CALCULUS I

(103101) WORKBOOK

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Limits

Recall: (Rules for limits)

$$\lim_{x\to a} x = a$$

$$\lim_{x\to a} c = c \qquad (c \text{ constant})$$

$$\lim_{x\to a}\big(f(x)\pm g(x)\big) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\big(c\,f(x)\big) = c\lim_{x\to a} f(x) \qquad (c \text{ constant})$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \qquad (\text{if } \lim_{x\to a} g(x)\neq 0)$$

$$\lim_{x\to a}\big(f(x)\big)^n = \left(\lim_{x\to a} f(x)\right)^n$$

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

Exercise 1: Find the following limits using the *rules for limits.*

1. $$\lim_{x\to 2}\left(x^2+2\right)\left(x^2+x\right) = \lim_{x\to 2}\left(\ldots\ldots\right)\cdot\lim_{x\to 2}\left(\ldots\ldots\right)$$

$$= \left(\left[\lim_{x\to 2}\ldots\ldots\right]^{\ldots\ldots} + \lim_{x\to 2}\ldots\ldots\right)\cdot\left(\left[\lim_{x\to 2}\ldots\ldots\right]^{\ldots\ldots} + \lim_{x\to 2}\ldots\ldots\right)$$

$$= \left(\ldots\ldots^{\ldots\ldots}+\ldots\ldots\right)\cdot\left(\ldots\ldots^{\ldots\ldots}+\ldots\ldots\right) = \ldots\ldots = \ldots\ldots$$

2. $$\lim_{y\to 3}\frac{3(8y^2-1)}{2y^2(y-1)^4} = \frac{\lim_{y\to 3}\left(\qquad\right)}{\lim_{y\to 3}\ldots\ldots} = \frac{3}{2}\,\frac{8\lim_{y\to 3}\qquad - \lim_{y\to 3}}{\lim_{y\to 3}\ldots\ldots\cdot\left(\ldots\ldots\right)^4}$$

$$= \frac{3}{2}\,\frac{8\left(\lim_{y\to 3}\qquad\right)^2 - \lim_{y\to 3}}{\left(\lim_{y\to 3}\ldots\right)^2\cdot\left(\lim_{y\to 3}\ldots - \lim_{y\to 3}\ldots\right)^4} = \frac{3}{2}\,\frac{8(\quad)^2 -}{(\ldots)^2\cdot(\ldots-\ldots)^4} = \ldots\ldots = \ldots$$

3. $\lim\limits_{x\to -2}\sqrt[3]{\dfrac{3x^2+4x}{2x+3}} = \sqrt[3]{\lim\limits_{x\to -2}\dfrac{\quad}{\ldots\ldots\ldots\ldots\ldots}} = \sqrt[3]{\dfrac{\lim\limits_{x\to -2}(\qquad\qquad)}{\lim\limits_{x\to -2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}}$

$= \sqrt[3]{\dfrac{3\left(\lim\limits_{x\to -2}\qquad\right)^{\ldots\ldots}+4\left(\lim\limits_{x\to -2}\qquad\right)}{2\left(\lim\limits_{x\to -2}\qquad\right)+\left(\lim\limits_{x\to -2}\qquad\right)}} = \sqrt[3]{\dfrac{3(\quad)^{\ldots\ldots}+4(\quad)}{2(\quad)+(\quad)}} = \sqrt[3]{\dfrac{\quad}{\quad}} = \ldots\ldots$

Exercise 2: The following limits are given:

$$\lim_{x\to 2} f(x)=3,\quad \lim_{x\to 2} g(x)=-1 \text{ and } \lim_{x\to 2} h(x)=2$$

Find the specified limits:

1. $\lim\limits_{x\to 2}[\,3f(x)-2g(x)\,] = \lim\limits_{x\to 2}[\,3f(x)] - \lim\limits_{x\to 2}[\ldots\ldots\ldots\ldots]$

$= 3\lim\limits_{x\to 2}\ldots\ldots\ldots\ldots - 2\lim\limits_{x\to 2}\ldots\ldots\ldots\ldots = (\ldots\ldots)(\ldots\ldots)-(\ldots\ldots)(\ldots\ldots) = \ldots\ldots\ldots\ldots$

2. $\lim\limits_{x\to 2}\left[\,f(x)g(x)+h(x)^2\,\right] = \lim\limits_{x\to 2}[\,f(x)g(x)] + \lim\limits_{x\to 2}[\ldots\ldots\ldots\ldots]$

$= \left[\lim\limits_{x\to 2}\ldots\ldots\ldots\right]\left[\lim\limits_{x\to 2}\ldots\ldots\ldots\right] + \left[\lim\limits_{x\to 2}\ldots\ldots\ldots\right]^2 = (\ldots\ldots)(\ldots\ldots)+(\ldots\ldots)^2 = \ldots\ldots\ldots$

3. $\lim\limits_{x\to 2}\dfrac{h(x)-3g(x)}{f(x)^3+1} = \dfrac{\lim\limits_{x\to 2}[\ldots\ldots\ldots\ldots\ldots\ldots]}{\lim\limits_{x\to 2}[\ldots\ldots\ldots\ldots\ldots\ldots]}$

$= \dfrac{\lim\limits_{x\to 2}\ldots\ldots\ldots\ldots - \ldots\lim\limits_{x\to 2}\ldots\ldots\ldots}{\left(\lim\limits_{x\to 2}\ldots\ldots\right)^{\ldots\ldots}+\lim\limits_{x\to 2}\ldots\ldots\ldots} = \dfrac{\quad}{\ldots\ldots\ldots\ldots\ldots\ldots} = \ldots\ldots\ldots\ldots\ldots$

4. $\lim\limits_{x\to 2}\sqrt{f(x)^2-g(x)^2} = \sqrt{\lim\limits_{x\to 2}\left[\ldots\ldots\quad\ldots\quad\ldots\ldots\ldots\ldots\right]}$

$= \sqrt{\left(\lim\limits_{x\to 2}\ldots\ldots\ldots\ldots\right)^2-\left(\lim\limits_{x\to 2}\ldots\ldots\ldots\ldots\right)^2} = \sqrt{(\ldots\ldots)^2-(\ldots\ldots)^2} = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

Exercise 3: Compute each of the following limits:

1. $\lim\limits_{x \to -3} \dfrac{x^2 - x - 12}{x + 3}$

If we substitute $x = -3$ then we obtain a fraction of the form $\dfrac{\quad}{\ldots\ldots}$

We therefore must simplify:

$$\lim_{x \to -3} \frac{x^2 - x - 12}{x + 3} = \lim_{x \to -3} \frac{(\qquad)(\qquad)}{x + 3} = \lim_{x \to -3} \ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

2. $\lim\limits_{x \to 0} \dfrac{x^3 - 4x^2 + 2x}{x}$

If we substitute $x = \ldots\ldots$ then we obtain a fraction of the form $\dfrac{\quad}{\ldots\ldots}$.

We therefore must simplify:

$$\lim_{x \to 0} \frac{x^3 - 4x^2 + 2x}{x} = \lim_{x \to 0} \ldots\ldots\ldots\ldots = \lim_{x \to 0} \ldots\ldots\ldots\ldots = \ldots\ldots$$

3. $\lim\limits_{x \to -2} \dfrac{x^3 + 8}{x^2 - 4}$

If we substitute $x = \ldots\ldots$ then we obtain a fraction of the form $\dfrac{\quad}{\ldots\ldots}$.

We therefore must simplify:

$$\lim_{x \to -2} \frac{x^3 + 8}{x^2 - 4} = \lim_{x \to -2} \frac{(\qquad)(\qquad)}{(\qquad)(\qquad)}$$

$$= \lim_{x \to -2} \frac{\quad}{\ldots\ldots\ldots\ldots} = \ldots\ldots\ldots\ldots = \ldots\ldots$$

4. $\lim\limits_{x \to 3} \dfrac{x^2 + 8x}{x}$

If we substitute $x = \ldots\ldots$ then we obtain $\dfrac{\quad}{\ldots\ldots}$ Therefore,

$$\lim_{x \to 3} \frac{x^2 + 8x}{x} = \ldots\ldots = \ldots\ldots$$

5. $\lim\limits_{t\to 0}\dfrac{\sqrt{3-t}-\sqrt{3}}{t}$

If we substitute $t=$ .... then we obtain $\dfrac{\quad}{\ldots\ldots}$. We therefore must simplify:

$$\lim_{t\to 0}\frac{\sqrt{3-t}-\sqrt{3}}{t} = \lim_{t\to 0}\frac{\sqrt{3-t}-\sqrt{3}}{t}\,\frac{(\qquad\qquad)}{(\qquad\qquad)}$$

$$= \lim_{t\to 0}\frac{\quad - \quad}{t(\qquad\qquad)} = \lim_{t\to 0}\frac{\qquad}{\ldots\ldots\ldots\ldots} = \ldots\ldots$$

6. $\lim\limits_{h\to 0}\dfrac{(2+h)^3-8}{h}$

If we substitute $h=$ .... then we obtain $\dfrac{\quad}{\ldots\ldots}$. We therefore must simplify:

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h} = \lim_{h\to 0}\frac{\qquad\qquad -8}{h}$$

$$= \lim_{h\to 0}\frac{\qquad}{h} = \lim_{h\to 0}\left(\ldots\ldots\ldots\ldots\right) = \ldots\ldots$$

7. $\lim\limits_{x\to 1}\left[\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right]$

If we substitute $x=$ .... then we obtain $\dfrac{1}{\ldots}-\dfrac{2}{\ldots}$ .We therefore simplify:

$$\lim_{x\to 1}\left[\frac{1}{x-1}-\frac{2}{x^2-1}\right] = \lim_{x\to 1}\left[\frac{\ldots\ldots\ldots}{(x-1)(\qquad)}-\frac{2}{x^2-1}\right]$$

$$= \lim_{x\to 1}\frac{\ldots\ldots\ldots\ldots}{x^2-1} = \lim_{x\to 1}\frac{\qquad}{x^2-1} = \lim_{x\to 1}\frac{\qquad}{(x-1)(\qquad)} = \ldots\ldots$$

8. $\lim\limits_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$

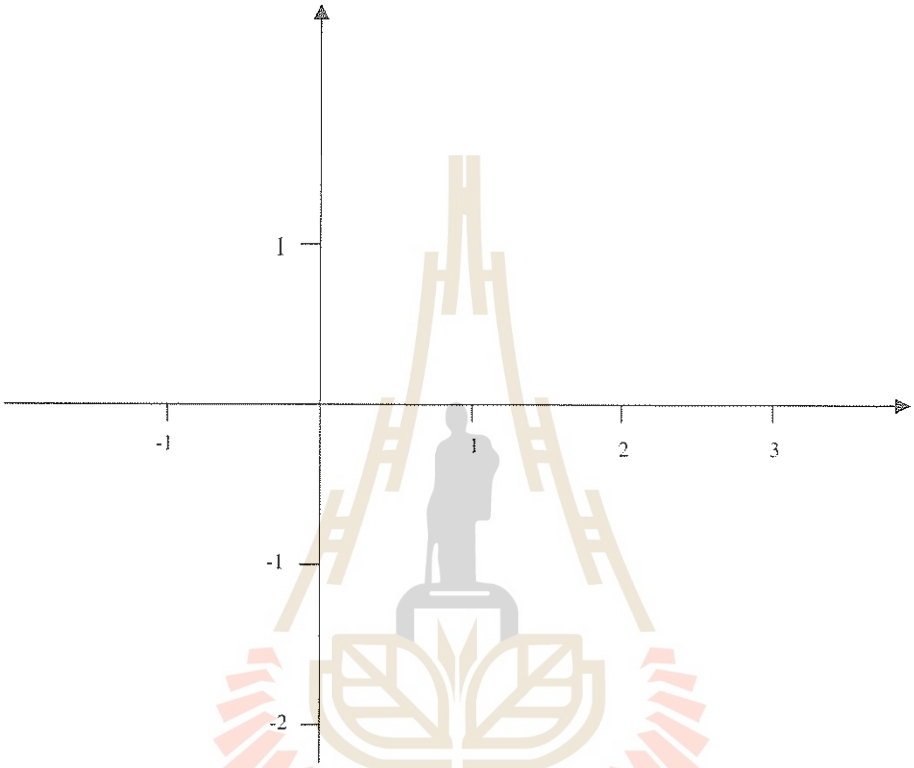
If we substitute $x=$ .... then we obtain $\dfrac{\quad}{\ldots}$. We therefore must simplify:

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim_{x\to 2}\frac{\frac{\quad}{2x}-\frac{\quad}{2x}}{x-2} = \lim_{x\to 2}\frac{\frac{\qquad}{2x}}{x-2} = \lim_{x\to 2}\frac{\qquad}{2x(x-2)} = \lim_{x\to 2}\frac{\qquad}{\ldots\ldots} = \ldots$$

Exercise 4: Consider the function

$$f(x) = \begin{cases} x+1 & (x<0) \\ 1-x^2 & (0 \le x \le 1) \\ x-2 & (x>1) \end{cases}$$

Sketch the graph of f :



Now find each of the following limits, if it exists.

1. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}$ .... = ...

2. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}$ .... = ...

3. $\lim_{x\to 0} f(x)$ ....

4. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}$ .... = ...

5. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}$ .... = ...

6. $\lim_{x\to 1} f(x)$ ....

Additional Exercises:

1) Find the following limits

a) $\lim_{x\to -1} \dfrac{x^2-x-2}{x+1}$

b) $\lim_{h\to 0} \dfrac{(h-4)^2-16}{h}$

c) $\lim_{t\to 2} \dfrac{t^3-2t-4}{t^2-4}$

d) $\lim_{h\to 0} \dfrac{\dfrac{2}{(3+h)^2}-\dfrac{2}{9}}{h}$

e) $\lim_{x\to -1} \dfrac{x^2-2x-2}{x-1}$

f) $\lim_{x\to 0} \left[\dfrac{1}{x\sqrt{x+1}}-\dfrac{1}{x}\right]$

g) $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

h) $\lim_{x\to 8} \dfrac{\sqrt[3]{x^2}-4}{x-8}$

i) $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$

2) Find the following one-sided limits

a) $\lim_{x\to 2^+} \dfrac{x-2}{|x-2|}$

b) $\lim_{x\to 2^-} \dfrac{x-2}{|x-2|}$

c) $\lim_{x\to 3^+} \dfrac{3-x}{|3-x|}$

d) $\lim_{x\to 3^-} \dfrac{3-x}{|3-x|}$

e) $\lim_{x\to 0^+} \left[\dfrac{1}{x}-\dfrac{1}{|x|}\right]$

f) $\lim_{x\to 0^-} \left[\dfrac{1}{x}-\dfrac{1}{|x|}\right]$

3) Sketch the graph of the function

$$f(x)=\begin{cases} x+4 & (x<0) \\ \sqrt{16-x^2} & (0\le x\le 4) \\ \sqrt{x-4} & (x>4) \end{cases}$$

Find each of the following limits, if it exists.

a) $\lim_{x\to 0^-} f(x)$

b) $\lim_{x\to 0^+} f(x)$

c) $\lim_{x\to 0} f(x)$

d) $\lim_{x\to 4^-} f(x)$

e) $\lim_{x\to 4^+} f(x)$

f) $\lim_{x\to 4} f(x)$

Limits Involving Trigonometric Functions

Recall:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Exercise 1: Find the following limits:

1. $$\lim_{x\to 0} \frac{\sin 3x}{5x} = \lim_{x\to 0} \frac{\cdots}{5}\,\frac{\sin 3x}{\ldots x} = \frac{\,}{5}\cdot\ldots\ldots = \frac{\qquad}{\ldots\ldots}$$

2. $$\lim_{t\to 0} \frac{\cos^2 t - 1}{t} = \lim_{t\to 0} \frac{(\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots)}{t} = \ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

3. $$\lim_{\theta\to 0} \frac{\sin^2 3\theta}{\theta} = \lim_{\theta\to 0} \ldots\ldots\ldots\ldots\ldots\ldots \frac{\qquad}{\theta} = \ldots\ldots\ldots \cdot \ldots\ldots = \ldots\ldots\ldots$$

4. $$\lim_{x\to 0} \frac{\tan^2 2x}{\tan^2 3x} = \left[\lim_{x\to 0} \frac{\tan 2x}{\tan 3x}\right]^2 = \left[\lim_{x\to 0} \frac{\dfrac{\sin 2x}{\ldots\ldots\ldots\ldots\ldots}}{\dfrac{\qquad}{\ldots\ldots\ldots\ldots\ldots}}\right]^2 = \left[\lim_{x\to 0} \frac{\sin 2x}{\qquad}\right]^2$$

$$= \left[\lim_{x\to 0} \frac{\sin 2x}{\ldots\ldots\ldots\ldots\ldots}\,\frac{\qquad}{\sin 3x}\,\frac{2x}{\ldots\ldots\ldots}\,\frac{\qquad}{\ldots\ldots\ldots}\right]^2$$

$$= [\ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots]^2 = \ldots\ldots$$

5. $$\lim_{x\to 0} \frac{\tan 3x}{x} = \lim_{x\to 0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

6. $$\lim_{x\to \pi/4} \frac{\tan 3x}{x} = \ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots \qquad !!!$$

7. $\lim\limits_{x \to 0} \dfrac{x^2}{\sin(x^2)} = \lim\limits_{u \to \ldots\ldots} \dfrac{}{\sin(\ldots\ldots)} = \lim\limits_{u \to \ldots\ldots} \dfrac{1}{\dfrac{\sin(\ldots\ldots)}{\ldots\ldots}} = \dfrac{1}{\ldots\ldots} = \ldots\ldots$

> $u = \ldots\ldots\ldots\ldots$
>
> If $x \to 0$ then $u \to \ldots\ldots\ldots\ldots$

8. $\lim\limits_{x \to \pi/2} \dfrac{1-\sin x}{x-\frac{\pi}{2}} = \lim\limits_{u \to \ldots\ldots} \dfrac{1-\sin(u+\ldots\ldots\ldots)}{\ldots\ldots\ldots} = \lim\limits_{u \to \ldots\ldots\ldots} \dfrac{1-}{\ldots\ldots\ldots} = \ldots\ldots$

> $u = \ldots\ldots\ldots\ldots$
>
> If $x \to \frac{\pi}{2}$ then $u \to \ldots\ldots\ldots\ldots$
>
> Also, $x = \ldots\ldots\ldots\ldots$

> $\sin\left(u+\frac{\pi}{2}\right)$
>
> $= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
>
> $= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

9. $\lim\limits_{\theta \to 0} \dfrac{\sin(\sin\theta)}{\sin\theta} = \lim\limits_{u \to \ldots\ldots} \dfrac{}{\ldots\ldots\ldots} = \ldots\ldots\ldots$

> $u = \ldots\ldots\ldots\ldots$
>
> If $\ldots \to 0$ then $u \to \ldots\ldots$

Additional Exercises:

1) Find the following limits:

a) $\lim\limits_{x \to 0} \dfrac{8x}{\sin 2x}$

b) $\lim\limits_{x \to 0} \dfrac{x \sin 3x}{\sin^2 9x}$

c) $\lim\limits_{h \to 0} \dfrac{\sin^2 2h}{1-\cos h}$

d) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{\tan x}$

e) $\lim\limits_{t \to 0} \dfrac{\cos t - 1}{\sin t}$

f) $\lim\limits_{t \to 0} \dfrac{\cos t - 1}{\sqrt[3]{t}}$

g) $\lim\limits_{\theta \to 0} \dfrac{\cos(\sin\theta)-1}{\sin\theta}$

h) $\lim\limits_{x \to \pi/4} \dfrac{1-\sin\left(x+\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}$

i) $\lim\limits_{x \to 0} \dfrac{2x^3}{\sin 3x - \tan 3x}$

j) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta + \tan\theta}$

k) $\lim\limits_{x \to 0} x \cos\dfrac{1}{x}$

l) $\lim\limits_{x \to 0} x^2 \sin\dfrac{1}{x^2}$

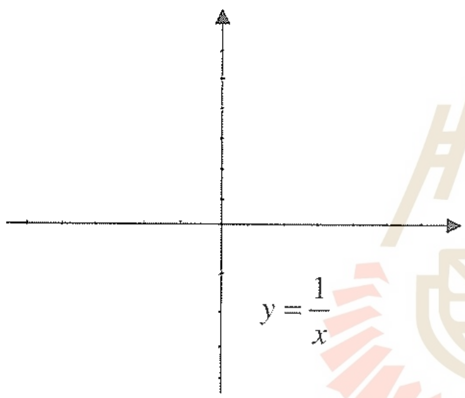
Limits Involving Infinity

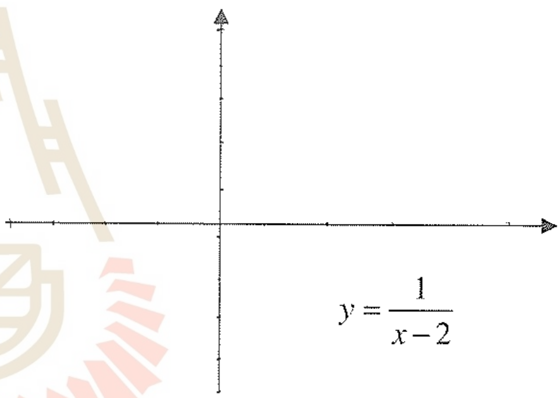
Recall:

$$\lim_{x\to 0^+}\frac{1}{x}=\infty \quad \text{and} \quad \lim_{x\to 0^-}\frac{1}{x}=-\infty$$

$$\lim_{x\to \infty}\frac{1}{x^n}=0 \quad \text{and} \quad \lim_{x\to -\infty}\frac{1}{x^n}=0 \qquad (n=1,2,\dots)$$

Exercise 1: Sketch the graph of $f(x)=\dfrac{1}{x-2}$. Then discuss $\displaystyle\lim_{x\to 2}\frac{1}{x-2}$





- If x is close to 2 but $x>2$ then $x-2$ is close to ……and is ……Therefore,

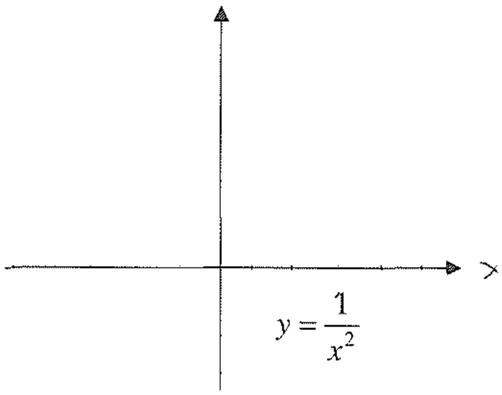
  $$\lim_{x\to 2^+}\frac{1}{x-2}=\dots\dots$$

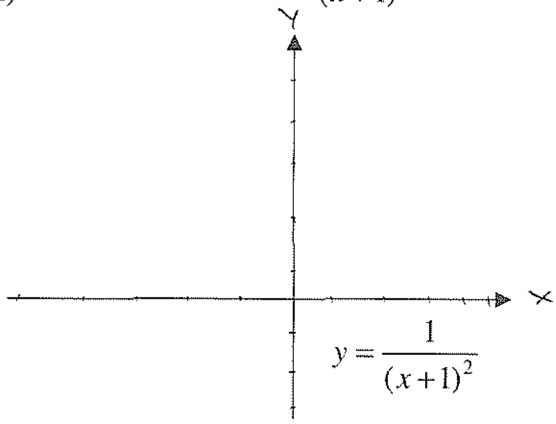
- If x is close to 2 but $x<2$ then $x-2$ is close to ..0..and is ..< 0 Therefore,

  $$\lim_{x\to 2^-}\frac{1}{x-2}=\dots\dots$$

- $$\lim_{x\to 2}\frac{1}{x-2} \quad \dots\dots\dots\dots\dots\dots$$

Exercise 2: Sketch the graph of $g(x) = \dfrac{1}{(x+1)^2}$. Then discuss $\lim\limits_{x \to -1} \dfrac{1}{(x+1)^2}$





- If x is close to -1 then $(x+1)^2$ is close to ……and is …… Therefore,

$$\lim_{x \to -1} \frac{1}{(x+1)^2} = \ldots\ldots$$

Exercise 3: Discuss $\lim\limits_{x \to -1} \dfrac{x-2}{x^2+4x+3}$

If we substitute $x = -1$ we obtain $\dfrac{\ldots\ldots}{\ldots\ldots} = \ldots\ldots$. Check the sign as $x \to -1$:

$$\frac{x-2}{x^2+4x+3} = \underline{\hspace{3cm}}$$

If $x \to -1^+$ then $\underline{\hspace{3cm}} \to \underline{\hspace{2cm}} = \ldots\ldots\ldots$

If $x \to -1^-$ then $\underline{\hspace{3cm}} \to \underline{\hspace{2cm}} = \ldots\ldots\ldots$

Therefore,

$$\lim_{x \to -1^+} \frac{x-2}{x^2+4x+3} = \ldots\ldots \quad \text{and} \quad \lim_{x \to -1^-} \frac{x-2}{x^2+4x+3} = \ldots\ldots$$

Exercise 4: Find all singular points of $f(x)=\dfrac{1+x}{x^2+x-6}$. Then find the limits at these points.

Factor the denominator:

$$f(x) \quad = \quad \frac{1+x}{x^2+x-6} \quad = \quad \frac{\qquad}{(\qquad)(\qquad)}$$

Singular points are $x=2$ and $x=$ ..-3... . Check the sign of $f(x)$ close to these points.

$1+x$				
$x+3$				
$x-2$				
$f(x)$				

Therefore,

$$\lim_{x\to 2^+}\frac{1+x}{x^2+x-6} \quad = \quad \ldots\ldots \qquad \text{and} \qquad \lim_{x\to 2^-}\frac{1+x}{x^2+x-6} \quad = \quad \ldots\ldots$$

$$\lim_{x\to -3^+}\frac{1+x}{x^2+x-6} \quad = \quad \ldots\ldots \qquad \text{and} \qquad \lim_{x\to -3^-}\frac{1+x}{x^2+x-6} \quad = \quad \ldots\ldots$$

Exercise 5: Discuss $\lim\limits_{x\to 3}\dfrac{1-2x}{x^2-6x+9}$

If we substitute $x=3$ we obtain $\dfrac{\ldots\ldots}{\ldots\ldots}=\ldots\ldots$. Check the sign as $x\to 3$:

$$\frac{1-2x}{x^2-6x+9} \quad = \quad \frac{1-2x}{(\qquad)} \quad = \quad (\ldots\ldots\ldots)\cdot\frac{1}{(\qquad)}$$

Therefore,

$$\lim_{x\to 3}\frac{1-2x}{x^2-6x+9} \quad = \quad \lim_{x\to 3}(\ldots\ldots\ldots)\cdot\frac{1}{(\qquad)} \quad = \quad \ldots\ldots\cdot\ldots\ldots \quad = \quad \ldots\ldots$$

Exercise 6: Find $\lim_{x\to\infty} \frac{5x^2-2x+1}{2x^2+3x-4}$ *without* using shortcuts.

$$\lim_{x\to\infty} \frac{5x^2-2x+1}{2x^2+3x-4} = \lim_{x\to\infty} \frac{\frac{1}{\dots\dots}\cdot(5x^2-2x+1)}{\frac{1}{\dots\dots}\cdot(2x^2+3x-4)}$$

$$= \lim_{x\to\infty} \frac{5-2\frac{1}{\dots}+\frac{1}{\dots\dots}}{2+3\frac{1}{\dots}-4\frac{1}{\dots\dots}} = \frac{5-2\cdot\dots+\dots}{2+3\cdot\dots-4\cdot\dots} = \frac{\dots\dots}{\dots\dots}$$

Exercise 7: Find the following limits quickly:

1. $\lim_{x\to\infty} \frac{2x^2-4x+3}{3-4x^2} = \lim_{x\to\infty} \frac{\dots\dots}{\dots\dots} = \lim_{x\to\infty} \frac{\dots\dots}{\dots\dots} = \dots\dots$

2. $\lim_{x\to-\infty} \frac{x^2+3x+1}{3x^4-2x+1} = \lim_{x\to\infty} \frac{\dots\dots}{\dots\dots} = \lim_{x\to\infty} \frac{\dots\dots}{\dots\dots} = \dots\dots$

3. $\lim_{x\to\infty} \frac{2x^2-2x+1}{1-3x^3} = \lim_{x\to\infty} \frac{2x^2}{\dots\dots\dots} = \lim_{x\to\infty} \frac{2}{\dots\dots\dots} = \frac{}{\dots\dots}$

4. $\lim_{x\to\infty} \frac{\pi+x-2\sqrt{x}}{2x^2-x+4} = \lim_{x\to\infty} \frac{\dots\dots\dots}{2x^2} = \lim_{x\to\infty} \frac{\dots\dots\dots}{\dots\dots\dots} = \dots\dots$

5. $\lim_{x\to-\infty} \frac{2x-4x^3}{2x^2+3x+4} = \lim_{x\to-\infty} \frac{\dots\dots\dots}{\dots\dots\dots} = \lim_{x\to-\infty} \dots\dots\dots\dots = \dots\dots$

6. $\lim_{x\to\infty} \frac{\sqrt{x^2+4}}{2x+6} = \lim_{x\to\infty} \frac{\dots\dots\dots}{\dots\dots\dots} = \lim_{x\to\infty} \frac{\dots\dots\dots}{\dots\dots\dots} \lim_{x\to\infty} \dots\dots\dots\dots = \dots\dots$

7. $\lim_{x\to-\infty} \frac{\sqrt{x^2+4}}{2x+6} = \lim_{x\to-\infty} \frac{\dots\dots\dots}{\dots\dots\dots} = \lim_{x\to-\infty} \frac{\dots\dots\dots}{\dots\dots\dots} \lim_{x\to-\infty} \dots\dots\dots\dots = \dots\dots$

(Note: $\sqrt{x^2} = \dots\dots$)

Exercise 8: Find the following trigonometric limits:

1. $\lim_{x\to\infty} \frac{\cos x^2}{\sqrt{x}}$

We estimate:

$$\ldots\ldots\ldots \leq \cos x^2 \leq \ldots\ldots\ldots \qquad \text{for all } x > 0 .$$

$$\ldots\ldots\ldots \leq \frac{\cos x^2}{\sqrt{x}} \leq \ldots\ldots\ldots \qquad \text{for all } x > 0 .$$

Now

$$\lim_{x\to\infty} \frac{-1}{\ldots\ldots\ldots} = \ldots\ldots \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{\ldots\ldots\ldots} = \ldots\ldots$$

By the ……………………….theorem, $\lim_{x\to\infty} \frac{\cos x^2}{\sqrt{x}} = \ldots\ldots$

2. $\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{u\to\ldots\ldots} \ldots\ldots \sin u = \lim_{u\to\ldots\ldots} \frac{\ldots\ldots\ldots}{\ldots\ldots} = \ldots\ldots$

$u = \ldots\ldots\ldots\ldots$

If $x \to \infty$ then $u \to \ldots\ldots$

Additional Exercises:

1) Discuss the following limits:

a) $\lim_{x\to 3} \frac{x^2-4}{3+2x-x^2}$

b) $\lim_{x\to -1} \frac{x^2-4}{3+2x-x^2}$

c) $\lim_{x\to 1} \frac{x-3}{x^4-1}$

d) $\lim_{x\to \pi} \frac{x}{\sin x}$

e) $\lim_{x\to 3} \frac{\sin x}{3-x}$

f) $\lim_{x\to \infty} \frac{x^2-4x+1}{3x-4x^2}$

g) $\lim_{x\to \infty} \frac{x^5+3x^2-1}{3x-4x^2}$

h) $\lim_{x\to -\infty} \frac{x^5+3x^2-1}{3x-4x^2}$

i) $\lim_{x\to \infty} \frac{x-4\sqrt{x}+1}{3x-4}$

j) $\lim_{x\to \infty} \frac{\sin x+2x}{x^2}$

Definition of the Derivative

Recall: The *derivative* of a function $y = f(x)$ at $x = a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Alternatively, setting $x = a + h$,

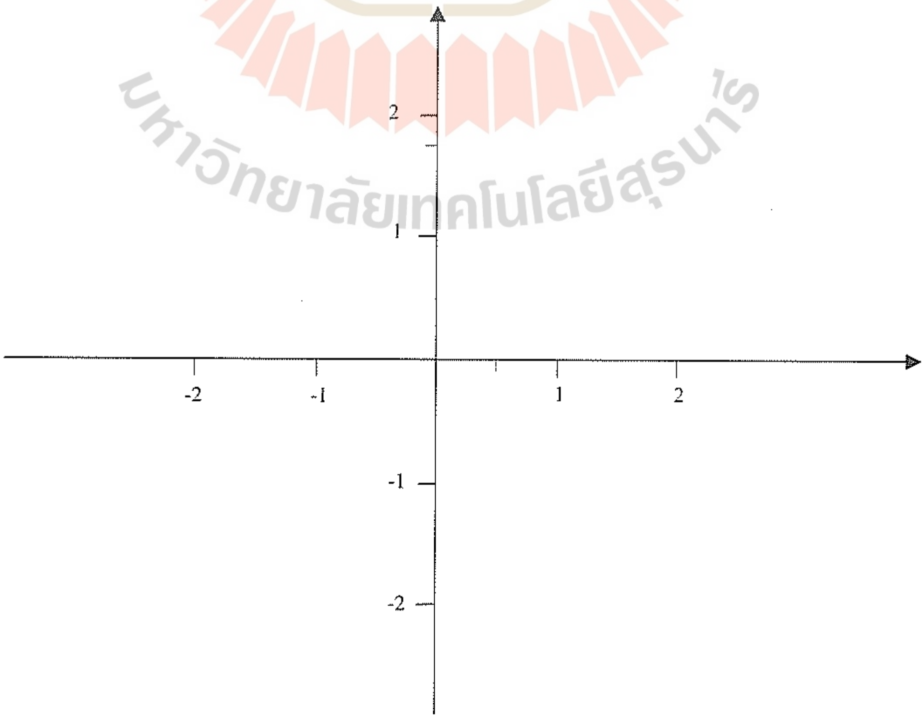
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If we compute the derivative at every point x in the domain of f we obtain a function $\frac{dy}{dx} = f'(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Exercise 1: Consider the function $f(x) = 2 - x^2$ and the point $P\left(\frac{1}{2}, \frac{7}{4}\right)$ on its graph.

1. Sketch the graph of f and the point P for $-2 \le x \le 2$



2. Sketch the secant line passing the points P and $Q(x, f(x))$, and compute its slope.

a) $x = 2$.

The slope is

$$m_{PQ} = \frac{\quad - f(0.5)}{\ldots - 0.5} = \frac{\quad - \quad}{\ldots - 0.5} = \ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

b) $x = 1$.

The slope is

$$m_{PQ} = \frac{\quad - f(0.5)}{\ldots - 0.5} = \frac{\quad - \quad}{\ldots - 0.5} = \ldots\ldots\ldots = \ldots\ldots = \ldots\ldots$$

c) $x = 0.6$

The slope is

$$m_{PQ} = \frac{\quad - f(0.5)}{\ldots - 0.5} = \frac{\quad - \quad}{\ldots - 0.5} = \ldots\ldots\ldots\ldots = \ldots\ldots$$

3. Sketch the tangent line to the graph of f at the point P. We expect this tangent line to have slope ....

4. Compute the slope of this tangent line:

$$m_{\tan} = f'(\ldots) = \lim_{x \to \ldots} \frac{\quad - f(0.5)}{x - 0.5} = \lim_{x \to \ldots} \frac{\quad - \quad}{x - 0.5} = \lim_{x \to \ldots} \frac{\quad}{2x - 1}$$

$$= \lim_{x \to \ldots} \frac{\quad}{2x - 1} = \lim_{x \to \ldots} \frac{\quad}{2x - 1} = \ldots\ldots\ldots$$

5. The equation of the tangent line at the point P is

$$y - \ldots\ldots = m(x - \ldots\ldots)$$

$$y - \ldots\ldots = \ldots\ldots(x - \ldots\ldots)$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 2: Find the equation of the tangent line to the graph of $f(x)=\dfrac{1}{x-1}$ at the point where $x=3$.

Solution:

The *slope* of the tangent line where $x=3$ is

$$m_{\tan} = f'(\ldots) = \lim_{x\to 3}\frac{f(\ldots\ldots)-f(3)}{x-3} = \lim_{x\to 3}\frac{\quad - \quad}{x-3}$$

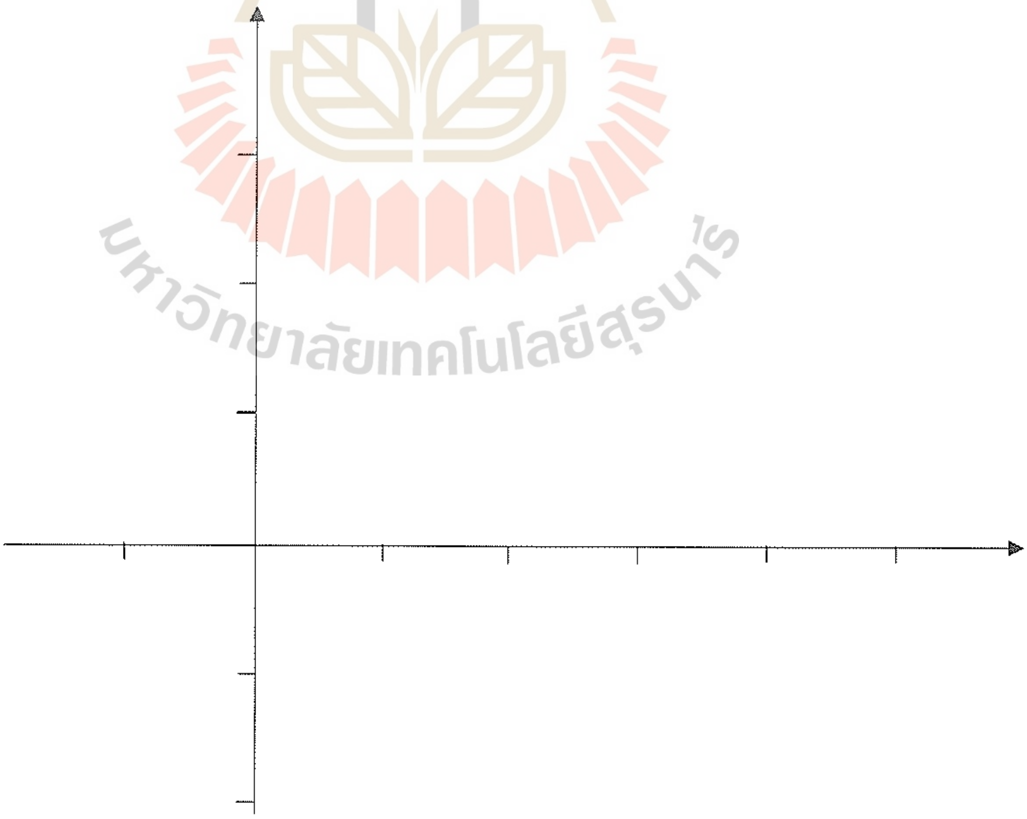
$$= \lim_{x\to 3}\frac{\overline{\quad\quad}}{x-3} = \lim_{x\to 3}\frac{\quad}{x-3} = \lim_{x\to 3}\frac{\quad}{2(x-1)(x-3)} = \ldots\ldots\ldots$$

The equation of the tangent line at the point where $x=3$ is

$$y-\ldots\ldots = m(x-\ldots\ldots)$$

$$y-\ldots\ldots = \ldots\ldots(x-\ldots\ldots)$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$



Exercise 3: Consider the function $f(x) = x^2 - 4x$. Find $f'(3)$ and find the equation of the tangent line to the graph of f at the point where $x = 3$.

Solution:

The *slope* of the tangent line where $x = 3$ is

$$m_{\tan} = f'(3) \quad = \quad \lim_{x \to 3} \frac{f(\ldots\ldots) - f(3)}{x-3} \quad = \quad \lim_{x \to 3} \frac{(\qquad\qquad) - (\qquad\qquad)}{x-3}$$

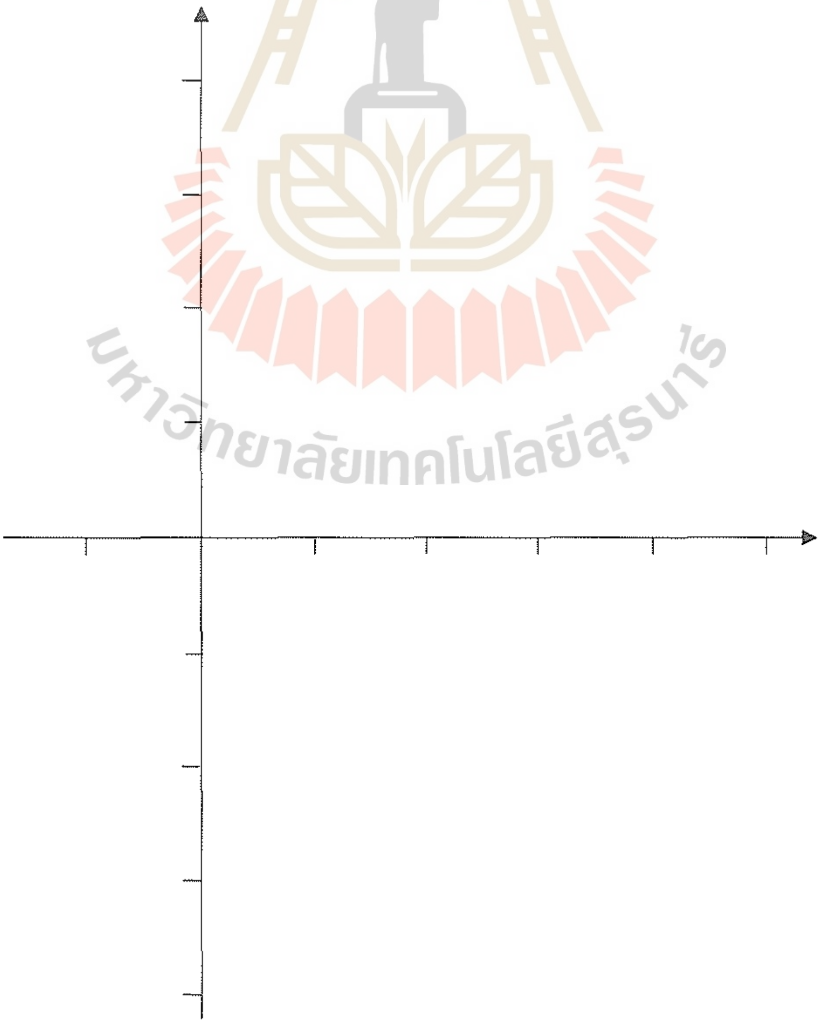
$$= \quad \lim_{x \to 3} \frac{\qquad\qquad}{x-3} \quad = \quad \lim_{x \to 3} \frac{\qquad\qquad}{x-3} \quad = \quad \ldots\ldots\ldots$$

The equation of the tangent line at the point where $x = 3$ is

$$y - \ldots\ldots \quad = \quad m(x - \ldots\ldots)$$

$$y - \ldots\ldots \quad = \quad \ldots\ldots(x - \ldots\ldots)$$

$$y \;=\; \ldots\ldots\ldots\ldots\ldots\ldots$$



Exercise 4: Find the derivative of each function using the *definition* of the derivative.

1. $f(x)=x^3-x^2+2x$. Sol: By definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\left[(\ldots\ldots\ldots)^{\ldots\ldots}-(\ldots\ldots\ldots)^{\ldots\ldots}+2(\ldots\ldots\ldots)\right]-\left[(\ldots\ldots)^{\ldots\ldots}-(\ldots\ldots)^{\ldots\ldots}+2\ldots\ldots\right]}{h}$$

$$= \lim_{h\to 0}\frac{\left[(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)-(\ldots\ldots\ldots\ldots\ldots\ldots)+(\ldots\ldots\ldots\ldots)\right]-\left[\ldots\ldots-\ldots\ldots+\ldots\ldots\right]}{h}$$

$$= \lim_{h\to 0}\frac{\qquad\qquad\qquad\qquad}{h}$$

$$= \lim_{h\to 0}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. $f(x)=\dfrac{1}{x^2}$ Sol: By definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{\qquad}-\dfrac{1}{\qquad}}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{\qquad}{(\quad)^{\ldots}(\quad)}-\dfrac{\qquad}{(\quad)^{\ldots}(\quad)}}{h} = \lim_{h\to 0}\frac{\dfrac{\qquad}{(\quad)^{\ldots}(\quad)}}{h}$$

$$= \lim_{h\to 0}\frac{\qquad}{(\quad)^{\ldots}} = \lim_{h\to 0}\frac{\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots} = \ldots\ldots\ldots = \ldots\ldots$$

3. $g(x)=\sqrt{1+2x}$ Sol: By definition of the derivative,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\sqrt{\qquad\qquad}-\sqrt{\qquad\qquad}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{\qquad\qquad}-\sqrt{\qquad\qquad}}{h}\cdot\frac{\sqrt{\qquad\qquad}+\sqrt{\qquad\qquad}}{\sqrt{\qquad\qquad}+\sqrt{\qquad\qquad}}$$

$$= \lim_{h\to 0}\frac{\qquad}{h\left(\sqrt{\qquad\qquad}+\sqrt{\qquad\qquad}\right)} = \lim_{h\to 0}\frac{\qquad}{h\left(\sqrt{\qquad\qquad}+\sqrt{\qquad\qquad}\right)}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Additional Exercises:

1) Find the slope of the tangent line at the point P. Then find the equation of the tangent line at P.

a) $f(x) = 1 - x^3, \quad P(1,0)$

b) $f(x) = 1 - x^3, \quad P(0,1)$

c) $g(x) = \dfrac{1}{2x-1}, \quad P\left(-1, -\tfrac{1}{3}\right)$

2) Each of the following limits represents the derivative of a function $f(x)$ at some number a. Find $f(x)$ and a.

a) $\displaystyle\lim_{h\to 0} \frac{(3+h)^2 - 9}{h}$

b) $\displaystyle\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$

c) $\displaystyle\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \frac{1}{2}}{h}$

d) $\displaystyle\lim_{x\to 1} \frac{x^9 - 1}{x - 1}$

e) $\displaystyle\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$

3) Find the derivative of each function using the *definition* of the derivative.

a) $f(x) = 3x + 4$

b) $g(x) = 5$

c) $f(x) = x + \dfrac{1}{x}$

d) $h(x) = \dfrac{x+1}{x-1}$

e) $s(t) = 3t^2 - 9t$

Rules for Derivatives

Recall:

1. Basic Derivatives:

$$\frac{d}{dx}(c) = 0 \qquad (c \text{ constant})$$

$$\frac{d}{dx}\left(x^n\right) = n\,x^{n-1} \qquad (n \text{ real})$$

2. Basic Rules for Derivatives:

$$(f \pm g)' = f' \pm g' \qquad \text{(Sum/Difference Rule)}$$

$$(cf)' = cf' \qquad (c \text{ constant})$$

$$(fg)' = fg' + gf' \qquad \text{(Product Rule)}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \qquad \text{(Quotient Rule)}$$

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$$

Exercise 1: Find the derivatives of the following functions:

1. $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots = \frac{1}{\ldots\ldots\ldots}$

2. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots = \frac{-1}{\ldots\ldots\ldots}$

Exercise 2: Find the derivatives.

1. If $f(x) = x^5$ then $f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

2. If $f(x) = \dfrac{1}{x^3}$ we write $f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

Then $f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

3. If $g(x) = \dfrac{1}{\sqrt{x^3}}$ we write $g(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

Then $g'(x) = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

4. If $y = 4t^3$ then $\dfrac{dy}{dt} = 4\cdot\ldots\ldots\ldots = \ldots\ldots\ldots$

5. If $f(x) = 3x^6 - 2x^2 + 2x - 4$

then $f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

6. If $y = 5x^4 - \sqrt{3}x^5 - 4x + \sqrt{5}$

then $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

7. If $f(x) = 2x - \pi + \dfrac{1}{2x} - \dfrac{2}{\sqrt[3]{x}} + \dfrac{\sqrt{2}}{x^3}$

we write

$f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Then $f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 3: Find the derivatives by

a) using the product rule

b) expanding the product before differentiating.

1. $y=(3x-1)(2x+9)$

1. Method: Product Rule.

$$\frac{dy}{dx} = (3x-1)(\ldots\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

$$= (3x-1)(\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. Method: Expand first.

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

2. $y=\left(t+\dfrac{1}{t}\right)\left(\sqrt{t}-\dfrac{1}{\sqrt{t}}\right)$

First write

$$y = (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

1. Method: Product Rule.

$$\frac{dy}{dt} = \left(t+t^{-1}\right)\frac{d}{dt}(\ldots\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots)\frac{d}{dt}\left(t+t^{-1}\right)$$

$$= \left(t+t^{-1}\right)(\ldots\ldots\ldots\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. Method: Expand first.

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then $\dfrac{dy}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 4: Find the derivatives using the quotient rule.

1. If $f(x) = \dfrac{x^2+1}{x^2-1}$ then

$$f'(x) = \frac{\qquad\qquad\qquad\qquad}{(\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad\qquad\qquad\qquad}{(\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad\qquad\qquad}{(\ldots\ldots\ldots)^2} = \frac{\qquad\qquad\qquad}{(\ldots\ldots\ldots)^2}$$

2. If $y = \dfrac{2}{x^2+x+1}$ then

$$\frac{dy}{dx} = \frac{-2(\qquad\qquad)}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2} = \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots\ldots)^2} = \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots\ldots)^2}$$

3. If $y = \dfrac{\sqrt{t}-2}{\sqrt{t}+2}$ then

$$\frac{dy}{dt} = \frac{\left(\sqrt{t}+2\right)\frac{d}{dt}(\qquad\qquad) - (\qquad\qquad)\frac{d}{dt}(\qquad\qquad)}{(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\left(\sqrt{t}+2\right)\qquad - (\qquad\qquad)}{(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{1}{2\sqrt{t}}\,\frac{\qquad\qquad\qquad\qquad}{(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad\qquad\qquad\qquad}{2\sqrt{t}\,(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad\qquad\qquad\qquad}{\sqrt{t}\,(\ldots\ldots\ldots\ldots)^2}$$

4. If $f(x) = \dfrac{\sqrt[3]{x}}{x^2 - x - 2}$ then write

$$f(x) = \frac{\qquad}{x^2 - x - 2}$$

Differentiate:

$$f'(x) = \frac{(\qquad)(\qquad) - (\qquad)(\qquad)}{(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{(\qquad) - (\qquad)(\qquad)}{\ldots\ldots\ldots(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad}{\ldots\ldots\ldots(\ldots\ldots\ldots\ldots)^2}$$

Exercise 5: Find the derivative in the simplest way.

1. If $y = \sqrt[3]{t}\left(t - 2 + \dfrac{1}{t}\right)$ then we write

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Differentiate,

$\dfrac{dy}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

2. If $f(x) = \dfrac{3 - 2x + x^3}{\sqrt{x}}$ then we write

$f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Differentiate,

$f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 6: Find the equation of the tangent line to the graph of $f(x) = x - \frac{1}{2x}$ when $x = -\frac{1}{2}$.

Solution: Write $f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

Then $f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

Also, $f\left(-\frac{1}{2}\right) = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

and $f'\left(-\frac{1}{2}\right) = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

Now the equation of the tangent line is

$$y - y_0 = \ldots\ldots(x - \ldots\ldots)$$

At $x = -\frac{1}{2}$ we obtain

$$y - \ldots\ldots\ldots = \ldots\ldots(x - \ldots\ldots)$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Additional Exercises:

1) Find the derivatives of

a) $f(x) = x^4 - 2x + \pi + \frac{\sqrt{5}}{x^2} - \frac{1}{2\sqrt{x}}$

b) $g(t) = \left(t^2 + t\right)\left(\sqrt{t} + 2\sqrt[3]{t} - 1\right)$

c) $y = \frac{4x-5}{2-3x}$

d) $y = \frac{x}{x - \frac{2}{x}}$

e) $f(x) = \frac{3}{4-x^2}$

f) $h(x) = \frac{(x-1)(x-4)}{(x-2)(x-3)}$

g) $y = (x+5)\left(x^2+7\right)(2-3x)$

2) Find the points on the graph of $y = f(x)$ where the tangent line

1. is horizontal
2. has slope m.

a) $f(x) = x^3 - 3x^2 + 9x + 1, \quad m = 6$

b) $f(x) = \frac{x}{x^2+1}, \quad m = {}^{12}\!/_{25}$

Derivatives of Trigonometric Functions

Recall:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = \ldots\ldots\ldots\ldots\ldots$$

$$\frac{d}{dx}(\tan x) = \ldots\ldots\ldots\ldots\ldots$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d}{dx}(\csc x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 1: Find the derivatives of the following functions:

1. If $f(x) = 3\sin x - 4\cos x$

then $f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

2. If $y = x^2 \csc x$, then by the _______________ rule,

$$\frac{dy}{dx} = \ldots\ldots\frac{d}{dx}(\ldots\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots)\frac{d}{dx}(\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. If $y = \dfrac{\sin x}{1+\cos x}$, then by the ______________ rule,

$$\frac{dy}{dx} = \frac{\qquad \frac{d}{dx} \qquad - \qquad \frac{d}{dx} \qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad - \qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2} = \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots\ldots)^2}$$

4. If $y = \dfrac{\tan x - 1}{\sec x}$, then by the ______________ rule,

$$y' = \frac{\qquad - \qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad - \qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad - \qquad}{\ldots\ldots\ldots\ldots\ldots\ldots} = \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots}$$

5. If $f(x) = x(\tan x - 1)\sec x$, then by the ______________ rule,

$$f'(x) = (\ldots x \ldots)'(\tan x - 1)(\sec x)$$

$$+ \; x(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots) \; + \; x(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

$$= (\ldots 1 \ldots)(\tan x - 1)(\sec x)$$

$$+ \; x(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots) \; + \; x(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 2: Find all values of x where the tangent line to the curve

$$f(x) = \frac{\cos x}{\sin x - 2} \quad \text{is horizontal.}$$

Solution: Compute the derivative.

$$f'(x) = \frac{(\qquad)(\qquad)' - (\qquad)(\qquad)'}{(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{(\qquad)(\qquad) - (\qquad)(\qquad)}{(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots)^2} = \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots)^2}$$

The tangent line is horizontal when $\ldots\ldots\ldots = 0$.

$$\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Additional Exercises:

1) Find the derivatives of

a) $y = 2\cos x - 3\tan x$

b) $y = \csc x \cot x$

c) $y = \dfrac{\tan x}{x}$

d) $f(x) = \dfrac{x^2 \tan x}{\sec x}$

e) $y = x^3 \sin x + 2x^2 \cos x - 6x \sin x$

f) $y = x^{-3} \sin x \tan x$

2) Find the equations of the tangent line and the normal line at the given point.

a) $f(x) = \sin x - \cos x, \quad P(\pi/4, 0)$

b) $f(x) = \sec x - 2\cos x, \quad P(\pi/3, 1)$

The Chain Rule

Recall: If $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We can also write this as

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot g'(x)$$

Exercise 1: Find $\frac{dy}{dx}$ and $\left.\frac{dy}{dx}\right|_{x=1}$ by

a) using the chain rule

b) directly by expressing y as a function of the variable x.

1. $y = u^2$ and $u = 2x^2 + 3x$

a) By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(\ldots\ldots\ldots)\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots\ldots)$$

$$= (\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots) = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots)$$

Then

$$\left.\frac{dy}{dx}\right|_{x=1} = (\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

b) Compose first,

$$y = u^2 = (\ldots\ldots\ldots\ldots\ldots)^2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

so that

$$\left.\frac{dy}{dx}\right|_{x=1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

2. $y = u - u^2$ and $u = \sqrt{x} + \sqrt[3]{x}$

a) By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\ldots\ldots\ldots = \frac{d}{du}(\ldots\ldots\ldots\ldots\ldots\ldots)\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots\ldots)$$

$$= (\ldots\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

$$= \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$$

Then

$$\left.\frac{dy}{dx}\right|_{x=1} = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

b) Compose first,

$$y = u - u^2 = (\ldots\ldots\ldots\ldots\ldots) - (\ldots\ldots\ldots\ldots\ldots)^2$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

so that

$$\left.\frac{dy}{dx}\right|_{x=1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots = \ldots\ldots$$

Exercise 2: Separate each function as $y = f(u)$ and $u = g(x)$, and differentiate using the chain rule.

1. $y = (4x+3)^7$

Here, $y = \ldots\ldots\ldots\ldots$ where $u = 4x+3$.

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\ldots\ldots\ldots = (\ldots\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. $y=\left(x^3-5x\right)^4$

Here, $y=$ .... where $u=$ ....

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\text{........} = (\text{.................})(\text{.................})$$

$$= \text{..}$$

3. $y=\sin\left(x^2+x-1\right)$

Here, $y=$ .... where $u=$ ....

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\text{........} = (\text{.................})(\text{.................})$$

$$= \left(\text{......................}\right)(\text{..............}) = (\text{..............})\cos(\text{..................})$$

Exercise 3: Find the derivatives of the following functions.

1. $f(x) = \left(x^3-4x^2+2x+1\right)^{-3}$

By the chain rule,

$$f'(x) = (-3)(\text{..........................})^{\text{.....}}\frac{d}{dx}(\text{..............................})$$

$$= (-3)(\text{..........................})^{\text{.....}}(\text{..............................})$$

$$= \text{..}$$

2. $g(x) = \sqrt{x^2+4x}$

By the chain rule,

$$g'(x) = \frac{1}{2\sqrt{\text{..........................}}}\frac{d}{dx}(\text{....................})$$

$$= \frac{1}{2\sqrt{\text{..........................}}}(\text{.................}) = \frac{}{\sqrt{\text{........................}}}$$

3. $F(z) = \dfrac{z-4}{z+2}^{\,3}$

By the chain rule,

$$F'(z) = \ldots\ldots\left(\frac{z-4}{z+2}\right)^{\ldots\ldots} \frac{d}{dz}\left(\frac{\rule{3cm}{0.4pt}}{\ldots\ldots\ldots\ldots\ldots\ldots}\right)$$

$$= \ldots\ldots\left(\frac{z-4}{z+2}\right)^{\ldots\ldots} \frac{(\ldots\ldots\ldots\ldots)(\ldots\ldots) - (\ldots\ldots\ldots\ldots)(\ldots\ldots)}{(\ldots\ldots\ldots\ldots)^{\ldots\ldots}}$$

$$= \ldots\ldots\left(\frac{z-4}{z+2}\right)^{\ldots\ldots} \frac{\rule{4cm}{0.4pt}}{(\ldots\ldots\ldots\ldots)^{\ldots\ldots}} = \frac{\ldots\ldots\ (\ldots\ldots\ldots\ldots)^{\ldots}}{(z+2)^{\ldots\ldots}}$$

4. $y = (3x-2)^{10}(5x^2-x+1)^{12}$

By the ______________ rule and the _________ rule,

$$\frac{dy}{dx} = (3x-2)^{10}\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots)^{\ldots\ldots} + (\ldots\ldots\ldots\ldots\ldots)^{\ldots\ldots}\frac{d}{dx}(\ldots\ldots\ldots\ldots)^{\ldots\ldots}$$

$$= (3x-2)^{10}\ \ldots\ldots(\ldots\ldots\ldots\ldots\ldots)^{\ldots\ldots}(\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots\ldots)^{\ldots\ldots}\ \ldots\ldots(\ldots\ldots\ldots)^{\ldots\ldots}(\ldots\ldots)$$

$$= (3x-2)^{\ldots\ldots}(5x^2-x+1)^{\ldots\ldots}\Big[\ldots\ldots(\ldots\ldots\ldots)(\ldots\ldots\ldots) + \ldots\ldots(\ldots\ldots\ldots\ldots)\Big]$$

$$= (3x-2)^{\ldots\ldots}(5x^2-x+1)^{\ldots\ldots}\Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big]$$

$$= (3x-2)^{\ldots\ldots}(5x^2-x+1)^{\ldots\ldots}\Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big]$$

5. $y = \sqrt{1+\sqrt{x}}$

First write $y = \left(1+\sqrt{x}\right)^{\cdots\cdots}$ Then by the chain rule,

$$\frac{dy}{dx} = \ldots\ldots(\ldots\ldots\ldots\ldots)^{\cdots\cdots}\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots(\ldots\ldots\ldots\ldots)^{\cdots\cdots}(\ldots\ldots\ldots\ldots\ldots)$$

$$= \ldots\ldots\frac{}{\sqrt{\ldots\ldots\ldots\ldots\ldots}}(\ldots\ldots\ldots) = \frac{1}{\sqrt{\ldots\ldots\ldots\ldots\ldots\ldots}}$$

6. $f(x) = \sin\sqrt{x^2+2}$

By the chain rule,

$$f'(x) = \ldots\ldots\ldots\sqrt{\ldots\ldots\ldots\ldots\ldots}\;\frac{d}{dx}\left(\sqrt{\ldots\ldots\ldots\ldots\ldots}\right)$$

$$= \ldots\ldots\ldots\sqrt{\ldots\ldots\ldots\ldots\ldots}\;\frac{1}{2\sqrt{\ldots\ldots\ldots\ldots\ldots}}\frac{d}{dx}\left(\ldots\ldots\ldots\ldots\ldots\right)$$

$$= \ldots\ldots\ldots\sqrt{\ldots\ldots\ldots\ldots\ldots}\;\frac{}{\sqrt{\ldots\ldots\ldots\ldots\ldots}}$$

7. $y = \sin^3 x + \cos x^3$

Write $y = (\sin x)^{\cdots\cdots} + \cos\left(x^{\cdots\cdots}\right)$ Apply the chain rule to each term,

$$\frac{dy}{dx} = \ldots\ldots(\ldots\ldots\ldots\ldots\ldots)^{\cdots\cdots}\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots) + \left[\ldots\ldots\ldots\left(x^3\right)\right]\frac{d}{dx}(\ldots\ldots\ldots)$$

$$= \ldots\ldots(\ldots\ldots\ldots\ldots\ldots)^{\cdots\cdots}\;\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\left(x^3\right)(\ldots\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\sin\left(x^3\right)$$

8. $y = \cos^2\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

Write $y = \left[\cos\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right]^{\cdots\cdots}$

The outermost function is $y = \ldots\ldots$. By the chain rule,

$$\frac{dy}{dx} = \ldots\ldots \left[\ldots\ldots\ldots\left(\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right)\right] \cdot \frac{d}{dx}\left[\ldots\ldots\ldots\left(\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right)\right]$$

$$= \ldots\ldots \ \ldots\ldots\ldots\left(\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right) \cdot \left[\ldots\ldots\ldots\left(\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right)\right] \cdot \frac{d}{dx}\left(\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right)$$

$$= \ldots\ldots \ \ldots\ldots\ldots\left(2\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right) \cdot \frac{(\qquad)(\qquad)-(\qquad)(\qquad)}{(\ldots\ldots\ldots\ldots)^2}$$

$$= \ldots\ldots \ \ldots\ldots\ldots\left(2\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right) \cdot \frac{\qquad\qquad}{2\sqrt{x}\,(\ldots\ldots\ldots\ldots)^2}$$

$$= \frac{\qquad\qquad}{\sqrt{x}\,(\ldots\ldots\ldots\ldots)^2} \cdot \sin\left(2\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right)$$

9. $y = 2\sec\sqrt{x}\tan\sqrt{x}$

By the ____________ rule and the ________ rule,

$$\frac{dy}{dx} = \left(2\sec\sqrt{x}\right)\cdot\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots) + \left(2\tan\sqrt{x}\right)\cdot\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots)$$

$$= \left(2\sec\sqrt{x}\right)\cdot(\ldots\ldots\ldots\ldots\ldots)\cdot\frac{d}{dx}(\ldots\ldots\ldots)$$

$$+ \left(2\tan\sqrt{x}\right)\cdot[\ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots]\cdot\frac{d}{dx}(\ldots\ldots\ldots)$$

$$= 2(\ldots\ldots\ldots\ldots\ldots)^{\cdots\cdots}\cdot(\ldots\ldots\ldots) + 2\left(\tan\sqrt{x}\right)^{\cdots\cdots}\cdot(\ldots\ldots\ldots\ldots\ldots)\cdot(\ldots\ldots\ldots)$$

$$= \frac{(\ldots\ldots\ldots\ldots\ldots) + \tan^{\cdots\cdots}\sqrt{x}\ \ldots\ldots\ldots\ldots\ldots}{\sqrt{x}}$$

$$= \frac{2\sec^3\ldots\ldots\ldots - \sec\sqrt{x}}{\sqrt{x}}$$

Exercise 4: The table below contains values of the functions f and g, and of their derivatives. Use it to find the specified derivatives.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	1	1/2	-2
4	4	-2	0	1

1. $\frac{d}{dx}\left(4f(x) \right)\Big|_{x=1}$ = ... = ... = ...

2. $\frac{d}{dx}\left(2f(x)-3g(x) \right)\Big|_{x=4}$ = ... = ... = ...

3. $\frac{d}{dx}\left(f(x)g(x) \right)\Big|_{x=1}$ = ... = ... = ...

4. $\frac{d}{dx}\left((f\circ g)(x) \right)\Big|_{x=1}$ = ... = ... = ...

5. $\frac{d}{dx}\left((g\circ g)(x) \right)\Big|_{x=1}$ = ... = ... = ...

6. $\frac{d}{dx}\left(\sqrt{f(x)^2+g(x)^2} \right)\Big|_{x=4}$ = $\dfrac{}{2\sqrt{\ldots\ldots\ldots\ldots\ldots\ldots}}\Big|_{x=4}$

= $\dfrac{}{2\sqrt{\ldots\ldots\ldots\ldots\ldots}}\Big|_{x=4}$ = $\dfrac{}{2\sqrt{\ldots\ldots\ldots}}$ = ... = ...

Additional Exercises:

1) Find the derivatives.

a) $f(x)=\left(x^3-4x\right)^5$

b) $y=\dfrac{x}{\sqrt{7-3x}}$

c) $y=\left(x-\dfrac{1}{x}\right)^{2/3}$

d) $s(t)=\sqrt[4]{\dfrac{t^3+1}{t^3-1}}$

e) $f(x)=\tan^2 x+\tan x^2$

f) $y=x\sin\dfrac{1}{x}$

g) $y=\sin^3\left(\cos\sqrt{x}\right)$

h) $f(\theta)=\left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$

i) $y=\sqrt{\sin x+\sqrt{1-\sin x}}$

j) $f(x)=\sqrt{x+\sqrt{x+\sqrt{x}}}$

Implicit Differentiation

Exercise 1: If $y = f(x)$ and

$$3y^2 + 4xy = 3x^2 + 1,$$

find $\frac{dy}{dx}$.

Solution: Take the derivative on both sides of the equation.

$$\frac{d}{dx}\left(3y^2 + 4xy\right) = \frac{d}{dx}\left(3x^2 + 1\right)$$

$$3\frac{d}{dx}\left(y^2\right) + 4\frac{d}{dx}(xy) = \ldots\ldots\ldots\ldots$$

By the product and chain rules,

$$\ldots\ldots\frac{dy}{dx} + 4(\ldots\ldots\ldots + \ldots\ldots\ldots) = \ldots\ldots\ldots$$

$$\ldots\ldots\frac{dy}{dx} + \ldots\ldots\ldots + \ldots\ldots\ldots = \ldots\ldots\ldots$$

Solve for $\frac{dy}{dx}$:

$$\ldots\ldots\frac{dy}{dx} + \ldots\ldots\ldots = \ldots\ldots\ldots - \ldots\ldots\ldots$$

$$\frac{dy}{dx}\left[\ldots\ldots\ldots\ldots\ldots\ldots\right] = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dy}{dx} = \frac{\ldots\ldots\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots\ldots\ldots} = \frac{\ldots\ldots\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 2: Find $\frac{dy}{dx}$ if

$$\cos(x-y) = y\sin x.$$

Solution: Take the derivative on both sides of the equation.

$$\frac{d}{dx}\left[\cos(x-y)\right] = \frac{d}{dx}\left[y\sin x\right]$$

By the product and chain rules,

$$\ldots\ldots(x-y)\frac{d}{dx}(\ldots\ldots\ldots\ldots) = \ldots\ldots\cos x + \sin x\ldots\ldots\ldots$$

$$\ldots\ldots(x-y)(\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots\cos x + \sin x\ldots\ldots\ldots$$

Solve for $\frac{dy}{dx}$:

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\cos x + \sin x\ldots\ldots\ldots$$

$$\frac{dy}{dx}\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dy}{dx} = \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 3: If $x^5 + xy^3 + x^2y + y^5 = 4$, find $\frac{dy}{dx}$ at the point $(1,1)$.

Solution: Take the derivative on both sides of the equation.

$$\frac{d}{dx}\left(x^5 + xy^3 + x^2y + y^5\right) = \frac{d}{dx}(\ldots\ldots)$$

$$\frac{d}{dx}\left(x^5\right) + \frac{d}{dx}\left(xy^3\right) + \frac{d}{dx}(\ldots\ldots\ldots) + \frac{d}{dx}\left(y^5\right) = \frac{d}{dx}(\ldots\ldots)$$

By the product and chain rules,

$$[\ldots\ldots\ldots] + \left[\ \ldots\ldots \frac{d}{dx}\left(y^3\right) + y^3 \ldots\ldots\ \right] + \left[\ \ldots\ldots \frac{dy}{dx} + y \ldots\ldots\ \right] + \frac{d}{dx}\left(y^5\right) \quad = \quad \ldots\ldots$$

$$[\ldots\ldots\ldots] + \left[\ \ldots\ldots\ldots\ldots\ldots \frac{dy}{dx} + y^3\ \right] + \left[\ \ldots\ldots \frac{dy}{dx} + y \ldots\ldots\ \right] + \ \ldots\ldots\ldots\ldots \quad = \quad \ldots\ldots$$

Substitute $(x, y) = \ldots\ldots\ldots\ldots$

$$[\ldots\ldots\ldots] + \left[\ \ldots\ldots\ldots\ldots\ldots \frac{dy}{dx} + \ldots\ldots\ \right] + \left[\ \ldots\ldots \frac{dy}{dx} + \ldots\ldots\ldots\ \right] + \ \ldots\ldots\ldots\ldots \quad = \quad \ldots\ldots$$

and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left[\ldots\ldots\ldots\ldots\ldots\right] \quad = \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad = \quad \ldots\ldots\ldots$$

$$\frac{dy}{dx} \quad = \quad \frac{\qquad\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 4: Find the equation of the tangent line to the curve

$$\sin^3(xy) + \cos(x+y) + x = \frac{\pi}{2},$$

at the point $(\pi/2, 0)$.

Solution:

1. Find the derivative $\frac{dy}{dx}$ at $(\pi/2, 0)$ by implicit differentiation:

$$\frac{d}{dx}\left(\sin^3(xy) + \cos(x+y) + x\right) \quad = \quad \frac{d}{dx}\left(\frac{\pi}{2}\right)$$

$$3\sin^2(xy)\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots\ldots) \ - \ \sin(x+y)\frac{d}{dx}(\ldots\ldots\ldots) \ + \ \ldots\ldots\ldots \quad = \quad \ldots\ldots$$

$$3\sin^2(xy)\,(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) \ - \ \sin(x+y)\,(\ldots\ldots\ldots\ldots\ldots) + \ \ldots\ldots \quad = \quad \ldots\ldots$$

Now substitute $(\pi/2, 0)$:

$3\sin^2(\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots) - \sin(\ldots\ldots+\ldots)(\ldots\ldots\ldots\ldots\ldots) + \ldots\ldots = \ldots\ldots$

$(\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots) - (\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots) + \ldots\ldots = \ldots\ldots$

$$\frac{dy}{dx}[\ldots\ldots\ldots] = \ldots\ldots\ldots\ldots$$

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots$$

2. The equation of the tangent line at $(\pi/2, 0)$ is

$$y - y_0 = m(x - x_0)$$

$$y - \ldots\ldots = \ldots\ldots(x - \ldots\ldots)$$

$$y = \ldots\ldots\ldots\ldots$$

Additional Exercises:

1) Find $\dfrac{dy}{dx}$ by implicit differentiation

a) $x^2 + xy - y^3 = 3$

b) $\sqrt{xy} - 2x = \sqrt{y}$

c) $2y^2 + \sqrt[3]{xy} = 3x^2 + 17$

d) $2xy = \left(x^2 + y^2\right)^{3/2}$

e) $x\sin y + \cos 2y = \cos y$

f) $\sec(2x + y) + \cos(2x - y) = x$

2) Find the equation of the tangent line to the curve at the given point.

a) $2xy + \pi\sin y = 2\pi$, $(1, \pi/2)$

b) $3\left(x^2 + y^2\right)^2 = 25\left(x^2 - y^2\right)$, $(2,1)$

3) Find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ and compare both, if

$$y^4 + x^2y^2 + x^4y = y + 1$$

Graphing

Recall:

Test for Increase/Decrease: Let f be differentiable on (a,b).

1. If $f'(x) > 0$ for all x in (a,b), then f is ***increasing*** on (a,b).
2. If $f'(x) < 0$ for all x in (a,b), then f is ***decreasing*** on (a,b).

Test for Concavity: Let f be twice differentiable on (a,b).

1. If $f''(x) > 0$ for all x in (a,b), then f is ***concave up*** on (a,b).
2. If $f''(x) < 0$ for all x in (a,b), then f is ***concave down*** on (a,b).

Critical Number: A *critical number of* f is a number c in the domain of f where

1. $f'(c) = 0$, or
2. $f'(c)$ does not exist.

Fermat's Theorem: If f has a relative extremum at c, then c must be a critical number of f.

First Derivative Test for Relative Extrema. Suppose f is continuous at the critical number c, and differentiable in some small open interval (a,b) around c (except possibly at c)

1. If $f'(x) > 0$ for all $a < x < c$ and $f'(x) < 0$ for all $c < x < b$, then f has a *relative maximum* at c.
2. If $f'(x) < 0$ for all $a < x < c$ and $f'(x) > 0$ for all $c < x < b$, then f has a *relative minimum* at c.
3. If $f'(x)$ does not change signs at c, then f has *no relative extremum* at c.

Second Derivative Test for Relative Extrema. Let c be a critical number of f of type $f'(c) = 0$. Suppose f is twice differentiable in some small open interval (a,b) around c.

1. If $f''(c) < 0$, then f has a *relative maximum* at c.
2. If $f''(c) > 0$, then f has a *relative minimum* at c.
3. If $f''(c) = 0$, then this test is inconclusive.

How to Find the Absolute Extrema. Suppose f is continuous on the *closed* interval $[a,b]$.

1. Find all critical numbers of f in $[a,b]$, and compute the value of f at each critical number .
2. Compute the values of f at the endpoints, namely $f'(a)$ and $f'(b)$.
3. The largest of the values computed in 1. and 2. is the absolute maximum of f, and the smallest of the values is the absolute minimum of f on $[a,b]$.

Exercise 1: Find all critical numbers of $f(x) = x^4 - 6x^2 - 3$.

Solution:

1. The domain of f is ____________________
2. Find the derivative of f.

$f'(x) =$ ____

$=$ ____

$f'(x) = 0$ when $x =$ ______________________________

$f'(x)$ is undefined when $x =$ ________________________

Answer: The critical numbers are ______________________

Exercise 2: Find all critical numbers of $f(x) = x^{4/3} - x^{1/3}$.

Solution:

1. The domain of f is ____________________
2. Find the derivative of f.

$f'(x) =$ ____

$=$ ____

$f'(x) = 0$ when $x =$ ______________________________

$f'(x)$ is undefined when $x =$ ________________________

Answer: The critical numbers are ______________________

Exercise 3: Find all critical numbers of $f(x) = |\sin x|$.

Solution:

1. The domain of f is ____________________

2. Find the derivative of f. Since $|x| = \sqrt{\ldots\ldots\ldots}$ we can rewrite f as

$$f(x) = \sqrt{\ldots\ldots\ldots\ldots}$$

Then

$$f'(x) = \rule{6cm}{0.4pt} = \rule{6cm}{0.4pt}$$

$$= \begin{cases} & (\sin x > 0) \\ & (\sin x < 0) \\ & (\sin x = 0) \end{cases}$$

$$= \begin{cases} & (\, 2n\pi < x < 2(n+1)\pi \,) \\ & (\qquad\qquad\qquad) \\ & (\qquad\qquad\qquad) \end{cases}$$

$f'(x) = 0$ when ____

or ____

$f'(x)$ is undefined when $x =$ ____

Answer: The critical numbers are ______________________________

Exercise 4: Consider $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$.

Find the intervals where f is increasing and decreasing. Then find the relative extrema, and sketch the graph of f

Solution:

1. Find the critical numbers.

   $f'(x) =$ ______________________________ = ______________________________

   = ______________________________

   $f'(x) = 0$ when $x =$ ______________________________

2. Check the sign of f'.

$f'(x)$				
$f(x)$				

f is increasing on ______________________________

f is decreasing on ______________________________

f has a relative maximum at ______________________________

f has a relative minimum at ______________________________

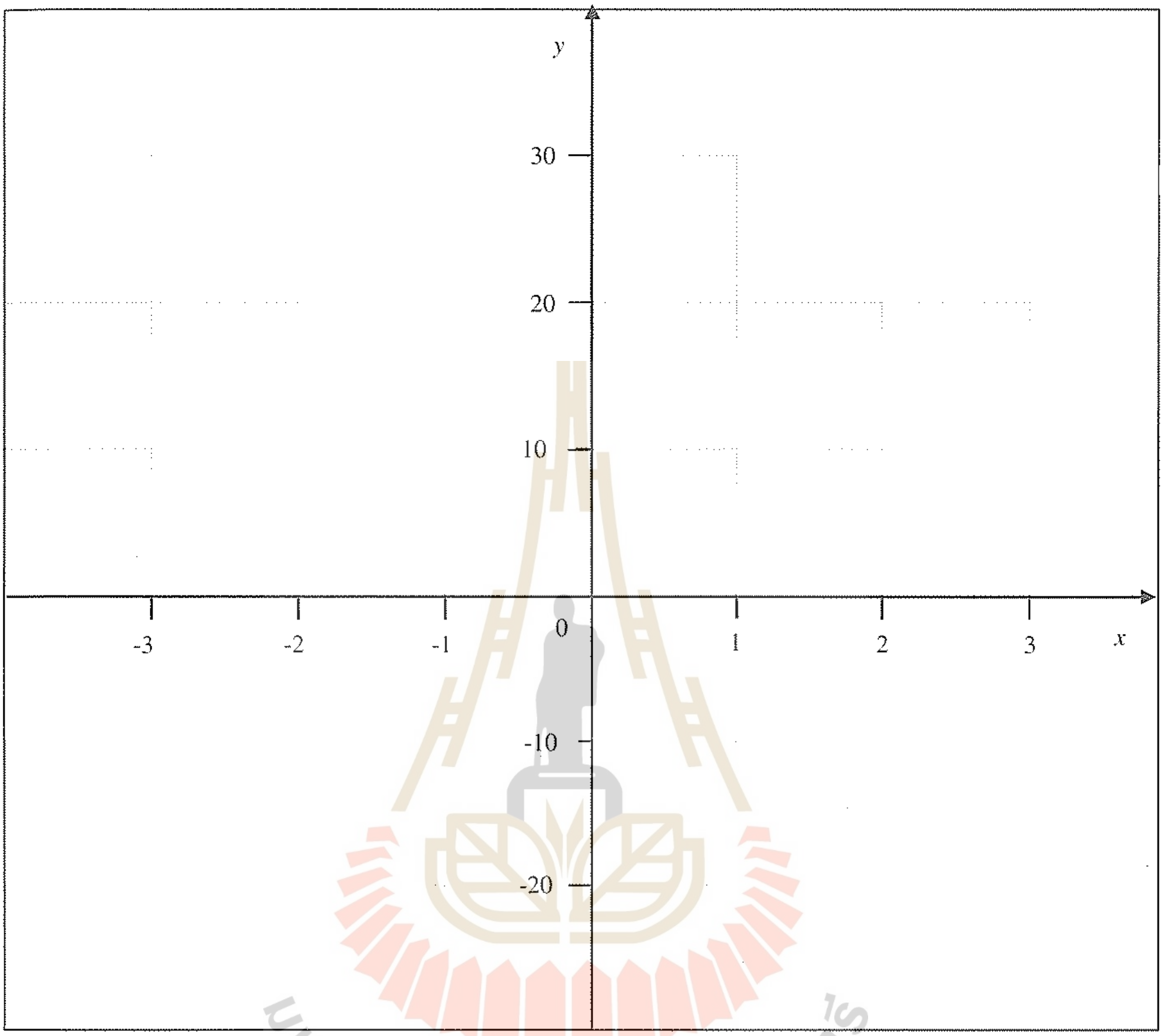
Table of values:

x	-3	-2	-1	0	1	2
$f(x)$						

The relative maximum values are ______________________________

The relative minimum values are ______________________________

Sketch the graph of f .



Exercise 5: Consider $f(x) = x^{2/3}(x^2 - 16)$.

Find the intervals where f is increasing and decreasing. Then find the relative extrema, and sketch the graph of f.

Solution:

1. Find the critical numbers. The domain of f is ____________________

   $f'(x) =$ ____

   $=$ ____

   $=$ ____

   $f'(x) = 0$ when $x =$ ____________________

   $f'(x)$ is undefined when $x =$ ____________________

   The critical numbers are: $x =$ ____________________

2. Check the sign of f'.

f'				
f				

f is increasing on ____________________

f is decreasing on ____________________

f has a relative maximum at ____________________

f has a relative minimum at ____________________

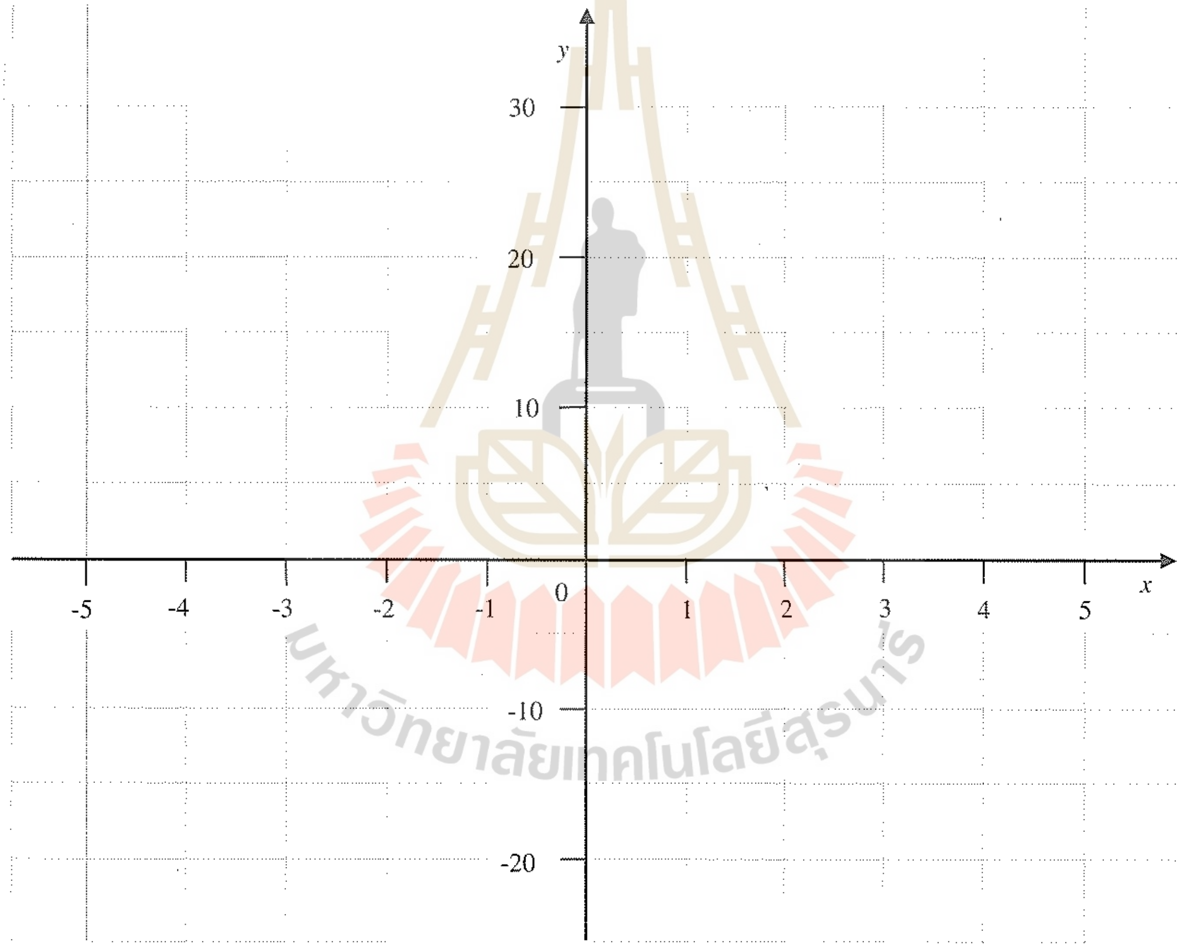
Table of values:

x	-5	-4	-2	-1	0	1	2	4	5
$f(x)$									

The relative maximum values are ______________________________

The relative minimum values are ______________________________

3. Sketch the graph of f .



There is a corner in the graph at $x =$ __________ because ___________

____

Exercise 6: Consider $f(x) = 2x^2 - x - x^3$.

Find the relative extreme values of f using the second derivative test.

Solution:

1. Find the critical numbers.

   $f'(x) =$ ______________________________ = ______________________________

   $f'(x) = 0$ when $x =$ ______________________________

   The critical numbers are $x =$ ______________________________

2. Look at f''

   $f''(x) =$ ____

   $f''(\ldots\ldots) = \ldots\ldots \Rightarrow f$ has a relative ______________ at $x =$ ________

   $f''(\ldots\ldots) = \ldots\ldots \Rightarrow f$ has a relative ______________ at $x =$ ________

Answer: The relative maximum value is $f(\ldots\ldots) =$ ____________________

The relative minimum value is $f(\ldots\ldots) =$ ____________________

Exercise 7: Consider $f(x) = x^4 - 4x^3 + 10$.

Find the intervals where f is increasing and decreasing, intervals where f is concave up or concave down, and the inflection points. Then sketch the graph of f.

Solution:

$f'(x) =$ ______________________________ = ______________________________

$f''(x) =$ ______________________________ = ______________________________

f, f' and f'' are all defined on ________________________

1. Find the intervals of increase/decrease

$f'(x) = 0$ when $x =$ ____________________________

The critical numbers are $x =$ ________________________

Check the sign of f' :

f'			
f			

f is increasing on ____________________________

f is decreasing on ____________________________

at $x =$ ________ f has a relative ________________________

at $x =$ ________ f has ________________________________

2. Find the intervals where f is concave up/down

$f''(x) = 0$ when $x =$ ____________________________

Check the sign of f'' :

f''			
f			

f is concave up on ______________________________

f is concave down on ______________________________

f has an inflection point at $x =$ ______________________

Table of values:

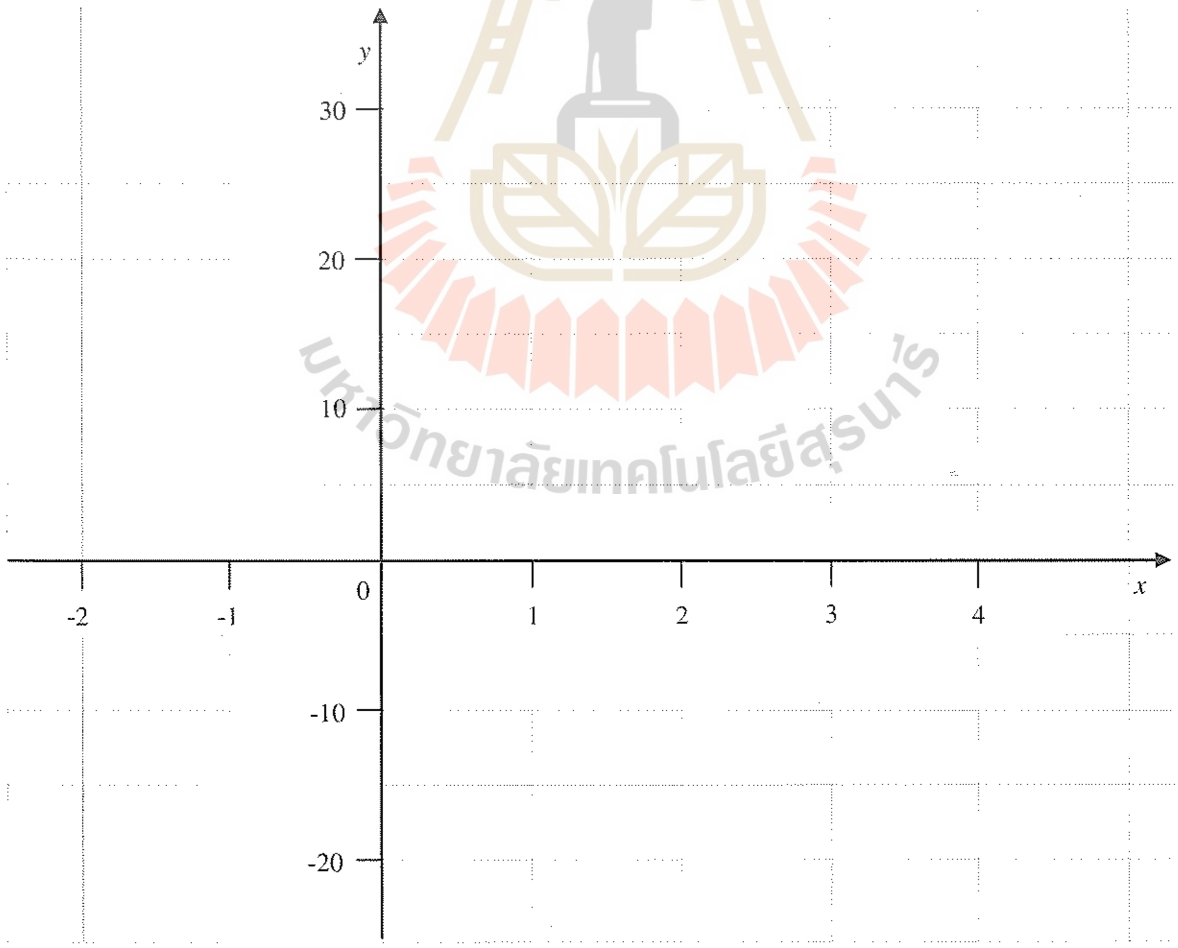
x	-2	-1	0	1	2	3	4	5
$f(x)$								

The relative maximum values are ____________________________________

The relative minimum values are ____________________________________

The inflection points are ____

3. Sketch the graph of f .



Additional Exercises:

1) Find the absolute maximum and minimum value of each function on the given closed interval.

   a) $f(x) = x^3 - x^2 - x + 2$ on $[0,2]$

   b) $f(x) = x + \sqrt{1-x}$ on $[0,1]$

   c) $g(x) = (x^2 + x)^{2/3}$ on $[-2,3]$

   d) $f(\theta) = \tan^2\theta - 2\tan\theta$ on $[-\frac{\pi}{3}, \frac{\pi}{3}]$

2) Find the absolute maximum of absolute minimum value of each function, if it exists.

   a) $f(x) = x^4 + 4x + 2$ on $(-\infty, \infty)$

   b) $f(x) = 4x^3 - 3x^4$ on $(-\infty, \infty)$

   c) $h(x) = \pi x^2 + \dfrac{1000}{x}$ on $(0, \infty)$

   d) $g(x) = \dfrac{x}{1+x^2}$ on $(-\infty, \infty)$

3) Find the relative extrema of the given functions by using
   i) the first derivative test
   ii) the second derivative test (where possible)
   Which of these are also absolute extrema ?

   a) $f(x) = 2x^3 - 9x^2 + 12x$

   b) $f(x) = x^4 + 3x^3 - 8$

   c) $g(t) = \sin^2 t$ on $[0, 2\pi]$

   d) $h(x) = |x^2 - 4|$

4) For each of the following functions
   i) find the intervals of increase / decrease
   ii) find the relative extrema
   iii) find intervals of concavity
   iv) find the inflection points
   v) sketch the graph

   a) $f(x) = \dfrac{x^3}{3} - 2x^2 + 3x - 2$

   b) $f(x) = 3x^5 - 25x^3 + 60x$

c) $f(x) = x^4 - 8x^2 + 16$

d) $f(x) = x^4 - 16x^3 + 96x^2 - 256x$

e) $f(x) = (10x - x^2)^4$

f) $f(x) = x^{4/3} - x^{1/3}$

g) $f(x) = x^{2/3}(x-4)^{1/3}$

h) $f(x) = 2\sin x - x, \qquad 0 \le x \le 2\pi$

i) $g(t) = 2\cos t + \sin^2 t, \qquad -\pi \le t \le \pi$

5) Sketch a continuous curve $y = f(x)$ with the stated properties.

a) $f(2) = 3, \quad f'(2) = 0, \quad f''(x) > 0$ for all x.

b) $f(-1) = 4, \quad f'(-1) = 0, \quad f''(x) < 0$ for all x.

c) $f(3) = -2, \quad f''(x) < 0$ for all $x \ne 3$ and $\lim_{x \to 3^+} f'(x) = \infty, \quad \lim_{x \to 3^-} f'(x) = -\infty$

6) Sketch a continuous curve $y = f(x)$ with the stated properties.

a) $f(1) = 0, \quad f(3) = 4, \quad f'(1) = f'(3) = 0$
$f'(x) < 0$ on $(-\infty, 1) \cup (3, \infty)$, $\quad f'(x) > 0$ on $(1, 3)$
$f''(x) > 0$ on $(-\infty, 2)$, $\quad f''(x) < 0$ on $(2, \infty)$

b) $f(-2) = 4, \quad f(2) = -1, \quad f'(-2) = 0, \quad f$ is not differentiable at 2.
$f'(x) < 0$ on $(-2, 2)$, $\quad f'(x) > 0$ on $(-\infty, -2) \cup (2, \infty)$
$f''(x) < 0$ for all $x \ne 2$

c) $f(0) = 0, \quad f(4) = -2, \quad f'(0) = f'(4) = 0$
$f'(x) < 0$ on $(0, 4)$, $\quad f'(x) > 0$ on $(4, \infty)$
$f''(x) < 0$ on $(0, 2) \cup (7, \infty)$, $\quad f''(x) > 0$ on $(2, 7)$
$\lim_{x \to \infty} f(x) = 2$
$f(-x) = f(x)$ for every x.

d) $f'(0) = 1, \quad f'(3) = 0$
$f'(x) > 0$ on $(0, 3)$, $\quad f'(x) < 0$ on $(3, \infty)$
$f''(x) < 0$ on $(0, 5)$, $\quad f''(x) > 0$ on $(5, \infty)$
$\lim_{x \to \infty} f(x) = 0$
$f(-x) = -f(x)$ for every x.

7) For each of the following functions,
   i) find the domain
   ii) find x and y intercepts
   iii) find symmetry (if any) find intervals of concavity
   iv) find asymptotes (if any)
   v) find the intervals of increase / decrease
   vi) find the relative extrema
   vii) find intervals of concavity
   viii) find the inflection points
   ix) sketch the graph

a) $f(x) = 5x^3 - 3x^5$

b) $f(x) = x^4 - 4x^3 + 4x^2$

c) $f(x) = 4x^5 + 80x^2 - 125$

d) $f(x) = \dfrac{x}{x^2 - 9}$

e) $f(x) = \dfrac{1}{x^2 + x - 2}$

f) $f(x) = \dfrac{x^2}{x^2 + 2}$

g) $f(x) = \dfrac{25 - 3x^2}{x^3}$

h) $f(x) = \sin x - \cos x$

Inverse Functions

Exercise 1: Consider the function

$$f(x) = x^2 + 2x \qquad (x \geq -1)$$

a) Show that $f(x)$ is one-to-one

b) Find its inverse function $f^{-1}(x)$

c) Sketch the graphs of $f(x)$ and $f^{-1}(x)$

d) Find $\frac{df}{dx}\Big|_{x=1}$ and $\frac{df^{-1}}{dx}\Big|_{x=f(1)}$. Compare the two.

e) Sketch the tangent lines to the graph of $f(x)$ at the point $(1,3)$, and to the graph of $f^{-1}(x)$ at the point $(3,1)$.

Solution:

a) Take the derivative.

$$f'(x) = \ldots\ldots\ldots\ldots\ldots = 2(\ldots\ldots\ldots)$$

On the interval $(-1,\infty)$, $f'(x) > \ldots\ldots\ldots$

That is, $f(x)$ is ______________ on $[-1,\infty)$

We conclude that $f(x)$ is ______________ on $[-1,\infty)$.

b) Write $y = x^2 + 2x = (x^2 + 2x + \ldots) - \ldots\ldots = (\ldots\ldots\ldots)^2 - \ldots\ldots$

Solve for x:

$$y + \ldots\ldots\ldots = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots)^2$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

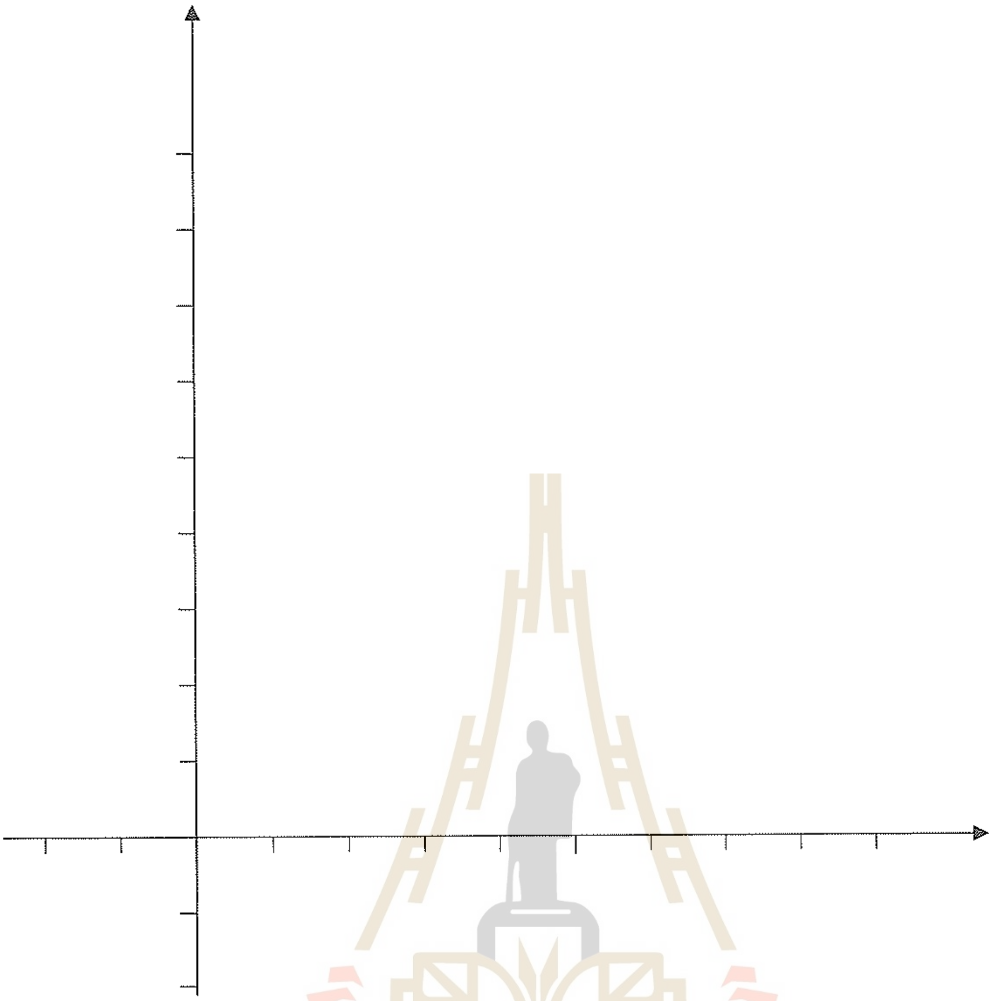
Since $x \geq \ldots\ldots$ always, then

$$x = f^{-1}(y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Exchange x and y,

$$y = f^{-1}(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

c)



d) Take the derivatives:

i) $\frac{df}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \Rightarrow \quad \frac{df}{dx}\Big|_{x=1} = \ldots\ldots\ldots\ldots\ldots\ldots$

ii) $\frac{df^{-1}}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ and $f(1) = \ldots\ldots\ldots\ldots$

$$\Rightarrow \frac{df^{-1}}{dx}\Big|_{x=f(1)} = \frac{df^{-1}}{dx}\Big|_{x=3} = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

Compare i) and ii). We see: $\frac{df^{-1}}{dx}\Big|_{x=3} = \frac{1}{\ldots\overline{\qquad}\ldots}$

e) Sketch in the above graph.

Exercise 2: Show that $f(x) = \dfrac{x-2}{x+1}$ is one-to-one. Then find $f^{-1}(x)$.

Solution: The domain of f is ....

Now

$$f'(x) = \frac{\text{—}}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2} = \frac{\qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)^2} > \ldots\ldots$$

That is,

$f(x)$ is ________________ on the interval ....

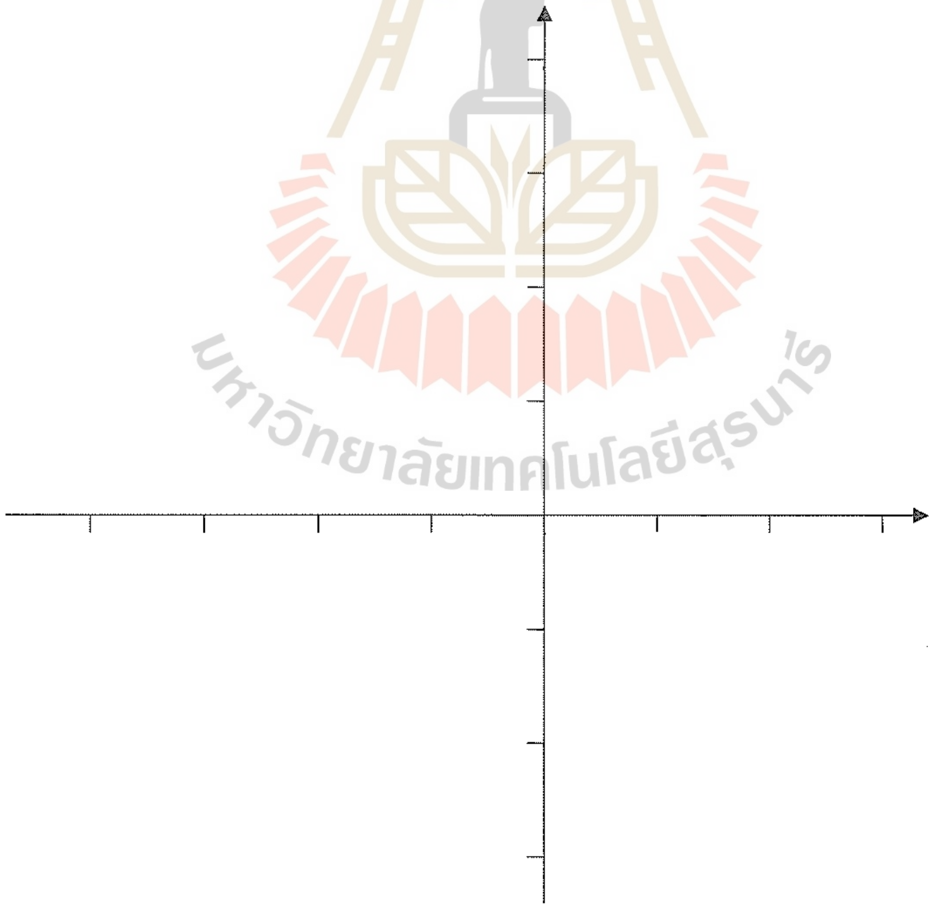
$f(x)$ is ________________ on the interval ....

Therefore,

f is ____one-to-one____ on the interval ....

f is ___________________ on the interval ....

Can we conclude that $f(x)$ is one-to-one on its domain ? Sketch the graph:



(Asympotes are $x = \ldots\ldots$ and $y = \ldots\ldots$)

We see: If $x < -1$ then $f(x) > \ldots\ldots$

If $x > -1$ then $f(x) < \ldots\ldots$

Therefore, $f(x)$ *is one-to-one on its domain.*

Now find $f^{-1}(x)$. Write

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and solve for x:

$$y(\ldots\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \ldots\ldots\ldots = \ldots\ldots\ldots - \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x = \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots} = \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exchange x and y. The inverse function is

$$y = f^{-1}(x) = \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 3: Consider $f(x) = x^5 - x^3 + 2x + 1$ (*)

Show that f is invertible and find $\frac{df^{-1}}{dx}\Big|_{x=3}$.

Solution: Check whether f is increasing / decreasing.

$$f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The discriminant is $b^2 - 4ac \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\Rightarrow$ $f'(x) \ldots\ldots\ldots\ldots$ on $(-\infty, \infty)$

$\Rightarrow$ f is ____________ on $(-\infty, \infty)$

$\Rightarrow$ f is ____________

Now find $\frac{df^{-1}}{dx}\Big|_{x=3}$.

It is not possible to find $f^{-1}(x)$ from (*). Instead, we use the formula

$$\frac{df^{-1}}{dx}\Big|_{x=f(a)} = \frac{1}{\frac{df}{dx}\Big|_{x=a}}$$

Now

1) $\frac{df}{dx} =$ ....

2) We want the derivative of $f^{-1}(x)$ when $x = f(a) = 3$. Looking at (*) we see that

$$f(a) = 3 \quad \text{when} \quad a^5 - a^3 + 2a + 1 = 3$$

$$a = \ldots\ldots\ldots$$

Then

$$\frac{df^{-1}}{dx}\Big|_{x=f(a)=3} = \frac{1}{\frac{df}{dx}\Big|_{x=a=\ldots}} = \frac{1}{\frac{df}{dx}\Big|_{x=\ldots}} = \frac{1}{\ldots\ldots\ldots\ldots\ldots\ldots} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 4: Consider $f(x) = 3 + x + e^x$.

Show that f is invertible and find $\frac{df^{-1}}{dx}\Big|_{x=4}$.

Solution: Check whether f is increasing / decreasing.

$f'(x) =$ ....

$\Rightarrow$ $f'(x)$ .... on $(-\infty, \infty)$

$\Rightarrow$ f is ____________ on $(-\infty, \infty)$

$\Rightarrow$ f is ____________

It is not possible to find $f^{-1}(x)$. Instead, we use the formula

$$\frac{df^{-1}}{dx}\bigg|_{x=f(a)} = \frac{1}{\dfrac{df}{dx}\bigg|_{x=a}}$$

1) $\dfrac{df}{dx} =$ ....

2) We want the derivative of $f^{-1}(x)$ when $x = f(a) = 4$. Now

$f(a) = 4$ when .... $= 4$

$a =$ ....

Therefore

$$\frac{df^{-1}}{dx}\bigg|_{x=f(a)=4} = \frac{1}{\dfrac{df}{dx}\bigg|_{x=a}} = \frac{1}{\dfrac{df}{dx}\bigg|_{x=\dots}} = \frac{1}{\dots\dots\dots\dots\dots\dots} = \dots\dots\dots\dots\dots\dots$$

Additional Exercises:

1) Find $f^{-1}(x)$

a) $f(x) = 3x - 7$

b) $f(x) = \sqrt{2+5x}$

c) $f(x) = \dfrac{1}{x+3}$

d) $f(x) = 5x^2 + 2, \quad x \geq 0$

2) Show that $f^{-1}(x)$ exists. Then find $\dfrac{df^{-1}}{dx}\bigg|_{x=a}$

a) $f(x) = 3 + 5x^3 + \sin(\pi x), \qquad -\infty < x < \infty, \quad a = 3$

b) $f(x) = \sin x + \cos x, \qquad -\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{4}, \quad a = 1$

Inverse Trigonometric Functions

Recall: (The definitions of the inverse trigonometric functions)

$$y=\sin^{-1}x \quad \Leftrightarrow \quad x=\sin y \qquad (\ldots\ldots \le x \le \ldots\ldots,\quad \ldots\ldots \le y \le \ldots\ldots)$$

$$y=\cos^{-1}x \quad \Leftrightarrow \quad x=\cos y \qquad (\ldots\ldots \le x \le \ldots\ldots,\quad \ldots\ldots \le y \le \ldots\ldots)$$

$$y=\tan^{-1}x \quad \Leftrightarrow \quad x=\tan y \qquad (\ldots\ldots < x < \ldots\ldots,\quad \ldots\ldots < y < \ldots\ldots)$$

The derivatives are:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

Exercise 1: Some typical values of $y=\sin^{-1}x$

$$\sin^{-1}(0) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = 0$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = \frac{1}{2}$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = \ldots\ldots\ldots$$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = \ldots\ldots\ldots$$

$$\sin^{-1}(1) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = \ldots\ldots\ldots$$

$$\sin^{-1}\left(-\frac{1}{2}\right) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = -\frac{1}{2}$$

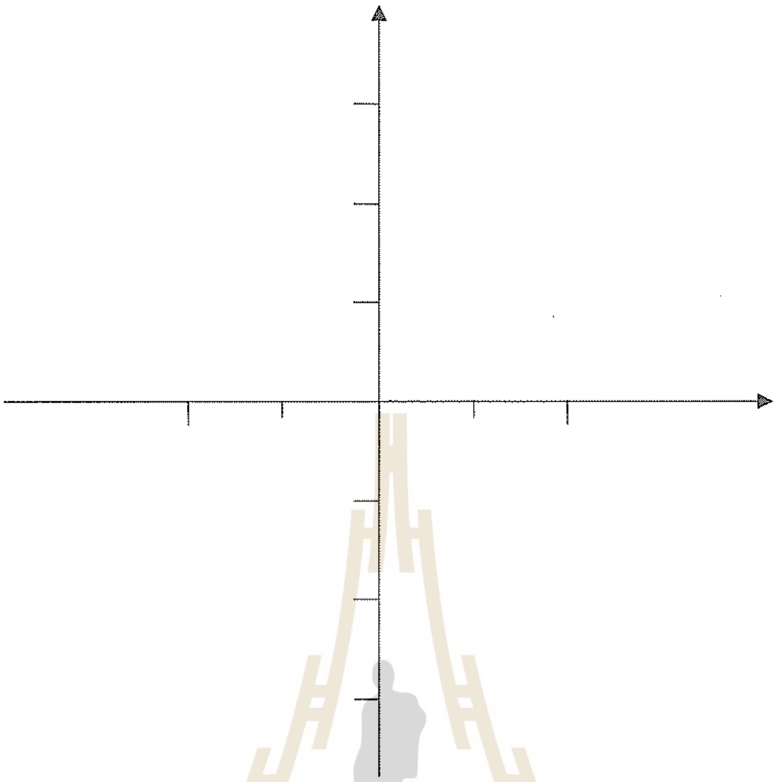
$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = \ldots\ldots\ldots$$

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = \ldots\ldots\ldots$$

$$\sin^{-1}(-1) = \ldots\ldots\ldots\ldots \quad \text{because} \quad \sin(\ldots\ldots\ldots) = -1$$

Exercise 2: Sketch the graph of $y = \sin^{-1} x$

Solution:



Exercise 3: Find the following values:

1. $\sin\left(\sin^{-1}\frac{1}{2}\right)$ 2. $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$

Solution:

1. $\sin^{-1}\frac{1}{2} = \ldots\ldots\ldots\ldots$

Therefore, $\sin\left(\sin^{-1}\frac{1}{2}\right) = \sin(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$

In general,

$$\boxed{\sin\left(\sin^{-1} x\right) = \ldots\ldots\ldots} \qquad (-1 \le x \le 1)$$

2. $\sin\frac{3\pi}{4} = \ldots\ldots\ldots\ldots$

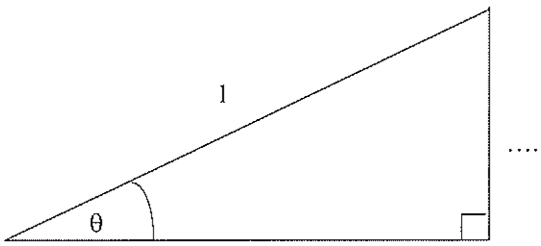
Therefore, $\sin^{-1}\left(\sin\frac{3\pi}{4}\right) = \sin^{-1}(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$

In general,

$$\boxed{\sin^{-1}(\sin x) = \ldots\ldots\ldots} \qquad \textit{only if} \quad \ldots\ldots \le x \le \ldots\ldots$$

Exercise 4: Find: $\tan\left(\sin^{-1} 0.4\right)$

Solution: Sketch a right triangle where $\theta = \sin^{-1} 0.4$, that is $\sin\theta = \ldots\ldots\ldots$.



The side adjacent to θ has length .... Therefore,

$$\tan\left(\sin^{-1} 0.4\right) = \tan\theta = \frac{\text{opp}}{\ldots\ldots} = \frac{\quad}{\ldots\ldots} = \ldots\ldots$$

Exercise 5: Some typical values of $y = \cos^{-1} x$

$\cos^{-1}(0) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = 0$

$\cos^{-1}\left(\frac{1}{2}\right) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \frac{1}{2}$

$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\cos^{-1}(1) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\cos^{-1}\left(-\frac{1}{2}\right) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \ldots\ldots\ldots$

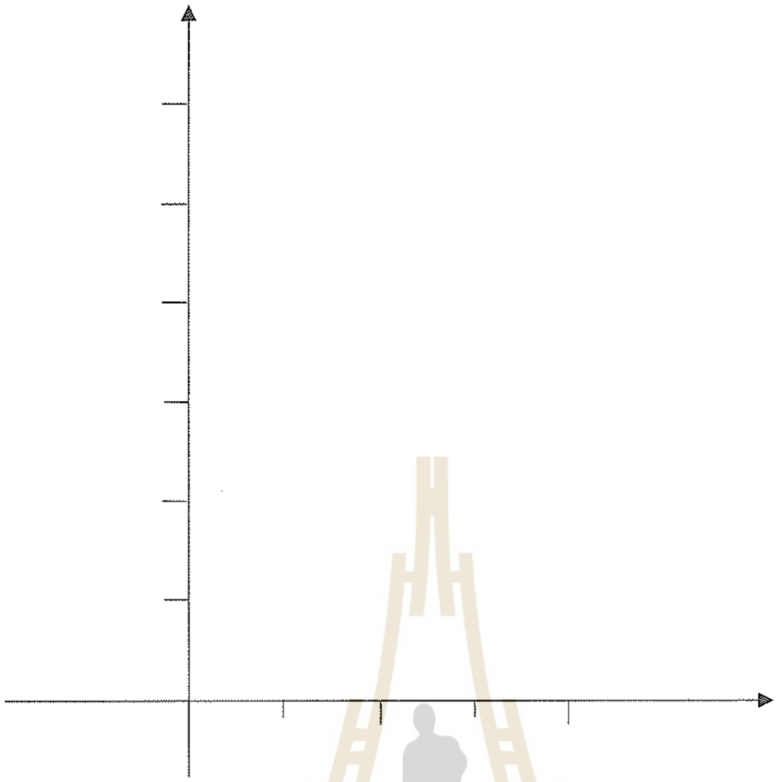
$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\cos^{-1}(-1) = \ldots\ldots\ldots\ldots$ because $\cos(\ldots\ldots\ldots) = \ldots\ldots\ldots$

Exercise 6: Sketch the graph of $y = \cos^{-1} x$

Solution:



Exercise 7: Find the following values

1. $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$ **2.** $\cos^{-1}\left[\cos\left(-\frac{\pi}{3}\right)\right]$

Solution:

1. $\cos^{-1}\frac{\sqrt{3}}{2} = \ldots\ldots\ldots\ldots$

Therefore, $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \cos(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$

In general,

$\cos\left(\cos^{-1} x\right) = \ldots\ldots\ldots$	$(-1 \le x \le 1)$

2. $\cos\left(-\frac{\pi}{3}\right) = \ldots\ldots\ldots\ldots$

Therefore, $\cos^{-1}\left[\cos\left(-\frac{\pi}{3}\right)\right] = \cos^{-1}(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$

In general,

$\cos^{-1}(\cos x) = \ldots\ldots\ldots$	*only if* $\ldots\ldots \le x \le \ldots\ldots$

Exercise 8: If x is any number, $-1 \le x \le 1$, find $\sin\left(\cos^{-1} x\right)$.

Solution:

1. Method: Change $\sin$ to $\cos$.

Set

$$\theta = \cos^{-1} x$$

From

$$\sin^2\theta + \cos^2\theta = 1$$

we obtain

$$\sin\theta = \pm\sqrt{\ldots\ldots\ldots\ldots\ldots\ldots}$$

so that

$$\sin\left(\cos^{-1} x\right) = \pm\sqrt{\ldots\ldots\ldots\ldots\ldots\ldots} = \pm\ldots\ldots\ldots\ldots\ldots\ldots$$

Because always $\ldots\ldots\ldots\ldots \le \cos^{-1} x \le \ldots\ldots\ldots\ldots$

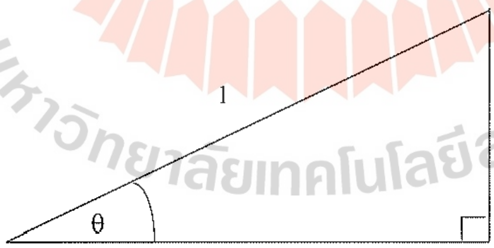
and

$$\sin\theta \ge \ldots\ldots\ldots \text{ on } [0, \ldots\ldots]$$

then

$$\sin\left(\cos^{-1} x\right) = \ldots\ldots\ldots\ldots$$

2. Method: Sketch a right triangle where $\theta = \cos^{-1} x$, that is $\cos\theta = \ldots\ldots\ldots$.



The side opposite to θ has length ………….. Therefore,

$$\sin\left(\cos^{-1} x\right) = \sin\theta = \frac{\text{opp}}{\ldots\ldots} = \frac{}{\ldots\ldots} = \ldots\ldots$$

Exercise 9: Some typical values of $y = \tan^{-1} x$.

$\tan^{-1}(0) = \ldots\ldots\ldots\ldots$ because $\tan(\ldots\ldots\ldots) = 0$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \ldots\ldots\ldots\ldots$ because $\tan(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\tan^{-1}(1) = \ldots\ldots\ldots\ldots$ because $\tan(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\tan^{-1}(\sqrt{3}) = \ldots\ldots\ldots\ldots$ because $\tan(\ldots\ldots\ldots) = \sqrt{3}$

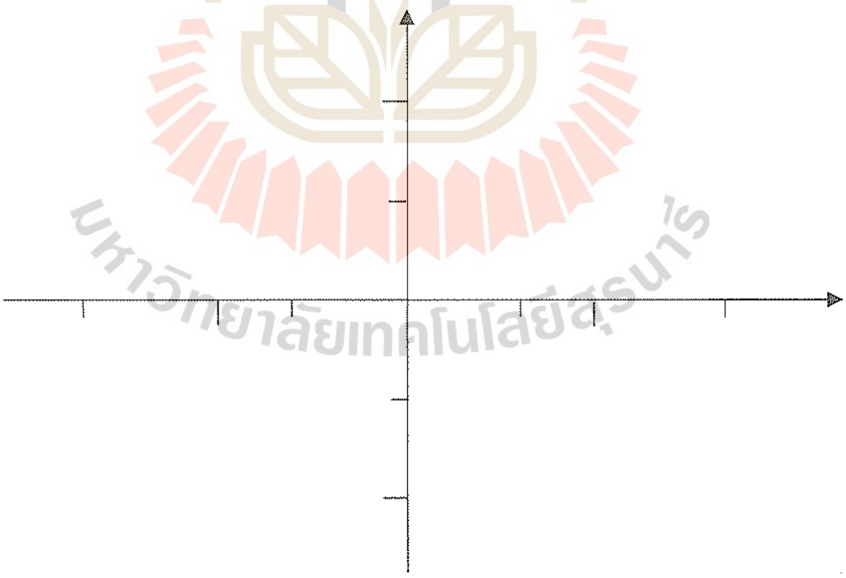
$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \ldots\ldots\ldots\ldots$ because $\tan(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\tan^{-1}(-1) = \ldots\ldots\ldots\ldots$ because $\tan(\ldots\ldots\ldots) = \ldots\ldots\ldots$

$\tan^{-1}(-\sqrt{3}) = \ldots\ldots\ldots\ldots$

Exercise 10: Sketch the graph of $y = \tan^{-1} x$.

Solution:



$\lim_{x\to\infty} \tan^{-1} x = \ldots\ldots\ldots\ldots\ldots$

$\lim_{x\to-\infty} \tan^{-1} x = \ldots\ldots\ldots\ldots\ldots$

Symmetry: $y = \tan^{-1} x$ is an ____________ function.

Exercise 11: Find the following values

1. $\tan\left(\tan^{-1}(-1)\right)$ **2.** $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Solution:

1. $\tan^{-1}(-1) = \ldots\ldots\ldots\ldots$

Therefore, $\tan\left(\tan^{-1}(-1)\right) = \tan(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$

In general,

$$\boxed{\tan\left(\tan^{-1}x\right) = \ldots\ldots\ldots} \qquad (-\infty \le x \le \infty)$$

2. $\tan\frac{7\pi}{6} = \tan\frac{\quad}{6} = \ldots\ldots\ldots$ (because $y = \tan x$ has period $\ldots\ldots$)

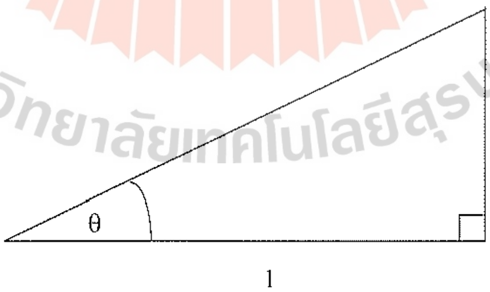
Therefore, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}(\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$

In general,

$$\boxed{\tan^{-1}(\tan x) = \ldots\ldots\ldots} \qquad \textit{only if} \quad \ldots\ldots \le x \le \ldots\ldots$$

Exercise 12: If x is any number, find $\sec\left(\tan^{-1}x\right)$.

Solution: Sketch a right triangle where $\theta = \tan^{-1}x$, that is $\tan\theta = \ldots\ldots = \frac{\quad}{1}$.



The hypotenuse has length $\ldots\ldots\ldots\ldots\ldots$ Therefore,

$$\sec\left(\tan^{-1}x\right) = \sec\theta = \frac{\text{hyp}}{\ldots\ldots} = \frac{\quad}{\ldots\ldots} = \ldots\ldots$$

Exercise 13: Find the derivative of $f(x) = \sin^{-1}(2x-1)$.

Solution: By the chain rule, with

$$f(u) = \sin^{-1}(u) \qquad \text{and} \qquad u = 2x-1$$

we have

$$f'(x) = \frac{1}{\sqrt{1-\left(\ldots\ldots\ldots\ldots\ldots\right)^2}} \cdot \frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots)$$

$$= \frac{1}{\sqrt{1-\left(\ldots\ldots\ldots\ldots\ldots\ldots\right)}} \cdot (\ldots\ldots) = \frac{}{\sqrt{\ldots\ldots\ldots\ldots\ldots\ldots}} = \frac{1}{\sqrt{\ldots\ldots\ldots\ldots\ldots}}$$

Exercise 14: Find the derivative of $y = \dfrac{\arcsin x}{x}$.

Solution: By the quotient rule,

$$\frac{dy}{dx} = \frac{\ldots\ldots(\ldots\ldots\ldots\ldots)' - \ldots\ldots\ldots\ldots\ldots(\ldots\ldots)'}{\ldots\ldots\ldots\ldots}$$

$$= \frac{\ldots\ldots\left(\dfrac{1}{\sqrt{\ldots\ldots\ldots\ldots}}\right) - \ldots\ldots\ldots\ldots\ldots(\ldots\ldots)}{\ldots\ldots\ldots\ldots} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 15: Find the derivative of $y = \tan^{-1}\left(x^3\right)$.

Solution: By the chain rule,

$$\frac{dy}{dx} = \frac{1}{\ldots\ldots\ldots\ldots\ldots\ldots} \cdot \frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots)$$

$$= \frac{1}{\ldots\ldots\ldots\ldots\ldots\ldots} \cdot (\ldots\ldots\ldots\ldots\ldots) = \frac{}{\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 16: Find the derivative of $y = \sec^{-1}\sqrt{1+x^2}$.

Solution: By the chain rule,

$$\frac{dy}{dx} = \frac{d}{du}\left(\sec^{-1} u\right)\cdot\frac{du}{dx} \qquad \left(u = \ldots\ldots\ldots\ldots\right)$$

$$= \frac{1}{\ldots\ldots\sqrt{\ldots\ldots\ldots -1}}\cdot\frac{d}{dx}\left(\ldots\ldots\ldots\ldots\ldots\right)$$

$$= \frac{1}{\ldots\ldots\ldots\ldots\ldots\sqrt{\left(\ldots\ldots\ldots\ldots\ldots\right)^2 -1}}\cdot\left(\frac{\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots}\right)$$

$$= \frac{x}{\left(\ldots\ldots\ldots\right)\sqrt{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}} = \frac{\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 17: Find $\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right)$.

Solution: If $x \to 0^+$ then $u = \frac{1}{x} \to \ldots\ldots\ldots\ldots$

Therefore, $\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right) = \lim_{u\to\ldots\ldots} \tan^{-1}(\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots$

Additional Exercises:

1) Find the following values

a) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

b) $\arccos\left(\cos\frac{5\pi}{4}\right)$

c) $\tan^{-1}\left(\tan\frac{7\pi}{4}\right)$

d) $\sin\left(\cos^{-1}\frac{1}{2}\right)$

e) $\sec\left(\tan^{-1}\left[-\frac{3}{5}\right]\right)$

f) $\tan\left(\arccos x\right)$

2) Find the derivatives of

a) $f(x) = \sin^{-1}\sqrt{x}$

b) $y = \frac{1}{\arctan x^2}$

c) $y = \sin^{-1}\left(\frac{1}{x}\right)$

d) $f(x) = \left(1+\cos^{-1}(3x)\right)^3$

e) $y = \cos(x^{-1}) + (\cos x)^{-1} + \cos^{-1} x$

f) $y = e^{-x}\sec^{-1}\left(e^{-x}\right)$

Exponential Functions

Recall:

1) If $a, b > 0$ then

$$a^x a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$a^x b^x = (ab)^x$$

2) If e is the number with

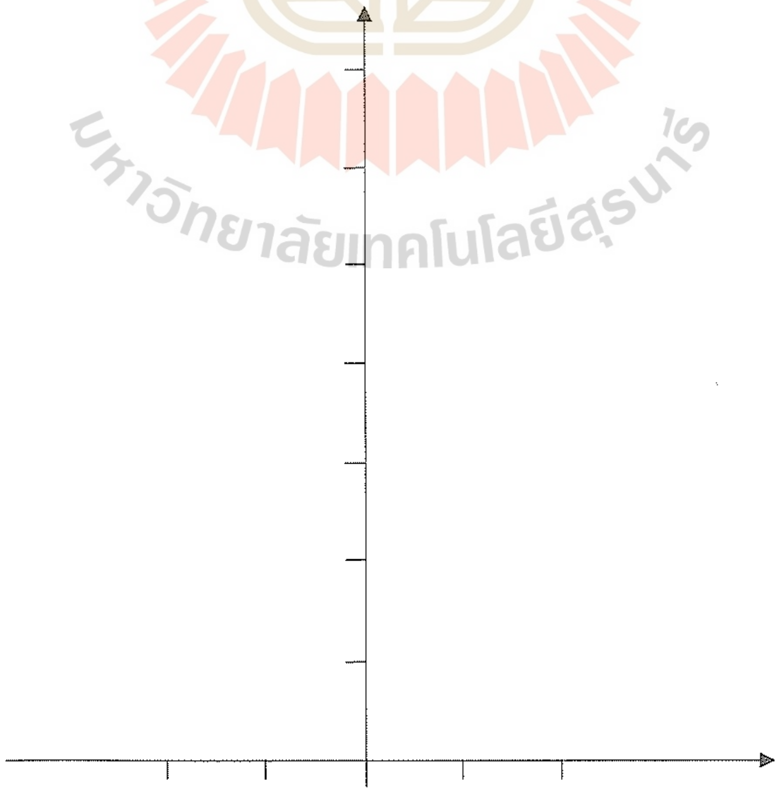
$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

then

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Exercise 1: Sketch the graphs of $y = 2^x$, $y = 5^x$, $y = 10^x$, $y = e^x$, $y = 2^{-x}$, $y = \left(\frac{1}{5}\right)^x$ in the same coordinate system.

Solution:



Exercise 2: Use the graphs in exercise 1 to find the following limits:

$$\lim_{x\to\infty} 2^{x} = \ldots\ldots\ldots\ldots$$

$$\lim_{x\to\infty} 2^{-x} = \ldots\ldots\ldots\ldots$$

$$\lim_{x\to-\infty} 2^{x} = \ldots\ldots\ldots\ldots$$

$$\lim_{x\to-\infty} e^{x} = \ldots\ldots\ldots\ldots$$

$$\lim_{x\to\infty} \left(\frac{1}{5}\right)^{x} = \ldots\ldots\ldots\ldots$$

Exercise 3: If $y = e^{kx}$ (k =constant), find $\frac{dy}{dx}$.

Solution: We must use the __________ rule, with $y = e^{u}$ and $u = kx$. We get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{u}\right) = \frac{d}{du}(\ldots\ldots)\cdot\frac{du}{dx} = (\ldots\ldots\ldots)(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$$

Exercise 4: If $f(x) = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$ find $f'(x)$.

Solution: We must use the ________________ rule and Exercise 3,

$$f'(x) = \frac{(\qquad)\frac{d}{dx}(\qquad\qquad) - (\qquad\qquad)\frac{d}{dx}(\qquad\qquad)}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

$$= \frac{(\qquad\qquad)(\qquad\qquad) - (\qquad\qquad)(\qquad\qquad)}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

$$= \frac{\qquad\qquad\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

$$= \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)^{2}}$$

Exercise 5: If $y = \sqrt{x}\, e^{-x^2}$ find $\dfrac{dy}{dx}$.

Solution: We must use the ____________________ rule.

$$\frac{dy}{dx} = (\ldots\ldots\ldots\ldots)\frac{d}{dx}(\ldots\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots)\frac{d}{dx}(\ldots\ldots\ldots\ldots)$$

$$= (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

$$= \left(\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \right)e^{-x^2} = \left(\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \right)e^{-x^2}$$

Exercise 6: Sketch the graph of $f(x) = x^2 e^x$

Solution:

1. Find the critical numbers.

$f'(x) =$ ______________________ $=$ ____________________ ____

$f'(x) = 0$ when $\ldots\ldots\ldots\ldots = 0 \quad \Rightarrow \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$

The critical numbers are: $x =$ ________

2. Check the sign of f'.

f'			
f			

f is increasing on ____________ and ____________

f is decreasing on ____________ and ____________

f has a relative maximum at ______________________________

f has a relative minimum at ______________________________

3. Check the sign of f''

$f''(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f''(x) = 0$ when $\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$

$x =$ ______________________ = ______________________

$f''(x)$			
x^2+4x+2			
f			

f is concave up on ____

f is concave down on ____

f has inflection point(s) at _______________________________________

4. Combine all information:

$f'(x)$					
$f''(x)$					
$f(x)$					

5. Table of values :

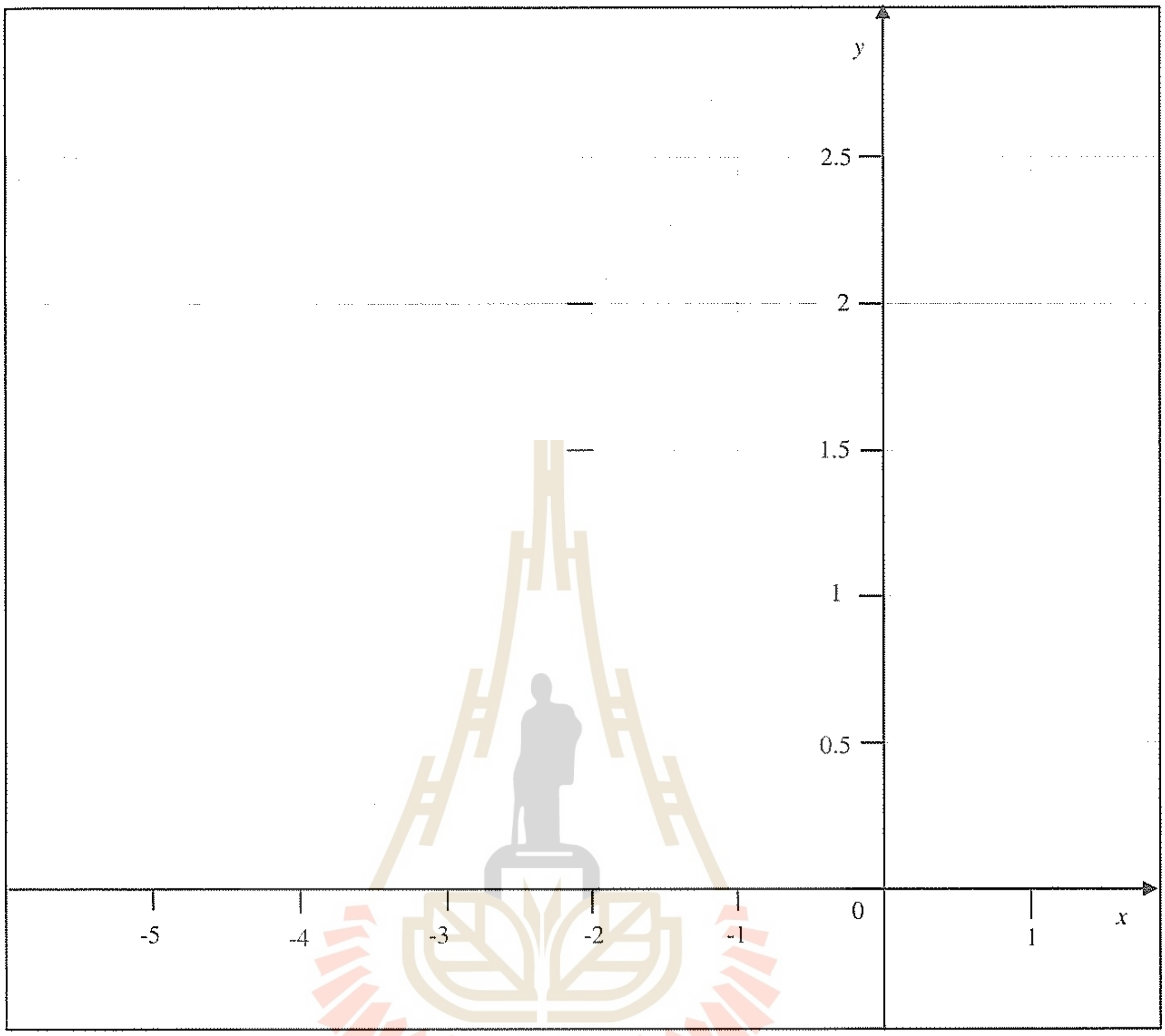
x	-5	$-2-\sqrt{2}$ (-3.4)	-2	$-2+\sqrt{2}$ (-0.6)	0	1
$f(x)$	~0.17	~0.38	~0.54	~0.19		

The relative maximum value is ______________________

The relative minimum value is ______________________

Inflection points are ______________________________

6. Sketch the graph of f :



Additional Exercises:

1) Differentiate.

a) $f(x) = e^{-3x} \sin 5x$

b) $y = \dfrac{e^{3x}}{1+e^{x}}$

c) $y = \tan\left(e^{3x+2}\right)$

d) $h(x) = \dfrac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

e) $y = e^{-x} \sec^{-1}\left(e^{-x}\right)$

2) Find the following limits.

a) $\lim\limits_{x \to \infty} e^{1/x}$

b) $\lim\limits_{x \to 0^+} e^{-1/x}$

c) $\lim\limits_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

The Natural Logarithm

Recall:

1) (The definition of the natural logarithm)

$$y = \ln x \quad \Leftrightarrow \quad x = e^y$$

Domain of $y = \ln x$: $\ldots\ldots < x < \ldots\ldots$

Range of $y = \ln x$: $\ldots\ldots < y < \ldots\ldots$

2) The rules of the logarithms are

$$\ln(xy) = \ln x + \ln \ldots$$

$$\ln \frac{x}{y} = \ln x - \ln \ldots$$

$$\ln x^y = y \ln x$$

3) The derivatives are:

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad (x > 0)$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x} \qquad (x \neq 0)$$

Exercise 1: Compute the following values:

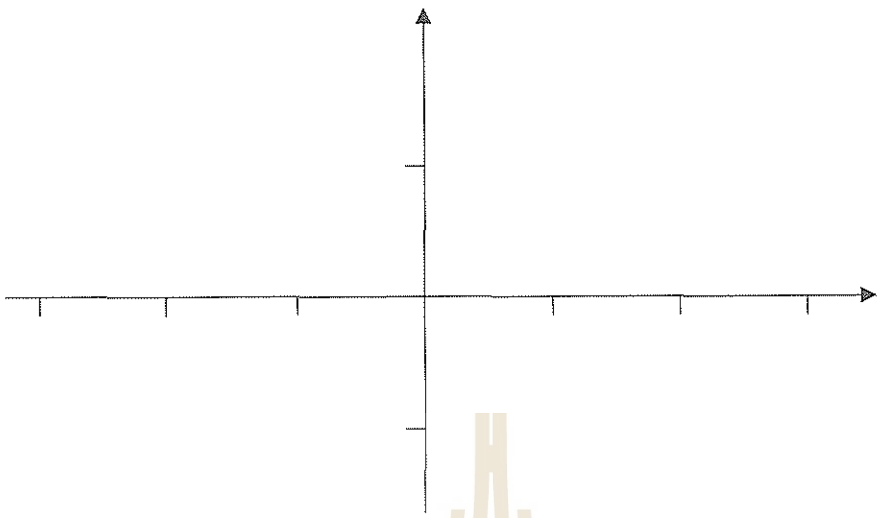
1. $\ln e^2 + \ln \frac{2}{e} - e^{\ln 3} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

2. $e^{2\ln 4} - \ln 8 + 3\ln 2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

3. $\ln x^2 - 2\ln e^{x-4} + e^{\ln(2x-8)} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

Exercise 2: Sketch the graphs of $y = \ln x$ and $y = \ln|x|$.

Solution:



Symmetry: We observe that $y = \ln|x|$ is an ____________ function, so its graph is symmetric about ________________

$\lim\limits_{x\to\infty} \ln x = \ldots\ldots\ldots$ $\qquad$ $\lim\limits_{x\to 0^+} \ln x = \ldots\ldots\ldots$

$\lim\limits_{x\to -\infty} \ln|x| = \ldots\ldots\ldots$ $\qquad$ $\lim\limits_{x\to 0} \ln|x| = \ldots\ldots\ldots$

Exercise 3: Find the equation of the tangent line to the graph of $y = x\ln x$ at the point $(1,0)$.

Solution: By the ____________ rule,

$$y' = x\frac{d}{dx}(\ldots\ldots\ldots) + \ln x\frac{d}{dx}(\ldots\ldots\ldots)$$

$$= (\ldots\ldots)(\ldots\ldots) + (\ldots\ldots)(\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Thus, the tangent line at point $(1,0)$ has slope $m = y'(\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The equation of the tangent line is

$$y - y_o = m(x - \ldots\ldots)$$

$$y - \ldots\ldots = m(x - \ldots\ldots)$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 4: If $y = \ln\left| x + \sqrt{x^2 - 1} \right|$ find $\dfrac{dy}{dx}$.

Solution: This is a composition of the functions

$$y = \ln|u| \quad \text{and} \quad u = \ldots\ldots\ldots\ldots\ldots\ldots$$

By the ____________ rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\ldots\ldots\ldots\ldots}\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots)$$

$$= \frac{1}{\ldots\ldots\ldots\ldots\ldots\ldots}(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

$$= \frac{1}{\ldots\ldots\ldots\ldots\ldots\ldots} \, \frac{\qquad\qquad}{\sqrt{x^2 - 1}}$$

$$= \frac{1}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 5: If $f(x) = \ln\left(\dfrac{x+1}{x-1}\right)^{3/5}$ find $f'(x)$.

Solution: Simplify first. By the rules for logarithms,

$$f(x) = \frac{3}{5}\left[\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots}\right] = \frac{3}{5}\left[\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\right]$$

Then

$$f'(x) = \frac{3}{5}\left[\frac{1}{\ldots\ldots\ldots\ldots} - \frac{1}{\ldots\ldots\ldots\ldots}\right]$$

$$= \frac{3}{5}\,\frac{\qquad\qquad}{(\ldots\ldots\ldots)(\ldots\ldots\ldots)} = \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 6: Sketch the graph of $f(x) = \ln\left| x^2 - 3 \right|$.

Solution:

1. Find the domain of f

We need $\left| x^2 - 3 \right|$..............

$x^2 - 3 \neq$ ....

$x \neq$ ....

The domain is ________________________________

2. Observe that

$$f(-x) = \ln\left|(\ldots\ldots)^2 - 3\right| = \ln\left|\ldots - 3\right| = \ldots\ldots$$

Therefore, f is an ____________ function.

Its graph is symmetric about _________________________

3. Check for asymptotes.

$$\lim_{x\to\ldots} \ln\left|x^2 - 3\right| = \lim_{u\to\ldots} \ldots\ldots\ldots = \ldots$$

$$\lim_{x\to\ldots} \ln\left|x^2 - 3\right| = \lim_{u\to\ldots} \ldots\ldots\ldots = \ldots$$

The lines _________________ are _____________ asymptotes.

4. Find the critical numbers.

$f'(x) =$ _______________________________

$f'(x) = 0$ when $x =$ ____________________________

$f'(x)$ is undefined when $x =$ ______________________

The critical numbers are: $x =$ _________ because ____________________

5. Check the sign of f'.

Sign of f'				
$f'(x)$				
Test value x				
f				

f is increasing on ____________ and ______________

f is decreasing on ____________ and ______________

f has a relative maximum at __________________________________

f has a relative minimum at __________________________________

6. Check the sign of f''.

$f''(x) =$ ______________________ = ________________ = ______________

We see that $f''(x) \ldots\ldots 0$ always.

Sign of f''			
f			

f is concave up on ____

f is concave down on ____

f has inflection point(s) _______________________________________

7. Find the x-intercepts:

$$y = f(x) = 0 \quad \text{when} \quad \left| x^2 - 3 \right| = \ldots\ldots$$

$$x^2 - 3 = \ldots\ldots \text{ or } \ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

$$x^2 = \ldots\ldots \text{ or } x^2 = \ldots\ldots$$

The x-intercepts are $x =$ __________________________

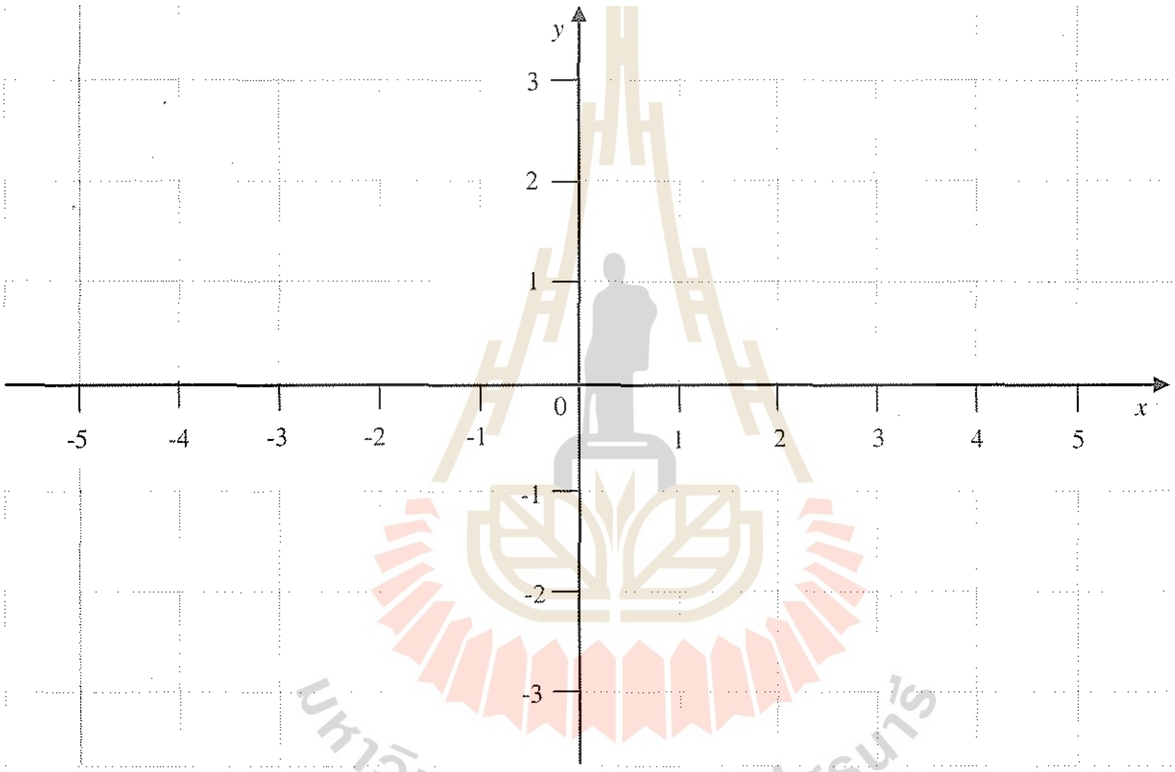
8. Table of values:

x	-4	-2	-1	0	1	2	4
$f(x)$				1.1	0.69		2.57

The relative maximum value is ______________________________

The relative minimum values is ______________________________

9. Sketch the graph of f.



Additional Exercises:

Compute the derivatives of:

1) $f(x) = \ln\left(4x^3 - 2x^2 + 3x - 1\right)$

2) $f(x) = \ln\left| 5x^2 + 3 \right|$

3) $f(x) = \ln\left| \sin^3 x \right|$

4) $y = \ln\left[(5x-7)^4 (2x+3)^3\right]$

5) $h(x) = \ln \sqrt[3]{6x+7}$

6) $f(x) = \ln\left(\tan^4(2x)\right)$

Arbitrary Logarithms and Exponentials

Recall:

1) (The definition of the logarithm)

$$y = \log_a x \quad \Leftrightarrow \quad x = a^y$$

Domain of $y = \log_a x$: $\ldots\ldots < x < \ldots\ldots$

Range of $y = \log_a x$: $\ldots\ldots < y < \ldots\ldots$

2) The rules of the logarithms are

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^y = y \log_a x$$

3) Relationship between various bases:

$$a^x = \left(e^{\cdots\cdots}\right)^x = e^{\cdots\cdots}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

4) The derivatives are

$$\frac{d}{dx} a^x = \ldots\ldots\ldots$$

$$\frac{d}{dx} \log_a x = \frac{1}{x\ldots\ldots\ldots} \qquad (x > 0)$$

$$\frac{d}{dx} \log_a |x| = \frac{1}{x\ldots\ldots\ldots} \qquad (x \neq 0)$$

Exercise 1: Express in terms of base e:

1. $3^{-(x^2+1)} = e^{\cdots\cdots\cdots}$

2. $\log_2\left(e^{\sin x}\right) = \ldots\ldots\ldots \log_2 e = \ldots\ldots\ldots\ldots \frac{\ln\ldots\ldots}{\ldots\ldots\ldots} = \frac{}{\ldots\ldots\ldots}$

Exercise 2: Compute the following derivatives.

1. $\dfrac{d}{dx}\left(5^{\tan x}\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \dfrac{d}{dx}\left(\ldots\ldots\ldots\right) = \ldots\ldots\ldots\ldots\ldots\ldots$

2. If $f(x) = 1.6^{x} + x^{1.6}$ then $f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

3. If $f(x) = \log_4(x^3 \sin x)$, first write

$$f(x) = \ldots\ldots \log_4 x + \log_4(\ldots\ldots\ldots)$$

Then $f'(x) = \ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

4. $\dfrac{d}{dx}\left(\log_{10}\left(x^2+x\right)\cdot\left(4^{x}-1\right)^{3}\right)$

$$= \frac{d}{dx}\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$= \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} + \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 3: Find $\dfrac{dy}{dx}$ if $y = \dfrac{(x+1)^4\, e^{x^2-1}}{(x^2+3)^{1/4}}$

Solution: We use ____________________ differentiation. Write

$$\ln y = \ln\left[\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}\right]$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots$$

Then by implicit differentiation,

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dy}{dx} = y\left[\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots} + \ldots\ldots\ldots - \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right]$$

$$= \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\left[\frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots} + \ldots\ldots\ldots - \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots}\right]$$

Exercise 4: Find the derivative of $y = x^{\sin x}$.

Solution 1: (Use the *definition* of $f(x)^{g(x)}$)

Write

$$y = x^{\sin x} = \left(e^{\cdots\cdots}\right)^{\sin x} = e^{\cdots\cdots\cdots\cdots}$$

Then by the ________ rule,

$$\frac{dy}{dx} = \cdots\cdots\cdots\cdots\cdots\frac{d}{dx}(\cdots\cdots\cdots\cdots\cdots)$$

$$= x^{\cdots\cdots}\left(\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\right)$$

Solution 2: (Use logarithmic differentiation)

Write

$$\ln y = \ln(\cdots\cdots\cdots) = \cdots\cdots\cdots\cdots$$

Then by implicit differentiation,

$$\cdots\cdots\frac{dy}{dx} = \cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots$$

$$\frac{dy}{dx} = y\left(\cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots\right)$$

$$= x^{\cdots\cdots}\left(\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\right)$$

Exercise 5: Find the derivative of $y = x^{1/x}$ in two ways.

Solution 1: (Use the *definition* of $f(x)^{g(x)}$)

$$y = x^{1/x} = \left(e^{\cdots\cdots}\right)^{1/x} = e^{\cdots\cdots\cdots\cdots}$$

Then by the ________ rule,

$$\frac{dy}{dx} = \cdots\cdots\cdots\cdots\cdots\frac{d}{dx}\left(\cdots\cdots\cdots\cdots\cdots\right)$$

$$= x^{\cdots\cdots}\left(\cdots\cdots\cdots\cdots\cdots\cdots\right) = x^{\cdots\cdots}\left(\cdots\cdots\cdots\cdots\cdots\cdots\right)$$

Solution 2: (Use logarithmic differentiation.) Write

$$\ln y = \ln\left(\ldots\ldots\ldots\ldots\right) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Then by implicit differentiation,

$$\ldots\ldots\ldots\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dy}{dx} = y\left(\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\right) = x^{\ldots\ldots}\left(\ldots\ldots\ldots\ldots\ldots\ldots\right)$$

Additional Exercises:

1) Compute the derivatives of

a) $f(x) = \log_4 |\tan 2x|$

b) $g(x) = \log_{10} \dfrac{x}{x-1}$

c) $y = x^{2/5}\left(x^2+8\right)^7 e^{-x^2}$

d) $y = \dfrac{\left(x^3+1\right)^4 \sin^2 x}{\sqrt[3]{x}}$

e) $y = x^{\pi} + \pi^{x} + x^{x} + \pi^{\pi}$

f) $f(x) = 2^{3^x}$

g) $y = x^{\ln x}$

h) $y = (\sin x)^x$

i) $y = \sqrt[3]{\dfrac{x^2+9}{x+9}}$

2) Sketch the following graphs in the *same* coordinate system.

a) $y = \ln x$

b) $y = \log_2 x$

c) $y = \log_5 x$

d) $y = \log_{10} x$

e) $y = \log_{1/2} x$

f) $y = \log_2(-x)$

3) Find the following limits.

a) $\lim\limits_{x\to 1^+} 3^{2/(x-1)}$

b) $\lim\limits_{x\to 1^-} 3^{2/(x-1)}$

c) $\lim\limits_{x\to\infty} \ln(1+e^{-x^2})$

d) $\lim\limits_{x\to 0^+} \ln(\sin x)$

e) $\lim\limits_{x\to\infty} \dfrac{\log_2 x}{\log_2 x + 1}$

f) $\lim\limits_{x\to\infty} \left[\, \log_{10}(x+1) - \log_{10}(2x+3) \,\right]$

4) Find the inverse function of

a) $f(x) = \log_2(x+2)$

b) $g(x) = \sqrt{\ln x}$

c) $h(x) = \dfrac{1+e^x}{1-e^x}$

Hyperbolic Functions

Recall:

1) (Definition of the hyperbolic functions)

$$\sinh x = \frac{e^x - \quad}{2}$$

$$\cosh x = \frac{e^x + \quad}{2}$$

$$\tanh x = \frac{\qquad}{\ldots\ldots\ldots} = \frac{e^x - \quad}{e^x + \quad}$$

2) The main identity is

$$\cosh^2 x - \sinh^2 x = \ldots\ldots\ldots$$

3) The derivatives are:

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \text{sech}\, x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$$

Exercise 1: Find the following values:

$\sinh(0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\cosh(0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\cosh(\ln 4) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots = \ldots\ldots$

$\tanh(\ln x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 2: Prove the following identities:

1. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
2. $\sinh(2x) = 2\sinh x \cosh x$

Solution:

1. Use the definition of $\sinh(x)$ and $\cosh(y)$

$$\sinh x \cosh y + \cosh x \sinh y = \frac{\qquad}{2}\,\frac{\qquad}{2} + \frac{\qquad}{2}\,\frac{\qquad}{2}$$

$$= \frac{\qquad\qquad}{4} + \frac{\qquad\qquad}{4}$$

$$= \frac{\qquad\qquad}{4}$$

$$= \frac{\qquad\qquad}{2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. Choose $y = x$ in 1.,

$\sinh(x+x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

or

$\sinh(2x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 3: Find the derivatives of the given functions

1. $f(x) = \sinh\left(x^2+1\right)$

By the ______________ rule,

$$f'(x) = \ldots\ldots\ldots\left(x^2+1\right)\frac{d}{dx}(\ldots\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. $f(x) = \cosh^3 x$

By the __________ rule,

$$f'(x) = \ldots\ldots\ldots\ldots\ldots \frac{d}{dx}(\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. $y = \tan^{-1}(\tanh x)$

By the __________ rule,

$$\frac{dy}{dx} = \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots} \frac{d}{dx}(\ldots\ldots\ldots\ldots)$$

$$= \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{\qquad\qquad}{\ldots\ldots + \ldots\ldots} = \frac{\qquad\qquad}{\ldots\ldots + \ldots\ldots\ldots}$$

Additional Exercises:

1) Sketch the graphs of

$y = \sinh x$, $y = \cosh x$ and $y = \tanh x$

2) Compute the derivatives of

a) $f(x) = e^x \sinh x$

b) $y = \tanh(e^t)$

c) $h(x) = \ln(\sinh x)$

d) $y = x^{\cosh x}$

3) Find the following limits.

a) $\lim_{x\to\infty} \tanh x$

b) $\lim_{x\to-\infty} \tanh x$

c) $\lim_{x\to-\infty} \sinh x$

d) $\lim_{x\to-\infty} \operatorname{sech} x$

L'Hôpital's Rule

Recall: If $\dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ at $x=a$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} \overset{H}{=} \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Exercise 1: Find $\displaystyle\lim_{x\to 0}\frac{x+\sin 3x}{x+\sin 5x}$.

Solution: This is of type ....

Therefore,

$$\lim_{x\to 0}\frac{x+\sin 3x}{x+\sin 5x} \overset{H}{=} \lim_{x\to 0}\frac{(x+\sin 3x)'}{(\ldots\ldots\ldots\ldots\ldots)'} = \lim_{x\to 0}\frac{\rule{3cm}{0.4pt}}{\ldots\ldots\ldots\ldots\ldots\ldots} = \ldots\ldots\ldots = \ldots\ldots$$

Exercise 2: Find $\displaystyle\lim_{x\to\infty}\frac{\ln(\ln x)}{\sqrt{x}}$.

Solution: This limit is of type ....

Therefore,

$$\lim_{x\to\infty}\frac{\ln(\ln x)}{\sqrt{x}} \overset{H}{=} \lim_{x\to\infty}\frac{\rule{3cm}{0.4pt}}{\rule{2cm}{0.4pt}} = \lim_{x\to\infty}\frac{\rule{3cm}{0.4pt}}{\ldots\ldots\ldots\ldots\ldots\ldots} = \ldots\ldots\ldots$$

Exercise 3: Find $\displaystyle\lim_{x\to\pi}\frac{\tan x}{x}$.

Solution: Careful ! This limit is ______ of type ....

Therefore,

$$\lim_{x\to\pi}\frac{\tan x}{x} = \frac{\rule{2cm}{0.4pt}}{\ldots\ldots\ldots} = \ldots\ldots\ldots$$

Exercise 4: Find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x - \dfrac{x^2}{2}}{x^3}$.

Solution: This is of type .... Therefore,

$$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3} \overset{H}{=} \lim_{x\to 0} \frac{\qquad\qquad}{} \qquad \text{(still type ....)}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\qquad\qquad}{} \qquad \text{(still type ....)}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\qquad\qquad}{} = ...$$

Exercise 5: Find $\lim\limits_{x\to\infty} e^{-x} \ln x$.

Solution: This is of type ....

Therefore, rewrite this product as a .... ,

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\qquad}{......} \overset{H}{=} \lim_{x\to\infty} \frac{\qquad}{......}$$

$$= \lim_{x\to\infty} \frac{\qquad}{......} = ...$$

Exercise 6: Find $\lim\limits_{x\to 0} \left[\dfrac{1}{\ln(x+1)} - \dfrac{1}{x}\right]$

Solution: This is of type ....

Therefore, rewrite this as a ______________,

$$\lim_{x\to 0}\left[\frac{1}{\ln(x+1)} - \frac{1}{x}\right] = \lim_{x\to 0}\left[\frac{\qquad}{x\ln(x+1)} - \frac{\qquad}{x\ln(x+1)}\right]$$

$$= \lim_{x\to 0} \frac{\qquad\qquad}{x\,\ln(x+1)} \qquad \text{(type ....)}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\qquad\qquad}{......} = \lim_{x\to 0} \frac{\qquad\qquad}{......} \qquad \text{(still type ......)}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{\qquad\qquad}{.........} = \lim_{x\to 0} \frac{\qquad\qquad}{.........} = ...$$

Exercise 7: Find $\lim_{x\to 0}(1-2x)^{1/x}$

Solution: This is of type ....

Therefore, write the function as $y =$ ....

Then

$$\ln y \;=\; \text{......................} \;=\; \text{......................}$$

We now have a limit of type ______ .

$$\lim_{x\to 0}\ln y \;=\; \lim_{x\to 0}\frac{\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}}{\text{..............}} \;\overset{H}{=}\; \lim_{x\to 0}\frac{\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}}{\text{..............}} \;=\; \text{........}$$

Exponentiate:

$$\lim_{x\to 0}(1-2x)^{1/x} \;=\; \lim_{x\to 0} y \;=\; \lim_{x\to 0} e^{\ln y} \;=\; e^{\cdots\cdots} \;=\; \text{........}$$

Exercise 8: Find $\lim_{x\to 0^+}(\sin x)^{\tan x}$

Solution: This is of type ....

Therefore, write the function as $y =$ ....

Consider $\ln y \;=\; \text{......................} \;=\; \text{......................}$

$$\lim_{x\to 0^+}\ln y \;=\; \lim_{x\to 0^+} \text{.........................} \qquad (\text{ type } \text{\_\_\_\_\_\_})$$

$$=\; \lim_{x\to 0^+}\frac{\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}}{\text{........}} \;\overset{H}{=}\; \lim_{x\to 0^+}\frac{\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}}{\text{........}}$$

$$=\; \lim_{x\to 0^+}\frac{\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}}{\text{........}} \;=\; \lim_{x\to 0^+} \text{......................} \;=\; \text{........}$$

Exponentiate:

$$\lim_{x\to 0^+}(\sin x)^{\tan x} \;=\; e^{\lim_{x\to 0^+}\text{......................}} \;=\; e^{\cdots\cdots} \;=\; \text{......}$$

Additional Exercises:

Find the following limits:

1) $\lim_{x \to -1} \frac{x^8 - 1}{x^6 - 1}$

2) $\lim_{x \to 0} \frac{e^{x-1}}{\sin x}$

3) $\lim_{x \to 0} \frac{\tan x}{x^3}$

4) $\lim_{x \to 0} \frac{6^x - 2^x}{x}$

5) $\lim_{x \to \infty} \frac{6^x - 2^x}{x}$

6) $\lim_{x \to \infty} \frac{\ln(1 + e^x)}{5x}$

7) $\lim_{x \to 0} \frac{\sin^2 x}{\tan(x^2)}$

8) $\lim_{x \to 0} \frac{x}{\sin^{-1}(3x)}$

9) $\lim_{x \to \infty} e^{-x} \ln x$

10) $\lim_{x \to \infty} x^3 e^{-2x}$

11) $\lim_{x \to 0^+} \sqrt{x} \csc x$

12) $\lim_{x \to \infty} \left(\frac{x+1}{x} \right)^{2x}$

13) $\lim_{x \to \infty} \left(\frac{1}{x} \right)^{2/x}$

14) $\lim_{x \to 0^+} \left(\frac{1}{x} \right)^{2x}$

Antiderivatives / The Indefinite Integral

Recall: If $F'(x) = f(x)$ on an interval I, then F is called an *antiderivative* of f on I.

Any other antiderivative of f on I is of the form

$$F(x) + C \qquad (C \text{ constant})$$

The function $F(x) + C$ is called the *general antiderivative*, or the *indefinite integral*, of $f(x)$ on I:

$$\boxed{\int f(x)\,dx = F(x) + C}$$

whenever $F'(x) = f(x)$.

Table of Basic Integrals:

$f(x)$	$\int f(x)\,dx = F(x) + C$
1	$x + C$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$

Rules for Integrals:

$$\boxed{\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx}$$

$$\boxed{\int kf(x)\,dx = k\int f(x)\,dx} \quad (k \text{ constant})$$

Exercise 1:

$$\int 2x\,dx = \ldots\ldots + C \quad \text{because} \quad \frac{d}{dx}(\ldots\ldots + C) = \ldots\ldots\ldots\ldots\ldots$$

$$\int 3x^2\,dx = \ldots\ldots + C \quad \text{because} \quad \frac{d}{dx}(\ldots\ldots + C) = \ldots\ldots\ldots\ldots\ldots$$

$$\int x^{-2}\,dx = \ldots\ldots + C \quad \text{because} \quad \frac{d}{dx}(\ldots\ldots + C) = \ldots\ldots\ldots\ldots\ldots$$

$$\int \sec^2 x\,dx = \ldots\ldots + C \quad \text{because} \quad \frac{d}{dx}(\ldots\ldots + C) = \ldots\ldots\ldots\ldots\ldots$$

Exercise 2:

1. Since

$$\frac{d}{dx}\left(x^4 - 2x\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

therefore

$$\int \ldots\ldots\ldots\ldots\ldots\ldots\ldots\,dx = x^4 - 2x + \ldots\ldots$$

2. Since

$$\frac{d}{dx}\left(\cos^3 x\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

therefore

$$\int \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,dx = \cos^3 x + \ldots\ldots$$

Exercise 3: Find $\int \left(12x^2 + 6x - 5 \right) dx$

Solution:

$$\int \left(12x^2 + 6x - 5 \right) dx \;=\; 12 \int x^2\, dx \;+\; 6 \int x\, dx \;-\; 5 \int 1\, dx$$

$$= 12 \frac{\qquad}{\ldots\ldots} + 6 \frac{\qquad}{\ldots\ldots} - 5\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

Check: $\frac{d}{dx}\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 4: Find $\int \left(2\sqrt{x} + 4\sqrt[3]{x} + \frac{2}{\sqrt{x}} \right) dx$

Solution:

$$\int \left(2\sqrt{x} + 4\sqrt[3]{x} + \frac{2}{\sqrt{x}} \right) dx \;=\; \int \left(2x^{\ldots\ldots} + 4x^{\ldots\ldots} + 2x^{\ldots\ldots} \right) dx$$

$$= 2 \frac{\qquad}{\ldots} + 4 \frac{\qquad}{\ldots} + 2 \frac{\qquad}{\ldots} + C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

Check:

$$\frac{d}{dx}\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 5: Evaluate the given integrals.

1. $$\int \left(\sec^2 t + t^2\right) dt \;=\; \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots + C$$

2. $$\int \left(z - \frac{1}{z}\right)^2 dz \;=\; \int \left(\ldots\ldots\ldots - \ldots\ldots\ldots + \ldots\ldots\ldots\right) dz$$

$$= \int \left(\ldots\ldots\ldots - \ldots\ldots\ldots + \ldots\ldots\ldots\right) dz$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. $$\int \frac{1}{\cos^2\theta}\, d\theta \;=\; \int \ldots\ldots\ldots\ldots\, d\theta \;=\; \ldots\ldots\ldots\ldots + C$$

4. $$\int \frac{\left(\sqrt{x}+3\right)^2}{x^3}\, dx \;=\; \int \frac{(\qquad\qquad)}{x^3}\, dx$$

$$= \int \left(\frac{\quad}{x^3} + \frac{\quad}{x^3} + \frac{\quad}{x^3}\right) dx \;=\; \int \left(\ldots\ldots\ldots + \ldots\ldots\ldots + \ldots\ldots\ldots\right) dx$$

$$= \ldots\ldots\ldots + \ldots\ldots\ldots + \ldots\ldots\ldots + C \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

5. $$\int \frac{\sin^3\theta}{\cos^2\theta}\, d\theta \;=\; \int \frac{\sin\theta(\qquad\qquad)}{\cos^2\theta}\, d\theta$$

$$= \int \left[\frac{\sin\theta}{\cos^2\theta} - \frac{\sin\theta}{\cos^2\theta}\right] d\theta$$

$$= \int \left[\ldots\ldots\ldots\ldots\ldots - \sin\theta\right] d\theta \;=\; \ldots\ldots\ldots\ldots\ldots\ldots + C$$

6. $$\int \left(2x^2 - 4x\right)\left(3x - x^3\right) dx \;=\; \int \left(6x^3 - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right) dx$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

Exercise 6: If $f''(x) = x + \sqrt{x}$ and $f(1) = 1$, $f'(1) = 2$, find $f(x)$.

Solution: Because $f'(x)$ is an antiderivative of $f''(x)$ we integrate.

$$f'(x) = \int f''(x)\,dx = \int (x + \ldots\ldots\ldots)\,dx = \int (x + \ldots\ldots\ldots)\,dx$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now we can find C: The condition $f'(1) = 2$ gives

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 2$$

$$C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

so that

$$f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now integrate once more:

$$f(x) = \int f'(x)\,dx = \int\left(\frac{x^2}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\right)dx$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C_1$$

We can find the value of C_1: The condition $f(1) = 1$ gives

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 1$$

$$C_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

so that

$$f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Additional Exercises:

1) Evaluate the following indefinite integrals:

a) $\int \left(4x^3 - 2x + \frac{3}{x^2}\right) dx$

b) $\int \left(7x^{3/4} - 3x^{1/2} - 4x^{1/3}\right) dx$

c) $\int (2y-4)(3y+2)\, dy$

d) $\int \frac{x^2+3x+2}{x+1}\, dx$

e) $\int (3x-4)^3\, dx$

f) $\int \left(\frac{2}{u^3} - \frac{4}{\sqrt[3]{u}} + 4 - \frac{5}{\sqrt{u^3}}\right) du$

g) $\int \frac{x^3+2x^2-4x+2}{\sqrt{x}}\, dx$

h) $\int \frac{1}{4\sec\phi}\, d\phi$

i) $\int \tan^2 x\, dx$

j) $\int (4\sin x + 3\cos x)\, dx$

2) Solve the differential equation: $f'(x) = 12x^2 - 6x + 3,\quad f(1) = 7$

3) Solve the differential equation: $\frac{dy}{dx} = 4x^{1/2},\quad y(4) = 21$

4) If $\frac{d^2y}{dt^2} = 4\cos t - 3\sin t$ and $y = 2,\ \ y' = 1$ when $t = 0$, find $y(t)$.

5) A particle is moving along a straight line, with given acceleration $a(t)$. Find velocity $v(t)$ and position $s(t)$ of the particle at time $t > 0$.

a) $a(t) = 2 - 6t,\quad v(0) = -5,\quad s(0) = 4$

b) $a(t) = 3t^2,\quad v(0) = 20,\quad s(0) = 5$

The Substitution Rule

Recall:

$$\boxed{\int f\left(g(x)\right) g'(x)\, dx \quad = \quad \int f(u)\, du}$$

where $u = g(x)$ and $du = g'(x)\,dx$.

Exercise 1: Find $\displaystyle\int 2x \sqrt[3]{x^2+1}\ dx$

Solution: We set

$u =$ ....

Then

$du =$ ....

and

$$\int 2x \sqrt[3]{x^2+1}\ dx = \int \sqrt[3]{\ldots\ldots\ldots\ldots}\ \ldots\ldots\ldots\ldots$$

$$= \int \ldots\ldots\ldots\ldots du \quad = \quad \ldots\ldots\ldots\ldots + C \quad = \quad \ldots\ldots\ldots\ldots\ldots + C$$

Check: $\dfrac{d}{dx}(\ldots\ldots\ldots\ldots\ldots\ldots) \quad = \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 2: Find $\displaystyle\int 3x^2 \sin(x^3)\, dx$

Solution: We set

$u =$ ....

Then

$du =$ ....

and

$$\int 3x^2 \sin(x^3)\, dx \quad = \quad \int \sin(\ldots\ldots)\ \ \ldots\ldots\ldots$$

$$= \quad \ldots\ldots\ldots\ldots + C \quad = \quad \ldots\ldots\ldots\ldots\ldots + C$$

Check: $\dfrac{d}{dx}(\ldots\ldots\ldots\ldots\ldots\ldots) \quad = \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad = \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 3: Evaluate the following integrals using the correct substitutions:

1. $$\int x\sqrt{x^2+7}\,dx = \frac{1}{2}\int 2x\sqrt{x^2+7}\,dx = \frac{1}{2}\int \ldots\ldots\ldots\ldots\, du$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$$= \frac{1}{2}\int \ldots\ldots\ldots\ldots\, du = \ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C$$

2. $$\int \sqrt{3x-2}\,dx = \frac{1}{\ldots}\int \ldots\ldots\ldots\ldots\, du = \ldots\ldots \int \ldots\ldots\ldots\ldots\, du$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$$= \ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C$$

3. $$\int x\left(x^2+4\right)^{99}\,dx = \int \ldots\ldots\ldots\ldots\, du$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$$= \ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C$$

4. $\int \frac{x+3}{(x^2+6x)^2}\,dx = \int \frac{\rule{3cm}{0.4pt}}{\ldots\ldots\ldots\ldots\ldots}\,dx = \int_{\ldots\ldots} du = \int_{\ldots\ldots} du$

$u = \ldots\ldots\ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots\ldots\ldots$

$= \ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$

5. $\int \frac{t^2}{\sqrt{1-t}}\,dt = \int \frac{(\qquad)^2}{\ldots\ldots\ldots\ldots}\,du = \int \frac{\rule{3cm}{0.4pt}}{\ldots\ldots\ldots\ldots}\,du$

$u = \ldots\ldots\ldots\ldots \Rightarrow t = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots \Rightarrow dt = \ldots\ldots\ldots\ldots$

$= \int (\qquad\qquad\qquad)\,du = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$

6. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \ldots\ldots\ldots\ldots\ldots\, du$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots \Rightarrow \ldots\ldots\ldots\ldots = 2\,du$

$= \ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C$

Exercise 4: Find $\int x^3\sqrt{1-x^2}\,dx$

Solution: We set

$u =$ ....

Then

$du =$ ....

so that

$x\,dx =$ ....

and

$x^2 =$ ....

We obtain

$$\int x^3\sqrt{1-x^2}\,dx = \int x^2\sqrt{1-x^2}\,(x\,dx) = \ldots\ldots \int (\ldots\ldots\ldots)\sqrt{\ldots\ldots}\,du$$

$$= \ldots\ldots \int \left(\ldots\ldots\ldots\ldots\ldots\right)\ldots\ldots du = \ldots\ldots \int \left(\ldots\ldots\ldots\ldots\ldots\ldots\right) du$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

Check: $\dfrac{d}{dx}\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$

= ....

= ....

= .... = ...

Exercise 5: Evaluate the following trigonometric integrals by substitution:

1. $\int \sin^3 \cos x\, dx$

Because the derivative of $\sin x$ is _________ and appears as a factor in the integrand, we substitute $u =$ ________

$$\int \sin^3 \cos x\, dx \quad = \quad \int \ldots\ldots\ldots du \quad = \quad \ldots\ldots\ldots\ldots + C \quad = \quad \ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

2. $\int \sin x\,(1+\cos x)^2\, dx$

Because the derivative of _______________ is _________ and appears as a factor in the integrand, we substitute $u =$ ___________

$$\int \sin x\,(1+\cos x)^2\, dx \quad = \quad \int \ldots\ldots\ldots\ldots du \quad = \quad \ldots\ldots\ldots + C$$

$$= \quad \ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

3. $\int \sec^2 x \tan^2 x \, dx$

Because the derivative of ________ is ________ and appears as a factor in the integrand, we substitute $u =$ ________

$$\int \sec^2 x \tan^2 x \, dx = \int \ldots\ldots\ldots \, du = \ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

4. $\int \sin x \sec^5 x \, dx$

Write in terms of $\sin x$ and $\cos x$.

$$\int \sin x \sec^5 x \, dx = \int \ldots\ldots\ldots\ldots \, dx = \int \ldots\ldots\ldots\ldots \, du = \int \ldots\ldots\ldots \, du$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

$$= \ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C$$

5. $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \, (\ldots\ldots\ldots) \cos x \, dx$

$$= \int \sin^2 x \, (\ldots\ldots\ldots\ldots) \cos x \, dx = \int \ldots\ldots\ldots\ldots\ldots\ldots \, du$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

$$= \int \ldots\ldots\ldots\ldots\ldots \, du = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

Exercise 6: If $\int f(x)\,dx = F(x)+C$, then what is $\int f(ax+b)\,dx$?

$(a, b$ constant, $a \neq 0)$

Solution: We set

$u =$ ....

Then

$du =$ ....

so that

$dx =$ ....

Then

$$\int f(ax+b)\,dx = \int \frac{f(\ldots)}{\ldots\ldots}\,du = \frac{1}{\ldots}\int f(\ldots)\,du$$

$$= \frac{1}{\ldots}F(\ldots)+C = \frac{1}{\ldots}F\left(\ldots\ldots\ldots\right)+C$$

We have shown:

$$\boxed{\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C}$$

Examples:

1. $\int \cos(3x)\,dx =$ ____________ $+C$

2. $\int \sec^2(5x-3)\,dx =$ ____________ $+C$

3. $\int \sec(\pi x)\tan(\pi x)\,dx =$ ____________ $+\ldots\ldots$

4. $\int \frac{1}{2\sqrt{3x+4}}\,dx =$ ____________ $+\ldots\ldots$

Additional Exercises: Evaluate the following indefinite integrals by substitution:

1) $\displaystyle\int \frac{1}{\sqrt{2-4x}}\,dx$

2) $\displaystyle\int \frac{3x}{\sqrt{x^2+4}}\,dx$

3) $\displaystyle\int \frac{3x^5}{\sqrt{x^2+4}}\,dx$

4) $\displaystyle\int \frac{x+1}{(x^2+2x-4)^7}\,dx$

5) $\displaystyle\int \frac{1}{(3x-4)^{10}}\,dx$

6) $\displaystyle\int v^2\sqrt[3]{v^3+1}\,dv$

7) $\displaystyle\int \cos 3x\,\sqrt[3]{\sin 3x}\,dx$

8) $\displaystyle\int \sin^2 x\,dx$

9) $\displaystyle\int \sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)dx$

10) $\displaystyle\int \sec^3 x\tan x\,dx$

11) $\displaystyle\int \cos^4 3x\sin 3x\,dx$

12) $\displaystyle\int \cos^2(\pi x)\sin^3(\pi x)\,dx$

13) $\displaystyle\int \cos^2(2\pi x)\sin^2(2\pi x)\,dx$

14) $\displaystyle\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx$

15) $\displaystyle\int \frac{x^4-x^2}{\sqrt{3x^5-5x^3+2}}\,dx$

Hint: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Definition of the Definite Integral

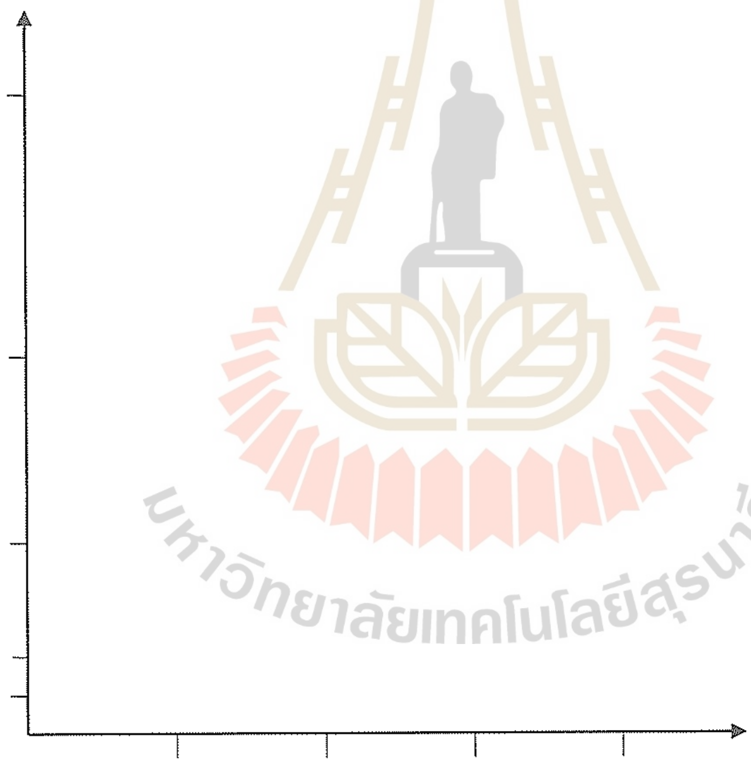
Recall:

$$\int_a^b f(x)\,dx \quad = \quad \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

where $P: a = x_0 \le x_1 \le x_2 \le \cdots\cdots \le x_{n-1} \le x_n = b$ is a partition of the interval $[a,b]$, x_k^* is an arbitrary point in $[x_{k-1}, x_k]$, and $\Delta x_k = x_k - x_{k-1}$.

Exercise 1: Find the area of the region bounded by the graph of $f(x) = x^2 + 1$ and the x-axis between $x = 1$ and $x = 4$ using limits of Riemann sums.

Solution: First sketch the graph of f.



Next we approximate the region by rectangles. For simplicity, partition $[1,4]$ into n intervals *of equal length.* Each interval must have length

$$\Delta x \quad = \quad \Delta x_k \quad = \quad \frac{\qquad}{n} \quad =$$

The partition points are then

$$x_0 = a = \ldots\ldots, \quad x_1 = \ldots\ldots, \quad x_2 = \ldots\ldots, \quad \cdots\cdots \quad x_k = \ldots\ldots, \quad \cdots\cdots \quad x_n = b = \ldots\ldots$$

For simplicity, we choose x_k^* the *right endpoint* of $[x_{k-1}, x_k]$. Then

$$x_1^* = \ldots\ldots, \quad x_2^* = \ldots\ldots, \quad x_3^* = \ldots\ldots, \quad \cdots\cdots \quad x_k^* = \ldots\ldots, \quad \cdots\cdots \quad x_n^* = \ldots\ldots$$

Sketch and consider the combined area of all rectangles whose base is the interval $[x_{k-1}, x_k]$ and whose height is $f\left(x_k^*\right)$. Its is

$$S_n = \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k = \sum_{k=1}^{n} f(\ldots\ldots\ldots)\ \ldots\ldots$$

This sum is also called a ______________________ sum. Let's compute it.

$$S_n = \sum_{k=1}^{n}\left[\left(\ldots\ldots\ldots\ldots\right)^2 + 1\right]\ldots\ldots = \sum_{k=1}^{n}\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]\ldots\ldots$$

$$= \sum_{k=1}^{n}\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$= \frac{6}{n}\left(\sum_{k=1}^{n}\ldots\ldots\right) + \frac{\quad}{n^2}\left(\sum_{k=1}^{n}\ldots\ldots\right) + \frac{\quad}{n^3}\left(\sum_{k=1}^{n}\ldots\ldots\right)$$

$$= 6 + \frac{\quad}{n^2}\,\frac{\qquad}{2} + \frac{\quad}{n^3}\,\frac{n(\quad)(\quad)}{6}$$

$$= 6 + 9\,\frac{\qquad}{n} + \frac{9}{2}\,\frac{(\quad)(\quad)}{n^2}$$

$$= 6 + 9\left(1+\frac{\quad}{\ldots}\right) + \frac{9}{2}\left(1+\frac{\quad}{\ldots}\right)\left(2+\frac{\quad}{\ldots}\right)$$

Now let $n \to \infty$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[6 + 9\left(1+\frac{\quad}{\ldots}\right) + \frac{9}{2}\left(1+\frac{\quad}{\ldots}\right)\left(2+\frac{\quad}{\ldots}\right)\right]$$

$$= 6 + 9(1+\ldots) + \frac{9}{2}(1+\ldots)(2+\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

Recall that this limit is also called the *definite integral.*

Answer: $\displaystyle\int_1^4 (x^2+1)\,dx = \ldots\ldots\ldots$

Exercise 2: Consider $f(x)=16-x^2$ on the interval $[0,4]$ with partition $P=\{\,0,\ 1,\ 2,\ 3,\ 3.6,\ 4\,\}$.

1. Find $\|P\|$
2. If x_k^* is the *right endpoint* of each interval $[x_{k-1},x_k]$, find the Riemann sum and sketch the rectangles.
3. If x_k^* is the *midpoint* of each interval $[x_{k-1},x_k]$, find the Riemann sum and sketch the rectangles.

Solution:

1. The partition points are

$x_0 =$ ......, $x_1 =$......, $x_2 =$......, $x_3 =$......, $x_4 =$......, $x_5 =$......

We have

Δx_1 = ...−......=...... Δx_4 = ...−......=......

Δx_2 = ...−......=...... Δx_5 = ...−......=......

Δx_3 = ...−......=......

Therefore $\|P\|$ = ...

2. Let x_k^* be the *right endpoint*.

Then

x_1^* = ... $f\left(x_1^*\right)$ = f(......) = (16−......) = ...

x_2^* = ... $f\left(x_2^*\right)$ = f(......) = (16−......) = ...

x_3^* = ... $f\left(x_3^*\right)$ = f(......) = (16−......) = ...

x_4^* = ... $f\left(x_4^*\right)$ = f(......) = (16−......) = ...

x_5^* = ... $f\left(x_5^*\right)$ = f(......) = (16−......) = ...

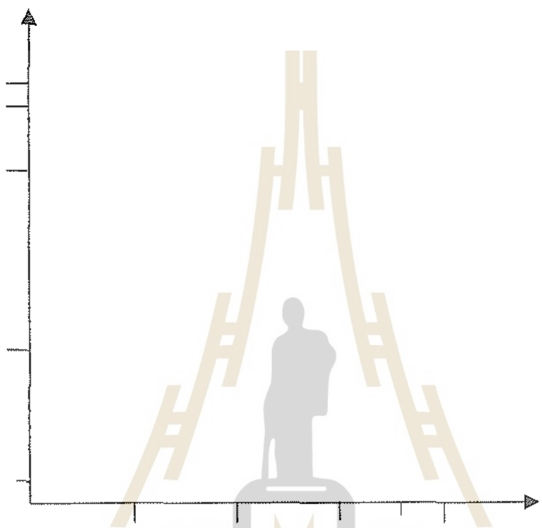
The Riemann sum is

$$S_5 = \sum_{k=1}^{5} f(x_k^*)\,\Delta x_k = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + f(x_3^*)\,\Delta x_3 + f(x_4^*)\,\Delta x_4 + f(x_5^*)\,\Delta x_5$$

$$= (\,\ldots\ldots\,)(\,\ldots\,) + (\,\ldots\ldots\,)(\,\ldots\,) + (\,\ldots\ldots\,)(\,\ldots\,) + (\,\ldots\ldots)(\,\ldots\,) + (\ldots\ldots)(\,\ldots\,)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. Let x_k^* be the *midpoint*.



Then

x_1^*	=	0.5	$f\left(x_1^*\right)$	=	$f(\,0.5\,)$	=	$\left(16-0.5^2\right)$	=	15.75
x_2^*	=	………	$f\left(x_2^*\right)$	=	$f(\ldots\ldots)$	=	$(16-\ldots\ldots)$	=	………
x_3^*	=	………	$f\left(x_3^*\right)$	=	$f(\ldots\ldots)$	=	$(16-\ldots\ldots)$	=	………
x_4^*	=	………	$f\left(x_4^*\right)$	=	$f(\ldots\ldots)$	=	$(16-\ldots\ldots)$	=	………
x_5^*	=	………	$f\left(x_5^*\right)$	=	$f(\ldots\ldots)$	=	$(16-\ldots\ldots)$	=	………

The Riemann sum is

$$S_5 = \sum_{k=1}^{5} f(x_k^*)\,\Delta x_k = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + f(x_3^*)\,\Delta x_3 + f(x_4^*)\,\Delta x_4 + f(x_5^*)\,\Delta x_5$$

$$= (\,\ldots\ldots\,)(\,\ldots\,) + (\,\ldots\ldots\,)(\,\ldots\,) + (\,\ldots\ldots\,)(\,\ldots\,) + (\,\ldots\ldots)(\,\ldots\,) + (\ldots\ldots)(\,\ldots\,)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 3: Consider $f(x)=3x-1$ on the interval $[-2,2]$ with partition $P=\{-2,\ -1.2,\ -0.6,\ 0,\ 0.8,\ 1.6,\ 2\}$. Find $\|P\|$. If x_k^* is the *midpoint* of each interval $[x_{k-1},x_k]$, find the Riemann sum and sketch the rectangles.

Solution: We have

$$\Delta x_1 = (\ldots\ldots-\ldots\ldots) = \ldots\ldots \qquad \Delta x_4 = (\ldots\ldots-\ldots\ldots) = \ldots\ldots$$

$$\Delta x_2 = (\ldots\ldots-\ldots\ldots) = \ldots\ldots \qquad \Delta x_5 = (\ldots\ldots-\ldots\ldots) = \ldots\ldots$$

$$\Delta x_3 = (\ldots\ldots-\ldots\ldots) = \ldots\ldots \qquad \Delta x_6 = (\ldots\ldots-\ldots\ldots) = \ldots\ldots$$

Therefore $\|P\| = \ldots\ldots\ldots$

Let x_k^* be the *midpoint* of each interval. Then

$$x_1^* = \ldots\ldots\ldots \qquad f\left(x_1^*\right) = f(\ldots\ldots) = (\ldots\ldots\ldots-1) = \ldots\ldots\ldots$$

$$x_2^* = \ldots\ldots\ldots \qquad f\left(x_2^*\right) = f(\ldots\ldots) = (\ldots\ldots\ldots-1) = \ldots\ldots\ldots$$

$$x_3^* = \ldots\ldots\ldots \qquad f\left(x_3^*\right) = f(\ldots\ldots) = (\ldots\ldots\ldots-1) = \ldots\ldots\ldots$$

$$x_4^* = \ldots\ldots\ldots \qquad f\left(x_4^*\right) = f(\ldots\ldots) = (\ldots\ldots\ldots-1) = \ldots\ldots\ldots$$

$$x_5^* = \ldots\ldots\ldots \qquad f\left(x_5^*\right) = f(\ldots\ldots) = (\ldots\ldots\ldots-1) = \ldots\ldots\ldots$$

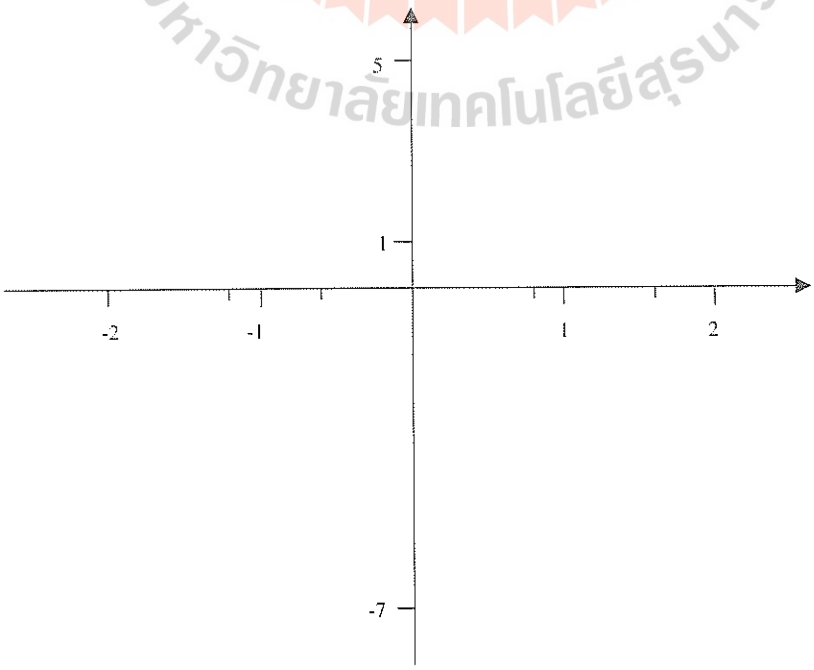
$$x_6^* = \ldots\ldots\ldots \qquad f\left(x_6^*\right) = f(\ldots\ldots) = (\ldots\ldots\ldots-1) = \ldots\ldots\ldots$$

The Riemann sum is

$$S_6=\sum_{k=1}^{6} f(x_k^*)\,\Delta x_k = f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + f(x_3^*)\,\Delta x_3 + f(x_4^*)\,\Delta x_4 + f(x_5^*)\,\Delta x_5 + f(x_6^*)\,\Delta x_6$$

$$=(\qquad)(\quad)+(\qquad)(\quad)+(\qquad)(\quad)+(\qquad)(\quad)+(\qquad)(\quad)+(\qquad)(\quad) = \ldots\ldots\ldots$$

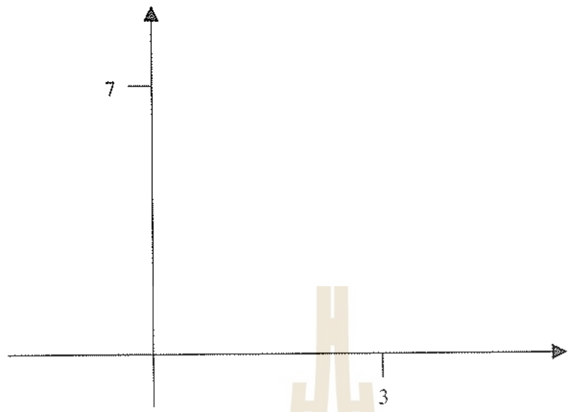
Sketch:



Exercise 4: Using geometry, find the following integrals.

1. $\int_0^3 (2x+1)\,dx$

Solution: Sketch the graph of $f(x) = 2x+1$:



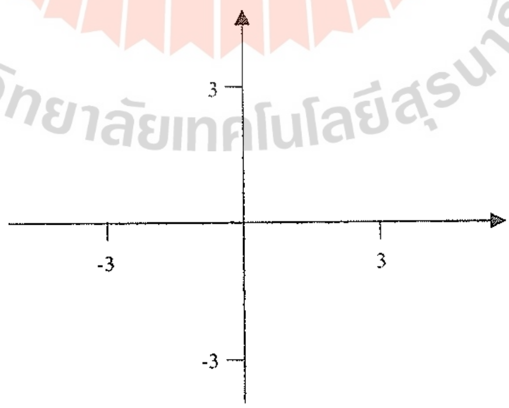
Because $f(x) \ge 0$ on $[0,3]$, the value of the integral equals the area below the graph of $f(x)$:

$$\int_0^3 (2x+1)\,dx \;=\; \text{area of the trapezoid\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$= \frac{1}{2}(\ldots\ldots+\ldots\ldots)(\ldots\ldots) \;=\; \ldots\ldots\ldots\ldots$$

2. $\int_0^3 \sqrt{9-x^2}\,dx$

Solution: Sketch the graph of $f(x) = \sqrt{9-x^2}$:

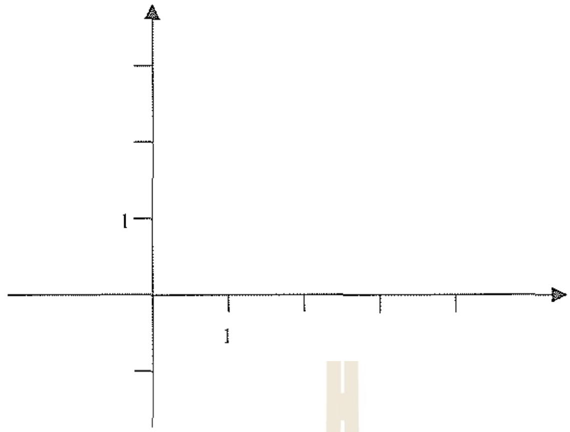


$$\int_0^3 \sqrt{9-x^2}\,dx \;=\; \text{area of 1/4 circle \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$= \frac{1}{4}\pi(\ldots\ldots)^2 \;=\; \ldots\ldots\ldots\ldots$$

3. $\int_1^4 (3-x)\,dx$

Solution: Sketch the graph:



$$\int_1^4 (3-x)\,dx = \text{area of the triangle \_\_\_\_ - area of the triangle \_\_\_\_\_\_}$$

$$= \frac{1}{2}(\ldots\ldots)(\ldots\ldots) - \frac{1}{2}(\ldots\ldots)(\ldots\ldots) = \ldots\ldots\ldots\ldots$$

Exercise 5: The following integrals are given:

$$\int_1^3 f(x)\,dx = 7, \quad \int_3^5 f(x)\,dx = 4 \quad \text{and} \quad \int_1^5 g(x)\,dx = 2\,.$$

Find the integrals indicated.

1. $\int_3^1 f(x)\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

2. $\int_1^5 f(x)\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

3. $\int_1^5 2f(x)\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

4. $\int_1^5 [\,2f(x) - 3g(x)\,]\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots = \ldots\ldots$

5. $\int_1^3 f(u)\,du = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

Exercise 6: Find the average value of $f(x) = 3 - x$ over the interval $[-1, 4]$

Solution: Recall that the average value of $f(x)$ on $[a, b]$ is

$$avg(f) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

1) Compute the integral.

$$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{4} (3-x)\,dx = \int_{-1}^{4} 3\,dx - \int_{-1}^{4} x\,dx$$

$$= (\ldots)(\ldots\ldots\ldots\ldots) - \frac{\ldots}{2} = \ldots\ldots\ldots\ldots = \ldots\ldots$$

2) The average value is

$$avg(f) = \frac{1}{\ldots\ldots} \int_{-1}^{4} f(x)\,dx = \frac{1}{\ldots\ldots}\ldots\ldots = \ldots\ldots$$

Exercise 7: Estimate $\int_0^2 \sqrt{x^3+1}\,dx$.

Solution: If $0 \le x \le 2$

then $0 \le x^3 \le 2^3$

or $\ldots\ldots \le x^3 + 1 \le \ldots\ldots$

so that $\ldots\ldots \le \sqrt{x^3+1} \le \ldots\ldots$

Integrate,

$$\int_0^2 \ldots\ldots\ldots\,dx \le \int_0^2 \sqrt{x^3+1}\,dx \le \int_0^2 \ldots\ldots\ldots\,dx$$

$$(\ldots\ldots)(\ldots\ldots) \le \int_0^2 \sqrt{x^3+1}\,dx \le (\ldots\ldots)(\ldots\ldots)$$

$$\ldots\ldots \le \int_0^2 \sqrt{x^3+1}\,dx \le \ldots\ldots$$

Exercise 8: Estimate $\int_0^3 \sqrt{x+1}\,dx$ without computing the integral.

Solution: If $\quad 0 \le x \le 3$

then $\quad \ldots\ldots \le x+1 \le \ldots\ldots$

so that $\quad \ldots\ldots \le \sqrt{x+1} \le \ldots\ldots$

Integrate,

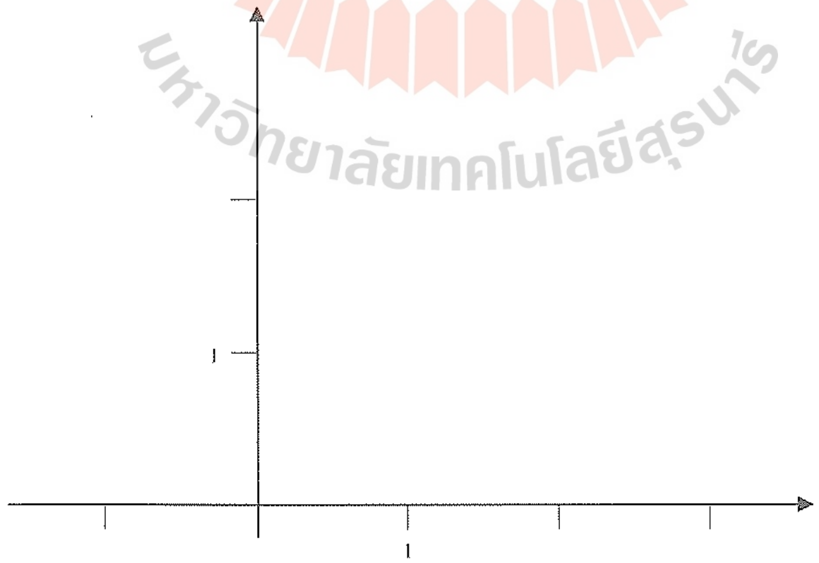
$$\int_0^3 \ldots\ldots\ldots\ldots\, dx \le \int_0^3 \sqrt{x+1}\, dx \le \int_0^3 \ldots\ldots\ldots\ldots\, dx$$

$$(\ldots\ldots\ldots)(\ldots\ldots\ldots) \le \int_0^3 \sqrt{x+1}\, dx \le (\ldots\ldots\ldots)(\ldots\ldots\ldots)$$

$$\ldots\ldots\ldots\ldots \le \int_0^3 \sqrt{x+1}\, dx \le \ldots\ldots\ldots\ldots$$

<u>This is not a very good estimate</u>. Upper and lower estimates differ by a very large amount. Let us try to give a better estimate.

Sketch the graph. (This is the graph of $y=\sqrt{x}$ shifted ____ units to the _________)



Lower estimate:

Sketch and find the equation of the line connecting the points $(0,\ldots)$ and $(3,\ldots)$:

$$\ldots\ldots\ldots \leq \sqrt{x+1} \qquad \text{on } [0,3].$$

Therefore,

$$\int_0^3 \ldots\ldots\ldots\ldots dx \leq \int_0^3 \sqrt{x+1}\,dx$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \leq \int_0^3 \sqrt{x+1}\,dx$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots \leq \int_0^3 \sqrt{x+1}\,dx$$

Upper estimate:

Sketch and find the equation of the tangent line at $x = 0$:

If $f(x) = \sqrt{1+x}$ then $f'(x) = \ldots\ldots\ldots\ldots$

The tangent line at $x = 0$ is given by

$$y - f(\ldots) = m(x - \ldots) \qquad \text{where } m = f'(\ldots) = \ldots\ldots$$

$$y - \ldots\ldots = (\ldots)(x - \ldots)$$

$$y = \ldots(x - \ldots) + \ldots\ldots = \ldots\ldots\ldots\ldots\ldots$$

Now because this tangent line is ___________ the graph, we have

$$\begin{cases} \sqrt{1+x} \leq \ldots\ldots\ldots\ldots\ldots & \text{on } [0,2] \\ \sqrt{1+x} \leq 2 & \text{on } [2,3] \end{cases}$$

Therefore,

$$\int_0^3 \sqrt{x+1}\,dx \leq \int_0^2 \ldots\ldots\ldots\ldots dx + \int_2^3 \ldots\ldots\ldots\ldots dx$$

$$\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

Answer: $\ldots\ldots\ldots\ldots \leq \int_0^3 \sqrt{x+1}\,dx \leq \ldots\ldots\ldots\ldots$

Additional Exercises:

1) Using geometry, find

a) $\int_{-2}^{2} |x+1| \, dx$

b) $\int_{0}^{2\sqrt{2}} \left(\sqrt{16-x^2} - x\right) dx$

2) Write as *one single* integral

a) $\int_{1}^{3} f(x)\, dx \quad + \quad \int_{3}^{6} f(x)\, dx \quad + \quad \int_{6}^{12} f(x)\, dx$

b) $\int_{-3}^{5} g(x)\, dx \quad - \quad \int_{-3}^{0} g(x)\, dx \quad + \quad \int_{5}^{6} g(x)\, dx$

3) Without computing the integrals, show that

a) $\int_{0}^{1} x^2 \, dx \quad \le \quad \int_{0}^{1} x \, dx$

b) $\int_{1}^{2} x \, dx \quad \le \quad \int_{1}^{2} x^2 \, dx$

c) $\int_{-2}^{8} (x^2 - 3x + 4)\, dx \quad \ge \quad 0$

d) $\int_{4}^{6} \frac{1}{x}\, dx \quad \le \quad \int_{4}^{6} \frac{1}{8-x}\, dx$

e) $\int_{0}^{\pi/2} \sin^3 x \, dx \quad \le \quad \int_{0}^{\pi/2} \sin x \, dx$

f) $\int_{1}^{3} \sqrt{x^2+1}\, dx \quad \ge \quad 4$

g) $\frac{1}{2} \quad \le \quad \int_{1}^{2} \frac{1}{x}\, dx \quad \le \quad \frac{3}{4}$

h) $\frac{\pi}{2} \quad \le \quad \int_{\pi/6}^{5\pi/6} \sin x \, dx \quad \le \quad \frac{2\pi}{3}$

The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus, Part 1: If $f(x)$ is continuous on $[a,b]$, and if

$$F(x) = \int_a^x f(t)\,dt,$$

then $F(x)$ is differentiable on $[a,b]$, and

$$F'(x) = f(x)$$

Fundamental Theorem of Calculus, Part 2: If $F(x)$ is any antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

Exercise 1: Consider $f(x) = 3x - 1$ on the interval $[1,4]$. For each x in $[1,4]$, find

$$F(x) = \int_1^x f(t)\,dt,$$

Then find $F'(x)$.

Solution: Sketch

$F(x) = \int_1^x (\ldots\ldots\ldots)\,dt$ = area trapezoid ____________ = ...

= .... = ...

Then $F'(x) = \dfrac{d}{dx}\Big(\ldots\ldots\ldots\ldots\ldots\ldots\Big) = \ldots\ldots\ldots\ldots\ldots\ldots$ Note that $F'(x) = f(x)$!

Exercise 2: If $F(x) = \int_1^x \sqrt{1+t^4}\, dt$

then $F'(x) = \ldots\ldots\ldots\ldots\ldots$

Exercise 3: If $G(x) = \int_2^x \frac{t+4}{t^3-2t}\, dt$

then $G'(x) = \ldots\ldots\ldots\ldots\ldots$

Exercise 4: $\frac{d}{dx}\left[\int_{-1000}^{x} \left(t^2-4t+2\right)^{99}\, dt\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 5: $\frac{d}{d\theta}\left[\int_{-\pi}^{\theta} \sin(u^2)\, du\right] = \ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 6: If $F(x) = \int_x^2 t^3\cos(t^2)\, dt$ find $F'(x)$.

Solution: Move x to the upper limit of integration:

$$F'(x) = \frac{d}{dx}\left[\int_x^2 t^3\cos(t^2)\, dt\right] = \frac{d}{dx}\left[-\int_2^x t^3\cos(t^2)\, dt\right] = \ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 7: If $\quad H(x) = \displaystyle\int_1^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds \quad$ find $H'(x)$.

Solution: This is a composition of two functions,

$$u = \sqrt{x} \qquad \text{and} \qquad H(u) = \int_1^{u} \frac{s^2}{s^2+1}\,ds .$$

By the chain rule,

$$\frac{dH}{dx} = \frac{dH}{du}\,\frac{du}{dx} = \frac{d}{du}\left[\int_1^{u} \frac{s^2}{s^2+1}\,ds\right] \frac{d}{dx}\sqrt{x}$$

$$= \frac{\cdots\cdots\cdots}{u^2+\cdots\cdots}\cdot\frac{1}{2\cdots\cdots\cdots} = \frac{\cdots\cdots\cdots}{\left(\sqrt{x}\right)^2+\cdots}\cdot\frac{1}{2\cdots\cdots\cdots} = \frac{}{(x+\cdots\cdots)}$$

Exercise 8: If $\quad F(x) = \displaystyle\int_x^{x^2} t^2\cos t\,dt \quad$ find $F'(x)$.

Solution: Split into 2 integrals:

$$F(x) = \int_x^{0} t^2\cos t\,dt + \int_0^{x^2} t^2\cos t\,dt = \int_0^{x^2} t^2\cos t\,dt - \cdots\cdots\cdots\cdots\cdots\cdots$$

By the chain rule,

$$F'(x) = \frac{d}{dx}\left[\int_0^{x^2} t^2\cos t\,dt\right] - \frac{d}{dx}\left[\int_0^{\cdots\cdots} \cdots\cdots\cdots\cdots\,dt\right]$$

$$= (\,\cdots\cdots)^2\,(\,\cdots\cdots\cdots\cdots)\cdot\frac{d}{dx}(\,\cdots\cdots\cdots) - \cdots\cdots\cdots\cdots\cdots\cdots$$

$$= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Exercise 9: Find the following integrals by using part 2 of the Fundamental Theorem.

1. $\displaystyle\int_1^3 3x^2\,dx = \ldots\ldots\ldots\ \ldots\ldots\Big]_1^3 = \ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

2. $\displaystyle\int_1^2 \left(5x^2-4x+3\right)dx = \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big]_1^2$

$= (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) - (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

$= \ldots\ldots\ldots\ldots\ldots = \ldots\ldots$

3. $\displaystyle\int_0^1 u\left(\sqrt{u}+\sqrt[3]{u}\right)du = \int_0^1 (\ldots\ldots\ldots\ldots\ldots\ldots)\,du = \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big]_0^1$

$= (\ldots\ldots\ldots\ldots\ldots) - (\ldots\ldots) = \ldots\ldots\ldots\ldots = \ldots\ldots$

4. $\displaystyle\int_1^2 \frac{t^6-t^2}{t^4}\,dt = \int_1^2 (\ldots\ldots\ldots\ldots\ldots\ldots)\,dt = \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big]_1^2$

$= (\ldots\ldots\ldots\ldots\ldots\ldots) - (\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

5. $\displaystyle\int_0^{\pi/2} \left(\cos\theta+2\sin\theta\right)d\theta = \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big]_0^{\pi/2}$

$= (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) - (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

$= (\ldots\ldots\ldots\ldots) - (\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots$

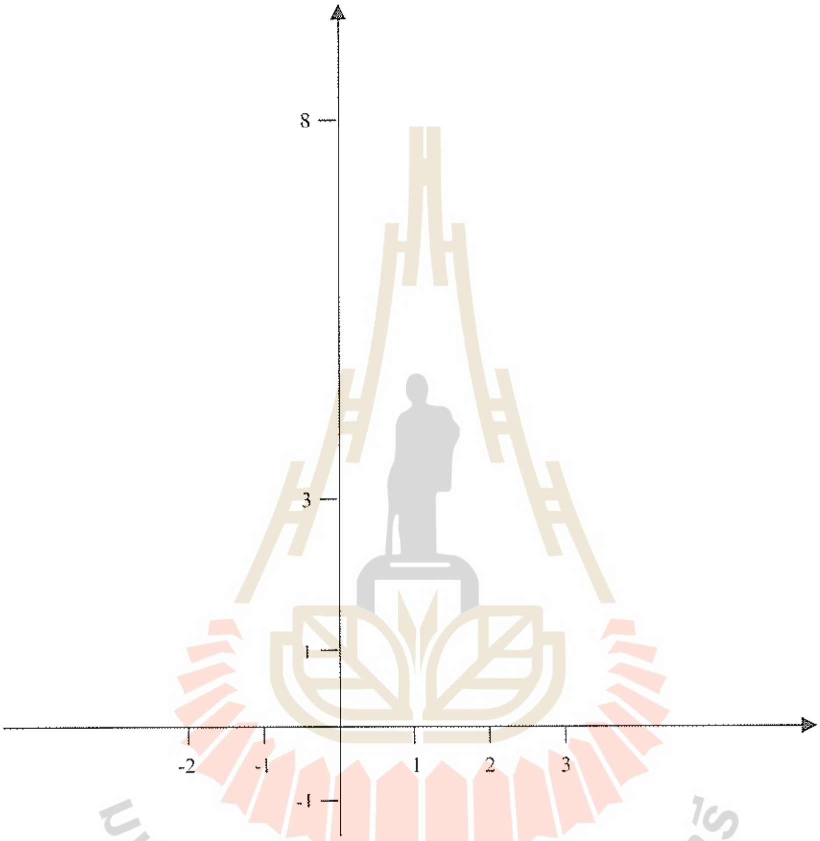
6. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = \Big[\ldots\ldots\ldots\ldots\Big]_0^{\pi/4} = (\ldots\ldots\ldots) - (\ldots\ldots\ldots)$

$= \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots$

Exercise 10: Find $\int_{-2}^{3} \left| x^2 - 1 \right| dx$.

Solution: First sketch the graph.

$$\left| x^2 - 1 \right| = \begin{cases} \ldots\ldots\ldots\ldots\ldots & \text{if } x^2 - 1 \geq \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots & \text{if } \ldots\ldots\ldots\ldots\ldots \end{cases} = \begin{cases} \ldots\ldots\ldots\ldots\ldots & \text{if } \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots & \text{if } \ldots\ldots\ldots\ldots\ldots \end{cases}$$



Therefore,

$$\int_{-2}^{3} \left| x^2 - 1 \right| dx = \int_{-2}^{-1} (\ldots\ldots\ldots\ldots)\, dx + \int_{-1}^{1} (\ldots\ldots\ldots\ldots)\, dx + \int_{\ldots\ldots}^{\ldots\ldots} (\ldots\ldots\ldots\ldots)\, dx$$

$$= \Big[\ldots\ldots\ldots\ldots \Big]_{-2}^{-1} + \Big[\ldots\ldots\ldots\ldots \Big]_{\ldots}^{\ldots} + \Big[\ldots\ldots\ldots\ldots \Big]_{\ldots}^{3}$$

$$= \Big[(\ldots\ldots\ldots) - (\ldots\ldots\ldots) \Big] + \Big[(\ldots\ldots\ldots) - (\ldots\ldots\ldots) \Big] + \Big[(\ldots\ldots\ldots) - (\ldots\ldots\ldots) \Big]$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Additional Exercises:

1) Find the derivatives of

a) $F(x) = \int_{-1}^{x} \left(t^3 - 2t\right)^{19} dt$

b) $G(x) = \int_{x}^{2} \sqrt{t}\cos t \, dt$

c) $H(x) = \int_{0}^{5x+1} \frac{1}{u^2-5} du$

d) $F(x) = \int_{\sqrt{x}}^{x^2} t\cos(t^3)\, dt$

e) $G(x) = \int_{\sin x}^{\cos x} \sec t \, dt$

f) $H(u) = \int_{u-1}^{u+1} \sqrt{x^2+1}\, dx$

2) Evaluate each definite integral.

a) $\int_{-3}^{7} \sqrt{5}\, dx$

b) $\int_{1}^{2} \frac{1}{x^2} dx$

c) $\int_{1}^{3} \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt$

d) $\int_{0}^{2} \left(x^3 - 1\right)^2 dx$

e) $\int_{-\pi/6}^{\pi/3} \left(\cos\theta - 2\sin\theta\right) d\theta$

f) $\int_{-5}^{-2} \frac{x^4-1}{x^2+1} dx$

g) $\int_{-2}^{-1} \frac{x-1}{\sqrt[3]{x^2}} dx$

h) $\int_{\pi/3}^{\pi/2} \cos u \cot u \, du$

3) Find the area of the region bounded by the given curves:

a) $y = 4x^2 - 4x + 3, \quad y = 0, \quad x = 0, \quad x = 2$

b) $y = \left| x - x^2 \right|, \quad y = 0, \quad x = -1, \quad x = 2.$

4) The position of a particle at time t is $s(t) = t^2 - 2t - 8$.

Find its average velocity over the time interval $[1,6]$

a) by using the definite integral

b) without using the integral.

Integration by Substitution in the Definite Integral

Recall:

$$\int_a^b f\left(g(x)\right) g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

where $u = g(x)$ and $du = g'(x)\,dx$.

Exercise 1: Find $\int_0^1 2x\sqrt{x^2+1}\,dx$

Solution: We set

$u =$ ....

Then

$du =$ ....

If $x = 0$ then $u =$ ....

If $x = 1$ then $u =$ ....

Therefore,

$$\int_0^1 2x\sqrt{x^2+1}\,dx = \int_{\cdots}^{\cdots} \cdots\cdots\, du = \int_{\cdots}^{\cdots} \cdots\cdots\, du$$

$$= \cdots\cdots \Big]_{\cdots}^{\cdots} = \cdots\cdots - \cdots\cdots = \cdots\cdots$$

Exercise 2: Find $\int_0^{\pi} \sin^5 x \cos x\,dx$

Solution: We set

$u =$ ....

Then

$du =$ ....

If $x = 0$ then $u =$ ....

If $x = \pi$ then $u =$ ....

$$\int_0^{\pi} \sin^5 x \cos x\,dx = \int_{\cdots}^{\cdots} \cdots\cdots\, du = \cdots\cdots \Big]_{\cdots}^{\cdots} = \cdots\cdots$$

Exercise 3: Compute the given integrals by choosing an appropriate substitution.

1. $\int_{0}^{1} x\left(x^{2}+1\right)^{9}\,dx = \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots\,du$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$= \left.\frac{}{\ldots\ldots\ldots}\right]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

2. $\int_{2}^{3} \frac{3x^{2}-1}{\left(x^{3}-x\right)^{2}}\,dx = \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots\,du = \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots\,du$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$= \left.\frac{}{\ldots\ldots\ldots}\right]_{\ldots\ldots}^{\ldots\ldots} = \left.\frac{}{\ldots\ldots\ldots}\right]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots$

3. $\int_{0}^{\pi/3} \frac{\sin\theta}{\cos^{2}\theta}\,d\theta = \int_{\ldots\ldots}^{\ldots\ldots} \frac{}{\ldots\ldots\ldots}\,du = \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots\,du$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$= \left.\frac{}{\ldots\ldots\ldots}\right]_{\ldots\ldots}^{\ldots\ldots} = \left.\frac{}{\ldots\ldots\ldots}\right]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots$

Exercise 4: Find the area of the region below the graph of $y = x\sin(x^2)$ between $x = 0$ and $x = \sqrt{\pi}$.

Solution: Since $x\sin(x^2) \geq \ldots\ldots$ on $\left[0, \sqrt{\pi}\right]$ we have

$$A = \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots\,du = \ldots\ldots\ldots\ldots\ldots\Big]_{\ldots\ldots}^{\ldots\ldots}$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

$$= \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots$$

Additional Exercises: Evaluate the following definite integrals by substitution:

1) $\displaystyle\int_0^1 (x+1)\left(x^2+2x\right)^{49}\,dx$

2) $\displaystyle\int_0^{\pi/2} \cos x\sqrt{\sin x}\,dx$

3) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx$

4) $\displaystyle\int_{\pi^2/16}^{\pi^2} \frac{\sin^2\sqrt{x}}{\sqrt{x}}\,dx$

5) $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\cos 2x}{\sin^2 2x}\,dx$

6) $\displaystyle\int_0^{\pi/3} \tan^3 x\sec^2 x\,dx$

Integrals Leading to Inverse Trigonometric Functions

Recall:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C \qquad \text{because} \qquad \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C \qquad \text{because} \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C \qquad \text{because} \qquad \frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

(check !)

Exercise 1: Evaluate the following integrals.

1. $$\int_0^{\sqrt{3}} \frac{8}{1+x^2}\,dx = 8\int_0^{\sqrt{3}} \frac{1}{\ldots\ldots\ldots}\,dx = \ldots\ldots\ldots\Big]_0^{\sqrt{3}}$$

$$= \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots = \ldots\ldots$$

2. $$\int_0^{1/2} \frac{4}{\sqrt{1-x^2}}\,dx = 4\int_0^{1/2} \frac{1}{\ldots\ldots\ldots}\,dx = \ldots\ldots\ldots\Big]_0^{1/2}$$

$$= \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots = \ldots\ldots$$

3. $$\int \frac{x^2}{\sqrt{1-x^6}}\,dx = \int \frac{\ldots\ldots}{\ldots\ldots\ldots}(\ldots\ldots) = \ldots\ldots\ldots + C$$

$$= \ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots$

$du = \ldots\ldots\ldots$

4. $$\int \frac{\tan^{-1}x}{1+x^2}\,dx = \int \ldots\ldots\,du = \ldots\ldots + C = \ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots$

$du = \ldots\ldots\ldots$

5. $$\int \frac{e^x}{1+e^{2x}}\,dx \;=\; \int \frac{e^x}{1+(\ldots)^2}\,dx \;=\; \int \ldots\ldots\ldots\, du \;=\; \ldots\ldots\ldots\ldots + C$$

$$= \;\ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

6. $$\int \frac{1}{\sqrt{16-x^2}}\,dx \;=\; \int \frac{1}{4\sqrt{1-(\ldots\ldots)^2}}\,dx \;=\; \int \frac{1}{4\sqrt{1-\ldots\ldots}}(\ldots du)$$

$$= \;\ldots\ldots\ldots\ldots\ldots + C$$

$$= \;\ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

7. $$\int \frac{1}{x^2+9}\,dx \;=\; \int \frac{1}{9\left[(\ldots\ldots)^2+\ldots\right]}\,dx \;=\; \frac{1}{9}\int \frac{1}{(\ldots\ldots)^2+\ldots}\,dx$$

$$= \;\frac{1}{9}\int \frac{1}{(\ldots\ldots)^2+\ldots}(\ldots\ldots du) \;=\; \frac{1}{\ldots\ldots}\;\ldots\ldots\ldots\ldots\ldots\ldots + C$$

$$= \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

The last two examples can be generalized: Let $a > 0$.

8. $$\int \frac{1}{\sqrt{a^2-x^2}}\,dx \;=\; \int \frac{1}{a\sqrt{1-(\ldots\ldots)^2}}\,dx \;=\; \frac{1}{a}\int \frac{1}{\sqrt{1-\ldots\ldots}}(\ldots du)$$

$$= \;\ldots\ldots\ldots\ldots\ldots + C$$

$$= \;\ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$

9. $$\int \frac{1}{x^2+a^2}\,dx \;=\; \int \frac{1}{a^2\left[(\ldots\ldots)^2+1\right]}\,dx \;=\; \frac{1}{a^2}\int \frac{1}{(\ldots\ldots)^2+1}\,dx$$

$$= \frac{1}{a^2}\int \frac{1}{(\ldots\ldots)^2+1}(\ldots\ldots du) \;=\; \frac{1}{\ldots\ldots}\;\ldots\ldots\ldots\ldots\ldots\ldots+C$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+C$$

We have shown:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx \;=\; \sin^{-1}\left(\frac{x}{a}\right)+C$$

and

$$\int \frac{1}{x^2+a^2}\,dx \;=\; \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

Exercise 2: Find $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$

Solution: Complete the square.

$$5-4x-x^2 \;=\; 5-[\ldots\ldots\ldots\ldots\ldots\ldots] \;=\; 5-[\ldots\ldots\ldots\ldots+\ldots\ldots-\ldots\ldots]$$

$$= 5-\left[(\ldots\ldots\ldots\ldots\ldots\ldots)^2-\ldots\ldots\ldots\right] \;=\; \ldots\ldots-(\ldots\ldots\ldots\ldots\ldots\ldots)^2$$

Therefore,

$$\int \frac{1}{\sqrt{5-4x-x^2}}\,dx \;=\; \int \frac{1}{\sqrt{\ldots\ldots-(\ldots\ldots\ldots\ldots)^2}}\,dx \;=\; \int \frac{1}{\sqrt{\ldots\ldots-\ldots\ldots\ldots}}\;\ldots\ldots$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

$$= \ldots\ldots\ldots\ldots\ldots+C$$

$$= \ldots\ldots\ldots\ldots\ldots+C$$

Exercise 3: Find $\displaystyle\int \frac{4}{x^2+6x+10}\,dx$

Solution: Complete the square.

$$\int \frac{4}{x^2+6x+10}\,dx = \int \frac{4}{\left(x^2+\ldots x+\ldots\right)+\ldots}\,dx = \int \frac{4}{\left(x+\ldots\right)^2+\ldots}\,dx$$

$$= \int \frac{4}{\ldots\ldots+\ldots\ldots}\,du = 4\ldots\ldots\ldots\ldots+C = \ldots\ldots\ldots\ldots\ldots\ldots+C$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

Exercise 4: Find $\displaystyle\int \frac{2}{x\sqrt{x^2-4}}\,dx$

Solution:

$$\int \frac{2}{x\sqrt{x^2-4}}\,dx = \int \frac{2}{2x\sqrt{(\ldots\ldots)^2-1}}\,dx = \ldots\int \frac{1}{\ldots\ldots\sqrt{\ldots\ldots-1}}\,du$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots+C$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots+C$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

Additional Exercises: Evaluate the following integrals:

1) $\displaystyle\int \frac{1}{x^2+25}\,dx$

2) $\displaystyle\int_0^1 \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

3) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

4) $\displaystyle\int \frac{1}{\sqrt{x}\,(1+x)}\,dx$

5) $\displaystyle\int \frac{\cos x}{\sqrt{16-\sin^2 x}}\,dx$

6) $\displaystyle\int \frac{1}{x\sqrt{x-1}}\,dx$

Integrals of the Natural Exponential Function

Recall:

$$\int e^x \, dx = e^x + C \qquad \text{because} \qquad \frac{d}{dx} e^x = e^x$$

Exercise 1: Find $\int e^{kx} \, dx$.

Solution: We substitute

$u = $ ....

Then

$du = $ ....

so that

$$\int e^{kx} \, dx = \int e^{\cdots\cdots} (\ldots\ldots du) = \ldots\ldots \int e^{\cdots\cdots} du = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

Exercise 2: Compute the following integrals by choosing appropriate substitutions.

1. $\int (3x^2 + 4x) e^{x^3 + 2x^2} \, dx = \int \ldots\ldots\ldots\ldots\ldots\ldots \, du = \ldots\ldots\ldots\ldots + C$

   $= \ldots\ldots\ldots\ldots\ldots\ldots + C$

   $u = \ldots\ldots\ldots\ldots\ldots$

   $du = \ldots\ldots\ldots\ldots\ldots$

2. $\int \cos x \, e^{\sin x} \, dx = \int \ldots\ldots\ldots \, du = \ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots + C$

   $u = \ldots\ldots\ldots\ldots$

   $du = \ldots\ldots\ldots\ldots$

3. $$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_{\dots\dots}^{\dots\dots} \dots\dots\,du = \dots\dots\Big]_{\dots\dots}^{\dots\dots} = \dots\dots - \dots\dots = \dots\dots\dots\dots\dots$$

$u = \dots\dots\dots\dots$	$x = 1 \Rightarrow u = \dots$
$du = \dots\dots\dots\dots$	$x = 4 \Rightarrow u = \dots$

4. $$\int \frac{e^x \cos\left(e^x\right)}{\sin^3\left(e^x\right)}\,dx = \int \frac{}{\dots\dots\dots}\,du = \int \dots\dots\,du = \dots\dots\dots + C$$

$$= \dots\dots\dots\dots\dots + C$$

$u = \dots\dots\dots\dots\dots\dots\dots$
$du = \dots\dots\dots\dots\dots\dots\dots$

Exercise 3: Find $\int \frac{\left(e^x+1\right)^2}{e^x}\,dx$.

Solution: Expand

$$\int \frac{\left(e^x+1\right)^2}{e^x}\,dx = \int \frac{\dots\dots\dots\dots\dots\dots\dots}{e^x}\,dx = \int\left(\frac{\dots\dots}{e^x} + \dots\dots\dots + \dots\dots\dots\right)dx$$

$$= \int\left(\dots\dots\dots + \dots\dots\dots + \dots\dots\dots\right)dx = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots + C$$

Additional Exercises: Evaluate the following integrals:

1) $\int e^{3x+4}\,dx$

2) $\int_{0.5}^{1} \frac{e^{1/x}}{x^2}\,dx$

3) $\int \frac{x}{e^{x^2}}\,dx$

4) $\int e^x \sin\left(e^x\right)\,dx$

Integrals Leading to the Natural Logarithm

Recall:

$$\int \frac{1}{x}\,dx = \ln|\,x\,| + C \qquad \text{because} \qquad \frac{d}{dx}\ln|\,x\,| = \frac{1}{x}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|\,f(x)\,| + C \qquad \text{(A)}$$

Exercise 1: Find $\displaystyle\int_{-e^2}^{-e} \frac{3}{x}\,dx$.

Solution:

$$\int_{-e^2}^{-e} \frac{3}{x}\,dx = 3\int_{-e^2}^{-e} \ldots\ldots\ldots\,dx = \ldots\ldots\ldots\Big]_{-e^2}^{-e} = \ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 2: Find $\displaystyle\int \frac{1}{2x-1}\,dx$.

Solution: We substitute

$u =$ ....

Then

$du =$ ....

so that

$$\int \frac{1}{2x-1}\,dx = \int \ldots\ldots\ldots\ldots\,du = \ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C$$

Exercise 3: Find a general formula for $\displaystyle\int \frac{1}{ax+b}\,dx \quad (a \neq 0)$

Solution: We substitute

$u =$ ....

Then

$du =$ ....

so that

$$\int \frac{1}{ax+b}\,dx = \int \ldots\ldots\ldots\ldots\,du = \ldots\ldots\ldots\ldots + C = \ldots\ldots\ldots\ldots\ldots\ldots + C$$

Exercise 4: Find $\int \frac{\ln(x^3)}{x}\,dx$

Solution: Simplify and substitute

$$\int \frac{\ln(x^3)}{x}\,dx = \ldots \int \frac{\ln(x)}{x}\,dx = \int \ldots\ldots\ldots\ldots\, du = \ldots\ldots\ldots\ldots + C$$

$$= \ldots\ldots\ldots\ldots\ldots + C$$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

Exercise 5: Evaluate the following integrals by using formula (A).

1. $\int \frac{x^2+1}{x^3+3x-4}\,dx = \int \frac{}{x^3+3x-4}\,dx = \ln|\ldots\ldots\ldots\ldots\ldots\ldots| + C$

2. $\int \frac{1}{x\ln x}\,dx = \int \frac{}{\ln x}\,dx = \ln|\ldots\ldots\ldots\ldots\ldots| + C$

3. $\int \frac{e^x}{e^x+3}\,dx = \int \frac{\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots)}{e^x+3}\,dx = \ln|\ldots\ldots\ldots\ldots\ldots\ldots| + C$

4. $\int \frac{\sin x}{1+\cos x}\,dx = \int \frac{\frac{d}{dx}(\ldots\ldots\ldots\ldots\ldots)}{1+\cos x}\,dx = \ln\left|\ldots\ldots\ldots\ldots\ldots\right| + C$

5. $\int \sec x\,dx = \int \sec x\,\frac{\sec x+\tan x}{\sec x+\tan x}\,dx = \int \frac{}{\sec x+\tan x}\,dx$

$$= \int \frac{\frac{d}{dx}\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)}{\sec x+\tan x}\,dx = \ln\left|\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right| + C$$

Additional Exercises:

Evaluate the following integrals:

1) $\int_{-1}^{0} \frac{1}{4-5x}\,dx$

2) $\int_{1}^{2} \frac{3x}{x^2+4}\,dx$

3) $\int \frac{(x+2)^2}{x}\,dx$

4) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

5) $\int \frac{1}{x(\ln x)^2}\,dx$

6) $\int \frac{3\cos x}{\pi+2\sin x}\,dx$

7) $\int \frac{\tan\left(e^{-3x}\right)}{e^{3x}}\,dx$

8) $\int \frac{\cos x\sin x}{\cos^2 x-4}\,dx$

Integrals of Exponential and Logarithmic Functions

Recall:

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = \frac{a^x \ldots\ldots}{\ln a} = \ldots\ldots\ldots$$

Exercise 1: Evaluate $\int \left(x^{10} + 10^{10} + 10^x\right) dx$.

Solution:

$$\int \left(x^{10} + 10^{10} + 10^x\right) dx = \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots + \ldots\ldots$$

Exercise 2: Find $\int_1^9 \frac{10^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution: Substitute.

$$\int_1^9 \frac{10^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots du = \left.\frac{}{\ldots\ldots\ldots}\right]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots\ldots - \ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

$u = \ldots\ldots\ldots\ldots$	$x = 1 \Rightarrow u = \ldots\ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$	$x = 9 \Rightarrow u = \ldots\ldots\ldots\ldots$

Exercise 3: Find $\int_{-1}^{1} 2^{3x-1}\, dx$.

Solution: Recall that

$$\int f(ax+b)\, dx = \frac{F(ax+b)}{a} + C$$

Therefore,

$$\int_{-1}^{1} 2^{3x-1}\, dx = \left.\frac{\qquad}{\dots\dots\dots}\right]_{\dots\dots}^{\dots\dots} = \dots\dots\dots\dots - \dots\dots\dots\dots = \dots\dots\dots\dots\dots\dots$$

$$= \dots\dots\dots\dots\dots\dots$$

Additional Exercises:

Evaluate the following integrals:

1) $\int \frac{\log_3 x}{x}\, dx$

2) $\int_{3}^{4} 5^t\, dt$

3) $\int x\, 2^{x^2-1}\, dx$

4) $\int \frac{10^{\tan x}}{\cos^2 x}\, dx$

Integrals of Hyperbolic Functions

Recall:

$$\int \sinh x \, dx = \cosh x + C \qquad \text{because} \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

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$$\int \operatorname{sech}^2 x \, dx = \tanh x + C \qquad \text{because} \qquad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

Exercise 1: Evaluate the given integrals..

1. $\int 5\cosh(3x-4)\, dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots$

(Here we use $\int f(ax+b)\, dx = \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots$)

2. $\int_0^{\ln 2} \operatorname{sech}^2\left(\frac{x}{5}\right) dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_0^{\ln 2}$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

3. $\int \tanh x \, dx = \int \frac{\qquad}{\ldots\ldots\ldots\ldots} dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots$

(Here we use $\int \frac{f'(x)}{f(x)} dx = \ldots\ldots\ldots\ldots\ldots + \ldots\ldots$)

4. $\int \frac{\sinh \sqrt{t}}{\sqrt{t}}\, dt = \int \ldots\ldots\ldots\ldots du = \ldots\ldots\ldots\ldots\ldots + \ldots = \ldots\ldots\ldots\ldots + \ldots$

$u = \ldots\ldots\ldots\ldots$

$du = \ldots\ldots\ldots\ldots$

Additional Exercises:

Evaluate the following integrals:

1) $\int \frac{\sinh x}{1+\cosh x}\, dx$

2) $\int e^t \operatorname{sech}^2(e^t)\, dt$