THE DISCREPANCY OF SPEACIAL SEQUENCES INDUCED BY GENERALIZED HALTON SEQUECNES AND BEATTY SEQUENCES

PRAPAPIT CHUTIMANTANON

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ลำดับฮาลตันแบบวางนัยทั่วไปและลำดับบีตตี

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วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for a Master's Degree.

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ประภาพิชญ์ ชุติมันตานนท์ : การวัดการกระจายตัวของลำดับพิเศษที่สร้างจากลำดับฮาลตันแบบวางนัยทั่วไปและลำดับบีตตี (THE DISCREPANCY OF SPECIAL SEQUENCES INDUCED BY GENERALIZED HALTON SEQUENCES AND BEATTY SEQUENCES) อาจารย์ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ ดร.พจน์ เลิศชูสกุล, 53 หน้า.

คำสำคัญ : การวัดการกระจายตัว/ลำดับฮาลตัน/ลำดับฮาลตันแบบวางนัยทั่วไป/ลำดับบีตตี

การประมาณค่าปริพันธ์เป็นปัญหาหนึ่งที่สำคัญในคณิตศาสตร์ โดยวิธีที่นิยมใช้ในการประมาณค่าได้แก่วิธีเสมือนมอนติคาร์โล วิธีนี้มีตัวแปรสำคัญในการประมาณค่าปริพันธ์คือ ลำดับที่มีการกระจายตัวระหว่างพจน์ต่ำ ดังนั้นวิทยานิพนธ์นี้สนใจสร้างลำดับพิเศษที่มีการกระจายตัวระหว่างพจน์ต่ำโดยการขยายงานของ Hofer (2018) ด้วยการเปลี่ยนจากการรวมลำดับฮาลตันกับลำดับบีตตีเป็นการรวมของลำดับฮาลตันแบบวางนัยทั่วไปกับลำดับบีตตีแทน จากนั้นทำการประมาณค่าการกระจายตัวของลำดับที่ถูกสร้างขึ้น โดยใช้ความรู้จากเรื่องการแจกแจงเอกรูปของลำดับ เศษส่วนต่อเนื่องของจำนวนอตรรกยะ และระบบพลวัต จากการศึกษาพบว่าการกระจายตัวของลำดับพิเศษที่ถูกสร้างนั้นได้ผลลัพธ์ที่คล้ายกับงานของ Hofer (2018) นั่นคือ เป็นลำดับที่เกือบจะมีการกระจายตัวระหว่างพจน์ต่ำ โดยมีคุณสมบัติใกล้เคียงกับลำดับที่มีการกระจายตัวระหว่างพจน์ต่ำ ลำดับพิเศษที่ถูกสร้างขึ้นนี้สามารถนำไปใช้ประโยชน์ต่อได้ในอีกหลายสาขา เช่น คณิตศาสตร์การเงิน สถิติ และฟิสิกส์

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Integral estimation is one of the most important problems in mathematics. A famous method for estimating integrals is the quasi-Monte Carlo method. This method depends on a crucial variable which is a low-discrepancy sequence. Therefore, this research focuses on constructing new sequences with low-discrepancy by extending the work of Hofer (2018), which used the combination of Halton sequences and Beatty sequences. Instead, our work uses the combination of generalized Halton sequences and Beatty sequences. The estimation of the discrepancy of these constructed sequences is achieved using techniques from the uniform distribution of sequences, continued fractions of irrational numbers, and dynamical systems. The study finds that the discrepancy of these sequences is similar to that in the work of Hofer (2018). Our discrepancy estimation of these sequences shows that they are almost low-discrepancy sequences, resembling low-discrepancy sequences. These constructed sequences with almost low-discrepancy properties are applicable in many fields such as financial mathematics, statistics, and physics.

School of Mathematics — Student's Signature ประภาพิศ
Academic 2022 — Advisor's Signature

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Prapapit Chutimantanon

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CHAPTER I

INTRODUCTION

Calculus is an important subject in mathematics and sciences. It has two major branches – differentiation and integration. Differentiation studies the rate of change of a function at a specific point. It is used in several topics such as limit, derivatives, and chain rule. Moreover, differentiation has many applications in other fields such as economics, statistics, and physics. On the other hand, integration studies the area under the curve of a function. In the late 17th century, Newton (1693) and Leibniz (1695) independently made important contributions to the development of calculus. Their independent work led to the discovery of the relation between differentiation and integration. In addition, Leibniz (1675) introduced the notation for integrals in the form of symbol $\int$. Integration has several applications in many fields such as finance, physics, and engineering. Nowadays, we can only integrate some basic functions, but some functions are too complicated to be integrated such as multi-dimensional functions, the Riemann zeta function, and the error function. This leads to the subject of numerical integration which studies how to estimate integrals. For example, the trapezoidal rule is a method used in numerical integration. In one-dimensional case, this method approximates the area under a curve of a function f by dividing the area into a number of trapezoids and then summing the area of each trapezoid. The area of each trapezoid can be calculated by using the formula $\frac{1}{2} \cdot h \cdot (f(x_1) + f(x_2))$, where h is the height of the trapezoid which is the width between points x_1 and x_2 on the x-axis. The accuracy of the trapezoidal rule increases with the number of trapezoids. If we use a large number of trapezoids, we obtain an approximation that is closer to the real value of the integral. The trapezoidal rule can be extended to higher dimensions. In multi-dimensional case, the trapezoidal rule works in a similar way as in one-dimensional case by dividing the integration domain of a function into a grid of smaller trapezoids. For example, in two-dimensional case, each trapezoid is represented by a two-dimensional array of points. The trapezoidal rule approximates the integral by summing up the areas of the trapezoids within the grid. By increasing the number of the trapezoids in the grid, the accuracy of the integral estimation improves. However, increasing the number of trapezoids makes the computation more complicated, especially in high-dimensional spaces. The curse of dimensionality refers to the phenomenon that the computational complicatedness of many numerical methods increases exponentially with the dimensionality of the problem.

In the 1940s, the developed method to overcome the curse of dimensionality was known as the Monte Carlo method. This method uses random samplings to estimate integrals. However, it is difficult to generate real random samples in practice. A more practical method is the quasi-Monte Carlo method, which uses low-discrepancy sequences instead of random samplings. This method has many applications in fields, including finance, physics, engineering, and computer graphics (Leobacher and Pillichshammer, 2014). The Koksma-Hlawka inequality contributes to the development of the quasi-Monte Carlo method in finding errors between the real value of the integral and the approximation using a sequence. In particular, it says that the accuracy of the numerical integration depends on two variables which are the variation of the function f in the sense of Hardy and Krause $V(f)$ and the star discrepancy of a sequence ω used as a sampling $D_N^*(\omega)$. According to the conditions of the Koksma-Hlawka inequality, we can establish a bound for $D_N^*(\omega)$ while $V(f)$ remains constant. Details of the Koksma-Hlawka inequality can be found on section 2.2. Moreover, $D_N^*(\omega) \to 0$ as $N \to \infty$ if and only if the sequence ω is uniformly distributed modulo 1 on the s-dimensional unit interval $[0,1)^s$. Note that the proof of this basic fact can be found in Appendix A.3. In our research, we are interested in the quantitative aspect of how quickly the star discrepancy of a given sequence ω converges to 0. Therefore, constructing low-discrepancy sequences is an interesting problem. Examples of low-discrepancy sequences include van der Corput sequences, Halton sequences, and Hammersley sequences. Much research has been performed to construct new low-discrepancy sequences and almost low-discrepancy sequences from known sequences. Almost low-discrepancy sequences are a good choice for many applications, as they are much faster to generate than low-discrepancy sequences. However, an almost low-discrepancy is a sequence which has a higher discrepancy than a low-discrepancy sequence. In our proposed research, we will extend the work of Hofer (2018) which studies constructed sequences induced by Halton sequences and Beatty sequences and proves that the discrepancy of these sequences is an almost low discrepancy. Here we construct new sequences induced by generalized Halton sequences and Beatty sequences. Moreover, our hypothesis is that our sequence is an almost low-discrepancy sequence. We will use techniques from uniform distribution of sequences, continued fractions of irrational numbers, and dynamical systems to estimate the discrepancy of these sequences.

1.1 Research objective

1. To provide new almost low-discrepancy sequences.
2. To get a rate of magnitude of upper bounds of the discrepancy of our constructed sequences.

1.2 Scope of the study

1. We apply the idea from Hofer (2018) and Haddley, Lertchoosakul, and Nair (2017) in constructing new almost low-discrepancy sequences induced by generalized Halton sequences and Beatty sequences.
2. We follow some of the ideas of Hofer (2018) to estimate the discrepancy of the newly constructed sequences. That is, we separate the estimation process of the discrepancy into three steps: firstly, we reduce the multi-dimensionality of the sequence to one-dimensional Kronecker sequences; secondly, we estimate the discrepancy of the one-dimensional Kronecker sequences; and finally, we apply a metric result of Hofer (2018).

CHAPTER II

MATHEMATICAL BACKGROUND AND TERMINOLOGY

In this section, we present some mathematical background and terminology, consisting of the main ideas of numerical integration, the Koksma-Hlawka inequality, low-discrepancy sequences, almost low-discrepancy sequences, van der Corput sequences, generalized van der Corput sequences, Halton sequences, generalized Halton sequences, Beatty sequences, Kronecker sequences, and continued fractions.

2.1 Numerical integration

Numerical integration deals with some mathematical techniques used to approximate the value of definite integrals of a function. Numerical integration is applied in several fields of sciences, engineering, and mathematics. In one-dimensional case, numerical integration involves dividing the integration interval into small subintervals and approximating the integral of the function over each subinterval using a suitable integration rule, such as the trapezoidal rule and Simpson's rule.

In multi-dimensional case, numerical integration becomes more complicated due to the increased number of independent variables and the complicatedness of the integration regions. The basic idea is similar to one-dimensional case. However, some rules that work in one-dimensional case may not be effective in multi-dimensional case because of the curse of dimensionality.

The accuracy of numerical integration can be measured by comparing the error between the true solution and the numerical solution. One tool used to find this accuracy is the Koksma-Hlawka inequality.

2.2 The Koksma-Hlawka inequality

The Koksma-Hlawka inequality tells us how close the real value of the integral and the approximation using a sequence is. To understand the inequality, we need the following definitions of variation and discrepancy.

Definition 2.1 (The variation in the sense of Hardy and Krause) Let f be a sufficiently smooth function, that is, $f \in C^s([0,1]^s)$. Then the *variation* $V(f)$ of f in *the sense of Hardy and Krause* is defined to be

$$V(f) = \sum_{k=1}^{s} \sum_{1 \le i_1 < \ldots < i_k \le s} V^{(k)}(f; i_1, \ldots, i_k).$$

Here, each term $V^{(k)}(f; i_1, \ldots, i_k)$ represents the variation of f when differentiating k components in each dimension; that is,

$$V^{(k)}(f; i_1, \ldots, i_k) = \int_0^1 \cdots \int_0^1 \left| \frac{\partial^k f|_{i_1,\ldots,i_k}}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| dx_{i_1} \cdots dx_{i_k},$$

where $f|_{i_1,\ldots,i_k}(x_{i_1}, \ldots, x_{i_k}) = f(\xi)$, and $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$ with

$$\xi_i = \begin{cases} x_i, & \text{if } i \in \{i_1, \ldots, i_k\}, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2 (The star discrepancy) Let $\omega = (x_n)_{n=0}^{\infty}$ be an infinite sequence in $[0,1)^s$. We define the *star discrepancy* of ω to be

$$D_N^*(\omega) = \sup_{J=\prod_{i=1}^{s}[0,z_i)\subseteq[0,1)^s} \left| \frac{A(J;N;\omega)}{N} - \lambda_s(J) \right|,$$

where $A(J;N;\omega) = \#\{0 \le n \le N-1 \colon x_n \in J\}$ is the counting function, and $\lambda_s(J) = \prod_{i=1}^{s} z_i$ is the s-dimensional Lebesgue measure of J.

Now we are ready to state the Koksma-Hlawka inequality. Note that its proof can be found on (Kuipers and Hiederreiter, 2006, pp. 147–150).

Theorem 2.1 (The Koksma-Hlawka inequality) *Let $\omega = (x_n)_{n=0}^{\infty}$ be a sequence in $[0,1)^s$ where s is a positive integer. Suppose that f is a function on $[0,1]^s$ which has bounded variation $V(f)$ in the sense of Hardy and Krause. Then, for any $N \in \mathbb{N}$, we have*

$$\left| \int_{[0,1]^s} f(x)\,dx - \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \right| \le V(f) D_N^*(\omega).$$

Consequently, we can establish a bound for $D_N^*(\omega)$ while $V(f)$ remains constant, as specified by the conditions of the Koksma-Hlawka inequality. Thus, in order to minimize the integration error between the real value of the integral and the approximation using a sequence, we have to use point sequences with small discrepancy, that is, sequences which are called low-discrepancy sequences.

2.3 Low-discrepancy sequences

A low-discrepancy sequence is a sequence with the property that for all values of N, its subsequence $x_0, \cdots, x_{N-1}$ has a low discrepancy. The convergence rate of low-discrepancy sequences can be described by using Big O notation, which indicates how quickly the star discrepancy approaches 0.

Definition 2.3 (Big O notation) Let $f(x)$ and $g(x)$ be two functions defined on a subset of the real numbers. We get

$$f(x) = O\left(g(x)\right),$$

if and only if there exists a positive real number C and a real number x_0 such that

$$|f(x)| \leq C|g(x)| \quad \text{for all} \quad x \geq x_0.$$

A low-discrepancy sequence is believed to have the best possible order of convergence which is $\frac{(\log N)^s}{N}$, where s is the number of dimensions.

Definition 2.4 (Low-discrepancy sequences) Let $s \in \mathbb{N}$, and ω be an infinite sequence in $[0,1)^s$. A sequence ω is a *low-discrepancy sequence* if

$$D_N^*(\omega) = O\left(\frac{(\log N)^s}{N}\right).$$

An almost low-discrepancy sequence is a sequence that has a small discrepancy, but not as small as a low-discrepancy sequence.

Definition 2.5 (Almost low-discrepancy sequences) Let ω be an infinite sequence in $[0,1)^s$. A sequence ω is an *almost low-discrepancy sequence*, if for any $\epsilon > 0$,

$$D_N^*(\omega) = O\left(\frac{(\log N)^{s+1+\epsilon}}{N}\right).$$

2.4 van der Corput sequences

van der Corput sequences are prototypes of low-discrepancy sequences which are commonly used in numerical analysis and the quasi-Monte Carlo method. Moreover, they are important for constructing other low-discrepancy sequences such as Halton sequences.

Definition 2.6 (van der Corput sequences) Let b be a nonnegative integer greater than 1. Then every nonnegative integer n has the unique b-adic representation of the form

$$n = \sum_{j=1}^{\infty} n_j b^{j-1} = n_1 + n_2 b + n_3 b^2 + n_4 b^3 + \cdots , \tag{2.1}$$

where $n_j \in \{0, 1, \ldots, b-1\}$. Note that the proof of existence and uniqueness of the b-adic representation can be found in Appendix A.1. The radical-inverse function

$$\phi_b : \mathbb{N}_0 \to [0, 1)$$

is defined by

$$\phi_b(n) = \phi_b\left(\sum_{j=1}^{\infty} n_j b^{j-1}\right) = \sum_{j=1}^{\infty} \frac{n_j}{b^j} = \frac{n_1}{b} + \frac{n_2}{b^2} + \frac{n_3}{b^3} + \cdots . \tag{2.2}$$

The *van der Corput sequence* in base b is defined to be

$$(\phi_b(n))_{n=0}^{\infty} = (\phi_b(0), \phi_b(1), \phi_b(2), \ldots). \tag{2.3}$$

Example 2.1 To construct the van der Corput sequence in base 2, in the first step, we write each nonnegative integer in base 2 by following the equation (2.1) to find digits $n_1, n_2, \ldots$. In the second step, we put the digits into the radical-inverse function ϕ_2 by following the equation (2.2). Finally, we write each value $\phi_2(0), \phi_2(1), \ldots$ in the form of the equation (2.3). Now we construct the van der Corput sequence in base 2 as follows:

- If $n = 0$, we get $0 = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(0) = 0$.
- If $n = 1$, we get $1 = 1 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(1) = \frac{1}{2}$.
- If $n = 2$, we get $2 = 0 + 1 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(2) = \frac{1}{2^2} = \frac{1}{4}$.

- If $n = 3$, we get $3 = 1 + 1 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(3) = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}$.
- If $n = 4$, we get $4 = 0 + 0 \cdot 2 + 1 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(4) = \frac{1}{2^3} = \frac{1}{8}$.
- If $n = 5$, we get $5 = 1 + 0 \cdot 2 + 1 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(5) = \frac{1}{2} + \frac{1}{2^3} = \frac{5}{8}$.
- If $n = 6$, we get $6 = 0 + 1 \cdot 2 + 1 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(6) = \frac{1}{4} + \frac{1}{2^3} = \frac{3}{8}$.
- If $n = 7$, we get $7 = 1 + 1 \cdot 2 + 1 \cdot 2^2 + 0 \cdot 2^3 + \cdots$, and so $\phi_2(7) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{7}{8}$.
- If $n = 8$, we get $8 = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 1 \cdot 2^3 + \cdots$, and so $\phi_2(8) = \frac{1}{2^4} = \frac{1}{16}$.
- Then we continue the process.

The van der Corput sequence in base 2 is

$$(\phi_2(n))_{n=0}^{\infty} = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \ldots\right).$$

2.5 Generalized van der Corput sequences

A generalized van der Corput sequence is a generalization of a van der Corput sequence. In this case, the base is a fixed sequence instead of a fixed number.

Definition 2.7 (Generalized van der Corput sequences) Let $\underline{b} = (b_j)_{j=1}^{\infty}$ be a sequence of nonnegative integers greater than 1. Then every nonnegative integer n has the unique $\underline{b}$-adic representation of the form

$$n = \sum_{j=1}^{\infty} n_j b_1 \cdots b_{j-1} = n_1 + n_2 b_1 + n_3 b_1 b_2 + n_4 b_1 b_2 b_3 + \cdots , \tag{2.4}$$

where $n_j \in \{0, 1, \ldots, b_j - 1\}$. Note that the proof of existence and uniqueness of the $\underline{b}$-adic representation can be found in Appendix A.2. This $\underline{b}$-adic representation is also called the Cantor expansion of n with respect to the Cantor base $\underline{b}$. Define the radical-inverse function by

$$\phi_{\underline{b}}(n) = \phi_{\underline{b}}\left(\sum_{j=1}^{\infty} n_j b_1 \cdots b_{j-1}\right) = \sum_{j=1}^{\infty} \frac{n_j}{b_1 \cdots b_j} = \frac{n_1}{b_1} + \frac{n_2}{b_1 b_2} + \frac{n_3}{b_1 b_2 b_3} + \cdots . \tag{2.5}$$

The *generalized van der Corput sequence* in base $\underline{b}$ is defined to be

$$(\phi_{\underline{b}})_{n=0}^{\infty} = (\phi_{\underline{b}}(0), \phi_{\underline{b}}(1), \phi_{\underline{b}}(2), \ldots). \tag{2.6}$$

Example 2.2 To construct the generalized van der Corput sequence in base $\underline{b} = (2, 3, 5, \dots)$, in the first step, we write each nonnegative integer in base $\underline{b}$ by following the equation (2.4) to find digits $n_1, n_2, \dots$. In the second step, we put the digits into the radical-inverse function $\phi_{\underline{b}}$ by following the equation (2.5). Finally, we write each value $\phi_{\underline{b}}(0), \phi_{\underline{b}}(1), \dots$ in the form of the equation (2.6). Now we construct the generalized van der Corput sequence in base $\underline{b} = (2, 3, 5, \dots)$ as follows:

- If $n = 0$, we get $0 = 0 + 0 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(0)) = 0$.
- If $n = 1$, we get $1 = 1 + 0 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(1)) = \frac{1}{2}$.
- If $n = 2$, we get $2 = 0 + 1 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(2)) = \frac{1}{2 \cdot 3} = \frac{1}{6}$.
- If $n = 3$, we get $3 = 1 + 1 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(3)) = \frac{1}{2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$.
- If $n = 4$, we get $4 = 0 + 2 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(4)) = \frac{2}{2 \cdot 3} = \frac{1}{3}$.
- If $n = 5$, we get $5 = 1 + 2 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(5)) = \frac{1}{2} + \frac{2}{2 \cdot 3} = \frac{5}{6}$.
- If $n = 6$, we get $6 = 0 + 0 \cdot 2 + 1 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(6)) = \frac{1}{2 \cdot 3 \cdot 5} = \frac{1}{30}$.
- If $n = 7$, we get $7 = 1 + 0 \cdot 2 + 1 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(7)) = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 5} = \frac{8}{15}$.
- If $n = 8$, we get $8 = 0 + 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 5 + \cdots$, and so $(\phi_{\underline{b}}(8)) = \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 5} = \frac{1}{5}$.
- Then we continue the process.

The generalized van der Corput sequence in base $\underline{b}$ is

$$(\phi_{\underline{b}}(n))_{n=0}^{\infty} = \left(0, \frac{1}{2}, \frac{1}{6}, \frac{2}{3}, \frac{1}{3}, \frac{5}{6}, \frac{1}{30}, \frac{8}{15}, \frac{1}{5}, \dots\right).$$

2.6 Halton sequences

Halton sequences are extensions of van der Corput sequences to higher dimensions.

Definition 2.8 (Halton sequences) Let $b_1, \dots, b_s$ be nonnegative integers greater than 1 which are pairwise coprime. For each i, where $1 \leq i \leq s$, every nonnegative integer n

has the unique b_i-adic representation of the form

$$n = \sum_{j=1}^{\infty} n_j b_i^{j-1} = n_1 + n_2 b_i + n_3 b_i^2 + n_4 b_i^3 + \cdots \,, \tag{2.7}$$

where $n_j \in \{0, 1, \ldots, b_i - 1\}$. The *Halton sequence* in bases $b_1, \ldots, b_s$ is defined to be

$$(\phi_{b_1}(n), \ldots, \phi_{b_s}(n))_{n=0}^{\infty} = ((\phi_{b_1}(0), \ldots, \phi_{b_s}(0)), (\phi_{b_1}(1), \ldots, \phi_{b_s}(1)), \ldots). \tag{2.8}$$

Example 2.3 To construct the Halton sequence in base 2 and base 3, in the first step, we write each nonnegative integer in base 2 and base 3 by following the equation (2.7) to find digits $n_1, n_2, \ldots$ in each base. In the second step, we put the digits into the radical-inverse functions ϕ_2 and ϕ_3 by following the equation (2.2). Finally, we write each value $\phi_2(0), \phi_2(1), \ldots$ and $\phi_3(0), \phi_3(1), \ldots$ in the form of the equation (2.8). Now we construct the Halton sequence in base 2 and base 3 as follows:

We represent n in base 2, as the van der Corput sequence in base 2 is

$$(\phi_2(n))_{n=0}^{\infty} = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \cdots\right).$$

We represent n in base 3 as follows:

- If $n = 0$, we get $0 = 0 + 0 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(0) = 0$.
- If $n = 1$, we get $1 = 1 + 0 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(1) = \frac{1}{3}$.
- If $n = 2$, we get $2 = 2 + 0 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(2) = \frac{2}{3}$.
- If $n = 3$, we get $3 = 0 + 1 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(3) = \frac{1}{3^2} = \frac{1}{9}$.
- If $n = 4$, we get $4 = 1 + 1 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(4) = \frac{1}{3} + \frac{1}{3^2} = \frac{4}{9}$.
- If $n = 5$, we get $5 = 2 + 1 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(5) = \frac{2}{3} + \frac{1}{3^2} = \frac{7}{9}$.
- If $n = 6$, we get $6 = 0 + 2 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(6) = \frac{2}{3^2} = \frac{2}{9}$.
- If $n = 7$, we get $7 = 1 + 2 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(7) = \frac{1}{3} + \frac{2}{3^2} = \frac{5}{9}$.
- If $n = 8$, we get $8 = 2 + 2 \cdot 3 + 0 \cdot 3^2 + \cdots$, and so $\phi_3(8) = \frac{2}{3} + \frac{2}{3^2} = \frac{8}{9}$.
- Then we continue the process.

The van der Corput sequence in base 3 is

$$(\phi_3(n))_{n=0}^{\infty} = \left(0, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \dots\right).$$

Therefore, the Halton sequence in base 2 and base 3 $(\phi_2(n), \phi_3(n))_{n=0}^{\infty}$ is

$$\left((0,0), \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{4}, \frac{2}{3}\right), \left(\frac{3}{4}, \frac{1}{9}\right), \left(\frac{1}{8}, \frac{4}{9}\right), \left(\frac{5}{8}, \frac{7}{9}\right), \left(\frac{3}{8}, \frac{2}{9}\right), \left(\frac{7}{8}, \frac{5}{9}\right), \dots\right).$$

2.7 Generalized Halton sequences

We would like to generalize Halton sequences by replacing fixed-number bases by fixed-sequence bases.

Definition 2.9 (Generalized Halton sequences) Let $\underline{b}_1 = (b_{1,j})_{j=1}^{\infty}, \dots, \underline{b}_s = (b_{s,j})_{j=1}^{\infty}$ be sequences of nonnegative integers greater than 1 such that for all $j_1, j_2 \in \mathbb{N}$ and all $1 \le i_1 < i_2 \le s$, we have b_{i_1,j_1} and b_{i_2,j_2} coprime. The *generalized Halton sequence* in bases $\underline{b}_1, \dots, \underline{b}_s$ is defined to be

$$(\phi_{\underline{b}_1}(n), \dots, \phi_{\underline{b}_s}(n))_{n=0}^{\infty} = ((\phi_{\underline{b}_1}(0), \dots, \phi_{\underline{b}_s}(0)), (\phi_{\underline{b}_1}(1), \dots, \phi_{\underline{b}_s}(1)), \dots). \tag{2.9}$$

Example 2.4 To construct the generalized Halton sequence in base $\underline{b}_1 = (2, 3, 7, \dots)$ and base $\underline{b}_2 = (5, 11, 13, \dots)$, in the first step, we write each nonnegative integer in base $\underline{b}_1$ and base $\underline{b}_2$ by following the equation (2.4) to find digits $n_1, n_2, \dots$ in each base. In the second step, we put the digits into the radical-inverse function $\phi_{\underline{b}_1}$ and $\phi_{\underline{b}_2}$ by following the equation (2.5). Finally, we write each value $\phi_{\underline{b}_1}(0), \phi_{\underline{b}_1}(1), \dots$ and $\phi_{\underline{b}_2}(0), \phi_{\underline{b}_2}(1), \dots$ in the form of the equation (2.9). Now we construct the generalized Halton sequence in base $\underline{b}_1$ and base $\underline{b}_2$ as follows:
We represent n in base $\underline{b}_1 = (2, 3, 7, \dots)$ as follows:

- If $n = 0$, we get $0 = 0 + 0 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 7 + \cdots$, and so $(\phi_{\underline{b}_1}(0)) = 0$.
- If $n = 1$, we get $1 = 1 + 0 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 7 + \cdots$, and so $(\phi_{\underline{b}_1}(1)) = \frac{1}{2}$.
- If $n = 2$, we get $2 = 0 + 1 \cdot 2 + 0 \cdot 2 \cdot 3 + 0 \cdot 2 \cdot 3 \cdot 7 + \cdots$, and so $(\phi_{\underline{b}_1}(2)) = \frac{1}{2 \cdot 3} = \frac{1}{6}$.

- If $n=3$, we get $3=1+1\cdot2+0\cdot2\cdot3+0\cdot2\cdot3\cdot7+\cdots$, and so $(\phi_{\underline{b}_1}(3))=\frac{1}{2}+\frac{1}{2\cdot3}=\frac{2}{3}$.
- If $n=4$, we get $4=0+2\cdot2+0\cdot2\cdot3+0\cdot2\cdot3\cdot7+\cdots$, and so $(\phi_{\underline{b}_1}(4))=\frac{2}{2\cdot3}=\frac{1}{3}$.
- If $n=5$, we get $5=1+2\cdot2+0\cdot2\cdot3+0\cdot2\cdot3\cdot7+\cdots$, and so $(\phi_{\underline{b}_1}(5))=\frac{1}{2}+\frac{2}{2\cdot3}=\frac{5}{6}$.
- If $n=6$, we get $6=0+0\cdot2+1\cdot2\cdot3+0\cdot2\cdot3\cdot7+\cdots$, and so $(\phi_{\underline{b}_1}(6))=\frac{1}{2\cdot3\cdot7}=\frac{1}{42}$.
- If $n=7$, we get $7=1+0\cdot2+1\cdot2\cdot3+0\cdot2\cdot3\cdot7+\cdots$, and so $(\phi_{\underline{b}_1}(7))=\frac{1}{2}+\frac{1}{2\cdot3\cdot7}=\frac{11}{21}$.
- If $n=8$, we get $8=0+1\cdot2+1\cdot2\cdot3+\cdots$, and so $(\phi_{\underline{b}_1}(8))=\frac{1}{2\cdot3}+\frac{1}{2\cdot3\cdot7}=\frac{4}{21}$.
- Then we continue the process.

The generalized Halton sequence in base $\underline{b}_1=(2,3,7,\dots)$ is

$$(\phi_{\underline{b}_1}(n))_{n=0}^{\infty}=\left(0,\frac{1}{2},\frac{1}{6},\frac{2}{3},\frac{1}{3},\frac{5}{6},\frac{1}{42},\frac{11}{21},\frac{4}{21},\frac{7}{10},\frac{11}{30}\dots\right).$$

We represent n in base $\underline{b}_2=(5,11,13,\dots)$ as follows:

- If $n=0$, we get $0=0+0\cdot5+0\cdot5\cdot11+0\cdot7\cdot11\cdot13+\cdots$, and so $(\phi_{\underline{b}_2}(0))=0$.
- If $n=1$, we get $1=1+0\cdot5+0\cdot5\cdot11+0\cdot7\cdot11\cdot13+\cdots$, and so $(\phi_{\underline{b}_2}(1))=\frac{1}{5}$.
- If $n=2$, we get $2=2+0\cdot5+0\cdot5\cdot11+0\cdot7\cdot11\cdot13+\cdots$, and so $(\phi_{\underline{b}_2}(2))=\frac{2}{5}$.
- If $n=3$, we get $3=3+0\cdot5+0\cdot5\cdot11+0\cdot7\cdot11\cdot13+\cdots$, and so $(\phi_{\underline{b}_2}(3))=\frac{3}{5}$.
- If $n=4$, we get $4=4+0\cdot5+0\cdot5\cdot11+0\cdot7\cdot11\cdot13+\cdots$, and so $(\phi_{\underline{b}_2}(4))=\frac{4}{5}$.
- If $n=5$, we get $5=0+1\cdot5+0\cdot5\cdot11+0\cdot7\cdot11\cdot13+\cdots$, and so $(\phi_{\underline{b}_2}(5))=\frac{1}{55}$.
- If $n=6$, we get $6=1+1\cdot5+0\cdot5\cdot11+\cdots$, and so $(\phi_{\underline{b}_2}(6))=\frac{1}{5}+\frac{1}{5\cdot11}=\frac{12}{55}$.
- If $n=7$, we get $7=2+1\cdot5+0\cdot5\cdot11+\cdots$, and so $(\phi_{\underline{b}_2}(7))=\frac{2}{5}+\frac{1}{5\cdot11}=\frac{23}{55}$.
- If $n=8$, we get $8=3+1\cdot5+0\cdot5\cdot11+\cdots$, and so $(\phi_{\underline{b}_2}(8))=\frac{3}{5}+\frac{1}{5\cdot11}=\frac{34}{55}$.
- Then we continue the process.

The generalized Halton sequence in base $\underline{b}_2 = (5, 11, 13, \dots)$ is

$$(\phi_{\underline{b}_2}(n))_{n=0}^{\infty} = \left(0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{55}, \frac{12}{55}, \frac{23}{55}, \frac{34}{55}, \dots\right).$$

Therefore, the generalized Halton sequence in base $\underline{b}_1$ and base $\underline{b}_2$ $(\phi_{\underline{b}_1}(n), \phi_{\underline{b}_2}(n))_{n=0}^{\infty}$ is

$$\left((0,0), \left(\frac{1}{2}, \frac{1}{5}\right), \left(\frac{1}{6}, \frac{2}{5}\right), \left(\frac{2}{3}, \frac{3}{5}\right), \left(\frac{1}{3}, \frac{4}{5}\right), \left(\frac{5}{6}, \frac{1}{55}\right), \left(\frac{1}{42}, \frac{12}{55}\right), \left(\frac{11}{21}, \frac{23}{55}\right), \dots\right).$$

2.8 Beatty sequences

A Beatty sequence is a sequence of integers. It has been studied in number theory and has applications in various fields, including computer science and physics (Masáková and Pelantová, 2007). The role of the floor function in defining a Beatty sequence is crucial, so we need the following definition of the floor function.

Definition 2.10 (The floor function) Let x be a real number. The *floor function* is defined by the equation $\lfloor x \rfloor =$ max$\{m \in \mathbb{Z} | m \leq x\}$.

Example 2.5 We give examples of how to use the floor function with other real numbers as follows:

- $n = 2$, we get $\lfloor 2 \rfloor = 2$.
- $n = 1.5$, we get $\lfloor 1.5 \rfloor = 1$.
- $n = -2.5$, we get $\lfloor -2.5 \rfloor = -3$.

The floor function is used to remove the fractional part of the product of a positive integer and an irrational number and leave only the integer part.

Definition 2.11 (Beatty sequences) Let α be any irrational number. Then $(\lfloor n\alpha \rfloor)_{n=0}^{\infty}$ is called the *Beatty sequence*.

Example 2.6 To construct the Beatty sequence where $\alpha = \sqrt{2}$, in the first step, we calculate the integral part of the product of each nonnegative integer and $\sqrt{2}$. Then, we put each integral part $\lfloor 0 \cdot \sqrt{2} \rfloor, \lfloor 1 \cdot \sqrt{2} \rfloor, \dots$ in the form of the Beatty sequence

$(\lfloor n\alpha \rfloor)_{n=0}^{\infty} = \left(\lfloor 0 \cdot \sqrt{2} \rfloor, \lfloor 1 \cdot \sqrt{2} \rfloor, \ldots\right)$. We now construct the Beatty sequence where $\alpha = \sqrt{2}$ as follows:

- If $n = 0$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 0 \cdot \sqrt{2} \rfloor = \lfloor 0 \rfloor = 0$.
- If $n = 1$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 1 \cdot \sqrt{2} \rfloor = \lfloor 1.414 \ldots \rfloor = 1$.
- If $n = 2$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 2 \cdot \sqrt{2} \rfloor = \lfloor 2.828 \ldots \rfloor = 2$.
- If $n = 3$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 3 \cdot \sqrt{2} \rfloor = \lfloor 4.242 \ldots \rfloor = 4$.
- If $n = 4$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 4 \cdot \sqrt{2} \rfloor = \lfloor 5.656 \ldots \rfloor = 5$.
- If $n = 5$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 5 \cdot \sqrt{2} \rfloor = \lfloor 7.071 \ldots \rfloor = 7$.
- If $n = 6$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 6 \cdot \sqrt{2} \rfloor = \lfloor 8.485 \ldots \rfloor = 8$.
- If $n = 7$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 7 \cdot \sqrt{2} \rfloor = \lfloor 9.899 \ldots \rfloor = 9$.
- If $n = 8$, we get $\lfloor 0 \cdot \alpha \rfloor = \lfloor 8 \cdot \sqrt{2} \rfloor = \lfloor 11.313 \ldots \rfloor = 11$.
- Then we continue the process.

The Beatty sequence where $\alpha = \sqrt{2}$ is

$$(\lfloor n\alpha \rfloor)_{n=0}^{\infty} = (0, 1, 2, 4, 5, 7, 8, 9, 11, \ldots).$$

2.9 Kronecker sequences

A Kronecker sequence is a sequence with points in $[0, 1)$. It has applications in number theory, dynamical systems, and signal processing (Patel, 2022). The role of the fractional part in defining a Kronecker sequence is crucial, we need the following definition of the fractional part.

Definition 2.12 (The fractional part) Let x be a real number. The *fractional part* of x is defined by the equation $\{x\} = x - \lfloor x \rfloor$.

Example 2.7 We give examples of how to use the fractional part with other real numbers as follows:

- $n = 2$, we get $\{2\} = 2 - \lfloor 2 \rfloor = 0$.

- $n = 1.5$, we get $\{1.05\} = 1.05 - \lfloor 1.05 \rfloor = 0.5$.

- $n = -9.45$, we get $\{-9.45\} = -9.45 - \lfloor 9.45 \rfloor = 0.55$.

Definition 2.13 (Kronecker sequences) Let α be any irrational number. Then $(\{n\alpha\})_{n=0}^{\infty}$ is called the *Kronecker sequence*.

Example 2.8 To construct the Kronecker sequence where $\alpha = \sqrt{2}$, in the first step, we calculate the fractional part of the product of each nonnegative integer and $\sqrt{2}$. Then, we put each fractional part $\{0 \cdot \sqrt{2}\}, \{1 \cdot \sqrt{2}\}, \ldots$ in the form of the Kronecker sequence $(\{n\alpha\})_{n=0}^{\infty} = (\{0 \cdot \sqrt{2}\}, \{1 \cdot \sqrt{2}\}, \ldots)$. We now construct the Kronecker sequence where $\alpha = \sqrt{2}$ as follows:

- If $n = 0$, we get $\{0 \cdot \alpha\} = \{0 \cdot \sqrt{2}\} = \{0\} = 0$.

- If $n = 1$, we get $\{1 \cdot \alpha\} = \{1 \cdot \sqrt{2}\} = \{1.414\ldots\} = 0.414\ldots$.

- If $n = 2$, we get $\{2 \cdot \alpha\} = \{2 \cdot \sqrt{2}\} = \{2.828\ldots\} = 0.828\ldots$.

- If $n = 3$, we get $\{3 \cdot \alpha\} = \{3 \cdot \sqrt{2}\} = \{4.242\ldots\} = 0.242\ldots$.

- If $n = 4$, we get $\{4 \cdot \alpha\} = \{4 \cdot \sqrt{2}\} = \{5.656\ldots\} = 0.656\ldots$.

- If $n = 5$, we get $\{5 \cdot \alpha\} = \{5 \cdot \sqrt{2}\} = \{7.071\ldots\} = 0.071\ldots$.

- If $n = 6$, we get $\{6 \cdot \alpha\} = \{6 \cdot \sqrt{2}\} = \{8.485\ldots\} = 0.485\ldots$.

- If $n = 7$, we get $\{7 \cdot \alpha\} = \{7 \cdot \sqrt{2}\} = \{9.899\ldots\} = 0.899\ldots$.

- If $n = 8$, we get $\{8 \cdot \alpha\} = \{8 \cdot \sqrt{2}\} = \{11.313\ldots\} = 0.313\ldots$.

- Then we continue the process.

The Kronecker sequence where $\alpha = \sqrt{2}$ is

$$(\{n\sqrt{2}\})_{n=0}^{\infty} = (0, 0.414\ldots, 0.828\ldots, 0.242\ldots, 0.656\ldots, 0.071\ldots, 0.485\ldots, \ldots).$$

2.10 Continued fractions

A continued fraction is an expression obtained through an iterative process of representing a number as its integer part plus a nested fraction.

Definition 2.14 (Continued fractions) A real number R can write in the form of the *continued fraction* as

$$R = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} = [a_0; a_1, a_2, a_3, \ldots],$$

where a_0 is an integer, and $a_1, a_2, a_3, \ldots$ are positive integers.

It is worth nothing that a real number has a unique continued fraction expansion if and only if it is an irrational number.

Example 2.9 (Continued fractions of rational numbers) We give an example of how to find the continued fraction form for $\frac{159}{46}$ to fraction part as follows:

- $\dfrac{159}{46} = 3 + \dfrac{21}{46}$, then $a_0 = 3$.
- $3 + \dfrac{21}{46} = 3 + \cfrac{1}{\cfrac{46}{21}} = 3 + \cfrac{1}{2 + \cfrac{4}{21}}$, then $a_1 = 2$.
- $3 + \cfrac{1}{2 + \cfrac{4}{21}} = 3 + \cfrac{1}{2 + \cfrac{1}{\cfrac{21}{4}}} = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{4}}}$, then $a_2 = 5$ and $a_3 = 4$.

Therefore, the continued fraction of $\dfrac{159}{46}$ is $[3; 2, 5, 4]$ or $[3; 2, 5, 3, 1]$.

Example 2.10 (Continued fractions of irrational numbers) We give an example of how to find the continued fraction form for $\sqrt{2}$. In the first step, we write that $\sqrt{2} = 1+(\sqrt{2}-1)$ and $\sqrt{2}-1 = \dfrac{1}{\sqrt{2}+1}$. In the second step, we write that $\sqrt{2}+1 = 2+(\sqrt{2}-1)$. Then, we continue to write $\dfrac{1}{\sqrt{2}+1}$ to $\dfrac{1}{2+\dfrac{1}{\sqrt{2}+1}}$ as follows:

- $\sqrt{2} = 1+(\sqrt{2}-1) = 1+\dfrac{1}{\sqrt{2}+1}$, then $a_0 = 1$.
- $1+\dfrac{1}{\sqrt{2}+1} = 1+\dfrac{1}{2+(\sqrt{2}-1)} = 1+\dfrac{1}{2+\dfrac{1}{\sqrt{2}+1}}$, then $a_1 = 2$.
- $1+\dfrac{1}{2+\dfrac{1}{\sqrt{2}+1}} = 1+\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{\sqrt{2}+1}}}$, then $a_2 = 2$.
- Then we continue the process to $2 = a_3 = a_4 = \ldots$.

Therefore, the continued fraction of $\sqrt{2}$ is $[1; 2, 2, 2, \ldots]$.

CHAPTER III

LITERATURE REVIEW

Low-discrepancy sequences are used in numerical integration, as they can improve the accuracy of the integral approximation. These sequences aim to have a more uniform distribution and lower discrepancy. In the late 19th century, Chebyshev (1866) developed the concept of uniform distribution as a measure of the randomness of a sequence, and also introduced the idea of constructing low-discrepancy sequences. van der Corput (1935) introduced van der Corput sequences, which are a type of a low-discrepancy sequence in one-dimensional sequences. They are constructed by reversing the digits in base b-adic expansion of nonnegative integers, where b is a nonnegative integer grater than 1. Moreover, van der Corput sequences are prototypes of low-discrepancy sequences. They are important for constructing other low-discrepancy sequences such as Halton sequences. Halton sequences are introduced by Halton (1960). They are multi-dimensional sequences that are constructed by a generalization of van der Corput sequences to higher dimensions with pairwise coprime bases. Considering whether a sequence is a low-discrepancy sequence, Niederreiter (1988) introduced the concept of digital nets: a type of a low-discrepancy sequence, and demonstrated that the discrepancy of digital nets, with a suitable choice of base sequence, is upper-bounded by $(\log N)^s/N$, where s represents the dimension of the sequence, and N denotes the number of terms in the sequence. Moreover, Niederreiter (1992) proved that the Halton sequence can be bounded by $C(\log N)^s/N$, where C is a constant number which depends on each base of the Halton sequence. It is believed that the formula $(\log N)^s/N$ is the best possible discrepancy bound for an arbitrary infinite sequence. This result when $s = 1$ is known as Schmidt's theorem, which was proved by Schmidt (1972). Schmidt (1972) proved the theorem by constructing a discontinuous function $f(x)$ at a point x_0, then using the Koksma-Klawka inequality to obtain a lower bound on the discrepancy of the sequence. The key idea for the proof is to choose the point x_0 such that the value of the value of $f(x_0)$ is large, but the variation of $f(x)$ over the unit cube is small. However, the generalization of Schmidt's theorem to arbitrary s is still an open problem.

Combinations of two sequences are frequently studied objects. For instance, Aistleitner, Hofer, and Larcher (2017) investigated the sequence induced by the Kronecker sequences of evil numbers and lacunary trigonometric products, where an evil number is a positive integer that has an even number of 1's in its binary expansion, and a lacunary trigonometric product is a mathematical function that is defined as the product of series of trigonometric functions. The sequence is a low-discrepancy sequence and has good quasi-random properties, making it useful for numerical integration and other applications. Hofer, Kritzer, Larcher, and Pillichshammer (2007) studied the estimation of the discrepancy of generalized van der Corput-Halton sequences and their subsequences. Hellekalek, and Niederreiter (2011) studied the uniform distribution of the sequence ω induced by Halton sequences and Beatty sequences

$$\omega = (\phi_{b_1}(\lfloor n\beta \rfloor), \ldots, \phi_{b_s}(\lfloor n\beta \rfloor))_{n=0}^{\infty},$$

with nonzero $\beta \in \mathbb{R}$ and pairwise coprime integers $b_1, \ldots, b_s$ all greater than 1. Hofer (2018) proved that the sequence ω is an almost low-discrepancy sequence by following three results. In the first result, Hofer (2018) found that the bound of the discrepancy of a multi-dimensional sequence ω was less than an one-dimensional Kronecker sequence by using generalized Halton sequence properties and the Chinese Remainder Theorem. Moreover, Hofer (2018) constructed an arbitrary interval J, and an interval J_ℓ which was constructed by generalized Halton sequences ideas to get the formula

$$|A(J;N;\omega) - N\lambda_s(J)| \leq |A(J_\ell;N;\omega) - N\lambda_s(J_\ell)| + \max\{A(J \setminus J_\ell;N;\omega), N\lambda_s(J \setminus J_\ell)\}.$$

For the Chinese Remainder Theorem, Hofer (2018) used it to count the number of a one-dimensional Kronecker sequence instead of a multi-dimensional sequence ω. In the second result, Hofer (2018) continued to estimate the discrepancy of a one-dimensional Kronecker sequence by using continued fractions. This process splits N into 2 parts to use the definition of discrepancy and the theorem of the estimation of the discrepancy Kronecker sequence to estimate discrepancy of one-dimensional Kronecker sequences.

The continued fraction expansion of an irrational number α provides a useful tool for studying the behavior of the one-dimensional Kronecker sequences $(\{n\alpha\})_{n=0}^{\infty}$, where $\{x\}$ denotes the fractional part of x. It can be shown that the upper bound on the discrepancy of the Kronecker sequence is related to the partial quotients of the continued fraction expansion of α. It was studied by Khintchine (1923), who proved that for all $\epsilon > 0$, the sequence $\omega = (\{n\alpha\})_{n=0}^{\infty}$ satisfies $D_N^*(\omega) = O((\log N)^{1+\epsilon})/N)$. This result used the idea of Kuipers and Niederraiter (2006) by using the division algorithm, Ostrowski expansion, and continued fractions to estimate the discrepancy of the one-dimensional Kronecker sequences. The last result, Hofer proved that the sequence ω is an almost low-discrepancy sequence by using dynamical system techniques. The process of this result is similar to lemma 2 of Larcher (2013).

In our work, we aim to construct new sequences induced by generalized Halton sequences and Beatty sequences. This will extend the work of Hofer (2018) from Halton sequences to generalized Halton sequences. It is worth noting that a generalized Halton sequence is a low-discrepancy sequence, as proved by Haddley, Lertchoosakul, and Nair (2017). They define generalized Halton sequences induced by generalized numeration systems. Moreover, a generalized Halton sequence is uniformly distributed modulo 1, as studied by Haddley, Lertchoosakul, and Nair (2013). By our hypothesis, we believe that our estimation of discrepancy of the new sequences induced by generalized Halton sequences and Beatty sequences would be almost low-discrepancy, which is similar to the result of Hofer (2018).

CHAPTER IV

RESEARCH METHODOLOGY

This chapter presents some elementary results which will be used throughout our work. Note that, at the end of this chapter, we provide a flowchart which summarizes the overview of our research methodology and our contributions.

Firstly, the bases of Halton sequences consist of pairwise coprime integers which can be proved by using the Chinese Remainder Theorem. This theorem is also used to reduce the calculation of the estimation of discrepancy of multi-dimensional sequences to the calculation of discrepancy of one-dimensional Kronecker sequences.

Theorem 4.1 (Chinese Remainder Theorem) *Let $s \in \mathbb{N}$, and $n_1, \ldots, n_s$ be pairwise coprime positive integers. If $a_1, \ldots, a_s$ are any integers, then the system of linear congruences*

$$x \equiv a_1 \pmod{n_1}, \quad x \equiv a_2 \pmod{n_2}, \quad \ldots, \quad x \equiv a_s \pmod{n_s},$$

has a solution which is unique modulo $n_1 n_2 \cdots n_s$.

Proof. We show the existence of solutions by using mathematical induction. In the base step $s = 2$, we will show that the system of the congruences

$$x \equiv a_1 \pmod{n_1} \tag{4.1}$$

$$x \equiv a_2 \pmod{n_2} \tag{4.2}$$

has a solution. By writing the congruence (4.1) as an equation, we obtain

$$x = a_1 + n_1 y, \quad \text{for some} \quad y \in \mathbb{Z}. \tag{4.3}$$

Then, we substitute x from the equation (4.3) into the congruence (4.2) as

$$a_1 + n_1 y \equiv a_2 \pmod{n_2}. \tag{4.4}$$

Subtracting by a_1 on both sides in the equation (4.4), we get

$$n_1 y \equiv a_2 - a_1 \pmod{n_2}. \tag{4.5}$$

Since $\gcd(n_1, n_2) = 1$, let n' be an inverse of $n_1 \pmod{n_2}$, we have

$$n' n_1 \equiv 1 \pmod{n_2}.$$

Multiplying through the congruence (4.5) by n', we have

$$y \equiv n'(a_2 - a_1) \pmod{n_2}. \tag{4.6}$$

Revising the congruence (4.6) as an equation, we get

$$y = n'(a_2 - a_1) + n_2 z \quad \text{for some} \quad z \in \mathbb{Z}. \tag{4.7}$$

Substituting y from the equation (4.7) into the congruence (4.4), the result is

$$x = a_1 + n_1 y = a_1 + n_1(n'(a_2 - a_1) + n_2 z) = a_1 + n_1 n'(a_2 - a_1) + n_1 n_2 z. \tag{4.8}$$

We can check that the form in the equation (4.8) satisfies the congruence (4.1) and the congruence (4.2).

Next, we show the inductive step. Suppose all congruences with s pairwise coprime moduli can be solved as

$$x \equiv a_1 \pmod{n_1}, \quad x \equiv a_2 \pmod{n_2}, \ldots, \quad x \equiv a_s \pmod{n_s}. \tag{4.9}$$

There exists a solution b to the first s congruences, we have

$$x \equiv b \pmod{n_1 n_2 \cdots n_s}, \tag{4.10}$$

We consider a system of congruences with $s+1$ pairwise coprime moduli as

$$x \equiv a_1 \pmod{n_1}, \ldots, \quad x \equiv a_s \pmod{n_s}, \quad x \equiv a_{s+1} \pmod{n_{s+1}}, \tag{4.11}$$

where $\gcd(n_i, n_j) = 1$ for all $i \neq j$. From the congruence (4.9) and the congruence (4.11), the two moduli $n_1 n_2 \cdots n_s$ and n_{s+1} are coprime because $\gcd(n_i, n_{i+1}) = 1$ for

$i = 1, 2, \ldots,$ and $\gcd(n_1 n_2 \cdots n_s, n_{s+1}) = 1$. Then, there exists c which is a solution to the congruence (4.10) and the congruence

$$x \equiv a_{s+1} \pmod{n_{s+1}}. \tag{4.12}$$

We get that $c \equiv b \pmod{n_1 n_2 \ldots n_s}$ and

$$c \equiv b \pmod{n_i} \quad \text{for} \quad i = 1, 2, \ldots, s. \tag{4.13}$$

By the inductive hypothesis, we have

$$b \equiv a_i \pmod{n_i} \quad \text{for} \quad i = 1, 2, \ldots, s. \tag{4.14}$$

By the congruence (4.13) and the congruence (4.14) through the transitivity property, we get

$$c \equiv a_i \pmod{n_i} \quad \text{for} \quad i = 1, 2, \ldots, s$$

and also

$$c \equiv a_{s+1} \pmod{n_{s+1}}.$$

Therefore, we see that c satisfies the $s+1$ given congruences.

Next, we show the uniqueness of the solution. Suppose that $x = c_1$ and $x = c_2$ are two solutions to the system of linear congruences, say

$$x \equiv a_1 \pmod{n_1}, \quad x \equiv a_2 \pmod{n_2}, \ldots, \quad x \equiv a_s \pmod{n_s}.$$

We have $c_1 \equiv c_2 \pmod{n_i}$ for $i = 1, 2, \ldots, s$. Then we obtain

$$n_i \mid (c_1 - c_2) \quad \text{for} \quad i = 1, 2, \ldots, s.$$

Because n_i are pairwise coprime, their product $n_1 n_2 \cdots n_s$ divides $c_1 - c_2$. Therefore, we get

$$c_1 \equiv c_2 \pmod{n_1 n_2 \cdots n_s}.$$

$\square$

Secondly, we show some basic properties of continued fractions. The discrepancy of the Kronecker sequence $(\{n\alpha\})_{n=0}^{\infty}$ is related to the continued fraction expansion of $\alpha = [\lfloor\alpha\rfloor; a_1(\alpha), a_2(\alpha), a_3(\alpha), \ldots]$. The convergents of $\frac{p_j}{q_j}$ to α are defined as the fractions $\frac{p_j}{q_j}$ that have the first j terms of the continued fraction expansion of α with $p_j, q_j \in \mathbb{Z}$ and $\gcd(p_j, q_j) = 1$. The following proof is a rewritten version of (Khinchin, 1964, p. 4).

Theorem 4.2 *For integer $j \geq 2$, we get*

$$p_j = a_j p_{j-1} + p_{j-2},$$

$$q_j = a_j q_{j-1} + q_{j-2},$$

where $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$, and $q_1 = a_1$.

Proof. We use mathematical induction to prove that for all integers j greater than or equal to 2, the following two equations hold

$$p_j = a_j p_{j-1} + p_{j-2}, \quad \text{and} \quad q_j = a_j q_{j-1} + q_{j-2}. \tag{4.15}$$

In the base step $j = 2$, the two equations in (4.15) become

$$\begin{aligned} \frac{p_2}{q_2} &= a_0 + \cfrac{1}{a_1 + \frac{1}{a_2}} \\ &= a_0 + \frac{a_2}{a_1 a_2 + 1} \\ &= \frac{a_2(a_1 a_0 + 1) + a_0}{a_1 a_2 + 1}. \end{aligned} \tag{4.16}$$

By definition, we know that $p_0 = a_0$, $q_0 = 1$, $q_1 = a_1$, and $p_1 = a_0 a_1 + 1$. Substituting these values into the equation (4.16), we get

$$\begin{aligned} \frac{p_2}{q_2} &= \frac{a_2(a_1 a_0 + 1) + a_0}{a_1 a_2 + 1} \\ &= \frac{a_2(a_1 a_0 + 1) + a_0}{a_1 a_2 + 1} \\ &= \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}. \end{aligned}$$

Hence, the base step is established. In the inductive step, let k be a fixed integer such that $k \geq 2$. Assume that the equation (4.15) holds for all j up to k. That is, we assume that

$$\frac{p_j}{q_j} = \frac{a_j p_{j-1} + p_{j-2}}{a_j q_{j-1} + q_{j-2}} \qquad \text{for all} \quad j \quad \text{such that} \quad 2 \leq j \leq k. \tag{4.17}$$

We show that the equation (4.17) also hold for $j = k+1$. Starting with $\dfrac{p_{k+1}}{q_{k+1}}$, we have

$$\begin{aligned}\frac{p_{k+1}}{q_{k+1}} &= [a_0; a_1, a_2, \cdots, a_k + \frac{1}{a_{k+1}}] \\ &= \frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}} \\ &= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} \\ &= \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}}.\end{aligned}$$

□

Continued fractions can be used to find good rational approximations of real numbers. This is because the continued fraction expansion of a real number consists of a sequence of integers that can be used to construct a sequence of rational numbers that increasingly well approximate the real numbers. For any positive integer j, and α is an irrational number. We have the inequality

$$\left|\alpha - \frac{p_j}{q_j}\right| < \frac{1}{q_j q_{j+1}}.$$

The following proof is a rewritten version of (Khinchin, 1964, p. 9).

Theorem 4.3 *For any irrational number α and any positive integer j, the convergent $\frac{p_j}{q_j}$ satisfies the inequality*

$$\left|\alpha - \frac{p_j}{q_j}\right| < \frac{1}{q_j q_{j+1}},$$

where p_j and q_j are the numerator and denominator of the jth convergent of the continued fraction expansion of α, respectively, and q_{j+1} is the denominator of the $(j+1)$th convergent.

Proof. The value of α lies between two rational numbers $\frac{p_0}{q_0}$ and $\frac{p_1}{q_1}$. As i increases, the denominators q_i and q_{i+1} get larger, so the bounds get closer to each other. The convergents become better and better approximation to α. In other words, $\frac{p_0}{q_0}, \frac{p_2}{q_2}, \ldots, \frac{p_i}{q_i}$, converges to α in the left-hand side, where i is an even number. $\frac{p_1}{q_1}, \frac{p_3}{q_3}, \ldots, \frac{p_{i+1}}{q_{i+1}}$, converges to α in the right-hand side. We suppose that the i-th convergent, $\frac{p_i}{q_i}$ is the closest rational number to α. Therefore, we have

$$\left|\alpha - \frac{p_j}{q_j}\right| < \left|\frac{p_{j+1}}{q_{j+1}} - \frac{p_j}{q_j}\right|. \tag{4.18}$$

Note that

$$q_j p_{j-1} - q_{j-1} p_j = (-1)^j \quad \text{for all} \quad j \in \mathbb{N}_0. \tag{4.19}$$

Hence, we have

$$\begin{aligned}\left|\alpha - \frac{p_j}{q_j}\right| &< \left|\frac{p_{j+1}}{q_{j+1}} - \frac{p_j}{q_j}\right| \\ &= \left|\frac{p_{j+1}q_j - p_j q_{j+1}}{q_{j+1}q_j}\right| \\ &= \left|\frac{(-1)^j}{q_{j+1}q_j}\right| \\ &= \frac{1}{q_j q_{j+1}}.\end{aligned}$$

□

Thirdly, we show some discrepancy theorems which are necessary to estimate the discrepancy of the one-dimensional Kronecker sequences. A fundamental result in discrepancy theory is that the estimation of discrepancy can be decomposed into a number of subsequences with small discrepancy using the triangle inequality, so we need the following definition of the discrepancy.

Definition 4.1 (The discrepancy) Let $\omega = (x_n)_{n=0}^{\infty}$ be an infinite sequence in $[0,1)^s$. We define the *discrepancy* of ω to be

$$D_N(\omega) = \sup_{J=\prod_{i=1}^{s}[a_i,b_i)\subseteq[0,1)^s} \left|\frac{A(J;N;\omega)}{N} - \lambda_s(J)\right|,$$

where $A(J;N;\omega) = \#\{0 \le n \le N-1 \colon x_n \in J\}$ is the counting function, and $\lambda_s(J) = \prod_{i=1}^{s}(b_i - a_i)$ is the s-dimensional Lebesgue measure of J.

Now we are ready to state the discrepancy estimation of a decoposed number into subsequences. This following proof is a rewritten version of (Kuipers and Hiederreiter, 2006, p. 115).

Theorem 4.4 (The discrepancy estimation of a decomposed number into subsequences) *For* $1 \leq i \leq s$, *let* ω_i *be a sequence of* N_i *elements from* $\mathbb{R}$ *with discrepancy* $D_{N_i}(\omega_i)$. *Let* ω_i *be a superposition of* $\omega_1, \ldots, \omega_s$, *which means that* ω *is a sequence obtained by listing the terms of* ω_i *in some orders. We set* $N = N_1 + \cdots + N_s$, *which is the number of elements of* ω. *Then,*

$$D_N(\omega) \leq \sum_{i=1}^{s} \frac{N_i}{N} D_{N_i}(\omega_i).$$

Proof. Let $J = [\alpha, \beta)$ be a subinterval of $[0,1)$. By the construction of N and ω, we have $A(J; N : \omega) = \sum_{i=1}^{s} A(J; N_i; \omega_i)$. Therefore

$$\begin{aligned}
D_N(\omega) &= \sup_{J \subseteq [0,1)} \left| \frac{A(J;N;\omega)}{N} - \lambda_s(J) \right| \\
&= \sup_{J \subseteq [0,1)} \left| \sum_{i=1}^{s} \frac{N_i}{N} \left(\frac{A(J;N_i;\omega_i)}{N_i} - \lambda_s(J) \right) \right| \\
&\leq \sum_{i=1}^{s} \frac{N_i}{N} \sup_{J \subseteq [0,1)} \left| \frac{A(J;N_i;\omega_i)}{N_i} - \lambda_s(J) \right| = \sum_{i=1}^{s} \frac{N_i}{N} D_{N_i}(\omega_i).
\end{aligned}$$

□

We specifically estimate the discrepancy of the Kronecker sequences. The following proof is a rewritten version of (Kuipers and Hiederreiter, 2006, pp. 125-126).

Theorem 4.5 (The discrepancy estimation of $(\{n\alpha\})_{n=0}^{\infty}$) *Suppose the irrational number* $\alpha = [a_0, a_1, \ldots]$ *has bounded partial quotients and* K *is an integer, with* $a_i \leq K$ *for* $i \geq 1$. *The discrepancy* $D_N(\omega)$ *of* $\omega = (\{n\alpha\})_{n=0}^{\infty}$ *satisfies* $ND_N(\omega) = O(\log N)$. *In particular, we have*

$$ND_N(\omega) \leq 3 + \left(\frac{1}{\log \xi} + \frac{K}{\log(K+1)} \right) \log N,$$

where $\xi = \dfrac{1+\sqrt{5}}{2}$.

Proof. We separate the process of estimating $ND_N^*(\omega)$ into three steps. In the first step, we construct $N = \sum_{i=0}^{s} b_i q_i$, by using the denominators $1 = q_0 \leq q_1 < q_2 < q_3 < \cdots$

of $\frac{p_i}{q_i}$, that converges to α. Let $N \geq 1$, there exists an $r \geq 0$ such that $q_r \leq N < q_{r+1}$. By using the division algorithm, we can simplify N to

$$N = b_r q_r + N_{r-1},$$

where $0 \leq N_{r-1} < q_r$. We note that $N < q_{r+1} = a_{r+1}q_r + q_{r-1} \leq (a_{r+1}+1)q_r$, then we get

$$\begin{aligned} N &< q_{r+1} \\ b_r q_r + N_{r-1} &\leq a_{r+1}q_r + q_r \\ b_r q_r + (N_{r-1} - q_r) &\leq a_{r+1}q_r \\ b_r q_r &\leq a_{r+1}q_r \\ b_r &\leq a_{r+1}. \end{aligned}$$

Therefore, we have that a representation of N is

$$\begin{aligned} N &= b_r q_r + N_{r-1} \\ &= b_r q_r + b_{r-1}q_{r-1} + N_{r-2} \\ &= b_r q_r + b_{r-1}q_{r-1} + \cdots + b_0 q_0 \\ &= \sum_{i=0}^{r} b_i q_i, \end{aligned}$$

where $0 \leq b_i \leq a_{i+1}$ for $0 \leq 1 \leq r$, and $b_r \geq 1$. In the second step, we find the discrepancy of D_{q_i} by decomposing the given sequence $(\{n\alpha\})_{n=0}^{N}$. This decomposition splits the sequence into b_r sequences, where n runs through q_r consecutive integers, then into b_{r-1} sequences, where n runs through q_{r-1} consecutive integers, and so on. After that, we estimate discrepancy in the first q_i terms by defining $n = n_0 + j$ with $1 \leq j \leq q_i$. By Theorem 4.3

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2},$$

we have

$$\alpha = \frac{p_i}{q_i} + \frac{\theta}{q_i + q_{i+1}},$$

where $|\theta| < 1$. Therefore, we get

$$\{n\alpha\} = \left\{n_0\alpha + \frac{jp_i}{q_i} + \frac{j\theta}{q_iq_{i+1}}\right\}.$$

Since $(p_i, q_i) = 1$, the numbers $n_0\alpha + \frac{jp_i}{q_i}$ with $1 \le j \le q_i$ are considered in modulo 1. Generate a sequence of q_i equidistant points, with a distance of $\frac{1}{q_i}$ between them. Considering $\left|\frac{j\theta}{q_iq_{i+1}}\right| < \frac{1}{q_{i+1}}$ for $1 \le j \le q_i$, the finite sequence $(\{n\alpha\})$, $n_0 + 1 \le n \le n_0 + q_i$, is obtained by shifting modulo 1, the elements $\{n_0\alpha + \left(\frac{jp_i}{q_i}\right)\}$, $1 \le j \le q_i$, either all to the right or all to the left by a distance less than $\frac{1}{q_{i+1}}$ which depends on the sign of θ. Therefore, we have that the discrepancy D_{q_i} of the finite sequence $(\{n\alpha\})$ with $n_0 + 1 \le n \le n_0 + q_i$ is

$$\begin{aligned}
D_{q_i} \le& \left|\frac{A((0, \frac{1}{q_i} - \frac{1}{q_{i+1}}]; q_i; \{n\alpha\})}{q_i} - \left(\frac{1}{q_i} - \frac{1}{q_{i+1}}\right)\right| \\
=& \left|\frac{2}{q_i} - \left(\frac{1}{q_i} - \frac{1}{q_{i+1}}\right)\right| \\
=& \frac{1}{q_i} + \frac{1}{q_{i+1}}.
\end{aligned}$$

In the last step, we estimate $ND_N(\omega)$, by Theorem 4.4 and the way in which we decomposed the original sequence $\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$, and $D_{q_i} \le \dfrac{1}{q_i} + \dfrac{1}{q_{i+1}}$, we have

$$\begin{aligned}
ND_N(\omega) \le& \sum_{i=0}^{r} q_ib_i\left(\frac{1}{q_i} + \frac{1}{q_{i+1}}\right) \\
=& \sum_{i=0}^{r} b_i\left(\frac{q_i}{q_{i+1}} + 1\right) \\
=& \sum_{i=0}^{r} b_i\frac{q_i}{q_{i+1}} + \sum_{i=0}^{r} b_i \\
=& \left(\frac{b_0q_0}{q_1} + \frac{b_1q_1}{q_2} + \cdots + \frac{b_rq_r}{q_{r+1}}\right) + \sum_{i=0}^{r} b_i \\
\le& \left(\frac{a_0q_0}{q_1} + \frac{a_1q_1}{q_2} + \cdots + \frac{a_rq_r}{q_{r+1}}\right) + \sum_{i=0}^{r} b_i \\
\le& r + 1 + \sum_{i=0}^{r} b_i.
\end{aligned}$$

Then, we estimate the first term $\sum_{i=0}^{r} b_i$, by proving that

$$\sum_{i=0}^{r} b_i \leq 1 + \frac{K}{\log(K+1)} \log N.$$

For given $N \geq 1$, we assume that $\sigma(N) = \sum_{i=0}^{r} b_i$. If $1 = q_0 < q_1$, then there exists $r \geq 0$ such that the smallest possible is $r = 0$, and a corresponding N satisfies $1 \leq \sigma(N) = b_0 = N < q_1 \leq K$, where K is an integer. If $q_0 = q_1 = 1$, then there exists $r \geq 0$ such that the smallest possible is $r = 1$, and a corresponding N satisfies $1 \leq \sigma(N) = N < q_2 \leq q_1 + 1 \leq K + 1$. Therefore, the case of $1 = q_0 \leq q_1$, we change $\sum_{i=0}^{r} b_r$ to N, and it suffices to show

$$N \leq 1 + \frac{K}{\log(K+1)} \log N, \tag{4.20}$$

for $1 \leq N < K + 1$. To prove the well-definedness of the inequality (4.20), we consider the function $f(x) = x - \dfrac{K}{\log(K+1)} \log x$ when $x \in [1, K+1]$, then f is concave upward on the entire interval by proving that $f(1) = f(K+1) = 1$. Now we consider an arbitrary N with $1 < q_r \leq N < q_{r+1}$, and $N = b_r q_r + N_{r-1}$ with $0 \leq N_{r-1} < q_r$. We suppose that $N_{r-1} > 0$, then $\sigma(N) = b_r + \sigma(N_{r-1})$, we get the inequality

$$\begin{aligned}
\sigma(N_{r-1}) \leq& 1 + \frac{K}{\log(K+1)} \log N_{r-1} \\
\sigma(N) - b_r \leq& 1 + \frac{K}{\log(K+1)} \log N_{r-1} \\
\sigma(N) \leq& 1 + b_r + \frac{K}{\log(K+1)} \log N_{r-1}.
\end{aligned}$$

By $N = b_r q_r + N_{r-1} > b_r N_{r-1} + N_{r-1} = (b_r + 1) N_{r-1}$, we get that

$$\sigma(N) \leq 1 + b_r + \frac{K}{\log(K+1)} \log \frac{N}{b_r + 1}. \tag{4.21}$$

We suppose that $N_{r-1} = 0$, then $\sigma(N) = b_r$ and $\dfrac{N}{b_r+1} = \dfrac{b_r q_r + N_{r-1}}{b_r + 1} \geq 1$ as $q_r \geq 1$. Therefore, $\log \dfrac{N}{b_r q_r}$ is well-defined in the case of $N_{r-1} \geq 1$. To compute the inequality (4.21), we will prove $b_r \leq \dfrac{K}{\log(K+1)} \log b_{r+1}$ by following an increasing function $g(x) =$

$\dfrac{x}{\log(x+1)}$ for $x > 0$ and by considering in interval $1 \leq b_r \leq a_{r+1} \leq K$. Then, we complete the bounding $\sigma(N)$ that

$$\begin{aligned}\sigma(N) = \sum_{i=0}^{r} b_i \leq& 1 + b_r + \frac{K}{\log(K+1)} \log \frac{N}{b_r+1} \\ \leq& 1 + \frac{K}{\log(K+1)} \log(b_r+1) + \frac{K}{\log(K+1)} \log \frac{N}{b_r+1} \\ =& 1 + \frac{K}{\log(K+1)} \log N.\end{aligned}$$

Therefore, we have

$$\sum_{i=0}^{r} b_i \leq 1 + \frac{K}{\log(K+1)} \log N. \tag{4.22}$$

To estimate the second term r, we use continued fractions that $q_i = a_i q_{i-1} + q_{i-2}$, where $q_0 = 1$. Then $q_1 = a_1 q_0$, so $q_2 \geq 2$ where $a_1 \geq 1$, and $q_2 = a_2 q_1 + q_0 = a_2 a_1 q_1 + 1 \geq 2$, where $a_2 \geq 2$. Therefore, the smallest value of q_i depends on the smallest $a_i = 1$. Considering $a_i = 1$, we get

$$[1; 1, 1, 1, \dots] = \frac{1+\sqrt{5}}{2}.$$

We now prove that $q_i \geq \xi^{i-1}$ for $i \geq 0$, where $\xi = \frac{1+\sqrt{5}}{2}$. Using mathematical induction, we assume that $q_i \geq \xi^{i-1}$, for $i \geq 0$. In the base step $i = 0$, we observe that $q_0 = 1 \geq \xi^{-1} = \left(\frac{1+\sqrt{5}}{2}\right)^{-1}$. In the inductive step, we assume that $q_{i-1} \geq \xi^{i-2}$ is true, for $i \geq 0$. We consider q_i as

$$\begin{aligned}q_i &= a_i q_{i-1} + q_{i-2} \\ &\geq a_i \xi^{i-2} + \xi^{i-3} \\ &\geq \xi^{i-2} + \xi^{i-3} \\ &= \xi^{i-2} \cdot \xi \\ &= \xi^{i-1}.\end{aligned}$$

Therefore, we get $N \geq q_r \geq \xi^{r-1}$, and

$$r \leq \frac{\log N}{\log \xi} + 1. \tag{4.23}$$

Finally, the result by following the inequality (4.22) and the inequality (4.23) is

$$
\begin{aligned}
ND_N(\omega) &\leq \sum_{i=0}^{r} b_i + r + 1 \\
&\leq 1 + \frac{K}{\log(K+1)} logN + \frac{\log N}{\log \xi} + 1 + 1 \\
&= 3 + \left(\frac{K}{\log(K+1)} + \frac{1}{\log \xi} \right) \log N.
\end{aligned}
$$

□

Moreover, the Koksma-Hlawka inequality is concerned with the star-discrepancy of sequence (D_N^*). Therefore, we now show the relationship between discrepancy and star discrepancy. The following proof is a rewritten version of (Kuipers and Hiederreiter, 2006, p. 91).

Theorem 4.6 *(The relation between discrepancy and star discrepancy) The discrepancies D_N and D_N^* are related by:*

$$D_N^* \leq D_N \leq 2D_N^*.$$

Proof. The inequality $D_N^* \leq D_N$ follows from the definition as

$$
\begin{aligned}
D_N &= \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([\alpha, \beta); N)}{N} - (\beta - \alpha) \right| \\
&= \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([0, \beta); N)}{N} - \frac{A([0, \alpha); N)}{N} - (\beta - \alpha) \right| \\
&\geq \sup_{0 < \alpha \leq 1} \left| \frac{A([0, \alpha); N)}{N} - \alpha \right| \\
&= D_N^*.
\end{aligned}
$$

Because the formula $D_N^* = \sup_{0<\alpha\leq 1} \left| \frac{A([0,\alpha);N)}{N} - \alpha \right|$ is a special case of the formula $D_N = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([0,\beta);N)}{N} - \frac{A([0,\alpha);N)}{N} - (\beta - \alpha) \right|$ by the definitions of the discrepancy and the star discrepancy. The inequality $D_N \leq 2D_N^*$ follows from the equality

$$A([\alpha, \beta); N) = A([0, \beta); N) - A([0, \alpha); N),$$

where $0 \leq \alpha < \beta \leq 1$. Therefore, we have

$$\begin{aligned}
D_N &= \sup_{0\leq\alpha<\beta\leq 1} \left| \frac{A([\alpha,\beta);N)}{N} - (\beta-\alpha) \right| \\
&= \sup_{0\leq\alpha<\beta\leq 1} \left| \frac{A([0,\beta);N)}{N} - \frac{A([0,\alpha);N)}{N} - (\beta-\alpha) \right| \\
&\leq \sup_{0<\beta\leq 1} \left| \frac{A([0,\beta);N)}{N} - \beta \right| + \sup_{0<\alpha\leq 1} \left| \frac{A([0,\alpha);N)}{N} - \alpha \right| \\
&= 2D_N^*.
\end{aligned}$$

□

Finally, we will apply the matric result of Hofer (2018) to show that a sequence is an almost low-discrepancy sequence. We call the metric result of Hofer (2018) as Lemma 4.7. To understand this lemma, we need the following definition of Lebesgue-almost all $\alpha \in \mathbb{R}$.

Definition 4.2 (Lebesgue-almost all) A property holds $Lebesgue\text{-}almost\ all$ if it holds for all elements in $\mathbb{R}$ except for a subset of Lebesgue measure zero.

We can say that Lebesgue-almost all real numbers in $[0,1)$ are irrational, which means that the set of irrational numbers has measure 1. The following lemma is based on (Hofer, 2018, p. 28).

Lemma 4.7 *Let $b_1,\ldots,b_s$ be positive integers. Then for Lebesgue-almost all $\alpha \in \mathbb{R}$, we have, for every $\epsilon > 0$,*

$$\sum_{j_1=0}^{L} \cdots \sum_{j_s=0}^{L} \sum_{k=1}^{L} a_k \left(\alpha b_1^{j_1} \cdots b_s^{j_s}\right) = O\left(L^{s+1+\epsilon}\right),$$

where $a_k \left(\alpha b_1^{j_1} \cdots b_s^{j_s}\right)$ is the continued fraction expansion of $\left(\alpha b_1^{j_1} \cdots b_s^{j_s}\right)$.

Overall, we have presented the workflow of our work in the following flowchart.

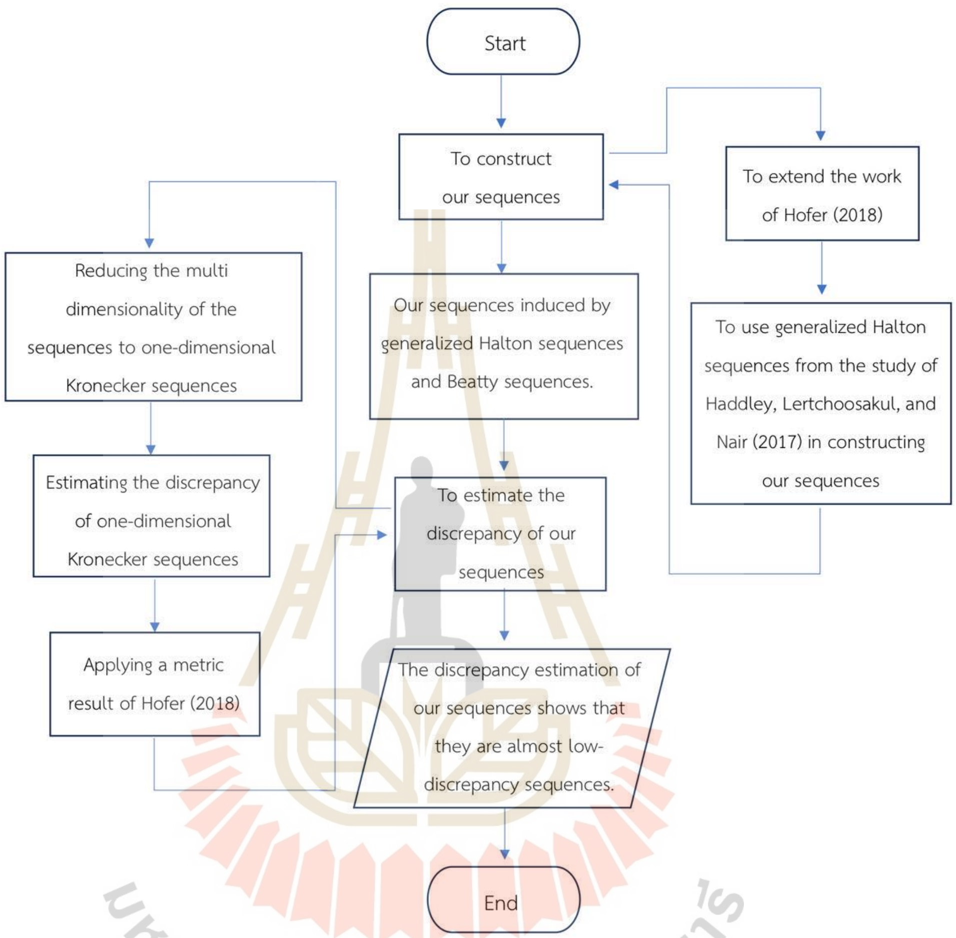

Figure 4.1 Flowchart of our work

CHAPTER V

RESULTS AND DISCUSSION

This chapter presents the main theorem's proof. We follow the idea of proof from Hofer (2018).

Theorem 5.1 (Main theorem) *Let* $\underline{b}_1 = (b_{1,j})_{j=1}^{\infty}, \ldots, \underline{b}_s = (b_{s,j})_{j=1}^{\infty}$ *be bounded sequences of nonnegative integers greater than* 1*, such that for all* $j_1, j_2 \in \mathbb{N}$ *and all* $1 \leq i_1 < i_2 \leq s$*, we have* b_{i_1,j_1} *and* b_{i_2,j_2} *are coprime. Then, for Lebesgue-almost all* $\alpha \in [0,1)$*, and for any* $\epsilon > 0$*, the sequence*

$$\omega = \left(\phi_{\underline{b}_1}(\lfloor n\alpha \rfloor), \ldots, \phi_{\underline{b}_s}(\lfloor n\alpha \rfloor)\right)_{n=0}^{\infty}$$

is an almost low-discrepancy sequence.

From now on, ω denotes our constructed sequence defined in the main theorem. We write $\beta = \frac{1}{\alpha}$, so that $\beta > 1$. Then the main theorem follows immediately from the following three auxiliary results.

The first lemma reduces the calculation of the discrepancy of a multi-dimensional sequence ω to the calculation of the discrepancy of one-dimensional Kronecker sequences.

Lemma 5.2 *For all* $N \in \mathbb{N}$*, we get this inequality*

$$ND_N^*(\omega) \leq M_1 \cdots M_s \sum_{K_1=0}^{k_1} \cdots \sum_{K_s=0}^{k_s} N\Delta_N^{(K_1,\ldots,K_s)} \left(\left(\left\{\frac{n}{\beta \Pi_{i=1}^s b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right) + s,$$

where

$$M_i = \max(b_{i,j})_{j=1}^{\infty},$$

$$m_i = \min(b_{i,j})_{j=1}^{\infty},$$

$$k_i = \left\lceil \log_{m_i} N \right\rceil,$$

$$b_{(i,K_i)} = b_{i,1} \cdots b_{i,K_i},$$

for all $i \in [1, s]$, *and*

$$N\Delta_N^{(K_1,\ldots,K_s)}(\omega) = \sup_{0 \le R < \prod_{i=1}^s b_{(i,K_i)}} \left| A\left(\left[\frac{R}{\prod_{i=1}^s b_{(i,K_i)}}, \frac{R+1}{\prod_{i=1}^s b_{(i,K_i)}} \right); N; \omega \right) - \frac{N}{\prod_{i=1}^s b_{(i,K_i)}} \right|.$$

Proof. For each $\ell = (\ell_1, \ldots, \ell_s) \in \mathbb{N}^s$ with $1 \le \ell_i \le b_{i,1} \cdots b_{i,k_i}$ for all $i \in [1, s]$, we can write each ℓ_i in form

$$\ell_i = l_{i,k_i} + l_{i,k_i-1} b_{i,k_i} + l_{i,k_i-2} b_{i,k_i} b_{i,k_i-1} + \cdots + l_{i,1} b_{i,k_i} \cdots b_{i,2},$$

where $l_{i,k_i-j} \in \{0, 1, \ldots, b_{i,k_i-j} - 1\}$ $(0 \le j \le k_i - 1)$. The first step in the proof is to estimate $A(J_\ell; N; \omega) - N\lambda_s(J_\ell)$, where $J_\ell \subseteq [0,1)^s$ is an interval of the form

$$J_\ell = \prod_{i=1}^s \left[0, \frac{\ell_i}{b_{(i,k_i)}} \right), \tag{5.1}$$

where $b_{(i,k_i)} = b_{i,1} \cdots b_{i,k_i}$, we can write J_ℓ as a disjoint union of the form

$$J_\ell = \prod_{i=1}^s \left(\bigcup_{K=1}^{k_i} \bigcup_{l=1}^{l_{i,K}} \left[\sum_{k=1}^{K-1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l-1}{b_{(i,K)}}, \sum_{k=1}^{K-1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l}{b_{(i,K)}} \right) \right),$$

with the term of disjoint intervals is bounded by $\prod_{i=1}^s (\sum_{k=1}^{k_i} l_{i,k}) \le \prod_{i=1}^s (M_i - 1) k_i$. The crucial step, we estimate $A(I_e; N; \omega) - N\lambda_s(I_e)$ for an elementary interval

$$I_e = \prod_{i=1}^s \left[\sum_{k=1}^{K_i-1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l_i - 1}{b_{(i,K_i)}}, \sum_{k=1}^{K_i-1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l_i}{b_{(i,K_i)}} \right),$$

where $1 \le K_i \le k_i$ and $1 \le l_i \le l_{i,K_i}$, by using properties of the generalized Halton sequence. For each nonnegative integer $n \in \mathbb{N}_0$, we denote the $\underline{b}_i$-adic expansion of $\lfloor \frac{n}{\beta} \rfloor$ by

$$\left\lfloor \frac{n}{\beta} \right\rfloor = n_{i,1} + n_{i,2} b_{i,1} + n_{i,3} b_{i,1} b_{i,2} + n_{i,4} b_{i,1} b_{i,2} b_{i,3} + \cdots,$$

where $n_{i,j} \in \{0, 1, \ldots, b_{i,j} - 1\}$ $(j \in \mathbb{N})$. Then

$$\left(\phi_{\underline{b}_1}\left(\left\lfloor \frac{n}{\beta} \right\rfloor \right), \ldots, \phi_{\underline{b}_s}\left(\left\lfloor \frac{n}{\beta} \right\rfloor \right) \right) \in \prod_{i=1}^s \left[\sum_{k=1}^{K_i-1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l_i - 1}{b_{(i,K_i)}}, \sum_{k=1}^{K_i-1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l_i}{b_{(i,K_i)}} \right),$$

if and only if, for all $i \in [1, s]$,

$$\frac{l_{i,1}}{b_{i,1}} + \cdots + \frac{l_{i,K_i-1}}{b_{(i,K_i-1)}} + \frac{l_i - 1}{b_{(i,K_i)}} \le \frac{n_{i,1}}{b_{i,1}} + \frac{n_{i,2}}{b_{i,1} b_{i,2}} + \cdots < \frac{l_{i,1}}{b_{i,1}} + \cdots + \frac{l_{i,K_i-1}}{b_{(i,K_i-1)}} + \frac{l_i}{b_{(i,K_i)}}.$$

This is equivalent to $n_{i,1} = l_{i,1}, \dots, n_{i,K_i-1} = l_{i,K_i-1}$, and $n_{i,K_i} = l_i - 1$ which in turn is equivalent to

$$\left\lfloor \frac{n}{\beta} \right\rfloor \equiv l_{i,1} + l_{i,2} b_{i,1} + \cdots + l_{i,K_i-1} b_{(i,K_i-2)} + (l_i - 1) b_{(i,K_i-1)} \pmod{b_{(i,K_i)}},$$

for all $i \in [1, s]$. As $b_{1,j_1}, \dots, b_{s,j_s}$ are pairwise coprime, it follows from the Chinese Remainder Theorem that there exists a unique integer R with $0 \le R < \prod_{i=1}^{s} b_{(i,K_i)}$ such that

$$\left\lfloor \frac{n}{\beta} \right\rfloor \equiv R \pmod{\prod_{i=1}^{s} b_{(i,K_i)}}.$$

Because

$$R \equiv \left\lfloor \frac{n}{\beta} \right\rfloor \equiv \left\lfloor \frac{n}{\beta} - \left(\prod_{i=1}^{s} b_{(i,K_i)} \right) \left\lfloor \frac{n}{\beta \prod_{i=1}^{s} b_{(i,K_i)}} \right\rfloor \right\rfloor \pmod{\prod_{i=1}^{s} b_{(i,K_i)}}, \tag{5.2}$$

we obtain the equivalence of the congruence (5.2) as

$$\left\{ \frac{n}{\beta \prod_{i=1}^{s} b_{(i,K_i)}} \right\} \in \left[\frac{R}{\prod_{i=1}^{s} b_{(i,K_i)}}, \frac{R+1}{\prod_{i=1}^{s} b_{(i,K_i)}} \right).$$

Therefore, we have

$$\begin{aligned}
&|A(J_\ell; N; \omega) - N\lambda_s(J_\ell)| \\
&= \left| \sum_{I_e} A(I_e; N; \omega) - N\lambda_s(I_e) \right| \\
&\le \sum_{I_e} |A(I_e; N; \omega) - N\lambda_s(I_e)| \\
&\le (M_1 - 1) \cdots (M_s - 1) \sum_{K_1=1}^{k_1} \cdots \sum_{K_s=1}^{k_s} N \Delta_N^{(K_1,\dots,K_s)} \left(\left(\left\{ \frac{n}{\beta \prod_{i=1}^{s} b_{(i,K_i)}} \right\} \right)_{n=0}^{\infty} \right).
\end{aligned}$$

An arbitrary interval $J = \prod_{i=1}^{s} [0, z_i) \subseteq [0,1)^s$ can be approximated by an interval J_ℓ of the form (5.1) by taking the nearest fraction to the left of z_i of the form $\ell_i / (b_{(i,k_i)})$. Then we have

$$\begin{aligned}
|A(J; N; \omega) - N\lambda_s(J)| &= |A(J_\ell; N; \omega) + A(J \setminus J_\ell; N; \omega) - N\lambda_s(J_\ell) - N\lambda_s(J \setminus J_\ell)| \\
&\le |A(J_\ell; N; \omega) - N\lambda_s(J_\ell)| + |A(J \setminus J_\ell; N; \omega) - N\lambda_s(J \setminus J_\ell)| \\
&\le |A(J_\ell; N; \omega) - N\lambda_s(J_\ell)| + \max(A(J \setminus J_\ell; N; \omega), N\lambda_s(J \setminus J_\ell)).
\end{aligned}$$

From the definition of $k_i = \lceil \log_{m_i} N \rceil$, we get $k_i \geq \log_{m_i} N$; that is, $m_i^{k_i} \geq N$. Hence, we have

$$
\begin{aligned}
N\lambda_s(J \setminus J_\ell) &\leq N \prod_{i=1}^{s} \left(z_i - \frac{\ell_i}{b_{(i,k_i)}} \right) \\
&\leq N \sum_{i=1}^{s} \frac{1}{b_{(i,k_i)}} \\
&\leq N \sum_{i=1}^{s} \frac{1}{m_i^{k_i}} \\
&\leq s.
\end{aligned}
$$

It remains to estimate $A(J \setminus J_\ell; N; \omega)$. Clearly, we have

$$
\begin{aligned}
A(J \setminus J_\ell; N; \omega) &\leq A(S; N; \omega) \\
&= |A(S; N; \omega) - N\lambda_s(S) + N\lambda_s(S)| \\
&\leq |A(S; N; \omega) - N\lambda_s(S)| + N\lambda_s(S),
\end{aligned}
$$

where S is an union of elementary intervals I_e of the form

$$
S = \bigcup_{i=1}^{s} \left(\prod_{j=1}^{i-1} [0,1) \times \left[\sum_{k=1}^{k_i - 1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l_{i,k_i}}{b_{(i,k_i)}}, \sum_{k=1}^{k_i - 1} \frac{l_{i,k}}{b_{(i,k)}} + \frac{l_{i,k_i} + 1}{b_{(i,k_i)}} \right) \times \prod_{j=i+1}^{s} [0,1) \right).
$$

By using again the definition of k_i, we have

$$
\begin{aligned}
N\lambda_s(S) &\leq N \sum_{i=1}^{s} \frac{1}{b_{(i,k_i)}} \\
&\leq N \sum_{i=1}^{s} \frac{1}{m_i^{k_i}} \\
&\leq s.
\end{aligned}
$$

Moreover, we get the estimation of $|A(S; N; \omega) - N\lambda_s(S)|$ as

$|A(S; N; \omega) - N\lambda_s(S)|$

$$
\begin{aligned}
&\leq N\Delta_N^{(k_1,0,\ldots,0)} \left(\left(\left\{ \frac{n}{\beta b_{(1,k_1)}} \right\} \right)_{n=0}^{\infty} \right) + N\Delta_N^{(0,k_2,\ldots,0)} \left(\left(\left\{ \frac{n}{\beta b_{(2,k_2)}} \right\} \right)_{n=0}^{\infty} \right) \\
&\quad + \cdots + N\Delta_N^{(0,\ldots,0,k_s)} \left(\left(\left\{ \frac{n}{\beta b_{(s,k_s)}} \right\} \right)_{n=0}^{\infty} \right).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&|A(J;N;\omega) - N\lambda_s(J)| \\
&\le |A(J_\ell;N;\omega) - N\lambda_s(J_\ell)| + \max(A(J\setminus J_\ell;N;\omega), N\lambda_s(J\setminus J_\ell)) \\
&\le |A(J_\ell;N;\omega) - N\lambda_s(J_\ell)| + |A(S;N;\omega) - N\lambda_s(S)| + N\lambda_s(S) \\
&\le (M_1-1)\cdots(M_s-1)\sum_{K_1=1}^{k_1}\cdots\sum_{K_s=1}^{k_s} N\Delta_N^{(K_1,\ldots,K_s)}\left(\left(\left\{\frac{n}{\beta\prod_{i=1}^s b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right) \\
&\quad + N\Delta_N^{(k_1,0,\ldots,0)}\left(\left(\left\{\frac{n}{\beta b_{(1,k_1)}}\right\}\right)_{n=0}^{\infty}\right) + N\Delta_N^{(0,k_2,\ldots,0)}\left(\left(\left\{\frac{n}{\beta b_{(2,k_2)}}\right\}\right)_{n=0}^{\infty}\right) \\
&\quad + \cdots + N\Delta_N^{(0,\ldots,0,k_s)}\left(\left(\left\{\frac{n}{\beta b_{(s,k_s)}}\right\}\right)_{n=0}^{\infty}\right) + s \\
&\le M_1\cdots M_s\sum_{K_1=0}^{k_1}\cdots\sum_{K_s=0}^{k_s} N\Delta_N^{(K_1,\ldots,K_s)}\left(\left(\left\{\frac{n}{\beta\prod_{i=1}^s b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right) + s.
\end{aligned}
$$

□

Next, the second lemma provides an estimation of the discrepancy of the one-dimensional Kronecker sequences.

Lemma 5.3 *For any* $(K_1,\ldots,K_s)\in\mathbb{N}_0^s$ *and* $\zeta\in[1,2]$, *we have*

$$N\Delta_N^{(K_1,\ldots,K_s)}\left(\left(\left\{\frac{n}{\beta\prod_{i=1}^s b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right) \le \beta+1+2\sum_{j=1}^{\lceil\log_\zeta N\rceil} a_j\left(\beta\prod_{i=1}^s b_{(i,K_i)}\right).$$

Proof. Let $1=q_0\le q_1<q_2<\cdots$ be the denominators of $\frac{p_i}{q_i}$ which converges to $\frac{1}{\beta\prod_{i=1}^s b_{(i,K_i)}}$ with $i\in\mathbb{N}_0$, $\gcd(p_i,q_i)=1$. Note that $\beta>1$. We have $\beta\prod_{i=1}^s b_{i,1}\cdots b_{i,K_i}>1$, and so, we get

$$a_1(1/(\beta\prod_{i=1}^s b_{i,1}\cdots b_{i,K_i})) = \lfloor\beta\prod_{i=1}^s b_{i,1}\cdots b_{i,K_i}\rfloor.$$

For given $N\ge1$, there exists $r\ge0$ such that $q_r\le N<q_{r+1}$. By the division algorithm, we have $N=N_rq_r+\bar{N}_{r-1}$ with $0\le\bar{N}_{r-1}<q_r$. By continued fractions, we note that

$$(a_{r+1}+1)q_r\ge q_{r+1}>N>N_rq_r,$$

and so, we have $N_r\le a_{r+1}$. If $r>0$, we write $\bar{N}_{r-1}=N_{r-1}q_{r-1}+\bar{N}_{r-2}$ with $0\le\bar{N}_{r-2}<q_{r-1}$. After that we find $N_{r-1}\le a_r$. We continue on this path, then we get

$$N=N_0+N_1q_1+N_2q_2+\cdots+N_rq_r,$$

with $q_r \le N < q_{r+1}$ and $0 \le N_i \le a_{i+1}(1/(\beta \prod_{i=1}^{s} b_{(i,K_i)}))$ $(0 \le i \le r)$. We start by splitting

$$N\Delta_N^{(K_1,\ldots,K_s)}\left(\left(\left\{\frac{n}{\beta \prod_{i=1}^{s} b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right)$$
$$\le N_0\Delta_{N_0}^{(K_1,\ldots,K_s)}\left(\left(\left\{\frac{n}{\beta \prod_{i=1}^{s} b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right)$$
$$+ (N-N_0)\Delta_{N-N_0}^{(K_1,\ldots,K_s)}\left(\left(\left\{\frac{n+N_0}{\beta \prod_{i=1}^{s} b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right).$$

For the first term, we use the definition of the discrepancy to get that

$$\begin{aligned} N_0\Delta_{N_0}^{(K_1,\ldots,K_s)}\left(\left(\left\{\frac{n}{\beta \prod_{i=1}^{s} b_{i,1}\cdots b_{i,K_i}}\right\}\right)_{n=0}^{\infty}\right) &\le \left|\lfloor\beta\rfloor + 1 - \frac{N_0}{\prod_{i=1}^{s} b_{i,1}\cdots b_{i,K_i}}\right| \\ &< \lfloor\beta\rfloor + 1 \\ &< \beta + 1, \end{aligned}$$

because $N_0 \le \lfloor \beta \prod_{i=1}^{s} b_{(i,K_i)} \rfloor$. For the second term, we proceed as in Theorem 4.5. We decompose the given sequence

$$\left(\left(\left\{\frac{n+N_0}{\beta \prod_{i=1}^{s} b_{(i,K_i)}}\right\}\right)_{n=0}^{n=N-N_0-1}\right) \tag{5.3}$$

into N_1 sequences, where n runs through q_1 consecutive integers, and so on, into N_r sequences, where n runs through q_r consecutive integers. We would like to estimate the first discrepancy q_i terms of such a finite sequence of the given sequence in (5.3) by assuming $n = n_0 + j$ with $1 \le j \le q_i$. By using Theorem 4.3 to approximate $\frac{1}{\beta \prod_{i=1}^{s} b_{(i,K_i)}}$, we get

$$\left|\frac{1}{\beta \prod_{i=1}^{s} b_{(i,K_i)}} - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}},$$

and we obtain

$$\frac{1}{\beta \prod_{i=1}^{s} b_{(i,K_i)}} = \frac{p_i}{q_i} + \frac{\theta}{q_i q_{i+1}} \quad \text{with} \quad |\theta| < 1.$$

Therefore, we have

$$\left\{\frac{n+N_0}{\beta \prod_{i=1}^{s} b_{(i,K_i)}}\right\} = \left\{\frac{N_0}{\beta \prod_{i=1}^{s} b_{(i,K_i)}} + \frac{jp_i}{q_i} + \frac{j\theta}{q_i q_{i+1}}\right\}.$$

To estimate the second term, we use Theorem 4.4 and Theorem 4.5 to obtain

$$
\begin{aligned}
&(N-N_0)\Delta_{N-N_0}^{(K_1,\dots,K_s)}\left(\left(\left\{\frac{n+N_0}{\beta\prod_{i=1}^{s}b_{(i,K_i)}}\right\}\right)_{n=0}^{\infty}\right)\\
&\le \sum_{i=1}^{r} N_i q_i\left(\frac{2}{q_i}-\left(\frac{1}{q_i}-\frac{1}{q_{i+1}}\right)\right)\\
&\le \sum_{i=1}^{r} N_i q_i\left(\frac{1}{q_i}+\frac{1}{q_{i+1}}\right)\\
&\le \sum_{i=1}^{r} N_i q_i\left(\frac{2}{q_i}\right)\\
&\le \sum_{i=1}^{r} a_{i+1}\left(\frac{1}{\beta\prod_{i=1}^{s}b_{(i,K_i)}}\right)q_i\left(\frac{2}{q_i}\right)\\
&= 2\sum_{i=1}^{r} a_{i+1}\left(\frac{1}{\beta\prod_{i=1}^{s}b_{(i,K_i)}}\right)\\
&\le 2\sum_{i=1}^{\lceil\log_\zeta N\rceil+1} a_{i+1}\left(\frac{1}{\beta\prod_{i=1}^{s}b_{(i,K_i)}}\right)\\
&= 2\sum_{j=1}^{\lceil\log_\zeta N\rceil} a_j\left(\beta\prod_{i=1}^{s}b_{(i,K_i)}\right),
\end{aligned}
$$

we estimate r by the fact that $N \ge q_r \ge \zeta^{r-1}$ for any $r \ge 0$. For any $a_i \ge 1$, we get $\zeta \in [1,2]$. □

Finally, we apply Lemma 4.7, and set the following the variable

$$L = \max\{k_1,\dots,k_s,\lceil\log_\zeta N\rceil\},$$

where K_i is a digit with $1 \le K_i \le k_i$ for all $i \in [1,s]$, and j is a digit with $1 \le j \le \lceil\log_\zeta N\rceil$. Then, we get the last lemma

Lemma 5.4 *Let $L \in \mathbb{N}$. Then, for Lebesgue-almost all $\beta \in \mathbb{R}$. For every $\epsilon > 0$, we have*

$$\sum_{K_1=0}^{L}\cdots\sum_{K_s=0}^{L}\sum_{j=1}^{L} a_j\left(\beta\prod_{i=1}^{s}b_{(i,K_i)}\right) = O\left(L^{s+1+\epsilon}\right).$$

Therefore, by Lemma 5.4, we can conclude that the sequence ω induced by generalized Halton sequneces and Beatty sequences is an almost low-discrepancy sequence.

CHAPTER VI

CONCLUSION AND RECOMMENDATION

In this research, we apply the idea of Hofer (2018) to estimate the discrepancy of our constructed sequence ω, induced by Beatty sequences and generalized Halton sequences. We have shown that the sequence ω is an almost low-discrepancy sequence. To prove this, we follow a similar approach as Hofer (2018), but we change the bases of Halton sequences to bases of generalized Halton sequences. Then, we reduce a multi-dimensional sequence ω to one-dimensional Kronecker sequences. To estimate the discrepancy of one-dimensional Kronecker sequences, we use continued fractions and adopt the theorem's idea for estimating the discrepancy of Kronecker sequences to establish bounds. Then, we apply our result in dynamical systems to estimate the discrepancy. In conclusion, we have shown that the discrepancy of our constructed sequence ω is bounded above by the following rate of magnitude

$$D_N^*(\omega) = O\left(\frac{(\log N)^{s+\epsilon+1}}{N}\right)$$

for every $\epsilon > 0$.

For further study, it would be interesting to explore the combination of Beatty sequences with other low-discrepancy sequences such as Sobol sequences, Faure sequences, and Hammersley sequences. Another recommendation is to investigate the substitution of subsequences in Beatty sequences with prime number terms, and then determine the discrepancy of these newly generated sequences.

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APPENDICES

APPENDIX A

SOME RELATED THEOREMS

This section presents some theorems, as mentioned in earlier chapters. It consists of the b-adic expansion, the $\underline{b}$-adic expansion, and the relationship between discrepancy and uniformly distributed modulo 1.

A.1 The b-adic expansion

Theorem 1.1 (The b-adic expansion) *Let $b > 1$ be an integer. Then, every nonnegative integer n has the unique b-adic representation of the form*

$$n = \sum_{j=1}^{\infty} n_j b^{j-1} = n_1 + n_2 b + n_3 b^2 + n_4 b^3 + \cdots ,$$

where $n_j \in \{0, 1, \ldots, b-1\}$.

Proof. We use the method of successive division by b. We start with N and repeatedly divide by b, keeping track of the remainders at each step. Firstly, we divide N by b and get a quotient q_1 and a remainder n_1

$$N = bq_1 + n_1, \quad \text{where} \quad 0 \le n_1 < b. \tag{A.1}$$

Next, we divide q_1 by b and get a quotient q_2 and a remainder n_2

$$q_1 = bq_2 + n_2, \quad \text{where} \quad 0 \le n_2 < b.$$

Substituting q_1 in the equation (A.1), we get

$$N = n_1 + b(bq_2 + n_2) = n_1 + n_2 b + q_2 b^2.$$

We can continue this process, dividing each quotient by b and getting a new remainder, until we get

$$N = n_1 + n_2 b + n_3 b^2 + \cdots ,$$

where n_i with $i \in \mathbb{N}$ is the remainder obtained in the successive divisions.

$\square$

A.2 The $\underline{b}$-adic expansion

Theorem 1.2 (The $\underline{b}$-adic expansion) *Let $\underline{b} = (b_j)_{j=1}^{\infty}$ be a sequence of nonnegative integers greater than 1. Then, every nonnegative integer n has the unique $\underline{b}$-adic representation of the form*

$$n = \sum_{j=1}^{\infty} n_j b_1 \cdots b_{j-1} = n_1 + n_2 b_1 + n_3 b_1 b_2 + n_4 b_1 b_2 b_3 + \cdots , \tag{A.2}$$

where $n_j \in \{0, 1, \ldots, b_j - 1\}$.

Proof. We use the method of successive division by b_1. We start with N and repeatedly divide by b_2, b_3, and so on, keeping track of the remainders at each step. Firstly, we divide N by b_1 and get a quotient q_1 and a remainder n_1

$$N = b_1 q_1 + n_1, \quad \text{where} \quad 0 \le q_1 < b_1. \tag{A.3}$$

Next, we divide q_1 by b_2 and get a quotient q_2 and a remainder n_2

$$q_1 = b_2 q_2 + n_2, \quad \text{where} \quad 0 \le n_2 < b_2.$$

Substituting q_1 in the equation (A.3), we get

$$N = n_1 + b_1(b_2 q_2 + n_2) = n_1 + n_2 b_1 + q_2 b_1 b_2.$$

We can continue this process, dividing each quotient by $b_3, b_4, \ldots$ and getting a new remainder, until we get

$$N = n_1 + n_2 b_1 + n_2 b_1 b_2 + \cdots ,$$

where n_i with $i \in \mathbb{N}$ is the remainder obtained in the successive divisions.

$\square$

A.3 The relationship between discrepancy and uniformly distributed modulo 1

Definition 1.1 (Uniformly distributed modulo 1) A sequence $\omega = (\{x_n\})_{n=0}^{\infty}$ of real numbers is uniformly distributed modulo 1 if for every pair a, b of real numbers with

$0 \leq a < b \leq 1$, we have

$$\lim_{N\to\infty} \frac{A([a,b);N;\omega)}{N} = b - a.$$

Theorem 1.3 (The relationship between discrepancy and uniformly distributed modulo 1) *A sequence* $\omega = (\{x_n\})_{n=0}^{\infty}$ *of real numbers is uniformly distributed modulo 1 if and only if* $\lim_{N\to\infty} D_N(\omega) = 0$.

Proof. Let m be an integer greater than or equal to 2, and $0 \leq k \leq m-1$. Let I_k denote the interval $I_k = \left[\frac{k}{m}, \frac{k+1}{m}\right)$. We assume that ω is uniformly distributed modulo 1. For every $\epsilon > 0$ there exists a positive integer N_0 dependent on m, such that for $N \geq N_0$ and for every $k = 0, 1, \ldots, m-1$, we have

$$-\frac{\epsilon}{2} < \frac{A(I_k;N;\omega)}{N} - \frac{1}{m} < \frac{\epsilon}{2}.$$

We consider an arbitrary interval $J = [\alpha, \beta) \subseteq [0,1)$, there exist intervals J_1 and J_2 are finite unions of intervals I_k, such that $J_1 \subseteq J \subseteq J_2$, with $\lambda(J) - \lambda(J_1) < \frac{2}{m}$ and $\lambda(J_2) - \lambda(J) < \frac{2}{m}$. We get for all $N \geq N_0$

$$\frac{A(J_1;N;\omega)}{N} - \lambda(J) < \frac{A(J;N;\omega)}{N} - \lambda(J) < \frac{A(J_2;N;\omega)}{N} - \lambda(J),$$

$$\lambda(J_1) - \frac{\epsilon}{2} - \lambda(J) < \frac{A(J;N;\omega)}{N} - \lambda(J) < \lambda(J_2) + \frac{\epsilon}{2} - \lambda(J).$$

By using the relationship of $\lambda(J_1), \lambda(J)$, and $\lambda(J_2)$, we get

$$-\frac{\epsilon}{2} - \frac{2}{m} < \frac{A(J;N;\omega)}{N} - \lambda(J) < \frac{\epsilon}{2} + \frac{2}{m} \quad \text{for all} \quad N \geq N_0.$$

Since the bounds are independent of interval J, we finally get $D_N(\omega) \leq \frac{\epsilon}{2} < \epsilon$ for all $N \geq N_0$.

Now, suppose that $\lim_{N\to\infty} D_N(\omega) = 0$. Suppose to the contrary that ω is not uniformly distributed modulo 1. Then, there exist a subinterval $[a,b]$ of $[0,1]$ and a positive constant ϵ such that

$$\left| \frac{A([a,b];N;\omega)}{N} - (b-a) \right| \geq \epsilon.$$

By the definition of discrepancy, we have

$$D_N(\omega) \geq \left| \frac{A([a,b];N;\omega)}{N} - (b-a) \right| \quad \text{for all} \quad N.$$

We have $D_N(\omega) \geq \epsilon$ for all N, which contradicts $\lim_{N\to\infty} D_N(\omega) = 0$. Therefore, ω must be uniformly distributed mod 1.

$\square$

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