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นายบุญเลิศ ศรีหิรัญ

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APPLICATION OF GROUP ANALYSIS TO STOCHASTIC DIFFERENTIAL EQUATIONS

Mr. Boonlert Srihirun

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STOCHASTIC PROCESS / DETERMINING EQUATIONS / RANDOM TIME CHANGE / ADMITTED LIE GROUP OF TRANSFORMATIONS /

This thesis deals with an application of group analysis to stochastic differential equations. A new definition of an admitted Lie group of transformations for stochastic differential equations involving Brownian motion is presented. The transformation of the dependent variables involves time as well, and it is proved that Brownian motion is transformed to Brownian motion. Applications to a variety of stochastic differential equations are presented.

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Boonlert Srihirun

Contents

Chapter I

Introduction

In general, almost all differential equations are very difficult to solve explicitly. Numerical methods are frequently used with much success for obtaining approximate solutions. However, exact solutions are interesting because with their help, one can analyze the properties of the equations studied. One of the methods used for finding exact solutions of differential equations is group analysis.

A general survey of the method of group analysis can be found in Ovsiannikov (1978) and Ibragimov (1999). This technique involves the study of symmetries of equations, by which one means a local group of transformations mapping a solution of a given system of equations to another solution of the same system. Symmetries make it possible to reduce the number of dependent and independent variables in a system, and also to construct new solutions from known solutions. There are two types of solutions which can be obtained by group analysis: invariant and partially invariant solutions.

In contrast to deterministic differential equations, there have been only few attempts to apply symmetry techniques to stochastic differential equations. They fall into two general groups as outlined in the following.

Consider a system of Itô equations,

$$dx_i = f_i(t,x)dt + g_{ik}(t,x)dB_k \quad (i = 1, ..., n; k = 1, ..., r) \tag{1.1}$$

where $f_i(t,x)$ is a drift vector, $g_{ik}(t,x)$ is a diffusion matrix and $B_k (k = 1, ..., r)$ are standard Brownian motions; the repeat index k means summation.

Recall that a Brownian motion or Wiener-Levy process is a real-valued stochastic process $\{B(t)\}_{t\geq 0}$ satisfying the properties :

(1) continuity: the map s $\longmapsto B(s,\omega)$ is continuous almost surely.

(2) independent increments: if $s \leq t$, then $B(t) - B(s)$ is independent of (the past) $\mathcal{F}_s = \sigma(B(u) : u \leq s)$.

(3) stationary increments: if $s \leq t$, then $B(t) - B(s)$ and $B(t-s) - B(0)$ have the same distribution functions.

A Brownian motion is said to be standard if it satisfies the following properties :

(a) $B(0,\omega) = 0$ almost surely.

(b) $E\{B(t)\} = 0$ for all $t \geq 0$.

(c) $E\{B(t)^2\} = t$ for all $t \geq 0$.

The first approach of applications of group analysis to stochastic differential equations (Misawa (1994), Albeverio and Fei (1995), Gaeta and Quintero (1999) and Gaeta (2004)) deals with fiber-preserving transformations

$$\overline{x}_i = \phi_i(t,x,\varepsilon), \quad \overline{t} = \varphi(t,\varepsilon) \quad (i = 1, ..., n). \tag{1.2}$$

The general form of the infinitesimal generator in this approach is

$$Y = \tau(t)\partial_t + \xi_i(t,x)\partial_{x_i}, \tag{1.3}$$

where $\tau(t) = \frac{\partial\varphi}{\partial\varepsilon}(t,\varepsilon)|_{\varepsilon=0}$ and $\xi_i(t,x) = \frac{\partial\phi_i}{\partial\varepsilon}(t,x,\varepsilon)|_{\varepsilon=0}$. By using Itô's formula, this transformation maps (1.1) into the system

$$d\overline{x}_i = \overline{f}_i(\overline{t},\overline{x})dt + \overline{g}_{ik}(\overline{t},\overline{x})dB_k. \tag{1.4}$$

Recall that according to Itô's formula (see Oksendal (1998)), the evolution of a scalar function $I(t,x)$ satisfies the condition

$$dI = (I_{,t} + f_j I_{,j} + \frac{1}{2} g_{jk} g_{lk} I_{,jl})dt + I_{,j} g_{jk} dB_k. \tag{1.5}$$

The requirement that an infinitesimal transformation map every solution of (1.1) to a solution of the same system gives the definition of an admitted Lie group for stochastic differential equations. This approach has been applied to stochastic dynamical systems (Misawa (1994) and Albeverio and Fei (1995)) and to the Fokker-Planck equation (Gaeta and Quintero (1999) and Gaeta (2004)). Its weakness is that it can only be applied to fiber-preserving transformations.

The second approach (Mahomed and Wafo Soh (2001), Mahomed, Pooe and Wafo Soh (2004), Unal (2003), Unal and Sun (2004), Ibragimov, Unal and Jogréus (2004)) deals with symmetry transformations for a system of Itô differential equations involving all the dependent variables in the transformation. The general form of the infinitesimal generator in this approach is

$$Y = \tau(t,x)\partial_t + \xi_i(t,x)\partial_{x_i}. \tag{1.6}$$

This approach has been applied to scalar second-order stochastic ordinary differential equations (Mahomed and Wafo Soh (2001) and Mahomed, Pooe and Wafo Soh (2004)), to the Hamiltonian-Stratonovich dynamical control system (Unal and Sun (2004)), to Itô and Stronovich Dynamical Systems (Unal (2003)) and to the Fokker-Planck equation (Unal and Sun (2004), Ibragimov, Unal and Jogréus (2004)). There have also been attempts to involve Brownian motion in the transformation. For the transformation of the Brownian motion in Unal (2003), the following formula is applied

$$d\overline{B}_k = dB_k + \frac{1}{2}\varepsilon(\tau_{,t} + f_j\tau_{,j} + \frac{1}{2}g_{jm}g_{lm}\tau_{,jl})dB_k, \quad (k = 1, ..., r). \tag{1.7}$$

Unfortunately, there is no a strict proof that the transformation of Brownian motion satisfies the properties of the Brownian motion.

In this thesis we construct a Lie group of transformations, involving both the dependent variables and the Brownian motion in the transformations. We carefully verify that Brownian motion is transformed to Brownian motion. This should allow obtaining a correct generalization of application of group analysis to stochastic differential equations.

This thesis is organized as follows. Chapter II introduces some necessary knowledge from stochastic processes. Transformations of Brownian motion are studied in Chapter III. Chapter IV considers notations of group analysis and provides references to known facts on application of group analysis to constructing determining equations for admitted Lie groups of transformations for stochastic differential equations. Chapter V is devoted to developing a new theory of application of group analysis to stochastic differential equations with one-dimension Brownian motion. Chapter VI extends the method developed in the thesis in case of multi-Brownian motion.

Chapter II

Stochastic Processes

This chapter is devoted to developing the tools for stochastic processes used throughout this thesis. In particular, it discusses stochastic integrals with respect to Brownian motion, martingales, alternative fields and changes of time.

2.1 Stochastic Integration of Processes

Let Ω be a given set of elementary events ω ; $\mathcal{F}$ a σ-algebra of subsets of Ω and $\mathcal{P}$ a probability (or probability measure) on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, \mathcal{P})$ is called a probability space. It is assumed that the σ-algebra $\mathcal{F}$ is generated by a family of σ-algebras $\mathcal{F}_t$ $(t \geq 0)$ such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \quad \forall s \leq t, \quad s, t \in I \ ,$$

where $I = [0, T]$, $T \in (0, \infty]$.

The nondecreasing family of σ-algebras $\mathcal{F}_t$ is also called a filtration and the σ-algebra $\mathcal{F}$ is denoted by $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$. The triple $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \mathcal{P})$ is called a filtrated probability space.

A stochastic process X on $(\Omega, \mathcal{F}, \mathcal{P})$ is a collection of random variables $\{X(t)\}_{t \geq 0}$. The process $\{X(t)\}_{t \geq 0}$ is said to be adapted to $(\mathcal{F}_t)_{t \geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each t. Denoting the Borel σ-algebra on $[0, \infty)$ by $\mathcal{B}$, the process X is called measurable if $(t, \omega) \longmapsto X(t, \omega)$ is a $\mathcal{B} \otimes \mathcal{F}$-measurable mapping. The process X is said to be continuous if the trajectories $t \longmapsto X(t, \omega)$

are continuous for almost all $\omega \in \Omega$. It is called progressively measurable if $X : [0,t] \times \Omega \longmapsto \mathbb{R}$ is a $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable mapping for each $0 \leq t < \infty$. Note that a progressively measurable process is measurable and adapted.

Proposition 2.1. An adapted process that is left-continuous or right-continuous is progressively measurable.

Proof. See Albin (2001), Theorem 8.8 p.58.

From now on, unless stated otherwise, we let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion and $\mathcal{F}_t = \sigma\big(B(r); 0 \leq r \leq t\big), \quad t \geq 0$. Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of $[0,T]$ and $Y_0, Y_1, ..., Y_{n-1}$ some random variables which are adapted to $\mathcal{F}_0$, $\mathcal{F}_{t_1}$,..., $\mathcal{F}_{t_{n-1}}$ respectively and satisfy the conditions $E\{Y_0^2\}$, ... , $E\{Y_{n-1}^2\} < \infty$. The process $\{X(t)\}_{t\geq 0}$ which is defined by

$$X(t) = Y_0 I_{\{0\}}(t) + \sum_{i=1}^{n} Y_{i-1} I_{(t_{i-1},t_i]}(t), \quad t \in [0,T]$$

is called a simple process. Here, I_S denotes the characteristic function of a set S. The set of simple processes forms the class S_T. In the case $T = \infty$, there is one more requirement for a simple process: $Y_{n-1} = 0$.

Definition 2.1. For a process $X \in S_T$, the Itô integral of $\{X(t)\}_{t\in[0,T]}$ is defined by

$$\int_0^t X(s)dB(s) = X(t_m)\big(B(t) - B(t_m)\big) + \sum_{i=1}^{m} X(t_{i-1})\big(B(t_i) - B(t_{i-1})\big),$$

$t \in (t_m, t_{m+1}]$.

A stochastic process $\{X(t)\}_{t\geq 0}$ is said to belong to the class E_T, $T \in (0,\infty]$

if it is measurable and adapted to $(\mathcal{F}_t)_{t\geq 0}$ with

$$E\{\int_0^T X^2(r)dr\} < \infty.$$

Definition 2.2. For a process $X \in E_T$, the Itô integral of $\{X(t)\}_{t\in[0,T]}$ is defined in the sense of convergence in the mean

$$\int_0^t X(s)dB(s) = \lim_{n\to\infty} \int_0^t X_n(s)dB(s), \quad t \in [0,T],$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequence of simple processes such that

$$\lim_{n\to\infty} \int_0^T E\big(X_n(s) - X(s)\big)ds = 0.$$

A stochastic process $\{X(t)\}_{t\geq 0}$ is said to belong to the class P_T of predictable processes on [0,T], $T \in (0,\infty]$ if it is measurable and adapted to $(\mathcal{F}_t)_{t\geq 0}$ with

$$\mathcal{P}\{\int_0^T X^2(r)dr < \infty\} = 1.$$

Note that $S_T \subset E_T \subset P_T$.

A stochastic process X is a nonanticipating functional if it is is measurable and adapted to $(\mathcal{F}_t)_{t\geq 0}$ with

$$\mathcal{P}\{\int_0^t X^2(r)dr < \infty, \quad t \geq 0\} = 1.$$

Definition 2.3. For a process $X \in P_T$, the Itô integral of $\{X(t)\}_{t\in[0,T]}$ is defined in the sense of convergence in probability,

$$\int_0^t X(s)dB(s) = \lim_{n\to\infty} \int_0^t X_n(s)dB(s), \quad t \in [0,T],$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequence of processes which belong to the class E_T such that

$$\lim_{n\to\infty} \int_0^T \big(X_n(s) - X(s)\big)ds = 0,$$

with limit in the sense of convergence in probability.

Remark 2.1. In Albin (2001) it is proven that processes $X, Y \in P_T$ satisfy the following;

a) The Itô integral $\int_0^t X(\tau)dB(\tau)$ is well-defined for $0 \leq t \leq T$,

b) $E\Big(\big(\int_0^t X(\tau)dB(\tau)\big)^2\Big) = \int_0^t E\big(X^2(\tau)\big)d\tau$ for $0 \leq t \leq T$ (Itô isometry property),

c) $E\Big(\int_0^t X(\tau)dB(\tau) \int_0^t Y(\tau)dB(\tau)\Big) = \int_0^t E\big(X(\tau)Y(\tau)\big)d\tau$ for $0 \leq t \leq T$,

d) $\int_0^t \big(aX(\tau) + bY(\tau)\big)dB(\tau) = a\int_0^t X(\tau)dB(\tau) + b\int_0^t Y(\tau)dB(\tau)$ a.s. for all $a, b \in \mathbb{R}$ and $0 \leq t \leq T$,

e) $\int_0^t X(\tau)dB(\tau)$ is $\mathcal{F}_t$-measurable for $0 \leq t \leq T$,

f) $\Big\{\int_0^t X(\tau)dB(\tau)\Big\}_{t\geq 0}$ is continuous and progressively measurable, with probability one.

g) $X(t)B(t) = \int_0^t X(\tau)dB(\tau) + \int_0^t B(\tau)dX(\tau)$ for $0 \leq t \leq T$.

2.2 Martingale in Continuous Time

We recall the definition of a martingale.

Definition 2.4 (Martingale). A stochastic process $\{X(t)\}_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a martingale if it is integrable for any t, i.e. $E|X(t)| < \infty$

and for any t and s with $0 \leq s \leq t < \infty$, and

$$E[X(t)\big|\mathcal{F}_s] = X(s) \quad \text{a.s.} \tag{2.1}$$

For us, the most important continuous-time martingales are those $\{X(t)\}$ for which there is a subset Ω_0 of Ω which $P(\Omega_0) = 1$ such that the trajectories $t \longmapsto X(t,\omega)$ are continuous for all $\omega \in \Omega_0$. Naturally, these will be called continuous martingales.

Definition 2.5 (Stopping time). A nonnegative random variable τ, which is allowed to take the value ∞, is called a stopping time with respect to the filtration $\mathcal{F}_t = (\mathcal{F}_s)_{s\leq t}$ if for each t, the event $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$.

Let $X(t)$ be any collection of random variables indexed by $t \in [0,\infty)$ and τ a stopping time. One defines the stopping time variable X_τ on the set $\{\omega : \tau(\omega) < \infty\}$ by taking

$$X_\tau(\omega) = X(t,\omega),$$

provided $\tau(\omega) = t$.

Definition 2.6. A family $\{X_\alpha\}_{\alpha\in\Lambda}$ of random variables is said to be uniformly integrable if

$$E\Big[|X_\alpha| I_{\{|X_\alpha|>n\}}\Big] \longrightarrow 0$$

uniformly in α as $n \to \infty$. That is

$$\lim_{n\to\infty} \sup_{\alpha\in\Lambda} E\Big[|X_\alpha| I_{\{|X_\alpha|>n\}}\Big] = 0.$$

Definition 2.7 (Local martingale). An adapted process $\{X(t)\}_{t\geq 0}$ is called a local martingale if there exists a sequence of stopping times $0 \leq \tau_1 \leq \tau_2 \leq ...$ with

$$\lim_{n\to\infty} \tau_n = \infty \quad \text{a.s.}$$

and for each n, the stopping process $\{X_{t\wedge\tau_n}\}_{t\geq 0}$ is a uniformly integrable martingale in t, where $X_{t\wedge\tau_n}(\omega) = X(s,\omega)$ and $s = \min\{t, \tau_n(\omega)\}$.

Lemma 2.1. A continuous local martingale $\{X(t)\}_{t\geq 0}$ such that

$$|X_t| \leq Z_T \quad \text{a.s. for all } t \in [0,T],$$

for some random variable Z_T with $E[Z_T] < \infty$ is a martingale on $[0,T]$ for each $T > 0$.

Proof. By the property of local martingale, there exists a sequence of stopping times $0 \leq \tau_1 \leq \tau_2 \leq ...$ with $\lim\limits_{k\to\infty} \tau_k = \infty \quad$ a.s. such that $\{X_{t\wedge\tau_n}\}_{t\geq 0}$ is a uniformly integrable martingale for every n. Then $X(t\wedge\tau_k) \to X(t)$ as $k \to \infty$ a.s. Since $|X_t| \leq Z_T$ a.s.,

$$\int_\Omega I_{\{|X(t\wedge\tau_k)|>n\}}|X(t\wedge\tau_k)|dP \leq \int_{\{\omega:Z_T(\omega)>n\}} Z_T dP, \text{ for all } k.$$

Then

$$\sup_k \int_\Omega I_{\{|X(t\wedge\tau_k)|>n\}}|X(t\wedge\tau_k)|dP \leq \int_{\{\omega:Z_T(\omega)>n\}} Z_T dP.$$

But $E[Z_T] < \infty$, so that $\displaystyle\int_{\{\omega:Z_T(\omega)>n\}} Z_T dP \to 0$ as $n \to \infty$, and hence $\{X_{t\wedge\tau_k}\}_{k=1}^\infty$ is uniformly integrable for every $t \in [0,T]$. To show that $X(t)$ is integrable, let

$$p(x) = \sup_k E[|X(t\wedge\tau_k)|I_{\{|X(t\wedge\tau_k)|>x\}}].$$

Since $\{X_{t\wedge\tau_k}\}_{k=1}^\infty$ is uniformly integrable, $p(x) \to 0$ as $x \to \infty$. But

$$|X(t\wedge\tau_k)|I_{\{|X(t\wedge\tau_k)|>x\}} \to |X(t)|I_{\{|X(t)|>x\}}$$

as $k \to \infty$ a.s. Then by Fatou's lemma,

$$E[|X(t)|I_{\{|X(t)|>x\}}] \leq \lim_{k\to\infty} \inf E[|X(t\wedge\tau_k)|I_{\{|X(t\wedge\tau_k)|>x\}}] \leq p(x). \tag{2.2}$$

Then $E[|X(t)|] = E[|X(t)|I_{\{|X(t)|>x\}}] + E[|X(t)|I_{\{|X(t)|\le x\}}] \le p(x) + x$, so $X(t)$ is integrable.

It is left to show that (2.1) holds. We first show that $E\Big[|X(t\wedge\tau_k) - X(t)|\Big] \to 0$ as $k \to \infty$. Since

$$|X(t\wedge\tau_k) - X(t)| \le |X(t\wedge\tau_k) - X(t)|I_{\{|X(t\wedge\tau_k)|\le x\}}$$

$$+|X(t\wedge\tau_k)|I_{\{|X(t\wedge\tau_k)|>x\}} + |X(t)|I_{\{|X(t\wedge\tau_k)|>x\}},$$

then $E[|X(t\wedge\tau_k) - X(t)|] \le E[|X(t\wedge\tau_k) - X(t)|I_{\{|X(t\wedge\tau_k)|\le x\}}]$

$$+E[|X(t\wedge\tau_k)|I_{\{|X(t\wedge\tau_k)|>x\}}] + E[|X(t)|I_{\{|X(t\wedge\tau_k)|>x\}}].$$

Since $|X(t\wedge\tau_k) - X(t)|I_{\{|X(t\wedge\tau_k)|\le x\}} \le 2Z_T$ and $|X(t\wedge\tau_k) - X(t)| \to 0$ a.s., by the Dominated Convergence Theorem, $E[|X(t\wedge\tau_k) - X(t)|I_{\{|X(t\wedge\tau_k)|\le x\}}] \to 0$ as $k \to \infty$. And since $|X(t)|I_{\{|X(t\wedge\tau_k)|>x\}} \le Z_T$ and $|X(t)|I_{\{|X(t\wedge\tau_k)|>x\}} \to |X(t)|I_{\{|X(t)|>x\}}$, by the Dominated Convergence Theorem, and (2.2)

$$E[|X(t)|I_{\{|X(t\wedge\tau_k)|>x\}}] \to E[|X(t)|I_{\{|X(t)|>x\}}] \le p(x)$$

as $k \to \infty$. Then $E\Big[|X(t\wedge\tau_k) - X(t)|\Big] \le 3p(x)$, for sufficiently large k. By choosing x to make $p(x)$ as small as we like, we see that

$$E\Big[|X(t\wedge\tau_k) - X(t)|\Big] \to 0 \text{ as } k \to \infty. \tag{2.3}$$

Next, let $0 \le s \le t \le T$. Since $E[X(t)]$, $E[X(t\wedge\tau_1)]$, ... are finite, $E[X(t)\big|\mathcal{F}_s]$, $E[X(t\wedge\tau_1)\big|\mathcal{F}_s]$, ... exist and unique. By (2.3), $E\Big[|E[X(t\wedge\tau_k)\big|\mathcal{F}_s] - E[X(t)\big|\mathcal{F}_s]|\Big] = E\Big[|E[(X(t\wedge\tau_k) - X(t)\big|\mathcal{F}_s|\Big] \le E\Big[E[|X(t\wedge\tau_k) - X(t)|\big|\mathcal{F}_s\Big] = E\Big[|X(t\wedge\tau_k) - X(t)|\Big] \to 0$ as $k \to \infty$. Then $E[X(t\wedge\tau_k)\big|\mathcal{F}_s] \to E[X(t)\big|\mathcal{F}_s]$ in L^1. But $X(t\wedge\tau_k)$ is a martingale, so that $E[X(t\wedge\tau_k)\big|\mathcal{F}_s] = X(s\wedge\tau_k)$ for all k. Then $X(s\wedge\tau_k) \to E[X(t)\big|\mathcal{F}_s]$ in L^1. On the other hand, $X(s\wedge\tau_k) \to X(s)$ a.s. and $|X(s\wedge\tau_k)| \le Z_T$, so by the

Dominated Convergence Theorem, $E[X(s \wedge \tau_k)] \to E[X(s)]$ as $k \to \infty$, that is $X(s \wedge \tau_k) \to X(s)$ in L^1. By uniqueness of limits, $E[X(t)\big|\mathcal{F}_s] = X(s)$ a.s. We have shown that $\{X(t)\}_{t \geq 0}$ is a martingale on $[0, T]$.

Definition 2.8. A stochastic process $\{Y(t)\}_{t \in [0,T]}$ is said to be bounded if $|Y(t)| \leq K$ for all $t \in [0, T]$, with probability one, for some constant $K > 0$.

Theorem 2.1. For $X \in \bigcup_{0<T<\infty} P_T$, there exists a sequence of stopping times $0 \leq \tau_1 \leq \tau_2 \leq ...$ with $\mathcal{P}\big(\lim_{n\to\infty} \tau_n = \infty\big) = 1$ such that

$$\Big\{ \int_0^{t \wedge \tau_n} X dB \Big\}_{t \geq 0} \text{ is a bounded martingale for every } n \in \mathbb{R}.$$

In particular, the process $\Big\{ \int_0^t X dB \Big\}_{t \geq 0}$ is a local martingale.

Proof. See Albin (2001), Theorem 12.10 p.76.

2.3 Alternative Fields and Time Change

2.3.1 Alternative Fields and Stopping Time Identity

We now introduce a filtration which will play an essential role in our investigation of the representation of a stochastic integral as a time change of Brownian motion.

Let τ be a $\{\mathcal{F}_t\}$-stopping time. Set

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, \text{for all } t \geq 0\}.$$

One easily shows that $\mathcal{F}_\tau$ is σ-algebra.

Proposition 2.2. Let τ be a $\{\mathcal{F}_t\}$-stopping time. Suppose that $\{X(t)\}_{t\geq 0}$ is progressively measurable. Then X_τ is $\mathcal{F}_\tau$-measurable.

Proof. First show that Then $\{X_\tau \leq y\} \cap \{\tau \leq t\} \in \mathcal{F}_t$, for all $y \in \mathbb{R}$. Fix $t \geq 0$, construct $\varphi : \{\omega : \tau(\omega) \leq t\} \longmapsto [0,t] \times \Omega$ by $\varphi(\omega) = (\tau(\omega), \omega)$. Since τ is a stopping time, $\tau : (\{\tau \leq t\}, \mathcal{F}_t \cap \{\tau \leq t\}) \longmapsto ([0,t], \mathcal{B}([0,t]))$ is measurable, so

$$\{\omega : \varphi(\omega) \in B \times A\} = A \cap \{\omega : \tau(\omega) \in B\} \text{ for all } A \in \mathcal{F}_t \text{ and } B \in \mathcal{B}([0,t]).$$

By construction of the product measure, it follows that the map φ from $(\{\omega : \tau(\omega) \leq t\}, \mathcal{F}_t \cap \{\tau \leq t\})$ into $([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t)$ is measurable. But the map X from $([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t)$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, hence $X_\tau = X \circ \varphi$ is measurable from $(\{\tau \leq t\}, \mathcal{F}_t \cap \{\tau \leq t\})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $\{X_\tau \leq y\} \cap \{\tau \leq t\} \in \mathcal{F}_t$, for all $y \in \mathbb{R}$.

It is left to show that for all $y \in \mathbb{R}$, $\{X_\tau \leq y\} \in \mathcal{F}_\tau$. Since $\{X_\tau \leq y\} \cap \{\tau \leq n\} \in \mathcal{F}_n \subset \mathcal{F}$ for all n,

$$\{X_\tau \leq y\} = \bigcup_{n=1}^{\infty} \{X_\tau \leq y\} \cap \{\tau \leq n\} \in \mathcal{F}.$$

Then $\{X_\tau \leq y\} \in \mathcal{F}_\tau$. Hence X_τ is $\mathcal{F}_\tau$-measurable.

2.3.2 Time Change of a Local Martingale

All of the prerequisites have been assembled, and we are ready to present the main result of this section, namely that a time change of a local martingale is also a local martingale.

Proposition 2.3. Suppose that $\{\mathcal{F}_t : 0 \leq t < \infty\}$ is a filtration and the process

$\{\tau_t : 0 \le t < \infty\}$ is monotone in the sense that $\tau_s \le \tau_t$ for all $0 \le s \le t < \infty$. If

$$\{\tau_t \le u\} \in \mathcal{F}_u, \quad \text{for all } t \ge 0 \text{ and } u \ge 0,$$

and if $\{X_t, \mathcal{F}_t\}$ is a continuous local martingale, then $\{X_{\tau_t}, \mathcal{F}_{\tau_t}\}$ is also a continuous local martingale.

Proof. See Michael Steele (2001), Proposition 7.13, p.108.

Chapter III

Stochastic Integrals as Time Change of Brownian Motion

In this chapter, we prepare the mathematical tools required for defining the transformation of Brownian motion.

The constructions below are similar to (Oksendal(1998), section 8.5). Let $\eta(t, x, a)$ be a sufficiently many times continuously differentiable function and $\{X(t)\}_{t\geq 0}$ a continuous and adapted stochastic process. Since $\eta^2(t, x, a)$ is continuous, $\eta^2(t, X(t, \omega), a)$ is also an adapted process. Define

$$\beta(t, \omega, a) = \int_0^t \eta^2(s, X(s, \omega), a)ds, \quad t \geq 0. \tag{3.1}$$

For brevity we write $\beta(t)$ instead of $\beta(t, \omega, a)$. The function $\beta(t)$ is called a random time change with time change rate $\eta^2(t, X(t, \omega), a)$. Note that $\beta(t)$ is an adapted process. Suppose now that $\eta(t, x, a) \neq 0$ for all (t, x, a). Then for each ω, the map $t \longmapsto \beta(t)$ is strictly increasing. Next define

$$\alpha(t, \omega, a) = \inf_{s\geq 0}\{s : \beta(s, \omega, a) > t\}, \tag{3.2}$$

and for brevity, write $\alpha(t)$ instead of $\alpha(t, \omega, a)$. For almost all ω, the map $t \longmapsto \alpha(t)$ is nondecreasing and continuous. One easily shows that for almost all ω, and for all $t \geq 0$,

$$\beta(\alpha(t)) = t = \alpha(\beta(t)). \tag{3.3}$$

Since $\beta(t)$ is an $\mathcal{F}_t$-adapted process, one has

$$\{\omega : \alpha(t) \leq s\} = \{\omega : t \leq \beta(s)\} \in \mathcal{F}_s, \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

Hence $t \longmapsto \alpha(t)$ is an $\mathcal{F}_s$-stopping time for each t.

The following theorem will be crucial for defining the transformation of a Brownian motion.

Theorem 3.1. Let $\eta(t,x,a)$ and $\{X(t)\}_{t\geq 0}$ be as above and $\{B(t)\}_{t\geq 0}$ a standard Brownian motion. Define

$$\bar{B}(t) = \int_0^t \eta(s, X(s,\omega), a)dB(s), \quad t \geq 0. \tag{3.4}$$

Then $(\bar{B}_{\alpha(t)}, \mathcal{F}_{\alpha(t)})$ is a standard Brownian motion, where

$$\mathcal{F}_{\alpha(t)} = \{A \in \mathcal{F} : A \cap \{\omega : \alpha(t) \leq s\} \in \mathcal{F}_s, \text{for all } s \geq 0\}.$$

Proof. Obviously, $\bar{B}(0) = 0$ a.s. Next we show that $\{\bar{B}_{\alpha(t)}\}_{t\geq 0}$ is continuous. Since the process $\{\eta^2(s, X(s,\omega), a)\}_{s\geq 0}$ is continuous,

$$\mathcal{P}(\{\omega : \int_0^T \eta^2(s, X(s,\omega), a)ds < \infty\}) = 1, \tag{3.5}$$

for each $0 \leq T < \infty$. Because $X(t)$ is $\mathcal{F}_t$-measurable and the function $\eta(t,x,a)$ is continuous, $\eta(t, X(t,\omega), a)$ is also $\mathcal{F}_t$-measurable. Hence $\{\eta(t, X(t,\omega), a)\}_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$. Since the process $\{\eta(t, X(t,\omega), a)\}_{t\geq 0}$ is continuous, it is progressively measurable, hence measurable and adapted. Thus $\eta \in P_T$, for all $T \geq 0$. By Remark 2.1, the Itô process $\left\{\int_0^t \eta(s, X(s,\omega), a)dB(s)\right\}_{t\geq 0}$ is continuous. Since for almost all ω, the map $t \longmapsto \alpha(t)$ is continuous, $\{\bar{B}_{\alpha(t)}\}_{t\geq 0}$ is also continuous.

It left to show that $\{\bar{B}_{\alpha(t)}\}_{t\geq 0}$ has independent and stationary increment. Let $Z_t = \exp\left(i\theta \int_0^t \eta(s, X(s,\omega), a)dB(s) + \frac{1}{2}\theta^2 \int_0^t \eta^2(s, X(s,\omega), a)ds\right)$. Then the process $\{Z_t\}_{t\geq 0}$ is continuous. Now write (3.4) as

$$\bar{B}(t) = \bar{B}(0) + \int_0^t \eta(s, X(s,\omega), a)dB(s).$$

Using the function $F(x) = e^{i\theta x}$ and Itô's formula, one has

$$e^{i\theta \bar{B}(t)} = e^{i\theta \bar{B}(0)} - \int_0^t \frac{1}{2}\theta^2 \eta e^{i\theta \bar{B}(s)} ds + \int_0^t i\theta\eta e^{i\theta \bar{B}(s)} dB(s).$$

Similarly, writing

$$\beta(t) = \beta(0) + \int_0^t \eta^2(s, X(s,\omega), a) ds,$$

and using the function $G(x) = e^{\frac{1}{2}\theta^2 x}$ and the Fundamental Theorem of Calculus, one has

$$e^{\frac{1}{2}\theta^2\beta(t)} = e^{\frac{1}{2}\theta^2\beta(0)} + \int_0^t \frac{1}{2}\theta^2\eta^2 e^{\frac{1}{2}\theta^2\beta(s)} ds.$$

Then by Itô's formula, one has

$$Z_t - Z_0 = e^{i\theta\bar{B}(t)} e^{\frac{1}{2}\theta^2\beta(t)} - e^{i\theta\bar{B}(0)} e^{\frac{1}{2}\theta^2\beta(0)}$$

$$= \int_0^t \frac{1}{2}\theta^2\eta^2 e^{i\theta\bar{B}(s)} e^{\frac{1}{2}\theta^2\beta(s)} ds - \int_0^t \frac{1}{2}\theta^2\eta^2 e^{i\theta\bar{B}(s)} e^{\frac{1}{2}\theta^2\beta(s)} ds$$
$$+ \int_0^t i\theta\eta e^{i\theta\bar{B}(s)} e^{\frac{1}{2}\theta^2\beta(s)} dB(s) = \int_0^t i\theta\eta Z_s dB(s),$$

which shows that Z_t is an $dB(t)$ integral. Hence Z_t is a continuous local martingale. Since for each ω, the map $t \longmapsto \alpha(t)$ is nondecreasing and continuous, and $t \longmapsto \alpha(t)$ is an $\mathcal{F}_s$-stopping time for each t, it follows from proposition 2.3 that $(Z_{\alpha(t)}, \mathcal{F}_{\alpha(t)})$ is also a continuous local martingale. Fix $T \in (0,\infty)$. Then for almost all ω, one has

$$| Z_{\alpha(t)} | = \left| \exp\Big(i\theta \int_0^{\alpha(t)} \eta(s, X(s,\omega), a) dB(s) + \frac{1}{2}\theta^2 \int_0^{\alpha(t)} \eta^2(s, X(s,\omega), a) ds \Big) \right|$$

$$= \exp\Big(\frac{1}{2}\theta^2 \int_0^{\alpha(t)} \eta^2(s, X(s,\omega), a) ds \Big) = \exp\Big(\frac{1}{2}\theta^2 \beta(\alpha(t)) \Big)$$
$$= \exp(\frac{1}{2}\theta^2 t) \leq \exp(\frac{1}{2}\theta^2 T),$$

for all $t \in [0,T]$. It follows from lemma 2.1 that $(Z_{\alpha(t)}, \mathcal{F}_{\alpha(t)})$ is a martingale on $[0,T]$. Since the process $\{\bar{B}_t\}_{t\geq 0}$ is continuous and adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, by proposition 2.1, it is progressively measurable. Because $t \longmapsto \alpha(t)$ is an $\mathcal{F}_s$-stopping time for each t, $\{\bar{B}_{\alpha(s)}\}_{s\geq 0}$ is adapted to $\{\mathcal{F}_{\alpha(s)}\}_{s\geq 0}$, hence $\exp(-i\theta\bar{B}_{\alpha(s)})$ is adapted to $\{\mathcal{F}_{\alpha(s)}\}_{s\geq 0}$. Thus $\exp(-i\theta\bar{B}_{\alpha(s)})$ is $\mathcal{F}_{\alpha(s)}$-measurable. Since $\exp(-\frac{1}{2}\theta^2 t)$ does not depend on ω, one has

$$E\Big[e^{-\frac{1}{2}\theta^2 t}e^{-i\theta\bar{B}_{\alpha(s)}}Z_{\alpha(t)}\Big|\mathcal{F}_{\alpha(s)}\Big] = e^{-\frac{1}{2}\theta^2 t}e^{-i\theta\bar{B}_{\alpha(s)}}E\Big[Z_{\alpha(t)}\Big|\mathcal{F}_{\alpha(s)}\Big],$$

a.s. for all $0 \leq s \leq t \leq T$. Now by the martingale property,

$$E\Big[Z_{\alpha(t)}\Big|\mathcal{F}_{\alpha(s)}\Big] = Z_{\alpha(s)} \quad \text{a.s. for all } 0 \leq s \leq t \leq T.$$

and hence for all $0 \leq s \leq t \leq T$,

$$E\Big[e^{i\theta(\bar{B}_{\alpha(t)}-\bar{B}_{\alpha(s)})}\Big|\mathcal{F}_{\alpha(s)}\Big] = E\Big[Z_{\alpha(t)}e^{-\frac{1}{2}\theta^2 t}e^{-i\theta\bar{B}_{\alpha(s)}}\Big|\mathcal{F}_{\alpha(s)}\Big]$$

$$= e^{-\frac{1}{2}\theta^2 t}e^{-i\theta\bar{B}_{\alpha(s)}}Z_{\alpha(s)} = e^{-\frac{1}{2}\theta^2 t}e^{-i\theta\bar{B}_{\alpha(s)}}e^{i\theta\bar{B}_{\alpha(s)}}e^{\frac{1}{2}\theta^2 s} = e^{-\frac{1}{2}\theta^2(t-s)} \quad \text{a.s.}$$

Claim. Let $\mathcal{F}'$ and $\mathcal{F}''$ be σ-algebras such that $\mathcal{F}' \subset \mathcal{F}''$. Suppose Y is an $\mathcal{F}''$-measurable random variable such that $E[\exp(i\theta Y)\big|\mathcal{F}'] = \exp(-\frac{1}{2}\theta^2\sigma^2)$. Then Y is independent of $\mathcal{F}'$ and it has the Gaussian distribution with mean zero and variance σ^2.

Proof of the claim : Let $A \in \mathcal{F}'$. Calculate $E[\exp(i\theta Y)I_A]$ as follows:

$$E[\exp(i\theta Y)I_A] = \int_\Omega \exp(i\theta Y)I_A dP = \int_A \exp(i\theta Y)dP = \int_A E[\exp(i\theta Y)\big|\mathcal{F}']dP$$

$$= \int_A \exp(-\frac{1}{2}\theta^2\sigma^2)dP = \exp(-\frac{1}{2}\theta^2\sigma^2)\int_A dP = \exp(-\frac{1}{2}\theta^2\sigma^2)P(A).$$

Consider the characteristic function of Y whose distribution function F_Y is given by

$$\phi(\theta) = \int_{\mathbb{R}} \exp(i\theta x)dF_Y(x) = E[\exp(i\theta Y)],$$

where $F_Y : \mathbb{R} \to [0,1]$ is defined by $F_Y(x) = P(Y \leq x)$ for all $x \in \mathbb{R}$. By taking $A = \Omega$, it is readily computed by

$$\phi(\theta) = E[\exp(i\theta Y)] = \exp(-\frac{1}{2}\theta^2\sigma^2).$$

This formula provides the characteristic of a Gaussian distribution with mean 0 and variance σ^2. To show that Y is independent of $\mathcal{F}'$, let $A \in \mathcal{F}'$ be such that $P(A) > 0$. Define $P_A : \mathcal{F}'' \to [0,1]$ by

$$P_A(B) = \frac{P(A \cap B)}{P(A)}, \quad \text{for } B \in \mathcal{F}''.$$

One easily shows that P_A is a probability measure, so that $(\Omega, \mathcal{F}'', P_A)$ is a probability space. Consider the characteristic function of Y whose distribution function G_Y is given by

$$\overline{\phi}(\theta) = \int_{\mathbb{R}} \exp(i\theta x) dG_Y(x) = E_A[\exp(i\theta Y)],$$

where $G_Y : \mathbb{R} \to [0,1]$ is defined by $G_Y(x) = P_A(Y \leq x)$ for all $x \in \mathbb{R}$. One easily shows that for an $\mathcal{F}''$-measurable random variable Z,

$$E_A(Z) = \frac{E(Z1_A)}{P_A}.$$

But $\exp(i\theta Y)$ is $\mathcal{F}''$-measurable, and

$$E_A(\exp(i\theta Y)) = \frac{E(\exp(i\theta Y)1_A)}{P_A} = \frac{\exp(-\frac{1}{2}\theta^2\sigma^2)P_A}{P_A} = \exp(-\frac{1}{2}\theta^2\sigma^2),$$

so

$$\overline{\phi}(\theta) = E_A[\exp(i\theta Y)] = \exp(-\frac{1}{2}\theta^2\sigma^2).$$

This formula provides the characteristic of a Gaussian distribution with mean 0 and variance σ^2. By uniqueness of the characteristic function, one gets

$$P_A(Y \leq x) = G_Y(x) = F_Y(x) = P(Y \leq x) \quad \text{for all } x \in \mathbb{R},$$

thus

$$P(A \cap \{Y \leq x\}) = P(A)P_A(Y \leq x) = P(A)P(Y \leq x) \quad \text{for all } x \in \mathbb{R}.$$

This show that Y is independent of any $A \in \mathcal{F}'$ with $P(A) > 0$. But Y is independent of any $A \in \mathcal{F}'$ with $P(A) = 0$, so Y is independent of $\mathcal{F}'$.

By the claim, $\bar{B}_{\alpha(t)} - \bar{B}_{\alpha(s)}$ is independent of $\mathcal{F}_{\alpha(s)}$ and it has Gaussian distribution with mean zero and variance $t - s$, for all $0 \leq s \leq t < \infty$. It follows that $(\bar{B}_{\alpha(t)}, \mathcal{F}_{\alpha(t)})$ is a standard Brownian motion.

Chapter IV

Group Analysis

In this chapter, the group analysis method is discussed as it applies to stochastic processes. A general introduction to this method can be found in various textbooks (cf. Ovsiannikov (1978), Handbook of Lie group analysis (1994), (1995), (1996)).

4.1 Lie Group of Transformations for Stochastic Processes

Assume that the set of transformations

$$\bar{t} = H(t, x, a), \quad \bar{x} = \varphi(t, x, a) \tag{4.1}$$

composes a Lie group. Let $h(t, x) = H_a(t, x, 0)$, $\xi(t, x) = \varphi_a(t, x, 0)$ be the coefficients of the infinitesimal generator

$$h(t, x)\partial_t + \xi(t, x)\partial_x.$$

According to Lie's theorem, the functions $H(t, x, a)$ and $\varphi(t, x, a)$ satisfy the Lie equations

$$\frac{\partial H}{\partial a} = h(H, \varphi), \quad \frac{\partial \varphi}{\partial a} = \xi(h, \varphi) \tag{4.2}$$

and the initial conditions for $a = 0$:

$$H(t, x, 0) = t, \quad \varphi(t, x, 0) = x. \tag{4.3}$$

Since $H_t(t, x, 0) = 1$, then $H_t(t, x, a) > 0$ in a neighborhood of $a = 0$, where one can find a function $\eta(t, x, a)$ such that

$$\eta^2(t, x, a) = H_t(t, x, a).$$

Using the function $\varphi(t,x,a)$, one can define a transformation $\bar{X}(\bar{t},\omega)$ of a stochastic process $X(t,\omega)$ by

$$\bar{X}(\bar{t},\omega) = \varphi\Big(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a\Big), \tag{4.4}$$

where the functions $\beta(t)$ and $\alpha(\bar{t})$ are as in formulae (3.1) and (3.2). This gives an action of Lie group (4.1) on the set of stochastic processes. Replacing $\bar{t}$ by $\beta(t)$, one gets

$$\bar{X}(\beta(t),\omega) = \varphi(t, X(t,\omega), a).$$

In calculations of an admitted Lie group of transformations[1] it is useful to introduce the function

$$\tau(t,x) = \eta_a(t,x,0).$$

Notice that the functions $h(t,x)$ and $\tau(t,x)$ are related by the formulae

$$\tau(t,x) = \frac{h_t(t,x)}{2}, \quad h(t,x) = 2\int_0^t \tau(s,x)ds.$$

Similar to partial differential equations, the functions $\tau(t,x)$ and $\xi(t,x)$ define a Lie group of transformations for stochastic processes. In fact, given $\tau(s,x)$ and $\xi(s,x)$, one sets

$$h(t,x) = 2\int_0^t \tau(s,x)ds.$$

Solving the Lie equations (4.2) with initial conditions (4.3), one then finds the functions $H(t,x,a)$ and $\varphi(t,x,a)$.

4.2 Determining Equations

This section is devoted to constructing determining equations of an admitted Lie group of transformations for stochastic differential equations.

[1] The proper definition of an admitted Lie group of transformation will be given in the next section.

Let the set of transformations (4.1) compose a Lie group. Assume that $X(t,\omega)$ is a continuous and adapted stochastic process satisfying the equation

$$X(t,\omega) = X(0,\omega) + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dB(s), \tag{4.5}$$

where the drift vector $f = (f_1, ..., f_n)$ and the diffusion matrix $g = (g_{ik})_{n\times r}$ are given functions, $B = (B_1, ..., B_r)$ is multi-Brownian motion, $\int_0^t f(s, X(s))dt$ is a Riemann integral and $\int_0^t g(s, X(s))dB(s)$ is an Itô integral.

Theorem 4.1 (Existence and Uniqueness). Suppose that the coefficients f and g of equation (4.5) with initial condition $X(0) = x_0$ satisfy a space-variable Lipchitz condition

$$|f(t,x) - f(t,y)|^2 + |g(t,x) - g(t,y)|^2 \leq k|x-y|^2,$$

and the spatial growth condition,

$$|f(t,x)|^2 + |g(t,x)|^2 \leq k\big(1 + |x|^2\big),$$

for some positive constant k. Then there exists a continuous adapted solution X_t of equation (4.5) that is uniformly bounded in $L^2(d\mathcal{P})$:

$$\sup_{0\leq t\leq T} E\big(X_t^2\big) < \infty.$$

Moreover, if X_t and Y_t are both continuous L^2 bounded solutions of equation (4.5) then

$$\mathcal{P}\big(X_t = Y_t \text{ for all } t \in [0,T]\big) = 1.$$

Proof. See Michael Steele (2001), Theorem 9.1 p.142.

In chapter III, it was proven that the processes

$$\bar{B}_k(t) = \int_0^{\alpha(t)} \eta(s, X(s,\omega), a)dB_k(s), \quad t \geq 0, \quad (k = 1, ..., r)$$

are standard Brownian motions. Consider

$$\psi(t,\omega) = \bar{X}(\beta(t),\omega),$$

where

$$\bar{X}(t,\omega) = \varphi(\alpha(t), X(\alpha(t),\omega), a)$$

is the transformation of the stochastic process $X(t,\omega)$ given by (4.4). In the previous section it was shown that for almost all ω, there is a relation

$$\psi(t,\omega) = \varphi(t, X(t,\omega), a).$$

Due to Itô's formula, one has

$$\begin{aligned}\psi(t,\omega) = \psi(0,\omega) + \int_0^t (\varphi_{,t} + f_j\varphi_{,j} + \frac{1}{2}g_{jk}g_{lk}\varphi_{,jl})(s, X(s,\omega), a)ds \\ + \int_0^t g_{jk}\varphi_{,j}(s, X(s,\omega), a)dB_k(s).\end{aligned} \tag{4.6}$$

Because $X(t,\omega)$ is a solution of (4.5) and $\varphi_x(t,x,a)$ is a continuous function, $g(t,X(t,\omega))\varphi_x(t,X(t,\omega),a)$ is a continuous process and $g\varphi_x$ is a nonanticipating functional. According to the time change formula for Itô integrals (McKean (1969)), a nonanticipating functional e with

$$\mathcal{P}\Big(\int_0^t e^2 ds + \int_0^t \eta^2 ds < \infty, \quad t \geq 0\Big) = 1$$

satisfies the formula

$$\int_0^{\alpha(t)} e(s,\omega)dB(s) = \int_0^t e(\alpha(s),\omega)\frac{1}{\eta(\alpha(s), X(\alpha(s),\omega), a)}d\bar{B}(s). \tag{4.7}$$

Applying this formula to the last term of equation (4.6), one obtains

$$\begin{aligned}\psi_i(t,\omega) &= \psi_i(0,\omega) \\ &+ \int_0^{\beta(t)} (\varphi_{i,t} + f_j\varphi_{i,j} + \frac{1}{2}g_{jk}g_{lk}\varphi_{i,jl})\Big(\alpha(s), X(\alpha(s),\omega), a\Big)\alpha_{\bar{t}}(s)ds \\ &+ \int_0^{\beta(t)} \frac{g_{jk}\varphi_{i,j}}{\eta}\Big(\alpha(s), X(\alpha(s),\omega), a\Big)d\bar{B}_k(s), \quad (i = 1, ..., n).\end{aligned} \tag{4.8}$$

Since $\beta(t,\omega,a)=\displaystyle\int_0^t \eta^2(s,X(s,\omega),a)ds$ and $\beta(\alpha(\bar t))=\bar t$ for almost all ω, then

$$\eta^2(\alpha(\bar t),X(\alpha(\bar t),\omega),a)\alpha_{\bar t}(\bar t)=1.$$

This gives

$$\alpha_{\bar t}(s)=\eta^{-2}(\alpha(s),X(\alpha(s),\omega),a). \tag{4.9}$$

Substitution of $\alpha_{\bar t}(s)$ into (4.8) leads to the equation

$$\begin{aligned}\psi_i(t,\omega)&=\psi_i(0,\omega)\\&+\int_0^{\beta(t)}\Big(\frac{\varphi_{i,t}+f_j\varphi_{i,j}+\frac{1}{2}g_{jk}g_{lk}\varphi_{i,jl}}{\eta^2}\Big)\Big(\alpha(s),X(\alpha(s),\omega),a\Big)ds\\&+\int_0^{\beta(t)}\frac{g_{jk}\varphi_{i,j}}{\eta}\Big(\alpha(s),X(\alpha(s),\omega),a\Big)d\bar B_k(s),\quad (i=1,...,n).\end{aligned}\tag{4.10}$$

Requiring that transformations (4.2) map a solution of equation (4.5) into a solution of the same equation, one obtains

$$\bar X(\bar t,\omega)=\bar X(0,\omega)+\int_0^{\bar t} f(s,\bar X(s,\omega))ds+\int_0^{\bar t} g(s,\bar X(s,\omega))d\bar B(s).$$

Substituting $\bar t=\beta(t)$ into this equation, one gets

$$\bar X(\beta(t),\omega)=\bar X(0,\omega)+\int_0^{\beta(t)} f(s,\bar X(s,\omega))ds+\int_0^{\beta(t)} g(s,\bar X(s,\omega))d\bar B(s). \tag{4.11}$$

Equations (4.10) and (4.11) will certainly be equal if the integrands of the two Riemann integrals as well those of the Itô integrals coincide. Comparing the Riemann and Itô integrals, respectively, one obtains,

$$\begin{aligned}(\varphi_{i,t}+f_j\varphi_{i,j}+\tfrac{1}{2}g_{jk}g_{lk}\varphi_{i,jl})&\Big(\alpha(t),X(\alpha(t),\omega),a\Big)\\&=f_i(t,\bar X(t,\omega))\eta^2\Big(\alpha(t),X(\alpha(t),\omega),a\Big),\end{aligned}\tag{4.12}$$

$$g_{jk}\varphi_{i,j}\Big(\alpha(t),X(\alpha(t),\omega),a\Big)=g_{ik}(t,\bar X(t,\omega))\eta\Big(\alpha(t),X(\alpha(t),\omega),a\Big), \tag{4.13}$$

$$(i=1,...,n;k=1,...,r).$$

Since $\bar{X}(\bar{t},\omega)=\varphi(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a)$, equations (4.12) and (4.13) become

$$\begin{aligned}(\varphi_{i,t}+f_j\varphi_{i,j}+\tfrac{1}{2}g_{jk}g_{lk}\varphi_{i,jl})&(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a)\\ &=f_i(\bar{t},\varphi(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a))\eta^2(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a),\end{aligned} \tag{4.14}$$

$$g_{jk}\varphi_{i,j}(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a)=g_{ik}(\bar{t},\varphi(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a))\eta(\alpha(\bar{t}),X(\alpha(\bar{t}),\omega),a), \tag{4.15}$$

$$(i=1,...,n;k=1,...,r).$$

Substituting $\bar{t}=\beta(t)$ into equations (4.14) and (4.15) and using (3.2), equations (4.14) and (4.15) can be rewritten as

$$(\varphi_{i,t}+f_j\varphi_{i,j}+\frac{1}{2}g_{jk}g_{lk}\varphi_{i,jl})(t,X(t,\omega),a)=f_i(\beta(t),\varphi(t,X(t,\omega),a))\eta^2(t,X(t,\omega),a), \tag{4.16}$$

$$g_{jk}\varphi_{i,j}(t,X(t,\omega),a)=g_{ik}(\beta(t),\varphi(t,X(t,\omega),a))\eta(t,X(t,\omega),a), \tag{4.17}$$

$$(i=1,...,n;k=1,...,r).$$

Differentiating equations (4.16) and (4.17) with respect to the parameter a, one obtains the equations

$$\begin{aligned}\Big(\varphi_{i,ta}+f_j\varphi_{i,ja}+\tfrac{1}{2}g_{jk}g_{lk}\varphi_{i,jla}\Big)&(t,X(t,\omega),a)\\ &=\Big(\eta^2(f_{i,t}\tfrac{\partial\beta}{\partial a}+f_{i,j}\varphi_{j,a})+2f_i\eta\eta_a\Big)(t,X(t,\omega),a),\end{aligned} \tag{4.18}$$

$$g_{jk}\varphi_{i,ja}(t,X(t,\omega),a)=\Big(g_{ik}\eta_a+\eta(g_{ik,t}\frac{\partial\beta}{\partial a}+g_{ik,j}\varphi_{j,a})\Big)(t,X(t,\omega),a), \tag{4.19}$$

$$(i=1,...,n;k=1,...,r).$$

Substituting $a=0$ into equations (4.18) and (4.19), one has

$$\begin{aligned}(\tfrac{\partial\varphi_i}{\partial a}\Big|_{a=0})_t+f_j(\tfrac{\partial\varphi_i}{\partial a}\Big|_{a=0})_{,j}&+\tfrac{1}{2}g_{jk}g_{lk}(\tfrac{\partial\varphi_i}{\partial a}\Big|_{a=0})_{,jl}\\ &=f_{i,t}\tfrac{\partial\beta}{\partial a}\Big|_{a=0}+f_{i,j}\tfrac{\partial\varphi_j}{\partial a}\Big|_{a=0}+2f_i\tfrac{\partial\eta}{\partial a}\Big|_{a=0},\end{aligned} \tag{4.20}$$

$$g_{jk}(\frac{\partial\varphi_i}{\partial a}\Big|_{a=0})_{,j}=g_{ik,t}\frac{\partial\beta}{\partial a}\Big|_{a=0}+g_{ik}\frac{\partial\eta}{\partial a}\Big|_{a=0}+g_{ik,j}\frac{\partial\varphi_j}{\partial a}\Big|_{a=0}, \tag{4.21}$$

$$(i = 1, ..., n; k = 1, ..., r).$$

Since $\beta(t, \omega, a) = \int_0^t \eta^2(s, X(s, \omega), a)ds$ for all $t \geq 0$, differentiating this with respect to a, one finds

$$\frac{\partial \beta}{\partial a}\Big|_{a=0} = 2\int_0^t \frac{\partial \eta}{\partial a}\Big|_{a=0} ds.$$

Substituting $\frac{\partial \beta}{\partial a}\big|_{a=0}$ into equations (4.20) and (4.21), one arrives at the following equations

$$\begin{gathered}
\xi_{i,t}(t, X(t,\omega)) + f_j \xi_{i,j}(t, X(t,\omega)) + \tfrac{1}{2} g_{jk} g_{lk} \xi_{i,jl}(t, X(t,\omega)) \\
-2f_{i,t}(t, X(t,\omega)) \int_0^t \tau(s, X(s,\omega))ds - f_{i,j}\xi_j(t, X(t,\omega)) - 2f_i\tau(t, X(t,\omega)) = 0, \\
g_{jk}\xi_{i,j}(t, X(t,\omega)) - 2g_{ik,t}(t, X(t,\omega)) \int_0^t \tau(s, X(s,\omega))ds \\
-g_{ik}\tau(t, X(t,\omega)) - g_{ik,j}\xi_j(t, X(t,\omega)) = 0, \\
(i = 1, ..., n; k = 1, ..., r).
\end{gathered} \tag{4.22}$$

Equations (4.22) are integro-differential equations for the functions $\tau(t, x)$ and $\xi(t, x)$. These equations have to be satisfied for any solution $X(t, \omega)$ of the stochastic differential equation (4.5). Similar to integro-differential equations (Grigoryev and Meleshko (1990) and Meleshko (2005)) one can define an admitted Lie group by using the determining equations (4.22).

Definition 4.1. A Lie group of transformations (4.1) is called admitted by stochastic differential equation (4.5), if for any solution $X(t, \omega)$ of (4.5) the functions $\xi(t, x)$ and $\tau(t, x)$ satisfy the determining equations (4.22).

The determining equations for an admitted Lie group of transformations were constructed under the assumption that transformations (4.1) transforms any solution of equation (4.5) into a solution of the same equation.

Assume that one has found the functions $\tau(t, x)$ and $\xi(t, x)$ which are solutions of the determining equations (4.22). Then the Lie group of transformations (4.1) is recovered by solving the Lie equations

$$\frac{\partial H}{\partial a}(t,x,a) = h(H,\varphi),$$

$$\frac{\partial \varphi}{\partial a}(t,x,a) = \xi(H,\varphi),$$

with the initial conditions for $a = 0$;

$$H = t, \quad \varphi = x,$$

where $h(t,x) = 2\int_0^t \tau(s,x)ds$, and

$$\eta^2 = \frac{\partial H}{\partial t}.$$

Chapter V

Applications to Itô Equations

In this chapter, the theory developed is applied to some stochastic differential equations.

5.1 Geometric Brownian Motion

Let μ and $\sigma > 0$ be constants. Consider the equation discussed in Chalasani and Jha (1996),

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t). \tag{5.1}$$

The solution of equation (5.1) with the initial condition $X(0) = X_0$ is called geometric Brownian motion. For equation (5.1), the functions in equation (4.22) are $f = \mu x$, $g = \sigma x$. The system of determining equations for equation (5.1) becomes

$$\begin{gathered}\xi_t + \mu x \xi_x + \tfrac{1}{2}\sigma^2 x^2 \xi_{xx} - \mu \xi - 2\mu x \tau = 0, \\ \sigma x \xi_x - \sigma x \tau - \sigma \xi = 0.\end{gathered} \tag{5.2}$$

From the second equation of (5.2) one finds

$$\tau = \frac{1}{x}(x\xi_x - \xi).$$

Substituting it into the first equation of (5.2), one obtains that the function $\xi(t, x)$ has to satisfy the equation

$$\xi_t - \mu x \xi_x + \frac{1}{2}\sigma^2 x^2 \xi_{xx} + \mu \xi = 0. \tag{5.3}$$

For the sake of simplicity we study the particular class of solutions of equation (5.3) defined by the additional assumption

$$\xi(t,x) = F(x).$$

In this case the determining equation (5.3) becomes

$$-\mu x F'(x) + \frac{1}{2}\sigma^2 x^2 F''(x) + \mu F(x) = 0.$$

The general solution of the last equation is

$$F(x) = C_1 x + C_2 x^{\gamma+1}, \tag{5.4}$$

where $\gamma = \frac{2\mu}{\sigma^2} - 1$. Hence $\tau = C_2\gamma x^\gamma$, and $h(t,x) = 2C_2\gamma x^\gamma t$. Thus, the admitted generators are

$$x\partial_x, \quad x^\gamma(x\partial_x + 2\gamma t\partial_t).$$

Notice that for $\gamma = 0$ the second generator coincides with the first.

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = xe^a, \quad \bar{t} = t.$$

Applying Itô's formula to the function $\varphi = xe^{2a}$, one has

$$X(t)e^{2a} = X(0)e^{2a} + \int_0^t \mu X(s,\omega)e^{2a}ds + \int_0^t \sigma X(s,\omega)e^{2a}dB(s).$$

Because of $\bar{X}(\bar{t},\omega) = \bar{X}(t,\omega) = \varphi(t, X(t,\omega), a) = X(t,\omega)e^{2a}$, one has

$$\bar{X}(\bar{t},\omega) = \bar{X}(0,\omega) + \int_0^{\bar{t}} \mu\bar{X}(s,\omega)ds + \int_0^{\bar{t}} \sigma\bar{X}(s,\omega)dB(s).$$

This confirms that this Lie group transforms a solution of (5.1) into a solution of the same equation.

Let us construct the Lie group of transformations corresponding to the second admitted generator

$$x^\gamma(\frac{1}{\gamma}x\partial_x + 2t\partial_t),$$

where $\gamma \neq 0$. One has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2\varphi^{\gamma} H, \quad \frac{\partial \varphi}{\partial a} = \frac{1}{\gamma}\varphi^{\gamma+1},$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi = x.$$

The solution of this Cauchy problem is

$$H = t(1 + ax^{\gamma})^{-2}, \; \varphi = (a + x^{-\gamma})^{-\frac{1}{\gamma}}.$$

Then $\eta^2 = H_t = (1 + ax^{\gamma})^{-2}$. The transformations of the independent variable t and the dependent variable x are

$$\bar{t} = H = t(1 + ax^{\gamma})^{-2}. \tag{5.5}$$

$$\bar{x} = \varphi = (a + x^{-\gamma})^{-\frac{1}{\gamma}}, \tag{5.6}$$

Let us show that the Lie group of transformations (5.5) and (5.6) transforms a solution of equation (5.1) into a solution of the same equation. Assume that $X(t)$ is a solution of equation (5.1). According to theorem 3.1, the Brownian motion $B(t)$ is transformed to the Brownian motion

$$\bar{B}(t) = \int_0^{\alpha(t)} (1 + aX^{\gamma}(s))^{-1} dB(s), \quad t \geq 0, \tag{5.7}$$

where

$$\beta(t) = \int_0^t (1 + aX^{\gamma}(s))^{-2} ds, \quad \alpha(t) = \inf_{s \geq 0}\{s : \beta(s) > t\}, \quad t \geq 0.$$

Applying Itô's formula to the function $\varphi(t, x, a) = (a + x^{-\gamma})^{-\frac{1}{\gamma}}$, one has

$$\begin{aligned} \varphi(t, X(t,\omega), a) = \varphi(0, X(0,\omega), a) + \int_0^t \mu(a + X^{-\gamma}(s,\omega))^{-\frac{1}{\gamma}}(1 + aX^{\gamma}(s,\omega))^{-2} ds \\ + \int_0^t \sigma(a + X^{-\gamma}(s,\omega))^{-\frac{1}{\gamma}}(1 + aX^{\gamma}(s,\omega))^{-1} dB(s). \end{aligned} \tag{5.8}$$

By virtue of (4.9),

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = (1 + aX^{\gamma}(\alpha(\bar{t})))^2.$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integral in (5.8), it becomes

$$\int_0^t \mu(a + X^{-\gamma}(s))^{-\frac{1}{\gamma}}(1 + aX^{\gamma}(s))^{-2} ds = \int_0^{\beta(t)} \mu[a + X^{-\gamma}(\alpha(\bar{s}))]^{-\frac{1}{\gamma}} d\bar{s}.$$

Because of the transformation of the Brownian motion (5.7) and the change of Brownian motion in the Itô integral (4.7), the Itô integral in (5.8) becomes

$$\int_0^t \sigma(a + X^{-\gamma}(s))^{-\frac{1}{\gamma}}(1 + aX^{\gamma}(s))^{-1} dB(s) = \int_0^{\beta(t)} \sigma[a + X^{-\gamma}(\alpha(\bar{s}))]^{-\frac{1}{\gamma}} d\bar{B}(\bar{s}).$$

Since $\Big(a + X^{-\gamma}(\alpha(\bar{t}), \omega)\Big)^{-\frac{1}{\gamma}} = \varphi\Big(\alpha(\bar{t}), X(\alpha(\bar{t}), \omega), a\Big) = \bar{X}(\bar{t}, \omega)$, one gets

$$\varphi(t, X(t, \omega), a) = \varphi(0, X(0, \omega), a) + \int_0^{\beta(t)} \mu \bar{X}(s, \omega) ds + \int_0^{\beta(t)} \sigma \bar{X}(s, \omega) d\bar{B}(s).$$

Because $\bar{X}(\beta(t), \omega) = \varphi(t, X(t, \omega), a)$ and $\bar{X}(0, \omega) = \varphi(0, X(0, \omega), a)$, one has

$$\bar{X}(\beta(t), \omega) = \bar{X}(0, \omega) + \int_0^{\beta(t)} \mu \bar{X}(s, \omega) ds + \int_0^{\beta(t)} \sigma \bar{X}(s, \omega) d\bar{B}(s).$$

This confirms that the Lie group of transformations (5.5) and (5.6) transforms any solution of equation (5.1) into a solution of the same equation.

5.2 Brownian Motion with Drift

Consider the equation discussed in Chalasani and Jha (1996),

$$dX(t) = \mu dt + dB(t), \tag{5.9}$$

where μ is constant. The solution of equation (5.9) with the initial condition $X(0) = X_0$ is called Brownian motion with drift. For equation (5.9), the functions in equation (4.22) are $f = \mu$, $g = 1$. The system of determining equations for equation (5.9) becomes

$$\begin{gathered} \xi_t + \mu\xi_x + \tfrac{1}{2}\xi_{xx} - 2\mu\tau = 0, \\ \xi_x - \tau = 0. \end{gathered} \tag{5.10}$$

From the second equation of (5.10) one finds

$$\tau = \xi_x.$$

Substituting it into the first equation of (5.10) one obtains that the function $\xi(t, x)$ has to satisfy the equation

$$\xi_t - \mu\xi_x + \frac{1}{2}\xi_{xx} = 0. \tag{5.11}$$

As in the previous example, for the sake of simplicity we study only the particular class of solutions of equation (5.11) defined by the assumption

$$\xi(t, x) = F(x).$$

In this case the determining equation (5.11) becomes

$$F'' - 2\mu F' = 0.$$

The general solution of the last equation depends on the value of μ. If $\mu \neq 0$, then

$$F(x) = C_1 + C_2 e^{2\mu x}. \tag{5.12}$$

Hence, $\tau = 2C_2\mu e^{2\mu x}$ and $h(t, x) = 4C_2 e^{2\mu x} t$, and a basis of generators corresponding to (5.12) is

$$\partial_x, \quad e^{2\mu x}(\partial_x + 4t\partial_t).$$

If $\mu = 0$, then

$$F(x) = C_1 + C_2 x, \tag{5.13}$$

and a basis of generators corresponding to (5.13) is ∂_x and $x\partial_x + 2t\partial_t$.

Let us construct the Lie group of transformations corresponding to the generator

$$e^{2\mu x}(\partial_x + 4t\partial_t), \quad (\mu \neq 0).$$

One has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 4\mu e^{2\mu\varphi} H, \quad \frac{\partial \varphi}{\partial a} = e^{2\mu\varphi},$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi = x.$$

The solution of this Cauchy problem is

$$H = t(1 - 2\mu a e^{2\mu x})^{-2}, \qquad \varphi = x - \frac{1}{2\mu}\ln(1 - 2\mu a e^{2\mu x}).$$

Then $\eta^2 = H_t = (1 - 2ae^{2\mu x})^{-2}$. The transformations of the independent variable t and the dependent variable x are

$$\bar{t} = H = t(1 - 2\mu a e^{2\mu x})^{-2}, \tag{5.14}$$

$$\bar{x} = \varphi = x - \frac{1}{2\mu}\ln(1 - 2\mu a e^{2\mu x}). \tag{5.15}$$

Let us show that the Lie group of transformations (5.14) and (5.15) transforms a solution of equation (5.9) into a solution of the same equation. Assume that $X(t)$ is a solution of equation (5.9). The Brownian motion $B(t)$ is transformed according to the formula

$$\bar{B}(t) = \int_0^{\alpha(t)} (1 - 2\mu a e^{2\mu X(s)})^{-1} dB(s), \quad t \geq 0, \tag{5.16}$$

where

$$\beta(t) = \int_0^t (1 - 2\mu a e^{2\mu X(s)})^{-2} ds, \quad \alpha(t) = \inf_{s \geq 0}\{s : \beta(s) > t\}, \quad t \geq 0.$$

Applying Itô's formula to the function $\varphi(t,x,a) = x - \frac{1}{2\mu}\ln(1-2\mu a e^{2\mu x})$, one has

$$\begin{aligned}\varphi(t,X(t,\omega),a) = \varphi(t,X(0,\omega),a) + \int_0^t \mu(1-2\mu a e^{2\mu X(s,\omega)})^{-2}ds \\ + \int_0^t (1-2\mu a e^{2\mu X(s,\omega)})^{-1}dB(s).\end{aligned} \tag{5.17}$$

By virtue of (4.9),

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = (1-2\mu a e^{2\mu X(s)})^2.$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integral in (5.17), it becomes

$$\int_0^t \mu(1-2\mu a e^{2\mu X(s)})^{-2}ds = \int_0^{\beta(t)} \mu d\alpha(\bar{s}).$$

Because of the transformation of the Brownian motion (5.16) and the change of Brownian motion in the Itô integral (4.7), the Itô integral in (5.17) becomes

$$\int_0^t (1-2\mu a e^{2\mu X(s)})^{-1}dB(s) = \int_0^{\beta(t)} d\bar{B}(\alpha(\bar{s})).$$

Because $\bar{X}(\beta(t),\omega) = \varphi(t,X(t,\omega),a)$ and $\bar{X}(0,\omega) = \varphi(0,X(0,\omega),a)$, one has

$$\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) + \int_0^{\beta(t)} \mu ds + \int_0^{\beta(t)} d\bar{B}(s).$$

This confirms that the Lie group of transformations (5.14) and (5.15) transforms any solution of equation (5.9) into a solution of the same equation.

Chapter VI

Applications to Systems of Itô Equations

In this chapter, we will discuss applications of group analysis for constructing determining equations for admitted Lie groups of transformations for some systems of stochastic differential equations.

6.1 One-dimensional Brownian motion

Let us consider the system of Itô equations

$$X_i(t,\omega) = X_i(0,\omega) + \int_0^t f_i(s, X(s,\omega))ds + \int_0^t g_i(s, X(s,\omega))dB(s), \quad (i = 1, ..., n) \tag{6.1}$$

where the drift rate f and the volatility g are given adapted stochastic processes and B is Brownian motion.

Definition. A Lie group of transformations (4.1) is called admitted by the stochastic differential equation (6.1), if for any solution $X(t,\omega)$ of (6.1) the functions $\xi(t,x)$ and $\tau(t,x)$ satisfy the system of determining equations

$$\begin{aligned}
&\xi_{i,t}(t,X(t,\omega)) + f_j\xi_{i,j}(t,X(t,\omega)) + \tfrac{1}{2}g_jg_k\xi_{i,jk}(t,X(t,\omega)) \\
&-2f_{i,t}(t,X(t,\omega))\int_0^t \tau(s,X(s,\omega))ds - f_{i,j}\xi_j(t,X(t,\omega)) - 2f_i\tau(t,X(t,\omega)) = 0, \\
&g_j\xi_{i,j}(t,X(t,\omega)) - 2g_{i,t}(t,X(t,\omega))\int_0^t \tau(s,X(s,\omega))ds \\
&\qquad\qquad -g_i\tau(t,X(t,\omega)) - g_{i,j}\xi_j(t,X(t,\omega)) = 0 \\
&\qquad (i = 1, ..., n).
\end{aligned} \tag{6.2}$$

The determining equations for the admitted Lie group of transformations were constructed under the assumption that the Lie group of transformations (4.1) transforms any solution of equation (6.1) into a solution of the same equation.

In the following, we present examples of systems of two equations involving a single Brownian motion. For convenience of notation, we will use the symbols X and Y instead of X_1 and X_2.

6.1.1 Narrow-sense Linear System

Let μ, ν and σ be constants, and $\sigma \neq 0$. Consider the system of equations discussed in Gard (1988),

$$\begin{aligned} dX(t) &= Y(t)dt, \\ dY(t) &= [-\nu^2 X(t) - \mu Y(t)]dt + \sigma dB(t). \end{aligned} \tag{6.3}$$

The system of equations (6.3) is called the narrow-sense linear system. For equations (6.3) the functions corresponding equations (6.2) are $f_1 = y$, $f_2 = -\nu^2 x - \mu y$, $g_1 = 0$ and $g_2 = \sigma$. The system of determining equations for system of equations (6.3) becomes

$$\begin{gathered} \xi_{1,t} + y\xi_{1,x} - (\nu^2 x + \mu y)\xi_{1,y} + \tfrac{1}{2}\sigma^2 \xi_{1,yy} - 2y\tau - \xi_2 = 0, \\ \xi_{2,t} + y\xi_{2,x} - (\nu^2 x + \mu y)\xi_{2,y} + \tfrac{1}{2}\sigma^2 \xi_{2,yy} + 2(\nu^2 x + \mu y)\tau + \nu^2 \xi_1 + \mu \xi_2 = 0, \\ \xi_{1,y} = 0, \quad \xi_{2,y} - \tau = 0. \end{gathered} \tag{6.4}$$

From the last equations of (6.4), one finds

$$\tau = \xi_{2,y}, \quad \xi_1 = \xi_1(t, x).$$

Substituting these into the remaining equations of (6.4), one obtains that the functions ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered} \xi_{1,t} + y\xi_{1,x} - 2y\xi_{2,y} - \xi_2 = 0, \\ \xi_{2,t} + y\xi_{2,x} + (\nu^2 x + \mu y)\xi_{2,y} + \tfrac{1}{2}\sigma^2 \xi_{2,yy} + \nu^2 \xi_1 + \mu \xi_2 = 0. \end{gathered} \tag{6.5}$$

Differentiating the first of equation in (6.5) with respect to y, one gets

$$\xi_{1,x} - 2y\xi_{2,yy} - 3\xi_{2,y} = 0. \tag{6.6}$$

Differentiating equation (6.6) with respect to y again, one finds

$$2y\xi_{2,yyy} + 5\xi_{2,yy} = 0,$$

thus

$$\xi_2 = H_1(t,x) + \frac{1}{\sqrt{y}}H_2(t,x) + yH_3(t,x).$$

Substituting this ξ_2 into the second equation of (6.5), one obtains

$$\frac{3}{8}\sigma^2 y^{-\frac{5}{2}}H_2 - \frac{1}{2}y^{-\frac{3}{2}}\nu^2 xH_2 + y^{-\frac{1}{2}}(\frac{1}{2}\mu H_2 + H_{2,t}) + y^{\frac{1}{2}}H_{2,x}$$

$$+y(H_{3,t} + H_{1,x} + 2\mu H_3) + y^2 H_{3,x} + H_{1,t} + \nu x^2 H_3 + \nu^2\xi_1 + \mu H_1 = 0.$$

Splitting the last equation with respect to y, one has

$$H_2 = 0, \quad H_{3,x} = 0, \quad H_{1,t} + \nu x^2 H_3 + \nu^2\xi_1 + \mu H_1 = 0, \tag{6.7}$$

so that $H_3 = f(t)$. Substituting ξ_2 into equation (6.6), one finds

$$\xi_{1,x} = 3f(t),$$

then $\xi_1 = f(t)x + g(t)$. Substituting it into the first equation of (6.5), one obtains

$$H_1 = 3xf' + g'.$$

Substituting this H_1 into the third equation of (6.7), one finds

$$x(3f'' + \nu^2 f + 3\mu f) + \nu x^2 f + g'' + \mu g' + \nu^2 g = 0.$$

Splitting the last equation with respect to x, one has

$$f(t) = 0, \quad g'' + \mu g' + \nu^2 g = 0,$$

and hence

$$g = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t},$$

where $\gamma_1 = -\frac{1}{2}(\mu + \sqrt{\mu^2 - 4\nu^2})$ and $\gamma_2 = -\frac{1}{2}(\mu - \sqrt{\mu^2 - 4\nu^2})$. Thus

$$\xi_1 = g = C_1 e^{\gamma_1 t} + C_2 e^{\gamma_2 t}, \quad \xi_2 = g' = C_1 \gamma_1 e^{\gamma_1 t} + C_2 \gamma_2 e^{\gamma_2 t}. \tag{6.8}$$

Hence $\tau = 0$, and $h = 0$. Thus, a basis of generator corresponding to (6.8) is

$$e^{\gamma_1 t}\partial_x + \gamma_1 e^{\gamma_1 t}\partial_y, \quad e^{\gamma_2 t}\partial_x + \gamma_2 e^{\gamma_2 t}\partial_y.$$

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x + ae^{\gamma_1 t}, \quad \bar{y} = y + a\gamma_1 e^{\gamma_1 t}, \quad \bar{t} = t. \tag{6.9}$$

Applying Itô's formula to the functions $\varphi_1 = x + ae^{\gamma_1 t}$ and $\varphi_2 = y + a\gamma_1 e^{\gamma_1 t}$, one has

$$\begin{aligned}
X(t) + ae^{\gamma_1 t} &= X(0) + a + \int_0^t (Y(s) + ae^{\gamma_1 s})ds,\\
Y(t) + ae^{\gamma_1 t} &= Y(0) + a + \int_0^t \big(-\nu^2(X(s) + ae^{\gamma_1 s}) - \mu(Y(s) + ae^{\gamma_1 s})\big)ds\\
&\qquad + \int_0^t \sigma dB(s).
\end{aligned}$$

Since $\bar{X}(\bar{t},\omega) = \bar{X}(t,\omega) = \varphi_1(t, X(t,\omega), a) = X(t,\omega) + ae^{\gamma_1 t}$ and $\bar{Y}(\bar{t},\omega) = \bar{Y}(t,\omega) = \varphi_2(t, X(t,\omega), a) = Y(t,\omega) + a\gamma_1 e^{\gamma_1 t}$, one has

$$\begin{aligned}
\bar{X}(\bar{t},\omega) &= \bar{X}(0,\omega) + \int_0^{\bar{t}} \bar{Y}(s,\omega)ds,\\
\bar{Y}(\bar{t},\omega) &= \bar{Y}(0,\omega) + \int_0^{\bar{t}} \big(-\nu^2\bar{X}(s,\omega) - \mu\bar{Y}(s,\omega)\big)ds + \int_0^{\bar{t}} \sigma dB(s).
\end{aligned}$$

This confirms that this Lie group transforms a solution of (6.3) into a solution of the same system.

The Lie group of transformations corresponding to the second admitted generator is

$$\bar{x} = x + ae^{\gamma_2 t}, \quad \bar{y} = y + a\gamma_2 e^{\gamma_2 t}, \quad \bar{t} = t. \tag{6.10}$$

We can show similarly as above that this Lie group transforms a solution of (6.3) into a solution of the same system.

6.1.2 Graph of Brownian Motion

Consider the system of equations discussed in Oksendal (1998),

$$\begin{aligned} dX(t) &= dt, \\ dY(t) &= dB(t). \end{aligned} \tag{6.11}$$

The solution of equations (6.11) with the initial condition $(X(0), Y(0)) = (t_0, y_0)$ may be regarded as the graph of Brownian motion. For equations (6.11) the corresponding functions of equations (6.2) are $f_1 = 1$, $f_2 = 0$, $g_1 = 0$ and $g_2 = 1$. The system of determining equations for equations (6.11) becomes

$$\begin{aligned} \xi_{1,t} + \xi_{1,x} + \tfrac{1}{2}\xi_{1,yy} - 2\tau = 0, \\ \xi_{2,t} + \xi_{2,x} + \tfrac{1}{2}\xi_{2,yy} = 0, \\ \xi_{1,y} = 0, \quad \xi_{2,y} - \tau = 0. \end{aligned} \tag{6.12}$$

From the last two equations of (6.12) one finds

$$\tau = \xi_{2,y}, \quad \xi_1 = \xi_1(t, x).$$

Substituting these values into the remaining equations of (6.12), one obtains that the functions ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{aligned} \xi_{1,t} + \xi_{1,x} - 2\xi_{2,y} = 0, \\ \xi_{2,t} + \xi_{2,x} + \tfrac{1}{2}\xi_{2,yy} = 0. \end{aligned} \tag{6.13}$$

Differentiating the first equation of (6.13) with respect to y, one gets

$$\xi_{2,yy} = 0,$$

thus

$$\xi_2 = F(t, x)y + G(t, x).$$

Substituting it into the remaining equation of (6.13), one obtains

$$y(F_t + F_x) + G_t + G_x = 0. \tag{6.14}$$

Splitting the last equation with respect to y, one has

$$F_t + F_x = 0, \quad G_t + G_x = 0.$$

The general solution of these equations is

$$F = F_1, \quad G = F_2.$$

where $F_1 = F_1(t-x)$ and $F_2 = F_2(t-x)$. Substituting ξ_2 into the first equation of (6.13), one obtains

$$\xi_{1,t} + \xi_{1,x} - 2F_1 = 0.$$

Hence

$$\xi_1 = 2xF_1 + F_3, \quad \xi_2 = yF_1 + F_2, \quad \tau = F_1, \tag{6.15}$$

where $F_3 = F_3(t-x)$. A basis of generators corresponding to (6.15) is

$$F_2\partial_y, \quad F_3\partial_x, \quad F_1(2x\partial_x + y\partial_y) + h\partial_t.$$

where $h = 2\int_0^t F_1(s-x)ds$.

Notice that if the first equation of (6.11) is considered as an ordinary differential equation (i.e., the function $X(t,\omega)$ does not depend on ω), then the functions F_1, F_2 and F_3 are constant.

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x, \quad \bar{y} = y + F_2 a, \quad \bar{t} = t. \tag{6.16}$$

It is not difficult to show that this Lie group transforms a solution of (6.11) into a solution of the same system.

Let us construct the Lie group of transformations corresponding to the third generator for the particular case defined by the assumption that $F_1 = k$, where k is constant. In this case the generator becomes

$$2x\partial_x + y\partial_y + 2t\partial_t.$$

For finding the Lie group of transformations corresponding to this generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2H, \quad \frac{\partial \varphi_1}{\partial a} = 2\varphi_1, \quad \frac{\partial \varphi_2}{\partial a} = \varphi_2,$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi_1 = x, \quad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y,

$$\bar{t} = H = te^{2a}, \quad \bar{x} = \varphi_1 = xe^{2a}, \quad \bar{y} = \varphi_2 = ye^{a}. \tag{6.17}$$

Hence $\eta^2 = H_t = e^{2a}$.

Let us show that the Lie group of transformation (6.17) transforms a solution of equations (6.11) into a solution of the same equations. Assume that $\big(X(t), Y(t)\big)$ is a solution of equations (6.11). According to theorem 3.1, the Brownian motion $B(t)$ is transformed to the Brownian motion

$$\bar{B}(t) = \int_0^{\alpha(t)} e^a dB(s), \tag{6.18}$$

where

$$\beta(t) = \int_0^t e^{2a} ds = te^{2a}, \quad \alpha(t) = \inf_{s\geq 0}\{s : \beta(s) > t\} = te^{-2a}, \quad t \geq 0.$$

Applying Itô's formula to the functions $\varphi_1(t,x,y,a) = xe^{2a}$ and $\varphi_2(t,x,y,a) = ye^a$, one has

$$\begin{aligned}\varphi_1(t,X(t,\omega),Y(t,\omega),a) &= \varphi_1(0,X(0,\omega),Y(0,\omega),a) + \int_0^t e^{2a}ds,\\ \varphi_2(t,X(t,\omega),Y(t,\omega),a) &= \varphi_2(0,X(0,\omega),Y(0,\omega),a) + \int_0^t e^{a}dB(s).\end{aligned} \tag{6.19}$$

By virtue of (4.9),

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = e^{-2a}.$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integral in (6.19), it becomes

$$\int_0^t e^{2a}ds = \int_0^{\beta(t)} d\bar{s}.$$

Because of the transformation of the Brownian motion (6.18), the Itô integral in (6.19) becomes

$$\int_0^t e^{a}dB(s) = \int_0^{\beta(t)} d\bar{B}(\bar{s}).$$

Since $\varphi_1(t,X(t,\omega),Y(t,\omega),a) = \bar{X}(\beta(t),\omega)$, and $\varphi_2(t,X(t,\omega),Y(t,\omega),a) = \bar{Y}(\beta(t),\omega)$, one has

$$\begin{aligned}\bar{X}(\beta(t),\omega) &= \bar{X}(0,\omega) + \int_0^{\beta(t)} ds,\\ \bar{Y}(\beta(t),\omega) &= \bar{Y}(0,\omega) + \int_0^{\beta(t)} d\bar{B}(s).\end{aligned}$$

This confirms that the Lie group of transformations (6.17) transforms any solution of system (6.11) into a solution of the same system.

6.1.3 Black and Scholes Market

Consider the system of equations discussed in Oksendal (1998),

$$\begin{aligned}dX(t) &= \rho X(t)dt,\\ dY(t) &= \mu Y(t)dt + \sigma Y(t)dB(t),\end{aligned} \tag{6.20}$$

where ρ, μ and σ are nonzero constants. The system of equations (6.20) with the initial condition $X(0) = 1$, $Y(0) = y > 0$ is called the Black and Scholes market. For equations (6.20) the corresponding functions of equations (6.2) are $f_1 = \rho x$, $f_2 = \mu y$, $g_1 = 0$ and $g_2 = \sigma y$. The system of determining equations for equations (6.20) becomes

$$\begin{gathered}
\xi_{1,t} + \rho x \xi_{1,x} + \mu y \xi_{1,y} + \tfrac{1}{2}\sigma^2 y^2 \xi_{1,yy} - 2\rho x \tau - \rho \xi_1 = 0,\\
\xi_{2,t} + \rho x \xi_{2,x} + \mu y \xi_{2,y} + \tfrac{1}{2}\sigma^2 y^2 \xi_{2,yy} - 2\mu y \tau - \mu \xi_2 = 0,\\
y\xi_{1,y} = 0, \quad y\xi_{2,y} - y\tau - \xi_2 = 0.
\end{gathered} \tag{6.21}$$

From the last pair of equations of (6.21), one finds

$$\xi_1 = \xi_1(t, x), \quad \tau = \xi_{2,y} - \frac{\xi_2}{y}.$$

Substituting these into the remaining equations of (6.21), one obtains that the functions ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered}
\xi_{1,t} + \rho x \xi_{1,x} - 2\rho x(\xi_{2,y} - \tfrac{\xi_2}{y}) - \rho \xi_1 = 0,\\
\xi_{2,t} + \rho x \xi_{2,x} - \mu y \xi_{2,y} + \tfrac{1}{2}\sigma^2 y^2 \xi_{2,yy} + \mu \xi_2 = 0.
\end{gathered} \tag{6.22}$$

Differentiating the first equation of (6.22) with respect to y, one gets

$$\xi_{2,yy} - \frac{1}{y}\xi_{2,y} + \frac{1}{y^2}\xi_2 = 0,$$

thus

$$\xi_2 = yF(t, x) + y \ln y G(t, x).$$

Substituting this ξ_2 into the second equation of (6.22), one obtains

$$y(F_t + \rho x F_x - (\mu - \frac{1}{2}\sigma^2)G) + y \ln y (G_t + \rho x G_x) = 0.$$

Splitting the last equation with respect to y, one has

$$G_t + \rho x G_x = 0, \quad F_t + \rho x F_x - (\mu - \frac{1}{2}\sigma^2)G = 0.$$

Thus

$$G = F_1, \quad F = \gamma \ln x F_1 + F_2$$

where $F_1 = F_1(t - \frac{\ln x}{\rho})$, $F_2 = F_2(t - \frac{\ln x}{\rho})$ and $\gamma = \frac{1}{\rho}(\mu - \frac{1}{2}\sigma^2)$. Substituting ξ_2 into the first equation of (6.22), one obtains

$$\xi_{1,t} + \rho x \xi_{1,x} - \rho \xi_1 - 2\rho x F_1 = 0.$$

Hence,

$$\xi_1 = 2x \ln x F_1 + x F_3, \quad \xi_2 = (y \ln y + \gamma y \ln x) F_1 + y F_2, \quad \tau = F_1, \tag{6.23}$$

where $F_3 = F_3(t - \frac{\ln x}{\rho})$. A basis of generators corresponding to (6.23) is

$$y F_2 \partial_y, \quad x F_3 \partial_x, \quad F_1(2x \ln x \partial_x + (y \ln y + \gamma y \ln x)\partial_y) + h\partial_t,$$

where $h = 2\displaystyle\int_0^t F_1(s - \frac{\ln x}{\rho})ds$.

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x, \quad \bar{y} = y \exp(F_2 a), \quad \bar{t} = t. \tag{6.24}$$

It is not difficult to show that this Lie group transforms a solution of (6.20) into a solution of the same system.

Let us construct the Lie group of transformations corresponding to the third generator for the particular case determined by the assumption $F_1 = k$, where k is constant. In this case the generator becomes

$$2x \ln x \partial_x + (y \ln y + \gamma y \ln x)\partial_y + 2t\partial_t.$$

For finding the Lie group of transformations corresponding to this generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2H, \quad \frac{\partial \varphi_1}{\partial a} = 2\varphi_1 \ln \varphi_1, \quad \frac{\partial \varphi_2}{\partial a} = \varphi_2 \ln \varphi_2 + \gamma \varphi_2 \ln \varphi_1,$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi_1 = x, \quad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y,

$$\bar{t} = H = te^{2a}, \quad \bar{x} = \varphi_1 = \exp\left(e^{2a}\ln x\right), \quad \bar{y} = \varphi_2 = \exp\left(\gamma e^{2a}\ln x + (\ln y - \gamma x)e^{a}\right). \tag{6.25}$$

Hence $\eta^2 = H_t = e^{2a}$.

Let us show that the Lie group of transformation (6.25) transforms a solution of equations (6.20) into a solution of the same equations. Assume that $\big(X(t), Y(t)\big)$ is a solution of equations (6.20). The Brownian motion $B(t)$ is transformed according to the formula

$$\bar{B}(t) = \int_0^{\alpha(t)} e^{a} dB(s), \tag{6.26}$$

where

$$\beta(t) = \int_0^t e^{2a} ds = te^{2a}, \quad \alpha(t) = \inf_{s\geq 0}\{s : \beta(s) > t\} = te^{-2a}, \quad t \geq 0.$$

Applying Itô's formula to the functions $\varphi_1(t, x, y, a) = \exp\left(e^{2a}\ln x\right)$ and $\varphi_2(t, x, y, a) = \exp\left(\gamma e^{2a}\ln x + (\ln y - \gamma x)e^{a}\right)$, one has

$$\begin{aligned}
\varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) \\
&\quad + \int_0^t \rho \exp\Big(\ln X(s,\omega) e^{2a}\Big) e^{2a} ds, \\
\varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a) \\
+ \int_0^t \mu \exp\Big(\gamma e^{2a}\ln X(s,\omega) &+ \big(\ln Y(s,\omega) - \gamma X(s,\omega)\big)e^{a}\Big) e^{2a} d(s) \\
+ \int_0^t \sigma \exp\Big(\gamma e^{2a}\ln X(s,\omega) &+ \big(\ln Y(s,\omega) - \gamma X(s,\omega)\big)e^{a}\Big) e^{a} dB(s).
\end{aligned} \tag{6.27}$$

By virtue of (4.9),

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = e^{-2a}.$$

Changing the variable of the integral $s=\alpha(\bar{s})$ in the Riemann integrals in (6.27), they become

$$\begin{aligned}
&\int_0^t \rho\exp\Big(e^{2a}\ln X(s)\Big)e^{2a}ds = \int_0^{\beta(t)} \rho\exp\Big(e^{2a}\ln X(\alpha(s))\Big)d\bar{s},\\
&\int_0^t \mu\exp\Big(\gamma e^{2a}\ln X(s) + \big(\ln Y(s) - \gamma X(s)\big)e^a\Big)e^{2a}d(s)\\
&= \int_0^t \mu\exp\Big(\gamma e^{2a}\ln X(\alpha(s)) + \big(\ln Y(\alpha(s)) - \gamma X(\alpha(s))\big)e^a\Big)d\bar{s}.
\end{aligned}$$

Because of the transformation of the Brownian motion (6.26) and the change of Brownian motion in the Itô integral (4.7), the Itô integral in (6.27) becomes

$$\begin{aligned}
&\int_0^t \sigma\exp\Big(\gamma e^{2a}\ln X(s) + (\ln Y(s) - \gamma X(s))e^a\Big)e^a dB(s)\\
&= \int_0^{\beta(t)} \sigma\exp\Big(\gamma e^{2a}\ln X(\alpha(s))e^{2a} + \big(\ln Y(\alpha(s)) - \gamma X(\alpha(s))\big)e^a\Big)d\bar{B}(\bar{s}).
\end{aligned}$$

Because $\exp\Big(\gamma e^{2a}\ln X(\alpha(s,\omega)) + \big(\ln Y(\alpha(s,\omega)) - \gamma X(\alpha(s,\omega))\big)e^a\Big) = \bar{Y}(s,\omega)$ and $\exp\big(e^{2a}\ln X(\alpha(s),\omega)\big) = \bar{X}(s,\omega)$, one gets

$$\begin{aligned}
\varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) + \int_0^{\beta(t)} \rho\bar{X}(s,\omega)ds,\\
\varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a)\\
&\quad + \int_0^{\beta(t)} \mu\bar{Y}(s,\omega)d(s) + \int_0^{\beta(t)} \sigma\bar{Y}(s,\omega)dB(s).
\end{aligned}$$

Since $\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \bar{X}(\beta(t),\omega)$, and $\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \bar{Y}(\beta(t),\omega)$, one has

$$\begin{aligned}
\bar{X}(\beta(t),\omega) &= \bar{X}(0,\omega) + \int_0^{\beta(t)} \rho\bar{X}(s,\omega)ds,\\
\bar{Y}(\beta(t),\omega) &= \bar{Y}(0,\omega) + \int_0^{\beta(t)} \mu\bar{Y}(s,\omega)d(s) + \int_0^{\beta(t)} \sigma\bar{Y}(s,\omega)dB(s).
\end{aligned}$$

This confirms that the Lie group of transformations (6.25) transforms any solution of system (6.20) into a solution of the same system.

6.1.4 Mean-reverting Ornstein-Uhlenbeck Process

Consider the system of equations discussed in Oksendal (1998),

$$\begin{aligned} dX(t) &= \rho X(t)dt, \\ dY(t) &= (m - Y(t))dt + \sigma dB(t), \end{aligned} \tag{6.28}$$

where $\rho > 0$, $m > 0$ and $\sigma \neq 0$ are constants. For equations (6.28) the functions corresponding equations of (6.2) are $f_1 = \rho x$, $f_2 = m - y$, $g_1 = 0$ and $g_2 = \sigma$. The system of determining equations for system of equations (6.28) becomes

$$\begin{gathered} \xi_{1,t} + \rho x \xi_{1,x} + (m - y)\xi_{1,y} + \tfrac{1}{2}\sigma^2 \xi_{1,yy} - 2\rho x \tau - \rho \xi_1 = 0, \\ \xi_{2,t} + \rho x \xi_{2,x} + (m - y)\xi_{2,y} + \tfrac{1}{2}\sigma^2 \xi_{2,yy} - 2(m - y)\tau + \xi_2 = 0, \\ \xi_{1,y} = 0, \quad \xi_{2,y} - \tau = 0. \end{gathered} \tag{6.29}$$

From the last pair of equations of (6.31), one finds

$$\xi_1 = \xi_1(t, x), \quad \tau = \xi_{2,y}.$$

Substituting them into the remaining equations of (6.21), one obtains that the function ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered} \xi_{1,t} + \rho x \xi_{1,x} - 2\rho x \xi_{2,y} - \rho \xi_1 = 0, \\ \xi_{2,t} + \rho x \xi_{2,x} - (m - y)\xi_{2,y} + \tfrac{1}{2}\sigma^2 y^2 \xi_{2,yy} + \xi_2 = 0. \end{gathered} \tag{6.30}$$

Differentiating the first equation of (6.31) with respect to y, one gets

$$\xi_{2,yy} = 0,$$

thus

$$\xi_2 = yF(t, x) + G(t, x).$$

Substituting it into the second equation of (6.31), one obtains

$$y(F_t + \rho x F_x + 2F) + G_t + \rho x G_x + G - mF = 0.$$

Splitting this equation with respect to y, one has

$$F_t + \rho x F_x + 2F = 0, \quad G_t + \rho x G_x + G - mF = 0,$$

thus

$$F = x^\gamma F_1, \quad G = -mx^\gamma F_1 + x^{\frac{\gamma}{2}} F_2$$

where $F_1 = F_1(t - \frac{\ln x}{\rho})$, $F_2 = F_2(t - \frac{\ln x}{\rho})$ and $\gamma = -\frac{2}{\rho}$. Substituting ξ_2 into the first equation of (6.31), one obtains

$$\xi_{1,t} + \rho x \xi_{1,x} - \rho \xi_1 - 2\rho x F_1 = 0.$$

Hence,

$$\xi_1 = \frac{2}{\gamma} x^{\gamma+1} F_1 + x F_3, \quad \xi_2 = (x^\gamma y - mx^\gamma) F_1 + x^{\frac{\gamma}{2}} F_2, \quad \tau = x^\gamma F_1, \tag{6.31}$$

where $F_3 = F_3(t - \frac{\ln x}{\rho})$. A basis of generators corresponding to (6.31) is

$$x^{\frac{\gamma}{2}-1} F_2 \partial_y, \quad x F_3 \partial_x, \quad F_1\big(\frac{2}{\gamma} x^{\gamma+1} \partial_x + (x^\gamma y - mx^\gamma)\partial_y\big) + h\partial_t,$$

where $h = 2\displaystyle\int_0^t x^\gamma F_1(s - \frac{\ln x}{\rho}) ds$.

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x, \quad \bar{y} = y + a x^{\frac{\gamma}{2}} F_2, \quad \bar{t} = t. \tag{6.32}$$

It is not difficult to show that this Lie group transforms a solution of (6.28) into a solution of the same system.

Let us construct the Lie group of transformations corresponding to the third generator for the particular case determined by the assumption $F_1 = k$, where k is constant. In this case the generator becomes

$$\frac{2}{\gamma} x^{\gamma+1} \partial_x + (x^\gamma y - mx^\gamma)\partial_y + 2x^\gamma t \partial_t.$$

For finding the Lie group of transformations corresponding to this generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2\varphi_2^\gamma H, \quad \frac{\partial \varphi_1}{\partial a} = \frac{2}{\gamma}\varphi_1^{\gamma+1}, \quad \frac{\partial \varphi_2}{\partial a} = \varphi_1^\gamma \varphi_2 - m\varphi_1^\gamma,$$

with the initial conditions $a = 0$:

$$H = t, \quad \varphi_1 = x, \quad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y,

$$\begin{aligned} &\bar{t} = H = tx^{-\gamma}(x^{-\gamma} - 2a)^{-1}, \quad \bar{x} = \varphi_1 = (x^{-\gamma} - 2a)^{-\frac{1}{\gamma}}, \\ &\bar{y} = \varphi_2 = (y - m)x^{-\frac{\gamma}{2}}(x^{-\gamma} - 2a)^{-\frac{1}{2}} + m. \end{aligned} \tag{6.33}$$

Hence $\eta^2 = H_t = x^{-\gamma}(x^{-\gamma} - 2a)^{-1}$.

Let us show that the Lie group of transformation (6.33) transforms a solution of equations (6.28) into a solution of the same equations. Assume that $(X(t), Y(t))$ is a solution of equations (6.28). The Brownian motion $B(t)$ is transformed according to the formula

$$\bar{B}(t) = \int_0^{\alpha(t)} X^{-\frac{\gamma}{2}}(s)(X^{-\gamma}(s) - 2a)^{-\frac{1}{2}} dB(s), \quad t \geq 0, \tag{6.34}$$

where

$$\beta(t) = \int_0^t X^{-\gamma}(s)(X^{-\gamma}(s) - 2a)^{-1} ds, \quad \alpha(t) = \inf_{s \geq 0}\{s : \beta(s) > t\}, \quad t \geq 0.$$

Applying Itô's formula to the functions $\varphi_1(t, x, y, a) = (x^{-\gamma} - 2a)^{-\frac{1}{\gamma}}$ and $\varphi_2(t, x, y, a) = (y - m)x^{-\frac{\gamma}{2}}(x^{-\gamma} - 2a)^{-\frac{1}{2}} + m$, one has

$$\begin{aligned} \varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) \\ &\quad + \int_0^t \rho X^{-\gamma-1}(s)(X^{-\gamma}(s) - 2a)^{-\frac{1}{\gamma}-1} ds, \\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a) \\ &\quad + \int_0^t (m - Y(s))X^{-\frac{\gamma}{2}}(s)X^{-\gamma}(s)(X(s)^{-\gamma} - 2a)^{-\frac{3}{2}} ds \\ &\quad + \int_0^t \sigma X^{-\frac{\gamma}{2}}(s)(X(s)^{-\gamma} - 2a)^{-\frac{1}{2}} dB(s). \end{aligned} \tag{6.35}$$

By virtue of (4.9),

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = X^{\gamma}(s)(X^{-\gamma}(s) - 2a).$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integrals in (6.35), they become

$$\begin{aligned}
&\int_0^t \rho X^{-\gamma-1}(s)(X^{-\gamma}(s) - 2a)^{-\frac{1}{\gamma}-1} ds = \int_0^{\beta(t)} \rho X^{-\gamma}\big(\alpha(s)) - 2a\big)^{-\frac{1}{\gamma}} d\bar{s},\\
&\int_0^t (m - Y(s))X^{-\frac{\gamma}{2}}(s)X^{-\gamma}(s)(X(s)^{-\gamma} - 2a)^{-\frac{3}{2}} ds\\
&= \int_0^t \big(m - Y(\alpha(s))\big)X^{-\frac{\gamma}{2}}(\alpha(s))\big(X(\alpha(s))^{-\gamma} - 2a\big)^{-\frac{1}{2}} d\bar{s}\\
&= \int_0^t \Big(m - \big(Y(\alpha(s) - m)\big)X^{-\frac{\gamma}{2}}(\alpha(s))\big(X(\alpha(s))^{-\gamma} - 2a\big)^{-\frac{1}{2}} - m\Big) d\bar{s}.
\end{aligned}$$

Because of the transformation of the Brownian motion (6.34), the Itô integral in (6.35) becomes

$$\int_0^t \sigma X^{-\frac{\gamma}{2}}(s)(X(s)^{-\gamma} - 2a)^{-\frac{1}{2}} dB(s) = \int_0^{\beta(t)} \sigma d\bar{B}(\bar{s}).$$

Since $\big(Y(\alpha(\bar{t}), \omega) - m\big)X^{-\frac{\gamma}{2}}(\alpha(\bar{t}), \omega)\big(X^{-\gamma}(\alpha(\bar{t}), \omega) - 2a\big)^{-\frac{1}{2}} + m = \bar{Y}(\bar{t}, \omega)$ and $\big(X^{-\gamma}(\alpha(\bar{t}), \omega) - 2a\big)^{-\frac{1}{\gamma}} = \bar{X}(\bar{t}, \omega)$, one gets

$$\begin{aligned}
\varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) + \int_0^{\beta(t)} \rho \bar{X}(s,\omega) ds,\\
\varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a)\\
&\quad + \int_0^{\beta(t)} \big(m - \bar{Y}(s,\omega)\big) ds + \int_0^{\beta(t)} \sigma d\bar{B}(s).
\end{aligned}$$

Because $\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \bar{X}(\beta(t), \omega)$, and $\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \bar{Y}(\beta(t), \omega)$, one has

$$\begin{aligned}
\bar{X}(\beta(t), \omega) &= \bar{X}(0, \omega) + \int_0^{\beta(t)} \rho \bar{X}(s,\omega) ds,\\
\bar{Y}(\beta(t), \omega) &= \bar{Y}(0, \omega) + \int_0^{\beta(t)} \big(m - \bar{Y}(s,\omega)\big) ds + \int_0^{\beta(t)} \sigma d\bar{B}(s).
\end{aligned}$$

This confirms that the Lie group of transformations (6.33) transforms any solution of system (6.60) into a solution of the same system.

6.1.5 Nonlinear Itô System

Consider the system of equations discussed in Oksendal (1998),

$$\begin{aligned} dX(t) &= dt, \\ dY(t) &= Y(t)dt + e^{X(t)}dB(t). \end{aligned} \tag{6.36}$$

For equations (6.36), the corresponding functions of equations (6.2) are $f_1 = 1$, $f_2 = y$, $g_1 = 0$ and $g_2 = e^x$. The system of determining equations for equations (6.36) becomes

$$\begin{gathered} \xi_{1,t} + \xi_{1,x} + y\xi_{1,y} + \tfrac{1}{2}e^x\xi_{1,yy} - 2\tau = 0, \\ \xi_{2,t} + \xi_{2,x} + y\xi_{2,y} + \tfrac{1}{2}e^x\xi_{2,yy} - 2y\tau - \xi_2 = 0, \\ \xi_{1,y} = 0, \quad \xi_{2,y} - \tau - \xi_1 = 0. \end{gathered} \tag{6.37}$$

From the last pair of equations of (6.37) one finds

$$\xi_1 = \xi_1(t,x), \quad \tau = \xi_{2,y} - \xi_1.$$

Substituting them into the remaining equations of (6.37), one obtains that the function ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered} \xi_{1,t} + \xi_{1,x} - 2(\xi_{2,y} - \xi_1) = 0, \\ \xi_{2,t} + \xi_{2,x} - y\xi_{2,y} + \tfrac{1}{2}e^x\xi_{2,yy} + 2y\xi_1 - \xi_2 = 0. \end{gathered} \tag{6.38}$$

Differentiating the first equation of (6.38) with respect to y, one obtains

$$\xi_{2,yy} = 0,$$

thus

$$\xi_2 = F(t,x)y + G(t,x).$$

Substituting it into equations (6.38), one finds

$$\begin{gathered} \xi_{1,t} + \xi_{1,x} - 2(F - \xi_1) = 0, \\ y(F_t + F_x - 2F + 2\xi_1) + G_t + G_x - G = 0. \end{gathered} \tag{6.39}$$

Splitting the last equation of (6.39) with respect to y, one has

$$F_t + F_x - 2F + 2\xi_1 = 0, \quad G_t + G_x - G = 0. \tag{6.40}$$

From the second of (6.40) one finds

$$G = e^x F_1,$$

where $F_1 = F_1(t-x)$. From the first equation of (6.39) one gets

$$F = \frac{1}{2}(2\xi_1 + \xi_{1,t} + \xi_{1,x}).$$

Substituting it into the first equation of (6.40), one obtains

$$\xi_{1,tt} + 2\xi_{1,tx} + \xi_{1,xx} = 0,$$

then

$$\xi_1 = xF_2 + F_3,$$

where $F_2 = F_2(t-x)$, and $F_3 = F_3(t-x)$. Substituting it into the first of equation of (6.39), one gets

$$F = F_2(\frac{1}{2} + x) + F_3.$$

Hence,

$$\xi_1 = F_2 x + F_3, \quad \xi_2 = F_1 e^x + F_2(\frac{1}{2} + x)y + F_3 y, \quad \tau = \frac{1}{2}F_2. \tag{6.41}$$

A basis of generators corresponding (6.42) is

$$F_1 e^x \partial_y, \quad F_3(\partial_x + y\partial_y), \quad F_2(x\partial_x + (\frac{1}{2} + x)y\partial_y) + h\partial_t,$$

where $h = 2\int_0^t F_2(s-x)ds$.

Notice that if the first equation of (6.36) is considered as an ordinary differential equation (i.e., the function $X(t,\omega)$ does not depend on ω), then the functions F_1, F_2 and F_3 are constant.

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x, \quad \bar{y} = y + e^x F_2 a, \quad \bar{t} = t. \tag{6.42}$$

It is not difficult to show that this Lie group transforms a solution of (6.36) into a solution of the same system.

Let us construct the Lie group of transformations corresponding to the third generator for the particular case determined by the assumption $F_2 = k$, where k is constant. In this case the generator becomes

$$2x\partial_x + (1 + 2x)y\partial_y + 2t\partial_t.$$

For finding the Lie group of transformations corresponding to this generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2H, \quad \frac{\partial \varphi_1}{\partial a} = 2\varphi_1, \quad \frac{\partial \varphi_2}{\partial a} = (1 + 2\varphi_1)\varphi_2,$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi_1 = x, \quad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y,

$$\bar{t} = H = te^{2a}, \quad \bar{x} = \varphi_1 = xe^{2a}, \quad \bar{y} = \varphi_2 = y\exp(a + xe^{2a} - x). \tag{6.43}$$

Hence $\eta^2 = H_t = e^{2a}$.

Let us show that the Lie group of transformation (6.43) transforms a solution of equations (6.36) into a solution of the same equations. Assume that $\big(X(t), Y(t)\big)$ is a solution of equations (6.36). The Brownian motion $B(t)$ is transformed according to the formula

$$\bar{B}(t) = \int_0^{\alpha(t)} e^a dB(s), \tag{6.44}$$

where

$$\beta(t) = \int_0^t e^{2a} ds = te^{2a}, \quad \alpha(t) = \inf_{s \geq 0}\{s : \beta(s) > t\} = te^{-2a}, \quad t \geq 0.$$

Applying Itô's formula to the functions $\varphi_1(t, x, y, a) = xe^{2a}$ and $\varphi_2(t, x, y, a) = y \exp(a + xe^{2a} - x)$, one has

$$\begin{aligned}
&\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \varphi_1(0, X(0,\omega), Y(0,\omega), a) + \int_0^t e^{2a} ds, \\
&\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \varphi_2(0, X(0,\omega), Y(0,\omega), a) \\
&+ \int_0^t Y(s) \exp\Big(a + X(s,\omega)e^{2a} - X(s,\omega)\Big) e^{2a} ds + \int_0^t \exp(a + X(s,\omega)e^{2a}) dB(s).
\end{aligned} \tag{6.45}$$

By virtue of (4.9),

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = e^{-2a}.$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integrals in (6.45), they become

$$\begin{aligned}
&\int_0^t e^{2a} ds = \int_0^{\beta(t)} d\bar{s}, \\
&\int_0^t Y(s) \exp\Big(a + X(s)e^{2a} - X(s)\Big) e^{2a} ds \\
&\qquad\qquad = \int_0^{\beta(t)} Y(\alpha(\bar{s})) \exp\Big(a + X(\alpha(\bar{s}))e^{2a} - X(\alpha(\bar{s}))\Big) d\bar{s}.
\end{aligned}$$

Because of the transformation of the Brownian motion (6.44) and the change of Brownian motion in the Itô integral (4.7), the Itô integral in (6.45) becomes

$$\int_0^t \exp(a + X(s)e^{2a}) dB(s) = \int_0^{\beta(t)} \exp(X(\alpha(\bar{s}))e^{2a}) d\bar{B}(\bar{s}).$$

Since $Y(\alpha(\bar{t}),\omega) \exp\big(a + X(\alpha(\bar{t}),\omega)e^{2a} - X(\alpha(\bar{t}),\omega)\big) = \bar{Y}(\bar{t},\omega)$ and $X(\alpha(\bar{t}),\omega)e^{2a} = \bar{X}(\bar{t},\omega)$, one gets

$$\begin{aligned}
&\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) + \int_0^{\beta(t)} ds, \\
&\bar{Y}(\beta(t),\omega) = \bar{Y}(0,\omega) + \int_0^{\beta(t)} \bar{Y}(s,\omega) ds + + \int_0^{\beta(t)} e^{\bar{X}(s,\omega)} d\bar{B}(s).
\end{aligned}$$

Because $\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \bar{X}(\beta(t),\omega)$, and $\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \bar{Y}(\beta(t),\omega)$, one has

$$\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) + \int_0^{\beta(t)} ds,$$
$$\bar{Y}(\beta(t),\omega) = \bar{Y}(0,\omega) + \int_0^{\beta(t)} \bar{Y}(s,\omega)ds + \int_0^{\beta(t)} e^{\bar{X}(s,\omega)} d\bar{B}(s).$$

This confirms that the Lie group of transformations (6.43) transforms any solution of system (6.36) into a solution of the same system.

6.2 Two-dimensional Brownian motion

6.2.1 System of Location and Motion

Let μ, σ_1 and σ_2 be nonzero constants. Consider the system of equations discussed in Steele (2001),

$$\begin{aligned} dX(t) &= Y(t)dt + \sigma_1 dB_1(t), \\ dY(t) &= -\mu Y(t)dt + \sigma_2 dB_2(t). \end{aligned} \tag{6.46}$$

For equations (6.46) the corresponding functions of equations (4.22) are $f_1 = y$, $f_2 = -\mu y$, $g_{11} = \sigma_1$, $g_{12} = 0$, $g_{21} = 0$ and $g_{22} = \sigma_2$. The system of determining equations for equations (6.46) becomes

$$\begin{gathered} \xi_{1,t} + y\xi_{1,x} - \mu y\xi_{1,y} + \tfrac{1}{2}\sigma_1^2\xi_{1,xx} + \tfrac{1}{2}\sigma_2^2\xi_{1,yy} - 2y\tau - \xi_2 = 0, \\ \xi_{2,t} + y\xi_{2,x} - \mu y\xi_{2,y} + \tfrac{1}{2}\sigma_1^2\xi_{2,xx} + \tfrac{1}{2}\sigma_2^2\xi_{2,yy} + 2\mu y\tau + \mu\xi_2 = 0, \\ \xi_{1,x} - \tau = 0, \quad \xi_{2,y} - \tau = 0, \quad \xi_{1,y} = 0, \quad \xi_{2,x} = 0. \end{gathered} \tag{6.47}$$

From the last equations of (6.47) one finds

$$\xi_1 = \xi_1(t,x), \quad \xi_2 = \xi_1(t,y), \quad \tau = \xi_{2,y}.$$

Substituting them into the remaining equations of (6.47), one obtains that the functions ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered}
\xi_{1,t} + y\xi_{1,x} + \tfrac{1}{2}\sigma_1^2\xi_{1,xx} - 2y\xi_{2,y} - \xi_2 = 0, \\
\xi_{2,t} + \mu y\xi_{2,y} + \tfrac{1}{2}\sigma_2^2\xi_{2,yy} + \mu\xi_2 = 0, \\
\xi_{1,x} - \xi_{2,y} = 0.
\end{gathered} \tag{6.48}$$

Differentiating the third equation of (6.48) with respect to x and y, respectively, one obtains

$$\xi_{1,xx} = 0, \quad \xi_{2,yy} = 0,$$

thus

$$\xi_1 = h_1(t)x + h_2(t), \quad \xi_2 = h_3(t)y + h_4(t).$$

Substituting them into other equations of (6.48), one obtains

$$\begin{gathered}
h_1' + h_2' - y(h_1 + 3h_3) - h_4 = 0, \\
h_3' + h_4' + 2\mu y h_3 + \mu h_4 = 0.
\end{gathered} \tag{6.49}$$

Splitting equations (6.49) with respect to y, one has

$$h_1 + 3h_3 = 0, \quad h_3 = 0,$$

thus

$$h_1 = 0, \quad h_3 = 0.$$

Substituting them into (6.49), one obtains

$$h_2' - h_4 = 0, \quad h_4' + \mu h_4 = 0, \tag{6.50}$$

so that

$$h_4 = C_1\mu e^{-\mu t}.$$

Substituting it into the first equation of (6.50), one obtains

$$h_2' + C_1\mu e^{-\mu t} = 0.$$

Then

$$h_2 = -C_1 e^{-\mu t} + C_2,$$

thus

$$\xi_1 = -C_1 e^{-\mu t} + C_2, \quad \xi_2 = C_1 \mu e^{-\mu t}. \tag{6.51}$$

Hence $\tau = 0$, and $h = 0$. Thus, a basis of generators corresponding to (6.51) is

$$e^{-\mu t}(\partial_x - \mu \partial_y), \quad \partial_x.$$

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x + ae^{-\mu t}, \quad \bar{y} = y - a\mu e^{-\mu t}, \quad \bar{t} = t. \tag{6.52}$$

Applying Itô's formula to the functions $\varphi_1 = x + ae^{-\mu t}$ and $\varphi_2 = y - a\mu e^{-\mu t}$, one has

$$\begin{aligned} X(t) + ae^{-\mu t} &= X(0) + a + \textstyle\int_0^t (Y(s) - a\mu e^{-\mu s})ds + \displaystyle\int_0^t \sigma_1 dB_1(s), \\ Y(t) - a\mu e^{-\mu t} &= Y(0) - \int_0^t \mu (Y(s) - a\mu e^{-\mu s})ds + \int_0^t \sigma_2 dB_2(s). \end{aligned}$$

Since $X(t,\omega) + ae^{-\mu t} = \bar{X}(t,\omega) = \bar{X}(\bar{t},\omega)$ and $Y(t,\omega) - a\mu e^{-\mu t} = \bar{Y}(t,\omega) = \bar{Y}(\bar{t},\omega)$, one has

$$\begin{aligned} \bar{X}(\bar{t},\omega) &= \bar{X}(0,\omega) + \int_0^{\bar{t}} \bar{Y}(s,\omega)ds + \int_0^{\bar{t}} \sigma_1 dB_1(s), \\ \bar{Y}(\bar{t},\omega) &= \bar{Y}(0,\omega) - \int_0^{\bar{t}} \mu \bar{Y}(s,\omega)ds + \int_0^{\bar{t}} \sigma_2 dB_2(s). \end{aligned}$$

This confirms that this Lie group transforms a solution of (6.46) into a solution of the same system.

The Lie group of transformations corresponding to the second admitted generator is

$$\bar{x} = x + a, \quad \bar{y} = y, \quad \bar{t} = t. \tag{6.53}$$

We can show similarly as above that this Lie group transforms a solution of (6.46) into a solution of the same system.

6.2.2 Vibrating String Model

Let μ and σ be nonzero constants. Consider the system of equations discussed in Oksendal (1998),

$$\begin{aligned} dX(t) &= Y(t)dt + \mu dB_1(t), \\ dY(t) &= -X(t)dt + \sigma dB_2(t). \end{aligned} \tag{6.54}$$

The system of equations (6.54) is a model for a vibrating string subject to a stochastic force. For equations (6.54), the corresponding functions of equations (4.22) are $f_1 = y$, $f_2 = -x$, $g_{11} = \mu$, $g_{12} = 0$, $g_{21} = 0$ and $g_{22} = \sigma$. The system of determining equations for equations (6.54) becomes

$$\begin{gathered} \xi_{1,t} + y\xi_{1,x} - x\xi_{1,y} + \tfrac{1}{2}\mu^2\xi_{1,xx} + \tfrac{1}{2}\sigma^2\xi_{1,yy} - 2y\tau - \xi_2 = 0, \\ \xi_{2,t} + y\xi_{2,x} - x\xi_{2,y} + \tfrac{1}{2}\mu^2\xi_{2,xx} + \tfrac{1}{2}\sigma^2\xi_{2,yy} + 2x\tau + \xi_1 = 0, \\ \xi_{1,x} - \tau = 0, \quad \xi_{2,y} - \tau = 0, \quad \sigma\xi_{1,y} = 0, \quad \xi_{2,x} = 0. \end{gathered} \tag{6.55}$$

From the last equations of (6.55) one finds

$$\xi_1 = \xi_1(t, x), \quad \xi_2 = \xi_2(t, y), \quad \tau = \xi_{2,y}.$$

Substituting them into the remaining equations of (6.55), one obtains that the function ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered} \xi_{1,t} + y\xi_{1,x} + \tfrac{1}{2}\mu^2\xi_{1,xx} - 2y\xi_{2,y} - \xi_2 = 0, \\ \xi_{2,t} + x\xi_{2,y} + \tfrac{1}{2}\sigma^2\xi_{2,yy} + \xi_1 = 0, \\ \xi_{1,x} - \xi_{2,y} = 0. \end{gathered} \tag{6.56}$$

Differentiating the third equation of (6.56) with respect to x and y, respectively, one obtains

$$\xi_{1,xx} = 0, \quad \xi_{2,yy} = 0,$$

thus

$$\xi_1 = h_1(t)x + h_2(t), \quad \xi_2 = h_3(t)y + h_4(t).$$

Substituting them into other equations of (6.56), one obtains

$$
\begin{aligned}
&h_1' + h_2' + y(h_1 + 3h_3) - h_4 = 0,\\
&h_3' + h_4' + x(h_3 + h_1) + h_2 = 0.
\end{aligned}
\tag{6.57}
$$

Splitting equations (6.57) with respect to y and x, respectively, one has

$$h_1 + 3h_3 = 0, \quad h_1 + h_3 = 0.$$

thus

$$h_1 = 0, \quad h_3 = 0.$$

Substituting them into equations (6.57), one obtains

$$h_2' - h_4 = 0, \quad h_4' + h_2 = 0. \tag{6.58}$$

Differentiating the second equation of (6.58), one obtains

$$h_4'' + h_2' = 0,$$

Substituting $h_2' = -h_4''$ into the first equation of (6.58), one gets

$$h_4'' + h_4 = 0,$$

thus

$$h_4 = C_1 \cos t - C_2 \sin t, \quad h_2 = -h_4' = C_1 \sin t + C_2 \cos t.$$

Hence,

$$\xi_1 = C_1 \sin t + C_2 \cos t, \quad \xi_2 = C_1 \cos t - C_2 \sin t. \tag{6.59}$$

Then $\tau = 0$ and $h = 0$. Thus, a basis of generators corresponding to (6.59) is

$$\sin t\partial_x + \cos t\partial_y, \quad \cos t\partial_x - \sin t\partial_y.$$

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x + a\sin t, \quad \bar{y} = y + a\cos t, \quad \bar{t} = t.$$

Applying Itô's formula to the functions $\varphi_1 = x + a\sin t$ and $\varphi_2 = y + a\cos t$, one has

$$
\begin{aligned}
X(t) + a\sin t &= X(0) + a + \int_0^t (Y(s) + a\cos s)ds + \int_0^t \mu dB_1(s),\\
Y(t) + a\cos t &= Y(0) - \int_0^t (X + a\sin s)ds + \int_0^t \sigma dB_2(s).
\end{aligned}
$$

Since $X(t,\omega) + a\sin t = \bar{X}(t,\omega) = \bar{X}(\bar{t},\omega)$ and $Y(t,\omega) + a\cos t = \bar{Y}(t,\omega) = \bar{Y}(\bar{t},\omega)$, one has

$$
\begin{aligned}
\bar{X}(\bar{t},\omega) &= \bar{X}(0,\omega) + \int_0^{\bar{t}} \bar{Y}(s,\omega)ds + \int_0^{\bar{t}} \mu dB_1(s),\\
\bar{Y}(\bar{t},\omega) &= \bar{Y}(0,\omega) - \int_0^{\bar{t}} \bar{X}(s,\omega)ds \int_0^{t} \sigma dB_2(s).
\end{aligned}
$$

This confirms that this Lie group transforms a solution of (6.54) into a solution of the same system.

The Lie group of transformations corresponding to the second admitted generator is

$$
\bar{x} = x + a\cos t, \quad \bar{y} = y - a\sin t, \quad \bar{t} = t.
$$

We can show similarly as above that this Lie group transforms a solution of (6.54) into a solution of the same system.

6.2.3 Ornstein-Uhlenbeck Process

Consider the system of equation discussed in Gaeta and Quintero, (1999),

$$
\begin{aligned}
dX(t) &= -X(t)dt,\\
dY(t) &= -Y(t)dt + dB_1(t) + dB_2(t).
\end{aligned}
\tag{6.60}
$$

This system represents an Ornstein-Uhlenbeck process and its corresponding Fokker-Planck equation is

$$
u_t + \frac{1}{2}u_{yy} - xu_x - yu_y - 2u = 0.
$$

For equations (6.60), the corresponding functions of equations (4.22) are $f_1 = -x$, $f_2 = -y$, $g_{11} = 0$, $g_{12} = 0$, $g_{21} = 1$ and $g_{22} = 1$. The system of determining equations for equations (6.60) becomes

$$\begin{gathered}
\xi_{1,t} - x\xi_{1,x} - y\xi_{1,y} + \xi_{1,yy} + 2x\tau + \xi_1 = 0, \\
\xi_{2,t} - x\xi_{2,x} - y\xi_{2,y} + \xi_{2,yy} + 2y\tau + \xi_2 = 0, \\
\xi_{1,y} = 0, \quad \xi_{2,y} - \tau = 0.
\end{gathered} \tag{6.61}$$

From the last two equations of (6.61) one finds

$$\xi_1 = \xi_1(t, x), \quad \tau = \xi_{2,y}.$$

Substituting them into the remaining equations of (6.61), one obtains that the function ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered}
\xi_{1,t} - x\xi_{1,x} + 2x\xi_{2,y} + \xi_1 = 0, \\
\xi_{2,t} - x\xi_{2,x} + y\xi_{2,y} + \xi_{2,yy} + \xi_2 = 0.
\end{gathered} \tag{6.62}$$

Differentiating the first equation of (6.62) with respect to y, one gets

$$\xi_{2,yy} = 0,$$

thus

$$\xi_2 = F(t, x)y + G(t, x).$$

Substituting it into the second equation of (6.62), one obtains

$$y(F_t + 2F - xF_x) + G_t - xG_x + G = 0.$$

Splitting the last equation with respect to y, one has

$$F_t + 2F - xF_x = 0, \quad G_t - xG_x + G = 0$$

thus

$$F = x^2F_1, \quad G = xF_2,$$

where $F_1 = F_1(xe^t)$ and $F_2 = F_2(xe^t)$. Substituting ξ_2 into the first equation of (6.62), one obtains

$$\xi_{1,t} - x\xi_{1,x} + 2x^3F_1 + \xi_1 = 0.$$

Hence,

$$\xi_1 = x^3F_1 + xF_3, \quad \xi_2 = x^2yF_1 + xF_2, \quad \tau = x^2F_1, \tag{6.63}$$

where $F_3 = F_3(xe^t)$. A basis of generators corresponding to (6.63) is

$$xF_2\partial_y, \quad xF_3\partial_x, \quad F_1x^2(x\partial_x + y\partial_y) + x^2h\partial_t,$$

where $h = 2\int_0^t F(xe^s)ds$.

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x, \quad \bar{y} = y + xF_2, \quad \bar{t} = t. \tag{6.64}$$

It is not difficult to show that this Lie group transforms a solution of (6.60) into a solution of the same equation.

Let us construct the Lie group of transformations corresponding to the third generator for the particular case determined by the assumption $F_1 = k$, where k is constant. In this case the generator becomes

$$x^3\partial_x + yx^2\partial_y + 2x^2t\partial_t.$$

For finding the Lie group of transformations corresponding to this generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2\varphi_1^2H, \quad \frac{\partial \varphi_1}{\partial a} = \varphi_1^3, \quad \frac{\partial \varphi_2}{\partial a} = \varphi_2\varphi_1^2,$$

with the initial conditions for $a = 0$;

$$H = t, \quad \varphi_1 = x, \quad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y,

$$\bar{t} = H = t(1-2ax^2)^{-1}, \quad \bar{x} = \varphi_1 = x(1-2ax^2)^{-\frac{1}{2}}, \quad \bar{y} = \varphi_2 = y(1-2ax^2)^{-\frac{1}{2}}. \tag{6.65}$$

Hence $\eta^2 = (1-2ax^2)^{-1}$.

Let us show that the Lie group of transformations (6.65) transforms a solution of equations (6.60) into a solution of the same equations. Assume that $(X(t), Y(t))$ is a solution of equations (6.60). The Brownian motions $B_1(t)$ and $B_2(t)$ are transformed according to the formula

$$\begin{aligned} \bar{B}_1(t) &= \int_0^{\alpha(t)} (1-2aX^2(s))^{-\frac{1}{2}} dB_1(s), \\ \bar{B}_2(t) &= \int_0^{\alpha(t)} (1-2aX^2(s))^{-\frac{1}{2}} dB_2(s), \quad t \geq 0, \end{aligned} \tag{6.66}$$

where

$$\beta(t) = \int_0^t (1-2aX^2(s))^{-1} ds, \quad \alpha(t) = \inf_{s\geq 0}\{s : \beta(s) > t\}, \quad t \geq 0.$$

Applying Itô's formula to the functions $\varphi_1(t,x,y,a) = x(1-2ax^2)^{-\frac{1}{2}}$ and $\varphi_2(t,x,y,a) = y(1-2ax^2)^{-\frac{1}{2}}$, one has

$$\begin{aligned} \varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) - \int_0^t X(s)(1-2aX^2(s))^{-\frac{3}{2}} ds \\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a) \\ &\quad - \int_0^t Y(s,\omega)(1-2aX^2(s))^{-\frac{3}{2}} ds + \int_0^t (1-2aX^2(s))^{-\frac{1}{2}} dB_1(s) \\ &\quad + \int_0^t (1-2aX^2(s))^{-\frac{1}{2}} dB_2(s). \end{aligned} \tag{6.67}$$

By virtue of (4.9),

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = (1-2aX^2(s)).$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integrals in (6.67), they become

$$\int_0^t X(s)(1-2aX^2(s))^{-\frac{3}{2}}ds = \int_0^{\beta(t)} X(\alpha(\bar{s}))\Big(1-2aX^2(\alpha(\bar{s}))\Big)^{-\frac{1}{2}}d\bar{s},$$

$$\int_0^t Y(s,\omega)\Big(1-2aX^2(s)\Big)^{-\frac{3}{2}}ds = \int_0^{\beta(t)} Y(\alpha(\bar{s}))\Big(1-2aX^2(\alpha(\bar{s}))\Big)^{-\frac{1}{2}}d\bar{s}.$$

Because of the transformation of the Brownian motions (6.66), the Itô integrals in (6.67) become

$$\int_0^t (1-2aX^2(s))^{-\frac{1}{2}}dB_1(s) = \int_0^{\beta(t)} d\bar{B}_1(\bar{s}).$$

$$\int_0^t (1-2aX^2(s))^{-\frac{1}{2}}dB_2(s) = \int_0^{\beta(t)} d\bar{B}_2(\bar{s}).$$

Since $X(\alpha(\bar{t}))\big(1-2aX^2(\alpha(\bar{t}))\big)^{-\frac{1}{2}} = \bar{X}(\bar{t},\omega)$ and $Y(\alpha(\bar{t}))\big(1-2aX^2(\alpha(\bar{t}))\big)^{-\frac{1}{2}} = \bar{Y}(\bar{t},\omega)$, one gets

$$\varphi_1(t,X(t,\omega),Y(t,\omega),a) = \varphi_1(0,X(0,\omega),Y(0,\omega),a) - \int_0^{\beta(t)} \bar{X}(s,\omega)d\bar{B}_1(s),$$

$$\begin{aligned}\varphi_2(t,X(t,\omega),Y(t,\omega),a) = \varphi_2(0,X(0,\omega),Y(0,\omega),a) - \int_0^{\beta(t)} \bar{Y}(s,\omega)d\bar{B}_1(s)\\ + \int_0^{\beta(t)} d\bar{B}_1(s) + \int_0^{\beta(t)} d\bar{B}_2(s).\end{aligned}$$

Because $\varphi_1(t,X(t,\omega),Y(t,\omega),a) = \bar{X}(\beta(t),\omega)$, and $\varphi_2(t,X(t,\omega),Y(t,\omega),a) = \bar{Y}(\beta(t),\omega)$, one has

$$\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) - \int_0^{\beta(t)} \bar{X}(s,\omega)ds,$$
$$\bar{Y}(\beta(t),\omega) = \bar{Y}(0,\omega) - \int_0^{\beta(t)} \bar{Y}(s,\omega)ds + \int_0^{\beta(t)} d\bar{B}_1(s) + \int_0^{\beta(t)} d\bar{B}_2(s).$$

This confirms that the Lie group of transformations (6.65) transforms any solution of equations (6.60) into a solution of the same equations.

6.2.4 Linear Itô Dynamical System

Consider the system of equations discussed in Unal (2003),

$$\begin{aligned} dX(t) &= -\tfrac{1}{2}X(t)dt - Y(t)dB_1(t), \\ dY(t) &= -\tfrac{1}{2}Y(t)dt + X(t)dB_2(t). \end{aligned} \tag{6.68}$$

For equations (6.68) the corresponding functions of equations (4.22) are $f_1 = -\frac{1}{2}x$, $f_2 = -\frac{1}{2}y$, $g_{11} = -y$, $g_{12} = 0$, $g_{21} = 0$ and $g_{22} = x$. The system of determining equations for equations (6.68) becomes

$$\begin{gathered} \xi_{1,t} - \tfrac{1}{2}x\xi_{1,x} - \tfrac{1}{2}y\xi_{1,y} + \tfrac{1}{2}y^2\xi_{1,xx} + \tfrac{1}{2}x^2\xi_{1,yy} + x\tau + \tfrac{1}{2}\xi_1 = 0, \\ \xi_{2,t} - \tfrac{1}{2}x\xi_{2,x} - \tfrac{1}{2}y\xi_{2,y} + \tfrac{1}{2}y^2\xi_{2,xx} + \tfrac{1}{2}x^2\xi_{2,yy} + y\tau + \tfrac{1}{2}\xi_2 = 0, \\ -y\xi_{1,x} + y\tau + \xi_2 = 0, \\ x\xi_{2,y} - x\tau - \xi_1 = 0, \quad \xi_{1,y} = 0, \quad \xi_{2,x} = 0. \end{gathered} \tag{6.69}$$

From the last three equations of (6.69) one finds

$$\xi_1 = \xi_1(t,x), \quad \xi_2 = \xi_2(t,y), \quad \tau = \xi_{2,y} - \frac{1}{x}\xi_1.$$

Substituting them into the remaining equations of (6.69), one obtains that the functions ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered} \xi_{1,t} - \tfrac{1}{2}x\xi_{1,x} + \tfrac{1}{2}y^2\xi_{1,xx} + x\xi_{2,y} - \tfrac{1}{2}\xi_1 = 0, \\ \xi_{2,t} - \tfrac{1}{2}y\xi_{2,y} + \tfrac{1}{2}x^2\xi_{2,yy} + y(\xi_{2,y} - \tfrac{1}{x}\xi_1) + \tfrac{1}{2}\xi_2 = 0, \\ -y\xi_{1,x} + y(\xi_{2,y} - \tfrac{1}{x}\xi_1) + \xi_2 = 0. \end{gathered} \tag{6.70}$$

Differentiating the third equation of (6.70) with respect to x, one obtains

$$\xi_{1,xx} + \frac{1}{x}\xi_{1,x} - \frac{1}{x^2}\xi_1 = 0,$$

thus

$$\xi_{1,x} = F(t)x + G(t)\frac{1}{x}.$$

Substituting it into the first equation of (6.70), one gets

$$x(F' - F + \xi_{2,y}) + \frac{1}{x^3}y^2G = 0.$$

Splitting this equation with respect to x and y, one has

$$F' - F + \xi_{2,y}, \quad G = 0. \tag{6.71}$$

Substituting $\xi_{1,x}$ into the second equation of (6.70), one gets

$$\xi_{2,t} - \frac{1}{2}y\xi_{2,y} + \frac{1}{2}x^2\xi_{2,yy} + y(\xi_{2,y} - F) + \frac{1}{2}\xi_2 = 0.$$

Splitting this equation with respect to x, one has

$$\xi_{2,yy} = 0,$$

thus

$$\xi_2 = H(t)y + K(t).$$

Substituting ξ_1 and ξ_2 into the third equation of (6.70), one gets

$$2y(F - H) + K = 0.$$

Splitting this equation with respect to y, one has

$$F = H, \quad K = 0,$$

thus

$$\xi_2 = F(t)y.$$

Substituting it into the first equation of (6.71), one gets

$$F' = 0,$$

thus $F = C$ for some constant C, and hence

$$\xi_1 = Cx, \quad \xi_2 = Cy. \tag{6.72}$$

It follows that $\tau = 0$, and $h = 0$. Thus, a basis of generators corresponding to (6.72) is

$$x\partial_x + y\partial_y. \tag{6.73}$$

The Lie group of transformations corresponding to the admitted generator (6.73) is

$$\bar{x} = xe^a, \quad \bar{y} = ye^a, \quad \bar{t} = t.$$

Applying Itô's formula to the functions $\varphi_1 = xe^a$ and $\varphi_2 = ye^a$, one has

$$\begin{aligned} X(t)e^a &= X(0)e^a - \frac{1}{2}\int_0^t X(s)e^a ds - \int_0^t Y(s)e^a dB_1(s), \\ Y(t)e^a &= Y(0)e^a - \frac{1}{2}\int_0^t Y(s)e^a ds + \int_0^t X(s)e^a dB_2(s). \end{aligned}$$

Since $X(t,\omega)e^a = \bar{X}(t,\omega) = \bar{X}(\bar{t},\omega)$ and $Y(t,\omega)e^a = \bar{Y}(t,\omega) = \bar{Y}(\bar{t},\omega)$, one has

$$\begin{aligned} \bar{X}(\bar{t},\omega) &= \bar{X}(0,\omega) - \frac{1}{2}\int_0^{\bar{t}} \bar{X}(s,\omega)ds - \int_0^{\bar{t}} \bar{Y}(s,\omega)dB_1(s), \\ \bar{Y}(\bar{t},\omega) &= \bar{Y}(0,\omega) - \frac{1}{2}\int_0^{\bar{t}} \bar{Y}(s,\omega)ds + \int_0^{\bar{t}} \bar{X}(s,\omega)dB_2(s). \end{aligned}$$

This confirms that this Lie group transforms a solution of (6.68) into a solution of the same system.

6.2.5 Nonlinear Itô System

Let μ_1 and μ_2 be constants. Consider the system of equations discussed in Gaeta and Quintero (1999),

$$\begin{aligned} dX(t) &= \tfrac{\mu_1}{X(t)}dt + dB_1(t), \\ dY(t) &= \mu_2 dt + dB_2(t). \end{aligned} \tag{6.74}$$

The associated Fokker-Planck equation is

$$u_t = \frac{1}{2}(u_{xx} + u_{yy}) + \frac{\mu_1}{x^2}u - \frac{\mu_1}{x}u_x - \mu_2 u_y,$$

which has been studied by Finkel (1999). For equations (6.74) the corresponding functions of equations (4.22) are $f_1 = \frac{\mu_1}{x}$, $f_2 = \mu_2$, $g_{11} = 1$, $g_{12} = 0$, $g_{21} = 0$ and $g_{22} = 1$. The system of determining equations for equations (6.74) becomes

$$\begin{aligned} &\xi_{1,t} + \tfrac{\mu_1}{x}\xi_{1,x} + \mu_2\xi_{1,y} + \tfrac{1}{2}\xi_{1,xx} + \tfrac{1}{2}\xi_{1,yy} - 2\tfrac{\mu_1}{x}\tau + \tfrac{\mu_1}{x^2}\xi_1 = 0, \\ &\xi_{2,t} + \tfrac{\mu_1}{x}\xi_{2,x} + \mu_2\xi_{2,y} + \tfrac{1}{2}\xi_{2,xx} + \tfrac{1}{2}\xi_{2,yy} - 2\mu_2\tau = 0, \\ &\xi_{1,x} - \tau = 0, \quad \xi_{2,y} - \tau = 0, \quad \xi_{1,y} = 0, \quad \xi_{2,x} = 0. \end{aligned} \tag{6.75}$$

From the last equations of (6.75) one can find

$$\xi_1 = \xi_1(t,x), \quad \xi_2 = \xi_2(t,y), \quad \tau = \xi_{2,y}.$$

Substituting these into the remaining equations of (6.75), one obtains that the function ξ_1 and ξ_2 have to satisfy the system of equations

$$\begin{gathered}\xi_{1,t} + \tfrac{\mu_1}{x}\xi_{1,x} + \tfrac{1}{2}\xi_{1,xx} - 2\tfrac{\mu_1}{x}\xi_{2,y} + \tfrac{\mu_1}{x^2}\xi_1 = 0,\\ \xi_{2,t} - \mu_2\xi_{2,y} + \tfrac{1}{2}\xi_{2,yy} = 0,\\ \xi_{1,x} - \xi_{2,y} = 0.\end{gathered} \tag{6.76}$$

Differentiating the third equation of (6.76) with respect to x and y, respectively, one obtains

$$\xi_{1,xx} = 0, \quad \xi_{2,yy} = 0,$$

thus

$$\xi_1 = h_1(t)x + h_2(t), \quad \xi_2 = h_3(t)y + h_4(t).$$

Substituting them into other equations of (6.76), one obtains

$$\begin{gathered}h_1' + h_2' + 2\tfrac{\mu_1}{x}h_1 - 2\tfrac{\mu_1}{x}h_3 + \tfrac{\mu_1}{x^2}h_2 = 0,\\ h_3' + h_4' - \mu_2 h_3 = 0.\end{gathered} \tag{6.77}$$

Splitting the first equation of (6.77) with respect to x, one has

$$h_1 - h_3 = 0, \quad h_2 = 0.$$

Substituting them into (6.77), one obtains

$$\begin{gathered}h_3' = 0,\\ h_3' + h_4' - \mu_2 h_3 = 0.\end{gathered} \tag{6.78}$$

From the first equation of (6.78) one finds

$$h_3 = C_1.$$

Substituting it into other equations of (6.77), one obtains

$$h_4' - \mu_2 C_1 = 0,$$

then

$$h_4 = \mu_2 C_1 t + C_2.$$

Hence

$$\xi_1 = C_1 x, \quad \xi_2 = C_1(y + \mu_2 t) + C_2, \quad \tau = C_1, \tag{6.79}$$

and $h = 2C_1 t$. Thus, a basis of generators corresponding to (6.79) is

$$\partial_y \quad x\partial_x + (y + \mu_2 t)\partial_y + 2t\partial_t.$$

The Lie group of transformations corresponding to the first admitted generator is

$$\bar{x} = x, \quad \bar{y} = y + a, \quad \bar{t} = t. \tag{6.80}$$

It is not difficult to show that this Lie group transforms a solution of (6.74) into a solution of the same system.

For finding the Lie group of transformations corresponding to the second admitted generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2H, \quad \frac{\partial \varphi_1}{\partial a} = \varphi_1, \quad \frac{\partial \varphi_2}{\partial a} = \varphi_2 + \mu_2 H,$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi_1 = x, \quad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y

$$\bar{t} = H = te^{2a}, \quad \bar{x} = \varphi_1 = xe^{a}, \quad \bar{y} = \varphi_2 = \mu_2 te^{2a} + (y - \mu_2 t)e^{a}. \tag{6.81}$$

Hence $\eta^2 = H_t = e^{2a}$.

Let us show that this Lie group of transformation transforms a solution of equations (6.74) into a solution of the same equation. Assume that $(X(t), Y(t))$ is a solution of equations (6.74). The Brownian motions $B_1(t)$ and $B_2(t)$ are transformed according to the formula

$$\begin{aligned} \bar{B}_1(t) &= \int_0^{\alpha(t)} e^a dB_1(s), \\ \bar{B}_2(t) &= \int_0^{\alpha(t)} e^a dB_2(s), \quad t \geq 0, \end{aligned} \tag{6.82}$$

where

$$\beta(t) = \int_0^t e^{2a} ds = te^{2a}, \quad \alpha(t) = \inf_{s \geq 0}\{s : \beta(s) > t\} = te^{-2a}, \quad t \geq 0.$$

Applying Itô's formula to the functions $\varphi_1(t, x, y, a) = xe^a$ and $\varphi_2(t, x, y, a) = \mu_2 te^{2a} + (y - \mu_2 t)e^a$, one has

$$\begin{aligned} \varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) \\ &\quad + \int_0^t \frac{\mu_1 e^a}{X(s,\omega)} ds + \int_0^t e^a dB_1(s), \\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a) + \int_0^t \mu_2 e^{2a} ds + \int_0^t e^a dB_2(s). \end{aligned} \tag{6.83}$$

By virtue of (4.9)

$$\frac{\partial \alpha}{\partial \bar{t}}(\bar{t}) = e^{-2a}.$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integrals in (6.83), they become

$$\int_0^t \frac{\mu_1 e^a}{X(s)} ds = \int_0^{\beta(t)} \frac{\mu_1}{X(\alpha(\bar{s}))e^a} d\bar{s},$$
$$\int_0^t \mu_2 e^{2a} ds = \int_0^{\beta(t)} \mu_2 d\bar{s}.$$

Because of the transformation of the Brownian motions (6.82), the Itô integrals in (6.83) become

$$\int_0^t e^a dB_1(s) = \int_0^{\beta(t)} d\bar{B}_1(\bar{s}).$$

$$\int_0^t e^a dB_2(s) = \int_0^{\beta(t)} d\bar{B}_2(\bar{s}).$$

Because $X(\alpha(\bar{t}), \omega)e^a = \bar{X}(\bar{t}, \omega)$, one gets

$$\begin{aligned}\varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) \\ &\quad + \int_0^{\beta(t)} \frac{\mu_1}{\bar{X}(s,\omega)} ds + \int_0^{\beta(t)} d\bar{B}_1(s), \\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a) + \int_0^{\beta(t)} \mu_2 ds + \int_0^{\beta(t)} d\bar{B}_2(s).\end{aligned}$$

Since $\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \bar{X}(\beta(t), \omega)$, and $\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \bar{Y}(\beta(t), \omega)$, one has

$$\begin{aligned}\bar{X}(\beta(t), \omega) &= \bar{X}(0,\omega) + \int_0^{\beta(t)} \frac{\mu_1}{\bar{X}(s,\omega)} ds + \int_0^{\beta(t)} d\bar{B}_1(s), \\ \bar{Y}(\beta(t), \omega) &= \bar{Y}(0,\omega) + \int_0^{\beta(t)} \mu_2 ds + \int_0^{\beta(t)} d\bar{B}_2(s).\end{aligned}$$

This confirms that the Lie group of transformations (6.81) transforms any solution of system (6.74) into a solution of the same system.

Chapter VII

Conclusion

This thesis is devoted to an application of group analysis to the stochastic differential equations.

7.1 Thesis Summary

A new definition of an admitted Lie group of transformations for stochastic differential equations was developed in this thesis. This approach includes dependent and independent variables in the transformation. The transformation of Brownian motion is defined by the transformation of dependent and independent variables. Correctness of all developed construction is strictly proven. Thus a correct approach for generalization of group analysis to stochastic differential equations has been developed. The developed theory was applied to a variety of stochastic differential equations. First, stochastic differential equations with one Brownian motion were studied. Then the theory was extended to stochastic differential equations with multi-Brownian motion. For one dimensional Itô stochastic differential equations, two applications were studied: an equation describing geometric Brownian motion and an equation describing Brownian motion with drift. For systems of stochastic differential equations with one-dimensional Brownian motion, five applications were studied: a system describing the graph of Brownian motion, a system describing the Black and Scholes market, a system describing a narrow-sense linear system, a system describing a mean-reverting Ornstein-Uhlenbeck process and a nonlinear Itô system. For systems of stochastic differential equations with multi-Brownian motion, four applications were studied: a system describing location and motion, a system describing a model for a vibrating string subject to a stochastic force, a system representing an Ornstein-Uhlenbeck process, a linear Itô dynamical system and a nonlinear Itô system.

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