}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\ \ .$$

This system of equations (3.9) are called the Navier-Stokes equations.

In the thesis we consider the Navier-Stokes equations in the case of free external body force ($f = 0$).

It is also useful to write the Navier-Stokes equations in dimensionless form. Let $\mathbf{u}^*, p^*, \mathbf{x}^*, t^*$ be dimensionless variables, and they are related by the formulae

$$\mathbf{u}^* = \frac{\mathbf{u}}{V},\ p^* = \frac{p}{Q},\ \mathbf{x}^* = \frac{\mathbf{x}}{L},\ t^* = \frac{t}{T},$$

where V, Q, L and T are velocity, pressure, length and time units, respectively. Denoting the dimensionless terms of $\triangledown$ and $\triangle$ by $\triangledown^* = L\triangledown$ and $\triangle^* = L^2 \triangle$. The Navier-Stokes equations are rewritten as

$$(S_t)\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \triangledown^*)\mathbf{u}^* = -(E_u)\triangledown^* p^* + (\frac{1}{R_e}) \triangle^* \mathbf{u}^*, \quad \triangledown^* \cdot \mathbf{u}^* = 0,$$

where $S_t = \frac{L}{VT},\ E_u = \frac{Q}{\rho V^2}, R_e = \frac{\rho V L}{\mu}$ are called the Strouhal Number, Euler Number and Reynolds Number, respectively. So by choosing the units L, V, T, Q such that $S_t = 1,\ E_u = 1,\ R_e = 1$, one obtains

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \triangledown^*)\mathbf{u}^* = -\triangledown^* p^* + \triangle^* \mathbf{u}^*, \quad \triangledown^* \cdot \mathbf{u}^* = 0.$$

After omitting $*$, one has

$$\mathbf{u}_t + (\mathbf{u} \cdot \triangledown)\mathbf{u} = -\triangledown p + \triangle \mathbf{u}, \quad \triangledown \cdot \mathbf{u} = 0, \tag{3.10}$$

where $\triangledown$ and $\triangle$ are the gradient and the Laplacian with respect to the space variables $\mathbf{x} = (x, y, z)$, respectively.

In component form the Navier-Stokes equations are:

$$u_t + uu_x + vu_y + wu_z = -p_x + u_{xx} + u_{yy} + u_{zz}, \tag{3.11}$$

$$v_t + uv_x + vv_y + wv_z = -p_y + v_{xx} + v_{yy} + v_{zz}, \tag{3.12}$$

$$w_t + uw_x + vw_y + ww_z = -p_x + w_{xx} + w_{yy} + w_{zz}, \tag{3.13}$$

$$u_x + v_y + w_z = 0. \tag{3.14}$$

The dependent variables u, v, w and p are functions of the space variables x, y, z and time t.

3.1.1 Admitted group of the Navier-Stokes equations

An admitted Lie group of point transformations of the Navier-Stokes equations in the three-dimensional case[1] was found in Bytev (1972). The Lie algebra admitted by the Navier-Stokes equations is infinite-dimensional. Its Lie algebra can be presented in the form of the direct sum $L^\infty \oplus L^5$, where the infinite-dimensional ideal L^∞ is generated by the operators[2]

$$\begin{aligned} X_0 &= \partial_t, X_{ij} = x_j\partial_{x_i} - x_i\partial_{x_j} + u^j\partial_{u^i} - u^i\partial_{u^j}, \\ Z &= 2t\partial_t + \sum_{k=1}^{3}(x_k\partial_{x_k} - u^k\partial_{u^k}) - 2p\partial_p, \\ \Psi_j &= \psi_j(t)\partial_{x_j} + \psi_j'(t)\partial_{u^j} - \rho x_j\psi_j''(t)\partial_p,\ \Phi = \phi(t)\partial_p, \end{aligned} \tag{3.15}$$

where $i = 1,2,\ j = 1,2,3,\ i < j$; ψ_j and ϕ are arbitrary functions of time, and the prime stands for differentiation with respect to t. The transformation corresponding to the generator X_0 is translation along the t-axis:

$$\overline{t} = t + a_1. \tag{3.16}$$

The transformation corresponding to the generator X_{ij} is a rotation of the coordinate system by the angle a_{i+j-1} in the (x_i, x_j) plane:

$$\begin{pmatrix}\overline{x}_i \\ \overline{x}_j\end{pmatrix} = A(a_{i+j-1})\begin{pmatrix}x_i \\ x_j\end{pmatrix}, \quad \begin{pmatrix}\overline{u}^i \\ \overline{u}^j\end{pmatrix} = A(a_{i+j-1})\begin{pmatrix}u^i \\ u^j\end{pmatrix}. \tag{3.17}$$

Here

$$A(a) = \begin{pmatrix}\cos a & \sin a \\ -\sin a & \cos a\end{pmatrix};\ i = 1,2;\ j = 1,2,3;\ i < j.$$

[1] The two-dimensional Navier-Stokes equations were studied by group analysis in Pukhnachov (1960).

[2] There is still no complete classification of the subalgebras of the Lie algebra $L^\infty \oplus L^5$. Classification of infinite–dimensional subalgebras of this algebra was studied in Khabirov (1992). An approach to classification of infinite dimensional Lie algebras was recently proposed in Ryzhkov (2004).

The transformation corresponding to the generator Z is a dilation:

$$\overline{t} = e^{2a_5}t,\ \overline{x}_k = e^{2a_5}x_k,\ \overline{u}^k = e^{-a_5}u^k,\ \overline{p} = e^{-2a_5}p,\ k = 1,2,3. \tag{3.18}$$

The transformation corresponding to the generator Ψ_j is a transition to a coordinate system moving transitionally along the x_j−axis which is noninertial in general:

$$\begin{aligned} \overline{x}_j &= x_j + a_{5+j}\psi_j(t),\ \overline{u}^j = u^j + a_{5+j}\psi_j'(t), \\ \overline{p} &= p - a_{5+j}x_j\psi_j''(t) - \frac{\rho}{2}a_{5+j}^2\psi_j(t)\psi_j''(t),\ j = 1,2,3. \end{aligned} \tag{3.19}$$

The transformation corresponding to the generator Φ is an addition of an arbitrary function of time to the pressure:

$$\overline{p} = p + a_9\phi(t). \tag{3.20}$$

Here $a_k \in R$ $(k = 1,...,9)$ are the parameters of the corresponding transformations. When the law of transformation is not indicated for some variables in (3.16) – (3.20), this means that the corresponding variables are transformed identically.

Let us consider the Navier-Stokes equations under the assumption $f = \nabla U$, which means that the field of external forces is potential. In this case the change $p = \overline{p} + \rho U$ of the function sought leads to equation (3.9) with $f = 0$ and the functions $\overline{u}$ and $\overline{p}$. Thus, the basis groups of the Navier-Stokes equations with $f = \nabla U$ and the system (3.9) with $f = 0$ are isomorphic. Observe that the fields of forces most important in applications, namely, the gravitational field and the field of inertia forces, posses the potentiality property. The specific property of Lie algebra (3.15) is its infinite dimensionality, due to the presence of arbitrary functions of t in the coefficients of the operators (3.15).

Successively putting $\psi_j = 1$ and $\psi_j = t$ in the coefficients of the operators Ψ_j one obtains the infinitesimal generators

$$X_j = \partial_{x_j},\ Y_j = t\partial_{x_j} + \partial_{u^j},\ j = 1, 2, 3.$$

The set of operators $X_0,\ X_j,\ Y_j,\ X_{ij}, (i = 1, 2;\ j = 1, 2, 3;\ i < j)$ generates a Lie algebra L^{10}, which corresponds to a 10−parameter group of transformations denoted G^{10}. The Group G^{10} is obviously a subgroup of G^{∞}. The Galilean algebra L^{10} is contained in $L^{\infty} \oplus L^5$. Several articles (Pukhnachov (1974), Cantwell (1978), Cantwell (2002), Lloyd (1981), Boisvert (1983), Steeb (1985), Ibragimov and Unal (1994), Lloyd (1981), Popovych (1995), Fushchich and Popovych (1994), Ludlow, Clarkson and Bassom (1999).) are devoted to invariant solutions of the Navier-Stokes equations[3]. While partially invariant solutions of the Navier–Stokes equations have been less studied[4], there has been substantial progress in studying such classes of solutions of inviscid gas dynamics equations (Ovsiannikov (1978), Ovsiannikov and Chupakhin (1996), Ovsiannikov (1995), Sidorov, Shapeev and Yanenko (1984), Meleshko (1991,1994), Ovsiannikov (1994), Chupakhin (1997), Grundland (1996).

3.2 Coordinate systems

For the sake of convenience, some problems require a special coordinate system. A coordinate transformation is a conversion from one system of the

[3]Short reviews devoted to invariant solutions of the Navier-Stokes equations can be found in Pukhnachov (1974), Cantwell (2002), Fushchich and Popovych (1994), Ludlow, Clarkson and Bassom (1999).

[4]The approach of partially invariant solutions to the Navier-Stokes equations was first applied in Pukhnachov (1974).

independent variables to another. In this thesis, the following coordinate systems are used.

The simplest one is the Cartesian coordinate system (D). In this system $\mathbf{x} = (x, y, z)$ is the vector of the independent variables, $\mathbf{u} = (u, v, w)$ is the vector of the dependent variables. Relations of other coordinate systems with the Cartesian coordinate system are given as follows.

The first system is the spherical coordinate system (S). In this system, the vector of the independent variables is (r, θ, φ), where

$$x = r \sin\theta \cos\varphi, \;\; y = r \sin\theta \sin\varphi, \;\; z = r \cos\theta.$$

The conversion of the Cartesian coordinate system into the spherical coordinate system is

$$r = \sqrt{x^2 + y^2 + z^2}, \;\; \theta = \arccos{(z/y)}, \;\; \varphi = \arctan{(y/x)}.$$

The corresponding physical components of the velocity vector in the Cartesian coordinate system $\mathbf{u} = (u, v, w)$ and in the spherical coordinate system $\mathbf{u} = (U, V, W)$ are related by the expressions

$$\begin{aligned} u &= U \sin\theta \cos\varphi + V \cos\theta \cos\varphi - W \sin\varphi, \\ v &= U \sin\theta \cos\varphi + V \cos\theta \sin\varphi - W \cos\varphi, \\ w &= U \sin\theta - V \sin\theta \end{aligned}$$

or, they can be written as follows

$$\begin{aligned} U &= u \sin\theta \cos\varphi + v \sin\theta \sin\varphi + w \cos\theta, \\ V &= u \cos\theta \cos\varphi + v \cos\theta \sin\varphi - w \sin\theta, \\ W &= -u \sin\varphi + v \cos\varphi. \end{aligned}$$

Note that the vector (V, W) can be described by its modulus H and the angle ω

$$V = H \cos\omega, \;\; W = H \sin\omega.$$

The second system is the cylindrical coordinate system (C). The relations between Cartesian and cylindrical coordinate systems (x, R, θ) are given by the formulae

$$x = x,\ y = R\cos\theta,\ z = R\sin\theta.$$

The conversion of the Cartesian coordinate system into the cylindrical coordinate system is

$$x = x,\ R = \sqrt{y^2 + z^2},\ \theta = \arctan(z/y).$$

The physical components of the velocity vector in the Cartesian coordinate system (u, v, w) and in the cylindrical coordinate system (U, V, W) are related by the expressions

$$u = U,\ v = V\cos\theta - W\sin\theta,\ w = V\sin\theta + W\cos\theta$$

or,

$$u = U,\ V = v\cos\theta + w\sin\theta,\ W = -v\sin\theta + w\cos\theta.$$

Introducing the modulus q and the angle φ, the vector (V, W) has components

$$V = q\cos\varphi,\ W = q\sin\varphi.$$

The third coordinate system is the polar coordinate system (P). The transformation from the Cartesian coordinate system to the polar coordinate system is presented by

$$\mathbf{x} = (x, y, z),\ \mathbf{u} = (u, q\cos\varphi, q\sin\varphi)$$

or

$$u = u,\ q = \sqrt{v^2 + w^2},\ \varphi = \arctan(w/v).$$

The fourth system is the polar conical coordinate system (PC). In this system, the relations are similar to the polar coordinate system. They are given by the formulae

$$\mathbf{x} = (x, y, z), \ \mathbf{u} = (u, yt^{-1} + q^* \cos\varphi^*, zt^{-1} + q^* \sin\varphi^*)$$

or

$$u = u, \ q^* = \sqrt{(v - yt^{-1})^2 + (w - zt^{-1})^2}, \ \varphi^* = \arctan\left((w - zt^{-1})/(v - yt^{-1})\right).$$

The next coordinate system is denoted by $(K1)$. In this system the coordinates are defined by the formulae

$$\mathbf{x} = (x, y, z), \ \mathbf{u} = \left(u, \frac{ty + z}{t^2 + 1} + V^* \cos\theta^*, \frac{tz - y}{t^2 + 1} + V^* \sin\theta^*\right)$$

or

$$\begin{aligned} &u = u, \\ &V^* = \sqrt{\left(v - \tfrac{ty+z}{t^2+1}\right)^2 + \left(w - \tfrac{tz-y}{t^2+1}\right)^2}, \\ &\theta^* = \arctan\left(\left(w - \tfrac{tz-y}{t^2+1}\right) / \left(v - \tfrac{ty+z}{t^2+1}\right)\right). \end{aligned}$$

The last coordinate system is denoted by $(K2)$. In this system the transformation is

$$\mathbf{x} = (x, y, z), \ \mathbf{u} = \left(u, \frac{j_1 - (\sigma t - \beta\tau)u + ty - \beta z}{t^2 - \alpha\beta}, \frac{j_2 - (\tau t - \alpha\sigma)u - \alpha y + tz}{t^2 - \alpha\beta}\right)$$

or

$$\begin{aligned} &u = u, \\ &j_1 = (t^2 - \alpha\beta)v + (\sigma t - \beta\tau)u - ty + \beta z, \\ &j_2 = (t^2 - \alpha\beta)v + (\tau t - \alpha\sigma)u + \alpha y - tz. \end{aligned}$$

These coordinate systems are used for convenience of writing a representation of partially invariant solutions.

Chapter IV

Analysis of Compatibility

Subgroups are taken from the optimal system of subalgebras (Ovsiannikov and Chupakhin (1996)) considered for the gas dynamics equations.

4.1 Three dimensional regular partially invariant submodels

In this thesis we study regular partially invariant solutions with defect 1 and rank 1 of the subalgebras presented in Table 4.1 where a list of the subalgebras with brief comments is given. These subalgebras were selected from the optimal system of subalgebras (cf. Ovsiannikov (1994)) of the algebra admitted by the gas dynamic equations. For gas dynamics equations, regular partially invariant solutions were studied in Ovsiannikov and Chupakhin (1996).

The basis of operators considered in the table is

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z,$$
$$X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_y + \partial_v, \quad X_6 = t\partial_z + \partial_w,$$
$$X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v, \quad X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w,$$
$$X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \; X_{10} = t\partial_t, \; X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z.$$

Note that the generator X_{11} is not admitted by the Navier-Stokes equations.

In Table 4.1 the number i in the first column denotes the number of the subalgebra according to the optimal system of subalgebras from Ovsiannikov (1994). The basis of generators of a subalgebra is given in the second column.

Table 4.1: Regular partially invariant submodels of the equation of the gas dynamics.

i	Basis L_i^4	Coordinate	Invariant: Independent	Invariant: Unknown	SF
1	$7,8,9,11$	S	r/t	U,H,p	ω
4	$1,4,10,7+\alpha 11$	C	$Re^{\alpha\theta}$	$q,\varphi-\theta,p$	u
5	$5,6,7,\beta 4+11$	PC	$x/t-\beta\ln t$	$u-x/t,q^*,p$	φ^*
6	$1,4,7,11$	C	R/t	$q,\varphi-\theta,p$	u
7	$2,3,7,\beta 4+11$	P	$x/t-\beta\ln t$	$u-x/t,q,p$	φ
9	$1,5,6,\beta 4+7$	PC	t	$u-\beta\varphi^*,q^*,p$	φ^*
10	$2,3,4,7$	P	t	$u-x/t,q,p$	φ
12	$1,2,3,\beta 4+7$	P	t	$u-\beta\varphi,q,p$	φ
13	$7,8,9,10$	S	r	U,H,p	ω
14	$2,3,7,10$	P	x	u,q,p	φ
16	$2,3,7,4+10$	P	$x-(1/2)t^2$	$u-t,q,p$	φ
17	$4,5,6,7$	PC	t	$u-x/t,q^*,p$	φ^*
18	$4,5,6,1+7$	PC	t	$u+(\varphi^*-x)/t,q^*,p$	φ^*
19	$4,3+5,2-6,\alpha 1+7$	K_1	t	$u+(\alpha\theta^*-x)/t,V^*,p$	θ^*
20	$1,3+5,2-6,\alpha 4+7$	K_1	t	$u-\alpha\theta^*,V^*,p$	θ^*
21	$2,3,4,1+7$	P	t	$u+(\varphi-x)/t,q,p$	φ
23	$1,4,10,11$	D	z/y	v,w,p	u
29	$1,4,6,\alpha 5+11$	D	$y/t-\alpha\ln t$	$u-y/t,w-z/t,p$	u
30	$2,3,6,\beta 4+\sigma 5+11$	D	$x/t-\beta\ln t$	$u-x/t,v-\sigma\ln t,p$	w
35	$2,3,5,4+\beta 6+10$	D	$x-(1/2)t^2$	$u-t,w-\beta t,p$	v
36	$2,3,5,6+10$	D	x	$u,w-t,p$	v
38	$2,3,5,10$	D	x	u,w,p	v
41	$1,\sigma 2+\tau 3+4,$ $\alpha 3+5,\beta 2+6$	K_2	t	j_1,j_2,p	u
42	$1,4,3+5,2-6$	K_1	t	V^*,θ^*,p	u
43	$1,4,5,6$	D	t	$v-y/t,w-z/t,p$	u
44	$2,\alpha 1+3,1+5,6$	D	t	$u,v-\alpha tw-x+\alpha z,p$	w
46	$2,\alpha 1+3,5,6$	D	t	$u,w+(x-\alpha z)/\alpha t,p$	v
48	$1,2,3+5,6$	D	t	$u,v+tw-z,p$	w
50	$1,2,3,4$	D	t	v,w,p	u

Each operator X_k is represented only by its number k. For example, the symbol $7+\alpha 11$, where α is a real number, denotes the operator $X_7+\alpha X_{11}$. The third column indicates the coordinate system in which the subalgebra is studied. The next two columns give invariants, where the first part represents an invariant only containing the independent variables and the second part contains the remaining invariants. The sixth column indicates the superfluous function (SF) used for constructing a partially invariant solution.

The construction of a partially invariant solution consists of several steps. First, choose a subgroup from Table 4.1. Then find a representation of a partially invariant solution. After that substitute the representation of the solution into the Navier-Stokes equations. Finally, one needs to study the compatibility of the obtained (reduced) system of equations. As a result one obtains an exact solution of the Navier-Stokes equations.

Recall that the Navier-Stokes equations define the pressure up to an arbitrary function of time. This property is essentially used in the next sections.

4.2 Analysis of compatibility of partially invariant solutions

In this section analysis of compatibility of the partially invariant solutions for typical representatives is presented in details. The analysis of compatibility of other subalgebras is similar. The final results are collected in tables at the end of this section.

4.2.1 Subalgebra generated by $L_{16}^4 = \{2, 3, 7, 4 + 10\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (P) are

$$u - t,\ q,\ p,\ x - 2^{-1}t^2.$$

The rank of the Jacobi matrix of the invariants with respect to the dependent variables is equal to three. Since this rank is less than the number of the dependent variables, there are no nonsingular invariant solutions that are invariant with respect to this group. The minimally possible defect of a partially invariant solution with respect to this group is equal to one. In this case a representation of a regular partially invariant solution is

$$u = U(s) + t,\ p = P(s),\ q = q(s),\ s = x - 2^{-1}t^2$$

while the function $\varphi(t, x, y, z)$ still depends on all independent variables. Substituting this representation of a solution into the Navier-Stokes equations, one obtains

$$P' - U'' + UU' + 1 = 0, \tag{4.1}$$

$$[(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - \varphi_z q \sin\varphi)q + (2q' - (t + U)q)\varphi_x] \sin\varphi$$
$$-[q'' - Uq' - q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + q^2\varphi_y \sin\varphi] \cos\varphi = 0, \tag{4.2}$$

$$[(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \varphi_t - \varphi_z q \sin\varphi)q + (2q' - (t + U)q)\varphi_x] \cos\varphi$$
$$+[q'' - Uq' - q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2)] \sin\varphi - q^2\varphi_y \cos^2\varphi] = 0, \tag{4.3}$$

$$q(\varphi_y \sin\varphi - \varphi_z \cos\varphi) - U' = 0. \tag{4.4}$$

Notice that for $q = 0$ the general solution of equations (4.1)-(4.4) is

$$U = C_1,\ P = -s + C_2,\ v = 0,\ w = 0,\ s = x - 2^{-1}t^2,$$

where C_1 and C_2 are constants. Since the pressure is defined up to an arbitrary function of time, one can assume $C_2 = 0$.

Further consideration is given for $q \neq 0$.

Integrating equation (4.1) with respect to s, one finds

$$P = U' - 2^{-1}U^2 - s + C.$$

Taking the combination of equations (4.2) and (4.3) by excluding the derivative φ_t, one has

$$q'' - Uq' - q(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) = 0. \tag{4.5}$$

Changing the independent variables (t, x, y, z) to (t, s, y, z), equations (4.2), (4.4) and (4.5) become

$$[(\varphi_{ss} + \varphi_{yy} + \varphi_{zz} - \varphi_t - \varphi_z q \sin\varphi)q + (2q' - Uq)\varphi_s]\sin\varphi$$
$$-[q'' - Uq' - q(\varphi_s^2 + \varphi_y^2 + \varphi_z^2) + q^2\varphi_y \sin\varphi]\cos\varphi = 0, \tag{4.6}$$

$$q'' - Uq' - q(\varphi_s^2 + \varphi_y^2 + \varphi_z^2) = 0, \tag{4.7}$$

$$q(\varphi_y \sin\varphi - \varphi_z \cos\varphi) - U' = 0. \tag{4.8}$$

For the compatibility analysis of systems (4.6)-(4.8), it is convenient to use an implicit representation of the function $\varphi(t, s, y, z)$. Assume that there is a function $F(\varphi, t, s, y, z)$ such that $F_\varphi \neq 0$ and

$$F(\varphi(t, s, y, z), t, s, y, z) = 0. \tag{4.9}$$

Taking the total derivatives of equation (4.9) with respect to t, s, y, z, one has

$$D_tF = F_\varphi\varphi_t + F_t = 0, \;\; D_sF = F_\varphi\varphi_s + F_s = 0,$$

$$D_yF = F_\varphi\varphi_y + F_y = 0, \;\; D_zF = F_\varphi\varphi_z + F_z = 0,$$

$$D_s^2F = F_\varphi\varphi_{ss} + \varphi_s^2F_{\varphi\varphi} + 2F_{s\varphi}\varphi_s + F_{ss} = 0,$$

$$D_y^2F = F_\varphi\varphi_{yy} + \varphi_y^2F_{\varphi\varphi} + 2F_{y\varphi}\varphi_y + F_{ss} = 0,$$

$$D_z^2F = F_\varphi\varphi_{zz} + \varphi_z^2F_{\varphi\varphi} + 2F_{z\varphi}\varphi_z + F_{ss} = 0.$$

All derivatives of the function $\varphi(t,s,y,z)$ can be found through the derivatives of the function $F(\varphi,t,s,y,z)$ from these equations:

$$\varphi_t = -F_t/F_\varphi,\ \varphi_s = -F_s/F_\varphi,\ \varphi_y = -F_y/F_\varphi,\ \varphi_z = -F_z/F_\varphi,$$
$$\varphi_{ss} = -(F_{\varphi\varphi}\varphi_s^2 + 2F_{\varphi s}\varphi_s + F_{ss})/F_\varphi,$$
$$\varphi_{yy} = -(F_{\varphi\varphi}\varphi_y^2 + 2F_{\varphi y}\varphi_y + F_{yy})/F_\varphi,$$
$$\varphi_{zz} = -(F_{\varphi\varphi}\varphi_z^2 + 2F_{\varphi z}\varphi_z + F_{zz})/F_\varphi.$$

Substituting these derivatives into equation (4.8), one obtains

$$F_y \sin\varphi - F_z \cos\varphi + aF_\varphi = 0, \tag{4.10}$$

where the function $a = q^{-1}U'$ only depends on s.

Case 1. Let $a = 0$, or $U = C_1$. Notice that if φ is constant, then the general solution of equations (4.1), (4.6)-(4.8) is

$$P = -s + \tilde{C},\ q = C_2 + C_3\mathsf{e}^{C_1 s},\ s = x - 2^{-1}t^2,$$

where C_1, C_2, C_3 and $\tilde{C}$ are constants. Since the pressure is defined up to an arbitrary function of time, one can assume $\tilde{C} = 0$.

Assume that φ is not constant. The universal invariant is $t, s, \hat{y}, \varphi$, where $\hat{y} = y\cos\varphi + z\sin\varphi$. Thus, the general solution of equation (4.10) is

$$F = \phi(t,s,\hat{y},\varphi).$$

Since $F_\varphi = (-y\sin\varphi + z\cos\varphi)\phi_{\hat{y}} + \phi_\varphi \neq 0$, this gives that $\phi_{\hat{y}}^2 + \phi_\varphi^2 \neq 0$. Substituting the function $F = \phi(t,s,\hat{y},\varphi)$ into equation (4.7), one obtains

$$a_1 z^2 + a_2 z + a_3 = 0 \tag{4.11}$$

where

$$
\begin{aligned}
a_1 &= -(q'' - C_1 q')\phi_{\hat{y}}^2, \\
a_2 &= -2(q'' - C_1 q')[\phi_\varphi \cos\varphi - \hat{y}\phi_{\hat{y}} \sin\varphi]\phi_{\hat{y}}, \\
a_3 &= -[(q'' - C_1 q')\phi_\varphi^2 - q\phi_s^2 - (\hat{y}^2(q'' - C_1 q') + q)\phi_{\hat{y}}^2]\cos^2\varphi \\
&\quad +(q'' - C_1 q')(2\phi_\varphi \cos\varphi \sin\varphi - \hat{y}\phi_{\hat{y}})\hat{y}\phi_{\hat{y}}.
\end{aligned}
$$

The coefficients a_1, a_2 and a_3 do not depend on z. Splitting (4.11) with respect to z, one finds

$$a_i = 0, \ \ (i = 1, 2, 3).$$

If $\phi_{\hat{y}} \neq 0$, then the equation $a_1 = 0$ implies that $q'' - C_1 q' = 0$, and the equation $a_3 = 0$ becomes

$$q(\phi_{\hat{y}}^2 + \phi_s^2) = 0.$$

Because of $\phi_{\hat{y}}^2 + \phi_s^2 \neq 0$, one has $q = 0$ which contradicts the assumption $q \neq 0$. If $\phi_{\hat{y}} = 0$, the equation $a_3 = 0$ implies that

$$(q'' - C_1 q')\phi_\varphi^2 - q\phi_s^2 = 0.$$

The general solution of the last equation is

$$\phi = \tilde{\phi}(t, \tilde{\varphi}),$$

where

$$\tilde{\varphi} = \varphi + H(s), \ \ H(s) = \int h(s)\mathrm{d}s, \ \ h = \pm((q'' - C_1 q')/q)^{1/2},$$

and $\phi_\varphi = \tilde{\phi}_{\tilde{\varphi}} \neq 0$.

Substituting $\phi = \tilde{\phi}(t, \tilde{\varphi})$ into equation (4.6), one obtains

$$b_1(s)\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + b_2(s)\tilde{\phi}_{\tilde{\varphi}} + \tilde{\phi}_t = 0, \tag{4.12}$$

where $b_1 = h^2, \ b_2 = -q^{-1}(2q' - C_1 q)h - h'$. Let us analyze this equation. Differentiating it with respect to s, one obtains

$$b_1' \tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + b_2' \tilde{\phi}_{\tilde{\varphi}} = 0.$$

Assume first that $b_1' \neq 0$. Then the last equation can be rewritten

$$\frac{\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}}}{\tilde{\phi}_{\tilde{\varphi}}} = -\frac{b_2'}{b_1'}.$$

Differentiating it with respect to s, one has $(-b_2'/\ b_1')' = 0$. Thus $b_2 = -C_2 b_1 + C_3$. This relation, after substituting representations for the coefficients b_1 and b_2 becomes

$$((q'' - C_1 q')q)^{1/2}(C_2(q'' - C_1 q') + C_3 q) + (q'' - C_1 q')(2q' - C_1 q)$$
$$- [(q''' - C_1 q'')q - (q'' - C_1 q')q']\, 2^{-1} = 0. \qquad (4.13)$$

In this case the solution of equation (4.12) is

$$\tilde{\phi} = C_4 \mathrm{e}^{C_2(\tilde{\varphi} - C_3 t)} + C_5.$$

Returning back to the solution of equations (4.6)-(4.8), one gets

$$\varphi = C_3 t - H(s) + C,$$

where $C = C_5(C_2 C_4)^{-1}$.

For the case $b_1' = 0$, then $b_2' = 0$, or $b_1 = C_2 \geq 0$ and $b_2 = C_3$. This means that

$$\frac{q'' - C_1 q'}{q} = C_2\ , \quad \frac{(2q' - C_1 q)C_2^{1/2}}{q} = \pm C_3.$$

Let $C_2 = 0$, then $C_3 = 0$, and, hence $q = C_4 + C_5 \mathrm{e}^{C_1 s}$. Equation (4.12) becomes $\tilde{\phi}_t = 0$. This means that the function $\tilde{\phi}$ only depends on $\tilde{\varphi}$. Returning back to the solution of equations (4.6)-(4.8), one has $\varphi = -H(s) + C_6$. If $C_2 > 0$, then $q = C_4 \mathrm{e}^{\tilde{C}s/2}$, where $\tilde{C} = C_1 \pm C_3 C_2^{-1/2}$ and the constants C_1, C_2 and C_3 are related by the equation $C_3^2 - C_2(C_1^2 + 4C_2) = 0$. The solution of equation (4.12) is $\phi = \tilde{\phi}(t, \tilde{\varphi})$ where the function $\tilde{\phi}(t, \tilde{\varphi})$ is a solution of the linear parabolic equation with constant coefficients

$$C_2 \tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + C_3 \tilde{\phi}_{\tilde{\varphi}} + \tilde{\phi}_t = 0.$$

Case 2. Assume that $a \neq 0$. The characteristic system of the equation (4.10) is

$$\frac{\mathrm{d}y}{\sin\varphi} = \frac{\mathrm{d}z}{-\cos\varphi} = \frac{\mathrm{d}\varphi}{a}.$$

Hence, the universal invariant of this system is t, s, y_1, y_2, where $y_1 = \cos\varphi + ay$, $y_2 = \sin\varphi + az$. Thus, the general solution of equation (4.8) is

$$F = \phi(t, s, y_1, y_2),$$

where $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$.

Substituting the derivatives of the function $\varphi(t, s, y, z)$ through the derivatives of the function $F(\varphi, t, s, y, z)$ into equation (4.7), one obtains

$$a_1 \cos^2\varphi + a_2 \cos\varphi \sin\varphi + a_3 \cos\varphi + a_4 \sin\varphi + a_5 = 0 \tag{4.14}$$

where

$$\begin{aligned}
a_1 &= [(q'' - Uq')a^2 + q(a')^2](\phi_{y_1} + \phi_{y_2})(\phi_{y_1} - \phi_{y_2}),\\
a_2 &= 2[(q'' - Uq')a^2 + q(a')^2]\phi_{y_1}\phi_{y_2},\\
a_3 &= -2[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s]qa'\phi_{y_1},\\
a_4 &= -2[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s]qa'\phi_{y_2},\\
a_5 &= [[2(y_1\phi_{y_1} + y_2\phi_{y_2})a'\phi_s + (a^2\phi_{y_2}^2 + \phi_s^2)a]q-\\
&\quad (q'' - Uq' - qa^2)a\phi_{y_1}^2]a + [(y_1\phi_{y_1} + y_2\phi_{y_2})^2 + \phi_{y_2}^2]q(a')^2.
\end{aligned}$$

Note that a_1, a_2, a_3, a_4 and a_5 do not depend on φ. Splitting (4.14) with respect to $\cos^2\varphi, \cos\varphi\sin\varphi, \cos\varphi, \sin\varphi$, one obtains

$$a_i = 0,\ (i = 1, 2, ..., 5).$$

Noting that $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$, the equations $a_1 = 0,\ a_2 = 0$ imply that $(q'' - Uq')a^2 + q(a')^2 = 0$, and the equations $a_3 = 0,\ a_4 = 0$ imply

$$[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s]a' = 0. \tag{4.15}$$

The last equation is split into two cases.

Case 2.1. Assume that $a' \neq 0$, then equation (4.15) gives

$$(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s = 0.$$

The general solution of this equation is $\phi = f(t, g_1, g_2)$, where $g_1 = y_2/y_1,\ g_2 = a/y_1$ and $f_{g_1}^2 + f_{g_2}^2 \neq 0$.

Substituting the function ϕ into the equation $a_5 = 0$, one obtains

$$(q'' - Uq' - qa^2)[(g_1 f_{g_1} + g_2 f_{g_2})^2 + f_{g_1}^2] = 0.$$

Since $f_{g_1}^2 + f_{g_2}^2 \neq 0$, then $(g_1 f_{g_1} + g_2 f_{g_2})^2 + f_{g_1}^2 \neq 0$. Hence, $q'' - Uq' - qa^2 = 0$. Finding q'' from this equation, and substituting it into the equation $(q'' - Uq')a^2 + q(a')^2 = 0$, one has $(a')^2 + a^4 = 0$. This gives $a = 0$, which contradicts the assumption $a \neq 0$.

Case 2.2 Assume that $a' = 0$. Substituting $a' = 0$ into the equation $a_5 = 0$, one obtains

$$(q'' - Uq' - qa^2)\phi_{y_1}^2 - q(a^2\phi_{y_2}^2 + \phi_s^2) = 0.$$

Using here the equation $(q'' - Uq')a^2 + q(a')^2 = 0$, one finds

$$a^2\phi_{y_1}^2 + a^2\phi_{y_2}^2 + \phi_s^2 = 0.$$

Hence, $\phi_{y_1} = \phi_{y_2} = \phi_s = 0$, which is a contradiction to the condition $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$. Therefore there is no partially invariant solutions in case $a \neq 0$.

4.2.2 Subalgebra generated by $L_5^4 = \{5, 6, 7, \beta 4 + 11\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (PC) are

$$u - x/t,\ q^*,\ p,\ s,\ x/t - \beta \ln t.$$

The representation of a regular partially invariant solution is

$$u = U(s) + x/t,\ \ p = P(s),\ \ q^* = q^*(s),\ \ s = x/t - \beta \ln t$$

and the function $\varphi^*(t, x, y, z)$ depends on all independent variables.

Substituting this representation of a solution into the Navier-Stokes equations, one obtains

$$U'' - t(P' - (\beta - U)U' + U) = 0, \tag{4.16}$$

$$\begin{aligned}
&[((\varphi^*_{xx} + \varphi^*_{yy} + \varphi^*_{zz} - \varphi^*_t - \varphi^*_z q^* \sin\varphi^*)t - (y\varphi^*_y + z\varphi^*_z))q^* \\
&+(2q^{*'} - (tU + x)q^*)\varphi^*_x]t \sin\varphi^* - [q^{*''} + (\beta - U)tq^{*'} \\
&-q^* t^2(\varphi^{*2}_x + \varphi^{*2}_y + \varphi^{*2}_z) + q^{*2}t^2\varphi^*_y \sin\varphi^*] \cos\varphi^* = 0,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&[((\varphi^*_{xx} + \varphi^*_{yy} + \varphi^*_{zz} - \varphi^*_t - \varphi^*_z q^* \sin\varphi^*)t - (y\varphi^*_y + z\varphi^*_z))q^* \\
&+(2q^{*'} - (tU + x)q^*)\varphi^*_x]t \cos\varphi^* + [q^{*''} + (\beta - U)tq^{*'} - q^* t \\
&-q^* t^2(\varphi^{*2}_x + \varphi^{*2}_y + \varphi^{*2}_z)] \sin\varphi^* - q^{*2}t^2\varphi^*_y \cos^2\varphi^*] = 0,
\end{aligned} \tag{4.18}$$

$$q^* t(\varphi^*_y \sin\varphi^* - \varphi^*_z \cos\varphi^*) - (U' + 3) = 0. \tag{4.19}$$

In the case $q^* = 0$, the general solution of equations (4.16)-(4.19) is

$$U = -3s + C_1,\ \ P = (2C_1 - 3(\beta + s))s + C_2,\ \ v = 0,\ \ w = 0,\ \ s = x/t - \beta \ln t,$$

where C_1 and C_2 are constants. Because the pressure is defined up to an arbitrary function of time, C_2 can be set to zero.

Further consideration is given for $q^* \neq 0$.

Equation (4.16) can be split with respect to t:

$$U'' = 0,\ \ P' - (\beta - U)U' + U = 0.$$

Integrating the last equations with respect to s, one finds

$$U = C_1 s + C_2,\ \ P = -C_1(C_1 + 1)\frac{s^2}{2} + ((\beta - C_2)C_1 - C_2)s + C_3.$$

Taking the combination of equations (4.17) and (4.18) by excluding the derivative φ_t^*, one has

$$q^{*''} - (C_1 s + C_2 - \beta)tq^{*'} - q^* t - q^* t^2(\varphi_x^{*2} + \varphi_y^{*2} + \varphi_z^{*2}) = 0. \tag{4.20}$$

Changing the independent variables (t, x, y, z) onto (t, s, y, z), equations (4.17), (4.20) and (4.19) become

$$\begin{aligned}
&[(\varphi_{ss}^* + t^2(\varphi_{yy}^* + \varphi_{zz}^* - \varphi_t^* - \varphi_z^* q^* \sin\varphi^*))q^* \\
&+(2q^{*'} - (C_1 s + C_2 - \beta)tq^*)\varphi_s^*] \sin\varphi^* - [q^{*''} - (C_1 s + C_2 - \beta)tq^{*'} \\
&-q^*\varphi_s^{*2} - q^{*2}t^2(\varphi_y^{*2} + \varphi_z^{*2}) + q^{*2}t^2\varphi_y^* \sin\varphi^*] \cos\varphi^* = 0, \qquad (4.21)
\end{aligned}$$

$$q^{*''} - (C_1 s + C_2 - \beta)tq^{*'} - q^* t - q^*\varphi_s^{*2} - q^{*2}t^2(\varphi_y^{*2} + \varphi_z^{*2}) = 0, \tag{4.22}$$

$$q^* t(\varphi_y^* \sin\varphi^* - \varphi_z^* \cos\varphi^*) - (C_1 + 3) = 0. \tag{4.23}$$

In the same way as for the previous subalgebra, for the compatibility analysis of systems (4.21)-(4.23) it is convenient to use an implicit representation of the function $\varphi^*(t, s, y, z)$. Assume that there is a function $F(\varphi^*, t, s, y, z)$ such that $F_\varphi^* \neq 0$ and

$$F(\varphi^*(t, s, y, z), t, s, y, z) = 0. \tag{4.24}$$

Taking the total derivatives of (4.24) with respect to t, x, y, z and substituting derivatives of the function φ^* into equation (4.23), it becomes

$$t(F_y \sin\varphi^* - F_z \cos\varphi^*) + aF_{\varphi^*} = 0, \tag{4.25}$$

where the function $a = (C_1 + 3)/q^*$ only depends on s.

Case 1. Let $a = 0$, or $C_1 = -3$. If φ^* is constant, then the equations (4.21) and (4.22) become

$$(q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t = 0.$$

Splitting the last equation with respect to t, one obtains $q^{*''} = 0$ and $(C_2 - 3s)q^{*'} + q^* = 0$. Solving them, one has $q^* = 0$ which contradicts the assumption $q^* \neq 0$.

So suppose φ^* is not constant. The universal invariant is $t, s, \hat{y}, \varphi^*$, where $\hat{y} = y\cos\varphi^* + z\sin\varphi^*$. Thus, the general solution of equation (4.25) is

$$F = \phi(t, s, \hat{y}, \varphi^*).$$

where $\phi_{\hat{y}}^2 + \phi_{\varphi^*}^2 \neq 0$. Substituting the function $F = \phi(t, s, \hat{y}, \varphi^*)$ into equation (4.22), one obtains

$$a_1 z^2 + a_2 z + a_3 = 0 \tag{4.26}$$

where

$$\begin{aligned}
a_1 &= -[(q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t]\phi_{\hat{y}}^2,\\
a_2 &= -2[(q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t][\phi_{\varphi}^* \cos\varphi^* - \hat{y}\phi_{\hat{y}} \sin\varphi^*]\phi_{\hat{y}},\\
a_3 &= -[[(q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t]\phi_{\varphi^*}^2 - q^*\phi_s^2\\
&\quad -[\hat{y}^2((q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t) + q^* t^2]\phi_{\hat{y}}^2] \cos^2\varphi^*\\
&\quad +[(q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t][2\phi_{\varphi}^* \cos\varphi^* \sin\varphi^* - \hat{y}\phi_{\hat{y}}]\hat{y}\phi_{\hat{y}}.
\end{aligned}$$

The coefficients a_1, a_2 and a_3 do not depend on z. Splitting (4.26) with respect to z, one finds

$$a_i = 0, \ (i = 1, 2, 3).$$

If $\phi_{\hat{y}} \neq 0$, then the equation $a_1 = 0$ implies that $(q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t = 0$, and the equation $a_3 = 0$ becomes

$$q^*(t^2\phi_{\hat{y}}^2 + \phi_s^2) = 0.$$

Because of $\phi_{\hat{y}}^2 + \phi_s^2 \neq 0$, one has $q = 0$ which contradicts the assumption $q \neq 0$. Hence, $\phi_{\hat{y}} = 0$. The equation $a_3 = 0$ implies that

$$((q^{*''} - q^* t) - (C_2 - 3s)q^{*'} t)\phi_{\varphi^*}^2 - q^*\phi_s^2 = 0.$$

The general solution of the last equation is

$$\phi = \tilde{\phi}(t, \tilde{\varphi}),$$

where

$$\tilde{\varphi} = \varphi + H(s),\ H(s) = \int h(s)\mathrm{d}s,\ h = \pm\left(\frac{(q^{*''} - q^*t) - (C_2 - 3s)q^{*'}t}{q}\right)^{1/2},$$

and $\tilde{\phi}_{\tilde{\varphi}} \neq 0$.

Substituting $\phi = \tilde{\phi}(t, \tilde{\varphi})$ into equation (4.21), one obtains

$$b_1(s)\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + b_2(s)\tilde{\phi}_{\tilde{\varphi}} + \tilde{\phi}_t = 0, \tag{4.27}$$

where $b_1 = h^2$, $b_2 = -(q^*)^{-1}(2q^{*'} - (C_2 - 3s)q^*t)h - h'$. Let us analyze this equation. Differentiating it with respect to s, one obtains

$$b_1'\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + b_2'\tilde{\phi}_{\tilde{\varphi}} = 0.$$

Assume first that $b_1' \neq 0$. Then the last equation can be rewritten

$$\frac{\tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}}}{\tilde{\phi}_{\tilde{\varphi}}} = -\frac{b_2'}{b_1'}.$$

Differentiating it with respect to s, one has $(-b_2'/\ b_1')' = 0$. Thus $b_2 = -C_4 b_1 + C_5$. This relation, after substituting representations for the coefficients b_1 and b_2 becomes

$$\frac{(2q^{*'} + (C_2 - 3s)q^*t)}{q^*}h + h' + C_4 h^2 - C_5 = 0.$$

Differentiating the last equation with respect to t, one obtains $(C_2 - 3s)h = 0$ and, hence $h = 0$. This give $(q^{*''} - q^*t) - (C_2 - 3s)q^{*'}t = 0$. After splitting it with respect to t and solving them, one obtains $q^* = 0$. This is a contradiction to the assumption $q^* \neq 0$.

In the case $b_1' = 0$, then $b_2' = 0$, or $b_1 = C_4 > 0$ and $b_2 = C_5$. This means that

$$h = \pm\sqrt{C_4}\ ,\ \frac{(2q^{*'} - (C_2 - 3s)q^* t)\sqrt{C_4}}{q^*} = \pm C_5.$$

Differentiating the last equation with respect to t, one has $(C_2 - 3s)\sqrt{C_4} = 0$ which also is a contradiction. Therefore partially invariant solutions do not exist in this case.

Case 2. Assume that $a \neq 0$. The characteristic system of the last equation is

$$\frac{\mathrm{d}y}{t \sin \varphi^*} = \frac{\mathrm{d}z}{-t\cos\varphi^*} = \frac{\mathrm{d}\varphi^*}{a}.$$

Hence, the universal invariant of this system is y_1, y_2, t, s, where $y_1 = t\cos\varphi^* + ay$, $y_2 = t\sin\varphi^* + az$. Thus, the general solution of equation (4.23) is

$$F = \phi(y_1, y_2, t, s),$$

where $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$.

Substituting the derivatives of the function $\varphi^*(t, s, y, z)$ through the derivatives of the function $F(\varphi^*, t, s, y, z)$ into equation (4.22), it becomes

$$a_1 \cos^2\varphi^* + a_2 \cos\varphi^* \sin\varphi^* + a_3\cos\varphi^* + a_4 \sin\varphi^* + a_5 = 0 \tag{4.28}$$

where

$$a_1 = [(q^{*''} - q^* t + (\beta - ((aq^* - 3)s + C_2))tq^{*'})a^2 + q^*(a')^2](\phi_{y_1} + \phi_{y_2})(\phi_{y_1} - \phi_{y_2}),$$
$$a_2 = 2[(q^{*''} - q^* t + (\beta - ((aq^* - 3)s + C_2))tq^{*'})a^2 + q^*(a')^2]\phi_{y_1}\phi_{y_2},$$
$$a_3 = -2[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s]q^* a'\phi_{y_1},$$
$$a_4 = -2[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s]q^* a'\phi_{y_2},$$
$$a_5 = [(2(y_1\phi_{y_1} + y_2\phi_{y_2})a'\phi_s + (a^2t^2\phi_{y_2}^2 + \phi_s^2)a)q^* - (q^{*''} + (\beta - ((aq^* - 3)s$$
$$+C_2))tq^{*'} - q^*(a^2 + t))at^2\phi_{y_1}^2]a + [(y_1\phi_{y_1} + y_2\phi_{y_2})^2 + t^2\phi_{y_2}^2]q^*(a')^2.$$

Note that a_1, a_2, a_3, a_4 and a_5 do not depend on φ^*. Splitting (4.28) with respect to $\cos^2\varphi^*, \cos\varphi^*\sin\varphi^*, \cos\varphi^*, \sin\varphi^*$, one obtains

$$a_i = 0, \ (i = 1, 2, ..., 5).$$

Noting that $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$, the equations $a_1 = 0, \ a_2 = 0$ imply that $(q^{*''} - q^*t + (\beta - ((aq^* - 3)s + C_2))tq^{*'})a^2 + q^*(a')^2 = 0$, and the equations $a_3 = 0, \ a_4 = 0$ imply

$$[(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s]a' = 0. \tag{4.29}$$

Consider the last equation.

Case 2.1. Assume that $a' \neq 0$, then equation (4.29) gives

$$(y_1\phi_{y_1} + y_2\phi_{y_2})a' + a\phi_s = 0.$$

The general solution of this equation is $\phi = f(t, g_1, g_2)$, where $g_1 = y_2/y_1, \ g_2 = a/y_1$ and $f_{g_1}^2 + f_{g_2}^2 \neq 0$.

Substituting the function ϕ into the equation $a_5 = 0$, one obtains

$$(q^{*''} + (\beta - ((aq^* - 3)s + C_2))tq^{*'} - q^*(a^2 + t))[(g_1f_{g_1} + g_2f_{g_2})^2 + f_{g_1}^2] = 0.$$

Because $f_{g_1}^2 + f_{g_2}^2 \neq 0$, one finds that $(g_1f_{g_1} + g_2f_{g_2})^2 + f_{g_1}^2 \neq 0$. Hence, $q^{*''} + (\beta - ((aq^* - 1)s + C_2))tq^{*'} - q^*(a^2 + t) = 0$. Finding $q^{*''}$ from this equation, and substituting it into the equation $(q^{*''} + (\beta - ((aq^* - 1)s + C_2))tq^{*'})a^2 + q^*((a')^2 + t) = 0$, one has $(a')^2 + a^4 = 0$. This gives $a = 0$, which contradicts the assumption $a \neq 0$.

Case 2. Assume that $a' = 0$. Substituting $a' = 0$ into the equation $a_5 = 0$, one obtains

$$(q^{*''} + (\beta - ((aq^* - 1)s + C_2))tq^{*'} - q^*(a^2 + t))t^2\phi_{y_1}^2 - q^*(a^2t^2\phi_{y_2}^2 + \phi_s^2) = 0.$$

Using here the equation $(q^{*''} + (\beta - ((aq^* - 1)s + C_2))tq^{*'})a^2 + q^*(a')^2 = 0$, one finds

$$a^2t^2\phi_{y_1}^2 + a^2t^2\phi_{y_2}^2 + \phi_s^2 = 0.$$

Hence, $\phi_{y_1} = \phi_{y_2} = \phi_s = 0$, which is a contradiction to the condition $\phi_{y_1}^2 + \phi_{y_2}^2 \neq 0$. Therefore there is no partially invariant solutions in case $q^* \neq 0$.

4.2.3 Subalgebra generated by $L_1^4 = \{7, 8, 9, 11\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (S) are

$$U,\ H,\ p,\ r/t.$$

The representation of a regular partially invariant solution is

$$U = U(s),\ H = H(s),\ p = P(s),\ s = r/t \tag{4.30}$$

while the function $\omega(t, r, \theta, \varphi)$ still depends on all independent variables. In Hematulin (2001) it is proven that even in the more general case there is no partially invariant solution in the case $H \neq 0$. The case $H = 0$ corresponds to spherically symmetric flows. After substituting the representation (4.30) into the Navier-Stokes equations one has

$$ts^2P' - (s^2U'' + (ts^3 - ts^2U + 4s)U' + 2U) = 0, \tag{4.31}$$

$$sU' + 2U = 0. \tag{4.32}$$

Solving the last equation, one obtain $U = s^{-2}C_1$, where C_1 is a constant. After substituting the function U into equation (4.31), one has

$$P's^5 - 2C_1(C_1 - s^3) = 0.$$

Therefore, the partially invariant solution of the Navier-Stokes equations in this case is

$$P = 2^{-1}s^{-4}C_1(4s^3 - C_1) + C_2, \ U = s^{-2}C_1, \ V = 0, \ W = 0, \tag{4.33}$$

where C_1 and C_2 are constants. Using the property that the pressure is defined up to an arbitrary function of time, one can set $C_2 = 0$.

Note that this partially invariant solution is obtained for a Lie group which is not admitted by the Navier-Stokes equations.

4.2.4 Subalgebra generated by $L_6^4 = \{1, 4, 7, 11\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (C) are

$$q, \ \varphi - \theta, \ p, \ R/t.$$

The representation of a regular partially invariant solution is

$$q = q(s), \ \varphi = \Psi(s) + \theta, \ p = P(s), \ s = R/t$$

and the function $U(t, x, y, z)$ depends on the all independent variables.

The Navier-Stokes equations can be written in the cylindrical coordinate system as

$$U_t + UU_x + VU_R + WU_\theta R^{-1} - U_R R^{-1} + p_x - U_{xx} - U_{RR} - U_{\theta\theta}R^{-2} = 0, \tag{4.34}$$

$$\begin{aligned} V_t + UV_x + VV_R + WV_\theta R^{-1} - V_R R^{-1} + p_R - V_{xx} - V_{RR} - V_{\theta\theta}R^{-2} \\ +(V + 2W_\theta - RW^2)R^{-2} = 0, \end{aligned} \tag{4.35}$$

$$\begin{aligned} (W_t + UW_x + VW_R + WW_\theta R^{-1} - W_R R^{-1} + p_\theta R^{-1} - W_{xx} - W_{RR})R^{-1} \\ -W_{\theta\theta}R^{-3} + (W - 2V_\theta + RVW)R^{-3} = 0, \end{aligned} \tag{4.36}$$

$$U_x + V_R + W_\theta R^{-1} + VR^{-1} = 0. \tag{4.37}$$

Substituting the representation of a solution into the Navier-Stokes equations, one obtains

$$[((U_{xx} - U_x U)R - qU_\theta \sin(\Psi+\theta)q)R + U_{\theta\theta} - U_t R^2]t^2$$
$$+U_{ss}R^2 - \cos(\Psi+\theta)U_s q R^2 t + (R^2+t)U_s R = 0, \quad (4.38)$$
$$(Rp_s - \sin(\Psi+\theta)^2 q^2 t + \cos(\Psi+\theta)^2 qRq_s)Rt$$
$$-(q_{ss}R^2 - qt^2 + (R^2+t)q_s R)\cos(\Psi+\theta) = 0, \quad (4.39)$$
$$(q_{ss}R^2 - qt^2 + (R^2+t)q_s R - (q_s R + qt)\cos(\Psi+\theta)qRt)\sin(\Psi+\theta) = 0, \quad (4.40)$$
$$(q_s R + qt)\cos(\Psi+\theta) + RtU_x = 0. \quad (4.41)$$

In the case $q = 0$ the general solution of equations (4.38)-(4.41) is

$$U = U(t,s,\theta), \ V = 0, \ W = 0, \ P = C_1, \ s = R/t,$$

where C_1 is constant and U satisfies the equation

$$U_{ss}s^2 + sU_s(s^2 t + 1) - U_t s^2 t^2 + U_{\theta\theta} = 0.$$

Since the pressure is defined up to an arbitrary function of time, one can assume $C_1 = 0$.

Further consideration is given for $q \neq 0$.

Expanding the expressions $\cos(\Psi+\theta)$ and $\sin(\Psi+\theta)$, equation (4.39) becomes

$$a_1 \sin^2\theta + a_2 \cos\theta \sin\theta + a_3 \cos\theta + a_4 \sin\theta + a_5 = 0 \qquad (4.42)$$

where

$$a_1 = qst^3[(q's + 2q)(2\sin^2\Psi - 1) + 2qs\Psi_s \cos\Psi \sin\Psi],$$
$$a_2 = -qst^3[2(q's+2q)\cos\Psi \sin\Psi - qs\Psi_s(2\sin^2\Psi - 1)],$$
$$a_3 = -t^2[((s^t+1)q's - 4q + s^2 q'' - qs^2\Psi_s^2)\cos\Psi -$$
$$(((s^t+1)q + 2q's)\Psi_s + qs\Psi_{ss})s\sin\Psi],$$

$$a_4 = t^2[((s^t+1)q's - 4q + s^2q'' - qs^2\Psi_s^2)\sin\Psi +$$
$$(((s^t+1)q + 2q's)\Psi_s + qs\Psi_{ss})s\sin\Psi],$$
$$a_5 = -st^3[(q's+2q)\sin^2\Psi - q's)q - sP' + sq^2\Psi_s\cos\Psi\sin\Psi].$$

Note that $a_i,\ (i = 1,...,5)$ do not depend on θ. Splitting (4.42) with respect to $\sin^2\theta,\ \cos\theta\sin\theta, \cos\theta, \sin\theta$, one obtains

$$a_i = 0,\ (i = 1,...,5).$$

Solving the equations $a_1 = 0$ and $a_2 = 0$, one has

$$q = \frac{C_2}{s^2},\ \Psi = C_3.$$

Substituting q, Ψ into equations $a_3 = 0$ and $a_4 = 0$, and solving them, one gets $2C_2t^3 = 0$. This gives $C_2 = 0$. It means that $q = 0$ which contradicts the assumption $q \neq 0$. Therefore, there is no partially invariant solution in case $q \neq 0$.

4.2.5 Subalgebra generated by $L_9^4 = \{4, 3+5, 2-6, \alpha 1 + 7\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (K_1) are

$$u + (\alpha\theta^* - x)/t,\ V^*,\ p,\ t.$$

The representation of a regular partially invariant solution is

$$u = U(t) - (\alpha\theta^* - x)/t,\ p = P(t),\ V^* = V^*(t),$$

and the function $\theta^*(t, x, y, z)$ still depends on all independent variables. Substituting this representation of a solution into the Navier-Stokes equations, one obtains

$$[(\alpha(\theta^*_{xx}+\theta^*_{yy}+\theta^*_{zz}-\theta^*_t-\theta^*_z V^*\cos\theta^*)+tU')(t^2+1)$$
$$+(t^2+1)U-\alpha(ty+z)\theta^*_y]t-\alpha(tU+x-\alpha\theta^*)(t^2+1)\theta^*_x$$
$$-\alpha((t^2+1)V^*\sin\theta^*+ty-z)t\theta^*_z=0, \quad (4.43)$$
$$[(\theta^*_{xx}+\theta^*_{yy}+\theta^*_{zz}-\theta^*_t)(t^2+1)^2V^*\sin\theta^*-(((ty+z)\theta^*_y-1)$$
$$(t^2+1)V^*\sin\theta^*-(t+1)(y-z))]t-(tU+x-\alpha\theta^*)$$
$$(t^2+1)\theta^*_x V^*\sin\theta^*-((t^2+1)V^*\sin\theta^*+ty-z)(t^2+1)$$
$$t\theta^*_z V^*\sin\theta^*+[(V^*(\theta^{*2}_x+\theta^{*2}_y+\theta^{*2}_z)+V^{*'}-V^{*2}\sin\theta^*)$$
$$(t^2+1)+V^*t](t^2+1)t\cos\theta^*=0, \quad (4.44)$$
$$[((\theta^*_{xx}+\theta^*_{yy}+\theta^*_{zz}-\theta^*_t)(t^2+1)-((ty+z)\theta^*_y+t))t-(tU+x-\alpha\theta^*)$$
$$(t^2+1)\theta^*_x-((t^2+1)V^*\sin\theta^*+ty-z)t\theta^*_z](t^2+1)V^*\cos\theta^*$$
$$-[((V^*(\theta^{*2}_x+\theta^{*2}_y+\theta^{*2}_z)+V^{*'})\sin\theta^*+V^{*2}\theta^*_y\cos^2\theta^*)(t^2+1)^2$$
$$+y+z-(y-3z)t-(t^2+1)V^*\sin\theta^*]t=0, \quad (4.45)$$
$$(t^2+1)(V^*t(\theta^*_y\sin\theta^*-\theta^*_z\cos\theta^*)+\alpha\theta^*_x)-(2t^2-t+1)=0. \quad (4.46)$$

Notice that for $V^*=0$ the equations (4.43)-(4.46) become

$$[(\alpha(\theta^*_{xx}+\theta^*_{yy}+\theta^*_{zz}-\theta^*_t)+tU')(t^2+1)+(t^2+1)U-\alpha(ty+z)\theta^*_y$$
$$-(ty-z)t\theta^*_z]t-\alpha(tU+x-\alpha\theta^*)(t^2+1)\theta^*_x=0, \quad (4.47)$$
$$(t+1)(y-z)=0, \quad (4.48)$$
$$y+z-(y-3z)t=0, \quad (4.49)$$
$$(2t^2-t+1)-\alpha(t^2+1)\theta^*_x=0. \quad (4.50)$$

Combining equations (4.48) and (4.51), one obtains $2(tz+y)=0$ and splitting it with respect to y, one gets $2=0$ which is a contradiction.

Further consideration is given for $V^*\neq 0$.

Taking the combination of equations (4.44) and (4.45) by excluding the derivative θ_t^*, one has

$$[(V^*(\theta_x^{*2}+\theta_y^{*2}+\theta_z^{*2})+V^{*'})(t^2+1)-((t^2+1)\sin^2\theta^*-t)V^*](t^2+1)$$
$$+(y+z-(y-3z)t)\sin\theta^*+((t^2+1)V^*\sin\theta^*+y-z)(t+1)\cos\theta^*=0. \quad (4.51)$$

For the compatibility analysis of systems (4.43), (4.44), (4.46) and (4.51), it is convenient to use an implicit representation of the function $\theta^*(t,x,y,z)$. Assume that there is a function $F(\theta^*,t,x,y,z)$ such that $F_\theta^*\neq 0$ and

$$F(\theta^*(t,x,y,z),t,x,y,z)=0. \quad (4.52)$$

In the same way as for the previous model, finding the derivatives of the function $\theta^*(t,x,y,z)$ from equation (4.52) and substituting them into equation (4.46), one obtains

$$V^*t(F_y\sin\theta^*-F_z\cos\theta^*)+\frac{2t^2-t+1}{t^2+1}F_\theta^*+\alpha F_x=0.$$

The characteristic system of the last equation is

$$\frac{\mathrm{d}y}{V^*t\sin\theta^*}=\frac{\mathrm{d}z}{-V^*t\cos\theta^*}=\frac{(t^2+1)\mathrm{d}\theta^*}{2t^2-t+1}=\frac{\mathrm{d}x}{\alpha}.$$

The universal invariant of this system is y_1,y_2,y_3,t, where $y_1=V^*t(t^2+1)\cos\theta^*+(2t^2-t+1)y$, $y_2=V^*t(t^2+1)\sin\theta^*+(2t^2-t+1)z$ and $y_3=\alpha(t^2+1)\theta^*-(2t^2-t+1)x$. The general solution of equation (4.46) is

$$F=\phi(y_1,y_2,y_3,t),$$

where $\phi_{y_1}^2+\phi_{y_2}^2+\phi_{y_3}^2\neq 0$.

Substituting the function $\phi(y_1,y_2,y_3,t)$ into equation (4.51), one obtains

$$a_1\cos^4\theta^*+a_2\cos^3\theta^*\sin\theta^*+a_3\cos^3\theta^*+a_4\cos^2\theta^*\sin\theta^*$$
$$+a_5\cos^2\theta^*+a_6\cos\theta^*\sin\theta^*+a_7\cos\theta^*+a_8\sin\theta^*+a_9=0, \quad (4.53)$$

where

$$
\begin{aligned}
a_1 &= -t^2(t^2+1)^3(2t^3+3t^2+1)V^{*3}(\phi_{y_1}^2 - 2\phi_{y_1}\phi_{y_2} - \phi_{y_2}^2),\\
a_2 &= -t^2(t^2+1)^3(2t^3+3t^2+1)V^{*3}(\phi_{y_1}^2 + 2\phi_{y_1}\phi_{y_2} - \phi_{y_2}^2),\\
a_3 &= t(t^2+1)^2V^{*2}[2\phi_{y_1}(t((3t+1)y_2-(t-1)y_1)\phi_{y_2} + \alpha(2t^3+3t^2+1)(t^2+1)\phi_{y_3})\\
&\quad -(t(t+1)(y_1-y_2)(\phi_{y_1}+\phi_{y_2})(\phi_{y_1}-\phi_{y_2}) - 2\alpha(t^2+1)(2t^3+3t^2+1)\phi_{y_2}\phi_{y_3})],\\
a_4 &= -t(t^2+1)^2V^{*2}[t((3t+1)y_2-(t-1)y_1)(\phi_{y_1}+\phi_{y_2})(\phi_{y_1}-\phi_{y_2})\\
&\quad -2\alpha(2t^3+3t^2+1)(t^2+1)\phi_{y_2}\phi_{y_3} + 2(t(t+1)(y_1-y_2)\phi_{y_2}\\
&\quad +\alpha(t^2+1)(2t^3+3t^2+1)\phi_{y_3})\phi_{y_1}],\\
a_5 &= (t^2+1)^2V^*[\alpha(2t(t+1)(y_1-y_2)\phi_{y_2} + \alpha(2t^3+3t^2+1)(t^2+1)\phi_{y_3})\phi_{y_3}\\
&\quad -t^2(t^2+1)V^*((5t^2+1)V^* - (2t^2-t+1)(t^2+1)(V^*)')\phi_{y_2}^2\\
&\quad +t^2(t^2+1)V^*(2(t^3+4t^2+1)V^* - (2t^2-t+1)(t^2+1)(V^*)')\phi_{y_1}^2\\
&\quad +2t(\alpha((3t+1)y_2-(t-1)y_1)\phi_{y_3} - t(t^2+1)(2t^3+3t^2+1)\phi_{y_2}))\phi_{y_1}],\\
a_6 &= (t^2+1)^2V^*[2t(t(t^2+1)V^*((5t^2+1)V^* - (2t^2-t+1)(t^2+1)(V^*)')\phi_{y_2}\\
&\quad -\alpha(t+1)(y_1-y_2)\phi_{y_3})\phi_{y_1} + 2\alpha t((3t+1)y_2-(t-1)y_1)\phi_{y_2}\phi_{y_3}\\
&\quad +(t^2+1)(2t^3+3t^2+1)(t^2V^{*2}\phi_{y_1}^2 + \alpha^2\phi_{y_3}^2)],\\
a_7 &= -(t^2+1)^2[2\alpha t(t^2+1)V^*((5t^2+1)V^* - (2t^2-t+1)(t^2+1)(V^*)')\phi_{y_2}\phi_{y_3}\\
&\quad -(t+1)(y_1-y_2)(t^2V^{*2}\phi_{y_1}^2 + \alpha^2\phi_{y_3}^2) + 2t(V^*)^2((3t+1)y_2-(t-1)y_1)t\phi_{y_2}\\
&\quad +\alpha(t^2+1)(2t^3+3t^2+1)\phi_{y_3})\phi_{y_1}],\\
a_8 &= (t^2+1)^2[2\alpha t(t^2+1)V^*((5t^2+1)V^* - (2t^2-t+1)(t^2+1)(V^*)')\phi_{y_1}\phi_{y_3}\\
&\quad +((3t+1)y_2-(t-1)y_1)(t^2V^{*2}\phi_{y_1}^2 + \alpha^2\phi_{y_3}^2)],\\
a_9 &= (t^2+1)^2[V^*(\phi_{y_1}^2[(2t^2-t+1)((2t^2-t+1)^2 + t^2(t^2+1)V^*(V^*)')\\
&\quad -t^2(t^2+1)(5t^2+1)V^{*2}] - [2\alpha t((3t+1)y_2-(t-1)y_1)\phi_{y_1}\phi_{y_3}\\
&\quad -(2t^2-t+1)^2\phi_{y_2}^2]) + [((2t^2-t+1)V^* + \alpha^2(t^2+1)^2(V^*)')(2t^2-t+1)\\
&\quad -\alpha^2(t^2+1)(5t^2+1)V^*]\phi_{y_3}^2].
\end{aligned}
$$

Note that $a_i, (i = 1, 2, ..., 9)$ do not depend on θ^*. Splitting (4.53) with respect to $\cos^4\theta^*$, $\cos^3\theta^*\sin\theta^*$, $\cos^3\theta^*$, $\cos^2\theta^*\sin\theta^*$, $\cos^2\theta^*$, $\cos\theta^*\sin\theta^*$, $\cos\theta^*$, $\sin\theta^*$, one obtains

$$a_i = 0, \ (i = 1, 2, ..., 9).$$

Equations $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$, give

$$-2(V^*)^3 t^2 (t^2+1)^3 (2t^3 + 3t^2 + 1)(\phi_{y_1} + \phi_{y_2})(\phi_{y_1} - \phi_{y_2}) = 0, \quad (4.54)$$

$$4(V^*)^3 t^2 (t^2+1)^3 (2t^3 + 3t^2 + 1)\phi_{y_1}\phi_{y_2} = 0. \quad (4.55)$$

Since $V^* \neq 0$ and the expression $t^2(t^2+1)^3(2t^3+3t^2+1)$ can not be zero, then $\phi_{y_1} = \phi_{y_2} = 0$. Equation $a_5 = 0$ becomes

$$\alpha^2 V^* (t^2+1)^3 (2t^3 + 3t^2 + 1)\phi_{y_3}^2 = 0.$$

The last equation implies that α has to be zero. From $a_9 = 0$, one has

$$V^*(t^2+1)^2(2t^2 - t + 1)^3 \phi_{y3}^2 = 0,$$

which also give $\phi_{y_3} = 0$. This is a contradiction to the condition $\phi_{y1}^2 + \phi_{y2}^2 + \phi_{y3}^2 \neq 0$. Therefore partially invariant solutions of the Navier-Stokes equations not exist in this case.

4.2.6 Subalgebra generated by $L_{23}^4 = \{1, 4, 10, 11\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (D) are

$$v, \ w, \ p, \ z/y.$$

The representation of a regular partially invariant solution is

$$v = V(s), \ w = W(s), \ p = P(s), \ s = z/y$$

while the function $u(t, x, y, z)$ depends on all independent variables. Substituting this representation of a solution into the Navier-Stokes equations, one obtains

$$u_t + uu_x + Vu_y + Wu_z - (u_{xx} + u_{yy} + u_{zz}) = 0, \tag{4.56}$$

$$((W - sV)V' - sP')y - ((s^2 + 1)V'' + 2sV') = 0, \tag{4.57}$$

$$((W - sV)W' + P')y - ((s^2 + 1)W'' + 2sW') = 0, \tag{4.58}$$

$$yu_x - (sV' + W') = 0. \tag{4.59}$$

Since V and W only depend on s, equations (4.57) and (4.58) can be split with respect to y:

$$(W - sV)V' - sP' = 0, \ (W - sV)W' + P' = 0, \tag{4.60}$$

$$(s^2 + 1)V'' + 2sV' = 0, \ (s^2 + 1)W'' + 2sW' = 0. \tag{4.61}$$

Solving equations (4.61), one has

$$V = C_1 \arctan(s) + C_2, \ W = C_3 \arctan(s) + C_4.$$

Multiplying the first equation by s and combining it with the second equation of (4.60), one obtains

$$(W - sV)(V' + sW') = 0.$$

Let $W - sV = 0$, then substituting V and W one gets

$$(C_4 - sC_2) + (C_3 - sC_1) \arctan(s) = 0.$$

Splitting the last equation with respect to s and $\arctan(s)$, one obtains $C_1 = C_2 = C_3 = C_4 = 0$. This means that $V = 0, W = 0$ and hence $P = C_5$. Substituting V and W into equation (4.59), one has $u_x = 0$ or $u = U(t, y, z)$. Equation (4.56) becomes

$$U_t - U_{yy} - U_{zz} = 0. \tag{4.62}$$

Thus, there is a solution of the Navier-Stokes equations of the type

$$u = U(t, y, z), \ V = 0, \ W = 0, \ P = C_5,$$

where the function $U(t, y, z)$ satisfies equation (4.62).

Similarly, in the case $V' + sW' = 0$, one obtains $c_1 = 0, \ C_3 = 0$, i.e., $V = C_2, W = C_4$. In this case $P = C_5$. Substituting V and W into equation (4.59), one has $u_x = 0$. This means that u depends on t, y, z or $u = U(t, s, y)$. Equation (4.56) becomes

$$U_t + C_2 U_y + C_4 U_z - U_{yy} + U_{zz} = 0. \tag{4.63}$$

Thus, there is a solution of the Navier-Stokes equations of the type

$$u = U(t, s, y), \ V = C_2, \ W = C_4, \ P = C_5,$$

where the function $U(t, s, y)$ satisfies equation (4.63). Because the pressure P is defined by the Navier-Stokes equations up to an arbitrary function of time, C_5 can be set to zero.

4.2.7 Subalgebra generated by $L^4_{35} = \{2, 3, 5, 4 + \beta 6 + 10\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (D) are

$$u - t, \ w - \beta t, \ p, \ x - 2^{-1}t^2.$$

The representation of a regular partially invariant solution is

$$u = U(s) + t, \ w = W(s) + \beta t, \ p = P(s), \ s = x - 2^{-1}t^2,$$

while the function $v = v(t, x, y, z)$ still depends on all independent variables. Substituting this representation of a solution into the Navier-Stokes equations, one obtains

$$U'' - UU' - P' - 1 = 0, \tag{4.64}$$

$$v_t + (U + t)v_x + vv_y + (W + \beta t)v_z - (v_{xx} + v_{yy} + v_{zz}) = 0, \tag{4.65}$$

$$W'' - UW' - \beta = 0, \tag{4.66}$$

$$U' + v_y = 0. \tag{4.67}$$

Integrating equations (4.64) and (4.67), one has

$$P = U' - 2^{-1}U^2 - s + C_1, \;\; v = -U'y + V(t, s, z).$$

The constant C_1 can be zero by using the property that the pressure p is defined up to an arbitrary function of time only.

Substituting v into equation (4.65), one arrives at the equation

$$V_t + UV_s - VU' + (W + \beta t)V_z - V_{ss} + V_{zz} + y(U''' - UU'' + U'^2) = 0.$$

Since U, V and W do not depend on y, the last equation can be split with respect to y:

$$U''' - UU'' + U'^2 = 0,$$

$$V_t + UV_s - VU' + (W + \beta t)V_z - V_{ss} + V_{zz} = 0.$$

Thus the studied partially invariant solution of the Navier-Stokes equations is

$$\begin{aligned} u \;\; &= U(s) + t, \;\; v = -U' + V(t, s, z), \;\; w = W(s) + \beta t, \\ P \;\; &= U' - 2^{-1}U^2 - s, \end{aligned}$$

where $s = x - 2^{-1}t^2$ and the function $U(s), W(s)$ and $V(t, s, z)$ satisfy the reduced system

$$\begin{aligned} U''' - UU'' + U'^2 &= 0, \\ W'' - UW' - \beta &= 0, \\ V_t + UV_s - VU' + (W + \beta t)V_z - V_{ss} + V_{zz} &= 0. \end{aligned} \tag{4.68}$$

4.2.8 Subalgebra generated by $L^4_{41} = \{1, \sigma 2 + \tau 3 + 4, \alpha 3 + 5, \beta 2 + 6\}$

Invariants of the Lie group corresponding to this algebra in the coordinate system (K_2) are

$$j_1,\ j_2,\ p,\ t.$$

The representation of a regular partially invariant solution is

$$v = \frac{j_1(t) - (\sigma t - \beta\tau)u + ty - \beta z}{t^2 - \alpha\beta},\ w = \frac{j_2(t) - (\tau t - \alpha\sigma)u - \alpha y + tz}{t^2 - \alpha\beta},\ p = P(t),$$

and the function $u(t, x, y, z)$ depends on all independent variables. Substituting this representation of a solution into the Navier-Stokes equations, one obtains

$$(\alpha\beta - t^2)(u_{xx} + u_{yy} + u_{zz} - uu_x - u_t) - ((\tau u - z)t - j_2$$
$$-\alpha(\sigma u - y))u_z - ((\sigma u - y)t - j_1 - \beta(\tau u - z))u_y = 0, \quad (4.69)$$
$$(\alpha\beta - t^2)((u_{xx} + u_{yy} + u_{zz} - uu_x - u_t)(\beta\tau - \sigma t) - j_1')$$
$$-(j_1 t + \beta j_2 + ((\sigma u - y)t - j_1 - \beta(\tau u - z))(\beta\tau - \sigma t)u_y$$
$$-((\tau u - z)t - j_2 - \alpha(\sigma u - y))(\beta\tau - \sigma t)u_z) = 0, \quad (4.70)$$
$$(\alpha\beta - t^2)((u_{xx} + u_{yy} + u_{zz} - uu_x - u_t)(\alpha\sigma - \tau t) - j_2')$$
$$-(j_2 t + \alpha j_1 + ((\tau u - z)t - j_2 - \alpha(\sigma u - y))(\alpha\sigma - \tau t)u_z)$$
$$-((\sigma u - y)t - j_1 - \beta(\tau u - z))(\alpha\sigma - \tau t)u_y = 0, \quad (4.71)$$
$$2t - (\alpha\beta - t^2)u_x + (\beta\tau - \sigma t)u_y + (\alpha\sigma - \tau t)u_z = 0. \quad (4.72)$$

Multiplying the equation (4.69) by $(\beta\tau - \sigma t)$ and adding to the equation (4.70), one has

$$\alpha\beta j_1' - t^2 j_1' + \beta j_2 + t j_1 = 0. \quad (4.73)$$

Multiplying the equation (4.69) by $(\alpha\sigma - \tau t)$ and adding to the equation (4.71), one has

$$\alpha\beta j_2' - t^2 j_2' + \alpha j_1 - t j_2 = 0. \quad (4.74)$$

After solving the equations (4.73) and (4.74), one obtains $j_1 = C_1 t + \beta C_2$, and $j_2 = -C_2 t - \alpha C_1$.

A characteristic system for equation (4.72) is

$$\frac{\mathrm{d}y}{\beta\tau - \sigma t} = \frac{\mathrm{d}z}{\alpha\sigma - \tau t} = \frac{\mathrm{d}x}{t^2 - \alpha\beta} = \frac{\mathrm{d}u}{-2t}.$$

Hence, the universal invariant of this system is $t, s_1, s_2, u - 2tx/(\alpha\beta - t^2)$, where $s_1 = (\beta\tau - \sigma t)x + (\alpha\beta - t^2)$, $s_2 = (\alpha\sigma - \tau t)x + (\alpha\beta - t^2)$. Thus, the general solution of equation (4.72) is

$$u = \frac{2tx}{\alpha\beta - t^2} + U(t, s_1, s_2).$$

Substituting j_1, j_2 and u into equation (4.69), it becomes a linear equation with respect to x. Splitting it with respect to x, one obtains two equations which are of the form

$$a_1(t)U_{s_1} + b_1(t)U_{s_2} + c_1(t) = 0, \tag{4.75}$$

$$F(U_{s_1 s_1}, U_{s_2 s_1}, U_{s_2 s_2}, U_{s_1}, U_{s_2}, U_t, U, s_1, s_2, t) = 0, \tag{4.76}$$

where

$$a_1 = (\alpha\beta - t^2)(\alpha^2\beta^2\sigma - 2\alpha\beta^2\tau t + 2\beta\tau t^3 - \sigma t^4),$$
$$b_1 = (\alpha\beta - t^2)(\alpha^2\beta^2\tau - 2\alpha^2\beta\sigma t + 2\alpha\sigma t^3 - \tau t^4),$$
$$c_1 = (\alpha\beta - t^2)(\alpha\beta + 3t^2).$$

For equation (4.75), a characteristic equation is

$$\frac{\mathrm{d}s_1}{a_1} = \frac{\mathrm{d}s_2}{a_2} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}U}{-c_1}.$$

The universal invariant of this system is $t, s_3, U + (c_1 s_1)/a_1$, where $s_3 = (b_1 s_1 - a_1 s_2)/a_1$. Thus, the general solution of equation (4.75) is

$$U = -\frac{c_1 s_1}{a_1} + \widetilde{U}(t, s_3).$$

Substituting U into equation (4.76), one obtains an equation which is linear with respect to s_1, and after splitting it with respect to s_1, one obtains two equations

$$a_2(t)\widetilde{U}_{s_3} + b_2(t) = 0, \tag{4.77}$$

$$F(\widetilde{U}_{s_3 s_3}, \widetilde{U}_{s_3}, \widetilde{U}_t, \widetilde{U}, s_3, t) = 0, \tag{4.78}$$

where

$$a_2 = (\alpha\beta - t^2)^8(\alpha\beta\sigma - 2\beta\tau t + \sigma t^2)(\alpha\sigma^2 - \beta\tau^2),$$
$$b_2 = (\alpha\beta - t^2)^5(\beta\tau + \sigma t).$$

After solving the equation (4.77), one has

$$\widetilde{U} = -\frac{b_2 s_3}{a_2} + H(t).$$

Substituting $\widetilde{U}$ into equation (4.78), one gets the first order nonhomogeneous linear equation:

$$(\alpha\beta - t^2)^4((2\beta\tau - \sigma t)t - \alpha\beta\sigma)^2[\alpha((C_1\tau + C_2\sigma)\beta + 2(C_1 - \sigma H)\sigma t) +$$
$$((C_1\tau + C_2\sigma)t + 2(C_2 + \tau H)\beta\tau)t - (\alpha\beta - t^2)(\alpha\sigma^2 - \beta\tau^2)H'] = 0.$$

Solving the last equation, one finds

$$H = \frac{\alpha(t(C_1\tau + C_2\sigma)\beta + C_1\sigma t) + \beta\tau C_2 t^2}{\alpha\beta(\alpha\sigma^2 - \beta\tau^2)} + (\alpha\beta - t^2)C_3.$$

Returning back to the partially invariant solution, one obtains the partially invariant solution of the Navier-Stokes equations

$$u = [\alpha((\sigma(z - C_2) - \tau(y + C_1))\beta t - ((C_3 t^2\tau - z)\beta^2\tau + C_1\sigma t^2))$$
$$+(\alpha^2(\sigma(C_3\sigma t^2 - y) + \beta^2\tau^2 C_3) - (\beta\alpha^3\sigma^2 C_3 + C_2\tau t^2))\beta]/\alpha\beta(\alpha\sigma^2 - \beta\tau^2),$$
$$v = [\alpha((\sigma(z - C_2) - \tau y)\beta\sigma + (\beta\tau - \sigma t)\beta^2\tau^2 C_3 - C_1\sigma^2 t)$$
$$-(\alpha^2\sigma^2 C_3 - C_2\tau)(\beta\tau - \sigma t)]/\alpha(\alpha\sigma^2 - \beta\tau^2),$$
$$w = [\alpha\tau((\sigma z - \tau(y + C_1))\beta - t(\beta^2\tau^2 C_3 + C_1\sigma))$$
$$-\beta(\alpha^3\sigma^3 C_3 + C_2\tau^2 t) - \sigma\alpha^2(\beta^2\tau^2 C_3 + \sigma C_1)]/\alpha\beta(\alpha\sigma^2 - \beta\tau^2),$$
$$p = P(t).$$

4.2.9 Result of compatibility analysis

The results of the study of regular partially invariant solutions corresponding to the remaining subalgebras of Table 4.1 are presented in this section. These results are considered according to the coordinate system in which a regular partially invariant solution is studied. For brevity, we omit details of their computations, and simply present the results which we have obtained. All results have been verified by substituting the solutions into the original systems using the REDUCE Program.

Table 4.2 is devoted to partially invariant solutions considered in the polar coordinate system. In each of these models the function φ in the representation of a partially invariant solution depends on all independent variables $\varphi = \varphi(t, x, y, z)$. The results of calculations show that such partially invariant solutions of the Navier-Stokes equations exist. These solutions depend on the form of the function $q(s)$.

Table 4.3 presents the results in the polar conical coordinate system. The superfluous function is $\varphi^* = \varphi^*(t, x, y, z)$. Only subalgebra number 5 is presented in this table. For this subalgebra, partially invariant solutions exist only in case $q^* = 0$. For other subalgebras considered in the polar conical coordinate system, there are no partially invariant solutions.

Table 4.4 is concerned with subalgebras considered in the spherical coordinate system. The superfluous function is $\omega = \omega(t, r, \theta, \varphi)$. Partially invariant solutions can only be found in the case $H = 0$.

Table 4.5 is devoted to partially invariant solutions considered in the cylindrical coordinate system. The function $u = u(t, x, R, \theta)$ is superfluous.

Table 4.2: Subalgebras considered in the polar coordinate system (P)

i	Representation of PIS	Partially invariant solution (PIS)
7	$u = U(s) + x/t$, $p = P(s),\ q = q(s)$, $s = x/t - \beta \ln t$, $\varphi = \varphi(t, x, y, z)$	$U = -s + C_1,\ P = -\beta s$, 1. $q = 0,\ \varphi = \varphi(t, s, y, z)$ 2. $q = C_2,\ \varphi = C_3$
12	$u = U(t) + \beta\varphi$, $p = P(t)$, $q = q(t)$, $\varphi = \varphi(t, x, y, z)$	$u = U(t) + \beta\varphi,\ p = 0$, 1. $q = 0,\ \varphi = \phi(t, y, z) - U/\beta$: $\phi_{yy} + \phi_{zz} - \phi_t = 0$ 2. $q = C_1,\ \varphi = C_2,\ U = C_3$ 3. $q = C_1 \mathrm{e}^{-C_2^2 t},\ \varphi = -C_2(x - C_3 t) + C_4$, $\beta = 0,\ U = C_3$
14	$u = U(x)$, $p = P(x)$, $q = q(x)$, $\varphi = \varphi(t, x, y, z)$	$U = C_1,\ P = 0$, 1. $q = 0,\ \varphi = \varphi(t, x, y, z)$ 2. $q = C_2 + C_3 \mathrm{e}^{C_1 x},\ \varphi = C_4$ 3. $q = q(x), \varphi = C_2 t - H(x) + C_3$: $h' + q^{-1}(2q' - C_1 q)h + C_2 h^2 - C_3 = 0$, $H = \int h(x)\mathrm{d}x,\ h = \pm((q'' - C_1 q')/q)^{1/2}$ 4. $q = C_4 + C_5 \mathrm{e}^{C_1 s}, \varphi = -H(x) + C_6$ 5. $q = C_4 \mathrm{e}^{\tilde{C}x/2},\ \varphi = -\tilde{\varphi} + H(x),\ \phi = \tilde{\phi}(t, \tilde{\varphi})$: $C_2 \tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + C_3 \tilde{\phi}_{\tilde{\varphi}} + \tilde{\phi}_t = 0,\ C_2 > 0$, $C_3 = \pm(C_2(C_1^2 + 4C_2))^{1/2},\ \tilde{C} = C_1 \pm C_3 C_2^{-1/2}$
16	$u = U(s) + t$, $p = P(s)$, $q = q(s)$, $s = x - 2^{-1}t^2$, $\varphi = \varphi(t, x, y, z)$	$U = C_1,\ P = -s$, 1. $q = 0,\ \varphi = \varphi(t, s, y, z)$ 2. $q = C_2 + C_3 \mathrm{e}^{C_1 s},\ \varphi = C_4$ 3. $q = q(s), \varphi = C_3 t - H(s) + C_4$: $h' + q^{-1}(2q' - C_1 q)h + C_2 h^2 - C_3 = 0$, $H = \int h(s)\mathrm{d}s,\ h = \pm((q'' - C_1 q')/q)^{1/2}$ 4. $q = C_4 + C_5 \mathrm{e}^{C_1 s}, \varphi = -H(s) + C_6$ 5. $q = C_4 \mathrm{e}^{\tilde{C}s/2},\ \varphi = -\tilde{\varphi} + H(s),\ \phi = \tilde{\phi}(t, \tilde{\varphi})$: $C_2 \tilde{\phi}_{\tilde{\varphi}\tilde{\varphi}} + C_3 \tilde{\phi}_{\tilde{\varphi}} + \tilde{\phi}_t = 0,\ C_2 > 0$, $C_3 = \pm(C_2(C_1^2 + 4C_2))^{1/2},\ \tilde{C} = C_1 \pm C_3 C_2^{-1/2}$
21	$u = U(t) - (\varphi - x)/t$, $p = P(t),\ q = q(t)$, $\varphi = \varphi(t, x, y, z)$	$u = U(t) - \varphi/t,\ p = 0$, $q = 0,\ \varphi = -\phi(t, y, z)/t$: $\phi_{yy} + \phi_{zz} - \phi_t + \phi = 0$

Table 4.3: Subalgebras considered in the polar conical coordinate system (PC)

i	Representation of PIS	Partially invariant solution (PIS)
5	$u = U(s) + x/t,\ p = P(s),$ $q^* = q^*(s),\ s = x/t - \beta \ln t,$ $\varphi^* = \varphi^*(t, x, y, z)$	$U = -3s + C_1,$ $P = (2C_1 - 3(\beta + s))s,$ $q^* = 0,\ \varphi^* = \varphi^*(t, s, y, z)$

Table 4.4: Subalgebras considered in the spherical coordinate system (S)

i	Representation of PIS	Partially invariant solution (PIS)
1	$U = U(s),\ p = P(s),$ $H = H(s),\ s = r/t,$ $\omega = \omega(t, r, \theta, \varphi)$	$U = s^{-2}C_1,$ $P = 2^{-1}s^{-4}C_1(4s^3 - C_1)$ $H = 0,\ \omega = \omega(t, s, \theta, \varphi)$
13	$U = U(r),\ p = P(r),$ $H = H(r),$ $\omega = \omega(t, r, \theta, \varphi)$	$U = r^{-2}C_1,$ $P = -2^{-1}r^{-4}C_1^2,$ $H = 0,\ \omega = \omega(t, r, \theta, \varphi)$

Table 4.5: Subalgebras considered in the cylindrical coordinate system (C)

i	Representation of PIS	Partially invariant solution (PIS)
4	$u = u(t, x, R, \theta),$ $p = P(s),\ q = q(s),$ $\varphi = \Psi(s) + \theta,\ s = Re^{-\alpha\theta}$	$u = C_4,\ P = -2^{-1}C_1^2 s^{2n},$ $q = C_1 s^n,\ \Psi = -\alpha \ln(s) + C_2,$ $n = -2(\alpha^2 + 1)^{-1}$
6	$u = u(t, x, R, \theta),$ $p = P(s),\ q = q(s),$ $\varphi = \Psi(s) + \theta,\ s = R/t$	$u = U(t, s, \theta),$ $P = 0,\ q = 0,$ $s^2 U_{ss} + s(s^2 t + 1)U_s - s^2 t^2 U_t + U_{\theta\theta} = 0$

Table 4.6: Subalgebras considered in the Cartesian coordinate system (D)

i	Representation of PIS	Partially invariant solution (PIS)
23	$u=u(t,x,y,z)$, $v=V(s)$, $w=W(s)$, $p=P(s),\ s=z/y$	1. $u=U(t,y,z),\ V=0,\ W=0,\ P=0:$ $U_t-U_{yy}-U_{zz}=0$ 2. $u=U(t,y,z),\ V=C_2,\ W=C_4,\ P=0:$ $U_t+C_2U_y+C_4U_z-U_{yy}+U_{zz}=0$
29	$u=u(t,x,y,z)$, $v=V(s)+y/t$, $w=W(s)+z/t$, $p=P(s)$, $s=y/t-\alpha\ln t$	1. $u=U(t,s,\tilde{z}),\ V=-2s+C_2,\ W=0,$ $P=(C_2-2\alpha-s)s,\ \tilde{z}=z/t:$ $t^2U_t+t(-2s+C_2-\alpha)U_s-U_{ss}-U_{\tilde{z}\tilde{z}}=0$ 2. $u=-(x-U(t,s,\tilde{z}))/t,\ V=-3s+C_2,\ W=0,$ $P=(C_2-3(\alpha+s))s,\ \tilde{z}=z/t:$ $t^3U_t+t^2(-3s+C_2-\alpha)U_s-U_{ss}-U_{\tilde{z}\tilde{z}}=0$
30	$u=U(s)+x/t$, $v=V(s)+\sigma\ln t$, $w=w(t,x,y,z)$, $p=P(s)$, $s=x/t-\beta\ln t$	1. $U=-2s+C_2,\ V=C_4,\ w=W(t,s,\tilde{y})/t,$ $P=(C_2-2\beta-s)s,\ \tilde{y}=y/t:$ $t^3W_t+t^2[(-2s+C_2-\beta)W_s+(C_4-\tilde{y})W_{\tilde{y}}]$ $-W_{ss}-W_{\tilde{y}\tilde{y}}=0$ 2. $U=-s+C_2,\ V=C_4,\ w=3z/t+W(t,s,\tilde{y}),$ $P=-\beta s,\ \tilde{y}=y/t:$ $t^2W_t+t[(-s+C_2-\beta)W_s+(C_4-\tilde{y})W_{\tilde{y}}]$ $-W_{ss}-W_{\tilde{y}\tilde{y}}=0$
35	$u=U(s)+t$, $v=v(t,x,y,z)$, $w=W(s)+\beta t$, $p=P(s)$, $s=x-t^2/2$	$u=U(s)+t,\ v=-U'+V(t,s,z),$ $w=W(s)+\beta t, P=U'-2^{-1}U^2-s:$ $U'''-UU''+U'^2=0,$ $W''-UW'-\beta=0,$ $V_t+UV_s-VU'+(W+\beta t)V_z-V_{ss}+V_{zz}=0$
36	$u=U(x)$, $v=v(t,x,y,z)$, $w=W(x)+t$, $p=P(x)$	$u=U(x),\ v=-U'+V(t,x,z),\ w=W(x)+t,$ $P=U'-2^{-1}U^2-x:$ $U'''-UU''+U'^2=0,$ $W''-UW'-1=0,$ $V_t+UV_x-VU'+(W+t)V_z-V_{xx}+V_{zz}=0$

Table 4.6: (Continued)

i	Representation of PIS	Partially invariant solution (PIS)
38	$u=U(x)$, $v=v(t,x,y,z)$, $w=W(x)$ $p=P(x)$	$u=U(x),\ v=-U'+V(t,x,z),\ w=W(x)$, $P=U'-2^{-1}U^2:$ $U'''-UU''+U'^2=0$, $W''-UW'=0$, $V_t+UV_x-VU'+WV_z-V_{xx}+V_{zz}=0$
44	$u=U(t)$, $v=V(t)+\alpha tw$ $+x-\alpha z$, $w=W(t,x,y,z)$, $p=P(t)$	$U=C_1,\ V=C_1t+C_2,\ w=W(t,x),\ p=0:$ $W_{xx}-C_1W_x-W_t=0$
48	$u=U(t)$, $v=V(t)-tw-z$, $w=W(t,x,y,z)$, $p=P(t)$	$U=C_1,\ V=C_2,\ w=W(t,x),\ p=0:$ $W_{xx}-C_1W_x-W_t=0$
50	$u=u(t,x,y,z)$, $v=V(t)$, $w=W(t)$, $p=P(t)$	$u=U(t,y,z),\ V=C_1,\ W=C_2,\ p=0:$ $U_t+C_1U_y+C_2U_z-(U_{yy}+U_{zz})=0$

Table 4.7: Subalgebras considered in the coordinate system $(K2)$

i	Representation of PIS	Partially invariant solution (PIS)
41	$u=u(t,x,y,z)$, $v=[j_1(t)-(\sigma t-\beta\tau)u$ $+ty-\beta z]/[t^2-\alpha\beta]$, $w=[j_2(t)-(\tau t-\alpha\sigma)u$ $-\alpha y+tz]/[t^2-\alpha\beta]$, $p=P(t)$	$u=[\alpha((\sigma(z-C_2)-\tau(y+C_1))\beta t-((C_3t^2\tau$ $-z)\beta^2\tau+C_1\sigma t^2))+\beta(\alpha^2(\sigma(C_3\sigma t^2-y)$ $+\beta^2\tau^2C_3)-(\beta\alpha^3\sigma^2C_3+C_2\tau t^2))]/$ $[\alpha\beta(\alpha\sigma^2-\beta\tau^2)]$, $j_1=C_1t+\beta C_2,\quad j_2=-\alpha C_1-C_2t$, $p=P(t)$

Table 4.6 is considered in the Cartesian coordinate system and the last Table 4.7 is devoted to partially invariant solutions considered in the coordinate system (K_2).

There are no regular partially invariant solutions of the Navier-Stokes equations for the subalgebras $9, 10, 17-20, 42, 43$.

Remark Notice that subalgebras $1, 4-7, 23, 29$ and 30 are not admitted by the Navier-Stokes equations. Nevertheless there exist solutions which are partially invariant with respect to them.

Chapter V

Optimal System

5.1 Admitted group of equation (4.62)

In this section, the Lie group admitted by equation (4.62) is studied. This is the heat equation, and it was obtained from the Navier-Stokes equations and gives rise to a partially invariant solutions of the Navier-Stokes equations

$$(F) \qquad u_t - u_{yy} - u_{zz} = 0,$$

where the function u depends on t, y, z.

Assume that the generator has a representation of the form

$$X = \xi^t(t,y,z,u)\partial_t + \xi^y(t,y,z,u)\partial_y + \xi^z(t,y,z,u)\partial_z + \zeta^u(t,y,z,u)\partial_u.$$

The second prolongation of the operator X is

$$\begin{aligned} X^{(2)} \;\; = \;\; & X + \zeta^{u_t}(t,y,z,u)\partial_{u_t} + \zeta^{u_y}(t,y,z,u)\partial_{u_y} + \zeta^{u_z}(t,y,z,u)\partial_{u_z} \\ & + \zeta^{u_{tt}}(t,y,z,u)\partial_{u_{tt}} + \zeta^{u_{ty}}(t,y,z,u)\partial_{u_{ty}} + \zeta^{u_{tz}}(t,y,z,u)\partial_{u_{tz}} \\ & + \zeta^{u_{yz}}(t,y,z,u)\partial_{u_{yz}} + \zeta^{u_{yy}}(t,y,z,u)\partial_{u_{yy}} + \zeta^{u_{zz}}(t,y,z,u)\partial_{u_{zz}}. \end{aligned}$$

The coefficients of the prolonged operator are defined by formulae (2.15). The determining equations are

$$X^{(2)}F\big|_{[F=0]} = 0. \tag{5.1}$$

All necessary calculations here as in the previous chapter were carried out on a computer using the symbolic manipulation program REDUCE (Hearn, 1999).

The result of the calculations is the admitted Lie group with the basis of the generators:

$$
\begin{aligned}
X_1 &= \partial_t,\ X_2 = \partial_z,\ X_3 = -\partial_y,\ X_4 = 2t\partial_t + y\partial_y + z\partial_z - u\partial_u, \\
X_5 &= 2t\partial_z - zu\partial_u,\ X_6 = y\partial_z - z\partial_y,\ X_7 = -2t\partial_y + yu\partial_u, \\
X_8 &= 4t^2\partial_t + 4ty\partial_y + 4tz\partial_z - (4t + y^2 + z^2)u\partial_u, \\
X_9 &= u\partial_u,\ X_{10} = b(t, y, z)\partial_u,
\end{aligned}
\tag{5.2}
$$

where $b(t, y, z)$ is an arbitrary solution of the heat equation

$$b_t - b_{yy} - b_{zz} = 0.$$

The problem is to construct subalgebras of the algebra L^{10}, which can be a source of invariant solutions of the heat equation. The classification of subalgebras can be done relatively easy for small dimensions. The optimal systems of subalgebras of the Lie algebra spanned by the generators $X_1, ..., X_9$ are constructed here.

The table of commutators $[X_i, X_j]$ is

$X_i \diagdown X_j$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1	0	0	0	$2X_1$	$2X_2$	0	$2X_3$	$4X_4$	0
X_2	0	0	0	X_2	$-X_9$	X_3	0	$2X_5$	0
X_3	0	0	0	X_3	0	$-X_2$	$-X_9$	$2X_7$	0
X_4	$-2X_1$	$-X_2$	$-X_3$	0	X_5	0	X_7	$2X_8$	0
X_5	$-2X_2$	X_9	0	$-X_5$	0	X_7	0	0	0
X_6	0	$-X_3$	X_2	0	$-X_7$	0	X_5	0	0
X_7	$-2X_3$	0	X_9	$-X_7$	0	$-X_5$	0	0	0
X_8	$-4X_4$	$-2X_5$	$-2X_7$	$-2X_8$	0	0	0	0	0
X_9	0	0	0	0	0	0	0	0	0

Inner automorphisms (cf. Ovsiannikov (1978)) are constructed with the help of the table of commutators.

To construct inner automorphisms, one has to solve the Lie equations. For example, for the automorphism A_1, one has the system of ordinary differential equations

$$\frac{d\overline{x}_1}{da} = -2\overline{x}_4, \quad \frac{d\overline{x}_2}{da} = -2\overline{x}_5, \quad \frac{d\overline{x}_3}{da} = -2\overline{x}_7, \quad \frac{d\overline{x}_4}{da} = -4\overline{x}_8.$$

and the initial values at $a = 0$

$$\overline{x}_1 = x_1, \ \overline{x}_2 = x_2, \ \overline{x}_3 = x_3, \ \overline{x}_4 = x_4.$$

Therefore, the automorphism A_1 only changes the coordinates x_1, x_2, x_3 and x_4 by the formulae

$$\overline{x}_1 = x_1 - 2a_1x_4 + 4a_1^2x_8, \ \overline{x}_2 = x_2 - 2a_1x_5,$$
$$\overline{x}_3 = x_3 - 2a_1x_7, \ \overline{x}_4 = x_4 - 4a_1x_8.$$

The remaining coordinates are unchanged.

In the same way, one obtains the automorphisms $A_i \quad (i = 2, \ldots, 9)$:

$$A_2: \ \overline{x}_2 = x_2 - a_2x_4, \ \overline{x}_3 = x_3 - a_2x_6, \ \overline{x}_5 = x_5 - 2a_2x_8, \ \overline{x}_9 = x_9 + a_2x_5 - a_2^2x_8$$

$$A_3: \ \overline{x}_2 = x_2 + a_3x_6, \ \overline{x}_3 = x_3 - a_3x_4, \ \overline{x}_7 = x_7 - 2a_2x_8, \ \overline{x}_9 = x_9 + a_3x_7 - a_3^2x_8$$

$$A_4: \ \overline{x}_1 = x_1\mathrm{e}^{2a_4}, \ \overline{x}_2 = x_2\mathrm{e}^{2a_4}, \ \overline{x}_3 = x_3\mathrm{e}^{2a_4},$$
$$\overline{x}_5 = x_5\mathrm{e}^{-2a_4}, \ \overline{x}_7 = x_7\mathrm{e}^{-2a_4}, \ \overline{x}_8 = x_8\mathrm{e}^{-2a_4}$$

$$A_5: \ \overline{x}_2 = x_2 + 2a_5x_1, \ \overline{x}_5 = x_5 + a_5x_4, \ \overline{x}_7 = x_7 - a_5x_6, \ \overline{x}_9 = x_9 - a_5x_2 - a_5^2x_1$$

$$A_6: \ \overline{x}_2 = x_2\cos(a_6) - x_3\sin(a_6), \ \overline{x}_3 = x_2\sin(a_6) + x_3\cos(a_6),$$
$$\overline{x}_5 = x_5\cos(a_6) - x_7\sin(a_6), \ \overline{x}_7 = x_5\sin(a_6) + x_7\cos(a_6)$$

$$A_7: \ \overline{x}_3 = x_3 + 2a_7x_1, \ \overline{x}_5 = x_5 + a_7x_6, \ \overline{x}_7 = x_7 + a_7x_4, \ \overline{x}_9 = x_9 - a_7x_3 - a_7^2x_1$$

$$A_8: \quad \overline{x}_4 = x_4 + 4a_8x_1, \quad \overline{x}_5 = x_5 + 2a_8x_2, \quad \overline{x}_7 = x_7 + 2a_8x_3,$$

$$\overline{x}_8 = x_8 + 2a_8x_4 + 4a_8^2x_1$$

$$A_9: \quad \overline{x}_1 = x_1, \quad \overline{x}_2 = x_2, \quad \overline{x}_3 = x_3, \quad \overline{x}_4 = x_4, \quad \overline{x}_5 = x_5, \quad \overline{x}_6 = x_6,$$

$$\overline{x}_7 = x_7, \quad \overline{x}_8 = x_8, \quad \overline{x}_9 = x_9.$$

Also there is the involution

$$E: \quad \overline{x}_3 = -x_3, \quad \overline{x}_6 = -x_6, \quad \overline{x}_7 = -x_7.$$

5.2 Decomposition of the algebra L^9

Before constructing an optimal system, let us study the algebraic structure of the algebra L^9. The algebra L^9 is decomposed as $I \oplus L^4$, where $I = \{X_2, X_3, X_5, X_7, X_9\}$ is an ideal and $L^4 = \{X_1, X_4, X_6, X_8\}$ is a subalgebra. According to the algorithm for constructing an optimal system of the algebra L^9, we use the two-step algorithm developed in Ovsiannikov (1994). First, an optimal system of subalgebras of the algebra L^4 is obtained. The next step is to glue the subalgebras from the optimal system of subalgebras of the algebra L^4 and the ideal I together.

Any subalgebra of a Lie algebra is completely defined by its basis generators. Any vector of the basis is a linear combination of the basis of generator of this Lie algebra. Hence, the subalgebra is completely defined by coefficients of these linear combinations. For example, let $L^k = \{Y_1, Y_2, ..., Y_k\}$ be a k-dimensional subalgebra of the algebra L^9. Operators Y_i, $(i = 1, 2, \ldots, k)$ are

$$Y_i = \sum_{\alpha=1}^{9} x_{i\alpha} X_\alpha, \ i = 1, ..., k.$$

Conditions for L^k to be a subalgebra are

$$[Y_i, Y_j] = \sum_{\alpha=1}^{k} C_{ij}^{\alpha} Y_\alpha; \quad i, j = 1, 2, \ldots, k.$$

For a classification of subalgebra, the coefficients C^{α}_{ij} have to be simplified by using the automorphism and subalgebra conditions.

5.3 Classification of the algebra L^4

Let us classify the algebra $L^4 = \{X_1, X_4, X_6, X_8\}$. The table of commutators of the algebra L^4 is

$X_i \diagdown X_j$	X_1	X_4	X_6	X_8
X_1	0	$2X_1$	0	$4X_4$
X_4	$-2X_1$	0	0	$2X_8$
X_6	0	0	0	0
X_8	$-4X_4$	$-2X_8$	0	0

Since the generator X_6 composes the center, the optimal system of subalgebras of $L^4 = \{X_1, X_4, X_6, X_8\}$ can be easily constructed by classifying the subalgebra $L^3 = \{X_1, X_4, X_8\}$ and gluing it with the center $\{X_6\}$. The idea of construction is as follows.

Let a subalgebra L^r of dimension $r \leq 4$ be formed by the operators

$$Y_i = a_{i1}X_1 + a_{i2}X_4 + a_{i3}X_6 + a_{i4}X_8, \;\; i = 1, ..., r$$

where a_{ij}, $(i = 1, ..., r;\; j = 1, 2, 3, 4)$ are arbitrary constants.

For the classification of L^4 we need to study two steps.

1. All coefficients a_{i3} are zero, $a_{i3} = 0$ $(i = 1, 2, 3, 4)$, it means that we will construct an optimal system of the subalgebra $L^3 = \{X_1, X_4, X_8\}$.

2. At least one of the coefficients of a_{i3} is not equal to zero.

Let us study the first step, and construct an optimal system of the subalgebra L^3. For convenience, we will denote the generators X_i by $\mathbf{i}$.

5.3.1 One-dimensional subalgebras of the algebra L^3

Let $Y = x_1\mathbf{1} + x_4\mathbf{4} + x_8\mathbf{8}$ which forms a one-dimensional subalgebra of the algebra L^3. The process of simplification of the coefficients of the operator Y is separated into the following cases.

Case 1. Assume that $x_8 \neq 0$. Then one can divide Y by x_8. Hence, without loss of generality one can consider

$$Y = x_1\mathbf{1} + x_4\mathbf{4} + \mathbf{8}.$$

By means of transformation A_1, it can transformed to an operator with $x_4 = 0$.

Case 1.1. Let $x_1 \neq 0$. By means of transformation A_4, one can transform it to $\varepsilon\mathbf{1} + \mathbf{8}$, where $\varepsilon = \pm 1$.

Case 1.2. Let $x_1 = 0$, then the representative of the class is the operator $\mathbf{8}$.

Case 2. Assume that $x_8 = 0$. Then one has $Y = x_1\mathbf{1} + x_4\mathbf{4}$.

Case 2.1. Let $x_4 \neq 0$. Dividing the operator Y by x_4, one obtains $Y = x_1\mathbf{1} + \mathbf{4}$. By using the automorphism A_1, the operator Y is transformed to X_4.

Case 2.2. Let $x_4 = 0$, then $Y = \mathbf{1}$.

5.3.2 Two-dimensional subalgebras of the algebra L^3

Let a subalgebra be formed by the operators

$$Y_i = a_{i1}\mathbf{1} + a_{i2}\mathbf{4} + a_{i3}\mathbf{8}, \ \ i = 1, 2$$

where a_{ij}, $(i = 1, 2; \ j = 1, 2, 3)$ are arbitrary constants. Note that the rank of the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ is equal to two.

Case 1. Assume that $a_{13} \neq 0$. We can divide Y_1 by a_{13}. Hence, by subtracting the operator $(a_{23}/a_{13})Y_1$ from Y_2, one can assume $a_{23} = 0$ and $a_{21}^2 + a_{22}^2 \neq 0$. Using the automorphisms A_1, the operator Y_1 is transformed to $Y_1 = a_{11}\mathbf{1} + \mathbf{8}$. The subalgebra condition gives

$$[\ a_{11}\mathbf{1} + \mathbf{8}\ ,\ a_{21}\mathbf{1} + a_{22}\mathbf{8}\ ] = \alpha(a_{11}\mathbf{1} + \mathbf{8}) + \beta(a_{21}\mathbf{1} + a_{22}\mathbf{8})$$

where α and β are arbitrary constants. Calculating the left hand side and comparing the coefficients on the left hand side with coefficients on the right hand side, one has

$$2a_{11}a_{22}\mathbf{1} - 4a_{21}\mathbf{4} - 2a_{22}\mathbf{8} = (\alpha a_{11} + \beta a_{21})\mathbf{1} + \beta a_{22}\mathbf{4} + \alpha\mathbf{8}.$$

Therefore

$$2a_{11}a_{22} = \alpha a_{11} + \beta a_{21},\ \ -4a_{21} = \beta a_{22},\ \ -2a_{22} = \alpha$$

Further consideration depends on values of the coefficients a_{11}, a_{21}, a_{22}. If $a_{22} = 0$, then $a_{21} = 0$ which is a contradiction to the condition $a_{21}^2 + a_{22}^2 \neq 0$. Hence, $a_{22} \neq 0$. One can assume that $a_{22} = 1$. Therefore $\alpha = -2$, $\beta = -4a_{11}$, and $a_{11} = -a_{21}^2$.

Case 1.1. If $a_{21} \neq 0$, then using the automorphism A_4, the operators Y_1 and Y_2 are transformed to $Y_1 = \mathbf{8} - \mathbf{1}$, $Y_2 = \mathbf{4} + \mathbf{1}$.

Case 1.2. If $a_{21} = 0$, then the operators Y_1 and Y_2 are $Y_1 = \mathbf{8}, Y_2 = \mathbf{4}$.

Case 2. Assume that $a_{13} = 0$. If $a_{23} \neq 0$, then by exchanging Y_1 and Y_2, this becomes the previous case. Hence, one can take $a_{23} = 0$. Therefore, the operators are $Y_1 = a_{11}\mathbf{1} + a_{12}\mathbf{4}, Y_2 = a_{21}\mathbf{1} + a_{22}\mathbf{4}$. Because the rank of the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is equal to 2, then by taking linear combinations of the operators Y_1 and Y_2 they can be transformed to $Y_1 = \mathbf{1}$ and $Y_2 = \mathbf{4}$.

5.3.3 Three-dimensional subalgebras of the algebra L^3

Let a subalgebra be formed by these operators

$$Y_i = a_{i1}\mathbf{1} + a_{i2}\mathbf{4} + a_{i3}\mathbf{8}, \ \ i = 1, 2, 3$$

where a_{ij}, $(i = 1, 2, 3;\ j = 1, 2, 3$ are arbitrary constants. Since the rank of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is equal to three, the basis if this subalgebra can be taken as

$$Y_1 = \mathbf{1}, Y_2 = \mathbf{4}, Y_3 = \mathbf{8}.$$

5.3.4 Optimal system of subalgebras of the algebra $L^3 = \{\mathbf{1},\ \mathbf{4},\ \mathbf{8}\}$

The result of classifying the algebra $L^3 = \{\mathbf{1},\ \mathbf{4},\ \mathbf{8}\}$ is the following:

Dimension		
1	2	3
$\mathbf{1}$	$\mathbf{1},\mathbf{4}$	$\mathbf{1},\mathbf{4},\mathbf{8}$
$\mathbf{4}$	$\mathbf{4},\mathbf{8}$	
$\mathbf{8}$	$\mathbf{1}-\mathbf{8},\mathbf{1}+\mathbf{4}$	
$\varepsilon\mathbf{1}+\mathbf{8}$		

where $\varepsilon = \pm 1$.

5.3.5 Optimal system of subalgebras of the algebra $L^4 = \{\mathbf{1},\ \mathbf{4},\ \mathbf{6},\ \mathbf{8}\}$

Let us consider the second step where at least one of the coefficients a_{i3} is not equal to zero. Without loss of generality one can assume

$$\begin{aligned} Y_1 &= \mathbf{6} + a_{11}\mathbf{1} + a_{12}\mathbf{4} + a_{14}\mathbf{8} \\ Y_i &= a_{i1}\mathbf{1} + a_{i2}\mathbf{4} + a_{i4}\mathbf{8},\ \ i = 2, ..., r,\ \ r \leq 4. \end{aligned}$$

Using the conditions for L^4 to be a subalgebra, one obtains

$$[Y_i, Y_j] = \alpha_{ij}\mathbf{6} + \beta_{ij}\mathbf{1} + \gamma_{ij}\mathbf{4} + \sigma_{ij}\mathbf{8}; \quad i, j = 1, 2, \dots, 4.$$

Because $L^3 = \{\mathbf{1},\ \mathbf{4},\ \mathbf{8}\}$ is a subalgebra and the generator $\mathbf{6}$ forms the center, then

$$[Y_i, Y_j] = \hat{\beta}_{ij}\mathbf{1} + \hat{\gamma}_{ij}\mathbf{4} + \hat{\sigma}_{ij}\mathbf{8}; \quad i, j = 1, 2, \dots, 4.$$

Comparing the coefficients, one obtains $\alpha_{ij} = 0;\ i, j = 1, 2, \dots, 4$. Because of these results and since the algebra $L^3 = \{\mathbf{1},\ \mathbf{4},\ \mathbf{8}\}$ has already been classified, therefore this allows simplifying the process of constructing the optimal system of the algebra L^4. This process construct by using the result of the optimal system of algebra L^3: we have to classify each optimal system of subalgebras of L^3 together with the generator $Y_1 = \mathbf{6} + a_{11}\mathbf{1} + a_{12}\mathbf{4} + a_{14}\mathbf{8}$. Here we give one example of this process. Other elements of the optimal system of the algebra L^4 are constructed in the similar way.

Let us consider the subalgebra $\{\mathbf{1} - \mathbf{8}, \mathbf{1} + \mathbf{4}\}$. For constructing three-dimensional subalgebras of the algebra L_4 one considers

$$Y_1 = \mathbf{6} + a_{11}\mathbf{1} + a_{12}\mathbf{4} + a_{14}\mathbf{8},\ Y_2 = \mathbf{1} - \mathbf{8},\ Y_3 = \mathbf{1} + \mathbf{4}.$$

Since Y_1 can be written as:

$$Y_1 = \mathbf{6} + (a_{11} - a_{12} + a_{14})\mathbf{1} + a_{12}(\mathbf{1} + \mathbf{4}) + a_{14}(\mathbf{8} - \mathbf{1}),$$

by forming a linear combination with Y_2 and Y_3, the operator Y_1 can be taken in the form $Y_1 = \mathbf{6} + \overline{a}_{11}\mathbf{1}$. The subalgebra conditions gives

$$[\mathbf{6} + \overline{a}_{11}\mathbf{1},\ \mathbf{1} - \mathbf{8}] = -4\overline{a}_{11}\mathbf{4} = \alpha(\mathbf{6} + \overline{a}_{11}\mathbf{1}) + \beta(\mathbf{1} - \mathbf{8}) + \gamma(\mathbf{1} + \mathbf{4})$$

where α, β and γ are arbitrary constants. Comparing the coefficients on the left side with the coefficients on the right side, one obtains

$$\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \overline{a}_{11} = 0.$$

Thus, one obtains that $Y_1 = \mathbf{6}$, and the subalgebra is $\{\mathbf{6},\ \mathbf{1} - \mathbf{8},\ \mathbf{1} + \mathbf{4}\}$.

The result of calculation is an optimal system of subalgebras of the algebra $L^4 = \{\mathbf{1},\ \mathbf{4},\ \mathbf{6},\ \mathbf{8}\}$ which is

Dimension			
1	2	3	4
$\mathbf{1}$	$\mathbf{1},\ \mathbf{4}$	$\mathbf{1},\ \mathbf{4},\ \mathbf{6}$	$\mathbf{1},\ \mathbf{4},\ \mathbf{6},\ \mathbf{8}$
$\mathbf{4}$	$\mathbf{4},\ \mathbf{6}$	$\mathbf{1},\ \mathbf{4},\ \mathbf{8}$	
$\mathbf{6}$	$\mathbf{4},\ \mathbf{8}$	$\mathbf{4},\ \mathbf{6},\ \mathbf{8}$	
$\mathbf{8}$	$\mathbf{1},\ \mathbf{6}+\beta\mathbf{4}$	$\mathbf{6},\ \mathbf{1}+\mathbf{4},\ \mathbf{1}-\mathbf{8}$	
$\mathbf{1}+\mathbf{6}$	$\mathbf{8},\ \mathbf{6}+\beta\mathbf{4}$		
$\varepsilon\mathbf{1}+\mathbf{8}$	$\varepsilon\mathbf{1}+\mathbf{8},\ \mathbf{6}$		
$\mathbf{4}+\mathbf{6}$	$\mathbf{1}+\mathbf{4},\ \mathbf{1}-\mathbf{8}$		
$\mathbf{8}+\mathbf{6}$			
$\varepsilon\mathbf{1}+\mathbf{6}+\mathbf{8}$			

where β is an arbitrary real parameter and $\varepsilon = \pm 1$.

5.3.6 Optimal system of subalgebras of the algebra L^9

After constructing an optimal system of subalgebras of the algebra L^4, the next step is the construction of an optimal system of subalgebras of the algebra $L^9 = \{\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6},\ \mathbf{7},\ \mathbf{8},\ \mathbf{9}\}$, by gluing subalgebras from the optimal system of subalgebras of the algebra L^4 and the ideal $I = \{\mathbf{2},\ \mathbf{3},\ \mathbf{5},\ \mathbf{7},\ \mathbf{9}\}$ together.

As it was seen for the algebra L^4, the process of constructing an optimal system of subalgebras of the algebra L^9 by gluing the algebra L^4 and the ideal I consists of the following steps. In the first step, the vectors

$$
\begin{aligned}
Y_i &= \sum_{j=\{2,3,5,7,9\}} a_{ij}X_j + \sum_{j=\{1,4,6,8\}} b_{ij}X_j, \quad (i = 1, 2, \ldots, k),\\
Y_{i+k} &= \sum_{j=\{2,3,5,7,9\}} c_{ij}X_j \qquad\qquad\qquad (i = 1, 2, \ldots, s),
\end{aligned}
$$

are composed. Here the vectors

$$\sum_{j=\{1,4,6,8\}} b_{ij} X_j$$

are basis elements from one of the k–dimensional subalgebras L^k of the optimal system of the algebra L^4. In matrix form, this step can be explained by the construction of the matrix

2 3 5 7 9	**1 4 6 8**
A	**B**
C	**0**

where the matrices **A**, **B** and **C** consist of the coefficients a_{ij}, $b_{i\alpha}$, $c_{\beta j}$, $(i = 1, 2, \ldots, k;\ j = 2, 3, 5, 7, 9;\ \alpha = 1, 4, 6, 8;\ \beta = 1, 2, \ldots, s)$. In this step, the matrix **A** is arbitrary. The rank of the matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix}$$

is equal to $k+s$ and this is the dimension of the subalgebra of the algebra L^9. The matrix **C** is chosen to be the simplest by taking linear combinations of it columns and has to take all possible values of the given rank s. Note also that the matrix **A** can be simplified with the help of the matrix **C**.

The next step is the process of checking the subalgebra conditions and checking linear dependence of commutators on the basis generators of the subalgebra.

In this thesis, we study only two-dimensional subalgebras of the algebra L^9, because the two-dimensional subalgebras allow obtaining invariant solutions which reduce the initial system of partial differential equations to a system of ordinary differential equations.

Let us give an example for constructing two-dimensional subalgebras, using the subalgebra $\{\varepsilon\mathbf{1}+\mathbf{8}\}$. The maximum possible dimension of a subalgebra of the algebra L^4 after gluing a subalgebra to the ideal I is two. In this case, the matrix $\mathbf{C}$ is a 1×5 matrix, the rank of which is equal to one:

2	**3**	**5**	**7**	**9**	**1**	**4**	**6**	**8**
a_{12}	a_{13}	a_{15}	a_{17}	a_{19}	ε	0	0	1
c_{22}	c_{23}	c_{25}	c_{27}	c_{29}	0	0	0	0

By virtue of the automorphism A_6:

$$\overline{x}_2 = x_2\cos(a_6) - x_3\sin(a_6),\ \ \overline{x}_3 = x_2\sin(a_6) + x_3\cos(a_6),$$

$$\overline{x}_5 = x_5\cos(a_6) - x_7\sin(a_6),\ \ \overline{x}_7 = x_5\sin(a_6) + x_7\cos(a_6).$$

We can consider three cases:

1. $c_{22}^2 + c_{23}^2 \neq 0$,
2. $c_{22}^2 + c_{23}^2 = 0,\ c_{25}^2 + c_{27}^2 \neq 0$,
3. $c_{22}^2 + c_{23}^2 = 0,\ c_{25}^2 + c_{27}^2 = 0,\ c_{29} \neq 0$.

Case 1. By using the automorphism A_6 one can assume $c_{22} = 1,\ c_{23} = 0$. In this case, by means of linear combinations and by the automorphisms A_2, A_3, A_5, A_7 the table of coefficients is transformed to

2	**3**	**5**	**7**	**9**	**1**	**4**	**6**	**8**
0	0	0	0	a_{19}	ε	0	0	1
1	0	c_{25}	c_{27}	c_{29}	0	0	0	0

The subalgebra conditions give

$$\begin{aligned}[\varepsilon\mathbf{1}+\mathbf{8}+a_{19}\mathbf{9},\ \mathbf{2}+c_{25}\mathbf{5}+c_{27}\mathbf{7}+c_{29}\mathbf{9}] &= 2\varepsilon c_{25}\mathbf{2} + 2\varepsilon c_{27}\mathbf{3} - \mathbf{25} \\ &= \alpha\left(\varepsilon\mathbf{1}+\mathbf{8}+a_{19}\mathbf{9}\right) \\ &\quad +\beta\left(\mathbf{2}+c_{25}\mathbf{5}+c_{27}\mathbf{7}+c_{29}\mathbf{9}\right),\end{aligned}$$

where the coefficients α and β are arbitrary constants. Comparing the coefficients, one obtains

$$\alpha = 0,\ \beta = \pm 2,\ \varepsilon = -1,\ c_{27} = 0,\ c_{29} = 0,\ c_{25} = \pm 1.$$

Therefore, in this case the subalgebra is $\{-\mathbf{1} + \mathbf{8} + a_{19}\mathbf{9},\ \mathbf{2} + \varepsilon\mathbf{5}\}$.

Case 2. Since $c_{22}^2 + c_{23}^2 = 0$, or $c_{22} = 0,\ c_{23} = 0$. Because of $c_{25}^2 + c_{27}^2 \neq 0$, by virtue of the automorphism A_6 one can take $c_{25} = 1,\ c_{27} = 0$. By means of linear combinations and by the automorphisms A_2, A_3, A_5, A_7, the coefficients are transformed to

2	**3**	**5**	**7**	**9**	**1**	**4**	**6**	**8**
0	0	0	0	a_{19}	ε	0	0	1
0	0	1	0	c_{29}	0	0	0	0

The subalgebra condition gives

$$\begin{aligned}[\varepsilon\mathbf{1} + \mathbf{8} + a_{19}\mathbf{9},\ \mathbf{5} + c_{29}\mathbf{9}] &= 2\varepsilon\mathbf{2} \\ &= \alpha\,(\varepsilon\mathbf{1} + \mathbf{8} + a_{19}\mathbf{9}) + \beta\,(\mathbf{5} + c_{29}\mathbf{9}),\end{aligned}$$

where the coefficients α and β are arbitrary constants. Comparing the coefficients, one obtains

$$\alpha = 0,\ \beta = 0,\ \varepsilon = 0.$$

This is a contradiction to $\varepsilon \neq 0$. Therefore, there exists no subalgebra in this case.

Case 3. Assume that $c_{22}^2 + c_{23}^2 = 0,\ c_{25}^2 + c_{27}^2 = 0$ and $c_{29} \neq 0$, or $c_{22} = 0$, $c_{23} = 0,\ c_{25} = 0,\ c_{27} = 0$. Since $c_{29} \neq 0$, without loss of generality one can choose $c_{29} = 1$. By taking linear combinations and by virtue of the automorphism A_2, A_3, A_5, A_7 the table of coefficients can be transformed to

2	**3**	**5**	**7**	**9**	**1**	**4**	**6**	**8**
0	0	0	0	0	ε	0	0	1
0	0	0	0	1	0	0	0	0

The subalgebra conditions give

$$[\varepsilon\mathbf{1}+\mathbf{8},\ \mathbf{9}] = 0 = \alpha\,(\varepsilon\mathbf{1}+\mathbf{8}) + \beta\,(\mathbf{9})\,,$$

which is satisfied with

$$\alpha = 0,\ \beta = 0.$$

Therefore, the subalgebra is $\{\varepsilon\mathbf{1}+\mathbf{8},\ \mathbf{9}\}$. Other elements of the optimal system of the algebra L^9 are constructed in the similar way.

The list of two-dimensional subalgebras of the optimal system of the algebra L^9 is presented in the Table 5.1.

Table 5.1: Two-dimensional subalgebras of the optimal system of the algebra L^9

N	Generator	N	Generator
1	$\mathbf{2}$, $\mathbf{3}$	11	$\mathbf{1}$, $\mathbf{2} + \alpha\mathbf{4} + \mathbf{6} + \beta\mathbf{9}$
2	$\mathbf{2}$, $\mathbf{7}$	12	$\mathbf{8}$, $\alpha\mathbf{4} + \mathbf{6} + \beta\mathbf{9}$
3	$\mathbf{5}$, $\mathbf{7}$	13	$\mathbf{8}$, $\alpha\mathbf{4} + \mathbf{5} + \mathbf{6} + \beta\mathbf{9}$
4	$\mathbf{2}$, $\mathbf{3} + \varepsilon\mathbf{7}$	14	$\mathbf{4} + \alpha\mathbf{9}$, $\mathbf{6} + \beta\mathbf{9}$
5	$\mathbf{1}$, $\mathbf{4} + \alpha\mathbf{9}$	15	$\mathbf{6} + \alpha\mathbf{9}$, $\varepsilon\mathbf{1} + \mathbf{8} + \beta\mathbf{9}$
6	$\mathbf{2}$, $\mathbf{4} + \alpha\mathbf{9}$	16	$\mathbf{2} + \mathbf{7}$, $\mathbf{3} + \mathbf{5} + \alpha\mathbf{7}$
7	$\mathbf{5}$, $\mathbf{4} + \alpha\mathbf{9}$	17	$\mathbf{2} + \alpha\mathbf{9}$, $\mathbf{1} + \beta\mathbf{7} + \gamma\mathbf{9}$
8	$\mathbf{8}$, $\mathbf{4} + \alpha\mathbf{9}$	18	$\mathbf{5} + \alpha\mathbf{9}$, $\beta\mathbf{3} + \mathbf{8} + \gamma\mathbf{9}$
9	$\mathbf{1}$, $\mathbf{2} + \mathbf{4} + \alpha\mathbf{9}$	19	$\mathbf{2} + \varepsilon\mathbf{5}$, $-\mathbf{1} + \mathbf{8} + \alpha\mathbf{9}$
10	$\mathbf{1}$, $\alpha\,\mathbf{4} + \mathbf{6} + \beta\mathbf{9}$	20	$\mathbf{1} - \mathbf{8} + \mathbf{2}\alpha\mathbf{9}$, $\mathbf{1} + \mathbf{4} + \alpha\mathbf{9}$

5.4 Invariant Solutions of the equation (4.62)

Invariant solutions of the equation (4.62) are presented in this section. Analysis of invariant solutions is presented in details for two examples. The analysis of the other cases is similar. The final results are collected in Table 5.2 in the next section.

5.4.1 Subalgebra 7 : $\{\mathbf{5}, \mathbf{4}+\alpha\mathbf{9}\}$

The basis of this subalgebra is

$$X_5 = 2t\partial_z - zu\partial_u,$$
$$X_4 + \alpha X_9 = 2t\partial_t + y\partial_y + z\partial_z + (\alpha-1)u\partial_u.$$

Let a function

$$f = f(t, y, z, u)$$

be an invariant of the generator X_5. This means that

$$2tf_z - zuf_u = 0.$$

The general solution of this equation is

$$f = F\left(t, y, \hat{u}\right), \ \hat{u} = u\mathrm{e}^{\frac{z^2}{4t}}.$$

After substituting it into the equation $(X_4+\alpha X_9)f = 0$, one obtains the equation

$$2tF_t + yF_y + (\alpha-1)\hat{u}F_{\hat{u}} = 0.$$

The characteristics system of the last equation is

$$\frac{\mathrm{d}t}{2t} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}\hat{u}}{(\alpha-1)\hat{u}}.$$

Thus the universal invariant of this subalgebras consist of invariants

$$\frac{y^2}{t}, \ \hat{u}y^{1-\alpha}, \ \ \hat{u} = u\mathrm{e}^{\frac{z^2}{4t}}.$$

Hence, a representation of the invariant solution is

$$u = y^{\alpha-1}\mathrm{e}^{-\frac{z^2}{4t}}\phi(q)$$

with arbitrary functions $\phi(q)$ and $q = y^2/t$. After substituting this representation into equation (4.62), one obtains the ordinary differential equation

$$8q^2\phi'' + 2q(q + 4\alpha - 2)\phi' + (2\alpha^2 - 6\alpha - q + 4)\phi = 0.$$

The general solution of the last equation is

$$\phi = \mathrm{e}^{-\frac{q}{8}}q^{\frac{2\alpha-3}{4}}\left[C_1W_1\left(\frac{2\alpha-1}{4}, \frac{1}{4}, \frac{q}{4}\right) + C_2W_2\left(\frac{2\alpha-1}{4}, \frac{1}{4}, \frac{q}{4}\right)\right],$$

where $W_1\left(\frac{2\alpha-1}{4}, \frac{1}{4}, \frac{q}{4}\right), W_2\left(\frac{2\alpha-1}{4}, \frac{1}{4}, \frac{q}{4}\right)$ are Whittaker functions and C_1, C_2 are arbitrary constants.

5.4.2 Subalgebra 16 : $\{\mathbf{2} + \mathbf{7}, \mathbf{3} + \mathbf{5} + \alpha\mathbf{7}\}$

The basis of this subalgebra consists of the generators

$$X_2 + X_7 = \partial_z - 2t\partial_y + yu\partial_u,$$
$$X_3 + X_5 + \alpha X_7 = -(1 + 2\alpha t)\partial_y + 2t\partial_z + (\alpha y - z)u\partial_u.$$

In order to find an invariant solution, one needs to find a universal invariant of this subalgebra. Let a function

$$f = f(t, y, z, u)$$

be an invariant of the generator $X_2 + X_7$. This means that

$$f_z - 2tf_y + yuf_u = 0.$$

The characteristics system of the last equation is

$$\frac{\mathrm{d}z}{1} = \frac{\mathrm{d}y}{-2t} = \frac{\mathrm{d}u}{yu} = \frac{\mathrm{d}t}{0}.$$

The general solution of this equation is

$$f = F\left(t, \hat{y}, \hat{u}\right), \ \hat{y} = y + 2tz, \ \hat{u} = ue^{\frac{y^2}{4t}}.$$

After substituting it into the equation

$$(X_3 + X_5 + \alpha X_7)f = 0$$

one obtains the equation

$$2t(1 + 2\alpha t - 4t^2)F_{\hat{y}} + \hat{y}\hat{u}F_{\hat{u}} = 0.$$

The characteristics system of this equation is

$$\frac{\mathrm{d}\hat{y}}{2t(1 + 2\alpha t - 4t^2)} = \frac{\mathrm{d}\hat{u}}{\hat{y}\hat{u}} = \frac{\mathrm{d}t}{0}.$$

Hence, the universal invariant of this subalgebras consist of invariants

$$t, \hat{u}e^{-\frac{\hat{y}^2}{4t(1+2\alpha t-4t^2)}}, \hat{y} = y + 2tz, \ \hat{u} = ue^{\frac{y^2}{4t}}.$$

A representation of the invariant solution of this subalgebra has the following form

$$u = e^{\frac{(y+2tz)^2}{4t(1+2\alpha t-4t^2)} - \frac{y^2}{4t}}\phi(t)$$

with an arbitrary function $\phi(t)$. After substituting the representation of the invariant solution into equation (4.62), the functions $\phi(t)$ has to satisfy the equation

$$(1 + 2\alpha t - 4t^2)\phi' + (\alpha - 4t)\phi = 0.$$

The general solution of the last equation is

$$\phi = C/\sqrt{1 + 2\alpha t - 4t^2}$$

where C is constant.

5.5 Result of invariant solutions of the equation (4.62)

The result of the study of invariant solutions of equation (4.62) corresponding to the subalgebras of Table 5.1 are presented in this section.

Table 5.2: Result of invariant solutions of the equation (4.62)

No.	Universal invariant	Invariant solution
1	$t\ ,\ u$	$u = C$
2	$t\ ,\ u\mathtt{e}^{y^2/4t}$	$u = (C\mathtt{e}^{-y^2/4t})/\sqrt{t}$
3	$t\ ,\ u\mathtt{e}^{(y^2+z^2)/4t}$	$u = (C\mathtt{e}^{-(y^2+z^2)/4t})/t$
4	1. $t\ ,\ u\mathtt{e}^{y^2/2(2t+1)}$	$u = (C\mathtt{e}^{-y^2/2(2t+1)})/\sqrt{2t+1}$
	2. $t\ ,\ u\mathtt{e}^{y^2/2(2t-1)}$	$u = (C\mathtt{e}^{-y^2/2(2t-1)})/\sqrt{2t-1}$
5	$z/y\ ,\ uz^{1-\alpha}$	$u = C_1(z-y\mathtt{i})^{\alpha-1} + C_2(z+y\mathtt{i})^{\alpha-1}$
6	$y^2/t\ ,\ uy^{1-\alpha}$	$u = t^{(2\alpha-1)/4}y^{-1/2}\mathtt{e}^{-y^2/8t}[C_1W_1(\frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t}) + C_2W_2(\frac{1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t})]$
7	$y^2/t\ ,\ uy^{1-\alpha}\mathtt{e}^{z^2/4t}$	$u = t^{(2\alpha-1)/4}y^{-1/2}\mathtt{e}^{-(y^2+2z^2)/8t}[C_1W_1(\frac{-1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t}) + C_2W_2(\frac{-1-2\alpha}{4}, \frac{1}{4}, \frac{y^2}{4t})]$
8	$z/y\ ,\ ut^{-\alpha}y^{1-\alpha}\mathtt{e}^{(y^2+z^2)/4t}$	$u = t^{\alpha}\mathtt{e}^{-(y^2+z^2)/4t}[C_1(z+y\mathtt{i})^{-\alpha-1} + C_2(z-y\mathtt{i})^{-\alpha-1}]$
9	$(z+1)/y\ ,\ uy^{1-\alpha}$	$u = C_1(z+1-y\mathtt{i})^{\alpha-1} + C_2(z+1+y\mathtt{i})^{\alpha-1}$
10	$q = \sqrt{y^2+z^2}\mathtt{e}^{-\alpha\arctan(z/y)}\ ,\ u\mathtt{e}^{(\alpha-\beta)\arctan(z/y)}$	$u = q^{-\frac{\alpha(\alpha-\beta)}{\alpha^2+1}}\mathtt{e}^{(\beta-\alpha)\arctan(z/y)}[C_1\sin(\frac{\alpha-\beta}{\alpha^2+1}\ln q) + C_2\cos(\frac{\alpha-\beta}{\alpha^2+1}\ln q)]$
11	$q = 2\alpha\arctan(\frac{\alpha+z+\alpha^2z}{1+y+\alpha^2y}) - \ln[(1+y+\alpha^2y)^2 + (\alpha+z+\alpha^2z)^2]\ ,$ $u\mathtt{e}^{(\alpha-\beta)\arctan(\frac{\alpha+z+\alpha^2z}{1+y+\alpha^2y})}$	$u = \mathtt{e}^{(\alpha-\beta)q/4}\mathtt{e}^{(\beta-\alpha)\arctan(\frac{\alpha+z+\alpha^2z}{1+y+\alpha^2y})}[C_1\sin(\frac{(\beta-\alpha)q}{4}) + C_2\cos(\frac{(\beta-\alpha)q}{4})]$
12	$q = (y^2+z^2)^{\frac{1}{2}}t^{-1}\mathtt{e}^{\alpha\arctan(z/y)}\ ,\ (y^2+z^2)^{\frac{1}{2}}u\mathtt{e}^{(\frac{y^2+z^2}{4t})-\beta\arctan(z/y)}$	$u = (y^2+z^2)^{-\frac{1}{2}}\mathtt{e}^{-(\frac{y^2+z^2}{4t})+\beta\arctan(z/y)}q^{\frac{1-\alpha\beta}{\alpha^2+1}}[C_1\sin(\frac{\alpha+\beta}{\alpha^2+1}\ln q)$ $+C_2\cos(\frac{\alpha+\beta}{\alpha^2+1}\ln q)]$
13	$q = (y^2+z^2)^{\frac{1}{2}}t^{-\frac{1}{2}}\ ,\ u(y^2+z^2)^{\frac{1-\alpha}{2}}\mathtt{e}^{\beta\arctan(y/z)}$	$u = (y^2+z^2)^{-\frac{1}{2}}t^{\frac{\alpha}{2}}\mathtt{e}^{-\beta\arctan(y/z)-\frac{q^2}{8}}[C_1W_1(\frac{-\alpha}{2}, \frac{\beta\mathtt{i}}{2}, \frac{q^2}{4}) + C_2W_2(\frac{-\alpha}{2}, \frac{\beta\mathtt{i}}{2}, \frac{q^2}{4})]$

Table 5.2: (Continued)

No.	Universal invariant	Invariant solution
14	$q=-[2\alpha\arctan(\frac{z+\alpha^2z-2\alpha t}{y+\alpha^2y+2t})+\ln[((z+\alpha^2z-2\alpha t)^2+(y+\alpha^2y+2t)^2)t^{-2}]$, $tu\mathrm{e}^{(\frac{y^2+z^2}{4t})-(\alpha+\beta)\arctan(\frac{z+\alpha^2z-2\alpha t}{y+\alpha^2y+2t})}$	$u=\mathrm{e}^{\alpha(\alpha+\beta)q/(2(\alpha^2+1))}\mathrm{e}^{(\alpha+\beta)\arctan(\frac{z+\alpha^2z-2\alpha t}{y+\alpha^2y+2t})-(\frac{y^2+z^2}{4t})}[C_1\sin(\frac{(\alpha+\beta)q}{2(\alpha^2+1)})$ $+C_2\cos(\frac{(\alpha+\beta)q}{2(\alpha^2+1)})]$
15	1. $q=(y^2+z^2)/(4t^2+1),\ u(4t^2+1)^{1/2}\mathrm{e}^{qt+\alpha\arctan(y/z)-(\beta\arctan(2t))/2}$	$u=(y^2+z^2)^{-1/2}\mathrm{e}^{(\beta\arctan(2t))/2-qt-\alpha\arctan(y/z)}[C_1W_1(\frac{\beta\mathtt{i}}{4},\frac{\alpha\mathtt{i}}{2},q\mathtt{i})$ $+C_2W_2(\frac{\beta\mathtt{i}}{4},\frac{\alpha\mathtt{i}}{2},q\mathtt{i})]$
	2. $q=(y^2+z^2)/(4t^2-1),\ u(4t^2+1)^{1/2}\mathrm{e}^{qt+\alpha\arctan(y/z)}((2t+1)/(2t-1))^{\beta/4}$	$u=(y^2+z^2)^{-1/2}\mathrm{e}^{-qt-\alpha\arctan(y/z)}((2t+1)/(2t-1))^{-\beta/4}[C_1W_1(\frac{-\beta}{4},\frac{\alpha\mathtt{i}}{2},q)$ $+C_2W_2(\frac{-\beta}{4},\frac{\alpha\mathtt{i}}{2},q)]$
16	t , $u\mathrm{e}^{\frac{y^2}{4t}-\frac{(y+2tz)^2}{4t(1+2\alpha t-4t^2)}}$	$u=(C\mathrm{e}^{\frac{(y+2tz)^2}{4t(1+2\alpha t-4t^2)}-\frac{y^2}{4t}})/\sqrt{1+2\alpha t-4t^2}$
17	$q=\beta t^2+y$, $u\mathrm{e}^{-\beta z-t(2\beta^2t^2+3(\beta y+\gamma))/3}$	$u=\mathrm{e}^{\beta z+t(2\beta^2t^2+3(\beta y+\gamma))/3}[C_1A_i(\frac{-\sqrt[3]{-\beta}(\beta q+\gamma-\alpha^2)}{\beta})+C_2B_i(\frac{-\sqrt[3]{-\beta}(\beta q+\gamma-\alpha^2)}{\beta})]$
18	$q=\frac{8ty-\beta}{8t^2}$, $ut\mathrm{e}^{\frac{(1-z)^2}{4t}+\frac{24t^2(y^2+\gamma-1)+\beta^2-12\beta ty}{96t^3}}$	$u=t^{-1}\mathrm{e}^{-\frac{24t^2(y^2+\gamma-1)+\beta^2-12\beta ty}{96t^3}-\frac{(1-z)^2}{4t}}[C_1A_i(\frac{2\gamma-\beta q-2}{2\beta^{2/3}})+C_2B_i(\frac{2\gamma-\beta q-2}{2\beta^{2/3}})]$
19	1. $q=\frac{(4t^2-1)^{1/2}}{y}$, $u(4t^2-1)^{1/2}((2t+1)/(2t-1))^{\alpha/4}\mathrm{e}^{\frac{z^2}{2(1+2t)}-\frac{y^2t}{4t^2-1}}$	$u=(4t^2-1)^{-1/2}((2t+1)/(2t-1))^{-\alpha/4}\mathrm{e}^{\frac{y^2t}{4t^2-1}+\frac{z^2}{2(1+2t)}}q^{1/2}[C_1W_1(\frac{1-\alpha}{4},\frac{1}{4},\frac{1}{q^2})$ $+C_2W_2(\frac{1-\alpha}{4},\frac{1}{4},\frac{1}{q^2})]$
	2. $q=\frac{(4t^2-1)^{1/2}}{y}$, $u(4t^2-1)^{1/2}((2t+1)/(2t-1))^{\alpha/4}\mathrm{e}^{-\frac{z^2}{2(1-2t)}-\frac{y^2t}{4t^2-1}}$	$u=(4t^2-1)^{-1/2}((2t+1)/(2t-1))^{-\alpha/4}\mathrm{e}^{\frac{y^2t}{4t^2-1}+\frac{z^2}{2(1-2t)}}q^{1/2}$ $[C_1W_1(\frac{-1-\alpha}{4},\frac{1}{4},\frac{1}{q^2})+C_2W_2(\frac{-1-\alpha}{4},\frac{1}{4},\frac{1}{q^2})]$
20	y/z , $u(1+2t)^{-\alpha}z^{1+\alpha}\mathrm{e}^{\frac{y^2+z^2}{2(1+2t)}}$	$u=(1+2t)^{\alpha}z^{-1-\alpha}\mathrm{e}^{-\frac{y^2+z^2}{2(1+2t)}}[C_1(y+z\mathtt{i})^{-\alpha-1}+C_2(y-z\mathtt{i})^{-\alpha-1}]$

where W_1,W_2 are Whittaker functions, A_i,B_i are Airy wave functions and C,C_1,C_2 are constants.

Chapter VI

Bäcklund Transformations

This part of the thesis is devoted to finding a Lie group of Bäcklund transformations admitted by the system of equation (4.68). First we recall the main knowledge from the theory of tangent transformations.

6.1 Lie Group of Finite Order Tangency

A point transformation is a transformation which involves transformations of the independent and the dependent variables only. On the other hand, a tangent transformation is a transformation which involves transformations of the independent variables, the dependent variables, as well as the derivatives. A natural generalization of the prolongation of point transformation leads to the tangent transformation. A tangent transformation which involves only the first order derivatives is called a contact transformation. A tangent transformation which involves derivatives up to finite order is called a Bäcklund transformation.

Consider the transformations of the independent variables x, dependent variables u and derivatives p:

$$\bar{x}_i = f^i(x, u, p; a), \ \bar{u}^k = \varphi^k(x, u, p; a), \ \bar{p}^k_\alpha = \psi^k_\alpha(x, u, p; a). \tag{6.1}$$

Here

$$p^k_\alpha \equiv \frac{\partial^{|\alpha|} u^k}{\partial x^\alpha} = \frac{\partial^{|\alpha|} u^k}{\partial x_1^{\alpha_1} ... x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n, \ , \ i = 1, ..., n, \ k = 1, ..., m, \ |\alpha| \in \{1, 2, ..., q\}$.

The functions f^i, φ^k and ψ^k_α depend on the independent variables x, dependent variables u and the derivatives p of order up to q ($|\alpha| \in \{1,2,...,q\}$). One extends the action of the group to the variables $\mathtt{d}x_i, \mathtt{d}u^k, \mathtt{d}p^k_\alpha$ by the usual formulae for the differentials

$$\begin{aligned} \mathrm{d}\bar{x}_i &= \tfrac{\partial f^i}{\partial x_j}\mathrm{d}x_j + \tfrac{\partial f^i}{\partial u^s}\mathrm{d}u^s + \sum_{|\beta|=1}^{q} \tfrac{\partial f^i}{\partial p^s_\beta}\mathrm{d}p^s_\beta, \\ \mathrm{d}\bar{u}^k &= \tfrac{\partial \varphi^k}{\partial x_j}\mathrm{d}x_j + \tfrac{\partial \varphi^k}{\partial u^s}\mathrm{d}u^s + \sum_{|\beta|=1}^{q} \tfrac{\partial \varphi^k}{\partial p^s_\beta}\mathrm{d}p^s_\beta, \\ \mathrm{d}\bar{p}^k_\alpha &= \tfrac{\partial \psi^k_\alpha}{\partial x_j}\mathrm{d}x_j + \tfrac{\partial \psi^k_\alpha}{\partial u^s}\mathrm{d}u^s + \sum_{|\beta|=1}^{q} \tfrac{\partial \psi^k_\alpha}{\partial p^s_\beta}\mathrm{d}p^s_\beta, \end{aligned} \tag{6.2}$$

where $i, j = 1, 2, ..., n,\ k, s = 1, 2, ..., m,\ |\alpha|, |\beta| \in \{1, 2, ..., q\}$.

The Lie group of transformations (6.1) is called a one-parameter group of tangent transformations if it preserves the tangent conditions

$$\begin{aligned} &\mathrm{d}u^k - p^k_i \mathrm{d}x_i = 0,\ \mathrm{d}p^k_\alpha - p^k_{\alpha,i}\mathrm{d}x_i = 0, \\ &i = 1, 2, ..., n,\ k = 1, 2, ..., m,\ |\alpha| \in \{1, 2, ..., q-1\}. \end{aligned} \tag{6.3}$$

This means that

$$\begin{aligned} &\mathrm{d}\bar{u}^k - \bar{p}^k_i \mathrm{d}\bar{x}_i = 0,\ \mathrm{d}\bar{p}^k_\alpha - \bar{p}^k_{\alpha,i}\mathrm{d}\bar{x}_i = 0, \\ &i = 1, 2, ..., n,\ k = 1, 2, ..., m,\ |\alpha| \in \{1, 2, ..., q-1\}. \end{aligned} \tag{6.4}$$

These conditions are strong, they provide very strong restrictions for the transformations (6.1).

The infinitesimal generator of this group is given by the vector field (ξ, η, ζ):

$$X = \xi^i \partial_{x_i} + \eta^k \partial_{u^k} + \sum_{|\alpha|=1}^{q} \zeta^k_\alpha \partial_{p^k_\alpha}, \tag{6.5}$$

where

$$\xi^i = \left.\frac{\partial f^i}{\partial a}\right|_{a=0},\ \eta^k = \left.\frac{\partial \varphi^k}{\partial a}\right|_{a=0},\ \zeta^k_\alpha = \left.\frac{\partial \psi^k_\alpha}{\partial a}\right|_{a=0}.$$

After substituting (6.2) into equations (6.4), one obtains

$$\left[\frac{\partial\varphi^k}{\partial x_j}+\frac{\partial\varphi^k}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial\varphi^k}{\partial p^s_\beta}p^s_{\beta,j}-p^k_i\left(\frac{\partial f^i}{\partial x_j}+\frac{\partial f^i}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial f^i}{\partial p^s_\beta}p^s_{\beta,j}\right)\right]\mathrm{d}x_j$$
$$+\sum_{|\beta|=q}\left(\frac{\partial\varphi^k}{\partial p^s_\beta}-p^k_i\frac{\partial f^i}{\partial p^s_\beta}\right)\mathrm{d}p^s_\beta=0\ ,$$
$$\left[\frac{\partial\psi^k_\alpha}{\partial x_j}+\frac{\partial\psi^k_\alpha}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial\psi^k_\alpha}{\partial p^s_\beta}p^s_{\beta,j}-p^k_{\alpha,i}\left(\frac{\partial f^i}{\partial x_j}+\frac{\partial f^i}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial f^i}{\partial p^s_\beta}p^s_{\beta,j}\right)\right]\mathrm{d}x_j \qquad (6.6)$$
$$+\sum_{|\beta|=q}\left(\frac{\partial\psi^k_\alpha}{\partial p^s_\beta}-p^k_{\alpha,i}\frac{\partial f^i}{\partial p^s_\beta}\right)\mathrm{d}p^s_\beta=0\ ,$$
$$(i,j=1,2,...,n,\ k,s=1,2,...,m,\ |\alpha|\in\{1,2,...,q-1\}\,).$$

The left side of these equations is a linear form with respect to the differentials $\mathrm{d}x_j,\ \mathrm{d}p^s_\beta,\ |\beta|=q$. Since $\mathrm{d}x_j,\ \mathrm{d}p^s_\beta,\ |\beta|=q$ are arbitrary, this implies that the coefficients of these forms have to be equal to zero. Differentiating these equations with respect to the parameter a and substituting $a=0$ into them, one obtains

$$\zeta^k_i=\frac{\partial\eta^k}{\partial x_j}+\frac{\partial\eta^k}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial\eta^k}{\partial p^s_\beta}p^s_{\beta,j}-p^k_i\left(\frac{\partial\xi^i}{\partial x_j}+\frac{\partial\xi^i}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial\xi^i}{\partial p^s_\beta}p^s_{\beta,j}\right)=0, \quad (6.7)$$

$$\frac{\partial\eta^k}{\partial p^s_\beta}-p^k_i\frac{\partial\xi^i}{\partial p^s_\beta}=0\ ,\quad (\ |\beta|=q\ ) \qquad (6.8)$$

$$\zeta^k_{\alpha,i}=\frac{\partial\zeta^k_\alpha}{\partial x_j}+\frac{\partial\zeta^k_\alpha}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial\zeta^k_\alpha}{\partial p^s_\beta}p^s_{\beta,j}-p^k_{\alpha,i}\left(\frac{\partial\xi^i}{\partial x_j}+\frac{\partial\xi^i}{\partial u^s}p^s_j+\sum_{|\beta|=1}^{q-1}\frac{\partial\xi^i}{\partial p^s_\beta}p^s_{\beta,j}\right)=0, \qquad (6.9)$$

$$\frac{\partial\zeta^k_\alpha}{\partial p^s_\beta}-p^k_{\alpha,i}\frac{\partial\xi^i}{\partial p^s_\beta}=0\ ,\quad (\ |\beta|=q\ ) \qquad (6.10)$$
$$(i,j=1,2,...,n,\ k,s=1,2,...,m,\ |\alpha|\in\{1,2,...,q-1\}\,).$$

Equations (6.7) and (6.9) can be written in the form

$$\zeta^k_i=D_j(\eta^k)-p^k_iD_j(\xi^i),\ \zeta^k_{\alpha,i}=D_j(\zeta^k_\alpha)-p^k_{\alpha,i}D_j(\xi^i),$$
$$(i=1,2,...,n,\ k=1,2,...,m,\ |\alpha|\in\{1,2,...,q-1\}\,) \qquad (6.11)$$

where the differential operators are

$$D_j=\frac{\partial}{\partial x_j}+p^s_j\frac{\partial}{\partial u^s}+\sum_{|\beta|=1}^{q-1}p^s_{\beta,j}\frac{\partial}{\partial p^s_\beta}.$$

It is convenient to rewrite equations (6.8) and (6.10) through the functions

$$W^k = \eta^k - \xi^i p_i^k,\ W_\alpha^k = \zeta_\alpha^k - \xi^i p_{\alpha,i}^k\ ,\ (k = 1, 2, ..., m,\ |\alpha| \in \{1, 2, ..., q-1\}).$$

Consider $q = 1$. Equation (6.8) becomes

$$\frac{\partial W^k}{\partial p_i^s} - \xi^i \delta_{ks} = 0\ ,\ (i = 1, 2, ..., n,\ k, s = 1, 2, ..., m). \tag{6.12}$$

If $m > 1$, choosing $k \neq s$ and $k = s$ in (6.12), one finds, respectively,

$$\xi^i = -\tfrac{\partial W^1}{\partial p_i^1} = -\tfrac{\partial W^2}{\partial p_i^2} = ... = -\tfrac{\partial W^m}{\partial p_i^m},\quad \tfrac{\partial W^k}{\partial p_i^s} = 0,$$
$$(i = 1, 2, ..., n,\ k, s = 1, 2, ..., m, k \neq s\,).$$

The general solution of these equations can be written in the form

$$\xi^i = V_i(x, u),\quad W^k = U^k - \xi^i p_i^k = U^k - V_i p_i^k,\ (i = 1, 2, ..., n,\ k = 1, 2, ..., m), \tag{6.13}$$

where the functions $V_i = V_i(x, u)$ and $U^k = U^k(x, u)$ do not depend on the derivatives. Thus the Lie group (6.1) is a prolongation of a Lie group of point transformations.

If $m = 1$, then (6.11) and (6.12) give

$$\xi^i = -\frac{\partial W}{\partial p_i},\quad \zeta_i = \frac{\partial W}{\partial x_j} + p_i \frac{\partial W}{\partial u},\quad (i = 1, 2, ..., n), \tag{6.14}$$

where $u = u^1,\ W = W^1,\ p_i = p_i^1,\ \zeta_i = \zeta_i^1$. Notice that from the definition of the function W and equation (6.14), one has

$$\eta = W - p_i \frac{\partial W}{\partial p_i}.$$

The function W is called a characteristic function. By induction one obtain the coefficients of the prolonged generator through the characteristic function

$$\zeta_\alpha = D_i W - p_{\alpha,i} \frac{\partial W}{\partial p_i},\quad (\,|\alpha| \geq 1).$$

Let $q > 1$. Equations (6.8) and (6.10) become

$$\frac{\partial W^k}{\partial p^s_\beta} = 0, \quad \frac{\partial W^k_\alpha}{\partial p^s_\beta} + \xi^i \frac{\partial p^k_{\alpha,i}}{\partial p^s_\beta} = 0, \quad (k, s = 1, 2, ..., m, \ |\alpha| \in \{1, 2, ..., q-1\}, \ |\beta| = q). \tag{6.15}$$

If $m > 1$, choosing $k \neq s$ in equation (6.15), one obtains

$$\frac{\partial W^k_\alpha}{\partial p^s_\beta} = 0, \quad (k, s = 1, 2, ..., m, \ |\alpha| \in \{1, 2, ..., q-1\}, \ |\beta| = q). \tag{6.16}$$

choosing $k = s$ and $\beta = \alpha, i$ in equation (6.8), one finds

$$\xi^i = -\frac{\partial W^k_\alpha}{\partial p^k_{\alpha,i}}, \quad (i = 1, 2, ..., n, \ k = 1, 2, ..., m, \ |\alpha| \in \{1, 2, ..., q-1\}). \tag{6.17}$$

Since in equation (6.15) the coordinates ξ^i do not depend on the derivatives $p^k_{\alpha,i}$, $(|\beta| = q, \ k = 1, 2, .., m)$, the general solution of equations (6.17) is

$$W^k_\alpha = U^k_\alpha - \xi^i p^k_{\alpha,i}, \quad (i = 1, 2, ..., n, \ k = 1, 2, ..., m, \ |\alpha| \in \{1, 2, ..., q-1\}).$$

where the functions U^k_α do not depend on the derivatives of order $q: \ p^s_\beta, \ (|\beta| = q, \ s = 1, 2, .., m)$. Because of equations (6.15) and (6.10), the coefficients η^k and $\zeta^k_\alpha, \ k = 1, 2, .., m, \ (|\alpha| = 1, ..., q-1)$ are also independent of the derivatives of order q. By induction on q one obtains (6.13). Hence, as in the case $q = 1$, the Lie group (6.1) is a prolongation of a Lie group of point transformations.

If $m = 1$, equations (6.15) become

$$\frac{\partial W}{\partial p_\beta} = 0, \ (|\beta| = q), \ \frac{\partial W_\alpha}{\partial p_\beta} + \xi^i \frac{\partial p_{\alpha,i}}{\partial p_\beta} = 0, \ (|\alpha| \in \{1, 2, ..., q-1\}, \ |\beta| = q). \tag{6.18}$$

If for given β and α there is i such that $\beta \neq \alpha, i$, then

$$\frac{\partial W}{\partial p_\beta} = 0.$$

Hence,

$$\frac{\partial}{\partial p_\beta}\left(\frac{\partial W_\alpha}{\partial p_{\alpha,i}} + \xi^i\right) = \frac{\partial}{\partial p_{\alpha,i}}\left(\frac{\partial W_\alpha}{\partial p_\beta}\right) + \frac{\partial \xi^i}{\partial p_\beta} = \frac{\partial \xi^i}{\partial p_\beta} = 0.$$

If $n > 1$, then for any $1 \leq i \leq n$ and β, $(|\beta| = q)$ there are α, $(|\alpha| = q-1)$ and k such that $k \neq i$, $\beta = \alpha, k$ and $\beta \neq \alpha, i$. Thus, the coefficients ξ^i, $i = 1, ..., n$ do not depend on p_β $(|\beta| = q)$, and

$$W_\alpha = U_\alpha - \xi^i p_{\alpha,i}, \ (|\alpha| = q-1),$$

where the functions U_α also do not depend on p_β, $(|\beta| = q)$. Similar to the case $m > 1$, one obtains that the group (6.1) is a prolongation of a Lie group of contact transformations.

If $n = 1$, equation (6.15) lead to

$$W = \eta - \xi p_1, \quad W_1 = \zeta_1 - \xi p_2, \quad W_2 = \zeta_2 - \xi p_3, \ \dots, \quad W_{q-1} = \zeta_{q-1} - \xi p_q,$$

and

$$\frac{\partial W}{\partial p_q} = 0, \quad \frac{\partial W_1}{\partial p_q} = 0, \quad \frac{\partial W_2}{\partial p_q} = 0, \ \dots, \quad \frac{\partial W_{q-2}}{\partial p_q} = 0, \quad \xi = -\frac{\partial W_{q-1}}{\partial p_q}. \qquad (6.19)$$

Because of (6.13), one finds

$$DW_{q-2} = D(\zeta_{q-2} - \xi p_{q-1}) = D\zeta_{q-2} - p_{q-1}D\xi - \xi p_q = \zeta_{q-1} - \xi p_q = W_{q-1}.$$

Since of (6.19)

$$\xi = -\frac{\partial W_{q-1}}{\partial p_q} = -\frac{\partial W_{q-2}}{\partial p_{q-1}}.$$

Hence,

$$\eta = W - p_1 \tfrac{\partial W_{q-2}}{\partial p_{q-1}}, \quad \zeta_1 = W_1 - p_2 \tfrac{\partial W_{q-2}}{\partial p_{q-1}}, \ \dots, \quad \zeta_{q-2} = W_{q-2} - p_{q-1} \tfrac{\partial W_{q-2}}{\partial p_{q-1}},$$

$$\zeta_{q-1} = DW_{q-2} + \xi p_q = DW_{q-2} - p_q \tfrac{\partial W_{q-2}}{\partial p_{q-1}}$$
$$= \tfrac{\partial W_{q-2}}{\partial x} + p_1 \tfrac{\partial W_{q-2}}{\partial u} + p_2 \tfrac{\partial W_{q-2}}{\partial p_1} + \dots + p_{q-1} \tfrac{\partial W_{q-2}}{\partial p_{q-2}}.$$

Thus, ξ, η, ζ_1, ζ_2, ..., ζ_{q-1} only depend on x, u, p_1, p_2, ..., p_{q-1}. By induction on q one obtains that the Lie group (6.1) is a prolongation of a Lie group of contact transformations.

Therefore, any Lie group of transformations (6.1) is not a prolongation of a Lie group of point transformations only for $m = 1$. For $m = 1$ any Lie group (6.1) is a prolongation of a Lie group of contact transformations, and there exists a characteristic function $W = W(x_1, x_2, ..., x_n, u, p_1, p_2, ..., p_n)$ such that

$$\xi^i = -\frac{\partial W}{\partial p_i}, \quad \eta = W - p_i\frac{\partial W}{\partial p_i}, \quad \zeta_\alpha = D^\alpha W - p_{\alpha,i}\frac{\partial W}{\partial p_i}, \ (i = 1, 2, ..., n, \ |\alpha| \geq 1). \tag{6.20}$$

This statement is known as the Lie Bäcklund theorem.

Theorem 5 (Bäcklund). *There are no tangent transformations of finite order N other than tangent transformations which are prolongations of contact* $(m = 1)$ *or point transformations* $(m > 1)$.

This means that for the case $m > 1$, a tangent transformation is the N-th prolongation of a group of point transformations, and for $m = 1$, a tangent transformation is the prolongation of a contact transformation.

6.2 An admitted Lie group of tangent transformations

The notion of a Lie group of point transformations has been generalized to involve derivatives in the transformations[1].

Assume that the transformations (6.1) form a one parameter Lie group G^1. The infinitesimal generator of the group G^1 is given by the equation (6.5). The coefficients of the infinitesimal generator for the derivatives of any order higher than q are defined by the recurrent prolongation formulae

$$\zeta^j_{\alpha,i} = D_i\zeta^j_\alpha - p^j_{\alpha,k}D_i, \ (|\alpha| = q, q+1, ...).$$

Here D_i is the operator of the total derivative with respect to x_i.

[1]For details one can see Ibragimov (1983,1999)

The infinitesimal generator X is prolonged for the differentials $\mathtt{d}u^j, \mathtt{d}x_i$ and $\mathtt{d}p^j_\alpha$:

$$\widetilde{X} = X + \widetilde{\xi^i}\,\partial_{\mathtt{d}x_i} + \widetilde{\eta^k}\,\partial_{\mathtt{d}u^k} + \widetilde{\zeta^j_\alpha}\,\partial_{\mathtt{d}p^j_\alpha}$$

by the usual formulae for the differentials

$$\begin{aligned}\widetilde{\xi^i} &= \mathtt{d}\xi^i = \tfrac{\partial \xi^i}{\partial x_l}\mathtt{d}x_l + \tfrac{\partial \xi^i}{\partial u^s}\mathtt{d}u^s + \tfrac{\partial \xi^i}{\partial p^s_\beta}\mathtt{d}p^s_\beta,\\ \widetilde{\eta^k} &= \mathtt{d}\eta^k = \tfrac{\partial \eta^k}{\partial x_l}\mathtt{d}x_l + \tfrac{\partial \eta^k}{\partial u^s}\mathtt{d}u^s + \tfrac{\partial \eta^k}{\partial p^s_\beta}\mathtt{d}p^s_\beta,\\ \widetilde{\zeta^j_\alpha} &= \mathtt{d}\zeta^j_\alpha = \tfrac{\partial \zeta^j_\alpha}{\partial x_l}\mathtt{d}x_l + \tfrac{\partial \zeta^j_\alpha}{\partial u^s}\mathtt{d}u^s + \tfrac{\partial \zeta^j_\alpha}{\partial p^s_\beta}\mathtt{d}p^s_\beta.\end{aligned}$$

A Lie group of tangent transformations has to satisfy the determining equations

$$\left(\widetilde{\eta^j} - \widetilde{\zeta^j_i}\,\mathtt{d}x_i - p^j_i\widetilde{\xi^i}\right)_{|(6.4)} = 0,\ \left(\widetilde{\zeta^j_\alpha} - \widetilde{\zeta^j_{\alpha,k}}\mathtt{d}x_k - p^j_{\alpha,k}\widetilde{\xi^k}\right)_{|(6.4)} = 0.$$

The Bäcklund theorem (Ibragimov, 1983) states that any tangent transformation is a prolongation of a Lie group of either contact transformations or point transformations. The first case is only possible for $m = 1$. Notice that the Bäcklund theorem was proven under the assumption that all derivatives only satisfy the tangent conditions.

A Lie group of tangent transformations with generator X is called admitted by a system of partial differential equations

$$(S) \qquad\qquad F(x,u,p) = 0$$

if the coefficients of the infinitesimal generator satisfy the determining equations

$$XF_{|(S)} = 0. \tag{6.21}$$

Invariance of the tangent conditions for the admitted Lie group of tangent transformations becomes

$$\begin{aligned}\left(\widetilde{\eta^j} - \widetilde{\zeta^j_i}\,\mathtt{d}x_i - p^j_i\widetilde{\xi^i}\right)_{|(6.4),(S)} &= 0,\\ \left(\widetilde{\zeta^j_\alpha} - \widetilde{\zeta^j_{\alpha,k}}\mathtt{d}x_k - p^j_{\alpha,k}\widetilde{\xi^k}\right)_{|(6.4),(S)} &= 0.\end{aligned}$$

In contrast to the Bäcklund theorem, admitted tangent transformations involve additional relations for the derivatives occurring in (S). This allows for the existence of Bäcklund transformations, namely tangent transformations of finite order.

6.3 Lie groups of Bäcklund transformations

Lie groups of Bäcklund transformations admitted by the system of equation (4.68) are presented in this section. Direct calculations show that the Lie group of transformations corresponding to the generators

$$Y_1 = U'\partial_V,\ Y_2 = (tU'+1)\partial_V,\ Y_3 = (\beta t + W + zU')\partial_V \tag{6.22}$$

is admitted by the system of equations (4.68). The corresponding transformations are:

$$\begin{aligned} Y_1\ :\ & \bar{t} = t,\ \ \overline{s} = s,\ \ \overline{z} = z, \\ & \overline{U} = U,\ \ \overline{U}' = U',\ \ \overline{U}'' = U'',\ \ \overline{U}''' = UU'' - (U')^2, \\ & \overline{W} = W,\ \ \overline{W}' = W',\ \ \overline{W}'' = W'' = UW' + \beta, \\ & \overline{V} = V + aU',\ \ \overline{V}_t = V_t,\ \ \overline{V}_z = V_z,\ \ \overline{V}_{zz} = V_{zz}, \\ & \overline{V}_s = V_s + aU'',\ \ \overline{V}_{ss} = V_{ss} + a(UU'' - (U')^2), \end{aligned}$$

$$\begin{aligned} Y_2\ :\ & \bar{t} = t,\ \ \overline{s} = s,\ \ \overline{z} = z, \\ & \overline{U} = U,\ \ \overline{U}' = U',\ \ \overline{U}'' = U'',\ \ \overline{U}''' = UU'' - (U')^2, \\ & \overline{W} = W,\ \ \overline{W}' = W',\ \ \overline{W}'' = W'' = UW' + \beta, \\ & \overline{V} = V + a(tU'+1),\ \ \overline{V}_t = V_t + aU',\ \ \overline{V}_z = V_z,\ \ \overline{V}_{zz} = V_{zz}, \\ & \overline{V}_s = V_s + atU'',\ \ \overline{V}_{ss} = V_{ss} + at(UU'' - (U')^2), \end{aligned}$$

$$Y_3 \ : \ \bar{t} = t, \ \ \bar{s} = s, \ \ \bar{z} = z,$$
$$\overline{U} = U, \ \ \overline{U}' = U', \ \ \overline{U}'' = U'', \ \ \overline{U}''' = UU'' - (U')^2,$$
$$\overline{W} = W, \ \ \overline{W}' = W', \ \ \overline{W}'' = W'' = UW' + \beta,$$
$$\overline{V} = V + a(\beta t + W + zU'), \ \ \overline{V}_t = V_t + a\beta,$$
$$\overline{V}_z = V_z + aU', \ \ \overline{V}_{zz} = V_{zz},$$
$$\overline{V}_s = V_s + aW' + azU'', \ \ \overline{V}_{ss} = V_{ss} + aW'' + az(UU'' - (U')^2).$$

The Lie group of transformations (6.22) was originally found by seeking an admitted Lie group of point transformations for the equivalent system

$$\widetilde{U}'' - U\widetilde{U}' + \widetilde{U}^2 = 0, \ \widetilde{U} = U',$$
$$W'' - UW' - \beta = 0, \tag{6.23}$$
$$V_t + UV_s - V\widetilde{U} + (W + \beta t)V_z - V_{ss} + V_{zz} = 0.$$

For system (6.23) the dependent variables are

$$u^1 = U, \ u^2 = V, \ u^3 = W, \ u^4 = U'.$$

The basis of the Lie algebra corresponding to the Lie group of point transformations admitted by system (6.23) is

$$X_1 = \partial_s, \ X_2 = \partial_z, \ X_3 = V\partial_V, \ X_4 = \partial_t + \beta t\partial_z, \ X_5 = t\partial_z + \partial_W,$$
$$X_6 = U'\partial_V, \ X_7 = (tU' + 1)\partial_V, \ X_8 = (\beta t + W + zU')\partial_V.$$

The generators $X_6, \ X_7, \ X_8$ coincide with the operators $Y_1, \ Y_2, \ Y_3$, respectively. Notice that if one looks for an admitted group by considering the dependent variables

$$u^1 = U, \ u^2 = V, \ u^3 = W, \ u^4 = U',$$
$$u^5 = W', \ u^6 = V_t, \ u^7 = V_s, \ u^8 = V_y,$$

then one obtains the following admitted generators

$$X_1 = \partial_s, \quad X_2 = \partial_z, \quad X_3 = V\partial_V + V_t\partial_{V_t} + V_s\partial_{V_s} + V_z\partial_{V_z},$$
$$X_4 = t\partial_z + \partial_W - V_z\partial_{V_t}, \quad X_5 = \partial_t - \beta t\partial_w.$$

Note that Y_1, Y_2, and Y_3 are no longer among these generators.

Chapter VII

Conclusion

7.1 Thesis Summary

This thesis is devoted to an application of group analysis to the Navier-Stokes equations.

7.1.1 Problems

Unsteady motion of incompressible viscous fluid is governed by the Navier-Stokes equations. These equations can be written in the compact form

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \triangle \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$ is the velocity field, p is the fluid pressure, ∇ is the gradient operator in the three-dimensional space, $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ and $\triangle$ is the Laplacian. A group classification of the Navier-Stokes equations in the three-dimensional case was done in Bytev (1972). The Lie algebra admitted by the Navier-Stokes equations is infinite-dimensional. Its Lie algebra can be presented in the form of the direct sum $L^\infty \oplus L^5$, where the infinite-dimensional ideal L^∞ is generated by the operators

$$\Psi_j = \psi_j(t)\partial_{x_j} + \psi_j'(t)\partial_{u^j} - \rho x_j \psi_j''(t)\partial_p, \ \Phi = \phi(t)\partial_p$$

with arbitrary functions $\psi_j(t)$, $(i = 1, 2, 3)$ and $\phi(t)$. The subalgebra L^5 has the basis:

$$\begin{aligned}
X_0 &= \partial_t, X_{ij} = x_j\partial_{x_i} - x_i\partial_{x_j} + u^j\partial_{u^i} - u^i\partial_{u^j} , \\
Z &= 2t\partial_t + \sum_{k=1}^{3}(x_k\partial_{x_k} - u^k\partial_{u^k}) - 2p\partial_p,
\end{aligned}$$

where $i = 1,2,\ j = 1,2,3,\ i < j$. The Galilean algebra L^{10} is contained in $L^\infty \oplus L^5$. There is still no classification of this algebra. Several articles are devoted to invariant solutions of the Navier-Stokes equations. Since partially invariant solutions of the Navier-Stokes equations have been less studied, therefore it is natural to investigate such partially invariant solutions.

Constructing partially invariant solutions consists of a sequence of steps: choosing a subgroup, finding a representation of a solution, substituting the representation into the studied system of equations, and studying compatibility of the obtained (reduced) system of equations.

The first problem that was studied in the thesis is to analyze all regular partially invariant solutions of the Navier-Stokes equations based on the subalgebras from Table 4.1.

It should be noted that the notion of compatibility plays the key role in constructing partially invariant solutions. During construction of a representation of a partially invariant solution, the property of the group to be admitted is not used. This fact gives rise to the assumption that one can construct partially invariant solutions with respect to a Lie group which is not necessary admitted. Earlier partially invariant solutions were only constructed with respect to admitted Lie groups. The subalgebras $1, 4-7, 23, 29$ and 30 from Table 4.1 are not admitted by the Navier-Stokes equations.

The second problem that is studied in the thesis is devoted to the construction of a Lie group of tangent transformations for a system of partial differential equations. The Bäcklund theorem states that in the general case there are no nontrivial tangent transformations of finite order except contact transformations. This theorem is proven under the assumption that all derivatives involved in the transformations are free: they only satisfy the tangent conditions. On the other hand, if the derivatives appearing in a system of partial differential equations satisfy additional relations other than the tangent conditions, then there may exist nontrivial tangent transformations of finite order. These transformations are called Bäcklund transformations.

After obtaining a reduced system of equations one can again apply group analysis to it: find an admitted group, construct optimal system of subalgebras, and obtain invariant and partially invariant solutions. Some subalgebras of Table 4.1 lead to the heat equation. The admitted algebra of the heat equation is spanned by the generators (5.2). The problem that was studied in the thesis is to construct all subalgebras of the algebra L^{10}, which can be a source of invariant solutions of the heat equation.

7.1.2 Results

1. All partially invariant solutions for the Navier-Stokes equations with respect to the subalgebras presented in Table 4.1 were studied. The final results are collected in Table 4.3-Table 4.7. There are no regular partially invariant solutions of the Navier-Stokes equations for the subalgebras $9, 10, 17-20, 42, 43$.

2. The subalgebras $1, 4-7, 23, 29$ and 30 from Table 4.1 are not admitted by the Navier-Stokes equations. Nevertheless, it is proven that there exist solutions which are partially invariant with respect to them.

3. The existence of Bäcklund transformations for a system of partial differential equations (4.68) which arises from the study of partially invariant solutions

of the Navier-Stokes equations is proven.

4. The optimal systems of two-dimensional subalgebras of the Lie algebra spanned by generators $X_1, ..., X_9$ (5.2) were obtained: there are 20 classes that have invariant solutions. The invariant solutions with respect to their subalgebras were presented.

7.1.3 Limitations

The thesis deals with regular partially invariant solutions of the Navier-Stokes equations with defect $\delta = 1$ and rank $\sigma = 1$. These solutions of this (σ, δ) type can be easily constructed. Subalgebras for studying were taken from part of the optimal system of subalgebras for the gas dynamics equations(Ovsiannikov and Chupakhin (1996)).

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Appendix

Special Functions

1 Gamma function

The Gamma function is defined for $Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty \mathtt{e}^{-t} t^{z-1} \mathtt{d}t,$$

and is extended to the rest of the complex plane, less the non-positive integers, by analytic continuation. Γ has a simple pole at each of the points $z = 0, -1, -2, ...$.

2 Kummer functions

The Kummer functions $\mathtt{KummerM}(\mu, \nu, s)$ and $\mathtt{KummerU}(\mu, \nu, s)$ solve the differential equation

$$sw'' + (\nu - s)w' - \mu w = 0.$$

3 Hypergeometric functions

The hypergeometric functions $\mathtt{hypergeom}(n; d; s)$, $n = (n_1, n_2, ..., n_p)$, $d = (d_1, d_2, ..., d_p)$ are solutions to the hypergeometric differential equation

$$s(1 - s)w'' + [c - (a + b + 1)s]w' - abw = 0.$$

The function $\mathtt{hypergeom}(n; d; s)$ is the generalized hypergeometric function $F(n; d; s)$. It is frequently denoted by ${}_pF_q(n; d; s)$. The definition of ${}_pF_q(n; d; s)$ is

$${}_pF_q(n; d; s) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(n_i + k)/\Gamma(n_i)}{\prod_{i=1}^{q} \Gamma(d_i + k)/\Gamma(d_i)} \frac{s^k}{k!}.$$

If $n_i = -m$, where m is a non-negative integer, the series is finite, stopping after $m+1$ terms. If $d_i = -m$, where m is a non-negative integer, the series is undefined, unless there is also a negative upper parameter of smaller absolute value, in which case the previous rule applies.

The complete solution to the hypergeometric differential equation is

$$w = C_1\ {}_2F_1(a, b; c, s) + C_2\ {}_2F_1(a + 1 - c, b + 1 - c; 2 - c, s)$$

where C_1 and C_2 are constants.

4 Whittaker functions

The Whittaker functions $W_1(\mu, \nu, s)$ and $W_2(\mu, \nu, s)$ solve the differential equation

$$\mathrm{w}'' + \left(-\frac{1}{4} + \frac{\mu}{s} + \frac{\frac{1}{4} - \nu^2}{s^2}\right) \mathrm{w} = 0.$$

They can be defined in terms of the hypergeometric and Kummer functions as follows:

$$\mathrm{W}_1(\mu,\nu,\mathrm{s}) = \mathrm{e}^{-\frac{\mathrm{s}}{2}} \mathrm{s}^{\frac{1}{2}+\nu} \mathrm{hypergeom}\left(\tfrac{1}{2}+\nu-\mu, 1 + 2\nu, \mathrm{s}\right),$$

$$\mathrm{W}_2(\mu,\nu,\mathrm{s}) = \mathrm{e}^{-\frac{\mathrm{s}}{2}} \mathrm{s}^{\frac{1}{2}+\nu} \mathrm{KummerU}\left(\tfrac{1}{2}+\nu-\mu, 1 + 2\nu, \mathrm{s}\right).$$

5 Airy wave functions

The Airy wave functions A_i and B_i are linearly independent solutions for w in the equation

$$w'' - sw = 0.$$

Specifically,

$$A_i(s) = c_1\ {}_0F_1(0, \tfrac{2}{3}, \tfrac{s^3}{9}) - c_2 s\ {}_0F_1(0, \tfrac{4}{3}, \tfrac{s^3}{9}),$$

$$B_i(s) = \sqrt{3}\left[c_1\ {}_0F_1(0, \tfrac{2}{3}, \tfrac{s^3}{9}) - c_2 s\ {}_0F_1(0, \tfrac{4}{3}, \tfrac{s^3}{9})\right]$$

where ${}_0F_1$ is the generalized hypergeometric function,

$$c_1 = A_i(0) \quad \texttt{and} \quad c_2 = -A_i'(0).$$

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