Lecture Notes
in
Measure Theory
(103 622)

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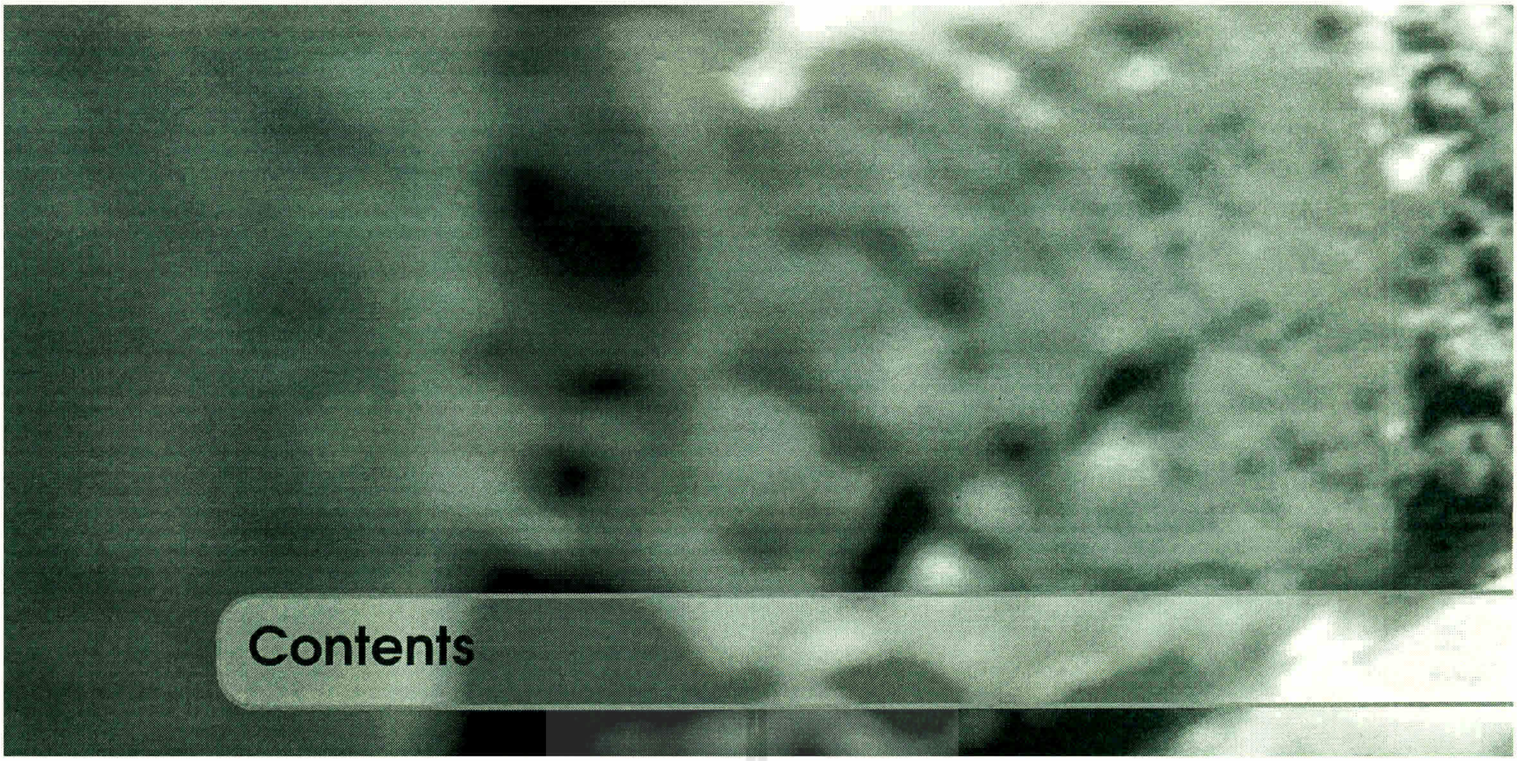

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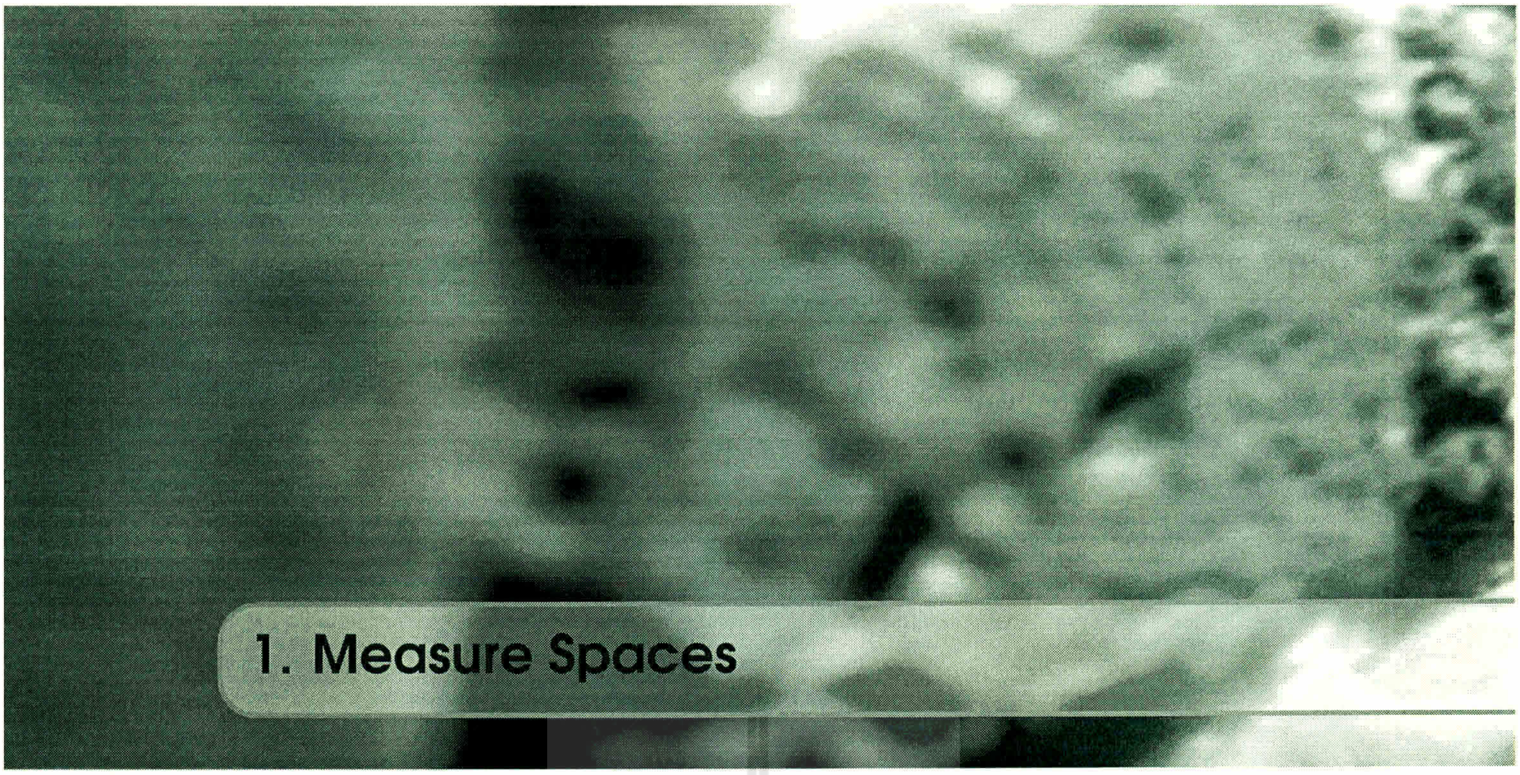


1. Measure Spaces

In this chapter we introduce the notion of measure of a set and discuss some of it properties. Measuring the size of a set is not really a new concept: we already have studied this idea in the case of Euclidean spaces $\mathbb{R}^n$. In one dimension, it is the length of an interval, in two dimensions it is the area of a bounded set while in three dimensions, it is the volume of a bounded set. Recall that not every bounded subset of the plane can be assigned an area: its boundary has to be sufficiently "nice". Thus when generalizing the concept of area or volume to arbitrary spaces, we first must introduce the class of sets to which we will assign such a measure; this leads to the concept of a σ-algebra.

Preliminaries

Let us first review and clarify some concepts and notations used throughout.

Given an arbitrary set Ω, we denote the collection of all subsets of Ω by $\mathscr{P}(\Omega)$ or 2^{Ω} and call it the *power set* of Ω.

Any collection $\mathscr{A}$ of sets can be indexed as $\mathscr{A} = \{A_\lambda\}_{\lambda\in\Lambda}$. Thus, we may denote the intersection of all set in $\mathscr{A}$ by

$$\bigcap_{\lambda\in\Lambda} A_\lambda$$

for convenience, and we can treat the union of all sets in $\mathscr{A}$ in a similar way.

A set E is called *countable* if there exists a surjection $f : \mathbb{N} \to E$. Thus, countable sets may be both, finite or infinite. In the latter case we will call E *countably infinite* or *denumerable*.

A *topology* on Ω is a collection $\tau \subseteq \mathscr{P}(\Omega)$ satisfying

1. $\emptyset \in \tau$ and $\Omega \in \tau$,
2. for every collection $\{U_\alpha\}_{\alpha\in A} \subseteq \tau$ we have $\bigcup_{\alpha\in A} U_\alpha \in \tau$,
3. for every finite collection $\{U_i\}_{i=1}^{n} \subseteq \tau$ we have $\bigcap_{i=1}^{n} U_\alpha \in \tau$.

The elements in τ are called *closed* sets, and $F \subseteq \Omega$ is called a *closed set* if F^c is open.

1.1 Algebras and Sigma-Algebras

Definition 1.1.1 Let Ω be an arbitrary set, and $\mathscr{F}$ a non-empty collection of subsets of Ω (i.e. $\mathscr{F} \subseteq \mathscr{P}(\Omega)$). Then

1. $\mathscr{F}$ is called an *algebra of subsets of* Ω (or an *algebra on* Ω) provided the following hold:
   (A1) Whenever $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$,
   (A2) Whenever $A_1, A_2, \ldots, A_N \in \mathscr{F}$, then $\bigcup_{n=1}^{N} A_n \in \mathscr{F}$.
   "$\mathscr{F}$ is closed under formation of complements and finite unions"
2. $\mathscr{F}$ is called a σ-*algebra of subsets of* Ω (or a σ-*algebra on* Ω) provided the following hold:
   (A1) Whenever $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$,
   (A2σ) Whenever $A_1, A_2, \ldots, A_n, \ldots \in \mathscr{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$.
   "$\mathscr{F}$ is closed under formation of complements and countable unions"

R Clearly, every σ-algebra $\mathscr{F}$ on Ω is also an algebra on Ω. Conversely, Example 1.1 shows that not every algebra is also a σ-algebra.

R From the above definitions, the following additional properties of algebras and σ-algebras follow immediately:

1. Let $\mathscr{F}$ be an algebra on Ω. Then
   (A3) Whenever $A_1, A_2, \ldots, A_N \in \mathscr{F}$, then $\bigcap_{n=1}^{N} A_n \in \mathscr{F}$.
   "$\mathscr{F}$ is closed under finite intersections"
   Proof. This follows from the fact that
   $$\bigcap_{n=1}^{N} A_n = \left[\bigcup_{n=1}^{N} A_n^c\right]^c,$$
   together with properties (A1) and (A2). ■
   (A4) Whenever $A, B \in \mathscr{F}$, then $A \backslash B \in \mathscr{F}$.
   "$\mathscr{F}$ is closed under formation of differences"
   Proof. This follows from the fact that
   $$A \backslash B = A \bigcap B^c,$$
   together with properties (A1) and (A3). ■
   (A5) $\emptyset \in \mathscr{F}$ and $\Omega \in \mathscr{F}$.
   Proof. By (A1) and (A2) above we have $\Omega = A \bigcup A^c \in \mathscr{F}$. It follows immediately that $\emptyset = \Omega^c \in \mathscr{F}$. ■
2. Let $\mathscr{F}$ be a σ-algebra on Ω. Then
   (A3σ) Whenever $A_1, A_2, \ldots, A_n, \ldots \in \mathscr{F}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathscr{F}$.
   "$\mathscr{F}$ is closed under countable intersections"
   Proof. This follows from the fact that
   $$\bigcap_{n=1}^{\infty} A_n = \left[\bigcup_{n=1}^{\infty} A_n^c\right]^c,$$
   together with properties (A1) and (A2σ). ■

Example 1.1 Let Ω be any set, and y $\mathscr{P}(\Omega)$ its power set.

1. $\mathscr{F}_s = \{\emptyset, \Omega\}$ is a σ-algebra. In fact, it is the smallest σ-algebra (and also the smallest algebra) on Ω.
2. $\mathscr{F}_l = \mathscr{P}(\Omega)$ is a σ-algebra. In fact, it is the largest σ-algebra (and also the largest algebra) on Ω.
3. Fix any $E \subseteq \Omega$. Then $\mathscr{F}_E = \{\emptyset, E, E^c, \Omega\}$ is a σ-algebra. In fact, it is the smallest σ-algebra (and also the smallest algebra) on Ω containing E.
4. Suppose that Ω is *infinite*. Then

$$\mathscr{F}_1 := \{E \subseteq \Omega : E \text{ is finite}\}$$

is *not* an algebra as (A1) does not hold. However,

$$\mathscr{F}_2 := \{E \subseteq \Omega : E \text{ is finite, or } E^c \text{ is finite}\}$$

is an algebra, as one easily verifies. Clearly, $\mathscr{F}_2$ is the smallest algebra on Ω containg all finite subsets of Ω.
On the other hand, $\mathscr{F}_2$ is *not* a σ-algebra. In fact, let $\{x_1, x_2, x_3, \dots\}$ be a countable subset of Ω. Set $E = \bigcup\limits_{k=1}^{\infty} \{x_{2k}\} = \{x_2, x_4, x_6, \dots\}$. Now each singleton $\{x_{2k}\}$ is in $\mathscr{F}_2$, while E and E^c are both infinite sets, an hence, $E \notin \mathscr{F}_2$. Thus, (A2σ) does not hold.
5. On the other hand,

$$\mathscr{F}_3 := \{F \subseteq \Omega : F \text{ is countable, or } F^c \text{ is countable}\}$$

is a σ-algebra for *any* Ω, as one easily checks. Clearly, $\mathscr{F}_3$ is the smallest algebra on Ω containg all countable subsets of Ω.
6. Let Ω be *infinite*, and $\{E_n\}_{n=1}^{\infty}$ be a countable family of pairwise disjoint subsets of Ω whose union is Ω. Set

$$\mathscr{F}_4 : = \{E \subseteq \Omega : E \text{ is the union of } \textit{some} \text{ of the sets } E_n\} = \left\{ \bigcup_{n \in S} E_n : S \subseteq \mathbb{N} \right\}.$$

It is left as an exercise to verify that $\mathscr{F}_4$ is a σ-algebra on Ω.

■

The next two exercises may be taken as alternative definitions of algebras and σ-algebras.

Exercise 1.1 Let Ω be a set, and let $\mathscr{F}$ be a non-empty collection of subsets of Ω satisfying

1. Whenever $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$,
2. Whenever $A, B \in \mathscr{F}$, then $A \bigcap B \in \mathscr{F}$.

Show that $\mathscr{F}$ is an algebra on Ω. ■

Exercise 1.2 Let Ω be a set, and let $\mathscr{F}$ be a non-empty collection of subsets of Ω. Suppose, $\mathscr{F}$ satisfies:

1. Whenever $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$,
2. Whenever $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$, then $\bigcap\limits_{n=1}^{\infty} A_n \in \mathscr{F}$.

Show that $\mathscr{F}$ is a σ-algebra on Ω. ■

We will often make use of the next theorem which allows us to replace any finite or countably infinite collection of sets in $\mathscr{F}$ with a collection of disjoint sets in $\mathscr{F}$.

Theorem 1.1.1 Let $\mathscr{F}$ be an algebra on Ω, and $\{A_n\}_{n=1}^{\infty}$ a countably infinite family of sets in $\mathscr{F}$. Then there exists a family $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathscr{F}$ satisfying

1. $B_n \subseteq A_n \ \forall n$, and
2. $\displaystyle\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i \quad \forall n.$

Furthermore, $\displaystyle\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$

Proof. We construct the sets B_n inductively. First, set $B_1 = A_1$. Then the assertion is true for $n = 1$. In general, suppose we have constructed pairwise disjoint sets $B_1, \ldots, B_n \in \mathscr{F}$ satisfying $B_i \subseteq A_i$ for all $i = 1, \ldots, n$ and

$$\bigcup_{i=1}^{n} B_n = \bigcup_{i=1}^{n} A_n. \tag{1.1}$$

Set $B_{n+1} = A_{n+1} \setminus \bigcup_{i=1}^{n} B_i \in \mathscr{F}$. Then by construction, the sets $B_1, \ldots, B_{n+1}$ are pairwise disjoint, and

$$\begin{aligned}\bigcup_{i=1}^{n+1} B_i &= B_{n+1} \bigcup \left(\bigcup_{i=1}^{n} B_i\right) = \left(A_{n+1} \setminus \bigcup_{i=1}^{n} B_i\right) \bigcup \left(\bigcup_{i=1}^{n} B_i\right) \\ &= A_{n+1} \bigcup \left(\bigcup_{i=1}^{n} B_i\right) \stackrel{(1.1)}{=} A_{n+1} \bigcup \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n+1} A_i.\end{aligned}$$

By induction, we thus obtain a family $\{B_n\}_{n=1}^{\infty}$ of pairwise disjoint sets, with $B_n \in \mathscr{F}$ for all n, so that 1. and 2. hold for all n.

Finally, by 2. we have

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{n=1}^{\infty} A_n.$$

This completes the proof. ■

R The above proof also shows that every *finite* collections $A_1, A_2, \ldots A_N \in \mathscr{F}$ can be modified to a pairwise *disjoint* collection $B_1, B_2, \ldots B_N \in \mathscr{F}$ so that 1. and 2. hold for all n, $1 \leq n \leq N$. We simply stop in the induction step when $n = N$.

Proposition 1.1.2 Let Λ be an index set, and for each $\lambda \in \Lambda$, let $\mathscr{F}_\lambda$ be a σ-algebra on Ω. Then

$$\bigcap_{\lambda \in \Lambda} \mathscr{F}_\lambda$$

is again a σ-algebra on Ω.

Proof. We need to show that properties (A1) and (A2σ) hold for $\mathscr{F} = \bigcap_{\lambda\in\Lambda} \mathscr{F}_\lambda$.

1. Let $A \in \bigcap\limits_{\lambda\in\Lambda} \mathscr{F}_\lambda$. Then $A \in \mathscr{F}_\lambda$ for all λ. Since each $\mathscr{F}_\lambda$ is a σ-algebra, then $A^c \in \mathscr{F}_\lambda$, for all λ, and hence

$$A^c \in \bigcap_{\lambda\in\Lambda} \mathscr{F}_\lambda.$$

2. Let $\{A_n\}_{n=1}^\infty \subseteq \bigcap\limits_{\lambda\in\Lambda} \mathscr{F}_\lambda$. Then $\{A_n\}_{n=1}^\infty \subseteq \mathscr{F}_\lambda$ for all λ. Since each $\mathscr{F}_\lambda$ is a σ-algebra, then $\bigcup\limits_{n=1}^\infty A_n \in \mathscr{F}_\lambda$, for all λ, and hence

$$\bigcup_{n=1}^\infty A_n \in \bigcap_{\lambda\in\Lambda} \mathscr{F}_\lambda.$$

We have shown that (A1) and (A2σ) both hold for $\mathscr{F}$ which hence is a σ-algebra. ■

Definition 1.1.2 Given a collection $\mathscr{K}$ of subsets of Ω (i.e. $\mathscr{K} \subseteq \mathscr{P}(\Omega)$), let $\{\mathscr{F}_\lambda\}_{\lambda\in\Lambda}$ denote the collection of *all* σ-algebras on Ω containing $\mathscr{K}$. That is,

1. Each $\mathscr{F}_\lambda$ is a σ-algebra on Ω,
2. $\mathscr{K} \subseteq \mathscr{F}_\lambda$ for all $\lambda \in \Lambda$,
3. If $\mathscr{F}$ is a σ-algebra on Ω with $\mathscr{K} \subseteq \mathscr{F}$, then $\exists\lambda \in \Lambda$ with $\mathscr{F} = \mathscr{F}_\lambda$.

Note that the collection $\{\mathscr{F}_\lambda\}_{\lambda\in\Lambda}$ is not empty, as $\mathscr{K} \subseteq \mathscr{P}(\Omega)$ and $\mathscr{P}(\Omega)$ is itself a σ-algebra. We set

$$\mathscr{F}_o := \bigcap_{\lambda\in\Lambda} \mathscr{F}_\lambda.$$

Then $\mathscr{K} \subseteq \mathscr{F}_o$, so that by Proposition 1.1.2, $\mathscr{F}_o$ is itself a σ-algebra containing $\mathscr{K}$. Furthermore, if $\mathscr{F}$ is *any* σ-algebra containing $\mathscr{K}$, then $\mathscr{F} = \mathscr{F}_\lambda$ for some $\lambda \in \Lambda$, so that $\mathscr{F}_o \subseteq \mathscr{F}$.

Thus $\mathscr{F}_o$ is the *smallest* σ-algebra containing $\mathscr{K}$, called the *σ-algebra generated by $\mathscr{K}$*, and denoted by $\sigma(\mathscr{K})$.

■ **Example 1.2** 1. Let Ω be any set. If $E \subseteq \Omega$, then

$$\sigma(\{E\}) = \{\emptyset, E, E^c, \Omega\}$$

(which is the σ-algebra $\mathscr{F}_E$ of Example 1.1).

2. Let Ω be any (possibly uncountable) set, and $\mathscr{A} = \{\{x\} : x \in \Omega\}$, the collection of all one-element subsets of Ω (*"singletons"*). Then

$$\mathscr{A} \subseteq \{F \subseteq \Omega : F \text{ is countable or } F^c \text{ is countable}\} \subseteq \sigma(\mathscr{A})$$

Since the set in the middle is nothing else but the σ-algebra $\mathscr{F}_3$ of Example 1.1, it follows that

$$\sigma(\mathscr{A}) = \mathscr{F}_3 = \{F \subseteq \Omega : F \text{ is countable or } F^c \text{ is countable}\}.$$

3. Let Ω be an infinite set, and $\mathscr{K} = \{E_n\}_{n=1}^\infty$ be a countable family of pairwise disjoint subsets of Ω whose union is Ω. Then $\sigma(\mathscr{K})$ is the σ-algebra $\mathscr{F}_4$ of Example 1.1 (The proof is left as an easy exercise).

■

(R) Let $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(\Omega)$ with $\mathscr{A} \subseteq \mathscr{B}$. Then $\mathscr{A} \subseteq \mathscr{B} \subseteq \sigma(\mathscr{B})$. Since $\sigma(\mathscr{A})$ is the smallest σ-algebra containing $\mathscr{A}$, it follows that $\sigma(\mathscr{A}) \subseteq \sigma(\mathscr{B})$.

E.g. Let Ω be any set, and let

$$\mathscr{A} = \{\{x\} : x \in \Omega\} \qquad \text{and} \qquad \mathscr{B} = \{E \subseteq \Omega : E \text{ is finite }\}.$$

Since $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{F}_3$ (the σ-algebra of Example 1.1), it follows from Example 1.2 that

$$\mathscr{F}_3 = \sigma(\mathscr{A}) \subseteq \sigma(\mathscr{B}) \subseteq \mathscr{F}_3,$$

and hence $\mathscr{F}_3 = \sigma(\mathscr{B})$ as well.

1.2 Borel Sigma-Algebras

Definition 1.2.1 Let Ω be a metric space (or more generally, a topological space), and let

$$\tau := \{ U \subseteq \Omega : U \text{ is open} \}$$

denote the collection of open sets. Then $\sigma(\tau)$, the σ-algebra generated by the open sets, is called the *Borel σ-algebra on Ω*, and is denoted by $\mathscr{B}(\Omega)$. The elements of $\mathscr{B}(\Omega)$ are called *Borel sets*.

(R) It is in general not possible to describe all Borel sets. However, the following subsets of Ω are always Borel sets:

1. If U is an *open subset* of Ω, then by definition, $U \in \sigma(\tau)$.
2. If F is a *closed subset* of Ω, then F^c is open, and hence by (A1), $F = (F^c)^c \in \sigma(\tau)$.
3. A set of the form $M = \bigcap_{i=1}^{\infty} G_i$, with G_i open for all i, is called a G_δ set. Note that a G_δ set need not be open. By (A3σ), *every G_δ subset of Ω is a Borel set.*
4. Similarly, a set of the form $M = \bigcup_{i=1}^{\infty} F_i$, with F_i closed for all i, is called an F_σ set. Note that an F_σ set need not be closed. By (A2σ), *every F_σ subset of Ω is a Borel set.*
5. If *in addition, Ω is a T_1 space* (this is always true for metric spaces), then singletons $\{x\}$ are closed, and thus they are Borel sets. It follows from (A2σ) that *all countable subsets of Ω are Borel sets.*

Next we want to study generating sets for the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ on the real line. The main tool will be Lindelöff's Theorem.

(R) Let $\mathscr{I}_0$ be the collection of all non-empty open intervals with rational endpoints,

$$\mathscr{I}_0 := \{J_{r,s} = (r,s) \; : \; r < s, \; r,s \in \mathbb{Q}\}.$$

The map

$$J_{r,s} \in \mathscr{I}_0 \mapsto (r,s) \in \mathbb{Q} \times \mathbb{Q}$$

clearly is injective. Since $\mathbb{Q} \times \mathbb{Q}$ is a countable set, it follows that $\mathscr{I}_0$ must be countable as well: There exist only countably many distinct open intervals with rational endpoints.

Lemma 1.2.1 (Lindelöff's Theorem for the real line)
Every non-empty open set $U \subseteq \mathbb{R}$ is the countable union of bounded, open intervals with rational endpoints. That is,

$$U = \bigcup_{n=1}^{N} (r_n, s_n) \qquad N \in \mathbb{N} \cup \{\infty\},\ r_n, s_n \in \mathbb{Q}.$$

Proof. Let $U \subseteq \mathbb{R}$ be open, $U \neq \emptyset$. Thus, for each $x \in U$, there exists $\varepsilon = \varepsilon_x > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \subseteq U.$$

Now by density of the rational numbers in $\mathbb{R}$, there exist $r = r_x, s = s_x \in \mathbb{Q}$ so that

$$x - \varepsilon < r < x < s < x + \varepsilon.$$

Then

$$J_x := (r,s) \subset (x-\varepsilon, x+\varepsilon) \subseteq U.$$

That is, each J_x is a bounded, open interval with rational endpoints (i.e. $J_x \in \mathscr{I}_0$), and $x \in J_x \subseteq U$. Now

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} J_x \subseteq U$$

shows that

$$U = \bigcup_{x \in U} J_x. \tag{1.2}$$

However, by the previous Remark only countably many of the intervals J_x are distinct, and we can list the distinct interval as $\{J_n\}_{n=1}^N$, with $N \in \mathbb{N}$ or $N = \infty$. Thus, the union in (1.2) is really a union of the intervals $\{J_n\}_{n=1}^N$,

$$U = \bigcup_{n=1}^{N} J_n,$$

which proves the lemma. ∎

Next we show that the Borel σ-algebra on $\mathbb{R}$ is generated by the collection of open intervals with rational endpoints.

Theorem 1.2.2 $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{I}_0)$, where

$$\mathscr{I}_0 := \{J_{r,s} = (r,s) \ :\ r < s,\ r,s \in \mathbb{Q}\}.$$

Proof. Let

$$\tau := \{\, U \subseteq \mathbb{R} \ :\ U \text{ is open} \,\}.$$

Clearly, $\mathscr{I}_0 \subset \tau$, and hence $\sigma(\mathscr{I}_0) \subseteq \sigma(\tau) = \mathscr{B}(\mathbb{R})$.

To prove the reverse inclusion, let $U \in \tau$ be arbitrary. By Lindelöff's Theorem, we can write

$$U = \bigcup_{n=1}^{N} J_n, \qquad n \in \mathbb{N} \cup \{\infty\}, \quad J_n \in \mathscr{I}_0.$$

Since $J_n \in \mathscr{I}_0 \subseteq \sigma(\mathscr{I}_0)$ for all n, it now follows from (A2) or (A2σ) that $U \in \sigma(\mathscr{I}_0)$. As $U \in \tau$ was arbitrary, we conclude that

$$\tau \subseteq \sigma(\mathscr{I}_0)$$

Now $\mathscr{B}(\mathbb{R})$ is the smallest σ-algebra containing τ; hence

$$\mathscr{B}(\mathbb{R}) \subseteq \sigma(\mathscr{I}_0).$$

Thus the theorem is proved ■

Exercise 1.3 (Additional generators of $\mathscr{B}(\mathbb{R})$.)

1. Show that $\mathscr{B}(\mathbb{R})$ contains *all* intervals (i.e. open / closed / half open – both bounded and unbounded – intervals).
2. Let

$$\mathscr{I}_1 := \{J_{a,b} = (a,b) \;:\; a < b,\; a,b \in \mathbb{R}\},$$

denote the collection of all bounded, open intervals. Show that $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{I}_1)$.

3. Let

$$\mathscr{I}_2 := \{J_{r,s} = [r,s] \;:\; r < s,\; r,s \in \mathbb{Q}\}$$
$$\mathscr{I}_3 := \{J_{r,s} = (r,s] \;:\; r < s,\; r,s \in \mathbb{Q}\}$$
$$\mathscr{I}_4 := \{J_{r,s} = [r,s) \;:\; r < s,\; r,s \in \mathbb{Q}\}.$$

Show that $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{I}_2) = \sigma(\mathscr{I}_3) = \sigma(\mathscr{I}_4)$.

4. Let

$$\mathscr{I}_5 := \{J_{a,b} = [a,b] \;:\; a < b,\; a,b \in \mathbb{R}\}$$
$$\mathscr{I}_6 := \{J_{a,b} = (a,b] \;:\; a < b,\; a,b \in \mathbb{R}\}$$
$$\mathscr{I}_7 := \{J_{a,b} = [a,b) \;:\; a < b,\; a,b \in \mathbb{R}\}.$$

Show that $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{I}_5) = \sigma(\mathscr{I}_6) = \sigma(\mathscr{I}_7)$.

■

1.3 The Extended Real Numbers

In measure theory and the theory of integration, it is very convenient to treat the symbols ∞ and $-\infty$ as if they were numbers:

Definition 1.3.1 The set

$$R^* := \mathbb{R} \cup \{\infty, -\infty\} \qquad (\text{also written } [-\infty, \infty])$$

is called the set of *extended real numbers*.

1. We extend addition from $\mathbb{R}$ to $\mathbb{R}^*$ as follows: For all $a \in \mathbb{R}$,

$$\begin{aligned} \infty + a &= a + \infty = \infty \\ -\infty + a &= a + (-\infty) = -\infty \\ \infty + \infty &= \infty \\ -\infty + (-\infty) &= -\infty \end{aligned}$$

Naturally, we define $\infty - a := \infty + (-a)$, etc. Observe that $\infty - \infty$ is *undefined* ! Similarly, we define multiplication by

$$\infty \cdot a = a \cdot \infty = \begin{cases} \infty & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -\infty & \text{if } a < 0 \end{cases} \quad \text{and} \quad (-\infty) \cdot a = a \cdot (-\infty) = \begin{cases} -\infty & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ \infty & \text{if } a < 0 \end{cases}$$

for $a \in \mathbb{R}^*$. Observe here that $0 \cdot \infty = 0$ is *defined* ! Division is defined as usual, and

$$\frac{\infty}{a} = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0, \end{cases}$$

while $\frac{a}{\infty} = 0$ and $\frac{\infty}{\infty}$ is undefined.

2. Next we extend the order to $\mathbb{R}^*$ by setting

$$-\infty < a < \infty \qquad \forall a \in \mathbb{R}.$$

By this definition, every set $E \subseteq \mathbb{R}^*$ is bounded above and bounded below. Furthermore, $\sup E$ and $\inf E$ always exist in $\mathbb{R}^*$:

(a) case $\infty \in E$: Then $\sup E = \infty$.

(b) case $\infty \notin E$, and $E \cap \mathbb{R}$ is *not* bounded above in $\mathbb{R}$: Then $\sup E = \infty$.

(c) case $\infty \notin E$, and $E \cap \mathbb{R} \neq \emptyset$ is bounded above in $\mathbb{R}$: Then $\sup E$ in $\mathbb{R}^*$ coincides with the usual supremum of E as a subset of $\mathbb{R}$.

(d) case $E = \emptyset$ or $E = \{-\infty\}$: Then $\sup E = -\infty$.

3. Limits of sequences (x_n) in $\mathbb{R}^*$ may now include $\pm\infty$.

(a) Finite limits: If $L \in \mathbb{R}$, then

$$\lim_{n\to\infty} x_n = L \Leftrightarrow \forall \varepsilon > 0 \quad \exists N \in \mathbb{N}, \quad |x_n - L| < \varepsilon \quad \forall n \geq N.$$

(b) Infinite limits:

$$\lim_{n\to\infty} x_n = \infty \Leftrightarrow \forall M > 0 \quad \exists N \in \mathbb{N}, \quad x_n > M \quad \forall n \geq N.$$

$$\lim_{n\to\infty} x_n = -\infty \Leftrightarrow \forall M > 0 \quad \exists N \in \mathbb{N}, \quad x_n < -M \quad \forall n \geq N.$$

Every *increasing* sequence $(x_n) \uparrow$ in $\mathbb{R}^*$ converges to its supremum:

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

Similarly, every *decreasing* sequence $(x_n) \downarrow$ in $\mathbb{R}^*$ converges to its infimum.

4. *Limit superior* and *limit inferior* are defined as usual. If (x_n) is any sequence in $\mathbb{R}^*$, then

$$\limsup_n x_n = \overline{\lim_n} x_n := \inf_n \sup_{k\geq n} x_k.$$

Since the sequence (y_n), $y_n := \sup_{k\geq n} x_k$, is decreasing, it converges to its infimum, and hence

$$\overline{\lim_n} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sup_{k\geq n} x_k. \tag{1.3}$$

Similarly,

$$\liminf_n x_n = \underline{\lim}_n x_n := \sup_n \inf_{k\geq n} x_k,$$

and setting $z_n := \inf_{k\geq n} x_k$ then

$$\underline{\lim}_n x_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} \inf_{k\geq n} x_k. \tag{1.4}$$

Since $z_n \leq y_n$ for all n, then by (1.3) and (1.4),

$$\underline{\lim}_n x_n \leq \overline{\lim_n} x_n.$$

Furthermore, the sequence (x_n) converges to a limit $L \in \mathbb{R}^*$ if and only if

$$\underline{\lim}_n x_n = \overline{\lim_n} x_n = L.$$

The details are left as an exercise.

5. Next consider an infinite series

$$\sum_{k=1}^{\infty} a_k, \qquad a_k \in \mathbb{R}^*.$$

As usual, we say that this series converges in $\mathbb{R}^*$, if the sequence of its partial sums converges, i.e. if

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

exists in $\mathbb{R}^*$, and we call S the *sum* of the series.

Recall: If $\sum a_k$ is a series in $\mathbb{R}$, $0 \leq a_k < \infty$ for all k, then we can freely rearrange the series: If $\alpha : \mathbb{N} \mapsto \mathbb{N}$ is any bijection, then

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\alpha(k)},$$

even in case of divergence to ∞. The same is true for series of non-negative terms in $\mathbb{R}^*$: If $a_k \in [0,\infty]$ for all k, then

(a) The series $\sum_k a_k$ always converges in $\mathbb{R}^*$, as its sequence of partial sums is increasing,

(b) for every bijection $\alpha : \mathbb{N} \mapsto \mathbb{N}$ (=*rearrangement*),

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\alpha(k)}. \tag{1.5}$$

The details are left as an exercise.

It thus makes sense to introduce *unordered sums*: let I be a countable set, and for each $i \in I$, let $0 \leq a_i \leq \infty$. We define the *unordered sum* $\sum_{i \in I} a_i$ by

$$\sum_{i \in I} a_i := \sum_{k=1}^{\infty} a_{\alpha(k)}$$

where $\alpha : \mathbb{N} \to I$ is any bijection. By (1.5) this sum is independent of the choice of α.

1.4 Measures

We are now ready to formally introduce the concept of a measure.

Definition 1.4.1 Let Ω be a set and $\mathscr{F}$ a σ-algebra on Ω.

The pair $(\Omega, \mathscr{F})$ is called a *measurable space*, and elements E of $\mathscr{F}$ are called *measurable sets*.

1. A *measure* on $(\Omega, \mathscr{F})$ is a function $\mu : \mathscr{F} \to [0, \infty]$ satisfying:
   (M1) $\mu(\emptyset) = 0$.
   (M2) Whenever $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ is a countable collection of pairwise *disjoint* sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) \qquad (\text{"}\sigma\text{-additivity"})$$

   The triple $(\Omega, \mathscr{F}, \mu)$ is called a *measure space*.
2. If $\mu(\Omega) < \infty$, then μ is called a *finite measure*, and $(\Omega, \mathscr{F}, \mu)$ a *finite measure space*.
3. If there exists a countable collection $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ satisfying
   (a) $\Omega = \bigcup_{n=1}^{\infty} E_n$, and
   (b) $\mu(E_n) < \infty \ \forall n$,
   then μ is called a *σ-finite measure*, and $(\Omega, \mathscr{F}, \mu)$ is called a *σ-finite measure space* (Note that the sets E_n need not be disjoint).

R Since unions of sets are unordered, the sum on the right-hand side of (M2) is really an unordered sum,

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n).$$

R Let $\{E_n\}_{n=1}^{N} \subseteq \mathscr{F}$ be a *finite* collection of pairwise *disjoint* sets. Set $E_{N+1} = E_{N+2} = \cdots = \emptyset$. Then $\{E_n\}_{n=1}^{\infty}$ is a countable collection of pairwise *disjoint* sets in $\mathscr{F}$, and applying (M1) and (M2) we obtain

$$\text{(M2')} \qquad \mu\left(\bigcup_{n=1}^{N} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{\text{(M2)}}{=} \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{N} \mu(E_n) + \sum_{n=N+1}^{N} \mu(\emptyset)$$
$$\overset{\text{(M1)}}{=} \sum_{n=1}^{N} \mu(E_n) + \sum_{n=N+1}^{N} 0 = \sum_{n=1}^{N} \mu(E_n).$$

This property is called *(finite) additivity*.

Example 1.3 The following measures can be defined on *any* measurable space $(\Omega, \mathscr{F})$:

1. A *trivial measure* is given by

$$\mu(E) = 0 \qquad \forall E \in \mathscr{F}.$$

2. Another *trivial measure* is given by

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ \infty & E \in \mathscr{F},\, E \neq \emptyset. \end{cases}$$

Note that

$$\nu(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 1 & E \in \mathscr{F},\, E \neq \emptyset \end{cases}$$

is *not* a measure (unless $\mathscr{F} = \{\emptyset, \Omega\}$). In fact, pick $E \in \mathscr{F}, E \neq \emptyset, E \neq \Omega$. Then

$$\nu(E) + \nu(E^c) = 1 + 1 = 2 \neq \nu(E \cup E^c) = \nu(\Omega) = 1,$$

which shows that ν is not even additive.

3. The *counting measure* is defined by

$$\mu_c(E) = \begin{cases} \text{card}(E) & \text{if } E \in \mathscr{F} \text{ is finite} \\ \infty & \text{if } E \in \mathscr{F} \text{ is infinite.} \end{cases}$$

Then

(a) μ_c is finite $\Leftrightarrow$ Ω is a finite set.

(b) μ_c is σ-finite $\Leftrightarrow$ Ω is a countable set.

Note: The counting measure is the natural measure when Ω is a countable set. It is not a "good" measure when Ω is an uncountable set. because it is not σ-finite in this case.

To illustrate the non-suitability of the counting measure in case of an uncountable set, consider the case where $\Omega = \mathbb{R}$ with $\mathscr{F} = \mathscr{P}(\Omega)$ and μ_c the counting measure. Set

$$E_1 = \left\{ \frac{1}{k} : k \in \mathbb{N} \right\} \qquad \text{and} \qquad E_2 = [0, 1].$$

Then $\mu_c(E_1) = \mu_c(E_2) = \infty$, which contradicts our intuition that the two sets have very different "sizes".

4. Fix a point $a \in \Omega$ and set

$$\delta_a(E) = \begin{cases} 0 & \text{if } a \notin E \\ 1 & \text{if } a \in E \end{cases}$$

for all $E \in \mathscr{F}$. Then δ_a is a finite measure on $(\Omega, \mathscr{F})$ called the *Dirac one-point measure*. ■

Exercise 1.4 Prove the assertions in 3. and 4. of Example 1.3 above. ■

■ **Example 1.4** *(The Lebesgue measure on the real line)* Consider the measurable space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. In Chapter 4 we will study the following:

1. There exists a unique measure λ on $\mathscr{B}(\mathbb{R})$ with the property

$$\lambda(I) = b - a$$

for any bounded interval I with endpoints $a \leq b$. This measure λ is called the *Lebesgue measure*.

2. Since

$$\mathbb{R} = \bigcup_{n=1}^{\infty} E_n, \qquad E_n = [-n, n]$$

and

$$\lambda(E_n) = n - (-n) = 2n < \infty,$$

it follows that λ is σ-finite.

3. The Lebesgue measure λ is compatible with the topology of $\mathbb{R}$ in the following way:
   (a) $\lambda(K) < \infty \quad \forall K \subseteq \mathbb{R}$ compact,
   (b) $\lambda(E) = \inf\{\, \lambda(U) : E \subseteq U,\, U \subseteq \mathbb{R} \text{ is open} \,\} \quad \forall E \in \mathscr{B}(\mathbb{R})$.
   ("outer regularity")
   (c) $\lambda(E) = \sup\{\, \lambda(K) : K \subseteq E,\, K \subseteq \mathbb{R} \text{ is compact} \,\} \quad \forall E \in \mathscr{B}(\mathbb{R})$.
   ("inner regularity")

   (Because these three properties hold, λ is called a *regular Borel measure*.)

■

Exercise 1.5 Given $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, set

$$x + A := \{x + y : y \in A\}$$
$$-A := \{-y : y \in A\}$$
$$xA := \{xy : y \in A\}.$$

Show:

1. $\mathscr{F}_1 := \{x + A : A \in \mathscr{B}(\mathbb{R})\}$ is a σ-algebra on $\mathbb{R}$.
2. $\mathscr{F}_1$ contains all open intervals.
3. $\mathscr{F}_1 = \mathscr{B}(\mathbb{R})$.

Thus $\mathscr{B}(\mathbb{R})$ is *invariant under translations*. Similarly show that

1. $\mathscr{F}_2 := \{-A : A \in \mathscr{B}(\mathbb{R})\} = \mathscr{B}(\mathbb{R})$.
2. If $x > 0$, then $\mathscr{F}_3 := \{xA : A \in \mathscr{B}(\mathbb{R})\} = \mathscr{B}(\mathbb{R})$.

In fact, in a later exercise in Chapter 4 you will show that $\forall A \in \mathscr{B}(\mathbb{R})$,

$\lambda(x + A) = \lambda(A)$	the Lebesgue measure λ is 'translation invariant'
$\lambda(-A) = \lambda(A)$	the Lebesgue measure λ is 'inversion invariant'
$\lambda(xA) = x\lambda(A)$	$(x > 0)$

■

Theorem 1.4.1 *(Properties of Measures).* Let $(\Omega, \mathscr{F}, \mu)$ be a measure space.

1. Whenever $A, B \in \mathscr{F}$ and $A \subseteq B$ then

$$\text{(M3)} \qquad \mu(A) \leq \mu(B). \qquad \text{("monotonicity")}$$

2. Whenever $A, B \in \mathscr{F}$ with $A \subseteq B$ and $\mu(B) < \infty$ then

$$\mu(B \backslash A) = \mu(B) - \mu(A).$$

3. Whenever $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ is a countable collection of measurable sets (not necessarily disjoint), then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n) \qquad \text{("}\sigma\text{-subadditivity")}$$

4. Whenever $\{E_n\}_{n=1}^{N} \subseteq \mathscr{F}$ is a finite collection of measurable sets (not necessarily disjoint), then

$$\mu\left(\bigcup_{n=1}^{N} E_n\right) \leq \sum_{n=1}^{N} \mu(E_n) \qquad \text{("subadditivity")}$$

Proof. 1. Decompose B as

$$B = (B \backslash A) \cup A,$$

a *disjoint* union. Since $0 \leq \mu(B \backslash A)$ and μ is additive, then

$$\mu(A) \leq \mu(B \backslash A) + \mu(A) = \mu(B). \tag{1.6}$$

2. If $\mu(B) < \infty$, then by monotonicity, $\mu(A) < \infty$ as well. We may thus subtract $\mu(A)$ from all sides of (1.6), and obtain

$$\mu(B) - \mu(A) = \mu(B \backslash A).$$

3. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ be given. By Theorem 1.1.1, there exists a collection $\{B_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ of *pairwise disjoint* sets satisfying
   (a) $B_n \subseteq E_n$ for all n,
   (b) $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} E_n$.
   Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \overset{\text{(M2)}}{=} \sum_{n=1}^{\infty} \mu(B_n) \overset{\text{(M3)}}{\leq} \sum_{n=1}^{\infty} \mu(E_n). \tag{1.7}$$

4. Additivity follows from (1.7) by setting $E_{N+1} = E_{N+2} = \cdots = \emptyset$.

■

(R) Inspection of the proof shows that condition $\mu(B) < \infty$ in part 2. can be weakend to $\mu(A) < \infty$. It cannot be removed completely, however. For example, Let $\Omega = \mathbb{N}$, $\mathscr{F} = \mathscr{P}(\Omega)$ and μ be the counting measure. Set

$$A = \{2k \,:\, k \in \mathbb{N}\}, \qquad B = \mathbb{N}.$$

Then $B \backslash A = \{2k - 1 \,:\, k \in \mathbb{N}\}$, the set of odd, positive integers. We have

$$\mu(B) - \mu(A) = \infty - \infty$$

which is undefined, while also $\mu(B \backslash A) = \infty$.

Exercise 1.6 For each $n \in \mathbb{N}$, let μ_n be a measure on $(\Omega, \mathscr{F})$. Choose a sequence (α_n), $0 \le \alpha_n < \infty \; \forall n$.

1. Define $\mu : \mathscr{F} \to [0,\infty]$ by $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$. That is,

$$\mu(E) = \alpha_1\mu_1(E) + \alpha_2\mu_2(E) \qquad \forall E \in \mathscr{F}.$$

   Show:
   (a) μ is a measure on $(\Omega, \mathscr{F})$.
   (b) If μ_1 and μ_2 are finite measures, then μ is a finite measure.
   (c) If μ_1 and μ_2 are σ-finite measures, then μ is a σ-finite measure.
   (Note: By induction, the above statements extend to finite sums $\mu = \sum_{n=1}^{N} \alpha_n\mu_n$.)
2. Next define $\mu : \mathscr{F} \to [0,\infty]$ by $\mu = \sum_{n=1}^{\infty} \alpha_n\mu_n$. That is,

$$\mu(E) = \sum_{n=1}^{\infty} \alpha_n\mu_n(E) \qquad \forall E \in \mathscr{F}.$$

   Show:
   (a) μ is a measure on $(\Omega, \mathscr{F})$.
   (b) If there exists $M < \infty$ so that $\mu_n(\Omega) \le M$ for all n, and if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then μ is a finite measure.
   (c) Show by example: Even when the μ_n are all finite measures and $\sum_{n=1}^{\infty} \alpha_n < \infty$, then μ need not be σ-finite.

■

Definition 1.4.2 A countable collection $\{A_n\}_{n=1}^{\infty}$ of sets is called
- *increasing*, if $A_n \subseteq A_{n+1}$ for all n. We write $\{A_n\} \uparrow$.
- *decreasing*, if $A_n \supseteq A_{n+1}$ for all n. We write $\{A_n\} \downarrow$.

Theorem 1.4.2 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$.

1. If $\{A_n\} \uparrow$, then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.
2. If $\{A_n\} \downarrow$, and $\mu(A_{n_o}) < \infty$ for some $n_o \in \mathbb{N}$, then $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

Proof. 1. Suppose that $\{A_n\} \uparrow$. By Theorem 1.1.1, there exists a collection $\{B_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ of *pairwise disjoint* sets satisfying
   (a) $B_n \subseteq A_n$ for all n,
   (b) $\bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} A_n$ for all $N \in \mathbb{N} \cup \{\infty\}$.

   Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \overset{(b)}{=} \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \overset{(M2)}{=} \sum_{n=1}^{\infty} \mu(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n)$$
$$\overset{(M2')}{=} \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} B_n\right) \overset{(b)}{=} \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} A_n\right) \overset{\{A_n\}\uparrow}{=} \lim_{N\to\infty} \mu(A_N)$$

   which was to be shown.

2. Suppose that $\{A_n\} \downarrow$ and $\mu(A_{n_o}) < \infty$. For each $n \in \mathbb{N}$, set

$$E_n := A_{n_o} \backslash A_n \in \mathscr{F}.$$

Then $\{E_n\} \uparrow$, so that by the first part,

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(A_{n_o} \backslash A_n) \\ &\overset{\text{Thm 1.4.1}}{=} \lim_{n\to\infty} \left[\mu(A_{n_o}) - \mu(A_n)\right] = \mu(A_{n_o}) - \lim_{n\to\infty} \mu(A_n). \end{aligned} \tag{1.8}$$

On the other hand, as

$$\begin{aligned} \bigcup_{n=1}^{\infty} E_n &= \bigcup_{n=1}^{\infty} (A_{n_o} \backslash A_n) = \bigcup_{n=1}^{\infty} (A_{n_o} \cap A_n^c) = A_{n_o} \cap \left(\bigcup_{n=1}^{\infty} A_n^c\right) \\ &= A_{n_o} \cap \left(\bigcap_{n=1}^{\infty} A_n\right)^c = A_{n_o} \backslash \left(\bigcap_{n=1}^{\infty} A_n\right) \end{aligned}$$

it follows that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(A_{n_o} \backslash \left(\bigcap_{n=1}^{\infty} A_n\right)\right) \overset{\text{Thm 1.4.1}}{=} \mu(A_{n_o}) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right). \tag{1.9}$$

Comparing (1.8) and (1.9) gives

$$\mu(A_{n_o}) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(A_{n_o}) - \lim_{n\to\infty} \mu(A_n).$$

Since $\mu(A_{n_o}) < \infty$, we conclude that

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Thus, the theorem is proved. ■

R In part 2. above, the condition $\mu(A_{n_o}) < \infty$ can not be removed.
For example, Let $\Omega = \mathbb{N}$, $\mathscr{F} = \mathscr{P}(\Omega)$ and μ be the counting measure. For each $n \in \mathbb{N}$, set

$$A_n = \{n, n+1, n+2, \dots\}.$$

Then $\{A_n\} \downarrow$ and $\mu(A_n) = \infty$ for all n. The fact that $\bigcap_{n=1}^{\infty} A_n = \emptyset$ gives

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(\emptyset) = 0 \neq \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \infty = \infty.$$

Corollary 1.4.3 *(Borel-Cantelli Lemma).* Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ a countable family of measurable sets. If

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty \tag{1.10}$$

then

$$\mu\Big(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i\Big) = 0.$$

Proof. For each n, set $E_n := \bigcup_{i=n}^{\infty} A_i$. Then

1. $\{E_n\}\downarrow$, and
2. $\mu(E_n) = \mu\Big(\bigcup_{i=n}^{\infty} A_i\Big) \underset{\text{thm 1.4.1}}{\leq} \sum_{i=n}^{\infty} \mu(A_i) \leq \sum_{i=1}^{\infty} \mu(A_i) < \infty$ for all n by assumption.

We can thus apply part 2. of Theorem 1.4.2 to the sets $\{E_n\}$ to obtain

$$\begin{aligned}\mu\Big(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i\Big) = \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) &\underset{\text{thm 1.4.2}}{=} \lim_{n\to\infty} \mu(E_n) \\ &= \lim_{n\to\infty} \mu\Big(\bigcup_{i=n}^{\infty} A_i\Big) \underset{\text{thm 1.4.1}}{\leq} \lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(A_i) = 0\end{aligned}$$

by assumption (1.10). ■

R One easily checks that

$$\omega \in \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i \quad \Leftrightarrow \quad \omega \in A_i \text{ for infinitely many } i.$$

So Borel-Cantelli's theorem says that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then

$$\{\omega \in \Omega : \omega \in A_n \text{ for } \textit{infinitely} \text{ many } n\}$$

has measure zero.

Exercise 1.7 Recall that $(\Omega, \mathscr{F}, \mu)$ is called σ-finite, if there exists a collection $\{E_n\} \subseteq \mathscr{F}$ with
(i) $\mu(E_n) < \infty \; \forall n$,
(ii) $\Omega = \bigcup_{n=1}^{\infty} E_n$.

Show:

1. the sets E_n above may be assumed to be disjoint.
2. the sequence of sets $\{E_n\}$ may be assumed to be increasing.

■

Null Sets

Definition 1.4.3 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. A set $E \in \mathscr{F}$ is called a *null set* (or a μ*-null set* or a *set of measure zero*) if $\mu(E) = 0$.

Example 1.5 Let $(\Omega, \mathscr{F})$ be any measurable space.

1. Fix $a \in \Omega$ and consider the *one-point Dirac measure* δ_a. Then
$$E \in \mathscr{F} \text{ is a null set} \quad \Leftrightarrow \quad \delta_a(E) = 0 \quad \Leftrightarrow \quad a \notin E.$$
2. Let μ_c denote the *counting measure*. Then
$$E \in \mathscr{F} \text{ is a null set} \quad \Leftrightarrow \quad \mu_c(E) = 0 \quad \Leftrightarrow \quad \text{card}(E) = 0 \quad \Leftrightarrow \quad E = \emptyset.$$
3. Let μ be *any* measure on $(\Omega, \mathscr{F})$.
   (a) If $E \in \mathscr{F}$ is a null set, and $A \in \mathscr{F}$ with $A \subseteq E$, then by monotonicity,
   $$0 \le \mu(A) \le \mu(E) = 0,$$
   so that $\mu(A) = 0$ also. ("*Measurable* subsets of null sets are null sets")
   (b) If $\{E_n\}_{n=1}^N \subseteq \mathscr{F}, N \in \mathbb{N} \cup \{\infty\}$ is a countable collection of null sets, then by subadditivity, respectively σ-subadditivity, by
   $$0 \le \mu\left(\bigcup_{n=1}^N E_n\right) \le \sum_{n=1}^N \mu(E_n) = \sum_{n=1}^N 0 = 0$$
   which shows that $\bigcup_{n=1}^N E_n$ is a null set. ("Countable unions of null sets are null sets")

■

Example 1.6 (Some λ-null sets in $\mathscr{B}(\mathbb{R})$)

1. Let $E = \{x\}$, $x \in \mathbb{R}$ be a singleton. Since $E = \bigcap_{n=1}^\infty I_n$, $I_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, then by Theorem 1.4.2, part 2,
$$\lambda(E) = \lim_{n\to\infty} \lambda(I_n) = \lim_{n\to\infty} \frac{2}{n} = 0.$$
Thus, all *singletons* are λ-null sets.
2. Let $E = \{x_n\}_{n=1}^N \subseteq \mathbb{R}$ be countable, where $N \in \mathbb{N} \cup \{\infty\}$. We can write E as a countable union of singletons, $E = \bigcup_{n=1}^N \{x_n\}$. It follows from Example 1.5, part 3, that E is a null set. Thus, all *countable* subsets of $\mathbb{R}$ are λ-null sets.
3. Recall the *Cantor set* $\mathscr{C}$ which is of the form
$$\mathscr{C} = \bigcap_{n=1}^\infty E_n,$$
where each E_n is the disjoint union of 2^n intervals of length $\frac{1}{3^n}$ each,
$$E_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$
$$E_2 = [0, \tfrac{1}{3^2}] \cup [\tfrac{2}{3^2}, \tfrac{3}{3^2}] \cup [\tfrac{6}{3^2}, \tfrac{7}{3^2}] \cup [\tfrac{8}{3^2}, 1]$$
$$E_3 = [0, \tfrac{1}{3^3}] \cup [\tfrac{2}{3^3}, \tfrac{3}{3^3}] \cup [\tfrac{6}{3^3}, \tfrac{7}{3^3}] \cup [\tfrac{8}{3^3}, \tfrac{9}{3^3}]$$
$$\cup [\tfrac{18}{3^3}, \tfrac{19}{3^3}] \cup [\tfrac{20}{3^3}, \tfrac{21}{3^3}] \cup [\tfrac{24}{3^3}, \tfrac{25}{3^3}] \cup [\tfrac{26}{3^3}, 1]$$

Since $\{E_n\}\downarrow$ and $\lambda(E_n) = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$ for all n, then by Theorem 1.4.2, part 2,

$$\lambda(\mathscr{C}) = \lim_{n\to\infty} \lambda(E_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

The Cantor set is an example of a λ-null subset of $\mathbb{R}$ which is *uncountable*.

∎

1.5 Measurable Functions

In the realm of topological spaces, one is naturally interested in mappings which are compatible with the topologies. These are the *continuous* maps. Recall that a map $f:\Omega\to\Pi$ between topological spaces (Ω,τ) and (Π,κ) is said to be continuous, if $f^{-1}(U)\in\tau \quad \forall U\in\kappa$. That is, pre-images of open sets are open.

In the realm of maps between measurable spaces, we impose a similar requirement:

Definition 1.5.1 Let $(\Omega,\mathscr{F})$ and $(\Pi,\mathscr{E})$ be measurable spaces. A mapping $f:\Omega\to\Pi$ is said to be *$(\mathscr{F},\mathscr{E})$-measurable*, if

$$f^{-1}(E)\in\mathscr{F} \qquad \forall E\in\mathscr{E}. \tag{1.11}$$

(That is, pre-images of measurable sets are measurable sets.)

When Π is a topological space and $\mathscr{E}=\mathscr{B}(\Pi)$ is the Borel σ-algebra on Π, then we simply call f an *$\mathscr{F}$-measurable* mapping.

When Ω and Π are both topological spaces, and $\mathscr{F}$ and $\mathscr{E}$ are their Borel σ-algebras, then we call f a *Borel mapping*.

(R) We are mainly interested in mappings $f:\Omega\to\mathbb{R}$ (or more generally, $\Omega\to\mathbb{R}^n$), that is, in functions. By the above definition, such a function f is $\mathscr{F}$-measurable, provided that $f^{-1}(E)\in\mathscr{F}$ for all Borel subsets E of $\mathbb{R}$.

The next theorem says that in (1.11) it suffices to only consider the generators of the σ-algebra $\mathscr{E}$.

Theorem 1.5.1 Let $(\Omega,\mathscr{F})$ and $(\Pi,\mathscr{E})$ be measurable spaces and $f:\Omega\to\Pi$. Suppose that $\mathscr{E}=\sigma(\mathscr{K})$ (i.e. $\mathscr{E}$ is the σ-algebra generated by a collection $\mathscr{K}$ of subsets of Π). Then

$$f \text{ is } (\mathscr{F},\mathscr{E})\text{-measurable} \quad\Leftrightarrow\quad f^{-1}(E)\in\mathscr{F} \quad \forall E\in\mathscr{K}$$

Proof. $\Rightarrow$: Obvious by (1.11).

$\Leftarrow$: Suppose that

$$f^{-1}(E)\in\mathscr{F} \qquad \forall E\in\mathscr{K}. \tag{1.12}$$

Let us set

$$\mathscr{E}_o := \left\{E\in\mathscr{E} : f^{-1}(E)\in\mathscr{F}\right\}.$$

Clearly, $\mathscr{K}\subseteq\mathscr{E}_o$ by assumption (1.12), and also $\mathscr{E}_o\subseteq\mathscr{E}$.

Claim: $\mathscr{E}_o$ is a σ-algebra.

(A1) Let $E \in \mathscr{E}_o$, so that $f^{-1}(E) \in \mathscr{F}$. Since $\mathscr{F}$ is a σ-algebra, we have

$$f^{-1}(E^c) = \left[f^{-1}(E)\right]^c \overset{(A1)}{\in} \mathscr{F},$$

which shows that $E^c \in \mathscr{E}_o$ as well.

(A2σ) Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{E}_o$, so that $f^{-1}(E_n) \in \mathscr{F}$ for all n. Since $\mathscr{F}$ is a σ-algebra, we have

$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(E_n) \overset{(A2\sigma)}{\in} \mathscr{F},$$

which shows that $\bigcup_{n=1}^{\infty} E_n \in \mathscr{E}_o$ as well.

Thus the claim is proved: $\mathscr{E}_o$ is a σ-algebra containing $\mathscr{K}$.

Now as $\sigma(\mathscr{K})$ is the smallest σ-algebra containing $\mathscr{K}$, it follows from the claim that $\sigma(\mathscr{K}) \subseteq \mathscr{E}_o$, and hence that

$$\mathscr{E} = \sigma(\mathscr{K}) \subseteq \mathscr{E}_o \subseteq \mathscr{E}.$$

This shows that $\mathscr{E}_o = \mathscr{E}$, so that (1.11) holds. That is, f is an $(\mathscr{F}, \mathscr{E})$-measurable function. ■

Corollary 1.5.2 Let (Ω, τ) and (Π, κ) be topological spaces (for example, metric spaces). Then every continuous function $f : \Omega \to \Pi$ is a Borel function.

Proof. Recall that the Borel σ-algebras are generated by the open sets: $\mathscr{F} := \mathscr{B}(\Omega) = \sigma(\tau)$ and $\mathscr{E} := \mathscr{B}(\Pi) = \sigma(\kappa)$, where τ and κ denote the collections of open sets ("topologies") on Ω, respectively Π.

Now let $E \in \kappa$. Since E is open and f is continuous, it follows that $f^{-1}(E)$ is also open, that is, $f^{-1}(E) \in \tau \subset \mathscr{B}(\Omega)$. We have shown that

$$f^{-1}(E) \in \mathscr{F} \qquad \forall E \in \kappa,$$

hence by Theorem 1.5.1, f is $(\mathscr{F}, \mathscr{E})$-measurable. ■

Theorem 1.5.1 allows us to give a simple characterization of real valued, $\mathscr{F}$-measurable functions:

Corollary 1.5.3 Let $(\Omega, \mathscr{F})$ be a measurable space and $f : \Omega \to \mathbb{R}$. Then

$$f \text{ is } \mathscr{F}\text{-measurable} \quad \Leftrightarrow \quad f^{-1}\big((a,\infty)\big) \in \mathscr{F} \qquad \forall a \in \mathbb{R}. \tag{1.13}$$

Proof. Let us set

$$\mathscr{K} := \{(a,\infty) : a \in \mathbb{R}\}.$$

Choosing $\mathscr{E} = \mathscr{B}(\mathbb{R})$, by Theorem 1.5.1 we only need to prove the following claim:

Claim: $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{K})$.

In fact, clearly $\mathscr{K} \subset \mathscr{B}(\mathbb{R})$, so that $\sigma(\mathscr{K}) \subseteq \mathscr{B}(\mathbb{R})$. To prove the reverse inclusion, recall that by Theorem 1.2.2,

$$\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{I}_0), \quad \text{where} \quad \mathscr{I}_0 = \{(r,s) \subseteq \mathbb{R} : r < s, \; r,s \in \mathbb{Q}\}.$$

We first show that $\mathscr{I}_0 \subseteq \sigma(\mathscr{K})$. To this end, let an arbitrary interval $(r,s) \in \mathscr{I}_0$ be given. We can write

$$(r,s) = (-\infty,s) \cap (r,\infty) = \left[\bigcap_{n=1}^{\infty} \left(s - \tfrac{1}{n}, \infty\right)\right]^c \cap (r,\infty).$$

Now since $(r,\infty) \in \mathscr{K}$ and $\left(s - \frac{1}{n}, \infty\right) \in \mathscr{K}$ for all n, it follows from properties (A1), (A3) and (A3σ) of σ-algebras that $(r,s) \in \sigma(\mathscr{K})$. This shows that $\mathscr{I}_0 \subseteq \sigma(\mathscr{K})$.
It follows immediately that

$$\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{I}_0) \subseteq \sigma(\mathscr{K}),$$

which proves the claim and the corollary.

∎

Observe that condition (1.13) can be restated as:

$$f : \Omega \to \mathbb{R} \text{ is } \mathscr{F}\text{-measurable} \quad \Leftrightarrow \quad \{\, \omega \in \Omega : f(\omega) > a \,\} \in \mathscr{F} \qquad \forall a \in \mathbb{R}.$$

Since we will work with *extended real-valued* functions, we can make use of this characterization to extend the concept of measurability to functions $f : \Omega \to \mathbb{R}^*$:

Definition 1.5.2 A function $f : \Omega \to \mathbb{R}^*$ is said to be *$\mathscr{F}$-measurable* if

$$f^{-1}\left((a,\infty]\right) = \{\, \omega \in \Omega : f(\omega) > a\} \in \mathscr{F} \qquad \forall a \in \mathbb{R}.$$

Theorem 1.5.4 Let $(\Omega, \mathscr{F})$ be a measurable space and $f : \Omega \to \mathbb{R}^*$.
T.F.A.E. ("The following are equivalent")

1. $\{\, \omega \in \Omega : f(\omega) > a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{R}$. (i.e. f is $\mathscr{F}$-measurable.)
2. $\{\, \omega \in \Omega : f(\omega) \geq a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{R}$.
3. $\{\, \omega \in \Omega : f(\omega) < a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{R}$.
4. $\{\, \omega \in \Omega : f(\omega) \leq a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{R}$.
5. $\{\, \omega \in \Omega : f(\omega) > a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{Q}$.
6. $\{\, \omega \in \Omega : f(\omega) \geq a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{Q}$.
7. $\{\, \omega \in \Omega : f(\omega) < a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{Q}$.
8. $\{\, \omega \in \Omega : f(\omega) \leq a \,\} \in \mathscr{F} \quad \forall a \in \mathbb{Q}$.

Proof. We will make use of properties (A1), (A2) and (A3) of σ-algebras.
First we show the implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Rightarrow$ 1.

1. $\Rightarrow$ 2.: Suppose that 1. holds. Then for every $a \in \mathbb{R}$,

$$\{\, \omega \in \Omega : f(\omega) \geq a \,\} \; = \; \bigcap_{n=1}^{\infty} \underbrace{\{\, \omega \in \Omega : f(\omega) > a - \tfrac{1}{n} \,\}}_{\in \mathscr{F} \text{ by } 1.} \in \mathscr{F},$$

which shows that 2. holds.

2. $\Rightarrow$ 3.: Suppose that 2. holds. Then for every $a \in \mathbb{R}$,

$$\{\, \omega \in \Omega : f(\omega) < a \,\} \; = \; \Big[\underbrace{\{\, \omega \in \Omega : f(\omega) \geq a \,\}}_{\in \mathscr{F} \text{ by } 2.}\Big]^c \in \mathscr{F},$$

which shows that 3. holds.

3. $\Rightarrow$ 4.: Suppose that 3. holds. Then for every $a \in \mathbb{R}$,

$$\{\,\omega \in \Omega : f(\omega) \le a\,\} = \bigcap_{n=1}^{\infty} \underbrace{\{\,\omega \in \Omega : f(\omega) < a + \tfrac{1}{n}\,\}}_{\in \mathscr{F} \text{ by 3.}} \in \mathscr{F},$$

which shows that 4. holds.

4. $\Rightarrow$ 1.: Suppose that 4. holds. Then for every $a \in \mathbb{R}$,

$$\{\,\omega \in \Omega : f(\omega) > a\,\} = \Big[\underbrace{\{\,\omega \in \Omega : f(\omega) \le a\,\}}_{\in \mathscr{F} \text{ by 4.}}\Big]^c \in \mathscr{F},$$

which shows that 1. holds.

The implications 5. $\Rightarrow$ 6. $\Rightarrow$ 7. $\Rightarrow$ 8. $\Rightarrow$ 5. are proved in exactly the same way.

It is left to show that 1. $\Leftrightarrow$ 5.

1. $\Rightarrow$ 5.: This is trivial.

5. $\Rightarrow$ 1.: Suppose, 5. holds. Let $a \in \mathbb{R}$ be arbitrary. By density of $\mathbb{Q}$ in $\mathbb{R}$, we can pick a sequence (q_n) in $\mathbb{Q}$ so that
   (i) $a < q_n$ for all n, and
   (ii) $q_n \to a$.
   So if $f(\omega) > a$ then $f(\omega) > q_n > a$ for sufficiently *large* n, and hence

$$\{\,\omega \in \Omega : f(\omega) > a\,\} = \bigcup_{n=1}^{\infty} \underbrace{\{\,\omega \in \Omega : f(\omega) > q_n\,\}}_{\in \mathscr{F} \text{ by 5.}} \in \mathscr{F},$$

which shows that 1. holds.

This completes the proof. ■

Example 1.7 Let $(\Omega, \mathscr{F})$ be any measurable space.

1. Given a subset A of Ω, we define a function $\mathbf{1}_A : \Omega \to \mathbb{R}$ by

$$\mathbf{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

It is called the *characteristic function* or *indicator function* of the set A.

Claim: $\mathbf{1}_A$ is an $\mathscr{F}$-measurable function $\Leftrightarrow$ $A \in \mathscr{F}$.

To see this, for each $a \in \mathbb{R}$, set

$$E_a := \{\,\omega \in \Omega : \mathbf{1}_A(\omega) > a\,\}.$$

Then

$$E_a = \begin{cases} \emptyset & \text{if } a \ge 1 \\ A & \text{if } 0 \le a < 1 \\ \Omega & \text{if } a < 0. \end{cases}$$

Since $\emptyset, \Omega \in \mathscr{F}$ always, we see that

$$E_a \in \mathscr{F} \quad \forall a \in \mathbb{R} \quad \Leftrightarrow \quad A \in \mathscr{F}$$

which, by Definition 1.5.2 of an $\mathscr{F}$-measurable function, proves the claim.

2. Every *constant function* $f(\omega) = c$ is $\mathscr{F}$-measurable.
   To see this, note that for every $a \in \mathbb{R}$,

$$E_a := \{\, \omega \in \Omega : f(\omega) > a \,\} = \begin{cases} \emptyset & \text{if } a \geq c \\ \Omega & \text{if } a < c \end{cases}$$

   Since $\emptyset, \Omega \in \mathscr{F}$, the assertion follows by Definition 1.5.2 of an $\mathscr{F}$-measurable function.

■

Exercise 1.8 Show:
1. For all $A, B \subseteq \Omega$ we have $\mathbf{1}_{A\cap B} = \mathbf{1}_A \mathbf{1}_B$.
2. For all $A, B \subseteq \Omega$ we have $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_A \mathbf{1}_B$.
3. For all $A \subseteq \Omega$ we have $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$.
4. For all $A, \ldots, A_n, \subseteq \Omega$ we have $\mathbf{1}_{\bigcap_{k=1}^n A_k} = \prod_{k=1}^n \mathbf{1}_{A_k}$.
5. For all $A, \ldots, A_n \subseteq \Omega$ which are pairwise disjoint (i.e. $A_k \cap A_j = \emptyset$ if $k \neq j$) we have $\mathbf{1}_{\bigcup_{k=1}^n A_k} = \sum_{k=1}^n \mathbf{1}_{A_k}$.
6. The last two assertion also hold for $n = \infty$.

■

Exercise 1.9 Show that every *monotone increasing* function $f : \mathbb{R} \to \mathbb{R}$ is a Borel function. (Increasing means: $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$). ■

Exercise 1.10 Let $\Omega = \mathbb{R}$ and $\mathscr{F} = \{E \subseteq \mathbb{R} : E$ is countable or E^c is countable$\}$. Show:
1. The function $\mathbf{1}_{\mathbb{Q}}$ is $\mathscr{F}$-measurable.
2. The function $f(x) = x$ is not $\mathscr{F}$-measurable.
3. If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then h is $\mathscr{F}$-measurable $\Leftrightarrow$ h is constant.

■

Theorem 1.5.5 Let $(\Omega, \mathscr{F})$ be a measurable space, and $f, g : \Omega \to \mathbb{R}^*$ be $\mathscr{F}$-measurable functions. Then
1. $\{\, \omega \in \Omega : f(\omega) < g(\omega) \,\} \in \mathscr{F}$,
2. $\{\, \omega \in \Omega : f(\omega) \leq g(\omega) \,\} \in \mathscr{F}$,
3. $\{\, \omega \in \Omega : f(\omega) = g(\omega) \,\} \in \mathscr{F}$.

Proof. 1. We make the following observation: Suppose that $f(\omega) < g(\omega)$ at some $\omega \in \Omega$. Then by density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ so that

$$f(\omega) < q < g(\omega).$$

Hence,

$$\begin{aligned} \{\, \omega \in \Omega : f(\omega) < g(\omega) \,\} &= \bigcup_{q \in \mathbb{Q}} \{\, \omega \in \Omega : f(\omega) < q \quad \text{and} \quad q < g(\omega) \,\} \\ &= \bigcup_{q \in \mathbb{Q}} \Big(\underbrace{\{\, \omega \in \Omega : f(\omega) < q \}}_{\in \mathscr{F} \text{ by Thm 1.5.4.}} \cap \underbrace{\{\, \omega \in \Omega : q < g(\omega) \}}_{\in \mathscr{F} \text{ by def. of meas. fn.}} \Big) \in \mathscr{F} \end{aligned}$$

by properties (A2σ) and (A3) of a σ-algebra.

2. By property (A1) of σ-algebras,

$$\{\omega\in\Omega : f(\omega)\le g(\omega)\} = \underbrace{\{\omega\in\Omega : g(\omega)\le f(\omega)\}}_{\in\mathscr{F}\text{ by Part 1.}}{}^{c} \underset{\substack{\uparrow\\ \text{(A1)}}}{\in} \mathscr{F}.$$

3. By property (A3) of σ-algebras,

$$\begin{aligned}&\{\omega\in\Omega : f(\omega)=g(\omega)\}\\ &\quad= \underbrace{\{\omega\in\Omega : f(\omega)\le g(\omega)\}}_{\in\mathscr{F}\text{ by Part 2.}}\cap\underbrace{\{\omega\in\Omega : g(\omega)\le f(\omega)\}}_{\in\mathscr{F}\text{ by Part 2.}} \underset{\substack{\uparrow\\ \text{(A3)}}}{\in} \mathscr{F}.\end{aligned}$$

■

Let $f,g:\Omega\to\mathbb{R}^*$ and $\alpha\in\mathbb{R}$. As usual, algebraic operations on these extended real valued functions are defined pointwise. Thus, we define functions

αf	by	$(\alpha f)(\omega)=\alpha f(\omega) \quad \forall\omega\in\Omega$
$f+g$	by	$(f+g)(\omega)=f(\omega)+g(\omega) \quad \forall\omega\in\Omega$
		(this requires that $f(\omega)+g(\omega)\neq\begin{cases}\infty-\infty\\ -\infty+\infty\end{cases} \quad \forall\omega$)
fg	by	$(fg)(\omega)=f(\omega)g(\omega) \quad \forall\omega\in\Omega$
$\frac{f}{g}$	by	$\left(\frac{f}{g}\right)(\omega)=\frac{f(\omega)}{g(\omega)} \quad \forall\omega\in\Omega$
		(this requires that $g(\omega)\neq 0$ and $\frac{f(\omega)}{g(\omega)}\neq\frac{\pm\infty}{\pm\infty} \quad \forall\omega$)
$\max(f,g)$	by	$\max(f,g)(\omega)=\max(f(\omega),g(\omega)) \quad \forall\omega\in\Omega$
$\min(f,g)$	by	$\min(f,g)(\omega)=\min(f(\omega),g(\omega)) \quad \forall\omega\in\Omega$

We then set

$f^+=\max(f,0)$	*"positive part of f"*
$f^-=-\min(f,0)$	*"negative part of f"*
$\lvert f\rvert=f^++f^-$	*"absolute value of f"*

Some observations:

1. For all $\omega\in\Omega$,

$$f^+(\omega)\overset{\text{def.}}{=}\max(f,0)(\omega) = \max(f(\omega),0) = \begin{cases}f(\omega) & \text{if } f(\omega)\ge 0\\ 0 & \text{if } f(\omega)<0.\end{cases}$$

Similarly,

$$f^-(\omega)\overset{\text{def.}}{=}-\min(f,0)(\omega) = -\min(f(\omega),0) = \begin{cases}0 & \text{if } f(\omega)\ge 0\\ -f(\omega) & \text{if } f(\omega)<0.\end{cases}$$

2. For all $\omega \in \Omega$, $f^+(\omega)f^-(\omega) = 0$, that is, $f^+f^- = 0$.
3. For all $\omega \in \Omega$,

$$f^+(\omega)+f^-(\omega) = \left\{ \begin{array}{ll} f(\omega)+0 = f(\omega) & \text{if } f(\omega) \geq 0 \\ 0 - f(\omega) = -f(\omega) & \text{if } f(\omega) < 0 \end{array} \right\} = |f(\omega)|$$

and hence,

$$|f|(\omega) \stackrel{\text{def.}}{=} (f^+ + f^-)(\omega) = f^+(\omega)+f^-(\omega) = |f(\omega)|,$$

so that $|f|$ is the usual pointwise defined absolute value function.
4. Similarly, for all $\omega \in \Omega$,

$$f^+(\omega)-f^-(\omega) = \left\{ \begin{array}{ll} f(\omega)-0 = f(\omega) & \text{if } f(\omega) \geq 0 \\ 0 - [-f(\omega)] = f(\omega) & \text{if } f(\omega) < 0 \end{array} \right\} = f(\omega),$$

and thus,

$$\boxed{f = f^+ - f^-.}$$

Theorem 1.5.6 Let $(\Omega, \mathscr{F})$ be a measurable space, $f, g : \Omega \to \mathbb{R}^*$ be $\mathscr{F}$-measurable functions, and $\alpha \in \mathbb{R}$. Then the following functions are all $\mathscr{F}$-measurable (provided that they are defined on Ω):

$$\alpha f, \quad f+g, \quad fg, \quad \frac{f}{g}, \quad \max(f,g), \quad \min(f,g), \quad f^+, \quad f^-, \quad |f|.$$

Proof. αf : Let α be any real number. Then for every $a \in \mathbb{R}$,

$$E_a := \{ \omega \in \Omega : \alpha f(\omega) > a \} = \begin{cases} \{ \omega \in \Omega : f(\omega) > \frac{a}{\alpha} \} & \text{if } \alpha > 0 \\ \{ \omega \in \Omega : f(\omega) < \frac{a}{\alpha} \} & \text{if } \alpha < 0 \\ \emptyset & \text{if } \alpha = 0 \text{ and } a \geq 0 \\ \Omega & \text{if } \alpha = 0 \text{ and } a < 0, \end{cases}$$

which, by Theorem 1.5.4, shows that $E_a \in \mathscr{F}$. It follows by Definition 1.5.2 that αf is $\mathscr{F}$-measurable. (Note that this argument works even when $f(\omega) = \pm\infty$ for some ω.)

$f+g$: First we claim: for each $a \in \mathbb{R}$ and $\omega \in \Omega$,

$$f(\omega)+g(\omega) > a \quad \Leftrightarrow \quad \exists q \in \mathbb{Q}, \quad f(\omega) > q \quad \text{and} \quad g(\omega) > a - q. \tag{1.14}$$

In fact the "$\Leftarrow$" part is obvious (even when $f(\omega) = \infty$ and/or $g(\omega) = \infty$).
To prove the "$\Rightarrow$" part, we assume that the left-hand side of (1.14) holds. Then in particular, $f(\omega) \neq -\infty$ and $g(\omega) \neq -\infty$, and

$$f(\omega) > a - g(\omega). \tag{1.15}$$

Now by (1.15) and density of $\mathbb{Q}$ in $\mathbb{R}$, we may pick $q \in \mathbb{Q}$ so that

$$f(\omega) > q > a - g(\omega).$$

from which the two inequalities on the right-hand side of (1.14) follow. (Note that the above argument works even in the case where $f(\omega)=\infty$ and/or $g(\omega)=\infty$.) Thus, the claim is proved.

Now for every $a \in \mathbb{R}$, we have by the claim and properties (A2σ) and (A3) of a σ-algebra that

$$\begin{aligned}
&\{\omega\in\Omega : f(\omega)+g(\omega)>a\} \\
&= \bigcup_{q\in\mathbb{Q}} \{\omega\in\Omega : f(\omega)>q \text{ and } g(\omega)>a-q\} \\
&= \bigcup_{q\in\mathbb{Q}} \Big(\underbrace{\{\omega\in\Omega : f(\omega)>q\}}_{\in\mathscr{F}\text{ by def. of meas. fn.}} \cap \underbrace{\{\omega\in\Omega : g(\omega)>a-q\}}_{\in\mathscr{F}\text{ by def. of meas. fn.}}\Big) \in \mathscr{F},
\end{aligned}$$

which, by definition, shows that $f+g$ is $\mathscr{F}$-measurable.

$\max(f,g)$: For every $a\in\mathbb{R}$,

$$\begin{aligned}
&\{\omega\in\Omega : \max[f(\omega),g(\omega)]>a\} \\
&= \{\omega\in\Omega : f(\omega)>a \text{ or } g(\omega)>a\} \\
&= \underbrace{\{\omega\in\Omega : f(\omega)>a\}}_{\in\mathscr{F}\text{ by def. of meas. fn.}} \cup \underbrace{\{\omega\in\Omega : g(\omega)>a\}}_{\in\mathscr{F}\text{ by def. of meas. fn.}} \in \mathscr{F},
\end{aligned}$$

by property (A2) of σ-algebras. This shows that $\max(f,g)$ is $\mathscr{F}$-measurable.

$\min(f,g)$: For every $a\in\mathbb{R}$,

$$\begin{aligned}
&\{\omega\in\Omega : \min[f(\omega),g(\omega)]>a\} \\
&= \{\omega\in\Omega : f(\omega)>a \text{ and } g(\omega)>a\} \\
&= \underbrace{\{\omega\in\Omega : f(\omega)>a\}}_{\in\mathscr{F}\text{ by def. of meas. fn.}} \cap \underbrace{\{\omega\in\Omega : g(\omega)>a\}}_{\in\mathscr{F}\text{ by def. of meas. fn.}} \in \mathscr{F},
\end{aligned}$$

by property (A3) of σ-algebras. This shows that $\min(f,g)$ is $\mathscr{F}$-measurable.

$f^+, f^-, |f|$: Measurability of these three functions follows from the fact that

$$f^+ = \max(f,0), \quad f^- = -\min(f,0), \quad |f| = f^+ + f^-,$$

that constant functions are $\mathscr{F}$-measurable, and from what has already been proved above.

fg : 1. First suppose that $f,g:\Omega\to[0,\infty]$. We claim: for each $0\le a<\infty$ and $\omega\in\Omega$,

$$f(\omega)g(\omega)>a \quad\Leftrightarrow\quad \exists q\in\mathbb{Q}^+, \quad f(\omega)>q \text{ and } g(\omega)>\frac{a}{q}. \tag{1.16}$$

In fact the "$\Leftarrow$" part is obvious (even when $f(\omega)=\infty$ and/or $g(\omega)=\infty$).

To prove the "$\Rightarrow$" part, we assume that the left-hand side of (1.16) holds. Then in particular, $f(\omega)>0$ and $g(\omega)>0$, and

$$f(\omega) > \frac{a}{g(\omega)}. \tag{1.17}$$

Now by (1.17) and density of $\mathbb{Q}$ in $\mathbb{R}$, we may pick $q\in\mathbb{Q}^+$ so that

$$f(\omega) > q > \frac{a}{g(\omega)}.$$

From here the two inequalities on the right-hand side of (1.16) follow. (Note that the above argument works even in the case where $f(\omega) = \infty$ and/or $g(\omega) = \infty$, since $\frac{a}{\infty} = 0$.) Thus, the claim is proved.

Thus, for every $0 \leq a < \infty$, we have by the claim and properties (A2σ) and (A3) of a sigma-algebra that

$$\begin{aligned} E_a &:= \{\omega \in \Omega \,:\, f(\omega)g(\omega) > a\} \\ &= \bigcup_{q \in \mathbb{Q}^+} \{\omega \in \Omega \,:\, f(\omega) > q \text{ and } g(\omega) > \tfrac{a}{q}\} \\ &= \bigcup_{q \in \mathbb{Q}^+} \Big(\underbrace{\{\omega \in \Omega \,:\, f(\omega) > q\}}_{\in \mathscr{F} \text{ by def. of meas. fn.}} \cap \underbrace{\{\omega \in \Omega \,:\, g(\omega) > \tfrac{a}{q}\}}_{\in \mathscr{F} \text{ by def. of meas. fn.}} \Big) \in \mathscr{F}. \end{aligned}$$

On the other hand, when $-\infty < a < 0$ then obviously, $E_a = \Omega \in \mathscr{F}$. This shows that $E_a \in \mathscr{F}$ for all $a \in \mathbb{R}$, that is, fg is $\mathscr{F}$-measurable.

2. Now let $f, g : \Omega \to \mathbb{R}^*$ be arbitrary. Then for all $\omega \in \Omega$,

$$\begin{aligned} f(\omega)g(\omega) &= \big(f^+(\omega) - f^-(\omega)\big)\big(g^+(\omega) - g^-(\omega)\big) \\ &= f^+(\omega)g^+(\omega) - f^+(\omega)g^-(\omega) - f^-(\omega)g^+(\omega) + f^-(\omega)g^-(\omega). \end{aligned}$$

Note that for a given ω, at most one single term on the right-hand side is nonzero (why ?), so we don't encounter $\infty - \infty$ and all expressions are defined. It follows from what has been already shown above that fg is $\mathscr{F}$-measurable.

$\frac{f}{g}$: This is Exercise 1.11 below.

■

Exercise 1.11 1. Complete the proof of the theorem by showing that
   (a) $1/g$ is $\mathscr{F}$-measurable (provided that $g(\omega) \neq 0$ for all ω),
   (b) f/g is $\mathscr{F}$-measurable (provided that f/g is defined for all $\omega \in \Omega$).
2. Suppose, $f, g : \Omega \to \mathbb{R}$ are $\mathscr{F}$-measurable. (i.e. both functions are finite valued.) Then measurability of fg can be proved in an easier way:
   (a) Use the definition of a measurable function to show that f^2 is $\mathscr{F}$-measurable.
   (b) Use the fact that

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

   and the results for $f + g$ and αf to show that fg is $\mathscr{F}$-measurable.

■

(R) It follows from the Theorem that the set of all *real valued* $\mathscr{F}$-measurable functions is a real vector space. However, the set of *extended real valued* $\mathscr{F}$-measurable functions is obviously not a vector space.

Sequences of Measurable Functions

Consider a sequence $(f_n)_{n=1}^{\infty}$ (also written $\{f_n\}_{n=1}^{\infty}$) of extended real valued functions,

$$f_n : \Omega \to \mathbb{R}^*.$$

As usual, we define new functions $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim_n} f_n$, $\underline{\lim_n} f_n : \Omega \to \mathbb{R}^*$ *pointwise* by

$$\left[\sup_n f_n\right](\omega) := \sup_n \left[f_n(\omega)\right] \tag{1.18}$$

$$\left[\inf_n f_n\right](\omega) := \inf_n \left[f_n(\omega)\right] \tag{1.19}$$

$$\left[\overline{\lim_n} f_n\right](\omega) := \overline{\lim_n} \left[f_n(\omega)\right] \tag{1.20}$$

$$\left[\underline{\lim_n} f_n\right](\omega) := \underline{\lim_n} \left[f_n(\omega)\right], \tag{1.21}$$

for each $\omega \in \Omega$. (These functions are defined for all $\omega \in \Omega$, since $\sup_n a_n$, $\inf_n a_n$, $\overline{\lim_n}\, a_n$, $\underline{\lim_n}\, a_n$ exist for all sequences (a_n) in $\mathbb{R}^*$.)

We also define $\lim_{n\to\infty} f_n : \Omega \to \mathbb{R}^*$ as a pointwise limit by

$$\left[\lim_{n\to\infty} f_n\right](\omega) := \lim_{n\to\infty} f_n(\omega)$$

provided that the right-hand side limit exists *for all* $\omega \in \Omega$.

Here are some remarks to put these definitions into perspective.

1. For all $\omega \in \Omega$ we have by the definition of $\overline{\lim_n}\, a_n$ that

$$\begin{aligned}\left[\overline{\lim_n} f_n\right](\omega) &\overset{(1.20)}{=} \overline{\lim_n}\left[f_n(\omega)\right] \overset{\text{def.}}{=} \inf_n \sup_{k\geq n}\left[f_k(\omega)\right] \\ &\overset{(1.18)}{=} \inf_n\left(\left[\sup_{k\geq n} f_k\right](\omega)\right) \overset{(1.19)}{=} \left[\inf_n \sup_{k\geq n} f_k\right](\omega),\end{aligned}$$

that is,

$$\boxed{\overline{\lim_n} f_n = \inf_n \sup_{k\geq n} f_k.} \tag{1.22}$$

In a similar way,

$$\boxed{\underline{\lim_n} f_n = \sup_n \inf_{k\geq n} f_k.} \tag{1.23}$$

2. If $f = \lim_{n\to\infty} f_n$ exists, then we say that f_n *converges pointwise to* f and write

$$f_n(\omega) \to f(\omega) \qquad \text{or} \qquad f_n \longrightarrow f \qquad \text{or} \qquad f_n \xrightarrow{\text{p.w.}} f.$$

Now recall: if $a_n, a \in \mathbb{R}^*$, then $a = \lim_{n\to\infty} a_n \Leftrightarrow a = \overline{\lim_n}\, a_n = \underline{\lim_n}\, a_n$. Thus,

$$\begin{aligned}
f_n \xrightarrow{\text{p.w.}} f \quad &\Leftrightarrow \quad f = \lim_{n\to\infty} f_n \\
&\Leftrightarrow \quad f(\omega) = \lim_{n\to\infty} f_n(\omega) \qquad \forall\, \omega \in \Omega \\
&\Leftrightarrow \quad f(\omega) = \overline{\lim_n}\,[f_n(\omega)] = \underline{\lim_n}\,[f_n(\omega)] \qquad \forall\, \omega \in \Omega \\
&\overset{(1.20)}{\underset{(1.21)}{\Leftrightarrow}} \quad f(\omega) = \left[\overline{\lim_n}\, f_n\right](\omega) = \left[\underline{\lim_n}\, f_n\right](\omega) \qquad \forall\, \omega \in \Omega \\
&\Leftrightarrow \quad f = \overline{\lim_n}\, f_n = \underline{\lim_n}\, f_n.
\end{aligned}$$

3. Suppose that $(f_n)\uparrow$. By this we mean that (f_n) is an increasing sequence:

$$f_1 \le f_2 \le f_3 \le \ldots \le f_n \le f_{n+1} \ldots,$$

which in turn means that

$$f_1(\omega) \le f_2(\omega) \le f_3(\omega) \le \ldots \le f_n(\omega) \le f_{n+1}(\omega) \ldots \qquad \forall\, \omega \in \Omega.$$

Since *every* increasing sequence $(a_n)\uparrow$ in $\mathbb{R}^*$ converges, then the sequence $(f_n(\omega))\uparrow$ converges for *every* $\omega \in \Omega$. That is, $\lim_n f_n$ exists.

4. We recall here the concept of *uniform convergence*. Let $f_n, f : \Omega \to \mathbb{R}$ be real valued functions. Then by definition,

$$\begin{aligned}
&(f_n) \text{ converges uniformly to } f \\
&\Leftrightarrow \quad \forall\, \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{so that } |f_n(\omega) - f(\omega)| < \varepsilon \quad \forall\, \omega \in \Omega, \forall\, n \ge N.
\end{aligned}$$

(The important point here is that the same $N = N(\varepsilon)$ can be chosen for all ω.) We write $f_n \rightrightarrows f$ to denote uniform convergence. It is known and easy to show that

$$f_n \rightrightarrows f \quad \Leftrightarrow \quad \lim_{n\to\infty} \sup_{\omega\in\Omega} |f_n(\omega) - f(\omega)| = 0.$$

Theorem 1.5.7 Let $(\Omega, \mathscr{F})$ be a measurable space, and (f_n) a sequence of $\mathscr{F}$-measurable functions, $f_n : \Omega \to \mathbb{R}^*$.

1. The functions $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim_n}\, f_n$, and $\underline{\lim_n}\, f_n$ are all $\mathscr{F}$-measurable.
2. If the sequence (f_n) converges pointwise, say $f_n \xrightarrow{\text{p.w.}} f$, then $f : \Omega \to R^*$ is also $\mathscr{F}$-measurable.

Proof. $\sup_n f_n$: For every $a \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned}
\sup_n f_n(\omega) > a \quad &\Leftrightarrow \quad a \text{ is not an upper bound of } \{f_n(\omega) : n \in \mathbb{N}\} \\
&\Leftrightarrow \quad \exists n \in \mathbb{N}, \quad f_n(\omega) > a \\
&\Leftrightarrow \quad \omega \in \bigcup_{n=1}^{\infty} \{\omega \in \Omega : f_n(\omega) > a\}.
\end{aligned}$$

Hence for all $a \in \mathbb{R}$,

$$\begin{aligned}
\left\{\omega \in \Omega : [\sup_n f_n](\omega) > a\right\} &= \left\{\omega \in \Omega : \sup_n [f_n(\omega)] > a\right\} \\
&= \bigcup_{n=1}^{\infty} \underbrace{\{\omega \in \Omega : f_n(\omega) > a\}}_{\in \mathscr{F} \text{ as } f_n \text{ is } \mathscr{F}\text{-meas.}} \in \mathscr{F}
\end{aligned}$$

by property (A2σ) of σ-algebras. This shows that $\sup_n f_n$ is $\mathscr{F}$-measurable.

$\inf_n f_n$: In a similar way, for every $a \in \mathbb{R}$,

$$\left\{w \in \Omega : \left[\inf_n f_n\right](\omega) < a\right\} = \left\{w \in \Omega : \inf_n \left[f_n(\omega)\right] < a\right\}$$
$$= \bigcup_{n=1}^{\infty} \underbrace{\{\omega \in \Omega : f_n(\omega) < a\}}_{\in \mathscr{F} \text{ as } f_n \text{ is } \mathscr{F}\text{-meas.}} \in \mathscr{F}$$

by property (A2σ) of σ-algebras. This shows that $\inf_n f_n$ is $\mathscr{F}$-measurable.

$\overline{\lim}_n f_n$, $\underline{\lim}_n f_n$: By the above, for each $n \in \mathbb{N}$, the functions

$$g_n := \sup_{k \geq n} f_n \qquad \text{and} \qquad h_n := \inf_{k \geq n} f_n$$

are all $\mathscr{F}$-measurable. Applying the above again, it follows that

$$\overline{\lim_n} f_n = \inf_n \sup_{k \geq n} f_k = \inf_n g_n \qquad \text{and} \qquad \underline{\lim_n} f_n = \sup_n \inf_{k \geq n} f_k = \sup_n h_n$$

are both $\mathscr{F}$-measurable.

$\lim_{n\to\infty} f_n$: Suppose, $f_n \xrightarrow{\text{p.w.}} f$. Since by Remark 1.5

$$f = \overline{\lim_n} f_n,$$

it follows from the above that f is $\mathscr{F}$-measurable.

■

Exercise 1.12 Let $(\Omega, \mathscr{F})$ be a measurable space, $\{E_n\} \subseteq \mathscr{F}$ and $f_n = \mathbf{1}_{E_n}$ for each n. Show:

1. If $\{E_n\} \uparrow$ and $E = \bigcup_{n=1}^{\infty} E_n$, then $\{f_n\} \uparrow$ and $\mathbf{1}_E = \lim_{n\to\infty} f_n$.
2. If $\{E_n\} \downarrow$ and $E = \bigcap_{n=1}^{\infty} E_n$, then $\{f_n\} \downarrow$ and $\mathbf{1}_E = \lim_{n\to\infty} f_n$.

■

1.6 Simple, Measurable Functions

In this section, we prove a theorem about measurable functions which will of fundamental importance when discussing the Lebesgue integral. It says that every measurable functions is the limit of a sequence of measurable functions with finite range.

Throughout this section, $(\Omega, \mathscr{F})$ will denote a measurable space.

Definition 1.6.1 A function $\varphi : \Omega \to \mathbb{R}$ whose range is a finite set is called a *simple function.*

(R) Here are some properties which will be used throughout.

1. Let $\varphi : \Omega \to \mathbb{R}$ be simple, say range$(\varphi) = \{a_1, \ldots a_n\}$. Set

$$A_k := \varphi^{-1}(\{a_k\}) = \{\omega \in \Omega : \varphi(\omega) = a_k\}, \qquad k = 1, \ldots, n \tag{1.24}$$

Then clearly

(a) $A_k \cap A_j = \emptyset$ for $k \neq j$

(b) $\bigcup_{k=1}^{n} A_k = \Omega.$

We can thus write

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}. \tag{1.25}$$

This is called the *canonical representation* (or *canonical form*) of φ.

2. If $a_k = 0$ for some k, say $a_{k_o} = 0$, then $a_{k_o}\mathbf{1}_{A_{k_o}}(\omega) = 0$ for all $\omega \in \Omega$. We may thus remove this term from the canonical representation (1.25), and write

$$\varphi = \sum_{\substack{k=1 \\ k \neq k_o}}^{n} a_k \mathbf{1}_{A_k}. \tag{1.26}$$

Note, however, that $\bigcup_{k \neq k_o} A_k \neq \Omega$.

3. If $A_k \in \mathscr{F}$ for all k, then by Example 1.7, each function $\mathbf{1}_{A_k}$ is $\mathscr{F}$-measurable, so that by Theorem 1.5.6, φ is $\mathscr{F}$-measurable.
Conversely, suppose φ is $\mathscr{F}$-measurable. Since each singleton $\{a_k\}$ is a Borel set, then by (1.24), $A_k \in \mathscr{F}$ for all k.
We have shown:

$$\boxed{\varphi \text{ is } \mathscr{F}\text{-measurable} \quad \Leftrightarrow \quad A_k \in \mathscr{F} \quad \forall k.}$$

when φ has the canonical representation.

4. Consider a function of the form

$$\varphi = \sum_{k=1}^{n} c_k \mathbf{1}_{C_k} \qquad (c_k \in \mathbb{R},\ C_k \subseteq \Omega). \tag{1.27}$$

Since $\operatorname{range}(\varphi) \subseteq \{\sum_{k=1}^{n} \alpha_k c_k \ : \ \alpha_k \in \{0,1\}\ \}$, which is a finite set, then φ is simple. However (1.27) is *not* its canonical representation unless the sets C_k are disjoint and the numbers c_k are distinct.
When $C_k \in \mathscr{F}$ for all k, then clearly, φ is $\mathscr{F}$-measurable. However, when φ is $\mathscr{F}$-measurable, we *cannot* conclude in general that $C_k \in \mathscr{F}$!

■ **Example 1.8** Consider $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

1. A function of the form

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{I_k} \qquad (I_k \text{ an interval})$$

is called a *step function*. Since $I_k \in \mathscr{B}(\mathbb{R})\ \forall k$ (intervals are Borel sets), then every step function is a simple, Borel-measurable function.

2. A function of the form

$$\varphi = \alpha \mathbf{1}_{\mathbb{Q}} + \beta \mathbf{1}_{\mathbb{Q}^c} \qquad (\alpha, \beta \in \mathbb{R},\ \alpha \neq \beta)$$

is a simple, Borel-measurable function which is *not* a step function.

■

Notation: Let us set

$$\mathscr{L} = \mathscr{L}(\Omega, \mathscr{F}) := \{f : \Omega \to \mathbb{R}^* \mid f \text{ is } \mathscr{F}\text{-measurable}\}$$
$$\mathscr{L}^+ = \mathscr{L}^+(\Omega, \mathscr{F}) := \{f \in \mathscr{L} \mid f \geq 0\}.$$

Note that these are not vector spaces. However, for all $f, g \in \mathscr{L}^+$ and real numbers $\alpha, \beta \geq 0$ we have that $\alpha f + \beta g \in \mathscr{L}^+$.

Next we consider the sets of simple functions,

$$\begin{aligned}\mathscr{S} &= \mathscr{S}(\Omega,\mathscr{F}) := \{\varphi : \Omega \to \mathbb{R} \,|\, \varphi \text{ is simple and } \mathscr{F}\text{-measurable}\}\\ \mathscr{S}^+ &= \mathscr{S}^+(\Omega,\mathscr{F}) := \{\varphi \in \mathscr{S} \,|\, \varphi \geq 0\}.\end{aligned}$$

Then

1. $\mathscr{S} \subseteq \mathscr{L}$ and $\mathscr{S}^+ \subseteq \mathscr{L}^+$.
2. Let $\varphi \in \mathscr{S}$ with canonical representation $\varphi = \sum a_k \mathbf{1}_{A_k}$. Then

$$\varphi \in \mathscr{S}^+ \quad \Leftrightarrow \quad a_k \geq 0 \quad \forall k.$$

3. $\mathscr{S}$ is a vector space. In fact, let $\varphi, \psi \in \mathscr{S}$ and $\alpha, \beta \in \mathbb{R}$. If

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}, \qquad \text{and} \qquad \psi = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}$$

are the canonical representations, then the *linear combination*

$$\alpha\varphi + \beta\psi = \sum_{k=1}^{n}\sum_{j=1}^{m} (\alpha a_k + \beta b_j)\,\mathbf{1}_{A_k \cap B_j}$$

is a simple and $\mathscr{F}$-measurable function, since $A_k, B_j \in \mathscr{F}$ for all k, j.

4. In a similar way, whenever $\varphi, \psi \in \mathscr{S}$, then $\varphi\psi \in \mathscr{S}$. This can be seen from

$$\varphi\psi = \left(\sum_{k=1}^{n} a_k \mathbf{1}_{A_k}\right)\left(\sum_{j=1}^{m} b_j \mathbf{1}_{B_j}\right) = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k b_j \mathbf{1}_{A_k}\mathbf{1}_{B_j} = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k b_j \mathbf{1}_{A_k \cap B_j}.$$

Theorem 1.6.1 *(Structure Theorem for non-negative, measurable functions.)*
1. Let $f \in \mathscr{L}^+$. Then there exists a sequence $(\varphi_n) \uparrow$ in $\mathscr{S}^+$ with $\varphi_n \xrightarrow{\text{p.w.}} f$.
2. If $f \in \mathscr{L}^+$ is finite valued and bounded in $\mathbb{R}$ (i.e. $0 \leq f \leq N$, $\exists N \in \mathbb{N}$), then the sequence $(\varphi_n) \uparrow$ can be chosen so that $\varphi_n \rightrightarrows f$.

Proof. 1. Let $f \in \mathscr{L}^+$ be given. The idea is to split range(f) into small intervals and define the functions φ_n using these intervals, as follows:
Let $n \in \mathbb{N}$ be given. For each k, $1 \leq k \leq n2^n$, we set

$$A_{(n,k)} := \left\{ \omega \in \Omega : \frac{k-1}{2^n} \leq f(\omega) < \frac{k}{2^n} \right\}$$

and we also set

$$A_{(n,0)} := \left\{ \omega \in \Omega : f(\omega) \geq n \right\}.$$

Note that $A_{(n,k)} \in \mathscr{F}$ for all k since the function f is $\mathscr{F}$-measurable, so that

$$\varphi_n := \sum_{k=1}^{n2^n} \frac{k-1}{2^n}\mathbf{1}_{A_{(n,k)}} + n\mathbf{1}_{A_{(n,0)}} \in \mathscr{S}^+,$$

and this is the canonical form of φ_n. We perform the following steps.

(a) By definition of φ_n, for all $\omega \in \Omega$ we have

$$\varphi_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \le f(\omega) < \frac{k}{2^n}, \quad 1 \le k \le n2^n \\ n & \text{if } f(\omega) \ge n. \end{cases} \tag{1.28}$$

In particular,

$$0 \le \varphi_n(\omega) \le n \qquad \forall\, \omega \in \Omega.$$

(b) Let us compare the values of $\varphi_n(\omega)$ and $\varphi_{n+1}(\omega)$ for $\omega \in \Omega$. As in (1.28),

$$\varphi_{n+1}(\omega) = \begin{cases} \frac{k-1}{2^{n+1}} & \text{if } \frac{k-1}{2^{n+1}} \le f(\omega) < \frac{k}{2^{n+1}}, \quad 1 \le k \le (n+1)2^{n+1} \\ n+1 & \text{if } f(\omega) \ge n+1. \end{cases}$$

Consider three cases:
Case 1: $0 \le f(\omega) < n$. In this case, $\exists k$, $1 \le k \le n2^n$ so that

$$f(\omega) \in \left[\tfrac{k-1}{2^n}, \tfrac{k}{2^n}\right), \qquad \text{and thus} \qquad \varphi(\omega) = \frac{k-1}{2^n}.$$

Now going from n to $n+1$, this interval is split into the two subintervals

$$\left[\tfrac{k-1}{2^n}, \tfrac{k-\frac{1}{2}}{2^n}\right) = \left[\tfrac{2k-2}{2^{n+1}}, \tfrac{2k-1}{2^{n+1}}\right) \qquad \text{and} \qquad \left[\tfrac{k-\frac{1}{2}}{2^n}, \tfrac{k}{2^n}\right) = \left[\tfrac{2k-1}{2^{n+1}}, \tfrac{2k}{2^{n-1}}\right).$$

Now if $f(\omega) \in \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right)$ then

$$\varphi_{n+1}(\omega) = \frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} = \varphi_n(\omega),$$

while if $f(\omega) \in \left[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right)$ then

$$\varphi_{n+1}(\omega) = \frac{2k-1}{2^{n+1}} > \frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} = \varphi_n(\omega).$$

Combining both possibilities, we see that

$$\varphi_n(\omega) \le \varphi_{n+1}(\omega).$$

Case 2: $n \le f(\omega) < n+1$. Then

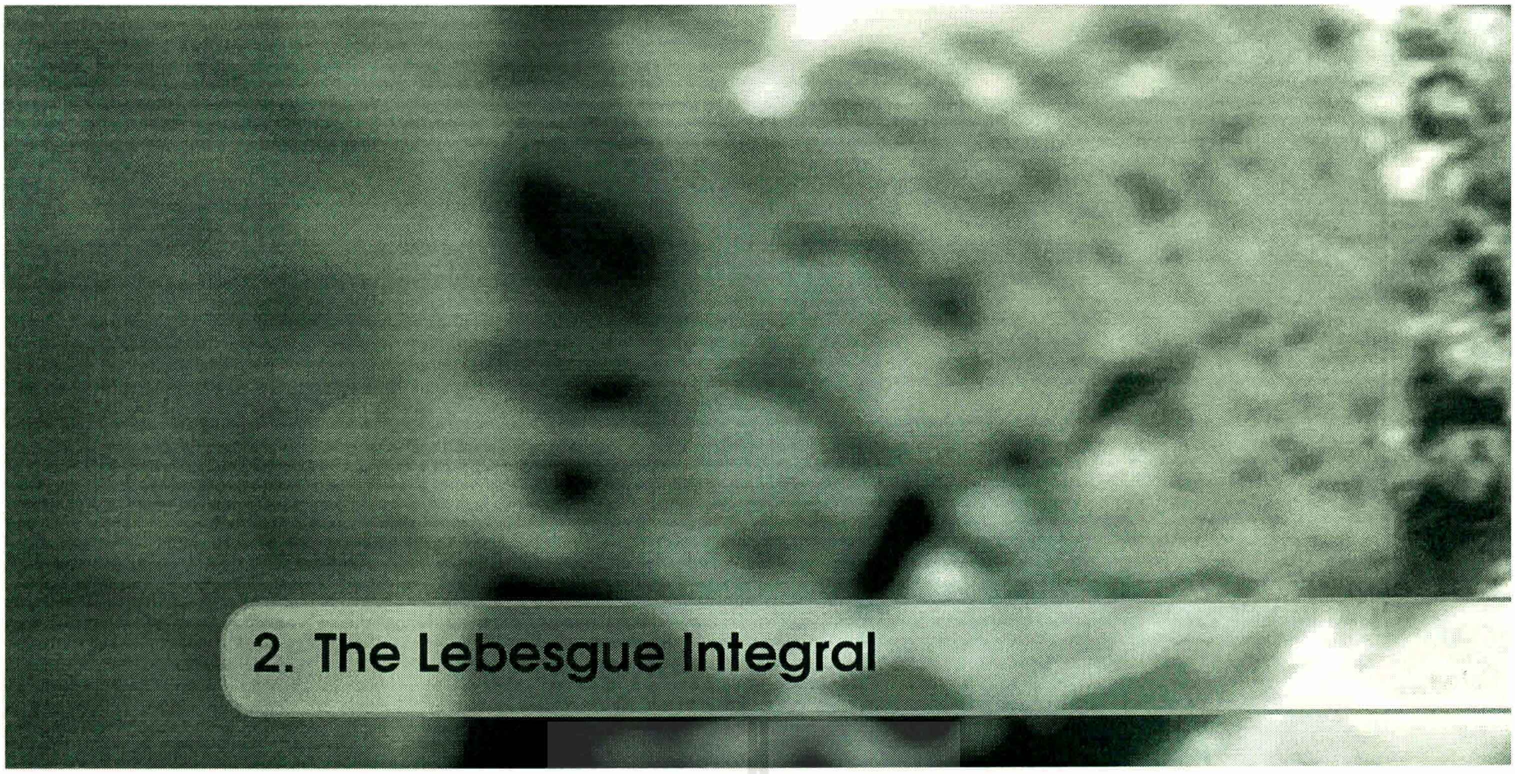
$$\varphi_n(\omega) = n \qquad \text{while} \qquad \varphi_{n+1}(\omega) = \frac{k-1}{2^{n+1}}$$

where k is the unique integer so that $\frac{k-1}{2^{n+1}} \le f(\omega) \le \frac{k}{2^{n+1}}$. Note that

$$\frac{n2^{n+1}}{2^{n+1}} = n \le f(\omega)$$

so that $n2^{n+1} \le k-1$. It follows that

$$\varphi_n(\omega) = n = \frac{n2^{n+1}}{2^{n+1}} \le \frac{k-1}{2^{n+1}} = \varphi_{n+1}(\omega).$$



2. The Lebesgue Integral

Having discussed measure spaces and measurable functions, we are now ready to introduce the Lebesgue integral on such spaces, and study its properties. Of particular interest is the behaviour of the integral with regards to limits of sequences of functions, and over sets of measure zero.

Throughout this chapter, $(\Omega, \mathscr{F}, \mu)$ will denote an arbitrary measure space.

2.1 The Integral of Simple, Nonnegative Measurable Functions

Definition 2.1.1 Let $\varphi \in \mathscr{S}^+$ be given, expressed in canonical form

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}, \qquad \text{where} \qquad A_k = \{\omega \in \Omega : \varphi(\omega) = a_k\} \in \mathscr{F}.$$

We define

$$\boxed{\int \varphi \, d\mu \; := \; \sum_{k=1}^{n} a_k \mu(A_k).} \tag{2.1}$$

R A couple of remarks:

1. Since $a_k \geq 0$ for all k, then $0 \leq \int \varphi \, du \leq \infty$ for $\varphi \in \mathscr{S}^+$.
2. When $a_k = 0$ then $a_k \mu(A_k) = 0 \cdot \mu(A_k) = 0$, even in case where $\mu(A_k) = \infty$! We may thus remove this term,

$$\int \varphi \, d\mu \; = \; \sum_{\substack{k=1 \\ a_k \neq 0}}^{n} a_k \mu(A_k). \tag{2.2}$$

Example 2.1 Let $(\Omega,\mathscr{F},\mu)=(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda)$.

1. If $\varphi=2\cdot\mathbf{1}_{[0,1]}+\mathbf{1}_{(1,2)}+3\cdot\mathbf{1}_{\{2\}}+\frac{1}{2}\cdot\mathbf{1}_{[3,5]}$ is a step function, then using (2.2),

$$\int\varphi\,d\lambda = 2\cdot\lambda([0,1])+1\cdot\lambda((1,2))+3\cdot\lambda(\{2\})+\frac{1}{2}\cdot\lambda([3,5])$$
$$=2\cdot1+1\cdot1+3\cdot0+\frac{1}{2}\cdot2=4.$$

2. If $\psi=\mathbf{1}_{\mathbb{Q}}\in\mathscr{S}^+$, then

$$\int\psi\,d\lambda=\int[1\cdot\mathbf{1}_{\mathbb{Q}}+0\cdot\mathbf{1}_{\mathbb{Q}^c}]\,d\lambda = 1\cdot\lambda(\mathbb{Q})+0\cdot\lambda(\mathbb{Q}^c)=1\cdot0+0=0.$$

3. If $f=c=\text{const}\geq0$, then $f\in\mathscr{S}^+$ and

$$\int f\,d\lambda=\int c\cdot\mathbf{1}_{\mathbb{R}}\,d\lambda=c\cdot\infty=\begin{cases}\infty & \text{if } c>0\\ 0 & \text{if } c=0.\end{cases}$$

■

Exercise 2.1 Let $(\Omega,\mathscr{F},\mu)=(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda)$. Find $\int\varphi\,d\lambda$ if

1. $\varphi(x)=\begin{cases}\lfloor x\rfloor & \text{if } 0\leq x\leq 10\\ 0 & \text{else.}\end{cases}$
2. $\varphi(x)=\begin{cases}\lfloor x^2\rfloor & \text{if } 0\leq x\leq 3\\ 0 & \text{else.}\end{cases}$
3. $\varphi(x)=\begin{cases}\lfloor 1+\sin x\rfloor & \text{if } 0\leq x\leq 2\pi\\ 0 & \text{else.}\end{cases}$

(Recall here that $\lfloor x\rfloor=\max\{n\in\mathbb{Z}:n\leq x\}$ is the greatest integer function. ■

Definition 2.1.2 A finite or countably infinite collection $\{B_j\}_{j=1}^m$ of subsets of Ω ($m\in\mathbb{N}$ or $m=\{\infty\}$) is called a (finite resp. countable) *partition* of Ω, if

(P1) the sets in $\{B_j\}_{j=1}^m$ are mutually disjoint: $B_k\cap B_j=\emptyset$ for $k\neq j$,

(P2) $\bigcup_{j=1}^m B_j=\Omega$.

If in addition, $B_j\in\mathscr{F}$ for all j, then $\{B_j\}_{j=1}^m$ is called a partition of Ω *by $\mathscr{F}$-measurable sets* or a *measurable partition*.

For example, the collection of intervals $\{I_k=[k,k+1)\}_{k\in\mathbb{Z}}$ is a countable partition of $\mathbb{R}$ by Borel sets.

R If $\{A_k\}_{k=1}^n$ and $\{B_j\}_{j=1}^m$ are partitions of Ω by $\mathscr{F}$-measurable sets, then

$$\{A_k\cap B_j:k=1,\dots n,\ j=1,\dots m\}$$

is also a partition of Ω by measurable sets. In fact by (A3), $A_k\cap B_j\in\mathscr{F}$ for all k,j.
Now if $\omega\in(A_k\cap B_j)\cap(A_{\tilde{k}}\cap B_{\tilde{j}})$, then $\omega\in A_k\cap A_{\tilde{k}}$ and $\omega\in B_j\cap B_{\tilde{j}}$, so that by (P1), $k=\tilde{k}$ and $j=\tilde{j}$. This shows that the sets $A_k\cap B_j$, $k=1,\dots n$, $j=1,\dots m$, are mutually disjoint.
On the other hand,

$$\bigcup_{k=1}^n\bigcup_{j=1}^m(A_k\cap B_j)=\bigcup_{k=1}^n\left(A_k\cap\left[\bigcup_{j=1}^m B_j\right]\right)\overset{\text{(P2)}}{=}\bigcup_{k=1}^n(A_k\cap\Omega)=\bigcup_{k=1}^n A_k\overset{\text{(P2)}}{=}\Omega.$$

Lemma 2.1.1 Let $\varphi \in \mathscr{S}^+$, and let

$$\varphi = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j} \tag{2.3}$$

be *any* representation of φ, where $\{B_1, \ldots B_m\}$ is a finite partition of Ω by measurable sets. Then

$$\int \varphi \, d\mu = \sum_{j=1}^{m} b_j \mu(B_j).$$

Proof. Let

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}$$

be the *canonical representation* of φ. We make the following important observation:

Observation. Let k, j be given, $1 \leq k \leq n$, $1 \leq j \leq m$. Suppose first that $A_k \cap B_j \neq \emptyset$. Then we can pick $\omega \in A_k \cap B_j$, and by (P1),

$$a_k = \varphi(\omega) = b_j.$$

It follows that

$$a_k \mu(A_k \cap B_j) = b_j \mu(A_k \cap B_j). \tag{2.4}$$

On the other hand, when $A_k \cap B_j = \emptyset$, then $\mu(A_k \cap B_j) = 0$, so that (2.4) holds trivially. Thus, (2.4) holds for all choices of k and j.

Now

$$\begin{aligned}
\int \varphi \, d\mu &\overset{\text{def}}{=} \sum_{k=1}^{n} a_k \mu(A_k) \\
&\overset{(P2)}{=} \sum_{k=1}^{n} a_k \mu\left(A_k \cap \left[\bigcup_{j=1}^{m} B_j\right]\right) = \sum_{k=1}^{n} a_k \mu\left(\bigcup_{j=1}^{m} [A_k \cap B_j]\right) \\
&\overset{(M2)}{=} \sum_{k=1}^{n} a_k \sum_{j=1}^{m} \mu(A_k \cap B_j) \qquad \{A_k \cap B_j\}_{j=1}^{m} \text{ are disjoint} \\
&= \sum_{j=1}^{m} \sum_{k=1}^{n} a_k \mu(A_k \cap B_j) \\
&\overset{(2.4)}{=} \sum_{j=1}^{m} \sum_{k=1}^{n} b_j \mu(A_k \cap B_j) \qquad \{A_k \cap B_j\}_{k=1}^{m} \text{ are disjoint} \\
&\overset{(M2)}{=} \sum_{j=1}^{m} b_j \mu\left(\bigcup_{k=1}^{n} [A_k \cap B_j]\right) = \sum_{j=1}^{m} b_j \mu\left(\left[\bigcup_{k=1}^{n} A_k\right] \cap B_j\right) \\
&\overset{(P2)}{=} \sum_{j=1}^{m} b_j \mu\left(\Omega \cap B_j\right) = \sum_{j=1}^{m} b_j \mu(B_j),
\end{aligned}$$

which proves the lemma. ■

Theorem 2.1.2 *(Properties of the integral).* Let $\varphi, \psi \in \mathscr{S}^+$ and $c \geq 0$.

1. $\int c\varphi \, d\mu = c \int \varphi \, d\mu.$ *("the integral is positive homogeneous")*
2. $\int (\varphi + \psi) \, d\mu = \int \varphi \, d\mu + \int \psi \, d\mu.$ *("the integral is additive")*
3. If $\varphi \leq \psi$ then $\int \varphi \, d\mu \leq \int \psi \, d\mu.$ *("the integral is monotone")*

Proof. Clearly, $c\varphi \in \mathscr{S}^+$ and $\varphi + \psi \in \mathscr{S}^+$. Furthermore, let

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k} \qquad \text{and} \qquad \psi = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}$$

denote the canonical representations.

1. Note that

$$c\varphi = c \sum_{k=1}^{n} a_k \mathbf{1}_{A_k} = \sum_{k=1}^{n} (ca_k) \mathbf{1}_{A_k}.$$

(This is not the canonical representation when $c = 0$!) Applying Lemma 2.1.1 we obtain that

$$\int c\varphi \, d\mu \overset{\text{lem}}{=} \sum_{k=1}^{n} (ca_k)\mu(A_k) = c \sum_{k=1}^{n} a_k \mu(A_k) \overset{\text{def}}{=} c \int \varphi \, d\mu.$$

2. Note that for each k,

$$A_k = A_k \cap \left[\bigcup_{j=1}^{m} B_j \right] = \bigcup_{j=1}^{m} (A_k \cap B_j),$$

a disjoint union. Thus,

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k} \overset{\text{exer}}{\underset{1.8}{=}} \sum_{k=1}^{n} a_k \sum_{j=1}^{m} \mathbf{1}_{A_k \cap B_j} = \sum_{k=1}^{n} \sum_{j=1}^{m} a_k \mathbf{1}_{A_k \cap B_j}.$$

In a similar way,

$$\psi = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j} \overset{\text{exer}}{\underset{1.8}{=}} \sum_{j=1}^{m} b_j \sum_{k=1}^{n} \mathbf{1}_{A_k \cap B_j} = \sum_{k=1}^{n} \sum_{j=1}^{m} b_j \mathbf{1}_{A_k \cap B_j}.$$

It follows by the distributive law that

$$\varphi + \psi = \sum_{k=1}^{n} \sum_{j=1}^{m} (a_k + b_j) \mathbf{1}_{A_k \cap B_j}.$$

Thus by Remark 2.1 and Lemma 2.1.1,

$$\begin{aligned}
\int (\varphi + \psi) \, d\mu &\overset{\text{lem}}{=} \sum_{k=1}^{n} \sum_{j=1}^{m} (a_k + bj)\mu(A_k \cap B_j) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{m} a_k \mu(A_k \cap B_j) + \sum_{k=1}^{n} \sum_{j=1}^{m} b_j \mu(A_k \cap B_j) \\
&\overset{\text{lem}}{=} \int \varphi \, d\mu + \int \psi \, d\mu.
\end{aligned}$$

3. Suppose, $\varphi \leq \psi$. We begin with an observation which is similar to that in the proof Lemma 2.1.1: If $A_k \cap B_j \neq \emptyset$, then we can pick $\omega \in A_k \cap B_j$, and by (P1),

$$a_k = \varphi(\omega) \leq \psi(\omega) = b_j.$$

It follows that

$$a_k \mu(A_k \cap B_j) \leq b_j \mu(A_k \cap B_j). \tag{2.5}$$

On the other hand, when $A_k \cap B_j = \emptyset$, then $\mu(A_k \cap B_j) = 0$, so that (2.5) holds as well. This shows that (2.5) holds for all choices of k and j. Then by Lemma 2.1.1,

$$\int \varphi \, d\mu \overset{\text{lem}}{=} \sum_{k=1}^{n} \sum_{j=1}^{m} a_k \mu(A_k \cap B_j) \leq \sum_{k=1}^{n} \sum_{j=1}^{m} b_j \mu(A_k \cap B_j) \overset{\text{lem}}{=} \int \psi \, d\mu.$$

Thus, the theorem is proved. ■

Corollary 2.1.3 Let $\varphi \in \mathscr{S}^+$ have an *arbitrary* representation

$$\varphi = \sum_{k=1}^{n} c_k \mathbf{1}_{C_k}, \qquad (c_k \geq 0,\ C_k \in \mathscr{F}).$$

Then

$$\int \varphi \, d\mu = \sum_{k=1}^{n} c_k \mu(C_k).$$

Proof. Applying the above Theorem and Lemma 2.1.1, we obtain

$$\begin{aligned} \int \varphi \, d\mu &= \int \left[\sum_{k=1}^{n} c_k \mathbf{1}_{C_k} \right] d\mu \overset{\text{thm}}{=} \sum_{k=1}^{n} \int c_k \mathbf{1}_{C_k} \, d\mu \\ &= \sum_{k=1}^{n} \left[\int c_k \cdot \mathbf{1}_{C_k} + 0 \cdot \mathbf{1}_{[C_k]^c} \right] d\mu \\ &\overset{\text{lem}}{=} \sum_{k=1}^{n} \left[c_k \cdot \mu(C_k) + 0 \cdot \mu\left([C_k]^c\right) \right] = \sum_{k=1}^{n} c_k \mu(C_k). \end{aligned}$$

■

We have a preliminary result relating limits of functions and the Lebesgue integral, and will be generalized later.

Lemma 2.1.4 *(Monotone Convergence Theorem for $\mathscr{S}^+$)* Let $(\varphi_n)\uparrow$ be an increasing sequence in $\mathscr{S}^+$, and $\varphi \in \mathscr{S}^+$. If

$$\varphi_n \xrightarrow{\text{p.w.}} \varphi$$

then

$$\int \varphi_n \, d\mu \longrightarrow \int \varphi \, d\mu.$$

(That is, $\lim_{n\to\infty} \int \varphi_n \, d\mu = \int \lim_{n\to\infty} \varphi_n \, d\mu$.)

Proof. By assumption,

$$\varphi_1 \le \varphi_2 \le \varphi_3 \le \ldots\ldots \le \varphi_n \le \varphi_{n+1} \le \ldots\ldots \le \varphi.$$

Hence by monotonicity of the integral (Theorem 2.1.2),

$$\int \varphi_1 \, d\mu \le \int \varphi_2 \, d\mu \le \int \varphi_3 \, d\mu \le \ldots \le \int \varphi_n \, d\mu \le \int \varphi_{n+1} \, d\mu \le \ldots \le \int \varphi \, d\mu.$$

Since every *increasing* sequence in $\mathbb{R}^*$ converges to its supremum,

$$\lim_{n\to\infty} \int \varphi_n \, d\mu \quad \text{exists, and} \quad \lim_{n\to\infty} \int \varphi_n \, d\mu \le \int \varphi \, d\mu.$$

It is thus left to show the reverse inequality, namely that

$$\int \varphi \, d\mu \le \lim_{n\to\infty} \int \varphi_n \, d\mu. \tag{2.6}$$

Claim: For *every* ε, $0 < \varepsilon < 1$, there exists a sequence (ψ_n) in $\mathscr{S}^+$ (which depends on ε) satisfying

(i) $\psi_n \le \varphi_n \quad \forall n$,

(ii) $\lim_{n\to\infty} \int \psi_n \, d\mu = (1-\varepsilon) \int \varphi \, d\mu$.

To prove the claim, write φ in canonical form,

$$\varphi = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k}.$$

(Recall that $a_k \ge 0 \; \forall k$, and that $\{A_1, \ldots, A_m\}$ is a partition of Ω by $\mathscr{F}$-measurable sets.) Now let ε be given, $0 < \varepsilon < 1$. For each $n \in \mathbb{N}$ and $1 \le k \le m$, we set

$$A_{k,n} := \{\, \omega \in A_k : (1-\varepsilon)a_k \le \varphi_n(\omega) \,\},$$

and we then set

$$\psi_n = \sum_{k=1}^{m} (1-\varepsilon) a_k \mathbf{1}_{A_{k,n}}.$$

Some observations:

(1) Clearly, $\psi_n \ge 0$ for all n.

(2) As φ_n is $\mathscr{F}$-measurable, and $A_{k,n} = A_k \cap \{\, \omega \in \Omega : \varphi_n(\omega) \ge (1-\varepsilon)a_k \,\}$, it follows that $A_{k,n} \in \mathscr{F}$ for all k, n. Hence, each ψ_n is $\mathscr{F}$-measurable. Together with (1) we conclude that $\psi_n \in \mathscr{S}^+$ for all n.

(3) $\psi_n \leq \varphi_n$ for all n. In fact, let $\omega \in \Omega$ and n be fixed. Then $\varphi(\omega) = a_k$ for some k, and hence $\omega \in A_k$.
If $\omega \in A_{k,n}$, then

$$\psi_n(\omega) = (1-\varepsilon)a_k \leq a_k = \varphi_n(\omega).$$

On the other hand, if $\omega \in A_k \backslash A_{k,n}$ then

$$\psi_n(\omega) = 0 \leq \varphi_n(\omega).$$

(4) For each k, the sequence of sets $\{A_{k,n}\}_{n=1}^{\infty}$ is increasing. In fact, since $\varphi_n \leq \varphi_{n+1}$, then

$$\begin{aligned} A_{k,n} &= \{\, \omega \in A_k : (1-\varepsilon)a_k \leq \varphi_n(\omega) \,\} \\ &\subseteq \{\, \omega \in A_k : (1-\varepsilon)a_k \leq \varphi_{n+1}(\omega) \,\} = A_{k,n+1}. \end{aligned}$$

(5) For each k, $A_k = \bigcup_{n=1}^{\infty} A_{k,n}$.
In fact, since by definition, $A_{k,n} \subseteq A_k$, then $\bigcup_{n=1}^{\infty} A_{k,n} \subseteq A_k$, for all k.
To show the reverse inclusion, let $\omega \in A_k$ be arbitrary. Since

$$\varphi_n(\omega) \to \varphi(\omega) = a_k \qquad \text{and} \qquad \varphi_n \leq \varphi,$$

$\exists N = N(\omega, \varepsilon)$ so that

$$(1-\varepsilon)a_k = (1-\varepsilon)\varphi(\omega) \leq \varphi_n(\omega) \leq \varphi(\omega) \qquad \forall n \geq N.$$

It follows that $\omega \in A_{k,n}$ for all $n \geq N$. This shows that $A_k \subseteq \bigcup_{n=1}^{\infty} A_{k,n}$.

Using these observations, we obtain

$$\begin{aligned} \lim_{n\to\infty} \int \psi_n \, d\mu &= \lim_{n\to\infty} \int \left[\sum_{k=1}^{m} (1-\varepsilon) a_k \mathbf{1}_{A_{k,n}} \right] d\mu \\ &\overset{\text{cor}}{\underset{2.1.3}{=}} \lim_{n\to\infty} \sum_{k=1}^{m} (1-\varepsilon) a_k \mu(A_{k,n}) \\ &= (1-\varepsilon) \sum_{k=1}^{m} a_k \lim_{n\to\infty} \mu(A_{k,n}) \\ &\overset{(4)}{=} (1-\varepsilon) \sum_{k=1}^{m} a_k \mu\left(\bigcup_{n=1}^{\infty} A_{k,n} \right) \\ &\overset{(5)}{=} (1-\varepsilon) \sum_{k=1}^{m} a_k \mu(A_k) \overset{\text{def}}{=} (1-\varepsilon) \int \varphi \, d\mu. \end{aligned}$$

Thus, the claim is proved.

Now for each ε, $0 < \varepsilon < 1$, let (ψ_n) be as in the claim. Then by (i) and monotonicity of the integral,

$$\int \psi_n \, d\mu \leq \int \varphi_n \, d\mu \qquad \forall n.$$

Letting $n \to \infty$ we obtain

$$(1-\varepsilon) \int \varphi \, d\mu \overset{\text{claim}}{=} \lim_{n\to\infty} \int \psi_n \, d\mu \leq \lim_{n\to\infty} \int \varphi_n \, d\mu \qquad \forall 0 < \varepsilon < 1.$$

Now letting $\varepsilon \to 0$ it follows that

$$\int \varphi \, d\mu \leq \lim_{n\to\infty} \int \varphi_n \, d\mu,$$

which proves (2.6) and the lemma. ∎

2.2 The Integral of Nonnegative, Measurable Functions

Let us recall that

$$\mathscr{L}^+ = \mathscr{L}^+(\Omega, \mathscr{F}, \mu) = \{ f : \Omega \to [0,\infty] \mid f \text{ is } \mathscr{F}\text{-measurable} \}.$$

Definition 2.2.1 For $f \in \mathscr{L}^+$, we define

$$\boxed{\int f\,d\mu := \sup E \qquad \text{where} \qquad E = \left\{ \int \varphi\,d\mu \;:\; \varphi \in \mathscr{S}^+,\ \varphi \le f \right\}.} \tag{2.7}$$

(R) Note that $0 \in E \subseteq [0,\infty]$, so that $0 \le \int f\,d\mu \le \infty$.

(R) Suppose $f \in \mathscr{S}^+$. Then we have two definitions of $\int f\,d\mu$: Definition (2.1) for the class $\mathscr{S}^+$, and the newer definition (2.7) for the class $\mathscr{L}^+$. We must show that both definitions are the same. In the following, all integrals will be according to definition (2.1).

Let $a \in E$. Then $a = \int \varphi\,d\mu$ for some $\varphi \in \mathscr{S}^+$ with $\varphi \le f$. Now by monotonicity of the integral in $\mathscr{S}^+$ (Theorem 2.1.2), then

$$a = \int \varphi\,d\mu \le \int f\,d\mu.$$

As $a \in E$ was arbitrary, it follows that

$$\sup E \le \int f\,d\mu. \tag{2.8}$$

On the other hand, as $f \in \mathscr{S}^+$, then $\int f\,d\mu \in E$ itself. It follows that

$$\int f\,d\mu \le \sup E. \tag{2.9}$$

Combine (2.8) and (2.9),

$$\int f\,d\mu = \sup E.$$

But the right-hand supremun is just $\int f\,d\mu$ according to definition (2.7). This shows that both definitions coincide.

In general, it is difficult to work with the definition of the integral $\int f\,d\mu$ given by (2.7). Instead, we prefer to work with limits of sequences. Recall that by the Structure Theorem for Measurable Functions, every $f \in \mathscr{L}^+$ is the pointwise limit of a sequence $(\varphi_n)\uparrow$ in $\mathscr{S}^+$.

Theorem 2.2.1 Let $f \in \mathscr{L}^+$ be given. If $(\varphi_n)\uparrow$ is any increasing sequence in $\mathscr{S}^+$ with $\varphi_n \xrightarrow{\text{p.w.}} f$, then

$$\int f\,d\mu = \lim_{n\to\infty} \int \varphi_n\,d\mu.$$

Proof. By definition of the integral,

$$\int f\,d\mu = \sup E, \qquad \text{where} \qquad E = \left\{ \int \varphi\,d\mu : \varphi \in \mathscr{S}^+,\ \varphi \le f \right\}. \tag{2.10}$$

Now let a sequence $(\varphi_n)\uparrow$ be given, $\varphi_n \in \mathscr{S}^+\ \forall n$, with $\varphi_n \xrightarrow{\text{p.w.}} f$. By (2.10) we must show that

$$\sup E = \lim_{n\to\infty} \int \varphi_n\,d\mu. \tag{2.11}$$

First Observations: (i) As $\varphi_n \leq \varphi_{n+1}$ for all n, then by monotonicity of the integral in $\mathscr{S}^+$,

$$\int \varphi_n \, d\mu \overset{\text{thm 2.1.2}}{\leq} \int \varphi_{n+1} \, d\mu$$

for all n, that is, $\{\int \varphi_n \, d\mu\}\uparrow$. Therefore, $\lim_{n\to\infty} \int \varphi_n \, d\mu$ exists in $[0,\infty]$.

(ii) Furthermore, as $\varphi_n \leq f$, then $\int \varphi_n \, d\mu \in E$ for all n, and hence,

$$\int \varphi_n \, d\mu \leq \sup E \qquad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int \varphi_n \, d\mu \leq \sup E.$$

It is left to prove the reverse inequality, namely that

$$\sup E \leq \lim_{n\to\infty} \int \varphi_n \, d\mu. \tag{2.12}$$

Claim: For each $a \in E$, there exists a sequence $(\psi_n)\uparrow$ in $\mathscr{S}^+$ with

(a) $\psi_n \leq \varphi_n \quad \forall n$,

(b) $\int \psi_n \, d\mu \to a$.

To prove the claim, let $a \in E$ be given. Then $a = \int \psi \, d\mu$ for some $\psi \in \mathscr{S}^+$ with $\psi \leq f$. Now for each $n \in \mathbb{N}$, we set

$$\psi_n := \min(\psi, \varphi_n) \leq \varphi_n.$$

Observations:

(i) Clearly, $\psi_n \leq \varphi_n$ for all n.

(ii) Each ψ_n is $\mathscr{F}$-measurable by Theorem 1.5.6.

(iii) Since $\text{range}(\psi_n) \subseteq \text{range}(\psi) \cup \text{range}(\varphi_n)$, it follows that each ψ_n is simple, and $\psi_n \geq 0$. Thus, $\psi_n \in \mathscr{S}^+$ for all n.

(iv) Since $\psi \leq f$ and $\varphi_n \leq f$, then $\psi_n \leq f$ for all n.

(v) Since $(\varphi_n)\uparrow$, then $(\psi_n)\uparrow$.

(vi) $\psi_n \overset{\text{p.w.}}{\longrightarrow} \psi$. To see this, let $\omega \in \Omega$ be given.

(a) Case 1: $\psi(\omega) < f(\omega)$. As $\varphi_n(\omega) \to f(\omega)$, $\exists n \in \mathbb{N}$ such that

$$\psi(\omega) < \varphi_n(\omega) \leq f(\omega) \qquad \forall n \geq N,$$

and hence

$$\psi_n(\omega) \overset{\text{def}}{=} \min\big(\psi(\omega), \varphi_n(\omega)\big) = \psi(\omega) \qquad \forall n \geq N,$$

so that trivially, $\psi_n(\omega) \to \psi(\omega)$.

(b) Case 2: $\psi(\omega) = f(\omega)$. Then as $\varphi_n(\omega) \leq f(\omega)$ we have

$$\psi_n(\omega) \overset{\text{def}}{=} \min\big(\psi(\omega), \varphi_n(\omega)\big) = \varphi_n(\omega) \qquad \forall n,$$

and hence, $\psi_n(\omega) = \varphi_n(\omega) \to f(\omega) = \psi(\omega)$.

By (v) and (vi) we can apply the Monotone Convergence Theorem for $\mathscr{S}^+$, and obtain

$$a = \int \psi \, d\mu \overset{\text{MCT}}{=} \lim_{n\to\infty} \int \psi_n \, d\mu.$$

Thus, the claim is proved.

Now given $a \in E$, let $(\psi_n)\uparrow$ be as in the claim. Since $\psi_n \leq \varphi_n$, then by monotonicity of the integral in $\mathscr{S}^+$

$$\int \psi_n \, d\mu \leq \int \varphi_n \, d\mu \qquad \forall n,$$

and letting $n \to \infty$,

$$a \overset{\text{claim}}{=} \lim_{n\to\infty} \int \psi_n \, d\mu \leq \lim_{n\to\infty} \int \varphi_n \, d\mu.$$

As $a \in E$ was arbitrary, then (2.12), and hence (2.11) follow. Thus, the theorem is proved ■

The properties of the integral discussed for nonnegative simple functions in Theorem 2.1.2 naturally carry over to the integral of arbitrary measurable functions:

Theorem 2.2.2 *(Properties of the integral).* Let $f, g \in \mathscr{L}^+$ and $c \geq 0$.

1. $\int cf \, d\mu = c \int f \, d\mu.$ *("the integral is positive homogeneous")*
2. $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$ *("the integral is additive")*
3. If $f \leq g$ then $\int f \, d\mu \leq \int g \, d\mu.$ *("the integral is monotone")*

Exercise 2.2 Prove Theorem 2.2.2. (Hint: use Theorems 2.1.2 and 2.2.1.) ■

Exercise 2.3 Consider the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$. Set

$$f(x) = x\mathbf{1}_{[0,1]}, \quad g(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1 \\ 0 & \text{else} \end{cases}, \quad h(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{else} \end{cases}.$$

Use Theorem 2.2.1 to find

$$\int f \, d\lambda, \qquad \int g \, d\lambda \qquad \text{and} \qquad \int h \, d\lambda.$$

■

Exercise 2.4 Consider the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, and let $f : \mathbb{R} \to [0,\infty)$ be continuous. Show: If $\int f \, d\lambda = 0$ then $f = 0$. ■

2.3 The Integral of Extended Real-Valued, Measurable Functions

We are now ready to define the integral of an arbitrary extended real-valued measurable function. Recall that

$$\mathscr{L} = \mathscr{L}(\Omega, \mathscr{F}, \mu) = \{ f : \Omega \to \mathbb{R}^* \mid f \text{ is } \mathscr{F}\text{-measurable} \}.$$

Now if $f \in \mathscr{L}$, then clearly, $f^+, f^- \in \mathscr{L}^+$, so that

$$\int f^+ \, d\mu \qquad \text{and} \qquad \int f^- \, d\mu$$

are both defined. Since $f = f^+ - f^-$, then the following definition is very natural.

Definition 2.3.1 Let $f \in \mathscr{L}$ be given. Then

$$\boxed{\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu} \tag{2.13}$$

provided that the right-hand side is *not* of the form $\infty - \infty$!

R For $f \in \mathscr{L}^+$ we now have two definitions of $\int f \, d\mu$: Definition (2.7) for the class $\mathscr{L}^+$, and the newer definition (2.13) for the class $\mathscr{L}$. It is easy to see that both definitions coincide. This is because $f^+ = f$ while $f^- = 0$. Thus (the integrals below are according to definition (2.7)),

$$\int f^+ \, d\mu - \int f^- \, d\mu = \int f \, d\mu - \int 0 \, d\mu = \int f \, d\mu.$$

Since the left-hand difference is the integral according to definition (2.13), while the right-hand side is the integral according to (2.7), it follows that both integrals coincide.

Definition 2.3.2 Let $f \in \mathscr{L}$ be given. Since $|f| = f^+ + f^-$, then

$$\int |f| \, d\mu \overset{\text{thm 2.2.2}}{=} \int f^+ \, d\mu + \int f^- \, d\mu \in [0, \infty].$$

We say that f *is integrable*, if

$$\int |f| \, d\mu < \infty.$$

■ **Example 2.2** Let $(\Omega, \mathscr{F}, \lambda) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$. Consider the step function

$$\varphi = 2 \cdot \mathbf{1}_{[0,1]} - 3 \cdot \mathbf{1}_{(1,2]} + 0.5 \cdot \mathbf{1}_{[4,\infty)} \in \mathscr{S}.$$

Then

$$\varphi^+ = 2 \cdot \mathbf{1}_{[0,1]} + 0.5 \cdot \mathbf{1}_{[4,\infty)} \qquad \text{and} \qquad \varphi^- = 3 \cdot \mathbf{1}_{(1,2]}$$

so that

$$\int \varphi^+ \, d\lambda = 2 \cdot \lambda([0,1]) + 0.5 \cdot \lambda([4,\infty)) = \infty, \qquad \int \varphi^- \, d\lambda = 3 \cdot \lambda((1,2]) = 3.$$

It follows that

$$\int \varphi \, d\lambda = \int \varphi^+ \, d\lambda - \int \varphi^- \, d\lambda = \infty - 3 = \infty.$$

Thus, $\int \varphi \, d\lambda$ exists while φ is not integrable. ■

(R) The above example shows that the two concepts "$\int f \, d\mu$ *exists*" and "f *is integrable*" are not the same !

1. By Definition 2.3.1

$$\int f \, d\mu \text{ exists} \quad \Leftrightarrow \quad \int f^+ \, d\mu < \infty \quad \underline{\textbf{or}} \quad \int f^- \, d\mu < \infty.$$

2. On the other hand, by Definition 2.3.2.

$$\begin{aligned} f \text{ is integrable} \quad &\overset{\text{def}}{\Leftrightarrow} \quad \int f^+ \, d\mu + \int f^- \, d\mu < \infty \\ &\Leftrightarrow \quad \int f^+ \, d\mu < \infty \quad \underline{\textbf{and}} \quad \int f^- \, d\mu < \infty. \end{aligned}$$

(R) Suppose, $\int f$ exists. By the triangle inequality and additivity of the integral in $\mathscr{L}^+$,

$$\begin{aligned} \left| \int f \right| &\overset{\text{def}}{=} \left| \int f^+ - \int f^- \right| \le \left| \int f^+ \right| + \left| \int f^- \right| \\ &= \int f^+ + \int f^- \overset{\text{thm 2.2.2}}{=} \int [f^+ + f^-] = \int |f|. \end{aligned}$$

That is,

$$\boxed{\left| \int f \, d\mu \right| \le \int |f| \, d\mu.}$$

(R) Let $f, g \in \mathscr{L}$ with $|f| \le |g|$ ("f is dominated by $|g|$"). If g is integrable, then by monotonicity of the integral in $\mathscr{L}^+$,

$$0 \le \int |f| \overset{\text{thm 2.2.2}}{\le} \int |g| < \infty.$$

Thus, f is also integrable.

Let us set

$$\mathscr{L}^1_{\mathbb{R}^*} = \mathscr{L}^1_{\mathbb{R}^*}(\Omega, \mathscr{F}, \mu) = \mathscr{L}^1 = \mathscr{L}^1(\Omega, \mathscr{F}, \mu) := \{ f \in \mathscr{L} \mid f \text{ is integrable} \}.$$

Thus,

$$\mathscr{L}^1_{\mathbb{R}^*} = \{ f : \Omega \to \mathbb{R}^* \mid f \text{ is } \mathscr{F}\text{-measurable and integrable} \}.$$

We also set

$$\mathscr{L}^1_{\mathbb{R}} = \mathscr{L}^1_{\mathbb{R}}(\Omega, \mathscr{F}, \mu) := \{ f : \Omega \to \mathbb{R} \mid f \text{ is } \mathscr{F}\text{-measurable and integrable} \}.$$

The next theorem implies that $\mathscr{L}^1_{\mathbb{R}}$ is a real vector space, and that the map

$$f \mapsto \int f \, d\mu$$

is a *monotone linear functional* on $\mathscr{L}^1_{\mathbb{R}}$.

Theorem 2.3.1 *(Properties of the integral).* Let $f, g \in \mathscr{L}^1_{\mathbb{R}^*}$ and $c \in \mathbb{R}$. Then

1. $cf \in \mathscr{L}^1_{\mathbb{R}^*}$ and $\int cf\, d\mu = c \int f\, d\mu$. *("the integral is homogeneous")*
2. If $f+g$ is defined, then $f+g \in \mathscr{L}^1_{\mathbb{R}^*}$ and
$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu. \qquad \textit{("the integral is additive")}$$
3. If $f \leq g$ then $\int f\, d\mu \leq \int g\, d\mu$. *("the integral is monotone")*

Proof. By assumption on f and g,

$$\begin{aligned} &\int |f| < \infty, \quad \int f^+ < \infty, \quad \int f^- < \infty, \\ &\int |g| < \infty, \quad \int g^+ < \infty, \quad \int g^- < \infty. \end{aligned} \tag{2.14}$$

1. There are three possibilities.
   (a) Case 1: $c = 0$. Then $cf = 0 \in \mathscr{L}^+$, so that clearly
   $$\int cf = \int 0 = 0 = 0 \cdot \int f = c \int f.$$
   (b) Case 2: $c > 0$. Then $(cf)^+ = cf^+$ and $(cf)^- = cf^-$, and hence,
   $$\begin{aligned} \int (cf)^+ &= \int cf^+ \overset{\text{thm 2.2.2}}{=} c \int f^+ < \infty \\ \int (cf)^- &= \int cf^- \overset{\text{thm 2.2.2}}{=} c \int f^- < \infty, \end{aligned}$$
   by assumption (2.14). As *both* of these integrals are finite, it follows that cf is integrable and that
   $$\begin{aligned} \int cf &\overset{\text{def}}{=} \int (cf)^+ - \int (cf)^- = c \int f^+ - c \int f^- \\ &= c \left[\int f^+ - \int f^- \right] \overset{\text{def}}{=} c \int f. \end{aligned}$$
   (c) Case 3: $c < 0$. Then $(cf)^+ = (-|c|f)^+ = |c|f^-$ and $(cf)^- = (-|c|f)^- = |c|f^+$, and hence,
   $$\begin{aligned} \int (cf)^+ &= \int |c|f^- \overset{\text{thm 2.2.2}}{=} |c| \int f^- < \infty \\ \int (cf)^- &= \int |c|f^+ \overset{\text{thm 2.2.2}}{=} |c| \int f^+ < \infty, \end{aligned}$$
   by assumption (2.14). As *both* of these integrals are finite, it follows that cf is integrable and that
   $$\begin{aligned} \int cf &\overset{\text{def}}{=} \int (cf)^+ - \int (cf)^- = |c| \int f^- - |c| \int f^+ \\ &= -|c| \left[\int f^+ - \int f^- \right] \overset{\text{def}}{=} c \int f. \end{aligned}$$

2. Suppose that $f(\omega)+g(\omega)$ is defined for all $\omega \in \Omega$. As

$$|f(\omega)+g(\omega)| \le |f(\omega)|+|g(\omega)|$$

(even when $f(\omega)=\pm\infty$ or $g(\omega)=\pm\infty$!), then by monotonicity and additivity of the integral in $\mathscr{L}^+$,

$$\int |f+g| \overset{\text{thm 2.2.2}}{\le} \int (|f|+|g|) \overset{\text{thm 2.2.2}}{=} \int |f| + \int |g| < \infty.$$

This shows that $f+g$ is also integrable, and in particular,

$$\int (f+g)^+ < \infty \qquad \text{and} \qquad \int (f+g)^- < \infty.$$

We now decompose

$$f+g = (f+g)^+ - (f+g)^-$$

while also

$$f+g = (f^+ - f^-) + (g^+ - g^-).$$

Equating both,

$$(f+g)^+(\omega) - (f+g)^-(\omega) = f^+(\omega) - f^-(\omega) + g^+(\omega) - g^-(\omega) \qquad \forall \omega \in \Omega.$$

We can add $(f+g)^-(\omega)$, $f^-(\omega)$ and $g^-(\omega)$ to both sides, provided they are finite valued, to obtain

$$(f+g)^+(\omega) + f^-(\omega) + g^-(\omega) = (f+g)^-(\omega) + f^+(\omega) + g^+(\omega).$$

Note that this identity is true even when $(f+g)^-(\omega)$, $f^-(\omega)$ or $g^-(\omega)$ take the value ∞, as can be seen by comparing both sides of the equation ! By additivity of the integral for $\mathscr{L}^+$ (Theorem 2.2.2),

$$\int (f+g)^+ + \int f^- + \int g^- = \int (f+g)^- + \int f^+ + \int g^+$$

Since all of these integrals are finite, we may subtract freely,

$$\int (f+g)^+ - \int (f+g)^- = \int f^+ - \int f^- + \int g^+ - \int g^-$$

that is,

$$\int (f+g) = \int f + \int g.$$

3. If $f \le g$ then

$$f^+ = \max(f,0) \le \max(g,0) = g^+$$
$$g^- = -\min(g,0) \le -\min(f,0) = f^-.$$

By monotonicity of the integral in $\mathscr{L}^+$ (Theorem 2.2.2),

$$\int f^+ \le \int g^+ \qquad \text{and} \qquad \int g^- \le \int f^-.$$

As all integrals are finite, we can subtract,

$$\int f = \int f^+ - \int f^- \le \int g^+ - \int g^- = \int g.$$

This completes the proof. ■

2.4 The Integral of Vector-Valued and Complex-Valued Functions

In this section, we will extend the concept of measurable functions to complex-valued functions. Since the set of complex numbers can be identified topologically with the real plane $\mathbb{R}^2$, it makes sense to consider complex-valued functions as vector-valued functions, and thus first discuss measurable functions of such type.

Vector-Valued, Measurable Functions

Let us briefly recall the basic concepts of the topology in $\mathbb{R}^d$. The *Euclidean norm* of a vector $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ is

$$\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2}.$$

Given $x \in \mathbb{R}^d$ and $\varepsilon > 0$, the set

$$B_\varepsilon(x) := \{y \in \mathbb{R}^d : \|y - x\| < \varepsilon\}$$

is called the *open ball with center x and radius* ε. If $U \subseteq \mathbb{R}^d$, then

$$U \text{ is } \textit{open} \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall x \in U \quad \exists \varepsilon = \varepsilon_x > 0 \quad \text{such that} \quad B_\varepsilon(x) \subseteq U.$$

It is well known and easy to verify that if $U_1, U_2, \ldots, U_d$ are *open* subsets of $\mathbb{R}$, then

$$U = U_1 \times U_2 \times \cdots \times U_d$$

is an open subset of $\mathbb{R}^d$.

First we give a version of Lindelöff's Theorem for $\mathbb{R}^d$. By an *open, bounded d-interval* we mean a subset

$$I = \prod_{i=1}^{d} (a_i, b_i) = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d)$$

of $\mathbb{R}^d$, where $a_i < b_i$, $a_i, b_i \in \mathbb{R}$, $i = 1 \ldots d$. We say that *I has rational endpoints* if $a_i, b_i \in \mathbb{Q}$ for all *i*. Let use set

$$\mathscr{I}_0^d = \left\{ I = \prod_{i=1}^{d} (r_i, s_i) : r_i < s_i,\ r_i, s_i \in \mathbb{Q},\ i = 1 \ldots d \right\},$$

the collection of all open d-intervals with rational endpoints. We observe that $\mathscr{I}_0^d$ is a countable set, as the mapping

$$\prod_{i=1}^{d} (r_i, s_i) \in \mathscr{I}_0^d \mapsto (r_1, s_2, r_2, s_2, \ldots, r_d, s_d) \in \mathbb{Q}^{2d}$$

is one-to-one.

Lemma 2.4.1 (Lindelöff's Theorem for $\mathbb{R}^d$)
Every non-empty open set $U \subseteq \mathbb{R}^d$ is the countable union of bounded, open d-intervals with rational endpoints. That is,

$$U = \bigcup_{n=1}^{N} J_n \qquad N \in \mathbb{N} \cup \{\infty\},\ J_n \in \mathscr{I}_0^d.$$

Proof. Let $U \subseteq \mathbb{R}$ be open, $U \neq \emptyset$. Thus, for each $x = (x_1, \dots, x_d) \in U$, there exists $\varepsilon = \varepsilon_x > 0$ such that

$$B_\varepsilon(x) \subseteq U, \qquad B_\varepsilon(x) := \{y = (y_1, \dots, y_d) \in \mathbb{R}^d : \|y - x\| < \varepsilon\}.$$

Set $\delta = \delta_x = \varepsilon/\sqrt{d}$. Now by density of $\mathbb{Q}$ in $\mathbb{R}$, for each $i = 1, \dots, d$, there exist $r_i,\ s_i \in \mathbb{Q}$ (depending on x), so that

$$x_i - \delta < r_i < x_i < s_i < x_i + \delta.$$

In particular,

$$|y_i - x_i| < \delta \qquad \forall y_i \in (r_i, s_i). \tag{2.15}$$

Set

$$J_x := \prod_{i=1}^{d} (r_i, s_i) \in \mathscr{I}_0^d.$$

Note that if $y = (y_1, \dots, y_d) \in J_x$ is arbitrary, then by (2.15),

$$\|y - x\|^2 = \sum_{i=1}^{d} |y_i - x_i|^2 \leq \sum_{i=1}^{d} \delta^2 = \sum_{i=1}^{d} \varepsilon^2/d = \varepsilon^2,$$

which shows that

$$J_x \subset B_\varepsilon(x) \subseteq U.$$

It follows that

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} J_x \subseteq U$$

which gives

$$U = \bigcup_{x \in U} J_x. \tag{2.16}$$

Now as $\mathscr{I}_0^d$ is a countable set, only countably many of the intervals J_x can be distinct, and we can list the distinct interval as $\{J_n\}_{n=1}^N$, with $N \in \mathbb{N}$ or $N = \infty$. Thus, the union in (2.16) is really a union of the intervals $\{J_n\}_{n=1}^N$,

$$U = \bigcup_{n=1}^{N} J_n,$$

which proves the lemma. ■

Next we show that the Borel σ-algebra on $\mathbb{R}^d$ is generated by the collection of open d-intervals with rational endpoints:

Corollary 2.4.2 $\mathscr{B}(\mathbb{R}^d) = \sigma(\mathscr{I}_0^d)$.

Proof. Let

$$\tau := \{\, U \subseteq \mathbb{R}^d \;:\; U \text{ is open} \,\}.$$

Since open d-intervals are open sets, then $\mathscr{I}_0^d \subset \tau$, and hence $\sigma(\mathscr{I}_0^d) \subseteq \sigma(\tau) = \mathscr{B}(\mathbb{R}^d)$.

To prove the reverse inclusion, let $U \in \tau$ be arbitrary. By Lindelöff's Theorem, we can write

$$U = \bigcup_{n=1}^{N} J_n, \qquad n \in \mathbb{N} \cup \{\infty\}, \quad J_n \in \mathscr{I}_0^d.$$

Since $J_n \in \mathscr{I}_0^d \subseteq \sigma(\mathscr{I}_0^d)$ for all n, it now follows from properties (A2) or (A2σ) of a σ-algebra that $U \in \sigma(\mathscr{I}_0^d)$. As $U \in \tau$ was arbitrary, we conclude that

$$\tau \subseteq \sigma(\mathscr{I}_0^d)$$

Now $\mathscr{B}(\mathbb{R}^d)$ is the smallest σ-algebra containing τ; hence

$$\mathscr{B}(\mathbb{R}^d) = \sigma(\tau) \subseteq \sigma(\mathscr{I}_0^d).$$

Thus the corollary is proved ∎

Let $f : \Omega \to \mathbb{R}^d$ be a vector-valued function. We write f in component form,

$$f = (f_1, f_2, \ldots, f_d) \qquad \text{where} \qquad f_i : \Omega \to \mathbb{R}, \quad i = 1, \ldots, d.$$

Theorem 2.4.3 Let $f : \Omega \to \mathbb{R}^d$. Then

$$f \text{ is } \mathscr{F}\text{-measurable} \quad \Leftrightarrow \quad \text{each } f_i \text{ is } \mathscr{F}\text{-measurable}, \qquad i = 1, \ldots d.$$

Proof. Observe that

$$\begin{aligned} f \text{ is } \mathscr{F}\text{-measurable} \quad &\overset{\text{def}}{\Leftrightarrow} \quad f^{-1}(E) \in \mathscr{F} \quad \forall\, E \in \mathscr{B}(\mathbb{R}^d) \\ &\underset{\text{+cor 2.4.2}}{\overset{\text{thm 1.5.1}}{\Leftrightarrow}} \quad f^{-1}(E) \in \mathscr{F} \quad \forall\, E \in \mathscr{I}_0^d. \end{aligned}$$

$\Rightarrow$: Suppose, f is $\mathscr{F}$-measurable. Then for each $i = 1, \ldots, d$ and $a \in \mathbb{R}$,

$$\begin{aligned} f_i^{-1}\big((a,\infty)\big) &= \{\omega \in \Omega : f_i(\omega) > a\} \\ &= f^{-1}\big(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{i-1 \text{ copies}} \times (a,\infty) \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d-i \text{ copies}}\big) \in \mathscr{F}, \end{aligned}$$

as $\mathbb{R}^{i-1} \times (a,\infty) \times \mathbb{R}^{d-i}$ is an open subset of $\mathbb{R}^d$. This shows that f_i is $\mathscr{F}$-measurable.

$\Leftarrow$: Suppose, each f_i is $\mathscr{F}$-measurable. Let

$$E = \prod_{i=1}^{d} (r_i, s_i) \in \mathscr{I}_0^d$$

be given. Then

$$\begin{aligned} f^{-1}(E) &= \{\omega \in \Omega : f(\omega) \in E\} \\ &= \{\omega \in \Omega : (f_1(\omega), f_2(\omega), \ldots, f_d(\omega)) \in (r_1, s_1) \times (r_2, s_2) \times \cdots \times (r_d, s_s)\} \\ &= \{\omega \in \Omega : f_i(\omega) \in (r_i, s_i), \quad i = 1, \ldots, d\} \\ &= \bigcap_{i=1}^{d} \underbrace{\{\omega \in \Omega : f_i(\omega) \in (r_i, s_i)\}}_{\in \mathscr{F} \text{ as } f_i \text{ is } \mathscr{F}\text{-meas.}} \in \mathscr{F}. \end{aligned}$$

This shows that f is $\mathscr{F}$-measurable. ■

Given $f, g : \Omega \to \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, the functions $f + g$ and αf are defined pointwise as usual by

$$(f+g)(\omega) = f(\omega) + g(\omega), \qquad (\alpha f)(\omega) = \alpha f(\omega)$$

for all $\omega \in \Omega$. We also define a function $\|f\| : \Omega \to [0, \infty)$ by

$$\|f\| = \sqrt{\sum_{i=1}^{d} f_i^2}.$$

Theorem 2.4.4 Let $f, g : \Omega \to \mathbb{R}^d$ be $\mathscr{F}$-measurable, and $\alpha \in \mathbb{R}$. Then

1. $\{\omega \in \Omega : f(\omega) = g(\omega)\} \in \mathscr{F}$.
2. $f + g$ and αf are $\mathscr{F}$-measurable.
3. $\|f\|$ is $\mathscr{F}$-measurable.

Exercise 2.5 Prove Theorem 2.4.4. (This is simply an application of Theorem 2.4.3, Theorem 1.5.5, Theorem 1.5.6 and Corollary 1.5.3.) ■

Recall: If (x_n) is a sequence in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, say $x_n = (x_1^{(n)}, x_2^{(n)}, \ldots, x_d^{(n)})$ and $x = (x_1, x_2, \ldots, x_d)$, then

$$x_n \to x \quad \overset{\text{def}}{\Leftrightarrow} \quad \|x_n - x\| \to 0.$$

From

$$|y_i| = \sqrt{y_i^2} \le \|y\| = \sqrt{\sum_{j=1}^{d} |y_j|^2} \le \sqrt{\left(\sum_{j=1}^{d} |y_j|\right)^2} = \sum_{j=1}^{d} |y_j| \qquad (i = 1, \ldots, d) \tag{2.17}$$

where $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$, we obtain, setting $y = x_n - x$, that

$$\begin{aligned} x_n \to x \quad &\overset{\text{def}}{\Leftrightarrow} \quad \|x_n - x\| \to 0 \\ &\Leftrightarrow \quad |x_i^{(n)} - x_i| \to 0 \qquad \forall i = 1, \ldots, d \\ &\Leftrightarrow \quad x_i^{(n)} \to x_i \qquad \forall i = 1, \ldots, d. \end{aligned} \tag{2.18}$$

That is, the sequence (x_n) converges to x if and only each component sequence $(x_i^{(n)})$ converges to x_i.

Definition 2.4.1 Now let (f_n) be a sequence of functions, $f_n : \Omega \to \mathbb{R}^d$, and $f : \Omega \to \mathbb{R}^d$. We say that (f_n) converges pointwise to f, if

$$f_n(\omega) \to f(\omega) \qquad \forall \omega \in \Omega,$$

in which case we write $f_n \xrightarrow{\text{p.w.}} f$. That is,

$$f_n \xrightarrow{\text{p.w.}} f \quad \overset{\text{def}}{\Leftrightarrow} \quad \|f_n(\omega) - f(\omega)\| \to 0 \qquad \forall\, \omega \in \Omega.$$

R Write the above functions in component form,

$$f_n = (f_1^{(n)}, f_2^{(n)}, \dots, f_d^{(n)}) \qquad \text{and also} \qquad f = (f_1, f_2, \dots, f_d),$$

where $f_i^{(n)}, f_i : \Omega \to \mathbb{R}$. Then by (2.18),

$$f_n \xrightarrow{\text{p.w.}} f \quad \Leftrightarrow \quad f_i^{(n)} \xrightarrow{\text{p.w.}} f_i \qquad \forall\, i = 1, \dots, d.$$

Now suppose that each f_n is $\mathscr{F}$-measurable, and $f_n \xrightarrow{\text{p.w.}} f$. By Theorem 2.4.3, each $f_i^{(n)}$ is $\mathscr{F}$-measurable, so that by Theorem 1.5.7, each component f_i of the limit function is $\mathscr{F}$-measurable. Applying Theorem 2.4.3 again, it follows that f is $\mathscr{F}$-measurable.

Definition 2.4.2 Let $f : \Omega \to \mathbb{R}^d$ be $\mathscr{F}$-measurable. We say that f *is integrable*, if each component function $f_i : \Omega \to \mathbb{R}$ is integrable. In this case we define the integral of f as the vector

$$\boxed{\int f\, d\mu := \left(\int f_1\, d\mu, \int f_2\, d\mu, \dots, \int f_d\, d\mu \right)}.$$

We set

$$\mathscr{L}^1_{\mathbb{R}^d} = \mathscr{L}^1_{\mathbb{R}^d}(\Omega, \mathscr{F}, \mu) := \{ f : \Omega \to \mathbb{R}^d \mid f \text{ is } \mathscr{F}\text{-measurable and integrable} \}.$$

Theorem 2.4.5 Let $f : \Omega \to \mathbb{R}^d$ be $\mathscr{F}$-measurable. Then

$$f \text{ is integrable} \quad \Leftrightarrow \quad \|f\| \text{ is integrable.}$$

Proof. From (2.17) we obtain that for each $1 \le i \le d$ and $\omega \in \Omega$,

$$|f_i(\omega)| \le \sqrt{\sum_{j=1}^{d} f_j(\omega)^2} \le \sum_{j=1}^{d} |f_j(\omega)|,$$

that is,

$$|f_i| \le \|f\| \le \sum_{j=1}^{d} |f_j|.$$

$\Rightarrow$: Suppose, f is integrable, that is, $\int |f_i| < \infty$ for all $i = 1,\ldots,d$. Then by additivity and monotonicity of the integral,

$$\int \|f\| \leq \int \left[\sum_{i=1}^{d} |f_i|\right] = \sum_{i=1}^{d} \int |f_i| < \infty,$$

which shows that $\|f\|$ is integrable.

$\Leftarrow$: If $\|f\|$ is integrable, then for each $i = 1,\ldots,d$, by monotonicity of the integral,

$$\int |f_i| \leq \int \|f\| < \infty,$$

which shows that f is integrable. ■

The next theorem says that $\mathscr{L}^1_{\mathbb{R}^d}$ is a vector space, and that the mapping $f \mapsto \int f\,d\mu$ is a linear map of $\mathscr{L}^1_{\mathbb{R}^d}$ into $\mathbb{R}^d$.

Theorem 2.4.6 *(Properties of the integral).* Let $f,g \in \mathscr{L}^1_{\mathbb{R}^d}$ and $\alpha \in \mathbb{R}$. Then

1. $\alpha f \in \mathscr{L}^1_{\mathbb{R}^d}$, and $\displaystyle\int \alpha f\,d\mu = \alpha \int f\,d\mu.$
2. $f+g \in \mathscr{L}^1_{\mathbb{R}^d}$, and $\displaystyle\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu.$

Exercise 2.6 Prove Theorem 2.4.6. (This is simply an application of Theorem 2.4.3, Theorem 1.5.5, Theorem 1.5.6 and Theorem 2.3.1.) ■

Complex-Valued, Measurable Functions

Recall that $\mathbb{C}$ can be identified with $\mathbb{R}^2$ topologically and as a real vector space, using the bijection

$$\Psi : z = x + iy \in \mathbb{C} \mapsto (x,y) \in \mathbb{R}^2.$$

The difference between $\mathbb{C}$ and $\mathbb{R}^2$ lies in their algebraic structure: vectors $(x,y) \in \mathbb{R}^d$ can only be multiplied by real numbers α, while complex numbers $z = x+iy \in \mathbb{C}$ can be multiplied by other complex numbers $c = \alpha + i\beta$.

Given $z = x + iy \in \mathbb{C}$, we write

$x = \mathrm{Re}(z)$ "real part of z"
$y = \mathrm{Im}(z)$ "imaginary part of z".

The Euclidean norm $\|(x,y)\|$ of $(x,y) \in \mathbb{R}^2$ now corresponds to the *absolute value* (or *modulus*) $|z|$ of $z = x+iy \in \mathbb{C}$, so that

$$|z| = \sqrt{x^2+y^2} = \|(x,y)\|.$$

The complex conjugate of $z \in \mathbb{C}$ is

$$\bar{z} = x - iy = \mathrm{Re}(z) - i\,\mathrm{Im}(z).$$

Then

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z-\bar{z}}{2i}, \qquad |z| = \sqrt{z\bar{z}}. \tag{2.19}$$

If (z_n) is a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$, say $z_n = x_n + iy_n \ \forall n$ and $z = x + iy$, then by (2.18),

$$z_n \to z \quad \Leftrightarrow \quad x_n \to x \qquad \text{and} \qquad y_n \to y.$$

Next let $f : \Omega \to \mathbb{C}$ be a complex-valued function. As usual, we define functions $\operatorname{Re} f, \operatorname{Im} f, \bar{f}, |f| : \Omega \to \mathbb{R}$ pointwise by

$$\begin{aligned} (\operatorname{Re} f)(\omega) &= \operatorname{Re}\big(f(\omega)\big) && \text{"real part of } f\text{"} \\ (\operatorname{Im} f)(\omega) &= \operatorname{Im}\big(f(\omega)\big) && \text{"imaginary part of } f\text{"} \\ \bar{f}(\omega) &= \overline{f(\omega)} && \text{"complex conjugate of } f\text{"} \\ |f|(\omega) &= |f(\omega)| && \text{"absolute value of } f\text{"} \end{aligned}$$

for $\omega \in \Omega$.

Using (2.19) one quickly verifies that

$$\operatorname{Re} f = \frac{f + \bar{f}}{2}, \qquad \operatorname{Im} f = \frac{f - \bar{f}}{2i}, \qquad f = \operatorname{Re} f + i \operatorname{Im} f$$

$$|f| = \sqrt{f\bar{f}}, \qquad \bar{f} = \operatorname{Re} f - i \operatorname{Im} f.$$

Continuing to identify $\mathbb{C}$ with $\mathbb{R}^2$, we can consider complex-valued functions as vector-valued functions: Given $f = \operatorname{Re} f + i \operatorname{Im} f : \Omega \to \mathbb{C}$ we identify f with the function $\hat{f} = (f_1, f_2) : \Omega \to \mathbb{R}^2$ by setting $f_1 = \operatorname{Re} f$ and $f_2 = \operatorname{Im} f$.

1. Applying Theorem 2.4.3, we obtain:

   $$f \text{ is } \mathscr{F}\text{-measurable} \quad \Leftrightarrow \quad \operatorname{Re} f \text{ and } \operatorname{Im} f \text{ are both } \mathscr{F}\text{-measurable}.$$

2. Remark 2.4 implies: Let $f_n, f : \Omega \to \mathbb{C}$. Then

   $$f_n \xrightarrow{\text{p.w.}} f \quad \Leftrightarrow \quad \operatorname{Re} f_n \xrightarrow{\text{p.w.}} \operatorname{Re} f \qquad \text{and} \qquad \operatorname{Im} f_n \xrightarrow{\text{p.w.}} \operatorname{Im} f.$$

   Furthermore, if each f_n is $\mathscr{F}$-measurable and $f_n \xrightarrow{\text{p.w.}} f$, then f is $\mathscr{F}$-measurable.

3. Definition 2.4.2 becomes: If f is $\mathscr{F}$-measurable, then

   $$f \text{ is integrable} \quad \Leftrightarrow \quad \operatorname{Re} f \text{ and } \operatorname{Im} f \text{ are both integrable.}$$

   In this case,

   $$\int f \, d\mu \;=\; \int \big[\operatorname{Re} f + i \operatorname{Im} f\big] \, d\mu \overset{\text{def}}{=} \int \operatorname{Re} f \, d\mu + i \int \operatorname{Im} f \, d\mu.$$

4. Theorem 2.4.5 becomes: If f is $\mathscr{F}$-measurable, then

   $$f \text{ is integrable} \quad \Leftrightarrow \quad |f| \text{ is integrable.}$$

5. Let us set

   $$\mathscr{L}^1_{\mathbb{C}} = \mathscr{L}^1_{\mathbb{C}}(\Omega, \mathscr{F}, \mu) \;:=\; \big\{ f : \Omega \to \mathbb{C} \;\big|\; f \text{ is } \mathscr{F}\text{-measurable and integrable} \big\}.$$

   Then $\mathscr{L}^1_{\mathbb{C}}$ can be identified with $\mathscr{L}^1_{\mathbb{R}^2}$ using the map

   $$f = \operatorname{Re} f + i \operatorname{Im} f \;\mapsto \hat{f} = \big(\operatorname{Re} f, \operatorname{Im} f\big).$$

As we already expect, the next theorem shows that $\mathscr{L}^1_{\mathbb{C}}$ is a complex vector space, and that the map

$$f \mapsto \int f \, d\mu$$

is a linear functional on $\mathscr{L}^1_{\mathbb{C}}$.

Theorem 2.4.7 *(Properties of the integral).* Let $f, g \in \mathscr{L}^1_{\mathbb{C}}$ and $c \in \mathbb{C}$. Then

1. $cf \in \mathscr{L}^1_{\mathbb{C}}$ and $\displaystyle\int cf \, d\mu = c \int f \, d\mu.$
   ("the integral is homogeneous")
2. $f + g \in \mathscr{L}^1_{\mathbb{C}}$ and $\displaystyle\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$
   ("the integral is additive")
3. $\bar{f} \in \mathscr{L}^1_{\mathbb{C}}$ and $\displaystyle\int \bar{f} \, d\mu = \overline{\int f \, d\mu}.$
4. $\displaystyle\left| \int f \, d\mu \right| \leq \int |f| \, d\mu.$

Proof. Since f is integrable, then so are the functions $\operatorname{Re} f$ and $\operatorname{Im} f$. We also note that by Theorem 2.4.6, properties 1. and 2. already hold, at least for *real* scalars c.

1. Thus, let $c = a + ib \in \mathbb{C}$. Then

$$cf = (a+ib)(\operatorname{Re} f + i \operatorname{Im} f) = (a \operatorname{Re} f - b \operatorname{Im} f) + i(a \operatorname{Im} f + b \operatorname{Re} f).$$

   Now by Theorem 2.3.1 ,

$$\operatorname{Re}(cf) = a \operatorname{Re} f - b \operatorname{Im} f \qquad \text{and} \qquad \operatorname{Im}(cf) = a \operatorname{Im} f + b \operatorname{Re} f$$

   are both integrable. Thus, cf is integrable, and

$$\begin{aligned} \int cf &= \int \operatorname{Re}(cf) + i \int \operatorname{Im}(cf) \\ &= \int (a \operatorname{Re} f - b \operatorname{Im} f) + i \int (a \operatorname{Im} f + b \operatorname{Re} f) \\ &\overset{\text{thm 2.3.1}}{=} a \int \operatorname{Re} f - b \int \operatorname{Im} f + i \left(a \int \operatorname{Im} f + b \int \operatorname{Re} f \right) \\ &= (a+ib) \left(\int \operatorname{Re} f + i \int \operatorname{Im} f \right) = c \int f. \end{aligned}$$

2. This is Theorem 2.4.6, part 2.
3. Clearly, the conjugate $\bar{f} = \operatorname{Re} f + i(-\operatorname{Im} f)$ is $\mathscr{F}$-measurable and integrable, because $\operatorname{Re} f$ and $\operatorname{Im} f$ are. Now

$$\begin{aligned} \int \bar{f} &= \int (\operatorname{Re} f + i(-\operatorname{Im} f)) = \int \operatorname{Re} f + i \int (-\operatorname{Im} f) \\ &\overset{\text{thm 2.3.1}}{=} \int \operatorname{Re} f - i \int \operatorname{Im} f = \overline{\left(\int \operatorname{Re} f + i \int \operatorname{Im} f \right)} = \overline{\int f}. \end{aligned}$$

4. Using the polar representation $z = re^{i\theta}$ of $z \in \mathbb{C}$, we can write

$$\int f = re^{i\theta} \qquad \text{for some } r > 0,\ 0 \le \theta < 2\pi.$$

Solve this identity for r,

$$r = e^{-i\theta}\int f \overset{\text{part 1.}}{=} \int e^{-i\theta} f = \int \operatorname{Re}\big(e^{-i\theta} f\big) + i\int \operatorname{Im}\big(e^{-i\theta} f\big).$$

Since r on the left is a real number, then last integral on the right must be zero,

$$\int \operatorname{Im}\big(e^{-i\theta} f\big) = 0.$$

Thus,

$$\begin{aligned}\left|\int f\right| &= \left|re^{i\theta}\right| = |r| = \left|\int \operatorname{Re}\big(e^{-i\theta} f\big)\right| \\ &\le \int \left|\operatorname{Re}\big(e^{-i\theta} f\big)\right| \le \int \left|e^{-i\theta} f\right| = \int |f|,\end{aligned}$$

where we have used the fact that $|\operatorname{Re} z| \le |z|$ for $z \in \mathbb{C}$, together with monotonicity of the integral.

■

2.5 The Integral over a Set

In the following, we will deal with both extended real-valued functions and with complex valued functions, although our discussion can easily be adapted to vector-valued functions. We thus let $\mathbb{K} = \mathbb{R}^*$ or $\mathbb{K} = \mathbb{C}$.

Definition 2.5.1 Let $f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable. Given $A \in \mathscr{F}$, we set

$$\int_A f\, d\mu := \int f\mathbf{1}_A\, d\mu, \tag{2.20}$$

provided that the right-hand integral is defined. We say that f *is integrable over* A, if $f\mathbf{1}_A$ is integrable. That is,

$$f \text{ is integrable over } A \overset{\text{def}}{\Leftrightarrow} \int |f|\,\mathbf{1}_A\, d\mu < \infty \overset{(2.20)}{\Leftrightarrow} \int_A |f|\, d\mu < \infty.$$

■ **Example 2.3** Let $(\Omega, \mathscr{F}, \mu) = \big(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda\big)$. If $f(x) = x$, then

$$\int_{[0,1]} x\, d\lambda = \int x\mathbf{1}_{[0,1]}\, d\lambda \overset{\text{exer 2.3}}{=} \frac{1}{2}.$$

Exercise 2.7 Let $A, B \in \mathscr{F}$ and $g : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable.

1. Suppose, $B \subseteq A$. Show:
   (a) If $g \in \mathscr{L}^+$, then $\int_B g\,d\mu \le \int_A g\,d\mu$.
   (b) If $\int_A g\,d\mu$ is defined, then $\int_B g\,d\mu$ is defined.
   (c) If g is integrable over A, then g is integrable over B.
2. Let A, B be arbitrary. Show:
   (a) If $\int_{A\cup B} g\,d\mu$ is defined, then $\int_A g\,d\mu$ is defined.
   (b) Even when $\int_A g\,d\mu$ and $\int_B g\,d\mu$ are both defined, then $\int_{A\cup B} g\,d\mu$ need not be defined.
   (c) If g is integrable over $A \cup B$, then g is integrable over A.
   (d) If g is integrable over A and integrable over B, then g is integrable over $A \cup B$.
3. Suppose that $A \cap B = \emptyset$. Show:

$$\int_{A\cup B} g\,d\mu = \int_A g\,d\mu + \int_B g\,d\mu$$

whenever the left-hand integral is defined.

Note: by induction, we obtain:

Let $A_1, \dots, A_n \in \mathscr{F}$ be mutually disjoint, and $A = \bigcup_{i=1}^n A_i$. If either $\int_A g\,du$ is defined or g is integrable over each A_i, then

$$\int_A g\,d\mu \;=\; \sum_{i=1}^n \int_{A_i} g\,d\mu.$$

■

Theorem 2.5.1 (The integral over a null set) Let $N \in \mathscr{F}$ be a null set. Then every $\mathscr{F}$-measurable function $f : \Omega \to \mathbb{K}$ is integrable over N, and

$$\int_N f\,d\mu = 0.$$

Proof. Let N be a given null set.

1. First let $f = \varphi \in \mathscr{S}^+$, say $\varphi = \sum_{k=1}^n a_k \mathbf{1}_{A_k}$. Then

$$\varphi \mathbf{1}_N = \left[\sum_{k=1}^n a_k \mathbf{1}_{A_k}\right] \mathbf{1}_N = \sum_{k=1}^n a_k \mathbf{1}_{A_k} \mathbf{1}_N = \sum_{k=1}^n a_k \mathbf{1}_{A_k \cap N} \in \mathscr{S}^+,$$

so that

$$\begin{aligned}\int_N \varphi\,d\mu &\overset{\text{def}}{=} \int \varphi \mathbf{1}_N\,d\mu = \int \left[\sum_{k=1}^n a_k \mathbf{1}_{A_k \cap N}\right] d\mu \\ &= \sum_{k=1}^n a_k \mu(A_k \cap N) = \sum_{k=1}^n a_k \cdot 0 = 0\end{aligned}$$

as measurable subsets of null sets are again null sets.

2. Next, let $f \in \mathscr{L}^+$. By the Structure Theorem, there exists a sequence $(\varphi_n) \uparrow$ in $\mathscr{S}^+$ with $\varphi_n \xrightarrow{\text{p.w.}} f$. By the first part, $\varphi_n \mathbf{1}_N \in \mathscr{S}^+$ for all n, and clearly, $(\varphi_n \mathbf{1}_N) \uparrow$ and $\varphi_n \mathbf{1}_N \xrightarrow{\text{p.w.}} f\mathbf{1}_N$. Hence,

$$\int_N f\,d\mu \overset{\text{def}}{=} \int f\mathbf{1}_N\,d\mu \overset{\text{thm 2.2.1}}{=} \lim_{n\to\infty} \int \varphi_n \mathbf{1}_N\,d\mu \overset{\text{part 1.}}{=} \lim_{n\to\infty} 0 = 0.$$

3. Now let $f : \Omega \to \mathbb{R}^*$ be $\mathscr{F}$-measurable. Clearly,

$$[f\mathbf{1}_N]^+ = f^+\mathbf{1}_N \qquad \text{and} \qquad [f\mathbf{1}_N]^- = f^-\mathbf{1}_N.$$

By the second part,

$$\int [f\mathbf{1}_N]^+\,d\mu = \int f^+\mathbf{1}_N\,d\mu \overset{\text{part 2.}}{=} 0 < \infty \qquad \text{and}$$
$$\int [f\mathbf{1}_N]^-\,d\mu = \int f^-\mathbf{1}_N\,d\mu \overset{\text{part 2.}}{=} 0 < \infty.$$

It follows that $f\mathbf{1}_N$ is integrable, and

$$\int_N f\,d\mu \overset{\text{def}}{=} \int f\mathbf{1}_N\,d\mu \overset{\text{def}}{=} \int [f\mathbf{1}_N]^+\,d\mu - \int [f\mathbf{1}_N]^-\,d\mu = 0 - 0 = 0.$$

4. Finally, let $f : \Omega \to \mathbb{C}$ be $\mathscr{F}$-measurable. As $\mathbf{1}_N$ is real-valued, then clearly,

$$\operatorname{Re}(f\mathbf{1}_N) = \operatorname{Re}(f)\mathbf{1}_N \qquad \text{and} \qquad \operatorname{Im}(f\mathbf{1}_N) = \operatorname{Im}(f)\mathbf{1}_N.$$

By part 3. these functions are integrable, and hence

$$\begin{aligned}\int_N f\,d\mu &\overset{\text{def}}{=} \int f\mathbf{1}_N\,d\mu \overset{\text{def}}{=} \int \operatorname{Re}(f\mathbf{1}_N)\,d\mu + i\int \operatorname{Im}(f\mathbf{1}_N)\,d\mu \\ &= \int (\operatorname{Re} f)\mathbf{1}_N\,d\mu + i\int (\operatorname{Im} f)\mathbf{1}_N\,d\mu \overset{\text{part 3.}}{=} 0 + i0 = 0.\end{aligned}$$

■

Example 2.4 Let $(\Omega, \mathscr{F}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

1. Let $f : R \to \mathbb{R}^*$ be any Borel function. Since $\mathbb{Q}$ is a countable set, it is a λ-null set, so that

$$\int_{\mathbb{Q}} f\,d\lambda = 0.$$

2. Now let $f \in \mathscr{L}^1_{\mathbb{K}}$. Then by Exercise 2.7, parts 2 and 3.,

$$\int_{\mathbb{R}} f\,d\lambda = \int_{\mathbb{R}\setminus\mathbb{Q}} f\,d\lambda + \int_{\mathbb{Q}} f\,d\lambda = \int_{\mathbb{R}\setminus\mathbb{Q}} f\,d\lambda + 0 = \int_{\mathbb{R}\setminus\mathbb{Q}} f\,d\lambda.$$

That is,

$$\int_{\mathbb{R}\setminus\mathbb{Q}} f\,d\lambda = \int_{\mathbb{R}} f\,d\lambda.$$

■

Exercise 2.8 (This exercise will show that $\int_A f\,d\mu$ is not really a new concept, but is the integral in some measure space $(A, \mathscr{F}_A, \mu_A)$.)

Let $(\Omega, \mathscr{F}, \mu)$ be a given measure space and $A \in \mathscr{F}$. Set

$$\mathscr{F}_A := \{E \in \mathscr{F} : E \subseteq A\}.$$

1. Show:
   (a) $\mathscr{F}_A = \{E \cap A : E \in \mathscr{F}\}$.
   (b) $\mathscr{F}_A$ is a σ-algebra on A. (Hint: To avoid confusion, denote by $A \backslash E$ the complement of E in A. Don't use the notation E^c.)

   We let μ_A denote the restriction of μ to A:

   $$\mu_A(E) := \mu(E) \quad \forall E \in \mathscr{F}_A.$$

   Clearly, μ_A is a measure on $(A, \mathscr{F}_A)$, so that $(A, \mathscr{F}_A, \mu_A)$ is a measure space.
2. Let $f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable. The *restriction of f to A* is the function $f_{|A} : A \to \mathbb{K}$ defined by

   $$f_{|A}(\omega) = f(\omega) \qquad \forall \omega \in A.$$

   Show: $f_{|A}$ is $\mathscr{F}_A$-measurable.
3. Conversely, let $f : A \to \mathbb{K}$ be $\mathscr{F}_A$-measurable. We extend f to a function $\tilde{f} : \Omega \to \mathbb{K}$ by setting

   $$\tilde{f}(\omega) := \begin{cases} f(\omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

   Show: $\tilde{f}$ is $\mathscr{F}$-measurable.
   (It is clear that $(\tilde{f})_{|A} = f$ for all $f : A \to \mathbb{K}$.)
4. Let $f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable. Show:
   (a) If $f \in \mathscr{L}^+(\Omega, \mathscr{F}, \mu)$, then $\displaystyle\int f_{|A}\,d\mu_A = \int_A f\,d\mu$.
   (b) $\displaystyle\int f_{|A}\,d\mu_A$ is defined $\Leftrightarrow$ $\displaystyle\int_A f\,d\mu$ is defined.
   In this case, both integrals coincide.
   (c) f is integrable over A $\Leftrightarrow$ $f_{|A} \in \mathscr{L}^1(A, \mathscr{F}_A, \mu_A)$.

■

2.6 Almost Everywhere

Let us briefly recall the following properties of null sets, which will be used repeatedly:

1. Let $N \in \mathscr{F}$ be a null set. If $A \in \mathscr{F}$ is a subset N, then by monotonicity, $0 \le \mu(A) \le \mu(N) = 0$, so that A is also a null set.
2. Let $N_1, N_2, N_3, \dots \in \mathscr{F}$ be a countable family of nulls sets. Then by σ-subadditivity,

   $$\mu\left(\bigcup_{k=1}^{\infty} N_k\right) \le \sum_{k=1}^{\infty} \mu(N_k) = 0,$$

   so that $\bigcup_{k=1}^{\infty} N_k$ is also a null set.

That is, measurable subsets of null sets are again null sets, and countable unions of null sets are null sets.

Definition 2.6.1 Let $E \in \mathscr{F}$, and let (S) be a statement about the elements of E. We write: *Statement (S) holds μ-a.e. ("almost everywhere") on E*, if there exists $N \in \mathscr{F}$ satisfying

1. $\mu(N) = 0$,
2. $\{\omega \in E : \text{statement } (S) \text{ does not hold }\} \subseteq N$.

When $E = \Omega$, we simply write: *Statement (S) holds μ-a.e.*

(R) Observe that in general. the set

$$B_{(S)} := \{\omega \in E : \text{statement } (S) \text{ does not hold }\}$$

need not be a measurable set (i.e. $B_{(S)} \notin \mathscr{F}$) !

Thus, the statement

"(S) holds μ-a.e. on E"

means the following:

1. Statement (S) holds for all $\omega \in E$ *outside* of some null set N.
2. For $\omega \in E \cap N$, the statement (S) may or may not hold.

Example 2.5 1. Let $f, g : \Omega \to \mathbb{R}^*$, and $E \in \mathscr{F}$. The statement

$$\text{"}f(\omega) = g(\omega) \quad \mu\text{-a.e. on } E\text{"}$$

means: There exists a null set $N \in \mathscr{F}$ with

$$B_{(S)} := \{\, \omega \in E : f(\omega) \neq g(\omega) \,\} \subseteq N.$$

Now if in additition, both f and g are $\mathscr{F}$-measurable, then by Theorem 1.5.5, $B_{(S)} \in \mathscr{F}$, and hence $B_{(S)}$ is itself a null set. Thus,

$$\text{"}f(\omega) = g(\omega) \quad \mu\text{-a.e. on } E\text{"} \quad \Leftrightarrow \quad \mu\big(\{\, \omega \in E : f(\omega) \neq g(\omega) \,\}\big) = 0.$$

2. Let $(\Omega, \mathscr{F}, \mu) = \big(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda\big)$. We have

$$\mathbf{1}_{\mathbb{Q}} = 0 \quad \lambda\text{-a.e.}$$

To see this, note that

$$B_{(S)} = \big\{\, \omega \in \Omega : \text{"}\mathbf{1}_{\mathbb{Q}}(\omega) = 0\text{" does not hold} \big\} = \big\{\, \omega \in \Omega : \mathbf{1}_{\mathbb{Q}}(\omega) \neq 0 \,\big\} = \mathbb{Q},$$

which is a λ-null set.

Now if we change the measure to the counting measure μ_c, then $\mu_c(\mathbb{Q}) = \infty$, and $B_{(S)}$ will no longer be a null set. That is, the statement

$$\mathbf{1}_{\mathbb{Q}} = 0 \quad \mu_c\text{-a.e.}$$

is not true.

3. Let $(\Omega, \mathscr{F}, \mu)$ be any measure space and $f_n, f : \Omega \to \mathbb{K}$. The statement

$$f_n \to f \quad \mu\text{-a.e.} \qquad \text{(or simply} \quad f_n \xrightarrow{\text{a.e.}} f)$$

means: There exists a null set $N \in \mathscr{F}$ so that

$$f_n(\omega) \to f(\omega) \quad \forall \omega \in N^c.$$

Note: When $\omega \in N$, the sequence $(f_n(\omega))$ may or may not converge to $f(\omega)$. Furthermore, f need not be $\mathscr{F}$-measurable even if each f_n is $\mathscr{F}$-measurable; see Example 2.6 below. However, changing the value of f on a null-set, we may assume that f is $\mathscr{F}$-measurable: ■

Theorem 2.6.1 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $f_n, f : \Omega \to \mathbb{K}$. Suppose that

1. each f_n is $\mathscr{F}$-measurable, and
2. $f_n \xrightarrow{\text{a.e.}} f$.

Then there exists an $\mathscr{F}$-measurable function $\tilde{f} : \Omega \to \mathbb{K}$ so that

$$f_n \xrightarrow{\text{a.e.}} \tilde{f} \qquad \text{and} \qquad f = \tilde{f} \text{ a.e.}$$

Proof. By assumption 2., there exists a null set $N \in \mathscr{F}$ so that

$$f_n(\omega) \to f(\omega) \quad \forall \omega \in N^c.$$

Set

$$\tilde{f}_n := f_n \mathbf{1}_{N^c} \qquad \text{and} \qquad \tilde{f} := f \mathbf{1}_{N^c}.$$

Then

1. $f_n = \tilde{f}_n$ a.e. and $f = \tilde{f}$ a.e.
2. $\tilde{f}_n$ is $\mathscr{F}$-measurable, for all n,
3. $\tilde{f}_n(\omega) \to \tilde{f}(\omega) \quad \forall \omega \in \Omega$,

Thus $\tilde{f}$ is also $\mathscr{F}$-measurable. Since for all $\omega \in N^c$,

$$f_n(\omega) = \tilde{f}_n(\omega) \to \tilde{f}(\omega),$$

then $f_n \xrightarrow{\text{a.e.}} \tilde{f}$. ■

(R) Replacing f with the function $\tilde{f}$ of the above proof if necessary, we may always assume that f is also $\mathscr{F}$-measurable.

■ **Example 2.6** Let $\Omega = \mathbb{R}$ and

$$\mathscr{F} = \{E \subseteq \mathbb{R} : \text{ is countable or } E^c \text{ is countable}\}.$$

Let $\mu = \delta_0$, the Dirac one-point measure at zero. Let $f_n = \mathbf{1}_{\mathbb{Q}}$ and $f = \mathbf{1}_{[0,1]}$. Then

1. each f_n is $\mathscr{F}$-measurable,
2. $f_n \xrightarrow{\text{a.e.}} f$. (Because $N := \mathbb{R} \setminus \{0\}$ is a null set, and $f_n(\omega) = 1 \to f(\omega) = 1 \; \forall \omega \in N^c = \{0\}$.)

Note, however, that f is *not* $\mathscr{F}$-measurable ! Nevertheless, if we set $\tilde{f} = f\mathbf{1}_{N^c} = f\mathbf{1}_{\{0\}} = \mathbf{1}_{\{0\}}$, then $\tilde{f}$ is $\mathscr{F}$-measurable and $f_n \xrightarrow{\text{a.e.}} \tilde{f}$. ■

Definition 2.6.2 A measure space $(\Omega, \mathscr{F}, \mu)$ is said to be *complete*, if it has the following property: Whenever $N \in \mathscr{F}$ is a null set, and $A \subseteq N$, then $A \in \mathscr{F}$ as well. ("subsets of null sets are measurable sets".) Then by monotonicity, A itself is a null set.

Theorem 2.6.2 Let $(\Omega, \mathscr{F}, \mu)$ be a *complete* measure space, and let $f, g : \Omega \to \mathbb{K}$ satisfy

1. f is $\mathscr{F}$-measurable,
2. $f = g$ a.e.

Then g is also $\mathscr{F}$-measurable.

Proof. By assumption 2., there exists a null set $N \in \mathscr{F}$ so that

$$f(\omega) = g(\omega) \qquad \forall \omega \in N^c.$$

1)) Assume first that $f, g : \Omega \to \mathbb{R}^*$. Then $\forall a \in \mathbb{R}$,

$$\begin{aligned}\{\omega \in \Omega \,|\, g(\omega) > a\} &= \{\omega \in N \,|\, g(\omega) > a\} \cup \{\omega \in N^c \,|\, g(\omega) > a\} \\ &= \{\omega \in N \,|\, g(\omega) > a\} \cup \{\omega \in N^c \,|\, f(\omega) > a\} \\ &= \underbrace{\{\omega \in N \,|\, g(\omega) > a\}}_{\in \mathscr{F} \text{ by completeness}} \cup \Big[\underbrace{\{\omega \in \Omega \,|\, f(\omega) > a\}}_{\in \mathscr{F} \text{ as } f \text{ is } \mathscr{F}\text{-meas.}} \cap \underbrace{N^c}_{\in \mathscr{F}}\Big] \in \mathscr{F},\end{aligned}$$

by properties (A2) and (A3) of a σ-algebra. This shows that g is $\mathscr{F}$-measurable.

2) Now let $f, g : \Omega \to \mathbb{C}$. Since $f = g$ on N^c, then $\operatorname{Re} f = \operatorname{Re} g$ and $\operatorname{Im} f = \operatorname{Im} g$ on N^c. That is, $\operatorname{Re} f = \operatorname{Re} g$ a.e. and $\operatorname{Im} f = \operatorname{Im} g$ a.e. Since $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathscr{F}$-measurable, then by part 1),

$$\operatorname{Re} g,\ \operatorname{Im} g : \Omega \to \mathbb{R}$$

are $\mathscr{F}$-measurable. It follows that g is $\mathscr{F}$-measurable. ∎

(R) Completeness can not be removed here. For let $(\Omega, \mathscr{F}, \mu)$ be as in Example 2.6. Set $f = 1$ and $g = \mathbf{1}_{[0,1]}$. Then f is $\mathscr{F}$-measureable and $f = g$ a.e. However g is not $\mathscr{F}$-measurable !

Corollary 2.6.3 Let $(\Omega, \mathscr{F}, \mu)$ be a *complete* measure space and $f_n, f : \Omega \to \mathbb{K}$. Suppose that

1. each f_n is $\mathscr{F}$-measurable, and
2. $f_n \xrightarrow{\text{a.e.}} f$.

Then f is also $\mathscr{F}$-measurable.

Proof. Let N, $\tilde{f}_n$ and $\tilde{f}$ be as in the proof of Theorem 2.6.2. Since $f = \tilde{f}$ a.e. and $\tilde{f}$ is $\mathscr{F}$-measurable, it follows from Theorem 2.6.2 that f is $\mathscr{F}$-measurable. ∎

Exercise 2.9 (Every measure space can be made complete.)
Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Set

$$\widetilde{\mathscr{F}} := \{E \cup A \,|\, E \in \mathscr{F},\ A \subseteq N \text{ for some null set } N \in \mathscr{F}\}.$$

1. Show: $\tilde{\mathscr{F}}$ is a σ-algebra on Ω.
2. Set $\tilde{\mu}(\widetilde{E}) := \mu(E) \quad \forall \widetilde{E} = E \cup A \in \widetilde{F}$. Show:
   (a) $\tilde{\mu}$ is well defined. (That is, if $\tilde{E} = E_1 \cup A_1 = E_2 \cup A_2$ for some $E_1, E_2 \in \mathscr{F}$ and subsets A_1, A_2 of null sets, then $\mu(E_1) = \mu(E_2)$.)

(b) $\tilde{\mu}(E) = \mu(E) \quad \forall E \in \mathscr{F}$.

(c) $\tilde{\mu}$ is a measure on $(\Omega, \widetilde{\mathscr{F}})$.

(d) The measure space $(\Omega, \widetilde{\mathscr{F}}, \tilde{\mu})$ is complete.

(e) If $(\Omega, \mathscr{F}, \mu)$ is complete, then $\widetilde{\mathscr{F}} = \mathscr{F}$.

We call $(\Omega, \widetilde{\mathscr{F}}, \tilde{\mu})$ the *completeion* of $(\Omega, \mathscr{F}, \mu)$. ∎

Example 2.7

1. The completion of the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$ is denoted by $(\mathbb{R}, \mathscr{M}, \lambda)$. $\mathscr{M}$ is called the *σ-algebra of Lebesgue measurable sets*. One can show that $\mathscr{B}(\mathbb{R}) \subsetneq \mathscr{M} \subsetneq \mathscr{P}(\mathbb{R})$.
2. Let $(\Omega, \mathscr{F})$ be any measurable space, and μ_c the counting measure. Since the empty set is the only μ_c-null set, it follows that $(\Omega, \mathscr{F}, \mu_c)$ is already complete.
3. Let $(\Omega, \mathscr{F})$ be any measurable space. Fix $a \in \Omega$ with $\{a\} \in \mathscr{F}$ and let δ_a denote the one-point Dirac measure on $(\Omega, \mathscr{F})$. Then $(\Omega, \mathscr{P}(\Omega), \delta_a)$ is the completetion of $(\Omega, \mathscr{F}, \delta_a)$, as can be easily checked.

∎

Theorem 2.6.4 Let $f, g : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable functions, with $f(\omega) = g(\omega)$ a.e. If $\int f \, d\mu$ is defined, then $\int g \, d\mu$ is also defined, and

$$\int f \, d\mu = \int g \, d\mu.$$

Proof. Set

$$N := \{\omega \in \Omega \,|\, f(\omega) \neq g(\omega)\}.$$

Then by Theorem 1.5.5, $N \in \mathscr{F}$, and then by assumption, $\mu(N) = 0$.

Case 1: $f, g \in \mathscr{L}^+$. Then $\int f, \int g$ are both defined, and

$$\int f \overset{\text{exer 2.7}}{=} \underbrace{\int_N f}_{=0 \text{ by thm 2.5.1}} + \int_{N^c} f \overset{\text{exer 2.7}}{=} \underbrace{\int_N g}_{=0 \text{ by thm 2.5.1}} + \underbrace{\int_{N^c} f}_{f=g \text{ on } N^c} \overset{\text{exer 2.7}}{=} \int g.$$

Case 2: $f, g : \Omega \to \mathbb{R}^*$, and suppose that $\int f$ is defined. Since $f(\omega) = g(\omega)$ for all $\omega \in \mathbb{N}^c$, then

$$f^+(\omega) = g^+(\omega) \qquad \text{and} \qquad f^-(\omega) = g^-(\omega) \qquad \text{for all } \omega \in N^c,$$

that is,

$$f^+ = g^+ \quad \mu\text{-a.e.} \qquad \text{and} \qquad f^- = g^- \quad \mu\text{-a.e.}$$

Then by case 1,

$$\int f^+ = \int g^+ \qquad \text{and} \qquad \int f^- = \int g^-$$

which shows that

1. $\int g$ is defined, and

Case 3: $f, g : \Omega \to \mathbb{C}$, , and suppose that $\int f$ is defined. (in the complex-valued case, this means "f is integrable"). Then again,

$$\mathrm{Re} f = \mathrm{Re} g \quad \text{a.e.}, \qquad \mathrm{Im} f = \mathrm{Im} g \quad \text{a.e.}, \qquad \text{and} \quad |f| = |g| \quad \text{a.e.},$$

namely at all $\omega \in N^c$. Now by case 1,

$$\int |g| = \int |f| < \infty,$$

which shows that g is integrable, and by case 2. that

$$\int g = \int \mathrm{Re} g + i \int \mathrm{Im} g = \int \mathrm{Re} f + i \int \mathrm{Im} f = \int f.$$

Thus the proof is complete. ∎

Corollary 2.6.5 Let $f, g : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable functions. If
1. g is integrable, and
2. $|f| \le |g|$ a.e. , ("f is dominated by g")

then $\int |f| \le \int |g|$. In particlular, f is also integrable.

Proof. Set

$$N := \{\omega \in \Omega : |f(\omega)| > |g(\omega)|\}.$$

. By Theorem 1.5.5, N is $\mathscr{F}$-measurable, and by assumption 2., $\mu(N) = 0$. Set $\tilde{f} = f \mathbf{1}_{N^c}$. Then
(a) $\tilde{f}$ is $\mathscr{F}$-measurable,
(b) $|f| = |\tilde{f}|$ a.e. ,
(c) $|\tilde{f}(\omega)| \le |g(\omega)|$ for all $\omega \in \Omega$.

It follows from monotonicity of the integral in $\mathscr{L}^+$ that

$$0 \le \int |f| \overset{\text{thm}}{\underset{2.6.4}{=}} \int |\tilde{f}| \le \int |g| < \infty$$

which proves the assertion. ∎

Theorem 2.6.6 Let $f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable. Then

$$\int |f| \, d\mu = 0 \quad \Leftrightarrow \quad f(\omega) = 0 \text{ a.e.}$$

Proof. $\Leftarrow$: If $f = 0$ a.e., then $|f| = 0$ a.e., so that by Theorem 2.6.4,

$$\int |f| = \int 0 = 0.$$

$\Rightarrow$: Supopose that $\int |f| = 0$. Let

$$N := \{\omega \in \Omega : f(\omega) \neq 0\} = \{\omega \in \Omega : |f(\omega)| > 0\} \in \mathscr{F}.$$

We can write

$$N = \bigcup_{n=1}^{\infty} A_n \qquad \text{where} \quad A_n := \left\{\omega \in \Omega \,:\, |f(\omega)| > \frac{1}{n}\right\} \in \mathscr{F}.$$

Now for each n, since $\frac{1}{n}\mathbf{1}_{A_n} \leq |f|$, we have by monotonicty of the integral that

$$0 \leq \mu(A_n) = \int \mathbf{1}_{A_n} = n \int \frac{1}{n}\mathbf{1}_{A_n} \leq n \int |f| = 0,$$

so that $\mu(A_n) = 0$. Thus, $\mu(N) = 0$ as well, which shows that $f(\omega) = 0$ a.e. ■

Theorem 2.6.7 Let $f : \Omega \to \mathbb{R}^*$ be integrable. Then f is finite valued almost everywhere. (That is, $|f(\omega)| < \infty$ a.e.).

Proof. Since f is integrable, then $M := \int |f| < \infty$. Set

$$N := \{\omega \in \Omega : |f(\omega| = \infty\} = \bigcap_{n=1}^{\infty} \{\omega \in \Omega : |f(\omega| > n\} \in \mathscr{F}.$$

Then for each $n \in N$,

$$n\mathbf{1}_N \leq |f|,$$

and hence by monotonicity of the integral in $\mathscr{L}^+$,

$$0 \leq n\mu(N) = n \int \mathbf{1}_N = \int n\mathbf{1}_N \leq \int |f| = M.$$

It follows that

$$0 \leq \mu(N) \leq \frac{M}{n} \qquad \forall n \in \mathbb{N},$$

from which we conclude that $\mu(N) = 0$. This proves the theorem. ■

(R) Given an integrable function $f : \Omega \to \mathbb{R}^*$, let N be as in the above proof. Set $\tilde{f} := f\mathbf{1}_{N^c}$. Then

1. $\tilde{f} = f$ a.e.
2. $\tilde{f} : \Omega \to \mathbb{R}$. (i.e. $\tilde{f}$ is *finite-valued.*)
3. $\tilde{f}$ is $\mathscr{F}$-measurable.
4. By Theorem 2.6.4, $\int \tilde{f} = \int f$. In particular, $\tilde{f}$ is integrable.

We have shown: Given $f \in \mathscr{L}^1_{\mathbb{R}^*}$, there exists $\tilde{f} \in \mathscr{L}^1_{\mathbb{R}}$ so that

$$\tilde{f}(\omega) = f(\omega) \text{ a.e.} \qquad \text{and} \qquad \int \tilde{f} = \int f.$$

For this reason, some authors consider the space $\mathscr{L}^1_{\mathbb{R}}$ only.

Exercise 2.10 Let $f, g \in \mathscr{L}^+$. Show:

1. If $f(\omega) \leq g(\omega)$ a.e., then $\int f\,d\mu \leq \int g\,d\mu$.
2. Suppose that $(\Omega, \mathscr{F}, \mu)$ satisfies the following property: *For each $E \in \mathscr{F}$, $\mu(E) = \infty$, there exists $A \in \mathscr{F}$ with $A \subseteq E$ and $0 < \mu(A) < \infty$.* (For example, σ-finite measure spaces have this property) Show:

$$\text{If} \quad \int_A f\,d\mu \leq \int_A g\,d\mu \quad \forall A \in \mathscr{F} \quad \text{then} f(\omega) \leq g(\omega) \text{ a.e. .}$$

2.7 Convergence Theorems

In this section, we give answers to the following question:

Let $f_n(\omega) \to f(\omega)$ a.e. If each f_n is integrable,

(i) will f be integrable ?

(ii) If yes, will $\int f\,d\mu = \lim_{n\to\infty} \int f_n d\mu$? That is, can we exchange the limit and the integral ?

In general, the answer to both questions is negative.

■ **Example 2.8** Let $(\Omega, \mathscr{F}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$.

1. Consider the functions $f_n(\omega) = \mathbf{1}_{[0,n]}(\omega) = \begin{cases} 1 & \text{if } 0 \leq \omega \leq n \\ 0 & \text{else.} \end{cases}$

   Then
   (a) $f_n(\omega) \xrightarrow{\text{p.w.}} f(\omega) = \mathbf{1}_{[0,\infty)}$, while
   (b) $\int f_n = \int \mathbf{1}_{[0,n]} = n \to \infty = \int f$.
   Note that f is not integrable ! Thus, (ii) holds, but (i) does not hold.
2. Consider the functions $f_n(\omega) = n\mathbf{1}_{(0,\frac{1}{n}]}(\omega) = \begin{cases} n & \text{if } 0 < \omega \leq \frac{1}{n} \\ 0 & \text{else.} \end{cases}$

   Then
   (a) $f_n(\omega) \xrightarrow{\text{p.w.}} f(\omega) = 0$, while
   (b) $\int f_n = n\int \mathbf{1}_{(0,\frac{1}{n}]} = n \cdot \frac{1}{n} = 1 \not\to \int f = \int 0 = 0.$
   Here (i) holds, but (ii) does not hold. Observe that the sequence (f_n) is unbounded.
3. Even when $f_n \rightrightarrows f$ the answer may be negative. For example, consider the functions $f_n = \frac{1}{n}\mathbf{1}_{[2^{n-1},2^n]}$. Then
   (a) $f_n(\omega) \rightrightarrows f(\omega) = 0$, and
   (b) $\int f_n = \frac{1}{n}\int \mathbf{1}_{[2^{n-1},2^n]} = \frac{1}{n} \cdot (2^n - 2^{n-1}) = \frac{2^{n-1}}{n} \to \infty$, while

   $\int f = \int 0 = 0$. That is, f is integrable, but

   $$\int f_n \not\to \int f.$$

   Again, (i) holds but (ii) does not hold.

Theorem 2.7.1 [Monotone Convergence Theorem for $\mathscr{L}^+$, MCT]
Let $(f_n)\uparrow$ be an increasing sequence in $\mathscr{L}^+$. Then

$$\int \left(\lim_{n\to\infty} f_n\right) d\mu \;=\; \lim_{n\to\infty} \int f_n \, d\mu.$$

Proof. First an observation: Since $(f_n)\uparrow$, then $f(\omega) := \lim_{n\to\infty} f_n(\omega)$ exists in $[0,\infty]$ for all $\omega \in \Omega$. Note that by Theorem 1.5.7, f is also $\mathscr{F}$-measurable. Furthermore, by monotonicity of the integral, $(\int f_n)\uparrow$, so that $\lim_{n\to\infty} \int f_n$ exists in $\mathbb{R}^*$.

Now by the Structure Theorem, for each f_n there exists a sequence $(\varphi_{n,k})_{k=1}^{\infty}\uparrow$ in $\mathscr{S}^+$ so that

$$f_n(\omega) = \lim_{k\to\infty} \varphi_{n,k}(\omega) \qquad \forall\, \omega \in \Omega.$$

We now construct a new sequence $(\psi_k)\uparrow$ in $\mathscr{S}^+$.

$$\begin{array}{ccccccccccccc}
\varphi_{11} & \leq & \varphi_{12} & \leq & \varphi_{13} & \leq & \varphi_{14} & \leq & \varphi_{15} & \leq & \cdots\cdots & \longrightarrow & f_1 \\
 & & & & & & & & & & & & \wedge\!| \\
\varphi_{21} & \leq & \varphi_{22} & \leq & \varphi_{23} & \leq & \varphi_{24} & \leq & \varphi_{25} & \leq & \cdots\cdots & \longrightarrow & f_2 \\
 & & & & & & & & & & & & \wedge\!| \\
\varphi_{31} & \leq & \varphi_{32} & \leq & \varphi_{33} & \leq & \varphi_{34} & \leq & \varphi_{35} & \leq & \cdots\cdots & \longrightarrow & f_3 \\
 & & & & & & & & & & & & \wedge\!| \\
\varphi_{41} & \leq & \varphi_{42} & \leq & \varphi_{43} & \leq & \varphi_{44} & \leq & \varphi_{45} & \leq & \cdots\cdots & \longrightarrow & f_4 \\
 & & & & & & & & & & & & \wedge\!| \\
\varphi_{51} & \leq & \varphi_{52} & \leq & \varphi_{53} & \leq & \varphi_{54} & \leq & \varphi_{55} & \leq & \cdots\cdots & \longrightarrow & f_5 \\
 & & & & & & & & & & & & \wedge\!| \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
 & & & & & & & & & & & & \downarrow \\
 & & & & & & & & & & & & f
\end{array}$$

For each $k \in \mathbb{N}$, set

$$\psi_k := \max\{\varphi_{1,k},\ \varphi_{2,k},\ \ldots,\ \varphi_{k,k}\}.$$

Then

(a) each ψ_k is simple, as $\operatorname{range}(\psi_k) \subseteq \bigcup_{n=1}^{k} \operatorname{range}(\varphi_{n,k})$, which is a finite set.
(b) each ψ_k is $\mathscr{F}$-measurable, by Theorem 1.5.6. Thus, $\psi_k \in \mathscr{S}^+$.
(c) the sequence (ψ_k) is increasing. In fact, for each k we have as $\varphi_{n,k} \leq \varphi_{n,k+1}$ that

$$\begin{array}{ccc}
\varphi_{1,k} & \leq & \varphi_{1,k+1} \\
\varphi_{2,k} & \leq & \varphi_{2,k+1} \\
\vdots & & \vdots \\
\varphi_{k,k} & \leq & \varphi_{k,k+1} \\
0 & \leq & \varphi_{k+1,k+1}
\end{array}$$

Taking the max, first over the right-hand column, and then over the left-hand column, we obtain

$$\begin{aligned}
\psi_k &= \max\{\varphi_{1,k},\ \varphi_{2,k}, \ldots,\ \varphi_{k,k}\} \\
&\leq \max\{\varphi_{1,k+1},\ \varphi_{2,k+1}, \ldots,\ \varphi_{k,k+1},\ \varphi_{k+1,k+1}\} = \psi_{k+1}.
\end{aligned}$$

(d) $\psi_k \xrightarrow{\text{p.w.}} f$. In fact, for each pair (n,k) of indices we have as $(\varphi_{n,k})_{k=1}^{\infty}\uparrow$ that

$$\varphi_{n,k} \le f_n \le f. \tag{2.21}$$

Hence for all $r \le k$, as $(f_n)\uparrow$,

$$\varphi_{r,k} \le \underbrace{\max_{1\le n\le k} \varphi_{n,k}}_{=\psi_k} \overset{(2.21)}{\le} \max_{1\le n\le k} f_n = f_k \le f. \tag{2.22}$$

Letting $k \to \infty$, then

$$f_r = \lim_{k\to\infty} \varphi_{r,k} \le \lim_{k\to\infty} \psi_k \le f$$

for all $r \in \mathbb{N}$. Next we let $r \to \infty$ to obtain

$$f = \lim_{r\to\infty} f_r \le \lim_{k\to\infty} \psi_k \le f$$

from which we conclude that

$$f = \lim_{k\to\infty} \psi_k.$$

We are now ready to compute $\int f$. In fact, by (2.22) and monotonicity of the integral in $\mathscr{L}^+$ we have for all k that

$$\int \psi_k \le \int f_k \le \int f.$$

Thus,

$$\int f \overset{\text{Thm}}{\underset{2.2.1}{=}} \lim_{k\to\infty} \int \psi_k \le \lim_{k\to\infty} \int f_k \le \int f,$$

which shows that

$$\int f\,d\mu = \lim_{k\to\infty} \int f_k\,d\mu$$

and completes the proof. ■

There are some generalizations of this Theorem. The first says that everywhere convergence may be replaced by a.e. convergence.

Corollary 2.7.2 Let (f_n) be a sequence in $\mathscr{L}^+$, and $f \in \mathscr{L}^+$. Suppose that
1. $(f_n(\omega))\uparrow$ a.e.
2. $f_n(\omega) \to f(\omega)$ a.e.

Then

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu.$$

The next Corollary removes the condition that $f_n \ge 0$ for all n. Instead, the sequence (f_n) needs to be bounded below by an integrable function.

Corollary 2.7.3 Let $f_n, f : \Omega \to \mathbb{R}^*$ be $\mathscr{F}$-measurable. Suppose that

1. $f_1 \in \mathscr{L}^1_{\mathbb{R}^*}$
2. $(f_n(\omega)) \uparrow$ a.e.
3. $f_n(\omega) \to f(\omega)$ a.e.

Then $\int f_n$ and $\int f$ are defined, and

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

We may also apply the MCT to the sequence of partial sums of a series:

Corollary 2.7.4 Let (f_n) be a sequence in $\mathscr{L}^+$. Then

$$\int \left[\sum_{n=1}^{\infty} f_n \right] d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

The last two Corollaries are an application of the MCT to integrals over sets.

Corollary 2.7.5 Let $f \in \mathscr{L}^+$, and $\{A_n\}_{n=1}^{\infty} \uparrow$ be an *increasing* sequence of sets in $\mathscr{F}$. Then

$$\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \lim_{n\to\infty} \int_{A_n} f \, d\mu.$$

Corollary 2.7.6 Let $f \in \mathscr{L}^+$, and $\{A_n\}_{n=1}^{\infty}$ be a collection of *mutually disjoint* sets in $\mathscr{F}$. Then

$$\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu.$$

Exercise 2.11 Prove the above five corollaries. ■

■ **Example 2.9** Let $f_n \in \mathscr{L}^+ \ \forall n$, and $f_n \xrightarrow{\text{p.w.}} f$. If we remove the assumption that $(f_n) \uparrow$, then

$$\lim_{n\to\infty} \int f_n \, d\mu$$

need no longer exists. Note, however, that

$$\underline{\lim}_n \int f_n \qquad \text{and} \qquad \overline{\lim}_n \int f_n$$

always exists.

For example, let $f_n = [2 + (-1)^n] \mathbf{1}_{[n,n+1]}$. Then $f_n \xrightarrow{\text{p.w.}} f = 0$. On the other hand,

$$\int f_n \, dm = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even,} \end{cases}$$

which shows that the sequence of integrals $(\int f_n)$ diverges. Observe that

$$\underline{\lim}_n \int f_n = 1 \qquad \text{and} \qquad \overline{\lim}_n \int f_n = 3.$$

so that

$$\int \lim_{n\to\infty} f_n = \int 0 = 0 < 1 = \liminf_n \int f_n.$$

In general we have: ■

Theorem 2.7.7 *[Fatou's Lemma]* Let (f_n) be a sequence in $\mathscr{L}^+$. Then

$$\int \left(\liminf_n f_n\right) d\mu \;\leq\; \liminf_n \int f_n \, d\mu.$$

Proof. As we want to apply the MCT, we set

$$g_n := \inf\{f_n, f_{n+1}, f_{n+2}, \dots\} \;=\; \inf_{k\geq n} f_k$$

for each $n \in \mathbb{N}$. Then

(a) $g_n \in \mathscr{L}^+$ by Theorem 1.5.7.
(b) $(g_n)\uparrow$.
(c) $\lim_{n\to\infty} g_n \;=\; \lim_{n\to\infty} \inf_{k\geq n} f_k \;=\; \liminf_n f_n$.

We can thus apply the MCT to the sequence (g_n) and obtain

$$\int \left(\lim_{n\to\infty} g_n\right) d\mu \;=\; \lim_{n\to\infty} \int g_n \, d\mu.$$

That is,

$$\int \left(\liminf_n f_n\right) d\mu \;=\; \lim_{n\to\infty} \int g_n \, d\mu \;=\; \liminf_n \int g_n \, d\mu \;\overset{g_n \leq f_n}{\leq}\; \liminf_n \int f_n \, d\mu,$$

where the last inequality follows from monotonicity of the integral. ■

R We usually don't have equality in Fatou's lemma as Example 2.9 above shows.

Theorem 2.7.8 *[Dominated Convergence Theorem, DCT]*
Let (f_n) be a sequence of $\mathscr{F}$-measurable functions, $f_n : \Omega \to \mathbb{K}$. Suppose that

1. there exists an $\mathscr{F}$-measurable function $f : \Omega \to \mathbb{K}$ so that

$$f_n(\omega) \to f(\omega) \text{ a.e.}$$

2. there exists an integrable function $g : \Omega \to [0,\infty]$ (i.e. $g \in \mathscr{L}^1 \cap \mathscr{L}^+$) so that

$$|f_n(\omega)| \leq g(\omega) \text{ a.e.}$$

Then

(a) f_n and f are all integrable, and
(b) $\displaystyle\int f \, d\mu \;=\; \lim_{n\to\infty} \int f_n \, d\mu$.

Proof. **Case 1:** We begin the proof under some stronger assumptions, namely that $f_n : \Omega \to \mathbb{R}$, $g : \Omega \to [0, \infty)$, and that assumptions 1. and 2. hold at *every* $\omega \in \Omega$.
We first note that since

$$|f_n(\omega)| \leq g(\omega) \qquad \forall\, \omega \in \Omega,$$

then also

$$|f(\omega)| = \lim_{n\to\infty} |f_n(\omega)| \leq g(\omega) < \infty \qquad \forall\, \omega \in \Omega.$$

That is, all f_n and f are dominated by g. It follows from Corollary 2.6.5 that all f_n and f are integrable. In addition, all functions involved are finite valued.
First consider the sequence of functions

$$(g + f_n)_{n=1}^{\infty}.$$

By the above note, $g + f_n \in \mathscr{L}^+ \cap \mathscr{L}^1_{\mathbb{R}}$ for all n. We can thus apply Fatou's Lemma and obtain

$$\begin{aligned}
\int g + \int f &= \int [g + f] = \int \left[g + \lim_{n\to\infty} f_n\right] \\
&= \int \lim_{n\to\infty} [g + f_n] = \int \underline{\lim}_n [g + f_n] \\
&\underset{\text{Fatou}}{\leq} \underline{\lim}_n \int [g + f_n] = \underline{\lim}_n \left[\int g + \int f_n\right] \\
&= \int g + \underline{\lim}_n \int f_n.
\end{aligned}$$

Since $\int g$ is finite, we can subtract it from both sides to obtain

$$\int f \leq \underline{\lim}_n \int f_n. \tag{2.23}$$

In a similar way, we consider the sequence of functions

$$(g - f_n)_{n=1}^{\infty} = (g + (-f_n))_{n=1}^{\infty}.$$

Again, by the above note, $g - f_n \in \mathscr{L}^+ \cap \mathscr{L}^1_{\mathbb{R}}$ for all n, and applying Fatou's Lemma we obtain as above (since $-f_n \to -f$) that

$$\int (-f) \leq \underline{\lim}_n \int (-f_n).$$

Recall that for any sequence (x_n) in $\mathbb{R}^*$ we have

(i) $\underline{\lim}_n (-x_n) = -\overline{\lim}_n x_n$.

(ii) $\lim_{n\to\infty} x_n$ exists $\Leftrightarrow$ $\underline{\lim}_n x_n = \overline{\lim}_n x_n$, in which case

$$\lim_{n\to\infty} x_n = \underline{\lim}_n x_n = \overline{\lim}_n x_n.$$

Hence,

$$-\int f = \int(-f) \underset{(i)}{\leq} -\overline{\lim_n}\int f_n.$$

Multiply by -1,

$$\overline{\lim_n}\int f_n \leq \int f. \tag{2.24}$$

Combining (2.23) and (2.24) we obtain

$$\int f \leq \underline{\lim_n}\int f_n \leq \overline{\lim_n}\int f_n \leq \int f,$$

from which we conclude that $\lim_{n\to\infty}\int f_n$ exists, and

$$\int f = \lim_{n\to\infty}\int f_n.$$

Case 2: Next we consider the general case of $f_n : \Omega \to \mathbb{R}^*$, removing the additional restrictions. By assumption 1.,

$$\{\omega \in \Omega : f_n(\omega) \not\to f(\omega)\} \subseteq K_0$$

for some null set K_0. Furthermore, by assumption 2., the sets

$$K_n := \{\omega \in \Omega : |f_n(\omega)| > g(\omega)\} \quad (n \in \mathbb{N})$$

are all null sets. Finally, since g is integrable, then the set

$$K_\infty := \{\omega \in \Omega : g(\omega) = \infty\}$$

is also a null set, by Theorem 2.6.7. It follows that the set

$$N := \left[\bigcup_{n=0}^{\infty} K_n\right] \cup K_\infty.$$

is a null set.

We now modify all functions involved on N, buy setting

$$\tilde{f}_n := f_n\mathbf{1}_{N^c}, \qquad \tilde{f} := f\mathbf{1}_{N^c} \qquad \text{and} \qquad \tilde{g} := g\mathbf{1}_{N^c}, \qquad .$$

These functions all satisfy the assumptions of Case 1, so that

(a) $\tilde{f}_n$ and $\tilde{f}$ are all integrable

(b) $\displaystyle\int \tilde{f}\,d\mu = \lim_{n\to\infty}\int \tilde{f}_n\,d\mu.$

Now as

$$f_n = \tilde{f}_n \text{ a.e.} \qquad \text{and} \qquad f = \tilde{f} \text{ a.e. },$$

it follows from Theorem 2.6.4 that f_n and f are integrable, and

$$\int f \underset{\text{Thm 2.6.4}}{=} \int \tilde{f} = \lim_{n\to\infty}\int \tilde{f}_n \underset{\text{Thm 2.6.4}}{=} \lim_{n\to\infty}\int f_n.$$

Case 3: It is left to consider the case $f_n : \Omega \to \mathbb{C}$. We simply split all functions into their real and imaginary parts:

(i) Since $f_n \to f$ a.e. then

$$\operatorname{Re}(f_n) \to \operatorname{Re}(f) \text{ a.e.} \qquad \text{and} \qquad \operatorname{Im}(f_n) \to \operatorname{Im}(f) \text{ a.e.}$$

(ii) Since $|\operatorname{Re}(z)|, |\operatorname{Im}(z)| \leq |z|$ for $z \in \mathbb{C}$, then by assumption 2.,

$$|\operatorname{Re}(f_n)| \leq |f| \leq g \text{ a.e.} \qquad \text{and} \qquad |\operatorname{Im}(f_n)| \leq |f| \leq g \text{ a.e.}$$

We thus can apply Case 2, to obtain that

(a) $\operatorname{Re}(f_n), \operatorname{Im}(f_n), \operatorname{Re}(f), \operatorname{Im}(f)$ are all integrable, and

(b) $\int \operatorname{Re}(f) = \lim_{n\to\infty} \int \operatorname{Re}(f_n)$ and $\int \operatorname{Im}(f) = \lim_{n\to\infty} \int \operatorname{Im}(f_n)$.

It follows that f_n and f are integrable, and

$$\int f \underset{\text{def}}{=} \int \operatorname{Re}(f) + i \int \operatorname{Im}(f) = \left[\lim_{n\to\infty} \int \operatorname{Re}(f_n)\right] + i \left[\lim_{n\to\infty} \operatorname{Im}(f_n)\right]$$
$$= \lim_{n\to\infty} \left[\int \operatorname{Re}(f_n) + i \int \operatorname{Im}(f_n)\right] \underset{\text{def}}{=} \lim_{n\to\infty} \int f_n.$$

This completes the proof. ■

Corollary 2.7.9 Let $f \in \mathscr{L}^1_{\mathbb{K}}$.

1. If $\{A_n\}_{n=1}^{\infty} \uparrow$ is an *increasing* sequence of sets in $\mathscr{F}$ then

$$\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \lim_{n\to\infty} \int_{A_n} f \, d\mu.$$

2. If $\{A_n\}_{n=1}^{\infty}$ is a collection of *mutually disjoint* sets in $\mathscr{F}$, then

$$\int_{\bigcup_{n=1}^{\infty} A_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu.$$

Exercise 2.12 Prove Corollary 2.7.9. ■

■ **Example 2.10** Let $(\Omega, \mathscr{F} \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, and consider the sequence of Borel functions

$$f_n(x) = \sin^n(x) \, \mathbf{1}_{[0,2\pi]}(x).$$

1. If $x \in [0, 2\pi]$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}$, then $|\sin(x)| < 1$. so that $\lim_{n\to\infty} \sin^n(x) = 0$. It follows that

$$f_n(x) \to f(x) = 0 \text{ a.e.}$$

2. For all $n \in \mathbb{N}$ we have that

$$|f_n(x)| \leq g(x) := \mathbf{1}_{[0,2\pi]}(x) \in \mathscr{L}^+ \cap L^1.$$

It follows from the DCT that

(a) Each f_n is integrable, and

(b) $\lim_{n\to\infty}\int_{[0,2\pi]} \sin^n(x)\,d\lambda = \lim_{n\to\infty}\int f_n\,d\lambda = \int f = d\lambda = \int 0\,d\lambda = 0.$

∎

We have already applied the MCT to series of non-negative, measurable functions by applying the MCT in Corollary 2.7.4. In a similar way, the DCT can be applied to series of arbitrary measurable functions:

Theorem 2.7.10 *[Beppo Levi]* Let (f_n) be a sequence of functions in $\mathscr{L}^1_{\mathbb{K}}$, and suppose that

$$\sum_{n=1}^{\infty}\int |f_n|\,d\mu < \infty.$$

Then the series $\sum_{n=1}^{\infty} f_n$ converges a.e. to some $f \in \mathscr{L}^1_{\mathbb{K}}$, and

$$\int f\,d\mu = \sum_{n=1}^{\infty}\int f_n\,d\mu. \qquad \left(\text{i.e.} \int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty}\int f_n\right) \tag{2.25}$$

Proof. 1. *Proof of convergence.* We want to make use of the fact that every absolutely convergent series in $\mathbb{R}$ or $\mathbb{C}$ converges. Let us first set

$$g := \sum_{n=1}^{\infty}\int |f_n| \in \mathscr{L}^+ \qquad \text{and} \qquad M := \int \sum_{n=1}^{\infty}\int |f_n| < \infty.$$

By Corollary 2.7.4 and the assumption we have

$$\int g\,d\mu = \int \left[\sum_{n=1}^{\infty}|f_n|\right] d\mu \underset{\text{cor 2.7.4}}{=} \sum_{n=1}^{\infty}\int |f_n|\,d\mu = M < \infty \tag{2.26}$$

so that by Theorem 2.6.7, g is finite-valued for all ω outside of some null set K. In particular, the functions $f_n(\omega)$ are finite valued for $\omega \notin K$, so that the partial sums

$$S_N(\omega) = \sum_{n=1}^{N} f_n(\omega)$$

are defined for all $\omega \notin K$. Since every absolutely convergent series in $\mathbb{R}$ or in $\mathbb{C}$ is convergent, then $\sum_{n=1}^{\infty} f_n(\omega)$ converges outside of K. That is, there exists $f:\Omega\to\mathbb{K}$ so that

$$f(\omega) = \sum_{n=1}^{\infty} f_n(\omega) = \lim_{N\to\infty} S_N(\omega) \ \text{ a.e.}$$

Finally, by Theorem 2.6.1 we may assume that f is $\mathscr{F}$-measurable.
2. *Proof that f is integrable and (2.25) holds.* For all $N \in \mathbb{N}$ we have

$$|S_N(\omega)| \leq \sum_{n=1}^{N}|f_n(\omega)| \leq \sum_{n=1}^{\infty}|f_n(\omega)| = g(\omega) \ \text{ a.e. .}$$

Since by (2.26), g is integrable, we may apply the DCT to the sequence (S_N) and obtain that

(a) f is integrable,

(b) $\int f \underset{\text{DCT}}{=} \lim_{N\to\infty}\int S_N = \lim_{N\to\infty}\int \sum_{n=1}^{N} f_n = \lim_{N\to\infty}\sum_{n=1}^{N}\int f_n = \sum_{n=1}^{\infty}\int f_n.$

This proves the theorem. ∎

Example 2.11 Let $(\Omega,\mathscr{F},\mu) = \big(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda\big)$, and consider the series

$$f_n(x) = \frac{(-1)^n}{\sqrt{n}}x^n.$$

We now have

$$\begin{aligned}\sum_{n=1}^{\infty}\int_{[0,1]}|f_n| &= \sum_{n=1}^{\infty}\int_{[0,1]}\frac{1}{\sqrt{n}}x^n \underset{(*)}{=} \sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{n}}\frac{x^{n+1}}{n+1}\right]_0^1 \\ &= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}(n+1)} \le \sum_{n=1}^{\infty}\frac{1}{n^{3/2}} < \infty\end{aligned}$$

by the p-series test.[1]

Thus by the Beppo Levi Theorem,

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}x^n$ converges a.e. on $[0,1]$ to an integrable function f, and

(b) $\displaystyle\int_{[0,1]}f = \sum_{n=1}^{\infty}\int_{[0,1]}f_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)\sqrt{n}}.$

Note: The above series is really a power series. If we apply the ratio test, we see that this series converges for $|x|<1$, but diverges for $|x|\ge 1$. Thus, the standard arguments for the integral of power series cannot be applied at the endpoint $x=1$. ■

Exercise 2.13 Fix $h\in\mathscr{L}^+$. For each $E\in\mathscr{F}$, set

$$\nu(E) := \int_E h\,d\mu.$$

Show:

1. ν is a measure on $(\Omega,\mathscr{F})$.
2. If $\mu(E)=0$ then $\nu(E)=0$.
3. ν is a finite measure $\Leftrightarrow$ h is integrable.
4. For each $f\in\mathscr{L}^+$, we have

$$\int f\,d\nu = \int fh\,d\mu \tag{2.27}$$

5. Let $f:\Omega\to\mathbb{R}^*$ be $\mathscr{F}$-measurable. Then

$$\int f\,d\nu \text{ is defined} \quad\Leftrightarrow\quad \int fh\,d\mu \text{ is defined.}$$

   If these integrals are defined, then (2.27) holds.
6. Let $f:\Omega\to\mathbb{K}$ be $\mathscr{F}$-measurable. Then

$$f\in\mathscr{L}^1_{\mathbb{K}}(\Omega,\mathscr{F},\nu) \quad\Leftrightarrow\quad fh\in\mathscr{L}^1_{\mathbb{K}}(\Omega,\mathscr{F},\mu).$$

   In this case, (2.27) holds.

■

[1](*): We will see later that for continuous integrands, the Lebesgue integral coincides with the Riemann integral.

2.8 The Connection Between the Riemann and the Lebesgue Integral

We let

$$R-\int_a^b f(x)\,dx$$

denote the Riemann integral over the interval $[a,b]$, and set

$$\mathscr{R}[a,b]=\{f:[a,b]\to\mathbb{R}: f \text{ is Riemann integrable}\}.$$

One can prove the following theorem:

Theorem 2.8.1 Let $f:[a,b]\to\mathbb{R}$. Then
1. $f\in\mathscr{R}[a,b] \quad\Leftrightarrow\quad$ f is continuous a.e.
2. If $f\in\mathscr{R}[a,b]$, then f is Lebesgue integrable, and

$$R-\int_a^b f(x)\,dx=\int_{[a,b]} f\,d\lambda.$$

For this reason, the Lebesgue integral over the interval $[a,b]$ is often also written as $\int_a^b f(x)\,dx$.

We now discuss the connection between the improper Riemann integral and the Lebesgue integral. It turns out that things are diefferent for nonnegative and arbitary integrands.

Thus, let I be any interval (bounded or unbounded).

1. *If $f:I\to[0,\infty)$ is improperly Riemann integrable on I, then f is also Lebesgue integrable over I, and both integrals coincide.*
   The above statement is a consequence of the various convergence theorems. For example, suppose $f:[a,b]\to[0,\infty)$ is continuous on $(a,b]$, but $\lim_{x\to a^+} f(x)$ does not exist in $\mathbb{R}$. Let (c_n) be any decreasing sequence in $(a,b]$ with $c_n\to a^+$. Then by definition of the improper integral,

$$R-\int_a^b f(x)\,dx=\lim_{n\to\infty} R-\int_{c_n}^b f(x)\,dx \underset{\text{thm 2.8.1}}{=} \lim_{n\to\infty}\int_{[c_n,b]} f\,dm$$
$$\underset{\text{cor 2.7.5}}{=} \int_{(a,b]} f\,dm \underset{\text{thm 2.5.1}}{=} \int_{[a,b]} f\,dm.$$

   An example of this situation would be the integral of $f(x)=\frac{1}{\sqrt{x}}$ over $[0,1]$. This function is not defined at 0, so we give it any value there. Then

$$R-\int_0^1 \frac{1}{\sqrt{x}}\,dx=\lim_{c\to0^+}\int_c^1\frac{1}{\sqrt{x}}\,dx=\lim_{c\to0^+}\left[2\sqrt{x}\right]_c^1=\lim_{c\to0^+}2\left(\sqrt{1}-\sqrt{c}\right)=2.$$

   Similarly

$$\int_{[0,1]}\frac{1}{\sqrt{x}}\,dm \underset{\text{cor 2.7.5}}{=} \lim_{n\to\infty}\int_{[\frac{1}{n},1]}\frac{1}{\sqrt{x}}\,dm$$
$$\underset{\text{thm 2.8.1}}{=} \lim_{n\to\infty} R-\int_{\frac{1}{n}}^1\frac{1}{\sqrt{x}}\,dx=\lim_{n\to\infty}2\left(1-\sqrt{\frac{1}{n}}\right)=2.$$

2. *If $f : I \to (-\infty, \infty)$ is improperly Riemann integrable on I, then f need not be Lebesgue integrable over I.*
   An example of this situation would be the integral of the function.

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathbf{1}_{[n-1,n)}$$

over $[0, \infty)$. Note that for a given $x \geq 0$,

$$f(x) = \frac{(-1)^n}{n},$$

where n is the unique positive integer with $x \in [n-1, n)$.

(a) First consider the improper Riemann integral $R-\int_0^{\infty} f(x)\,dx$. Given $b > 0$, there exists a unique $N \in \mathbb{N}$ so that $N < b \leq N+1$. Then

$$\begin{aligned} R-\int_0^b f(x)\,dx &= R-\int_0^N f(x)\,dx + R-\int_N^b f(x)\,dx \\ &= \underbrace{\sum_{n=1}^{N} \frac{(-1)^n}{n}}_{:=S_N} + \underbrace{\frac{(-1)^{N+1}}{N+1}[b-N]}_{:=R_N,\ |R_N| \leq \frac{1}{N+1}}. \end{aligned}$$

Now if $b \to \infty$, then $b - 1 < N \to \infty$ as well. Since $\lim_{N\to\infty} S_N$ exists (alternating harmonic series), and $\lim_{N\to\infty} R_N = 0$, it follows that

i. $R-\int_0^{\infty} f(x)\,dx = \lim_{b\to\infty} R-\int_0^b f(x)\,dx$ converges, and

ii. $R-\int_0^{\infty} f(x)\,dx = \lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(b) Next consider the Lebesgue integral.

$$\int_{[0,\infty)} |f|\,dm \underset{\text{cor 2.7.5}}{=} \lim_{N\to\infty} \int_{[0,N)} |f|\,dm = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

(the harmonic series.) This shows that f is not Lebesgue integrable on $[0, \infty)$.

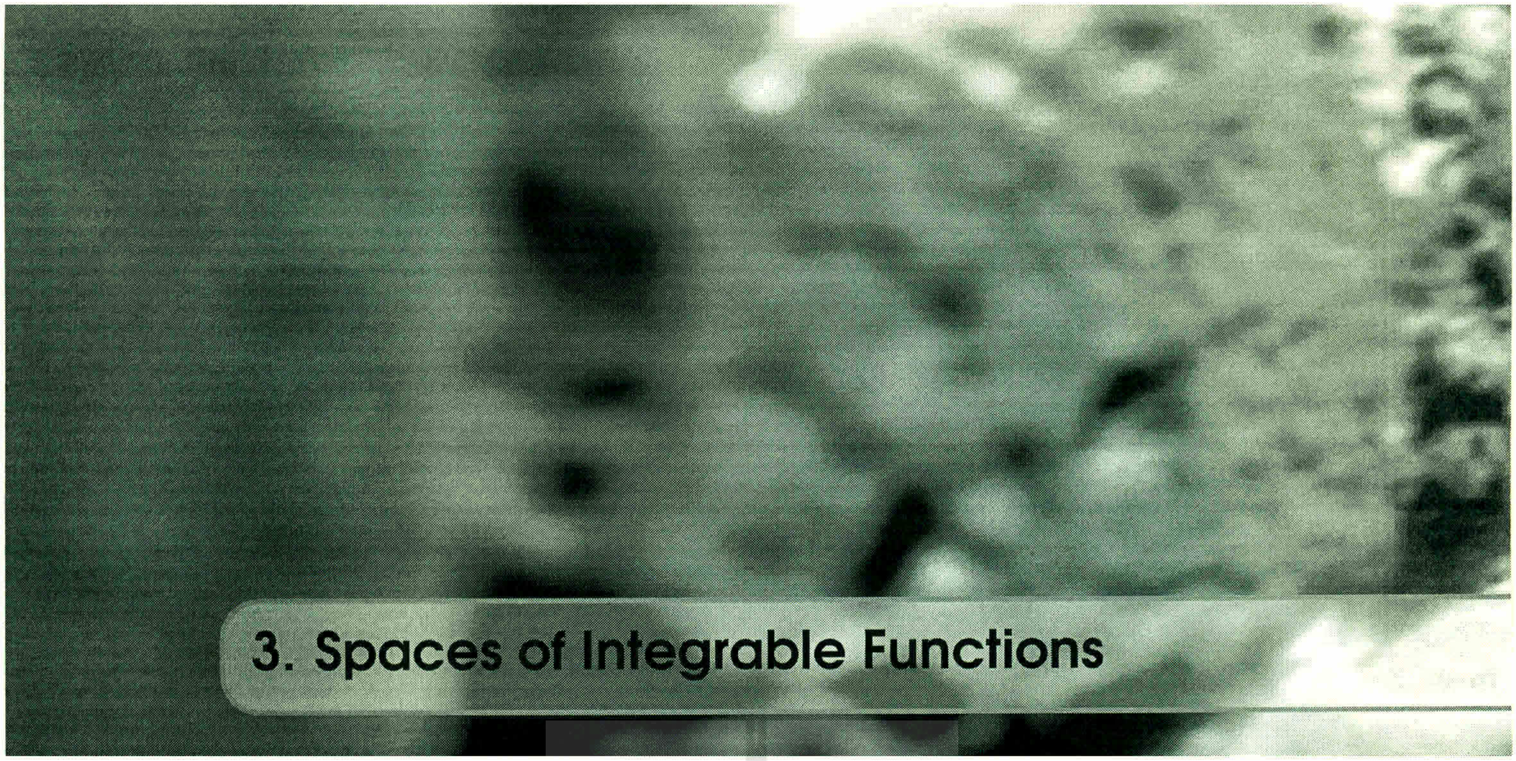
Exercise 2.14 Consider the "sinc"-function

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show: The improper Riemann integral

$$R-\int_0^{\infty} f(x)\,dx$$

converges, but f is not Lebesgue integrable on $[0, \infty)$. ■



3. Spaces of Integrable Functions

Throughout this chapter, $(\Omega, \mathscr{F}, \mu)$ will be a measure space. We also let $\mathbb{K} = \mathbb{R}$, $\mathbb{R}^*$ or $\mathbb{C}$.

When $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ then by Theorem 2.3.1, respectively 2.4.7,

$$\mathscr{L}^1 = \mathscr{L}^1_{\mathbb{K}} = \left\{ f : \Omega \to \mathbb{K} \,\middle|\, f \text{ is } \mathscr{F}\text{-measurable and integrable} \right\}$$

is a vector space over $\mathbb{K}$. In addition, it is easy to see that

$$\|f\|_1 = \int |f| \, d\mu$$

defines a seminorm on this linear space which in general is not a norm. Indeed, Theorem 2.6.6 implies that

$$\|f\|_1 = 0 \quad \Leftrightarrow \quad f(\omega) = 0 \text{ a.e.}$$

In this chapter we will modify the space $\mathscr{L}^1$ to obtain a normed linear space, by using the quotient space construction. It turns out that a larger classes of spaces are actually of interest here, and we begin by introducing these first.

3.1 The L^p-spaces

Definition 3.1.1 Given a number p, $1 \le p < \infty$, a measurable function $f : \Omega \to \mathbb{K}$ is called *p-integrable*, if

$$\int |f|^p \, d\mu < \infty.$$

We set

$$\mathscr{L}^p = \mathscr{L}^p_{\mathbb{K}} = \mathscr{L}^p_{\mathbb{K}}(\Omega, \mathscr{F}, \mu) := \left\{ f : \Omega \to \mathbb{K} : f \text{ is } \mathscr{F}\text{-measurable and } p\text{-integrable} \right\}.$$

For each $f \in \mathscr{L}^p$, we also set

$$\|f\|_p := \left[\int |f|^p \, d\mu\right]^{1/p} \qquad \text{"}p\text{-seminorm".}$$

(R) The case $p = 1$ has already been covered above. Here,

$$\mathscr{L}^1_{\mathbb{K}} = \{f : \Omega \to \mathbb{K} : f \text{ is } \mathscr{F}\text{-measurable and integrable}\}$$

is the *space of integrable functions*. Note that this is not a vector space when $\mathbb{K} = \mathbb{R}^*$. Nevertheless,

$$\|f\|_1 = \int |f| \, d\mu$$

is always defined.
The case $p = 2$ is also of particular interest. Here,

$$\mathscr{L}^2_{\mathbb{K}} = \{f : \Omega \to \mathbb{K} : f \text{ is } \mathscr{F}\text{-measurable and } \int |f|^2 d\mu < \infty\}$$

is called the *space of square-integrable functions*. Then

$$\|f\|_2 = \sqrt{\int |f|^2 \, d\mu}.$$

We are also interested in the space of bounded functions. However, we will employ a modified concept of bounded functions, which allows us to disregard the function values on null sets.

Definition 3.1.2 An $\mathscr{F}$-measurable function $f : \Omega \to \mathbb{K}$ is called *essentially bounded*, if there exists $M \geq 0$ so that

$$E_M := \{\omega \in \Omega : |f(\omega)| > M\}$$

is a null set. Such a number M is called an *essential bound* of f.

▪ **Example 3.1** Let $(\Omega, \mathscr{F}, \mu) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, and consider

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \\ p & \text{if } x = \frac{p}{q} \in \mathbb{Q},\ (p,q) = 1,\ q > 0. \end{cases}$$

As $|\sin(\pi x)| \leq 1$ for all x, and $\mathbb{Q}$ is a λ-null set, it follows that every $M \geq 1$ is an essential bound of f.

Next we show that any $0 \leq M < 1$ cannot be an essential bound. In fact, pick $x_o \in [0, \frac{1}{2})$ with $\sin(\pi x_o) = M$. Then

$$E_M = \{x \in \mathbb{R} : |f(x)| > M\} \supset \{x \in [0, \tfrac{1}{2}) : f(x) > M\} \supset (x_o, \tfrac{1}{2}) \cap \mathbb{Q}^c$$

which shows that E_M is not a null set.

It follows that $M = 1$ is the smallest essential bound of f. We write

$$\|f\|_\infty := 1.$$

■

Definition 3.1.3 We set

$$\mathscr{L}^\infty = \mathscr{L}^\infty_{\mathbb{K}} = \mathscr{L}^\infty_{\mathbb{K}}(\Omega, \mathscr{F}, \mu)$$
$$:= \{f : \Omega \to \mathbb{K} : f \text{ is } \mathscr{F}\text{-measurable and essentially bounded}\}.$$

For each $f \in \mathscr{L}^\infty$, we also set

$$\|f\|_\infty = \text{ess-sup} f := \inf\underbrace{\{M \mid M \text{ is an essential bound of } f\}}_{\text{call this set } S_f}. \tag{3.1}$$

(R) Let $f \in \mathscr{L}^\infty$ be given.

1. *If M is an essential bound of f and $L > M$, then L is also an essential bound of f.* In fact, whenever $L > M$, then
$$E_L = \{\omega \in \Omega : |f(\omega)| > L\} \subseteq E_M = \{\omega \in \Omega : |f(\omega)| > M\}.$$
So if E_M is a null set, then E_L will also be a null set.
2. *$\|f\|_\infty$ is itself an essential bound of f.* To see this, let $\varepsilon > 0$ be arbitrary. Then by (3.1), $\|f\|_\infty + \varepsilon$ is not a lower bound of S_f, hence there exists $M \in S_f$ (i.e. an essential bound M of f) so that
$$\|f\|_\infty \le M < \|f\|_\infty + \varepsilon.$$
It follows by part 1. that $\|f\|_\infty + \varepsilon$ is an essential bound for f, that is
$$\{\omega \in \Omega : |f(\omega)| > \|f\|_\infty + \varepsilon\}$$
is a null set. Now choosing $\varepsilon = \frac{1}{n}$, we obtain that
$$E_{\|f\|_\infty} := \{\omega \in \Omega : |f(\omega)| > \|f\|_\infty\} = \bigcup_{n=1}^{\infty}\left\{\omega \in \Omega : |f(\omega)| > \|f\|_\infty + \frac{1}{n}\right\}$$
is a countable union of null sets, and thus is itself a null set. This shows that $\|f\|_\infty$ is an essential bound for f.

Thus by (3.1), *$\|f\|_\infty$ is the smallest essential bound for f.*

(R) Let $f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable, and $1 \le p < \infty$.

1. Since $|f|^p \ge 0$, then
$$\int |f|^p \, d\mu$$
is always defined (possibly $= \infty$). Thus,
$$\|f\|_p = \left[\int |f|^p\right]^{1/p}$$
is always defined (possibly $= \infty$). Hence
$$f \in \mathscr{L}^p \quad \Leftrightarrow \quad \int |f|^p < \infty \quad \Leftrightarrow \quad \|f\|_p < \infty.$$
Also,
$$\|f\|_p^p = \int |f|^p = \| \, |f|^p \, \|_1. \tag{3.2}$$

2. Similarly, $\|f\|_\infty$ is always defined (possibly ∞ when $S_f = \emptyset$). Hence,

$$f \in \mathscr{L}^\infty \quad \Leftrightarrow \quad S_f \neq \emptyset \quad \Leftrightarrow \quad \|f\|_\infty < \infty.$$

3. For $1 \le p < \infty$ we have

$$\|f\|_p = 0 \quad \overset{\text{def}}{\Leftrightarrow} \quad \int |f|^p = 0 \quad \underset{2.6.6}{\overset{\text{thm}}{\Leftrightarrow}} \quad |f(\omega)|^p = 0 \text{ a.e.} \quad \Leftrightarrow \quad f(\omega) = 0 \text{ a.e.}$$

Similarly,

$$\begin{aligned} \|f\|_\infty = 0 \quad &\Leftrightarrow \quad 0 \text{ is an essential bound of } f \\ &\Leftrightarrow \quad \{\omega \in \Omega : |f(\omega)| > 0\} \text{ is a null set} \\ &\Leftrightarrow \quad \{\omega \in \Omega : |f(\omega)| \neq 0\} \text{ is a null set} \\ &\Leftrightarrow \quad f(\omega) = 0 \text{ a.e.} \end{aligned}$$

Definition 3.1.4 Given $1 < p < \infty$, set

$$q := \frac{p}{p-1}. \tag{3.3}$$

Then $1 < q < \infty$ as well.

When $p = 1$ we set $q = \infty$, and when $p = \infty$ we set $q = 1$. The number q is called the *conjugate* of p.

(R) The following properties will be used throughout.

1. When $1 < p < \infty$, then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p} + \frac{p-1}{q} = 1.$$

2. Agreeing that $\frac{1}{\infty} = 0$, this identity remains valid for $p = 1$ or $p = \infty$:

$$\boxed{\frac{1}{p} + \frac{1}{q} = 1} \qquad (1 \le p \le \infty).$$

3. When $p = 2$, then $q = 2$ as well. ($p = 2$ is *self-conjugate*.)
4. By (3.3) we have for $1 < p < \infty$ that

$$q(p-1) = p.$$

Thus for any $\mathscr{F}$-measurable function $f : \Omega \to \mathbb{K}$,

$$\int |f|^p = \int \left[|f|^{p-1}\right]^q.$$

This shows that

(a) $f \in \mathscr{L}^p \quad \Leftrightarrow \quad |f|^{p-1} \in \mathscr{L}^q$

(b) $$\|f\|_p^p = \| \, |f|^{p-1} \|_q^q. \tag{3.4}$$

Lemma 3.1.1 Let $1 < p < \infty$ be fixed, and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \forall a,b \geq 0. \tag{3.5}$$

Proof. We note that when $a = 0$ or $b = 0$ then the above inequality holds trivially. We thus may assume that $a, b > 0$ in (3.5).

Let us derive an equivalent statement. Setting $u = a^p$ and $v = b^q$, then (3.5) is equivalent to

$$u^{1/p}v^{1/q} \leq \frac{u}{p} + \frac{v}{q} \qquad \forall u,v > 0.$$

Dividing by $v \neq 0$, this is equivalent to

$$\left[\frac{u}{v}\right]^{1/p} = \frac{u^{1/p}}{v^{1-(1/q)}} \leq \frac{1}{p}\left[\frac{u}{v}\right] + \frac{1}{q} \qquad \forall u,v > 0.$$

Setting $t = \frac{u}{v}$, this is equivalent to

$$t^{1/p} \leq \frac{t}{p} + \frac{1}{q} \qquad \forall t > 0. \tag{3.6}$$

To prove this statement, consider the function

$$f(t) = \frac{t}{p} + \frac{1}{q} - t^{1/p}, \qquad t \in (0, \infty).$$

Then

$$f'(t) = \frac{1}{p} - \frac{1}{p}t^{(1/p)-1} = \frac{1}{p}\left[1 - \frac{1}{t^q}\right].$$

Since

$$f'(t) < 0 \text{ on } (0,1) \qquad \text{and} \qquad f'(t) > 0 \text{ on } (1, \infty)$$

it follows that

$$f(t) \geq f(1) = 0 \qquad \forall t \in (0, \infty).$$

This shows that (3.6) holds, and proves the lemma. ∎

Theorem 3.1.2 *[Hölder's Inequality; $p = 2$: Cauchy-Schwarz Inequality]*
Let $1 \leq p \leq \infty$. If $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^q$, then
1. $fg \in \mathscr{L}^1$, and
2. $\int |fg|\, d\mu \leq \|f\|_p \|g\|_q$.

Proof. Note that it suffices to prove assertion 2.; then 1. will follow immediately. We distinguish three cases.

Case 1: $1 < p < \infty$.

(i) Suppose $\|f\|_p = 0$. Then $f(\omega) = 0$ a.e.

$$\Rightarrow |f(\omega)g(\omega)| = 0 \text{ a.e.}$$
$$\Rightarrow \int |fg| = 0,$$

so that assertion 2. holds trivially.

(ii) Similarly, if $\|g\|_q = 0$, then assertion 2. holds trivially.

(iii) We may thus assume that $\|f\|_p > 0$ and $\|g\|_q > 0$. Applying Lemma 3.1.1 with

$$a = \frac{|f(\omega)|}{\|f\|_p} \qquad \text{and} \qquad b = \frac{|g(\omega)|}{\|g\|_q}$$

we obtain

$$\frac{|f(\omega)|}{\|f\|_p}\frac{|g(\omega)|}{\|g\|_q} \leq \frac{1}{p}\frac{|f(\omega)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(\omega)|^q}{\|g\|_q^q}$$

Now we integrate. By linearity and monotonicity of the integral,

$$\frac{1}{\|f\|_p\|g\|_q}\int |fg| \leq \frac{1}{p}\frac{1}{\|f\|_p^p}\underbrace{\int |f|^p}_{=\|f\|_p^p} + \frac{1}{q}\frac{1}{\|g\|_q^q}\underbrace{\int |g|^q}_{=\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Finally, multiplying by $\|f\|_p^p\|g\|_q^q$, we arrive at assertion 2.

Case 2: $p = 1$. Then $q = \infty$. Now since $\|g\|_\infty$ is an essential bound for g, we have

$$|f(\omega)|\,|g(\omega)| \leq |f(\omega)|\,\|g\|_\infty \text{ a.e.}$$

By Corollary 2.6.5 and linearity of the integral,

$$\int |fg| \leq \int |f|\,\|g\|_\infty = \|g\|_\infty \int |f| = \|f\|_1\,\|g\|_\infty,$$

which proves assertion 2.

Case 3: $p = \infty$. Then $q = 1$, and assertion 2. follows from Case 2, by symmetry. ■

Theorem 3.1.3 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and $1 \leq p \leq \infty$.

1. $\mathscr{L}^p = \mathscr{L}^p_{\mathbb{K}}(\Omega, \mathscr{F}, \mu)$ is a vector space over $\mathbb{K}$,
2. $\|\cdot\|_p$ is a seminorm on $\mathscr{L}^p$. That is, for all $f, g \in \mathscr{L}^p$ and $c \in \mathbb{K}$ we have

(N1): $\|f\|_p \geq 0$

(N3): $\|cf\|_p = |c|\,\|f\|_p$

(N4): $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ *("Minkowski's Inequality")*

Proof. Since the set of all $\mathbb{K}$-valued functions,

$$V_{\mathbb{K}} := \{f : \Omega \to \mathbb{K}\}$$

is a vector space, in order to prove 1. we only need to show that $\mathscr{L}^p$ is a subspace of $V_{\mathbb{K}}$. That is, we need to show that $cf \in \mathscr{L}^p$ and $f + g \in \mathscr{L}^p$ for all $f, g \in \mathscr{L}^p$ and $c \in \mathbb{K}$. Clearly by definition, $\|f\|_p \geq 0$ for all $f \in \mathscr{L}^p$.

Case 1: $p = 1$. We already know from Theorems 2.3.1 and 2.4.6 that $\mathscr{L}^1_{\mathbb{K}}$ is a vector space. Now let $f, g \in \mathscr{L}^1_{\mathbb{K}}$ and $c \in \mathbb{K}$. Then by linearity of the integral,

$$\|cf\|_1 \underset{\text{def}}{=} \int |cf| = \int |c|\,|f| = |c| \int |f| = |c|\,\|f\|_1,$$

which shows that (N3) holds. Furthermore, as

$$|(f+g)(\omega)| \leq |f(\omega)| + |g(\omega)| \qquad \forall\, \omega \in \Omega,$$

then by monotonicity and linearity of the integral,

$$\|f+g\|_1 \underset{\text{def}}{=} \int |f+g| \leq \int \big(|f| + |g|\big) = \int |f| + \int |g| = \|f\|_1 + \|g\|_1,$$

which shows that (N4) holds.

Case 2: $1 < p < \infty$. Let $f, g \in \mathscr{L}^p$ and $c \in \mathbb{K}$.

(a) Since the integral in $\mathscr{L}^+$ is homogeneous, then

$$\|cf\|_p^p \underset{\text{def}}{=} \int |cf|^p = \int |c|^p |f|^p = |c|^p \int |f|^p = |c|^p \|f\|_p^p < \infty.$$

It follows that

(i) $cf \in \mathscr{L}^p$, and

(ii) $\|cf\|_p = |c|\,\|f\|_p$.

(b) Furthermore, for all $\omega \in \Omega$ we have

$$\begin{aligned}
|f(\omega) + g(\omega)|^p &\leq \Big[|f(\omega)| + |g(\omega)|\Big]^p \\
&\leq \Big[2\max\Big(|f(\omega)|, |g(\omega)|\Big)\Big]^p \\
&= 2^p \Big[\max\Big(|f(\omega)|, |g(\omega|\Big)\Big]^p \\
&= 2^p \max\Big(|f(\omega)|^p, |g(\omega|^p\Big) \leq 2^p \Big[|f(\omega)|^p + |g(\omega|^p\Big].
\end{aligned}$$

Hence by monotonicity and additivity of the integral in $\mathscr{L}^+$,

$$\begin{aligned}
\int |f+g|^p &\leq \int 2^p \big[|f|^p + |g|^p\big] \\
&= 2^p \left[\int |f|^p + \int |g|^p\right] < \infty,
\end{aligned}$$

because $f, g \in \mathscr{L}^p$. This shows that $f + g \in \mathscr{L}^p$ also. Furthermore,

$$\begin{aligned}
\|f+g\|_p^p &\underset{\text{def}}{=} \int |f+g|^p = \int |f+g| \cdot |f+g|^{p-1} \\
&\leq \int \big[|f| + |g|\big] \cdot |f+g|^{p-1} \\
&= \int |f| \cdot \underbrace{|f+g|^{p-1}}_{\in \mathscr{L}^q \text{ by } (3.4)} + \int |g| \cdot \underbrace{|f+g|^{p-1}}_{\in \mathscr{L}^q \text{ by } (3.4)} \\
&\underset{\text{Hölder}}{\leq} \|f\|_p \,\| |f+g|^{p-1} \|_q + \|g\|_p \,\| |f+g|^{p-1} \|_q \\
&\underset{(3.4)}{=} \|f\|_p \,\|f+g\|_p^{p/q} + \|g\|_p \,\|f+g\|_p^{p/q}.
\end{aligned}$$

When $\|f+g\|_p \neq 0$ we can divide by $\|f+g\|_p^{p/q}$ to obtain

$$\|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p.$$

Since $p - \frac{p}{q} = p\left[1 - \frac{1}{q}\right] = p\frac{1}{p} = 1$, it follows that

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

When $\|f+g\|_p = 0$, this last inequality certainly holds as well.

Case 3: $p = \infty$. Let $f, g \in \mathscr{L}^\infty$ and $c \in \mathbb{K}$. Then

$$N_f := \{\omega \in \Omega : |f(\omega)| \geq \|f\|_\infty\}$$
$$N_g := \{\omega \in \Omega : |g(\omega)| \geq \|g\|_\infty\}$$

are both null sets.

(a) For all $\omega \notin N_f$ we have

$$\left|(cf)(\omega)\right| = |c|\,|f(\omega)| \leq |c|\,\|f\|_\infty,$$

which shows that cf is essentially bounded, and

$$\|cf\|_\infty \leq |c|\,\|f\|_\infty. \tag{3.7}$$

Now if $c \neq 0$ we use the same argument to show that

$$\left\|\frac{1}{c}(cf)\right\|_\infty \leq \frac{1}{|c|}\|cf\|_\infty$$

From the above two inequalities we obtain that

$$\|f\|_\infty = \left\|\frac{1}{c}(cf)\right\|_\infty \leq \frac{1}{|c|}\|cf\|_\infty \underset{(3.7)}{\leq} \frac{1}{|c|}|c|\,\|f\|_\infty = \|f\|_\infty,$$

so that

$$\|cf\|_\infty = |c|\,\|f\|_\infty.$$

On the other hand, when $c = 0$, then clearly

$$\|cf\|_\infty = \|0\|_\infty = 0 = |c|\|f\|_\infty.$$

(b) Furthermore, for all $\omega \notin N_f \cup N_g$ we have

$$|f(\omega)+g(\omega)| \leq |f(\omega)| + |g(\omega)| \leq \|f\|_\infty + \|g\|_\infty.$$

This shows that

(i) $f+g$ is essentially bounded, that is, $f+g \in \mathscr{L}^\infty$, and

(ii) $\|f+g\|_\infty = \text{ess-sup}\,(f+g) \leq \|f\|_\infty + \|g\|_\infty$.

The proof of the theorem is thus complete. ■

(R) Clearly, $\mathscr{L}^p_{\mathbb{R}^*}$ is not a vector space, as $f+g$ need not be defined when f and g are extended real-valued. However, the above proof shows that (N3) always holds, and (N4) holds whenever $f+g$ is defined.

We can now modify the spaces $\mathscr{L}^p$ in order to make each $\|\cdot\|_p$ a norm. For this, we will use the quotient space construction, as laid out in the next Exercise:

Exercise 3.1 Let X be a (real or complex) vector space and N a linear subspace. Define a relation $\sim$ on X by

$$x \sim y \quad \Leftrightarrow \quad x - y \in N \qquad (x, y \in X).$$

1. Show: $\sim$ is an equivalence relation.
2. We denote the equivalence class of $x \in X$ by $[x]$ and also by $\tilde{x}$. Set

   $$\tilde{X} := \{[x] : x \in X\},$$

   the set of equivalence classes.
   (a) Show that the following operations on $\tilde{X}$ are well defined for all $[x], [y] \in \tilde{X}$ and scalars α:

   $$[x] + [y] := [x + y] \qquad \alpha[x] := [\alpha x]$$

   (b) Show that $\tilde{X}$ is a vector space with these operations,
   (c) Show that the *quotient map* $q : X \to \tilde{X}$ given by $q(x) = [x]$ is linear and surjective.
3. Next let X carry a seminorm $\|\cdot\|_s$. We set

   $$N = \{x \in X : \|x\|_s = 0\}.$$

   (a) Show that N is a linear subspace of X.
   (b) Show that $\|[x]\| := \|x\|$ well defines a map $\tilde{X} \to \mathbb{R}$.
   (c) Show that $\|\cdot\|$ is a norm on $\tilde{X}$.

Now let $X = \mathscr{L}^p_{\mathbb{K}}(\Omega, \mathscr{F}, \mu)$ for some fixed $1 \le p \le \infty$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Thus we let

$$N = \{f \in \mathscr{L}^p : \|f\|_p = 0\}.$$

Since $\|f\|_p = 0 \Leftrightarrow f(\omega) = 0$ a.e. we thus have

$$N = \{f \in \mathscr{L}^p : f(\omega) = 0 \text{ a.e.}\}$$

and

$$f \sim g \quad \Leftrightarrow \quad f - g \in N \quad \Leftrightarrow \quad f(\omega) - g(\omega) = 0 \text{ a.e.} \quad \Leftrightarrow \quad f(\omega) = g(\omega) \text{ a.e.}$$

We denote the normed linear space space $\tilde{X}$ by $L^p_{\mathbb{K}}(\Omega, \mathscr{F}, \mu)$ or simply by $L^p(\Omega, \mathscr{F}, \mu)$, and keep using the symbol $\|\cdot\|_p$ for its norm.

R Strictly speaking, the elements of $L^p(\Omega, \mathscr{F}, \mu)$ are *equivalence classes* of functions. However, as any two elements in the same equivalence class are equal a.e., and the integral does not distinguish these functions, we 'confuse' the equivalence class $[f]$ of f with f itself. That is, we treat every element of $L^p(\Omega, \mathscr{F}, \mu)$ as a function, *which is uniquely defined up to a null set only*.

Thus, when we say

"let $f \in L^p$"

we really mean

"let $f \in \mathscr{L}^p$ be any representative of $[f] \in L^p$".

Clearly, Hölder's and Minkowski's inequalities hold for the spaces L^p as well.

R Even though $\mathscr{L}^p_{\mathbb{R}^*}(\Omega,\mathscr{F},\mu)$ is not a vector space, we can still introduce an equivalence relation by

$$f \sim g \quad \Leftrightarrow \quad f(\omega) = g(\omega) \text{ a.e.}$$

and obtain a set of equivalence classes $L^p_{\mathbb{R}^*}(\Omega,\mathscr{F},\mu)$. However, since every element f of $\mathscr{L}^p_{\mathbb{R}^*}(\Omega,\mathscr{F},\mu)$ is finite valued a.e., then its equivalence class $[f] \in \mathscr{L}^p_{\mathbb{R}^*}$ will contain functions which are finite valued *everywhere*, that is, elements of $\mathscr{L}^p_{\mathbb{R}}(\Omega,\mathscr{F},\mu)$. It follows that the collections of equivalence classes $L^p_{\mathbb{R}^*}(\Omega,\mathscr{F},\mu)$ and $L^p_{\mathbb{R}}(\Omega,\mathscr{F},\mu)$ are identical ! In this manner, $L^p_{\mathbb{R}^*}$ becomes a real normed linear space which is identical with $L^p_{\mathbb{R}}$.

For this reason, we will treat all functions as finite valued from now on. In fact, when $f:\Omega \to \mathbb{R}^*$ is $\mathscr{F}$-measurable and finite valued a.e., then we will implicitly modify the values of f on a null set so that f becomes finite valued everywhere.

Exercise 3.2 Consider the measure space $(\mathbb{N},\mathscr{P}(\mathbb{N}),\mu_c)$ where μ_c denotes the counting measure. Since $\mathscr{F} = \mathscr{P}(\mathbb{N})$, every function $f:\mathbb{N}\to\mathbb{K}$ is $\mathscr{F}$-measurable. In addition, every function $f:\mathbb{N}\to\mathbb{K}$ can be identified with a sequence $(x_k)_{k=1}^{\infty} = (x_1,x_2,\ldots,x_k,\ldots)$ in $\mathbb{K}$, by setting

$$x_k = f(k) \qquad (k \in \mathbb{N}).$$

The map $f \mapsto (x_k)$ thus constitutes a linear isomorphism between the set of $\mathbb{K}$-valued functions on $\mathbb{N}$ onto the vector space of all sequences in $\mathbb{K}$.

1. Let $f:\mathbb{N}\to[0,\infty)$ be non-negative, simple. Show:

$$\int f\,d\mu = \sum_{k=1}^{\infty} f(k). \tag{3.8}$$

2. Let $f:\mathbb{N}\to[0,\infty)$. Show that (3.8) holds.
3. Let $f:\mathbb{N}\to\mathbb{K}$. Show:

$$f \text{ is integrable } \Leftrightarrow \sum_{k=1}^{\infty} |f(k)| < \infty.$$

   In addition, if f is integrable, then (3.8) holds.
4. Let $f,g:\mathbb{N}\to\mathbb{K}$. Show: $\quad f = g \text{ a.e.} \Leftrightarrow f = g.$
5. We set $\quad \ell^p := \mathscr{L}^p(\mathbb{N},\mathscr{P}(\mathbb{N}),\mu_c) = L^p(\mathbb{N},\mathscr{P}(\mathbb{N}),\mu_c)$. Show:
   (a) If $1 \le p < \infty$, then

$$f \in \ell^p \Leftrightarrow \sum_{k=1}^{\infty} |f(k)|^p < \infty, \quad \text{and} \quad \|f\|_p = \left[\sum_{k=1}^{\infty} |f(k)|^p\right]^{1/p}.$$

   (b) If $p = \infty$, then

$$f \in \ell^\infty \Leftrightarrow \sup_{k\in\mathbb{N}} |f(k)| < \infty, \quad \text{and} \quad \|f\|_\infty = \sup_{k\in\mathbb{N}} |f(k)|.$$

6. Show that $\ell_p \subset \ell_q$ but $\ell_p \neq \ell_q$ for $1 \le p < q \le \infty$.

■

Exercise 3.3 Let $(\Omega, \mathscr{F}, \mu)$ be a *finite* measure space, and $1 \leq p < q \leq \infty$. Show:

1. $L^q(\Omega, \mathscr{F}, \mu) \subset L^p(\Omega, \mathscr{F}, \mu)$
2. $\exists k = k(p,q)$ so that $\|f\|_p \leq k\|f\|_q \qquad \forall f \in L^q(\Omega, \mathscr{F}, \mu)$.

Compare this to Exercise 3.2 !!

Exercise 3.4 Let $\Omega = (0,\infty)$, $\mathscr{F} = \mathscr{B}(0,\infty)$, $\mu = \lambda$. Set

$$g(x) = \frac{1}{\sqrt{x}(1 + |\ln x|)}.$$

1. Show: $g \in L^2(0,\infty)$, but $g \notin L^q(0,\infty)$ for $p \neq q$, $1 \leq q \leq \infty$.
2. Use the above to show: If $1 \leq p < q \leq \infty$, then $L^p(0,\infty) \neq L^q(0,\infty)$.
3. What if μ is the Dirac one-point measure ?

Compare with Exercises 3.2 and 3.3 !!

Exercise 3.5 Let $f_n, f \in L^\infty(\Omega, \mathscr{F}, \mu)$. Show: If $f_n \xrightarrow{\|\cdot\|_\infty} f$, then

1. $f_n(\omega) \to f(\omega)$ a.e.
2. $\exists g \in L^\infty(\Omega, \mathscr{F}, \mu)$ so that $|f_n(\omega)| \leq g(\omega)$ a.e.

Exercise 3.6 Let $\Omega = \mathbb{R}$, $\mathscr{F} = \mathscr{P}(\mathbb{R})$ and $\mu = \delta_a$, the one-point Dirac measure at $x = a$.

1. Given $f \geq 0$, find $\displaystyle\int f\,d\mu$.
2. Describe the elements and the norm in the spaces $\mathscr{L}^p$ and L^p, $1 \leq p < \infty$.
3. Describe the elements and the norm in the spaces $\mathscr{L}^\infty$ and L^∞.

3.2 Completeness of the L^p-spaces

In the previous section, we have defined the spaces $L^p(\Omega, \mathscr{F}, \mu)$, and shown that they are normed linear spaces. In this section we will show that they are complete, that is, Banach spaces.

Let us first discuss convergence of sequences in $L^p(\Omega, \mathscr{F}, \mu)$. So far, we are familiar with two types of convergence. Given $f_n, f \in L^p(\Omega, \mathscr{F}, \mu)$,

1. the sequence (f_n) converges *(pointwise) a.e.* to f, written $f_n \xrightarrow{\text{a.e.}} f$, if there exists a null set $N \in \mathscr{F}$ so that

$$f_n(\omega) \to f(\omega) \qquad \forall \omega \in \Omega \backslash N.$$

Note that it does not make sense to talk about *everywhere* convergence in L^p, because elements of L^p, when considered as measurable functions, are uniquely defined up to a null set only. To be precise, the notion of a.e.-convergence should be defined as follows: Let $f_n, f \in L^p$. Pick arbitrary representatives $g_n, g \in \mathscr{L}^p$ of each equivalence class, that is, $f_n = [g_n]$ and $[f] = g$. Then

$$f_n \xrightarrow{\text{a.e.}} f \quad \underset{\text{def}}{\Leftrightarrow} \quad g_n(\omega) \xrightarrow{\text{a.e.}} g(\omega).$$

This definition makes sense as changing the values of g_n and g on null sets does not change the property of a.e. convergence.

2. the sequence (f_n) converges to f in the *p-th mean*, written $f_n \xrightarrow{\|\cdot\|_p} f$, if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

In general the two types of convergence are different:

Example 3.2 1. Let $\Omega = [0,1]$, $\mathscr{F} = \mathscr{B}([0,1])$ and $\mu = \lambda$, the Lebesgue measure.
Set $f_n = n^2 \mathbf{1}_{(\frac{1}{n+1},\frac{1}{n}]}$, $n = 1,2,3,\ldots$. Since $([0,1], \mathscr{B}([0,1]), \lambda)$ is a finite measure space, and each f_n is simple, then clearly $f_n \in L^p$ $\forall n$, $\forall 1 \le p \le \infty$. It is also easy to see that

$$f_n(\omega) \to 0 \qquad \forall\, \omega \in [0,1]$$

(as functions), that is, $f_n \xrightarrow{\text{a.e.}} f = 0$ as elements of L^p.
Claim: $f_n \not\to f = 0$ in the p-th mean, $\forall 1 \le p \le \infty$.
Case 1: $1 \le p < \infty$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned}\|f_n - f\|_p^p &= \left\| n^2 \mathbf{1}_{(\frac{1}{n+1},\frac{1}{n}]} - 0 \right\|_p^p = \int n^{2p} \mathbf{1}_{(\frac{1}{n+1},\frac{1}{n}]}\, d\lambda \\ &= n^{2p}\left(\frac{1}{n} - \frac{1}{n+1}\right) = n^{2p}\frac{1}{n(n+1)} = \frac{n^{2p-1}}{n+1} \ge \frac{1}{2}\end{aligned}$$

which shows that (f_n) cannot converge to $f = 0$ in the p-th mean.
Case 2: $p = \infty$. Since $\|f_n - f\|_\infty = \operatorname*{ess\text{-}sup}_{\omega\in[0,1]} |f_n(\omega) - 0| = n^2 \ge 1$ for all n, then (f_n) cannot converge to $f = 0$ in the essential supremum norm.
This proves the claim.
(We observe that Corollary 3.2.4 below implies that if (f_n) converges in the p-th mean to some function f, then $f = 0$ as an element of L^p. Thus, (f_n) cannot converge in the p-th mean at all.)

2. Let $\Omega = \mathbb{R}$, $\mathscr{F} = \mathscr{B}(\mathbb{R})$ and $\mu = m$, the Lebesgue measure.
Fix $r \ge 1$ and set $f_n = \frac{1}{n}\mathbf{1}_{[0,n^r]}$, $n = 1,2,3,\ldots$. Then clearly $f_n \in L^p$ $\forall n$, $\forall 1 \le p \le \infty$.
Since $|f_n(\omega) - 0| = |f_n(\omega)| \le \frac{1}{n}$ for all $\omega \in \mathbb{R}$, it follows that

$$f_n \underset{\mathbb{R}}{\rightrightarrows} f = 0.$$

(that is, $f_n \to 0$ uniformly .) In particular, $f_n \xrightarrow{\text{a.e.}} f = 0$ as elements of L^p.

Claim: $f_n \not\to f = 0$ in the p-th mean, for all $1 \le p \le r$, while $f_n \xrightarrow{\|\cdot\|_p} f = 0$ for all $r < p \le \infty$.
Case 1: $1 \le p < \infty$. Then

$$\begin{aligned}\|f_n - f\|_p^p &= \left\| \frac{1}{n}\mathbf{1}_{[0,n^r]} - 0 \right\|_p^p = \int \frac{1}{n^p}\mathbf{1}_{[0,n^r]}\, d\lambda = \frac{1}{n^p} n^r \\ &= n^{r-p} \to \begin{cases} \infty & \text{if } 1 \le p < r \\ 1 & \text{if } p = r \\ 0 & \text{if } p > r \end{cases} \qquad \text{as } n \to \infty.\end{aligned}$$

Case 2: $p = \infty$. Clearly,

$$\|f_n - f\|_\infty = \left\| \frac{1}{n}\mathbf{1}_{[0,n^r]} - 0 \right\|_\infty = \frac{1}{n} \to 0 \qquad \text{as } n \to \infty.$$

Both cases show that $\|f_n - f\|_p \to 0$ precisely when $p > r$ and thus prove the claim. ■

'he above examples show that if $f_n \xrightarrow{\text{a.e.}} f$, then (f_n) may or may not converge to f in the p-th mean. Conversely, Example 5.2 in Section 5.1 below will show that in general, $f_n \xrightarrow{\|\cdot\|_p} f$ does not imply that $f_n \xrightarrow{\text{ae.}} f$ when $1 \leq p < \infty$. However, Corollary 3.2.3 will show that there exists at least a subsequence (f_{n_k}) with $f_{n_k} \xrightarrow{\text{ae.}} f$. On the other hand, when $p = \infty$ we have:

Exercise 3.7 Let $f_n, f \in L^\infty(\Omega, \mathscr{F}, \mu)$. Show: If $f_n \xrightarrow{\|\cdot\|_\infty} f$ then

1. $f_n(\omega) \xrightarrow{\text{a.e}} f(\omega)$,
2. There exists $g \in L^\infty(\Omega, \mathscr{F}, \mu)$ with $|f_n(\omega)| \leq g(\omega)$ *a.e.*

(R) In case $p = 1$, let $f_n, f \in L^1(\Omega, \mathscr{F}, \mu)$ and suppose that $f_n \xrightarrow{\|\cdot\|_1} f$. Then

$$\left|\int f_n \, d\mu - \int f \, d\mu\right| \leq \int |f_n - f| \, d\mu = \|f_n - f\|_1 \to 0 \qquad \text{as } n \to \infty,$$

so that

$$\int f_n \, d\mu \to \int f \, d\mu.$$

There is a variation of the Dominated Convergence Theorem:

Theorem 3.2.1 *(Dominated Convergence Theorem (DCT) for the L^p-spaces)*
Let $1 \leq p < \infty$, and let f_n, f be $\mathscr{F}$-measurable functions. Suppose that

(1) $f_n(\omega) \xrightarrow{\text{a.e}} f(\omega)$,
(2) there exists $g \in L^p(\Omega, \mathscr{F}, \mu)$ with $|f_n(\omega)| \leq g(\omega)$ *a.e.*

Then

(a) $f_n, f \in L^p(\Omega, \mathscr{F}, \mu)$, and
(b) $f_n \xrightarrow{\|\cdot\|_p} f$.

Proof. Let us first show that $f_n, f \in L^p$. (To be precise, we show that $f_n, f \in \mathscr{L}^p$.) In fact, assumptions (1) and (2) together imply that

$$|f(\omega)| \leq g(\omega) \quad a.e. \tag{3.9}$$

as well. It follows from assumptions (1) and (2), from (3.9) and monotonicity of the integral that

$$\int |f_n|^p \leq \int g^p < \infty \qquad \text{and} \qquad \int |f_n|^p \leq \int g^p < \infty,$$

which shows that $f_n, f \in L^p$.

Next we prove that $f_n \xrightarrow{\|\cdot\|_p} f$. By assumption (1),

$$|f_n(\omega) - f(\omega)|^p \to 0 \quad a.e.$$

On the other hand,

$$|f_n(\omega) - f(\omega)|^p \leq \Big[|f_n(\omega)| + |f(\omega)|\Big]^p \leq \Big[g(\omega) + g(\omega)\Big]^p = 2^p g(\omega)^p \quad a.e.$$

for all n. Since $g \in L^p$ by assumption, then the function $2^p g^p$ is integrable. We can thus apply the usual DCT (Theorem 2.7.8) to the sequence of integrable functions $\{|f_n(\omega) - f(\omega)|^p\}$ and obtain

$$\lim_{n\to\infty} \|f_n - f\|_p^p = \lim_{n\to\infty} \int |f_n - f|^p \underset{\text{DCT}}{=} \int \lim_{n\to\infty} |f_n - f|^p = \int 0 = 0,$$

which shows that $f_n \xrightarrow{\|\cdot\|_p} f$ as $n \to \infty$. Thus the Theorem is proved. ■

R This theorem does not apply when $p = \infty$. For example, let $(\Omega, \mathscr{F}, \mu) = ([0,1], \mathscr{B}([0,1]), \lambda)$, let $f_n = \mathbf{1}_{(\frac{1}{n},1]}$ and $f = g = \mathbf{1}_{(0,1]}$. Then

(1) $f_n(\omega) \to f(\omega) \quad \forall\, \omega \in [0,1]$,
(2) $|f_n(\omega)| \le g(\omega) \quad \forall\, \omega \in [0,1]$.

However,

$$\|f_n - f\|_\infty = \|\mathbf{1}_{(0,\frac{1}{n}]}\|_\infty = 1 \not\to 0 \qquad \text{as } n \to \infty.$$

Theorem 3.2.2 *(Completeness of $L^p(\Omega, \mathscr{F}, \mu)$ for $p \neq \infty$)*
Let $1 \le p < \infty$, and let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $L^p(\Omega, \mathscr{F}, \mu)$. Then there exist a subsequence $(f_{n_k})_{k=1}^\infty$ and $f, g \in L^p(\Omega, \mathscr{F}, \mu)$ satisfying

(1) $|f_{n_k}(\omega)| \le g(\omega)$ a.e. $\forall k$,
(2) $f_{n_k}(\omega) \xrightarrow{\text{a.e.}} f(\omega)$ as $k \to \infty$,
(3) $f_n \xrightarrow{\|\cdot\|_p} f$ as $n \to \infty$.

In particular, $L^p(\Omega, \mathscr{F}, \mu)$ is complete.

Proof. The proof proceeds in four major steps.

Step 1: Extract a subsequence (f_{n_k}) satisfying $\|f_{n_{k+1}} - f_{n_k}\|_p < \frac{1}{2^k}$ $\forall k$.
Step 2: Show that $\sum_k [f_{n_{k+1}} - f_{n_k}]$ converges a.e.; conclude that $f(\omega) = \lim_{k\to\infty} f_{n_k}(\omega)$ exists a.e.
Step 3: Show that $f \in L^p$.
Step 4: Show that $f_n \xrightarrow{\|\cdot\|_p} f$.

The extraction of the subsequence (f_{n_k}) is a standard process for Cauchy sequences, and is done by induction. Since (f_n) is Cauchy, then for $\varepsilon = \frac{1}{2}$ there exists $n_1 \in \mathbb{N}$ so that

$$\|f_n - f_{n_1}\|_p < \frac{1}{2} \quad \forall n > n_1.$$

Similarly, for $\varepsilon = \frac{1}{2^2}$ there exists $n_2 \in \mathbb{N}$ so that

$$\|f_n - f_{n_2}\|_p < \frac{1}{2^2} \quad \forall n > n_2.$$

Increasing n_2 if necessary we may assume that $n_1 < n_2$. In general, suppose we have already chosen $n_1 < n_2 < \cdots < n_{k-1}$ as desired. Since (f_n) is Cauchy, for $\varepsilon = \frac{1}{2^k}$ there exists $n_k \in \mathbb{N}$ so that

$$\|f_n - f_{n_k}\|_p < \frac{1}{2^k} \quad \forall n > n_k.$$

Increasing n_k if necessary we may assume that $n_{k-1} < n_k$. Continuing this way, we thus obtain a subsequence (f_{n_k}) of (f_n) satisfying

$$\left\|f_{n_{k+1}} - f_{n_k}\right\|_p < \frac{1}{2^k} \quad \forall k. \tag{3.10}$$

This completes step 1.

To show a.e.-convergence of this subsequence, let us set

$$g_k := \left|f_{n_1}\right| + \sum_{i=1}^{k} \left|f_{n_{i+1}} - f_{n_i}\right|, \qquad k = 1, 2, \ldots . \tag{3.11}$$

Then (i) $g_k \in \mathscr{L}^+ \ \forall k$, and (ii) $(g_k) \uparrow$. Hence $g := \lim_{k\to\infty} g_k$ exists (as an extended real-valued function), and $g \in \mathscr{L}^+$.

Claim: $g \in L^p$. In fact, as $\left(g_k^p\right) \uparrow$, then

$$\begin{aligned}
\int g^p &= \int \left[\lim_{k\to\infty} g_k^p\right] \underset{\text{MCT}}{=} \lim_{k\to\infty} \int g_k^p \\
&= \lim_{k\to\infty} \left\| \left|f_{n_1}\right| + \sum_{i=1}^{k} \left|f_{n_{i+1}} - f_{n_i}\right| \right\|_p^p \\
&\underset{\text{Minkowski}}{\leq} \lim_{k\to\infty} \left[\left\|f_{n_1}\right\|_p + \sum_{i=1}^{k} \left\|f_{n_{i+1}} - f_{n_i}\right\|_p \right]^p \\
&\underset{(3.10)}{<} \lim_{k\to\infty} \left[\left\|f_{n_1}\right\|_p + \sum_{i=1}^{k} \frac{1}{2^i} \right]^p = \left[\left\|f_{n_1}\right\|_p + 1 \right]^p < \infty,
\end{aligned}$$

which proves the claim.

By the claim, g is finite valued a.e., that is,

$$g(\omega) = \left|f_{n_1}(\omega)\right| + \sum_{i=1}^{\infty} \left|f_{n_{i+1}}(\omega) - f_{n_i}(\omega)\right| < \infty \quad \text{a.e.} \tag{3.12}$$

Since every absolutely convergent series in $\mathbb{K}$ converges, there exists an $\mathscr{F}$-measurable function f so that

$$\begin{aligned}
f(\omega) &= \lim_{k\to\infty} \left[f_{n_1}(\omega) + \underbrace{\sum_{i=1}^{k} \left[f_{n_{i+1}}(\omega) - f_{n_i}(\omega)\right]}_{\text{telescoping sum}} \right] \\
&= f_{n_1}(\omega) + \lim_{k\to\infty} \left[f_{n_{k+1}}(\omega) - f_{n_1}(\omega) \right] \\
&= \lim_{k\to\infty} f_{n_{k+1}}(\omega) = \lim_{k\to\infty} f_{n_k}(\omega) \quad \text{a.e.}
\end{aligned}$$

This proves assertion (2) and completes step 2.

Note that by the triangle inequality,

$$\left|f_{n_{k+1}}\right| = \left| f_{n_1} + \sum_{i=1}^{k} \left[f_{n_{i+1}} - f_{n_i}\right] \right| \leq \left|f_{n_1}(\omega)\right| + \sum_{i=1}^{\infty} \left|f_{n_{i+1}}(\omega) - f_{n_i}(\omega)\right| = g,$$

for all k, which shows that assertion (1) holds. Furthermore, going to limits,

$$|f(\omega)| = \lim_{k\to\infty} |f_{n_k}(\omega)| \le g(\omega) \quad \text{a.e.},$$

and as $g \in L^p$ it follows that $f \in L^p$ as well. This completes step 3.

Finally by step 3,

$$|f_{n_k}(\omega) - f(\omega)| \to 0 \quad \text{a.e}$$

while also

$$|f_{n_k}(\omega) - f(\omega)| \le |f_{n_k}(\omega) + f(\omega)| \le 2g(\omega) \quad \text{a.e.}$$

Since $g \in L^p$, we may apply the DCT for the L^p spaces to the sequence $\{|f_{n_k} - f|\}_{k=1}^{\infty}$ and obtain that

$$\|f_{n_k} - f\|_p = \| |f_{n_k} - f| - 0 \|_p \to 0,$$

that is,

$$f_{n_k} \xrightarrow{\|\cdot\|_p} f \quad \text{as } k \to \infty.$$

Then by a standard property of Cauchy sequences,

$$f_n \xrightarrow{\|\cdot\|_p} f \quad \text{as } n \to \infty.$$

This completes step 4 and the proof of the theorem. ■

Corollary 3.2.3 Let $1 \le p < \infty$ and let $(f_n)_{n=1}^{\infty}$ be a convergent sequence in $L^p(\Omega, \mathscr{F}, \mu)$, say $f_n \xrightarrow{\|\cdot\|_p} h$. Then

1. there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ with $f_{n_k}(\omega) \xrightarrow{\text{a.e}} h(\omega)$,
2. there exists $g \in L^p(\Omega, \mathscr{F}, \mu)$ with $|f_{n_k}(\omega)| \le g(\omega)$ a.e.

Proof. Since (f_n) converges, it is Cauchy. We let f_{n_k}, f and g be as in Theorem 3.2.2. Now since $f_n \xrightarrow{\|\cdot\|_p} h$ as $n \to \infty$, then also $f_{n_k} \xrightarrow{\|\cdot\|_p} h$ as $k \to \infty$. On the other hand, $f_{n_k} \xrightarrow{\|\cdot\|_p} f$ as $k \to \infty$ by Theorem 3.2.2. It now follows from uniqueness of limits that $h = f$ in L^p, that is, $h = f$ a.e., and the proof is complete. ■

We note that the case $p = \infty$ has already been covered in Exercise 3.7.

Corollary 3.2.4 *(Uniqueness of Limits)* Let $1 \le p \le \infty$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^p(\Omega, \mathscr{F}, \mu)$. Suppose that

1. $f_n \xrightarrow{\|\cdot\|_p} f$ for some $f \in L^p(\Omega, \mathscr{F}, \mu)$, and
2. $f_n(\omega) \xrightarrow{\text{a.e}} h(\omega)$ for some $\mathscr{F}$-measurable function h.

Then $f(\omega) = h(\omega)$ a.e.

Proof. By Corollary 3.2.3 (respectively Exercise 3.7 in case $p = \infty$), there exists a subsequence (f_{n_k}) so that $f_{n_k}(\omega) \xrightarrow{\text{a.e}} f(\omega)$. Since by assumption, $f_{n_k}(\omega) \xrightarrow{\text{a.e}} h(\omega)$, it now follows from uniqueness of limits in $\mathbb{K}$ that $f(\omega) = h(\omega)$ a.e. ■

Theorem 3.2.5 $L^\infty(\Omega, \mathscr{F}, \mu)$ is a Banach space.

Proof. Let (f_n) be a given Cauchy sequence in L^∞. We first find a limit function f. For each $m > n$, set

$$A_{m,n} := \{\, \omega \in \Omega \;:\; |f_m(\omega) - f_n(\omega)| > \|f_m - f_n\|_\infty \,\},$$

and set

$$A := \bigcup_{n=1}^{\infty} \bigcup_{m=n+1}^{\infty} A_{m,n}.$$

Then A is a null set. Furthermore, for all $\omega \in A^c$ and all $m > n$ we have

$$|f_m(\omega) - f_n(\omega)| \le \|f_m - f_n\|_\infty. \tag{3.13}$$

Since (f_n) is Cauchy, it follows from this inequality that the sequence $(f_n(\omega))$ is Cauchy in $\mathbb{K}$ for all $\omega \in A^c$, and thus converges by completeness of $\mathbb{K}$. That is, there exists an $\mathscr{F}$-measurable function f such that $f_n(\omega) \to f(\omega)\ \forall\, \omega \in A^c$.

Next we must show that $f \in L^\infty$ and $f_n \xrightarrow{\|\cdot\|_\infty} f$. Let $\varepsilon > 0$ be given. Since (f_n) is Cauchy in L^∞, by (3.13) there exists $N \in \mathbb{N}$ so that

$$|f_m(\omega) - f_n(\omega)| \le \|f_m - f_n\|_\infty < \frac{\varepsilon}{2} \qquad \forall\, \omega \in A^c,\ \ \forall m > n \ge N,$$

and thus letting $m \to \infty$,

$$|f(\omega) - f_n(\omega)| \le \frac{\varepsilon}{2} \qquad \forall\, \omega \in A^c,\ \ \forall n \ge N.$$

Since A is a null set, it follows that

$$\|f - f_n\|_\infty = \operatorname*{ess\text{-}sup}_{\omega\in\Omega} |f(\omega) - f_n(\omega)| \le \frac{\varepsilon}{2} < \varepsilon \qquad \forall n \ge N.$$

In particular, $f - f_N \in L^\infty$, so that $f = (f - f_N) + f_N \in L^\infty$. Furthermore, as ε was arbitrary, we conclude that that $f_n \xrightarrow{\|\cdot\|_\infty} f$. Thus the proof is complete. ■

Recall that a *simple function* is a function whose range is finite. Let us set here

$$\mathscr{S} = \mathscr{S}_{\mathbb{K}} = \mathscr{S}_{\mathbb{K}}(\Omega, \mathscr{F}, \mu) := \{\, \varphi : \Omega \to \mathbb{K} \,|\, \varphi \text{ is } \mathscr{F}\text{-measurable and simple} \,\}$$

which is a vector space over $\mathbb{K}$. When $\varphi \in \mathscr{S}$ has range $\{a_1, \dots a_n\}$, then as usual we consider its canonical representation

$$\varphi = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k} \qquad \text{where} \quad A_k = \{\omega \in \Omega : \varphi(\omega) = a_k\} \in \mathscr{F}.$$

Now let $1 \le p < \infty$. Since the sets A_k are disjoint, then clearly,

$$|\,\varphi\,|^p = \sum_{k=1}^{n} |a_k|^p \mathbf{1}_{A_k}.$$

Hence,

$$\int |\varphi|^p = \int \sum_{k=1}^{n} |a_k|^p \mathbf{1}_{A_k} = \sum_{k=1}^{n} |a_k|^p \mu(A_k),$$

which shows that $\varphi \in L^p \quad \Leftrightarrow \quad \mu(A_k) < \infty \quad \forall k$ with $a_k \neq 0$. On the other hand, clearly $\varphi \in L^\infty(\Omega, \mathscr{F}, \mu)$.

The Structure Theorem for Measurable Functions has an analogue for the space of p-integrable functions:

Theorem 3.2.6 *(Density of Simple Functions in L^p)* Let $1 \leq p \leq \infty$. For each $f \in L^p(\Omega, \mathscr{F}, \mu)$ there exists a sequence (φ_n) in $L^p(\Omega, \mathscr{F}, \mu) \cap \mathscr{S}(\Omega, \mathscr{F}, \mu)$ so that

(1) $|\varphi_n(\omega)| \leq |f(\omega)|$ a.e.

(2) $\varphi_n \xrightarrow{\|\cdot\|_p} f$.

Proof.

Case 1: $f \in \mathscr{L}^+$.

First suppose that $1 \leq p < \infty$. By the Structure Theorem (Theorem 1.6.1) there exists a sequence $(\varphi_n) \uparrow$ in $\mathscr{S}^+$ so that

$$\varphi_n(\omega) \to f(\omega) \quad \text{a.e.} \tag{3.14}$$

(We have only *a.e.* convergence as f is uniquely defined up to null sets only) In particular,

$$0 \leq \varphi_n(\omega) \leq f(\omega) \quad \text{a.e.} \tag{3.15}$$

so that (1) holds. Now since $f \in L^p$, by (3.14) and (3.15) we may apply the DCT for the L^p spaces, and obtain that $\varphi_n \in L^p$ for all n, and $\varphi_n \xrightarrow{\|\cdot\|_p} f$.

Now suppose that $p = \infty$. Set $A = \{\omega \in \Omega : f(\omega) > \|f\|_\infty\}$. Then $f\mathbf{1}_{A^c}$ is bounded, so by the Structure Theorem there exists a sequence $\{\varphi_n\} \uparrow$ in $\mathscr{S}^+$ so that

$$\varphi_n(\omega) \rightrightarrows f(\omega)\mathbf{1}_{A^c} \quad \text{on } \Omega. \tag{3.16}$$

In particular,

$$0 \leq \varphi_n(\omega) \leq f(\omega) \quad \text{a.e. } \omega, \tag{3.17}$$

and again, (1) holds. Now by uniform convergence,

$$\|f - \varphi_n\|_\infty = \|f\mathbf{1}_{A^c} - \varphi_n\|_\infty \leq \sup_{\omega \in \Omega} \left|(f\mathbf{1}_{A^c})(\omega) - \varphi_n(\omega)\right| \to 0 \quad \text{as } n \to \infty,$$

which shows that $\varphi_n \xrightarrow{\|\cdot\|_\infty} f$.

Case 2: $f \in \mathscr{L}^p$ is arbitrary. (The following computations are for $\mathbb{K} = \mathbb{C}$; they naturally simplify when $\mathbb{K} = \mathbb{R}$.) We may decompose f as

$$f = \text{Re}(f) + i\,\text{Im}(f) = [f_1 - f_2] + i[f_3 - f_4] \tag{3.18}$$

where $f_1 = \text{Re}(f)^+$, $f_2 = \text{Re}(f)^-$, $f_3 = \text{Im}(f)^+$ and $f_4 = \text{Im}(f)^-$. Since

$$0 \leq f_1, f_2 \leq f_1 + f_2 = |\text{Re}(f)| \leq |f| \in L^p$$
$$0 \leq f_3, f_4 \leq f_3 + f_4 = |\text{Im}(f)| \leq |f| \in L^p,$$

it follows that $f_j \in L^p$ for all $j = 1, \ldots, 4$. Now by Case 1, there exist sequences $\left(\varphi_j^{(n)}\right)_{n=1}^{\infty}$ in $L^p \cap \mathscr{S}^+$, $j = 1, \ldots, 4$ so that

1. $0 \leq \varphi_j^{(n)}(\omega) \leq f_j(\omega)$ a.e. $\forall n,\ j = 1,\dots,4,$
2. $\left\|\varphi_j^{(n)} - f_j\right\|_p \to 0$ as $n \to \infty,\ j = 1,\dots,4.$

Set

$$\varphi_n := \left[\varphi_1^{(n)} - \varphi_2^{(n)}\right] + i\left[\varphi_3^{(n)} - \varphi_4^{(n)}\right]$$

Then $\varphi_n \in L^p \cap \mathscr{S}$ for all n, and

$$|\mathrm{Re}(\varphi_n)| = \left|\varphi_1^{(n)} - \varphi_2^{(n)}\right| \leq \varphi_1^{(n)} + \varphi_2^{(n)} \leq f_1 + f_2 = |\mathrm{Re}(f)| \quad \text{a.e.}$$

$$|\mathrm{Im}(\varphi_n)| = \left|\varphi_3^{(n)} - \varphi_4^{(n)}\right| \leq \varphi_3^{(n)} + \varphi_4^{(n)} \leq f_3 + f_4 = |\mathrm{Im}(f)| \quad \text{a.e.}$$

so that

$$|\varphi_n| = \left[|\mathrm{Re}(\varphi_n)|^2 + |\mathrm{Im}(\varphi_n)|^2\right]^{1/2} \leq \left[|\mathrm{Re}(f)|^2 + |\mathrm{Im}(f)|^2\right]^{1/2} = |f| \quad \text{a.e.}$$

Finally, by Minkowski's inequality,

$$\begin{aligned}\|f - \varphi_n\|_p &= \left\|\Big([f_1 - f_2] + i[f_3 - f_4]\Big) - \left(\left[\varphi_1^{(n)} - \varphi_2^{(n)}\right] + i\left[\varphi_3^{(n)} - \varphi_4^{(n)}\right]\right)\right\|_p \\ &\leq \left\|f_1 - \varphi_1^{(n)}\right\|_p + \left\|\varphi_2^{(n)} - f_2\right\|_p + \left\|f_3 - \varphi_3^{(n)}\right\|_p + \left\|\varphi_4^{(n)} - f_4\right\|_p \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus the proof of the theorem is complete. ■

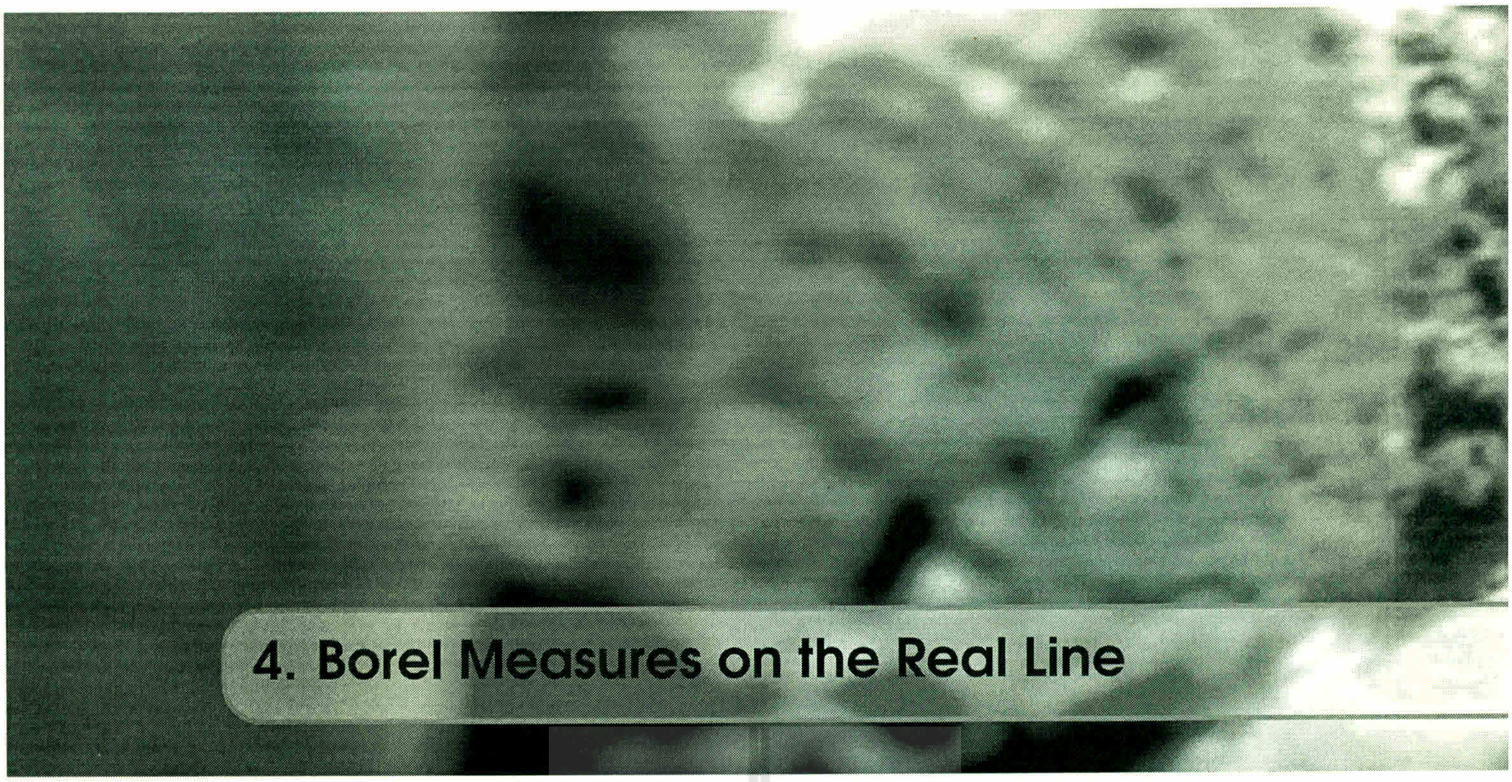
R In the case of the Lebesgue measure on the real line one can show:
Let I be *any* interval, and consider $L^p(I, \mathscr{B}(I), \lambda)$ for $1 \leq p < \infty$.

1. For each $f \in L^p(I, \mathscr{B}(I), \lambda)$ there exists a sequence $\{\varphi_n\}$ of *step functions* such that
   (a) $|\varphi_n(\omega)| \leq |f(\omega)|$ a.e. $\omega \in I$,
   (b) $\varphi_n \xrightarrow{\|\cdot\|_p} f$.
2. For each $f \in L^p(I, \mathscr{B}(I), \lambda)$ there exists a sequence $\{g_n\}$ in $C_c(I)$ such that
   (a) $|g_n(\omega)| \leq |f(\omega)|$ a.e. $\omega \in I$,
   (b) $g_n \xrightarrow{\|\cdot\|_p} f$.

   (We recall here that

$$C_c(I) = \left\{f : I \to \mathbb{K} \,|\, g \text{ is continuous},\ \exists [a,b] \subseteq I,\ f(\omega) = 0\ \forall \omega \in I \setminus [a,b]\right\}$$

   is the set of *continuous functions with compact support on I*.)



4. Borel Measures on the Real Line

In this chapter, we will proceed with the construction of finite and σ-finite Borel measures, that is, measures on the measurable space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. We will see how these measures relate with distributions and density functions which the reader may already be familiar with from a basic probability course.

4.1 Distribution Functions

(R) Let us briefly recall the concepts of one-sided limits and one-sided continuity at points on the real line usually taught in an undergraduate analysis course. For this, let $x_o \in \mathbb{R}$, I an open interval containing x_o, and $f : I \to \mathbb{R}$.

1. (right-hand limit at x_o) The following are equivalent:
   (a) $\lim_{x \to x_o^+} f(x) = L$. That is, for all $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $x \in I \cap (x_o, x_o + \delta)$.
   (b) if (x_n) is any sequence in I with $x_n > x_o$ for all n and $x_n \to x_o$, then $f(x_n) \to L$.
   (c) if $(x_n) \downarrow$ is any decreasing sequence in I with $x_n \to x_o$, then $f(x_n) \to L$.
2. A similar statement holds for the left-hand limit $\lim_{x \to x_o^-} f(x)$, if it exists.
3. f is right-continuous at x_o if and only if $\lim_{x \to x_o^+} f(x) = f(x_o)$.
4. Similarly, f is left-continuous at x_o if and only if $\lim_{x \to x_o^-} f(x) = f(x_o)$.
5. f is continuous at x_o if and only if f is both, left- and right continuous at x_o.
6. f is right-continuous / left-continuous / continuous on I if and only if f is right-continuous / left-continuous / continuous at every $x_o \in I$.

We also need to review some properties of monotone functions. Recall here that $f : I \to \mathbb{R}$ is called (monotone) increasing on I, if for all $x_1, x_2 \in I$ with $x_1 < x_2$ we have $f(x_1) \le f(x_2)$. Such a function has the following properties:

1. $f(x_o^+) = \lim_{x \to x_o^+} f(x)$ and $f(x_o^-) = \lim_{x \to x_o^-} f(x)$ both exist at all $x_o \in I$. That is, every discontinuity of f is a jump discontinuity.
2. It follows that f has only countably many discontinuities.
3. Hence, f is Borel-measurable.

Given a *finite* Borel measure μ on the real line, we define a function $F:\mathbb{R}\to[0,\infty)$ by

$$F(x)=F_\mu(x):=\mu\left((-\infty,x]\right). \tag{4.1}$$

This function has the following properties:

(D1) F is *increasing*: In fact, let $x_1,x_2\in\mathbb{R}$ with $x_1<x_2$. Then $(-\infty,x_1]\subset(-\infty,x_2]$, and hence by monotonicity of the measure,

$$F(x_1)=\mu\left((-\infty,x_1]\right)\le\mu\left((-\infty,x_2]\right)=F(x_2).$$

(D2) F is *right-continuous*: For let $x_o\in\mathbb{R}$, and $(x_n)\downarrow$ be a sequence in $\mathbb{R}$ with $x_n\to x_o$. Then

1. $\left\{(-\infty,x_n]\right\}_{n=1}^{\infty}$ is a decreasing sequence of sets, while also
2. $\bigcap\limits_{n=1}^{\infty}(-\infty,x_n]=(-\infty,x_o]$.

It follows from Theorem 1.4.2 that

$$F(x_o)=\mu\left((-\infty,x_o]\right)=\mu\left(\bigcap_{n=1}^{\infty}(-\infty,x_n]\right)\underset{\text{Thm 1.4.2}}{=}\lim_{n\to\infty}\mu\left((-\infty,x_n]\right)=\lim_{n\to\infty}F(x_n),$$

that is, $\lim_{x\to x_o^+}F(x)=F(x_o)$.

(D3) $\lim\limits_{x\to-\infty}F(x)=0$. To see this, let $(x_n)\downarrow$ be a sequence in $\mathbb{R}$ with $x_n\to-\infty$. Then

1. $\left\{(-\infty,x_n]\right\}_{n=1}^{\infty}$ is a decreasing sequence of sets, while also
2. $\bigcap\limits_{n=1}^{\infty}(-\infty,x_n]=\emptyset$.

It follows from Theorem 1.4.2 that

$$0=\mu(\emptyset)=\mu\left(\bigcap_{n=1}^{\infty}(-\infty,x_n]\right)\underset{\text{Thm 1.4.2}}{=}\lim_{n\to\infty}\mu\left((-\infty,x_n]\right)=\lim_{n\to\infty}F(x_n).$$

(D4) $\lim\limits_{x\to\infty}F(x)=\mu(\mathbb{R})$. To see this, let $(x_n)\uparrow$ be a sequence in $\mathbb{R}$ with $x_n\to\infty$. Then

1. $\left\{(-\infty,x_n]\right\}_{n=1}^{\infty}$ is an increasing sequence of sets, while also
2. $\bigcup\limits_{n=1}^{\infty}(-\infty,x_n]=\mathbb{R}$.

It follows from Theorem 1.4.2 that

$$\mu(\mathbb{R})=\mu\left(\bigcup_{n=1}^{\infty}(-\infty,x_n]\right)\underset{\text{Thm 1.4.2}}{=}\lim_{n\to\infty}\mu\left((-\infty,x_n]\right)=\lim_{n\to\infty}F(x_n).$$

(D5) If $(a,b]$ is any half-open interval, then

$$\mu\left((a,b]\right)=\mu\left((-\infty,b]\backslash(-\infty,a]\right)\underset{\text{Thm 1.4.2}}{=}\mu\left((-\infty,b]\right]-\mu\left((-\infty,a]\right)=F(b)-F(a).$$

F is called the *(cumulative) distribution function* of the Borel measure μ.

R Note that F need not be left-continuous at x_o. For example, let δ_{x_o} denote the Dirac one-point measure with mass at x_o. Then

$$F(x_o)=\delta_{x_o}\left((-\infty,x_o]\right)=1$$

while whenever $x<x_o$ then

$$F(x)=\delta_{x_o}\left((-\infty,x]\right)=0$$

so that

$$\lim_{x\to x_o^-}F(x)=0\neq F(x_o).$$

Exercise 4.1 An alternative definition of the distribution function would be

$$F(x) := \mu\left((-\infty, x)\right).$$

Show that (D1) and (D3)-(D5) still hold, but this function is now left-continuous.

When the Borel measure μ is no longer finite, but still finite on bounded sets, one can define a (cumulative) distribution function $F : \mathbb{R} \to (-\infty, \infty)$ of μ by fixing $c \in \mathbb{R}$ (usually $c = 0$) and setting

$$F(x) = F_c(x) := \begin{cases} \mu\left((c,x]\right) & \text{if } x \geq c, \\ -\mu\left((x,c]\right) & \text{if } x < c \end{cases} \tag{4.2}$$

The reader my easily verify that properties (D1), (D2) and (D5) still hold. Furthermore, when μ is a finite measure, then

$$F_c(x) = F_\mu(x) - F_\mu(c)$$

where F_μ denotes the distribution function as defined in (4.1).

We therefore define:

Definition 4.1.1 A function $F : \mathbb{R} \to \mathbb{R}$ satisfying
(D1) F is increasing, and
(D2) F is right-continuous
is called a *distribution function*.

4.2 Outer Measures

Considering the previous discussion, the question arises: Given a distribution function F, does there exist a Borel measure μ on $\left(\mathbb{R}, \mathscr{B}(\mathbb{R})\right)$ so that

$$\mu\left((a,b]\right) = F(b) - F(a) \qquad \forall a < b \ ?$$

Our goal is to give an affirmative answer to this question. In this section, we will construct what is called an *outer measure* from a distribution function. The next section will show how the outer measure becomes a measure, when restricted to an appropriate collection of subsets.

Thus, let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function.

Step 1: Given a bounded, half-open interval $(a,b]$ with $a \leq b$, we set

$$L_F(a,b] := F(b) - F(a). \tag{4.3}$$

Because of the following properties, we may consider L_F as specifying a generalized length of the half-open intervals $(a,b]$.

1. In the special case where $F(x) = x$ then $L_F(a,b] = b - a$, which is the length of the interval $(a,b]$.
2. If $a \leq c \leq b$, then

$$L_F(a,b] = F(b) - F(a) = \left[F(b) - F(c)\right] + \left[F(c) - F(a)\right] = L_F(a,c] + L_F(c,b]. \tag{4.4}$$

3. If $I_1 = (a,b]$, $I_2 = (c,d]$ and $I_2 \subseteq I_1$, then $a \le c \le d \le b$, so that since F is increasing,

$$L_F(I_2) = F(d) - F(c) \le F(b) - F(a) = L_F(I_1). \qquad \text{("monotonicity")}$$

4. Let $a_n \to a^+$ and $b_n \to b^+$. Then by right-continuity of F,

$$\begin{aligned} L_F(a,b] = F(b) - F(a) &= \lim_{n\to\infty} F(b_n) - \lim_{n\to\infty} F(a_n) \\ &= \lim_{n\to\infty} \left[F(b_n) - F(a_n)\right] = \lim_{n\to\infty} L_F(a_n,b_n]. \qquad \text{("right continuity")} \end{aligned}$$

Step 2: For any $A \subseteq \mathbb{R}$ we now set

$$\mu_F^*(A) := \inf S_A \qquad \text{where} \qquad S_A = \left\{ \sum_{n=1}^{\infty} L_F(I_n) \ : \ I_n = (a_n,b_n],\ A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}. \tag{4.5}$$

(Loosely speaking, if $\{I_n\}_{n=1}^{\infty}$ is a countable cover of A by half-open intervals, then the sum of the (generalized) lengths of these intervals will be an element of S_A.) Clearly, $S_A \neq \emptyset$ and $S_A \subseteq [0,\infty]$. Hence,

$$0 \le \mu_F^*(A) \le \infty.$$

We discuss the properties of μ_F^* in the following four propositions:

Proposition A If $A = (a,b]$ is a bounded, half-open interval, then $\mu_F^*(a,b] = L_F(a,b]$.

Proof. We first show that $\mu_F^*(a,b] \le L_F(a,b]$. In fact, choose $I_1 = (a,b]$ and $I_n = (a,a] = \emptyset$ for all $n \ge 2$. Then clearly, $A = (a,b] \subseteq \bigcup_{n=1}^{\infty} I_n$, so that $L_F(I_1) = \sum_{n=1}^{\infty} L_F(I_n) \in S_A$, and hence

$$\mu_F^*(A) = \inf S_A \le L_F(a,b].$$

Next we show that

$$L_F(a,b] \le \mu_F^*(a,b]. \tag{4.6}$$

In fact, we will show that for *every* $\varepsilon > 0$,

$$L_F(a,b] < \mu_F^*(a,b] + \varepsilon. \tag{4.7}$$

Letting $\varepsilon \to 0^+$, then (4.6) will follow.

Thus, let $\varepsilon > 0$ be given. Then $\mu_F^*(a,b] + \varepsilon$ is not a lower bound of the set S_A; hence there exists $s \in S_A$ with

$$\mu_F^*(a,b] \le s < \mu_F^*(a,b] + \frac{\varepsilon}{2}. \tag{4.8}$$

Now since $s \in S_A$, there exist intervals $I_n = (a_n,b_n]$ with

$$(a,b] \subseteq \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad s = \sum_{n=1}^{\infty} L_F(I_n). \tag{4.9}$$

Next we modify $(a,b]$ to a closed interval and each I_n to an open interval, so that we can use a compactness argument. By right-continuity of F, there exists $\tilde{a}$, $a < \tilde{a} < b$ with

$$L_F(a,b] < L_F(\tilde{a},b] + \frac{\varepsilon}{4}.$$

Similarly, by right continuity of F, for each b_n there exists $\tilde{b}_n > b_n$, so that

$$L_F(a_n, \tilde{b}_n] < L_F(a_n, b_n] + \frac{\varepsilon}{2^{n+2}}.$$

Now set $J_n = (a_n, \tilde{b}_n)$ so that $I_n \subset J_n$, for all n. It thus follows from (4.9) that $\{J_n\}_{n=1}^{\infty}$ is an open cover of the closed subinterval $[\tilde{a}, b]$ of $(a, b]$. By compactness, there exists a finite subcover $\{J_{n_1}, \ldots, Jn_N\}$ for $[\tilde{a}, b]$.

Now if $J_{n_k} \subseteq J_{n_l}$ for some pair of indices, we may remove the interval J_{n_k} from this subcover, and still have a cover of $[\tilde{a}, b]$; thus we may assume that the sets J_{n_k} are not contained in another.

We claim that $L_F(\tilde{a}, b] \leq \sum_{k=1}^{N} L_F(a_{n_k}, \tilde{b}_{n_k}]$. In fact, let us first relabel all the intervals J_n so that $n_k = k$ for $k = 1, \ldots N$, that is, we can write $J_k = (a_k, \tilde{b}_k)$ instead of $J_{n_k} = \left(a_{n_k}, \tilde{b}_{n_k}\right)$. Furthermore, since no interval is contained in another, we can do this relabeling so that

$$a_1 < a_2 < a_3 \cdots < a_k.$$

Observe that by the very same property, $a_{k+1} \leq \tilde{b}_k$ for $k = 1, \ldots N-1$. In addition, $a_1 < \tilde{a}$ while $b < \tilde{b}_N$.

Then since F is increasing,

$$\begin{aligned} L_F(\tilde{a}, b] &= F(b) - F(\tilde{a}) \leq F(\tilde{b}_N) - F(a_1) \\ &\leq F(\tilde{b}_N) + \sum_{k=1}^{N-1} \left[F(\tilde{b}_k) - F(a_{k+1})\right] - F(a_1) = \sum_{k=1}^{N} \left[F(\tilde{b}_k) - F(a_k)\right] = \sum_{k=1}^{N} L_F(a_k, \tilde{b}_k], \end{aligned}$$

and the claim is proved.

It now follows by (4.8) that

$$\begin{aligned} L_F(a, b] &< L_F(\tilde{a}, b] + \frac{\varepsilon}{4} \underset{\text{claim}}{\leq} \left[\sum_{k=1}^{N} L_F(a_{n_k}, \tilde{b}_{n_k}]\right] + \frac{\varepsilon}{4} \leq \left[\sum_{k=1}^{\infty} L_F(a_{n_k}, \tilde{b}_{n_k}]\right] + \frac{\varepsilon}{4} \\ &\leq \left[\sum_{k=1}^{\infty} L_F(a_k, b_k] + \frac{\varepsilon}{2^{k+2}}\right] + \frac{\varepsilon}{4} = \left[\sum_{k=1}^{\infty} L_F(a_k, b_k]\right] + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = s + \frac{\varepsilon}{2} \\ &< \mu_F^*(a, b] + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mu_F^*(a, b] + \varepsilon, \end{aligned}$$

which proves the proposition. ■

Proposition B $\mu_F^*(\emptyset) = 0$.

Proof. Simply choose $b = a$ in the previous proposition. ■

Proposition C *(monotonicity)* If $A \subseteq B$ then $\mu_F^*(A) \leq \mu_F^*(B)$.

Proof. Too see this, we first note that $S_B \subseteq S_A$. For if $s \in S_B$, then there exists intervals $\{I_n\}$, $I_n = (a_n, b_n]$, s.t.

$$B \subseteq \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad s = \sum_{n=1}^{\infty} L_F(I_n).$$

But as $A \subseteq B$, then also $A \subseteq \bigcup_{n=1}^{\infty} I_n$, and hence $s \in S_A$ as well.

It now follows that

$$\mu_F^*(A) \;=\; \inf S_A \underset{S_B \subseteq S_A}{\leq} \inf S_B \;=\; \mu_F^*(B).$$

■

Proposition D *(σ-subadditivity)* Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of subsets of $\mathbb{R}$. Then

$$\mu_F^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu_F^*(A_n). \tag{4.10}$$

Proof. Set $A = \bigcup_{n=1}^{\infty} A_n$. We will first show that for *every* $\varepsilon > 0$,

$$\mu_F^*(A) \leq \left[\sum_{n=1}^{\infty} \mu_F^*(A_n)\right] + \varepsilon. \tag{4.11}$$

Letting $\varepsilon \to 0^+$, then (4.10) will follow.

Thus, let $\varepsilon > 0$ be given. By definition of μ_F^*, for each n there exists a collection $\{I_k^{(n)}\}_{k=1}^{\infty}$ of half-open intervals so that

$$A_n \subseteq \bigcup_{k=1}^{\infty} I_k^{(n)} \qquad \text{and} \qquad \sum_{k=1}^{\infty} L_F\left(I_k^{(n)}\right) \leq \mu_F^*(A_n) + \frac{\varepsilon}{2^n}.$$

We now have that

$$A = \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(n)}.$$

a countable union of half-open intervals, so that by definition of μ_F^*,

$$\mu_F^*(A) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} L_F\left(I_k^{(n)}\right) \leq \sum_{n=1}^{\infty} \left[\mu_F^*(A_n) + \frac{\varepsilon}{2^n}\right] = \left[\sum_{n=1}^{\infty} \mu_F^*(A_n)\right] + \varepsilon.$$

That is, (4.11) holds. Thus the assertion follows. ■

(R) Observe that in the above proof we don't have strict inequalities in general. For example, the statement

$$\mu_F^*(A) < \left[\sum_{n=1}^{\infty} \mu_F^*(A_n)\right] + \varepsilon$$

is incorrect when $\mu_F^*(A) = \infty$.

Unfortunately, μ_F^* is not yet a measure because it is not σ-additive. We will, however, see that its restriction to an appropriate σ-subalgebra of $\mathscr{P}(\mathbb{R})$ is indeed σ-additive. But first a definition.

Definition 4.2.1 Let Ω be any set. A function

$$\mu^* : \mathscr{P}(\Omega) \to [0,\infty]$$

satisfying

(OM1) $\mu^*(\emptyset) = 0$,

(OM2) μ^* is monotone: if $A \subseteq B$ then $u^*(A) \leq u^*(B)$,

(OM3) μ^* is σ-subadditive: $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$,

is called an *outer measure* on Ω.

(R) Properties (OM1) and (OM3) imply, using standard arguments, that every outer measure is also *subadditive*:

$$\text{(OM3')} \qquad \mu^*\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} \mu_*(A_n)$$

for all $N \in N$ and $A_1, \ldots, A_N \subseteq \Omega$.

(R) Given a distribution function $F : \mathbb{R} \to \mathbb{R}$, then μ_F^* defined as in (4.5) is an outer measure on $\mathbb{R}$, by Propositions B–D.

Above we have constructed an outer measure on $\Omega = \mathbb{R}$ by using "generalized lengths" of intervals. It turns out that a similar construction can be used for arbitrary spaces Ω:

Exercise 4.2 Let Ω be a non-empty set, $\mathscr{E} \subseteq \mathscr{P}(\Omega)$ with $\emptyset \in \mathscr{E}$ and $\Omega \in \mathscr{E}$. Given a function $\rho : \mathscr{E} \to [0,\infty]$ satisfying $\rho(\emptyset) = 0$, define $\mu^* : \mathscr{P}(\Omega) \to [0,\infty]$ by

$$\mu^*(A) := \inf\left\{\sum_{n=1}^{\infty} \rho(I_n) \; : \; I_n \in \mathscr{E},\, A \subseteq \bigcup_{n=1}^{\infty} I_n\right\} \qquad (A \subseteq \Omega).$$

Show that μ^* is an outer measure. Why do we require $\Omega \in \mathscr{E}$?
(Hint: Verify that the proofs of propositions C and D still hold.) ■

4.3 From Outer Measure to Measure

Let an outer measure μ on a set Ω be given. We will consider subsets E of Ω with the following special property:

Property (PM)

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \qquad \forall A \subseteq \Omega$$

(R) Note that by subadditivity, we always have

$$\mu^*(A) = \mu^*\left([A \cap E] \cup [A \cap E^c]\right) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c) \qquad \forall A, E \subseteq \Omega.$$

Therefore, Property (PM) is equivalent to:

Property (PM1)

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \qquad \forall A \subseteq \Omega$$

Let us set

$$\mathscr{F}_\mu = \{E \subseteq \Omega \,:\, E \text{ satisfies condition (PM)}\},$$

called the set of μ^*-measurable subsets of Ω.

Theorem 4.3.1 *(Carathéodory)* $\mathscr{F}_\mu$ is a σ-algebra, and μ^* is a measure on $(\Omega, \mathscr{F}_\mu)$.

Proof. Clearly, $\emptyset \in \mathscr{F}_\mu$ and $\Omega \in \mathscr{F}_\mu$. By (OM1) we thus only need to show:

(S1): If $E \in \mathscr{F}_\mu$ then $E^c \in \mathscr{F}_\mu$.
(S2): If $E_1, E_2, \cdots \in \mathscr{F}_\mu$ then $\bigcup_{n=1}^\infty E_n \in \mathscr{F}_\mu$.
(M2): μ^* is σ-additive on $\mathscr{F}_\mu$.

Proof of (S1): Let $E \in \mathscr{F}_\mu$ be given. That is,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \qquad \forall A \subseteq \Omega.$$

Since $(E^c)^c = E$, this identity becomes

$$\mu^*(A) = \mu^*\left(A \cap (E^c)^c\right) + \mu^*\left(A \cap E^c\right) \qquad \forall A \subseteq \Omega,$$

which shows that (PM) holds for E^c as well, that is, $E^c \in \mathscr{F}_\mu$.

Proof of (S2) and (M2): We proceed in stages.

Step 1: Let $E_1, E_2 \in \mathscr{F}_\mu$. We show that $E_1 \cup E_2 \in \mathscr{F}_\mu$.
In fact, let $A \subseteq \Omega$ be arbitrary. Then

$$\begin{aligned} \mu^*(A) &\leq \mu^*\left(A \cap [E_1 \cup E_2]\right) + \mu^*\left(A \cap [E_1 \cup E_2]^c\right) \\ &= \mu^*\left([A \cap E_1] \cup [A \cap E_1^c \cap E_2]\right) + \mu^*\left(A \cap E_1^c \cap E_2^c\right) \\ &\underset{\text{(OM3')}}{\leq} \mu^*\left(A \cap E_1\right) + \mu^*\left([A \cap E_1^c] \cap E_2\right) + \mu^*\left([A \cap E_1^c] \cap E_2^c\right) \\ &\underset{\text{(PM) holds for } E_2}{=} \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) \underset{\text{(PM) holds for } E_1}{=} \mu^*(A). \end{aligned} \tag{4.12}$$

This shows that (PM) holds for $E_1 \cup E_2$, and hence $E_1 \cup E_2 \in \mathscr{F}_\mu$.

Step 2: It now follows by induction that if $E_1, E_2, \ldots, E_N \in \mathscr{F}_\mu$, then $\bigcup_{n=1}^N E_n \in \mathscr{F}_\mu$. Furthermore by step 1, $\bigcap_{n=1}^N E_n = \left[\bigcup_{n=1}^N E_n^c\right]^c \in \mathscr{F}_\mu$ as well.

Step 3: Let $E_1, E_2 \in \mathscr{F}_\mu$ be *disjoint*, that is, $E_1 \cap E_2 = \emptyset$. Then $E_1^c \cap E_2 = E_2$, so that (4.12) yields

$$\mu^*(A) = \mu^*\left(A \cap E_1\right) + \mu^*\left(A \cap E_2\right) + \mu^*\left(A \cap [E_1 \cup E_2]^c\right) \qquad \forall A \subseteq \Omega. \tag{4.13}$$

Step 4: Let $E_1, E_2, \ldots \in \mathscr{F}_\mu$ be *pairwise disjoint*. Generalizing (4.13), we show using induction: For each $N \in \mathbb{N}$,

$$\mu^*(A) = \sum_{n=1}^{N} \mu^*(A \cap E_n) + \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} E_n\right]^c\right) \qquad \forall A \subseteq \Omega. \tag{4.14}$$

In fact, this assertion is trivial when $N = 1$. Next suppose that we have shown that (4.14) holds for some N. Since $\bigcup_{n=1}^{N} E_n$ and E_{N+1} are disjoint, then applying (4.13) we obtain for all $A \subseteq \Omega$ that

$$\mu^*(A) = \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} E_n\right]\right) + \mu^*(A \cap E_{N+1}) + \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} E_n \cup E_{N+1}\right]^c\right).$$

Now replacing A in (4.14) with $A \cap \left[\bigcup_{k=1}^{N} E_k\right]$, the first term on the right-hand side above changes to a sum,

$$\mu^*(A) = \sum_{n=1}^{N} \mu^*\left(A \cap \left[\bigcup_{k=1}^{N} E_k\right] \cap E_n\right) + \mu^*\left(A \cap \left[\bigcup_{k=1}^{N} E_k\right] \cap \left[\bigcup_{n=1}^{N} E_n\right]^c\right)$$
$$+ \mu^*(A \cap E_{N+1}) + \mu^*\left(A \cap \left[\bigcup_{n=1}^{N+1} E_n\right]^c\right)$$

That is,

$$\mu^*(A) = \sum_{n=1}^{N} \mu^*(A \cap E_n) + \mu^*(\emptyset) + \mu^*(A \cap E_{N+1}) + \mu^*\left(A \cap \left[\bigcup_{n=1}^{N+1} E_n\right]^c\right)$$

which shows that (4.14) holds for $N+1$ as well. By induction, (4.14) holds for all N.

Step 5: Let $E_1, E_2, \ldots \in \mathscr{F}_\mu$ be *pairwise disjoint*. We show:

$$\text{(i)} \quad E := \bigcup_{n=1}^{\infty} E_n \in \mathscr{F}_\mu \qquad \text{and} \qquad \text{(ii)} \quad \mu^*(E) = \sum_{n=1}^{\infty} \mu^*(E_n) \quad \text{(i.e. (M2) holds).}$$

To see this, let $A \subseteq \Omega$ be arbitrary. Then by step 4, for each $N \in \mathbb{N}$,

$$\mu^*(A) = \sum_{n=1}^{N} \mu^*(A \cap E_n) + \mu^*\left(A \cap \left[\bigcup_{n=1}^{N} E_n\right]^c\right). \tag{4.15}$$

Now since

$$E^c = \left[\bigcup_{n=1}^{\infty} E_n\right]^c \subseteq \left[\bigcup_{n=1}^{N} E_n\right]^c,$$

then from (4.15) we obtain by by monotonicity of μ^*,

$$\sum_{n=1}^{N} \mu^*(A \cap E_n) + \mu^*(A \cap E^c) \leq \mu^*(A)$$

for all N. As the terms in the sum are all non-negative, we may let $N \to \infty$ to obtain

$$\sum_{n=1}^{\infty} \mu^*(A \cap E_n) + \mu^*(A \cap E^c) \leq \mu^*(A). \tag{4.16}$$

Thus, by σ-subadditivity of μ^*,

$$\mu^*(A\cap E)+\mu^*(A\cap E^c) = \mu^*\left(\bigcup_{n=1}^{\infty}[A\cap E_n]\right)+\mu^*(A\cap E^c)$$
$$\leq \sum_{n=1}^{\infty}\mu^*(A\cap E_n)+\mu^*(A\cap E^c)\leq \mu^*(A).$$

That is, (PM1) holds for E, so that $E\in\mathscr{F}_\mu$.

In addition, choosing $A=E$ in (4.16) we obtain as $\mu^*(\emptyset)=0$ that

$$\mu^*(E)\leq\sum_{n=1}^{\infty}\mu^*(E_n)\leq\mu^*(E),$$

which shows that μ^* is σ-additive on $\mathscr{F}_\mu$.

Step 6: It is left to prove (S2) for arbitrary, not necessarily disjoint sets $E_1,E_2,\ldots\in\mathscr{F}_\mu$, that is, we must show that

$$E:=\bigcup_{n=1}^{\infty}E_n\in\mathscr{F}_\mu.$$

In fact, by Theorem 1.1.1 there exist disjoint subsets B_n of Ω so that

$$E=\bigcup_{n=1}^{\infty}E_n=\bigcup_{n=1}^{\infty}B_n, \tag{4.17}$$

and furthermore, by the construction in its proof,

$$B_1=E_1, \qquad B_n=E_n\setminus\bigcup_{k=1}^{n-1}E_k=E_n\cap\left[\bigcup_{k=1}^{n-1}E_k\right]^c \quad (n\geq 2).$$

It follows by step 2 that $B_n\in\mathscr{F}_\mu$ for all n. Now as the sets B_n are mutually disjoint, then by (4.17) and step 6, $E\in\mathscr{F}_\mu$ as well.

Thus the proof is complete. ■

For ease of notation, let us set $\mu(E):=\mu^*(E)$ for each $E\in\mathscr{F}_\mu$. It follows from the Theorem that $(\Omega,\mathscr{F}_\mu,\mu)$ is a measure space.

R Let $E\subseteq\Omega$ be such that $\mu^*(E)=0$. Then by monotonicity of μ^*, for each $A\subseteq\Omega$ we have

$$\mu^*(A\cap E)+\mu^*(A\cap E^c)\leq\mu^*(E)+\mu^*(A)=0+\mu^*(A)=\mu^*(A),$$

that is, (PM1) holds for E, so that $E\in\mathscr{F}_\mu$. Thus, every μ^*-null subset of Ω is measurable. In particular, if $F\in\mathscr{F}_\mu$ is a null set, and $E\subseteq F$, then by monotonicity of μ^*,

$$0\leq\mu^*(E)\leq\mu^*(F)=\mu(F)=0,$$

that is, E is also a μ^* null set, and hence $E\in\mathscr{F}_\mu$.

This shows that the measure space $(\Omega,\mathscr{F}_\mu,\mu)$ obtained from the outer measure μ^* is complete.

4.4 Lebesgue-Stieltjes Measures

Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function, $\mu^* = \mu_F^*$ the outer measure on $\mathbb{R}$ determined by F (as in (4.3) and (4.5)), and $\mathscr{F}_\mu$ the σ-algebra of μ^*-measurable subsets of $\mathbb{R}$, so that by the previous theorem, $(\mathbb{R}, \mathscr{F}_\mu, \mu)$ is a measure space, where μ denotes the restriction of $\mu^* = \mu_F^*$ to $\mathscr{F}_\mu$.

Theorem 4.4.1 $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{F}_\mu$.

Proof. Since $\mathscr{B}(\mathbb{R})$ is generated by the collection $\mathscr{I}_6$ of half-open intervals (see Exercise 1.3), it suffices to show that every finite, half-open interval $E = (a,b]$ is an element of $\mathscr{F}_\mu$.

Indeed, we will show that for each $A \subseteq \Omega$ and each $\varepsilon > 0$,

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) < \mu^*(A) + \varepsilon \tag{4.18}$$

Letting $\varepsilon \to 0^+$ then (PM1) will follow, so that $E \in \mathscr{F}_\mu$.

Thus, let $A \subseteq \Omega$ and $\varepsilon > 0$ be given. By definition of μ^*, there exist half-open intervals $I_n = (a_n, b_n]$ so that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} L_F(I_n) < \mu^*(A) + \varepsilon.$$

For each $n \in \mathbb{N}$, set

$$I_n^{(1)} = I_n \cap (-\infty, a], \qquad I_n^{(2)} = I_n \cap (a,b], \qquad I_n^{(3)} = I_n \cap (b, \infty),$$

the parts of I_n lying to the left of E, inside E, respectively to the right of E. Note that each of these intervals is again half-open, possibly empty, and

$$I_n = (a_n, b_n] = I_n^{(1)} \cup I_n^{(2)} \cup I_n^{(3)}, \tag{4.19}$$

a disjoint union.

Observe that

$$A \cap E = A \cap (a,b] \subseteq \bigcup_{n=1}^{\infty} I_n \cap (a,b] = \bigcup_{n=1}^{\infty} I_n^{(2)}$$

while also

$$A \cap E^c = A \cap \left[(-\infty, a] \cup (b, \infty)\right] \subseteq \left[\bigcup_{n=1}^{\infty} I_n\right] \cap \left[(-\infty, a] \cup (b, \infty)\right] = \left[\bigcup_{n=1}^{\infty} I_n^{(1)}\right] \cup \left[\bigcup_{n=1}^{\infty} I_n^{(3)}\right].$$

Thus by definition of $\mu^* = \mu_F^*$,

$$\mu^*(A \cap E) \le \sum_{n=1}^{\infty} L_F\left(I_n^{(2)}\right)$$

$$\mu^*(A \cap E^c) \le \sum_{n=1}^{\infty} L_F\left(I_n^{(1)}\right) + \sum_{n=1}^{\infty} L_F\left(I_n^{(3)}\right)$$

Because the union in (4.19) is a disjoint union of adjacent intervals, then by property (4.4) of L_F,

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \le \sum_{n=1}^{\infty} \left[L_F\left(I_n^{(1)}\right) + L_F\left(I_n^{(2)}\right) + L_F\left(I_n^{(3)}\right)\right] = \sum_{n=1}^{\infty} L_F(I_n) < \mu^*(A) + \varepsilon,$$

so that (4.18) holds. Thus, the theorem is proved. ■

R The measure $\mu = \mu_F$ discussed above is called the *Lebesgue-Stieltjes measure* determined by the distribution function F. The theorem states that μ is a Borel measure.

Proposition 4.4.2 *(Properties of the Lebesgue-Stieltjes measure)*

1. If $\{a\}$ is a singleton, then
$$\mu_F(\{a\}) = 0 \quad \Leftrightarrow \quad F \text{ is continuous at } a.$$
2. μ_F is a σ-finite measure.
3. μ_F is a finite measure $\quad \Leftrightarrow \quad F$ is bounded.

Proof. 1. Since F is right continuous by definition, we only need to consider left-continuity of F at a. Thus, let $(x_n) \uparrow$ be a sequence in $\mathbb{R}$ with $x_n \to a$. Since bounded intervals have finite measure, then by Theorem 1.4.2,
$$\begin{aligned}\mu_F(\{a\}) = \mu_F\left(\bigcap_{n=1}^{\infty}(x_n, a]\right) &\underset{\text{Thm 1.4.2}}{=} \lim_{n\to\infty} \mu_F\big((x_n, a]\big)\\ &\underset{\text{Prop A}}{=} \lim_{n\to\infty}\big[F(a) - F(x_n)\big] = F(a) - F(a^-).\end{aligned}$$
It follows that
$$\mu_F(\{a\}) = 0 \quad \Leftrightarrow \quad F(a^-) = F(a) \quad \Leftrightarrow \quad F \text{ is continuous at } a.$$

2. Since $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n]$ and $0 \le \mu_F\big((-n, n]\big) = F(n) - F(-n) < \infty$ for each n, it follows that μ_F is σ-finite.
3. Now since F is increasing, then $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$ both exist in R^*, and
$$F \text{ is bounded above in } \mathbb{R} \quad \Leftrightarrow \quad \lim_{x\to\infty} F(x) \text{ is finite} \quad \Leftrightarrow \quad \lim_{n\to\infty} F(n) \text{ is finite.}$$
Similarly,
$$F \text{ is bounded below in } \mathbb{R} \quad \Leftrightarrow \quad \lim_{x\to-\infty} F(x) \text{ is finite} \quad \Leftrightarrow \quad \lim_{n\to\infty} F(-n) \text{ is finite.}$$
Now
$$\begin{aligned}\mu_F(\mathbb{R}) = \mu_F\big((-\infty, 0]\big) + \mu_F\big((0, \infty)\big) &= \mu_F\left(\bigcup_{n=1}^{\infty}(-n, 0]\right) + \mu_F\left(\bigcup_{n=1}^{\infty}(0, n)\right)\\ &\underset{\text{Thm 1.4.2}}{=} \lim_{n\to\infty}\mu_F\big((-n, 0]\big) + \lim_{n\to\infty}\mu_F\big((0, n)\big)\\ &\ge \lim_{n\to\infty}\mu_F\big((-n, 0]\big) + \lim_{n\to\infty}\mu_F\big((0, n-1]\big)\\ &= \lim_{n\to\infty}[F(0) - F(-n)] + \lim_{n\to\infty}[F(n-1) - F(0)] = \lim_{n\to\infty} F(n) - \lim_{n\to\infty} F(-n)\end{aligned}$$
which shows that
$$\mu_F(\mathbb{R}) < \infty \quad \Leftrightarrow \quad \lim_{n\to\infty} F(n) \text{ and } \lim_{n\to\infty} F(-n) \text{ are both finite} \quad \Leftrightarrow \quad F \text{ is bounded.}$$

■

R Observe that by the proof of the last part,
$$\mu_F(\mathbb{R}) = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x). \tag{4.20}$$

■ **Example 4.1** Let $F(x) = x$. The outer measure μ_F^* determined by this distribution function is called the *Lebesgue outer measure* and denoted by λ^*. The corresponding measure μ_F is called the *Lebesgue measure* and denoted by λ, while the σ-algebra $\mathscr{F}_\mu$ is denoted by $\mathscr{M}$ and called the *σ-algebra of Lebesgue measurable sets*. Furthermore, $\mathscr{M}$-measurable functions are called *Lebesgue-measurable functions*.

Since $F(x) = x$ is continuous, then singletons have zero measure, hence by σ-additivity, countable subsets of $\mathbb{R}$ are null sets. Furthermore,

$$\lambda\big((a,b]\big) = \mu_F\big((a,b]\big) = L_F(a,b] = F(b) - F(a) = b - a.$$

For arbitrary bounded intervals with endpoints $a < b$ we have

$$\begin{aligned}
\lambda\big([a,b]\big) &= \lambda\big(\{a\}\big) + \lambda\big([a,b]\big) = 0 + \lambda\big([a,b]\big) = b - a,\\
\lambda\big((a,b)\big) &= \lambda\big([a,b]\big) - \lambda\big(\{b\}\big) = \lambda\big([a,b]\big) - 0 = b - a,\\
\lambda\big([a,b)\big) &= \lambda\big(\{a\}\big) + \lambda\big((a,b]\big) = 0 + \lambda\big((a,b]\big) = b - a.
\end{aligned}$$

Thus, the Lebesgue measure of any bounded interval coincides with its length. ■

Exercise 4.3 Observe that for all bounded open intervals $I = (a,b)$,

$$\begin{aligned}
&\lambda(y+I) = \lambda\big((y+a,y+b)\big) = (y+b) - (y+a) = b - a = \lambda\big((a,b)\big) = \lambda(I) \quad (y \in \mathbb{R}),\\
&\lambda(-I) = \lambda\big((-b,-a)\big) = (-a) - (-b) = b - a = \lambda(I),\\
&\lambda(\alpha I) = \lambda\big((\alpha a, \alpha b)\big) = (\alpha b - \alpha a) = \alpha(b-a) = \alpha\lambda(I) \quad (\alpha > 0).
\end{aligned}$$

1. Let $E \subseteq R$. Show:
   a) $E \in \mathscr{M} \quad \Leftrightarrow \quad y + E \in \mathscr{M}$
   b) $E \in \mathscr{M} \quad \Leftrightarrow \quad -E \in \mathscr{M}$
   c) $E \in \mathscr{M} \quad \Leftrightarrow \quad \alpha E \in \mathscr{M} \quad (\alpha > 0)$
2. Show that for all $E \in \mathscr{M}$,
   $\lambda(y+E) = \lambda(E) \quad (y \in \mathbb{R})$ (λ is translation invariant)
   $\lambda(-E) = \lambda(E)$ (λ is reflection invariant)
   $\lambda(\alpha E) = \alpha\lambda(E) \quad (\alpha > 0)$ (λ is positive homogeneous).
   (In fact these properties hold for the Lebesgue outer measure λ^* and all $E \subseteq \mathbb{R}$.)
3. Given $f : \mathbb{R} \to \mathbb{R}$, let us define functions $f_y, f^*, {}_\alpha f$ by
   $f_y(x) = f(x-y)$ ($y \in \mathbb{R}$ fixed)
   $f^*(x) = f(-x)$
   ${}_\alpha f(x) = f(\alpha x)$ ($\alpha > 0$ fixed).
   Show:
   a) f is Lebesgue measurable iff f_y is Lebesgue measurable.
   b) f is Lebesgue measurable iff f^* is Lebesgue measurable.
   c) f is Lebesgue measurable iff ${}_\alpha f$ is Lebesgue measurable.
   Furthermore,
   a) $\int f\,d\lambda$ is defined iff $\int f_y\,d\lambda$ is defined, in which case $\int f\,d\lambda = \int f_y\,d\lambda$.
   b) $\int f\,d\lambda$ is defined iff $\int f^*\,d\lambda$ is defined, in which case $\int f\,d\lambda = \int f^*\,d\lambda$.
   c) $\int f\,d\lambda$ is defined iff $\int {}_\alpha f\,d\lambda$ is defined, in which case $\int f\,d\lambda = \alpha \int {}_\alpha f\,d\lambda$.

■ **Example 4.2** Fix $a < b$, and let

$$F(x) = \frac{1}{b-a} \int_{(-\infty,x]} \mathbf{1}_{(a,b]}\, d\lambda = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b. \end{cases}$$

Clearly, $F(x)$ is a distribution function, called the ***uniform distribution***. Furthermore, the corresponding measure μ_F is a probability measure (i.e. $\mu(\mathbb{R}) = 1$) by (4.20). ■

■ **Example 4.3** Fix a strictly increasing sequence of points in $\mathbb{R}$, $a_1 < a_2 < a_3 < \dots$, and fix a sequence $\{p_n\}_{n=1}^{\infty}$ of non-negative numbers with $\sum_{n=1}^{\infty} p_n = 1$. Now set

$$F(x) = \sum_{n=1}^{\infty} p_n \mathbf{1}_{[a_n,\infty]}.$$

Note that

$$F(x) = \begin{cases} 0, & x < a_1 \\ \sum_{n=1}^{m} p_n, & a_m \le x < a_{m+1} \\ 1, & x \ge \lim_{m\to\infty} a_m \text{ (if this limit is finite).} \end{cases}$$

It is easy to see that F is a distribution function. Furthermore, as

$$\lim_{x\to\infty} F(x) = \lim_{m\to\infty} \sum_{n=1}^{m} p_n = 1,$$

then by (4.20), the corresponding measure μ_F is a probability measure.

In the special case where $p_n = \frac{\lambda^n}{n!} e^{-\lambda}$ for some $\lambda > 0$, then F is called the *Poisson distribution function*. ■

(R) The map $F \to \mu_F$ is not one-to-one. For if F is a distribution function and c is a constant, then $\tilde{F} := F + c$ clearly is also a distribution function. Furthermore, as for all half-open intervals $(a,b]$,

$$L_{\tilde{F}}(a,b] = \tilde{F}(b) - \tilde{F}(a) = [F(b)+c] - [F(a)+c] = F(b) - F(a) = L_F(a,b],$$

then by definition of the outer measure,

$$\mu^*_{\tilde{F}} = \mu^*_F$$

so that

$$\mathscr{F}_{\mu_{\tilde{F}}} = \mathscr{F}_{\mu_F} \qquad \text{and} \qquad \mu_{\tilde{F}} = \mu_F.$$

However this is the only possibility to obtain the same measure as μ_F. To see this, let F and $\tilde{F}$ be two distribution functions whose measures μ_F and $\mu_{\tilde{F}}$ coincide on the Borel sets. Then for all $x \ge 0$,

$$\tilde{F}(x) - \tilde{F}(0) = \mu_{\tilde{F}}\big((0,x]\big) = \mu_F\big((0,x]\big) = F(x) - F(0), \tag{4.21}$$

while for $x < 0$,

$$\tilde{F}(x) - \tilde{F}(0) = -\mu_{\tilde{F}}\big((x,0]\big) = -\mu_F\big((x,0]\big) = F(x) - F(0). \tag{4.22}$$

It follows that

$$\tilde{F} = F + c, \qquad \text{where } c = \tilde{F}(0) - F(0).$$

For this reason, one can normalize F:

1. When F is bounded, i.e. μ_F is a finite measure, one can choose c so that

$$\lim_{x\to-\infty} F(x) = 0.$$

This type of normalization is chosen in probability theory. Then for all $x \in \mathbb{R}$,

$$\begin{aligned} F(x) &= F(x) - 0 = F(x) - \lim_{n\to\infty} F(-n) = \lim_{n\to\infty} [F(x) - F(-n)] \\ &= \lim_{n\to\infty} \mu_F((-n,x]) \underset{\text{Thm 1.4.2}}{=} \mu_F\left(\bigcup_{n=1}^{\infty}(-n,x]\right) = \mu_F((-\infty,x]). \end{aligned}$$

2. In the general case, one can choose c so that $F(0) = 0$. Then as shown on the right sides of (4.21) and (4.22),

$$F(x) = \begin{cases} \mu_F((0,x]), & x \geq 0 \\ -\mu_F((x,0]), & x < 0. \end{cases}$$

Another question still remains: If μ and $\tilde{\mu}$ are two measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ having the same distribution function, will then necessarily $\mu = \tilde{\mu}$? The answer is affirmative, as a consequence of the following Theorem.

Theorem 4.4.3 Let μ be a measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ which is finite on bounded sets. Furthermore, let F be the distribution function determined by μ as in (4.2). Then $\mu(E) = \mu_F(E)$ for all $E \in \mathscr{B}(\mathbb{R})$.

Proof. First some observations:

1. Let $\{I_k\}_{k=1}^{n}$ be finite collection of half-open intervals, $I_k = (a_k, b_k]$. Since the union of overlapping half-open intervals is again a half-open interval of the same type, we may assume that the intervals I_k are mutually disjoint. Now as μ and μ_F are Borel measures, then

$$\mu\left(\bigcup_{k=1}^{n} I_k\right) = \sum_{k=1}^{n} \mu(I_k) \underset{\text{(D5)}}{=} \sum_{k=1}^{n} L_F(I_k) = \sum_{k=1}^{n} \mu_F^*(I_k) = \sum_{k=1}^{n} \mu_F(I_k) = \mu_F\left(\bigcup_{k=1}^{n} I_k\right).$$

2. Thus if $\{I_n\}_{n=1}^{\infty}$ is any countably infinite collection of half-open intervals, $I_n = (a_n, b_n]$, then

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} I_n\right) &= \mu\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} I_k\right) \underset{\text{Thm 1.4.2}}{=} \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} I_k\right) = \lim_{n\to\infty} \mu_F\left(\bigcup_{k=1}^{n} I_k\right) \\ &\underset{\text{Thm 1.4.2}}{=} \mu_F\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} I_k\right) = \mu_F\left(\bigcup_{n=1}^{\infty} I_n\right). \end{aligned} \tag{4.23}$$

3. Now let $E \in \mathscr{B}(\mathbb{R})$ be given. Then for every cover $\{I_n\}_{n=1}^{\infty}$ of E by half-open intervals, $I_n = (a_n, b_n]$, we have by monotonicity and σ-subadditivity of μ,

$$\mu(E) \leq \mu\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} \mu(I_n) \underset{\text{(D5)}}{=} \sum_{n=1}^{\infty} [F(b_n) - F(a_n)] = \sum_{n=1}^{\infty} L_F(I_n).$$

It thus follows from the definition of $\mu(E) = \mu_F^*(E)$ that

$$\mu(E) \leq \mu_F(E) \qquad \forall E \in \mathscr{B}(\mathbb{R}). \tag{4.24}$$

Now to prove the reverse inequality, first suppose that $E \in \mathscr{B}(\mathbb{R})$ is such that $\mu_F(E) < \infty$. By definition of $\mu_F(E) = \mu_F^*(E)$, for each $\varepsilon > 0$ there exists a collection of half-open intervals $\{I_n\}_{n=1}^{\infty}$, $I_n = (a_n, b_n]$, covering E with

$$\sum_{n=1}^{\infty} L_F(I_n) < \mu_F(E) + \varepsilon.$$

Let us set $A = \bigcup_{n=1}^{\infty} I_n$. Then as $E \subseteq A$ and $\mu_F = \mu_F^*$ is a measure on $\mathscr{B}(\mathbb{R})$, we have by subadditivity,

$$\mu_F(A \backslash E) = \mu_F(A) - \mu_F(E) \leq \sum_{n=1}^{\infty} \mu_F(I_n) - \mu_F(E) = \sum_{n=1}^{\infty} L_F(I_n) - \mu_F(E) < \varepsilon.$$

Hence,

$$\mu_F(E) \leq \mu_F(A) \underset{(4.23)}{=} \mu(A) = \mu(E) + \mu(A \backslash E) \leq \mu(E) + \mu_F^*(A \backslash E) < \mu(E) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, it follows that

$$\mu_F(E) \leq \mu(E).$$

Together with (4.24) we obtain that

$$\mu_F(E) = \mu(E) \qquad \forall E \in \mathscr{B}(\mathbb{R}).$$

Now if $E \in \mathscr{B}(\mathbb{R})$ is arbitrary, set $E_n = E \cap (n, n+1]$ for each $n \in \mathbb{Z}$. Since $\{E_n\}_{n \in \mathbb{Z}}$ is a collection of disjoint Borel sets of finite measure, then by the above and σ-additivity,

$$\mu_F(E) = \mu_F\Big(\bigcup_{n \in \mathbb{Z}} E_n\Big) = \sum_{n \in \mathbb{Z}}^{\infty} \mu_F(E_n) = \sum_{n \in \mathbb{Z}} \mu(E_n) = \mu\Big(\bigcup_{n \in \mathbb{Z}} E_n\Big) = \mu(E).$$

Thus, the theorem is proved. ■

(R) It follows immediately that

1. there is a one-to-one correspondence between finite Borel measures on $\mathbb{R}$ and bounded distribution functions F satisfying $\lim_{x \to -\infty} F(x) = 0$, and
2. there is a one-to-one correspondence between Borel measures on $\mathbb{R}$ which are finite on bounded sets and distribution functions F satisfying $F(0) = 0$.

We note that Borel measures on $\mathbb{R}^n$ can be constructed by following the general procedure outlined above.

4.5 Regularity

Recall that the Borel σ-algebra on $\mathbb{R}^d$ is generated by the open sets, and hence also by the closed sets. While in general it is not possible to describe all the Borel sets, we nevertheless are interested in measures where this is not an obstacle, the regular Borel measures. Loosely speaking, these measures allow one to arbitrarily approximate any Borel set by an open set, respectively a closed set in terms of measure.

Furthermore, any sensible Borel measure on $\mathbb{R}^d$ should be finite on bounded sets. Since every bounded set is contained in a compact set, this property can also be expressed as compact sets having finite measure. There are Borel measures which do not possess this property, for example the counting measure. In fact, the counting measure is not a natural measure on $\mathbb{R}$ as both, the unit interval $I = [0,1]$ as well as the set of rationals in this interval, $I \cap \mathbb{Q}$, have the same infinite measure, whereas the two sets have different cardinalities: I is uncountable, while $I \cap \mathbb{Q}$ is countable.

The above concepts can be made precise as follows:

Definition 4.5.1 Let μ be a measure on $(\mathbb{R}^d, \mathscr{F})$, where $\mathscr{F}$ is a σ-algebra on $\mathbb{R}^d$ containing all Borel sets. Then μ is called *regular* if its satisfies:
(R1) $\mu(K) < \infty$ for all *compact* subsets K of $\mathbb{R}^d$,
(R2) for each Borel set A,

$$\mu(A) = \inf\{\, \mu(V) \,|\, A \subset V,\ V \text{ is open} \,\} \qquad \text{"outer regularity"},$$

(R3) for each Borel set A,

$$\mu(A) = \sup\{\, \mu(K) \,|\, K \subset A,\ K \text{ is compact} \,\} \qquad \text{"inner regularity"}$$

R One can define the concept of regularity for general measurable spaces of the form $(\Omega, \mathscr{F})$, where Ω is a topological space and $\mathscr{F}$ a σ-algebra containing the Borel subsets of Ω. In this case, one often requires inner regularity in (R3) to apply to *open* sets A only.

■ **Example 4.4** As already stated, the counting measure μ_c on $\mathbb{R}$ does not satisfy (R1). It is not outer regular either: Any nonempty open set V is uncountable and thus has infinite measure, so that for every Borel set $A \neq \emptyset$,

$$\inf\{\, \mu_c(V) \,|\, A \subset U,\ V \text{ is open} \,\} = \infty.$$

It follows that outer regularity does not apply to finite sets A. On the other hand, μ_c is inner regular as the reader may easily verify. ■

First a lemma which will be needed later.

Lemma 4.5.1 Let μ be a finite measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ Then each $A \in \mathscr{B}(\mathbb{R}^d)$ has the following property:
Property (P): For every $\varepsilon > 0$ there exist a closed set F and an open set U so that
1. $F \subseteq A \subseteq U$, and
2. $\mu(U \backslash F) < \varepsilon$.

Proof. Let us first set

$$\mathscr{F} = \left\{ A \in \mathscr{B}(\mathbb{R}^d) \;\middle|\; A \text{ satisfies property (P)} \right\}.$$

Step 1: We claim that $\mathscr{F}$ is a σ-algebra.

In fact, clearly $\emptyset \in \mathscr{F}$ and $\mathbb{R}^d \in \mathscr{F}$ as these are both open and closed, so that in particular, $\mathscr{F} \neq \emptyset$.

Next let $A \in \mathscr{F}$ be arbitrary. Given $\varepsilon > 0$, we choose F and U as in property (P). Then taking complements,

$$U^c \subseteq A^c \subseteq F^c$$

with U^c closed and F^c open. Now note that

$$F^c \backslash U^c = F^c \cap [U^c]^c = U \cap F^c = U \backslash F$$

and hence,

$$\mu(F^c \backslash U^c) = \mu(U \backslash F) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and $\mathscr{B}(\mathbb{R}^d)$ is a σ-algebra, it follows that $A^c \in \mathscr{F}$ also.

Now let $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$. Then by property (P), given $\varepsilon > 0$, for each $n \in \mathbb{N}$ there exist a closed set F_n and an open set U_n with

$$F_n \subseteq A_n \subseteq U_n \qquad \text{and} \qquad \mu(U_n \backslash F_n) < \frac{\varepsilon}{2^n}.$$

We set

$$A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{B}(\mathbb{R}^d), \qquad U = \bigcup_{n=1}^{\infty} U_n \qquad \text{and} \qquad \tilde{F} = \bigcup_{n=1}^{\infty} F_n.$$

Then clearly, $\tilde{F} \subseteq A \subseteq U$ and also

$$U \backslash \tilde{F} = \left[\bigcup_{n=1}^{\infty} U_n \right] \backslash \tilde{F} = \bigcup_{n=1}^{\infty} U_n \backslash \tilde{F} \subseteq \bigcup_{n=1}^{\infty} U_n \backslash F_n,$$

so that by σ-subadditivity,

$$\mu(U \backslash \tilde{F}) \leq \sum_{n=1}^{\infty} \mu\left(U_n \backslash F_n\right) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

While the set U is clearly open, the set $\tilde{F}$ need not be closed. Note however that the sequence of sets $\left\{U \backslash \bigcup_{n=1}^{N} F_n\right\}_{N=1}^{\infty}$ is decreasing, hence by Theorem 1.4.2,

$$\mu(U \backslash \tilde{F}) = \lim_{N \to \infty} \mu\left(U \backslash \bigcup_{n=1}^{N} F_n\right).$$

Hence choosing N sufficiently large and setting $F = \bigcup_{n=1}^{N} F_n$, a closed set, we still have

$$\mu(U \backslash F) < \varepsilon$$

with $F \subseteq \tilde{F} \subseteq A \subseteq U$. This shows that $A \in \mathscr{F}$ and proves the claim.

Step 2: We show that $\mathscr{F}$ contains all nonempty open, bounded d-intervals of the form

$$A = \prod_{i=1}^{d} (a_i, b_i).$$

In fact, note that

$$A = \bigcup_{n=1}^{\infty} I_n, \qquad \text{where} \qquad I_n = \prod_{i=1}^{d} \left[a_i + \frac{1}{n}, b_i - \frac{1}{n}\right] \text{ is closed and } \{I_n\} \uparrow.$$

Now by Theorem 1.4.2 again,

$$\mu(A) = \lim_{n \to \infty} \mu(I_n).$$

That is, choosing n sufficiently large we have

$$\mu(A) - \varepsilon < \mu(I_n).$$

We therefore may set $F = I_n$ and $U = A$ so that $F \subseteq A \subseteq U$ while also

$$\mu(U \backslash F) = \mu(A \backslash I_n) = \mu(A) - \mu(I_n) < \varepsilon;$$

hence it follows that $A \in \mathscr{F}$. This shows that $\mathscr{F}$ contains all the open, bounded d-intervals.

Finally, since $\mathscr{B}(\mathbb{R}^d)$ is generated by these open d-intervals (see Corollary 2.4.2) we conclude that $\mathscr{B}(\mathbb{R}^d) \subseteq \mathscr{F}$, and the assertion of the lemma follows. ■

The next theorem applies in particular to the Lebesgue measure:

Theorem 4.5.2 Let μ be measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}^d))$ which is finite on compact sets. Then μ is regular.

Proof. By assumption, (R1) holds already.

As we want to apply the previous lemma to prove inner and outer regularity, we need to modify μ to a finite measure. Thus for each $n \in \mathbb{N}$, we set

$$\mu_n(E) = \mu(E \cap I_n) \qquad \forall E \in \mathscr{B}(\mathbb{R}^d),$$

where

$$I_n = \prod_{i=1}^{d}(-n,n) = (-n,n) \times (-n,n) \times \cdots \times (-n,n)$$

is an open, bounded d-interval. Clearly, μ_n is a finite Borel measure on $\mathbb{R}^d$.

Now let $A \in \mathscr{B}(\mathbb{R}^d)$ be given. For each n, we set

$$A_n = A \cap I_n,$$

so that $\{A_n\} \uparrow$ and $A = \bigcup_{n=1}^{\infty} A_n$.

Next let $\varepsilon > 0$ be given. Applying the previous lemma, for each n there exist a closed set F_n and an open set U_n so that

$$F_n \subseteq A_n \subseteq U_n, \qquad \text{and} \qquad \mu_n\big(U_n \backslash F_n\big) < \frac{\varepsilon}{2^n}. \tag{4.25}$$

Replacing U_n with $U_n \cap I_n$, then (4.25) still holds, and in addition, $U_n \subseteq I_n$. Note also that the sets F_n are bounded, and hence compact. Let us set

$$F = \bigcup_{n=1}^{\infty} F_n \qquad \text{and} \qquad U = \bigcup_{n=1}^{\infty} U_n.$$

Then U is open and by (4.25), $F \subseteq A \subseteq U$ while also

$$U \backslash F = \left[\bigcup_{n=1}^{\infty} U_n\right] \backslash F = \bigcup_{n=1}^{\infty} U_n \backslash F \subseteq \bigcup_{n=1}^{\infty} U_n \backslash F_n,$$

so that by σ-subadditivity and since $U_n \subseteq I_n$,

$$\mu(U \backslash F) \le \sum_{n=1}^{\infty} \mu\big(U_n \backslash F_n\big) = \sum_{n=1}^{\infty} \mu_n\big(U_n \backslash F_n\big) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

It follows that

$$\mu(U) \le \mu(A) + \mu(U \backslash A) \le \mu(A) + \mu(U \backslash F) \le \mu(A) + \varepsilon \tag{4.26}$$

and also

$$\mu(A) \le \mu(F) + \mu(A \backslash F) \le \mu(F) + \mu(U \backslash F) \le \mu(F) + \varepsilon. \tag{4.27}$$

(Note that on the right-hand sides we don't have strict inequality as the sets A and F may have infinite measure.)

From (4.26) we obtain that

$$\inf\{\,\mu(V)\,|\,A\subset V,\ V \text{ is open}\,\}\le \mu(A)+\varepsilon.$$

But as $\varepsilon>0$ was arbitrary, then

$$\inf\{\,\mu(V)\,|\,A\subset V,\ V \text{ is open}\,\}\le \mu(A).$$

Since the reverse inequality is obvious and $A\in\mathscr{B}(\mathbb{R}^d)$ was arbitrary, then outer regularity of μ follows.

Now as the set F may not be compact, we modify inequality (4.27) similar to what was done in the proof of the previous lemma: For each n, set

$$K_n=\bigcup_{k=1}^{n}F_k.$$

Then each K_n is compact, $K_n\subseteq A_n$, $\{K_n\}\uparrow$, and $F=\bigcup_{n=1}^{\infty}K_n$, so that

$$\big(\mu(K_n)\big)\uparrow \qquad \text{and} \qquad \mu(K_n)\to\mu(F)\quad \text{as } n\to\infty. \tag{4.28}$$

Assume first that $\mu(A)$ is finite, so that $\mu(F)$ is finite as well. Thus, when n is sufficiently large,

$$\mu(F)<\mu(K_n)+\varepsilon$$

and hence by (4.27),

$$\mu(A)<\mu(K_n)+2\varepsilon.$$

It follows that

$$\mu(A)<\sup\{\,\mu(K)\,|\,K\subset A,\ K \text{ is compact}\,\}+2\varepsilon.$$

As $\varepsilon>0$ was arbitrary, then

$$\mu(A)\le\sup\{\,\mu(K)\,|\,K\subset A,\ K \text{ is compact}\,\}.$$

Again, since the reverse inequality is obvious, then

$$\mu(A)=\sup\{\,\mu(K)\,|\,K\subset A,\ K \text{ is compact}\,\}. \tag{4.29}$$

On the other hand, when $\mu(A)=\infty$ then by (4.27), $\mu(F)=\infty$ as well. Furthermore, by (4.28),

$$\sup\{\,\mu(K)\,|\,K\subset A,\ K \text{ is compact}\,\}\ge\sup\{\,\mu(K_n)\,|\,n\in\mathbb{N}\,\}=\lim_{n\to\infty}\mu(K_n)=\mu(F)=\infty,$$

from which (4.29) follows. Since $A\in\mathscr{B}(\mathbb{R}^d)$ was arbitrary, we have proved inner regularity of μ, and the proof of the theorem is complete. ■

Corollary 4.5.3 Let μ_F be the Borel measure on $\mathbb{R}$ determined by a distribution function $F : \mathbb{R} \to \mathbb{R}$, Then μ_F is regular.

(R) Theorem 4.5.2 generalizes to locally compact spaces Ω which have a countable base, with essentially the same proof.

In the above, we have considered the regularity properties for elements of the Borel σ-algebra only. However, many measures are defined and regular on larger σ-algebras:

Exercise 4.4 Recall from the discussion of Section 4.4 that every distribution function $F : \mathbb{R} \to \mathbb{R}$ gives rise to a Borel measure $\mu = \mu_F$ on $\mathbb{R}$ which is finite on bounded sets. However, the σ-algebra $\mathscr{F}_u$ of μ-measurable sets may be larger than the Borel σ-algebra. Our goal is to show that μ is regular on $\mathscr{F}_\mu$, without using Lemma 4.5.1.

1. Show that for all $A \subseteq \mathbb{R}$,

$$\mu_F^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \mu(J_n) \mid J_n = (a_n, b_n),\ A \subseteq \bigcup_{n=1}^{\infty} J_n \right\}.$$

2. Show that for all $A \subseteq \mathbb{R}$,

$$\mu_F^*(A) := \inf \left\{ \mu(U) \mid U \text{ is open, } A \subseteq U \right\}.$$

3. Show that for all bounded sets $A \in \mathscr{F}_\mu$,

$$\mu_F(A) := \sup \left\{ \mu(K) \mid K \text{ is compact, } K \subseteq A \right\}.$$

4. Show that for all sets $A \in \mathscr{F}_\mu$,

$$\mu_F(A) := \sup \left\{ \mu(K) \mid K \text{ is compact, } K \subseteq A \right\}.$$

5. Show that every $A \in \mathscr{F}_\mu$ is of the form $A = V \bigcap N$ where V is a G_δ set, and N a μ-null set in $\mathscr{F}_\mu$.
6. Show that every $A \in \mathscr{F}_\mu$ is of the form $A = K \bigcup N$ where K is a F_σ set, and N a μ-null set in $\mathscr{F}_\mu$.

5. Advanced Properties

5.1 Modes of Convergence

Let (f_n) be a sequence of integrable functions. We have already seen two types of convergence determined by the measure: *Almost everywhere convergence* and *convergence in the p-th mean*. In this section we will introduce an even weaker notion, that of *convergence in measure*, and discuss the relationship between the various modes of convergence.

As usual, we fix a measure space $(\Omega, \mathscr{F}, \mu)$. Furthermore, we let (f_n) be a sequence of $\mathscr{F}$-measurable functions, $f_n : \Omega \to \mathbb{K}$ and $f : \Omega \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We are already familiar with the following *modes of convergence*:

1. *pointwise convergence*:

$$f_n \longrightarrow f \quad \Leftrightarrow \quad \lim_{n\to\infty} f_n(\omega) = f(\omega) \quad \forall \omega \in \Omega.$$

In this case, f is also $\mathscr{F}$measurable by Theorem 1.5.7

2. *a.e. convergence*:

$$f_n \longrightarrow f \text{ a.e.} \quad \Leftrightarrow \quad \exists N \in \mathscr{F},\ \mu(N) = 0 \text{ s.t. } \lim_{n\to\infty} f_n(\omega) = f(\omega) \quad \forall w \in N^c.$$

In this case, f is equal a.e. to a measurable function (so we may assume f to be measurable) by Remark 1.5.

3. *uniform convergence*:

$$f_n \rightrightarrows f \quad \Leftrightarrow \quad \lim_{n\to\infty} \sup_{\omega\in\Omega} |f_n(\omega) - f(\omega)| = 0.$$

Obviously, 3. $\Rightarrow$ 1. $\Rightarrow$ 2.

4. *convergence in the p-th mean*: Suppose, $f_n, f \in L^p(\Omega, \mathscr{F}, \mu)$, $1 \le p \le \infty$. Then

$$f_n \xrightarrow{\|\cdot\|_p} f \quad \Leftrightarrow \quad \lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Clearly, when $p = \infty$, then 3. $\to$ 4.

Definition 5.1.1 Let $f_n,\ f:\Omega\to\mathbb{K}$ be $\mathscr{F}$-measurable functions. We say that $\{f_n\}$ *converges to f in measure* and write

$$f_n \overset{\text{meas}}{\longrightarrow} f$$

if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\big(\{\omega\in\Omega : |f_n(\omega)-f(\omega)|>\varepsilon\}\big)=0.$$

That is,

$$f_n \overset{\text{meas}}{\longrightarrow} f \quad \Leftrightarrow \quad \forall\varepsilon>0 \quad \mu\big(E_{n,\varepsilon}\big)\to 0 \quad \text{as } n\to\infty,$$

where $E_{n,\varepsilon}=\{\omega\in\Omega : |f_n(\omega)-f(\omega)|>\varepsilon\}$.

Exercise 5.1 Show:

1. In the above definition, we may replace "$|f_n(\omega)-f(\omega)|>\varepsilon$" with "$|f_n(\omega)-f(\omega)|\geq\varepsilon$".
2. Let $f_n \overset{\text{meas}}{\longrightarrow} f$ and let $g:\Omega\to\mathbb{K}$ be $\mathscr{F}$-measurable with $f(\omega)=g(\omega)$ a.e. Show that $f_n \overset{\text{meas}}{\longrightarrow} g$ as well.
3. The limit in measure is essentially unique: If f,g are $\mathscr{F}$-measurable and
$$f_n \overset{\text{meas}}{\longrightarrow} f \qquad \text{and} \qquad f_n \overset{\text{meas}}{\longrightarrow} g$$
then $f(\omega)=g(\omega)$ a.e.
4. Let $f_n, f, g_n, g:\Omega\to\mathbb{K}$ be $\mathscr{F}$-measurable, and α,β real numbers. If
$$f_n \overset{\text{meas}}{\longrightarrow} f \qquad \text{and} \qquad g_n \overset{\text{meas}}{\longrightarrow} g$$
then
$$\alpha f_n+\beta g_n \overset{\text{meas}}{\longrightarrow} \alpha f+\beta g.$$

■

■ **Example 5.1** Consider the measure space $(\mathbb{R},\mathscr{B}(\mathbb{R}),\lambda)$. We consider various sequences (f_n) of $\mathscr{B}(\mathbb{R})$-measurable functions.

1. Let $f_n=\frac{1}{n}\mathbf{1}_{(0,n)}$.
   (a) *(Uniform convergence)* Since for all $x\in\mathbb{R}$, $|f_n(x)-0|\leq\frac{1}{n}\to 0$ then $f_n\rightrightarrows 0$ on $\mathbb{R}$.
   (b) *(Convergence in the p-th mean)* Obviously, $f_n\in L^p(\mathbb{R})$ for all $1\leq p<\infty$. Now for all $m>n$,
$$\begin{aligned}
||f_m-f_n||_1 &= \int\left|\frac{1}{m}\mathbf{1}_{(0,m)}-\frac{1}{n}\mathbf{1}_{(0,n)}\right|d\lambda\\
&= \int\left|\frac{1}{m}-\frac{1}{n}\right|\mathbf{1}_{(0,n]}\,d\lambda+\int\frac{1}{m}\mathbf{1}_{(n,m)}\,d\lambda\\
&= \left|\frac{n-m}{nm}\right|n+\frac{1}{m}(m-n)=\frac{m-n}{m}+\frac{m-n}{m}=2-2\frac{n}{m}
\end{aligned}$$
which shows that
$$||f_m-f_n||_1\geq 1$$

whenever $m \geq 2n$. Hence, (f_n) is not Cauchy and thus does not converge in $L^1(\mathbb{R})$.
On the other hand, when $p > 1$ then

$$\|f_n - 0\|_p^p = \int \left|\frac{1}{n}\mathbf{1}_{(0,n)}\right|^p d\lambda = \frac{1}{n^p}\int \mathbf{1}_{(0,n)}\,d\lambda = \frac{1}{n^p}\cdot n = \frac{1}{n^{p-1}} \to 0$$

as $n \to \infty$. This shows that $f_n \xrightarrow{\|\cdot\|_p} 0$ in $L^p(\mathbb{R})$ for all $1 < p < \infty$.

(c) *(Convergence in measure)* For fixed $\varepsilon > 0$, set

$$E_{n,\varepsilon} = \{x \in \mathbb{R} : |f_n(x) - 0| > \varepsilon\}.$$

Let us choose N such that $\frac{1}{N} < \varepsilon$. Then for all $n \geq N$ and $x \in \mathbb{R}$ we have

$$|f_n(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

that is, $E_{n,\varepsilon} = \emptyset$. Hence

$$\lim_{n\to\infty} \lambda\left(E_{n,\varepsilon}\right) = \lim_{n\to\infty} \lambda(\emptyset) = \lim_{n\to\infty} 0 = 0.$$

which shows that $f_n \xrightarrow{\text{meas}} 0$.

2. Now let $f_n = \mathbf{1}_{(n,n+1)}$.

(a) *(Uniform convergence)* For $m \neq n$ we have

$$\sup_{x\in\mathbb{R}} |f_m(x) - f_n(x)| = 1$$

which shows that $\{f_n\}$ is not uniformly Cauchy, hence does not converge uniformly.

(b) *(Pointwise convergence)* On the other hand, for each $x \in \mathbb{R}$, we can pick $N \in \mathbb{N}$ with $x < N$. Then $x \notin (n, n+1)$ for all $n \geq N$, that is, $f_n(x) = 0$. This shows that $f_n(x) \to 0$ pointwise on $\mathbb{R}$ (and hence trivially, $f_n(x) \to 0$ a.e on $\mathbb{R}$).

(c) *(Convergence in the p-th mean)* Obviously, $f_n \in L^p(\mathbb{R})$ for all $1 \leq p < \infty$. Now for all $m \neq n$,

$$\begin{aligned}\|f_m - f_n\|_p &= \left[\int \left|\mathbf{1}_{(m,m+1)} - \mathbf{1}_{(n,n+1)}\right|^p d\lambda\right]^{1/p}\\ &= \left[\int \left[\mathbf{1}_{(m,m+1)} + \mathbf{1}_{(n,n+1)}\right] d\lambda\right]^{1/p} = 2^{1/p} \geq 1.\end{aligned}$$

Hence, $\{f_n\}$ is not Cauchy and thus does not converge in $L^p(\mathbb{R})$.

(d) *(Convergence in measure)* We claim that the sequence $\{f_n\}$ does not converge in measure.

For suppose to the contrary, that there exists an $\mathscr{F}$-measurable function f so that $f_n \xrightarrow{\text{meas}} f$. Let us first show that then $f(x) = 0$ a.e. In fact, for each $\varepsilon > 0$ we have by assumption that $\lambda(E_{n,\varepsilon}) \to 0$ as $n \to \infty$, where

$$E_{n,\varepsilon} = \{x \in \mathbb{R} : |f_n(x) - f| > \varepsilon\}.$$

Then in particular, by monotonicity of the measure, for each positive integer k,

$$\lambda\left((-\infty, k) \cap E_{n,\varepsilon}\right) \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

Since $f_n(x) = 0$ on $(-\infty, k)$ whenever $n > k$, then

$$(-\infty,k) \cap E_{n,\varepsilon} = \{x \in (-\infty,k) : |f(x)| > \varepsilon\} \qquad \text{whenever } n > k.$$

Thus by (5.1),

$$\{x \in (-\infty,k) : |f(x)| > \varepsilon\}$$

is a null set, and hence

$$\{x \in \mathbb{R} : |f(x)| > \varepsilon\} = \bigcup_{k=1}^{\infty} \{x \in (-\infty,k) : |f(x)| > \varepsilon\}$$

is a null set as well. Now as $\varepsilon > 0$ was arbitrary it follows from a standard argument that $f = 0$ a.e.

Choose $\varepsilon = \frac{1}{2}$. Then $E_{n,\varepsilon} = \{x \in \mathbb{R} : |f_n(x)| > \frac{1}{2}\} = (n, n+1)$ up to a null set, and thus

$$\lim_{n\to\infty} \lambda\big(E_{n,\varepsilon}\big) = \lim_{n\to\infty} \lambda\big((n,n+1)\big) = \lim_{n\to\infty} 1 = 1,$$

contradicting the fact that $f_n \stackrel{\text{meas}}{\longrightarrow} f = 0$.

■

The second example above shows that in general, almost-everywhere convergence does not imply convergence in measure. However, for finite measure spaces this implication is true:

Theorem 5.1.1 Suppose that $\mu(\Omega) < \infty$, and let $f_n, f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable functions with $f_n \stackrel{\text{a.e.}}{\longrightarrow} f$. Then also $f_n \stackrel{\text{meas}}{\longrightarrow} f$.

Proof. Let $\varepsilon > 0$ be arbitrary, but fixed. For ease of notation, we set

$$E_n = E_{n,\varepsilon} := \{\omega \in \Omega : |f_n(\omega) - f| > \varepsilon\}.$$

We need to show that $\mu(E_n) \to 0$ as $n \to \infty$.

To do so, for each $k \in \mathbb{N}$ we let

$$A_k := \bigcup_{n=k}^{\infty} E_n \qquad \big(= \{\omega : |f_n(\omega) - f(\omega)| > \varepsilon \text{ for some } n \geq k\big).$$

Observe that $\{A_k\}\downarrow$ and

$$\begin{aligned} \omega \in \bigcap_{k=1}^{\infty} A_k \quad &\underset{\text{Rem. 1.4}}{\iff} \quad \omega \text{ is contained in infinitely many } E_n \\ &\iff \quad |f_n(\omega) - f(\omega)| > \varepsilon \quad \text{for infinitely many } n \\ &\implies \quad f_n(\omega) \not\to f(\omega). \end{aligned}$$

Thus,

$$\bigcap_{k=1}^{\infty} A_k \subseteq \{\omega : f_n(\omega) \not\to f(\omega)\}.$$

By assumption, the right-hand set is contained in some null set, hence

$$\mu\Big(\bigcap_{k=1}^{\infty} A_k\Big) = 0.$$

Now as $\mu(\Omega) < \infty$ we have by Theorem 1.4.2 that

$$\lim_{k\to\infty} \mu(A_k) = \mu\Big(\bigcap_{k=1}^{\infty} A_k\Big) = 0.$$

Since $E_k \subseteq A_k$, then $0 \leq \mu(E_k) \leq \mu(A_k)$ for all k, and we can apply the Sandwich theorem to obtain that

$$\lim_{k\to\infty} \mu(E_k) = 0$$

as well. Thus, the theorem is proved. ■

The next example shows that the converse statement of this theorem is wrong. In fact, it shows that if $f_n \xrightarrow{\text{meas}} f$ then (f_n) need not converge a.e. Thus, at least in the class of finite measure spaces, convergence in measure is a weaker notion than almost-everywhere convergence.

■ **Example 5.2** We consider the measure space $\big([0,1], \mathscr{B}[0,1], \lambda\big)$. Recall that each $n \in \mathbb{N}$, can be expressed uniquely as

$$n = 2^k + m$$

with $k = k(n) \in \mathbb{N}_0$ and $m = m(n) \in \mathbb{N}$, where $0 \leq m < 2^k$. For each $n \in \mathbb{N}$, we set

$$f_n := \mathbf{1}_{\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)}.$$

For example,

$$f_1 = \mathbf{1}_{[0,1)},\ f_2 = \mathbf{1}_{[0,\frac{1}{2})},\ f_3 = \mathbf{1}_{[\frac{1}{2},1)},\ f_4 = \mathbf{1}_{[0,\frac{1}{4})},\ f_5 = \mathbf{1}_{[\frac{1}{4},\frac{2}{4})},\ f_6 = \mathbf{1}_{[\frac{2}{4},\frac{3}{4})},$$
$$f_7 = \mathbf{1}_{[\frac{3}{4},1)},\ f_8 = \mathbf{1}_{[0,\frac{1}{8})},\ f_9 = \mathbf{1}_{[\frac{1}{8},\frac{2}{8})},\ f_{10} = \mathbf{1}_{[\frac{2}{8},\frac{3}{8})},\ f_{11} = \mathbf{1}_{[\frac{3}{8},\frac{4}{8})},\ \ldots$$

Noting that $k = k(n) \to \infty$ as $n \to \infty$, we now consider various modes of convergence for the sequence (f_n).

1. *(Convergence in measure)* We claim that $f_n \xrightarrow{\text{meas}} 0$.
   In fact, for each ε, $0 \leq \varepsilon < 1$, we have

   $$E_{n,\varepsilon} := \{x \in \mathbb{R} : |f_n(x) - 0| > \varepsilon\} = \Big[\frac{m}{2^k}, \frac{m+1}{2^k}\Big).$$

   Hence,

   $$\lambda(E_{n,\varepsilon}) = \lambda\Big(\Big[\frac{m}{2^k}, \frac{m+1}{2^k}\Big)\Big) = \frac{1}{2^k} \to 0 \qquad \text{as } n \to \infty,$$

   which proves the claim.

2. *(Convergence in the p-th mean)* Clearly, $f_n \in L^p[0,1]$ for all $1 \le p < \infty$. We claim that $f_n \xrightarrow{\|\cdot\|_p} 0$.

   In fact,

$$\|f_n - 0\|_p^p = \int |f_n|^p \, d\lambda = \int \mathbf{1}_{\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)} \, d\lambda = \lambda\left(\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]\right) = \frac{1}{2^k} \to 0 \quad \text{as } n \to \infty,$$

   which proves the claim.

3. *(Almost-everywhere convergence)* We claim that (f_n) does not converge a.e.

   Suppose to the contrary, that there exists a Borel-measurable function f so that $f_n \xrightarrow{\text{a.e.}} f$. Clearly, f must be finite valued a.e. Applying Theorem 5.1.1 and part 1., it immediately follows that $f = 0$ a.e.

   Next let $x \in [0,1)$ be arbitrary. Since for each k,

$$\left\{\frac{0}{2^k}, \frac{1}{2^k}, \frac{2}{2^k}, \frac{3}{2^k}, \ldots, \frac{2^k}{2^k}\right\}$$

   is a partition of $[0,1]$ into subintervals of equal length, then for each k there exists a unique integer m_k, $0 \le m_k < 2^k$ so that $x \in \left[\frac{m_k}{2^k}, \frac{m_k+1}{2^k}\right)$. Set

$$n_k := 2^k + m_k \qquad (k = 1,2,3,\ldots).$$

   Then

$$f_{n_k}(x) = \mathbf{1}_{\left[\frac{m_k}{2^k}, \frac{m_k+1}{2^k}\right)}(x) = 1$$

   for all k. We have shown that for each $x \in [0,1)$, there exists a subsequence $(f_{n_k}(x))$ of $(f_n(x))$ (which of course depends on x) with

$$\lim_{k\to\infty} f_{n_k}(x) = \lim_{k\to\infty} 1,$$

   contradicting fact that $f_n(x) \to 0$ a.e. and thus proving the claim.

∎

R Observe that in the above example, one can construct a great variety of subsequences of $\{f_{n_k}\}$ which all converge to $f = 0$ a.e.

For example, choosing $n_k = 2^k$ we have $f_{n_k} = \mathbf{1}_{\left[0, \frac{1}{2^k}\right)}$, and clearly, $f_{n_k}(x) \to \mathbf{1}_{\{0\}}$ as $k \to \infty$, that is, $f_{n_k} \xrightarrow{\text{a.e.}} 0$.

In fact, we have in general:

Theorem 5.1.2 Let $f_n, f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable functions, with $f_n \xrightarrow{\text{meas}} f$. Then there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$.

Proof. By assumption, for each $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu(E_{n,\varepsilon}) = 0 \qquad \text{where} \quad E_{n,\varepsilon} := \{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \varepsilon\}.$$

That is, given $\delta > 0$ there exists $N = N(\varepsilon, \delta)$ such that

$$\mu\Big(\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \varepsilon\}\Big) < \delta \qquad \forall n \geq N. \tag{5.2}$$

Now we extract a subsequence of (f_n) inductively. By (5.2), choosing $\varepsilon = 1$ and $\delta = \frac{1}{2}$, there exists $n_1 \in \mathbb{N}$ such that

$$\mu\Big(\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > 1\}\Big) < \frac{1}{2} \qquad \forall n \geq n_1.$$

Next choosing $\varepsilon = \frac{1}{2}$ and $\delta = \frac{1}{4}$, there exists $n_2 > n_1$ such that

$$\mu\Big(\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \frac{1}{2}\}\Big) < \frac{1}{4} \qquad \forall n \geq n_2.$$

Suppose we have picked a positive integer n_k such that

$$\mu\Big(\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \frac{1}{k}\}\Big) < \frac{1}{2^k} \qquad \forall n \geq n_k. \tag{5.3}$$

Then by (5.2), choosing $\varepsilon = \frac{1}{k+1}$ and $\delta = \frac{1}{2^{k+1}}$, there exists $n_{k+1} > n_k$ such that

$$\mu\Big(\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \frac{1}{k+1}\}\Big) < \frac{1}{2^{k+1}} \qquad \forall n \geq n_{k+1}.$$

By induction, we thus obtain a subsequence (f_{n_k}) of (f_n) satisfying

$$\mu\Big(\underbrace{\{\omega \in \Omega : |f_{n_k}(\omega) - f(\omega)| > \frac{1}{k}\}}_{\text{call this set } A_k}\Big) \leq \frac{1}{2^k} \qquad \forall k.$$

Set

$$A := \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k.$$

Since

$$\sum_{k=1}^{\infty} \mu(A_k) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty,$$

then $\mu(A) = 0$ by the Borel-Cantelli Theorem.

Now let $\omega \in A^c$ be arbitrary. Then $\omega \notin \bigcup_{k=j}^{\infty} A_k$ for some j, and hence $\omega \notin A_k$ for all $k \geq j$. Equivalently,

$$|f_{n_k}(\omega) - f(\omega)| \leq \frac{1}{k} \qquad \forall k \geq j.$$

It follows that

$$f_{n_k}(\omega) \to f(\omega) \qquad \text{as } k \to \infty.$$

Since A is a null set, the assertion has been proved. ■

The next theorem states that every p-integrable function is bounded outside some set of arbitrarily small measure.

Theorem 5.1.3 *(Chebychev Inequality)* Let $1 \leq p < \infty$ and suppose $f \in L^p(\Omega, \mathscr{F}, \mu)$. Then for all $M > 0$,

$$\mu\Big(\{\omega \in \Omega : |f(\omega)| \geq M\}\Big) \leq M^{-p}\|f\|_p^p.$$

Proof. Given $M > 0$, set

$$E_M := \{\omega \in \Omega : |f(\omega)| \geq M\}.$$

Then by monotonicity of the integral,

$$\|f\|_p^p = \int |f|^p \, d\mu \geq \int_{E_M} |f|^p \, d\mu \geq \int_{E_M} M^p \, d\mu = M^p \mu(E_M),$$

from which the assertion follows immediately. ■

Corollary 5.1.4 Let $1 \leq p < \infty$ and suppose $f_n, f \in L^p(\Omega, \mathscr{F}, \mu)$ for all n.
If $f_n \xrightarrow{\|\cdot\|_p} f$ then $f_n \xrightarrow{\text{meas}} f$.

Proof. Let $\varepsilon > 0$ be arbitrary. Then by Chebychev's inequality,

$$\begin{aligned}\mu\Big(\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \varepsilon\}\Big) &= \mu\Big(\{\omega \in \Omega : |(f_n - f)(\omega)| > \varepsilon\}\Big) \\ &\leq \varepsilon^{-p}\|f_n - f\|_p^p \to 0 \quad \text{as } n \to \infty\end{aligned}$$

by assumption. This shows that $f_n \xrightarrow{\text{meas}} f$. ■

For the converse statement we have:

Theorem 5.1.5 *(DCT for Convergence in Measure)* Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, let $f_n, f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable, and $1 \leq p < \infty$. Suppose that

1. $f_n \xrightarrow{\text{meas}} f$,
2. there exists $g \in L^p(\Omega, \mathscr{F}, \mu)$ with $|f_n(\omega)| \leq g(\omega)$ a.e.

Then

(i) $f_n, f \in L^p(\Omega, \mathscr{F}, \mu)$ for all n, and

(ii) $f_n \xrightarrow{\|\cdot\|_p} f$.

Proof. (i): Note first that condition 2. guarantees that $f_n \in L^p(\Omega, \mathscr{F}, \mu)$ for all n. Furthermore, since $f_n \xrightarrow{\text{meas}} f$, by Theorem 5.1.2 there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$, and hence by 2.,

$$|f(\omega)| \leq g(\omega) \quad \text{a.e.}$$

as well, which shows that $f \in L^p(\Omega, \mathscr{F}, \mu)$.

(ii): Suppose to the contrary that $\|f_n - f\|_p \not\to 0$ as $n \to \infty$. Then there exist $\varepsilon > 0$ and a subsequence f_{n_k} so that

$$\|f_{n_k} - f\|_p \geq \varepsilon \qquad \forall k. \tag{5.4}$$

Now as $f_n \xrightarrow{\text{meas}} f$ then clearly, by definition of convergence in measure, $f_{n_k} \xrightarrow{\text{meas}} f$ as well. Applying Theorem 5.1.2 once more, there now exists a subsequence $\left(f_{n_{k_l}}\right)_{l=1}^{\infty}$ of $\left(f_{n_k}\right)_{k=1}^{\infty}$ such that

$$f_{n_{k_l}} \xrightarrow{\text{a.e.}} f \qquad \text{as } l \to \infty.$$

Applying the DCT for L^p-spaces (Theorem 3.2.1), it follows that

$$\|f_{n_{k_l}} - f\|_p \to 0 \qquad \text{as } l \to \infty,$$

which is impossible by (5.4). ■

Exercise 5.2 Let $f_n, f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable, $1 \leq p < \infty$, and suppose that $f_n \rightrightarrows f$ on Ω. Show:

1. $f_n \xrightarrow{\text{meas}} f$
2. If $\mu(\Omega) < \infty$ and $f_n \in L^p(\Omega, \mathscr{F}, \mu)$ for all n, then $f \in L^p(\Omega, \mathscr{F}, \mu)$ and $f_n \xrightarrow{\|\cdot\|_p} f$.
3. If $\mu(\Omega) = \infty$ then $\{f_n\}$ need not converge in $\|\cdot\|_p$.

■

It is obvious that $f_n \xrightarrow{\text{a.e.}} f$ does not imply that $f_n \rightrightarrows f$. There is, however, a result which says that we have uniform convergence outside of sets of small measure. Let us first make this concept precise.

Definition 5.1.2 Let $f_n, f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable. We say that $\{f_n\}$ *converges to f almost uniformly*, if for every $\varepsilon > 0$ there exists a set $B = B_\varepsilon \in \mathscr{F}$ with

(i) $\mu(B) < \varepsilon$,

(ii) $f_n \underset{\Omega \setminus B}{\rightrightarrows} f$.

■ **Example 5.3** Consider as usual the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$. Let

$$f_n(x) = \begin{cases} \frac{1}{x^2} & \text{if } |x| \geq \frac{1}{n} \\ 0 & \text{if } |x| < \frac{1}{n} \end{cases}, \qquad f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Now for every $\varepsilon > 0$, $f_n \underset{\mathbb{R} \setminus (-\varepsilon, \varepsilon)}{\rightrightarrows} f$ which shows $f_n \to f$ almost uniformly. On the other hand, convergence cannot be uniform because f is unbounded while each f_n is bounded. ■

Exercise 5.3 Show: If $f_n \xrightarrow{\text{a. unif.}} f$, then $f_n \xrightarrow{\text{a.e.}} f$ and $f_n \xrightarrow{\text{meas}} f$. ■

Theorem 5.1.6 *(Egoroff)* Let $(\Omega, \mathscr{F}, \mu)$ be a *finite* measure space, and $f_n, f : \Omega \to \mathbb{K}$ be $\mathscr{F}$-measurable. If $f_n \xrightarrow{\text{a.e.}} f$ then $f_n \xrightarrow{\text{a. unif.}} f$.

Proof. 1. Suppose first that $f_n(\omega) \longrightarrow f(\omega)$ for all $\omega \in \Omega$, and let $\varepsilon > 0$ be given.

For each $n \in \mathbb{N}$, we set

$$g_n = \sup_{j \geq n} |f_j - f|, \tag{5.5}$$

which is well defined as all functions are finite valued. Then

(a) Each g_n is $\mathscr{F}$-measurable by Theorem 1.5.7,
(b) The sequence (g_n) is monotone decreasing,
(c) $g_n(\omega) \to 0$ for each $\omega \in \Omega$.

We need to find a set $B \in \mathscr{F}$, $\mu(B) < \varepsilon$, so that $g_n \underset{\Omega\setminus B}{\rightrightarrows} 0$.

Now since $\mu(\Omega) < \infty$, then also $g_n \xrightarrow{\text{meas.}} 0$ by Theorem 5.1.1. That is, for every $\tilde{\varepsilon} > 0$ we have

$$\lim_{n\to\infty} \mu(E_{n,\tilde{\varepsilon}}) = 0 \qquad \text{where} \quad E_{n,\tilde{\varepsilon}} := \{\omega \in \Omega : |f_n(\omega)| > \tilde{\varepsilon}\}.$$

In particular, for every $\tilde{\varepsilon} = \frac{1}{k}$ there exists $n = n_k$ such that

$$\mu\Big(\underbrace{\{\omega \in \Omega : |g_{n_k}(\omega)| > \frac{1}{k}\}}_{\text{call this set } B_k}\Big) < \frac{\varepsilon}{2^k}. \tag{5.6}$$

Set

$$B = \bigcup_{k=1}^{\infty} B_k \in \mathscr{F}.$$

Then by σ-subadditivity,

$$\mu(B) \leq \sum_{k=1}^{\infty} \mu(B_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

We now claim that $g_n \underset{\Omega\setminus B}{\rightrightarrows} 0$. For this, given $\delta > 0$ we pick $k \in \mathbb{N}$ with $\frac{1}{k} < \delta$. Now if $\omega \in \Omega\setminus B$, then $\omega \notin B_k$, and hence

$$0 \leq g_{n_k}(\omega) \leq \frac{1}{k} < \delta.$$

Since $\{g_n\} \downarrow$, it follows that

$$0 \leq g_n(\omega) < \delta \qquad \forall\, \omega \in \Omega\setminus B,\ n \geq n_k,$$

which proves the claim. Now since

$$|f_n - f| \leq g_n \quad \forall n,$$

the assertion of the theorem follows for this particular case.

2. Next suppose that $f_n(\omega) \xrightarrow{\text{a.e.}} f(\omega)$. Then there exits a null set N with

$$f_n(\omega) \to f(\omega) \quad \forall\, \omega \in N^c.$$

Now let $\varepsilon > 0$ be given. By the first part, there exists $B \in \mathscr{F}$, $B \subseteq N^c$, so that $\mu(B) < \varepsilon$ and $f_n \underset{N^c\setminus B}{\rightrightarrows} f$.

We set $\tilde{B} = B \cup N$. Then
1. $\mu(\tilde{B}) = \mu(B) + \mu(N) = \mu(B) < \varepsilon$,
2. while also $\Omega\setminus\tilde{B} = N^c\setminus B$, so that $f_n \underset{\Omega\setminus\tilde{B}}{\rightrightarrows} f$.

Thus, the proof is complete. ■

(R) The various limits of (f_n) are independent of the mode of convergence. For example, if

$$f_n \overset{\text{meas.}}{\longrightarrow} f \quad \text{and} \quad f_n \overset{\text{a.e.}}{\longrightarrow} g$$

then by Theorem 5.1.2, there exists a subsequence f_{n_k} with $f_{n_k} \overset{\text{a.e.}}{\longrightarrow} f$. Hence, $f = g$ a.e.

(R) Suppose, $f_n, f : \Omega \to \mathbb{R}^*$ are $\mathscr{F}$-measurable and finite-valued a.e. The notions of *convergence in measure* and *almost uniform convergence* can be extended these functions in a natural way: If we let

$$N := \{\omega \in \Omega \mid |f|(\omega) < \infty, \ |f_n(\omega)| < \infty \ \forall n\},$$

which is a null set, then we can define

$$f_n \overset{\text{meas.}}{\longrightarrow} f \quad \Leftrightarrow \quad f_n \mathbf{1}_{E^c} \overset{\text{meas.}}{\longrightarrow} f \mathbf{1}_{E^c}$$

and

$$f_n \overset{\text{a. unif.}}{\longrightarrow} f \quad \Leftrightarrow \quad f_n \mathbf{1}_{E^c} \overset{\text{a. unif.}}{\longrightarrow} f \mathbf{1}_{E^c}.$$

It is easy to see that the above theorems still apply.

Exercise 5.4 Let $(\Omega, \mathscr{F}, \mu) = (\mathbb{R}, B(\mathbb{R}), \lambda)$ and $f_n = n \mathbf{1}_{[n, n+\frac{1}{n^3}]}$. Discuss all types of convergence of this sequence of function. ■

5.2 The Radon-Nikodym Theorem

Since measures are actually a class of functions, there is a natural way to compare two measures μ and ν on a measurable space $(\Omega, \mathscr{F})$:

$$\nu \leq \mu \quad \Leftrightarrow \quad \nu(E) \leq \mu(E) \quad \forall E \in \mathscr{F}.$$

As we will see in this section, a much weaker way of comparing two measures is also meaningful, which only compares the null sets:

Definition 5.2.1 Let μ and ν be two measures on a measurable space $(\Omega, \mathscr{F})$. If

$$\mu(E) = 0 \quad \text{implies} \quad \nu(E) = 0 \qquad \forall E \in \mathscr{F},$$

then we write "$\nu \ll \mu$" and say that ν *is absolutely continuous with respect to* μ.

(R) Careful: $\nu \ll \mu$ does *not* mean that $\nu(E) \leq \mu(E)$ for all $E \in \mathscr{F}$. It merely means that every μ-null is also a ν-null set !

■ **Example 5.4** Let us consider the following measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$:

a) the *Lebesgue measure* λ,
b) the *counting measure* μ_c,
c) the *Dirac one-point measure* δ_a, for some fixed $a \in \mathbb{R}$,
d) the sum of two distinct Dirac one-point measures, $\delta = \delta_a + \delta_b$, with $a \neq b$,

and compare any two of these measures.

1. If ν is *any* measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, then $\nu \ll \mu_c$. To see this, note that $\mu_c(E) = 0$ implies that $E = \emptyset$ so that $\nu(E) = 0$ also. (Clearly, this property holds for an arbitrary measurable space $(\Omega, \mathscr{F})$ as well.)
2. $\mu_c \not\ll \lambda$. To see this, let $E = \{a\}$ be a singleton. Then $\lambda(E) = 0$ while $\mu_c(E) = 1 \neq 0$.
3. $\mu_c \not\ll \delta_a$. To see this, pick $b \in \mathbb{R}$, $b \neq a$ and set $E = \{b\}$. Then $\delta_a(E) = 0$ while $\mu_c(E) = 1 \neq 0$.
4. $\lambda \not\ll \delta_a$. To see this, set $E = (a, a+1)$. Then $\delta_a(E) = 0$ while $\lambda(E) = 1 \neq 0$.
5. $\delta_a \not\ll \lambda$. To see this, set $E = \{a\}$. Then $\lambda(E) = 0$ while $\delta_a(E) = 1 \neq 0$.
6. Since $\delta_a \leq \delta$, then clearly, $\delta_a \ll \delta$.
7. $\delta \not\ll \delta_a$. In fact, let $E = \{b\}$. Then $\delta_a(E) = 0$ while $\delta(E) = \delta_b(E) = 1 \neq 0$.

■

■ **Example 5.5** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Exercise 2.13 shows how one can construct new measures which are absolutely continuous with respect to μ: Fix $h \in \mathscr{L}^+$. Then

$$\nu(E) := \int_E h \, d\mu \qquad (E \in \mathscr{F})$$

defines a measure on $(\Omega, \mathscr{F})$ with the following properties:

(a) $\nu \ll \mu$

(b) If $f : \Omega \to \mathbb{R}^*$ is $\mathscr{F}$-measurable, then

$$\int f \, d\nu \ \text{ is defined} \quad \Leftrightarrow \quad \int fh \, d\mu \ \text{ is defined.}$$

Furthermore, if any of these integrals is defined, then

$$\int f \, d\nu = \int fh \, d\mu. \tag{5.7}$$

■

The goal of this section is to show that every measure ν on $(\Omega, \mathscr{F})$ which is absolutely continuous with respect to μ arises in this way. But first some more introductory remarks and intermediate results.

Exercise 5.5 Let $(\Omega, \mathscr{F})$ be a measurable space. Show:

1. If μ, ν_1, ν_2 are measures on $(\Omega, \mathscr{F})$ with $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then for all $\alpha, \beta \geq 0$,

$$\alpha \nu_1 + \beta \nu_2 \ll \mu.$$

2. The relation "$\ll$" has the following properties:
   (i) $\mu \ll \mu$ (*Reflexivity*).
   (ii) If $\nu \ll \mu$ and $\mu \ll \sigma$ then $\nu \ll \sigma$ (*transitivity*).

   Here, μ, ν, σ are arbitrary measures on $(\Omega, \mathscr{F})$. Show by example that *antisymmetry*:

   If $\nu \ll \mu$ and also $\mu \ll \nu$ then $\mu = \nu$

   need not hold in general. (The two properties [(i)] and [(ii)] show that "$\ll$" is a *preorder* on the collection of measures on $(\Omega, \mathscr{F})$. However, it is not a partial order by lack of antisymmetry.)

■

Motivated by the second part of this exercise, we define:

Definition 5.2.2 Two measures μ and ν on $(\Omega, \mathscr{F})$ are said to be *equivalent*, written $\mu \approx \nu$, if

$$\mu \ll \nu \qquad \text{and} \qquad \nu \ll \mu.$$

(R) By applying part 2. of Exercise 5.5 one easily verifies that "$\approx$" is an equivalence relation on the collection of measures on $(\Omega, \mathscr{F})$.

Exercise 5.6 (Continuation of Exercise 2.13) Let ν denote the measure on $(\Omega, \mathscr{F}, \mu)$ defined as in Exercise 2.13. Show:

$$\nu \approx \mu \quad \Leftrightarrow \quad h > 0 \quad \text{a.e.}$$

Proposition 5.2.1 Let μ and ν be two measures on a measurable space $(\Omega, \mathscr{F})$, with $\nu(\Omega) < \infty$. Then

$$\nu \ll \mu \quad \Leftrightarrow \quad \forall \varepsilon > 0 \; \exists \delta > 0 \quad \text{so that} \quad \mu(E) < \delta \text{ implies } \nu(E) < \varepsilon, \quad \forall E \in \mathscr{F}. \tag{5.8}$$

Proof. $\Rightarrow$ Suppose to the contrary, that $\nu \ll \mu$, but there exists an $\varepsilon > 0$ so that no matter what $\delta > 0$, one can find a set $E \in \mathscr{F}$ with $\mu(E) < \delta$ but $\nu(E) \geq \varepsilon$. Then in particular, for each $\delta = \frac{1}{2^n}$ one can find sets $E_n \in \mathscr{F}$ with

$$\mu(E_n) < \frac{1}{2^n} \quad \text{but} \quad \nu(E_n) \geq \varepsilon.$$

Observe that

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} < 1.$$

Thus by the Borel-Cantelli Lemma,

$$\mu(A) = 0, \qquad \text{where} \quad A = \bigcap_{n=1}^{\infty} A_n, \quad A_n = \bigcup_{k=n}^{\infty} E_k.$$

Hence by assumption of absolute continuity, $\nu(A) = 0$. On the other hand, as ν is a finite measure, then by Theorem 1.4.2,

$$\nu(A) = \nu\left(\bigcap_{n=1}^{\infty} A_n\right) \underset{\text{Thm 1.4.2}}{=} \lim_{n\to\infty} \nu(A_n) \geq \varepsilon, \qquad \text{since } \nu(A_n) \geq \nu(E_n) \geq \varepsilon \text{ for all } n,$$

which is a contradiction.

$\Leftarrow$ Suppose, the right-hand statement in (5.8) holds. Let $E \in \mathscr{F}$ be a μ-null set, that is, $\mu(E) = 0$. Then by assumption, $0 \leq \nu(E) < \varepsilon$ for any $\varepsilon > 0$. However, this is only possible when $\nu(E) = 0$. ■

(R) Loosely speaking, the above proposition states that a finite measure ν is absolutely continuous with respect to a measure μ if and only if "μ-small sets" are also "ν-small sets". This property gave rise to the notion "absolutely continuous".

Corollary 5.2.2 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and $g \in L^1_{\mathbb{K}}(\Omega, \mathscr{F}, \mu)$. Then

$$\forall \varepsilon > 0 \ \exists \delta > 0 \quad \text{so that} \quad \mu(E) < \delta \ \text{implies} \ \left| \int_E g \, d\mu \right| < \varepsilon, \quad \forall E \in \mathscr{F}.$$

(That is, "integrals over small sets are small".)

Exercise 5.7 Prove Corollary 5.2.2. (Hint. Use the above proposition and Exercise 2.13.) ■

Corollary 5.2.3 Let $g \in L^1(\mathbb{R}, \mathscr{M}, \lambda)$ be given, and set

$$G(t) = \int_{(-\infty, t]} g \, d\lambda.$$

Then G is uniformly continuous on $\mathbb{R}$.

Proof. Given $\varepsilon > 0$, pick δ as in Corollary 5.2.2 for $\mu = \lambda$. Now let $s, t \in \mathbb{R}$ be arbitrary with $|t - s| < \delta$. Without loss of generality we may assume that $s \leq t$. Then

$$\lambda([s,t]) = t - s < \delta,$$

and hence,

$$|G(t) - G(s)| = \left| \int_{(-\infty,t]} g \, d\lambda - \int_{(-\infty,s]} g \, d\lambda \right| = \left| \int_{[s,t]} g \, d\lambda \right| < \varepsilon,$$

which was to be shown. ■

When discussing the Riemann integral, one usually introduces the notation of a *partition* of an interval $[a,b]$ and refinement of such a partition. This concept generalizes to measurable spaces:

Definition 5.2.3 Let $(\Omega, \mathscr{F})$ be a measurable space and $\mathscr{P} = \{A_i\}_{i=1}^M \subseteq \mathscr{F}$ a finite measurable partition of Ω. That is,

1. the sets A_i are mutually disjoint,
2. $\Omega = \bigcup_{i=1}^M A_i$.

(see Definition 2.1.2.)

A (measurable) partition $\mathscr{P}' = \{B_j\}_{j=1}^N$ is called a *refinement* of the partition $\{A_i\}_{k=1}^M$, if for each j $(1 \leq j \leq N)$ there exists a i $(1 \leq i \leq M)$ so that $B_j \subseteq A_i$.

Below, by "partition" we will always mean a finite, measurable partition.

(R) Let $\mathscr{P}$ and $\mathscr{P}'$ be as above. Since the sets B_j are pairwise disjoint, then each A_i is the union of the sets B_j contained in it:

$$A_i = \bigcup_{\{j : B_j \subseteq A_i\}} B_j.$$

(R) Let $\mathscr{P}_1 = \{A_i\}_{i=1}^M$ and $\mathscr{P}_2 = \{B_j\}_{j=1}^N$ be two partitions of Ω. Then

$$\mathscr{P}_1 \wedge \mathscr{P}_2 := \{A_i \cap B_j \mid i = 1 \dots M, \ j = 1 \dots N\}$$

clearly is a refinement of both, $\mathscr{P}_1$ and $\mathscr{P}_2$.

The following proposition leads us half way to the desired result.

Proposition 5.2.4 Let $(\Omega, \mathscr{F}, \mu)$ be a *finite* measure space, and ν a measure on $(\Omega, \mathscr{F})$ with $\nu \leq \mu$. Then there exists an $\mathscr{F}$-measurable function $h : \Omega \to [0,1]$ satisfying

$$\nu(E) = \int_E h\, d\mu \qquad \forall E \in \mathscr{F}. \tag{5.9}$$

Proof. Let us first construct the function h in steps.

Step 1: Given a partition $\mathscr{P} = \{A_i\}_{i=1}^M$ of Ω, let us set

$$h_P = \sum_{i=1}^M c_i \mathbf{1}_{A_i}, \qquad \text{where} \qquad c_i = \begin{cases} \frac{\nu(A_i)}{\mu(A_i)} & \text{if } \mu(A_i) \neq 0 \\ 0 & \text{if } \mu(A_i) = 0. \end{cases}$$

Clearly, h_P is $\mathscr{F}$-measurable and simple, and since $\nu \leq \mu$, then $0 \leq h_P \leq 1$. In particular, $h_P \in L^1(\Omega, \mathscr{F}, \mu)$.

We make the following observation: Let $A \in \mathscr{F}$ be the union of some of the partition sets, say

$$A = \bigcup_{i \in I} A_i, \qquad I \subseteq \{1, \ldots, M\}. \tag{5.10}$$

Then

$$\begin{aligned} \int_A h_P\, d\mu &= \int h_P \mathbf{1}_A\, d\mu = \int \left[\sum_{i=1}^M c_i \mathbf{1}_{A_i}\right] \mathbf{1}_A\, d\mu = \int \left[\sum_{i=1}^M c_i \mathbf{1}_{A_i \cap A}\right] d\mu \\ &= \int \left[\sum_{i \in I} c_i \mathbf{1}_{A_i}\right] d\mu = \sum_{i \in I} c_i \mu(A_i) = \sum_{i \in I} \nu(A_i) = \nu(A). \end{aligned}$$

Here we have used the fact that $\nu(A_i) = c_i \mu(A_i)$, even when $\mu(A_i) = 0$, since $\nu \leq \mu$. That is,

$$\nu(A) = \int_A h_P\, d\mu \tag{5.11}$$

for all sets of form (5.10).

Step 2: Next let $\mathscr{P}' = \{B_j\}_{j=1}^N$ be any refinement of a partition $\mathscr{P}$ as in Step 1, and let $h_{P'}$ be the function constructed for this refinement as in Step 1. That is,

$$\nu(B) = \int_B h_{P'}\, d\mu \tag{5.12}$$

for all sets B which are unions of some of the sets B_j. Since each set $A_i \in \mathscr{P}$ is such a union, then

$$\int_{A_i} h_P\, d\mu \underset{(5.11)}{=} \nu(A_i) \underset{(5.12)}{=} \int_{A_i} h_{P'}\, d\mu. \tag{5.13}$$

In addition, since $h_P(\omega) = c_i$ on the set A_i, then

$$\int_{A_i} h_P h_{P'}\, d\mu = \int_{A_i} c_i h_{P'}\, d\mu = c_i \int_{A_i} h_{P'}\, d\mu \underset{(5.13)}{=} c_i \int_{A_i} h_P\, d\mu = \int_{A_i} h_P h_P\, d\mu,$$

that is,

$$\int_{A_i} h_P h_{P'}\, d\mu = \int_{A_i} h_P^2\, d\mu. \tag{5.14}$$

It follows that

$$\int_{A_i} [h_{P'} - h_P]^2 \, d\mu = \int_{A_i} \left[h_{P'}^2 - 2h_P h_{P'} + h_P^2\right] d\mu \underset{(5.14)}{=} \int_{A_i} \left[h_{P'}^2 - h_P^2\right] d\mu.$$

Thus, if A is a union of some of the sets A_i as in (5.10), then

$$0 \leq \int_A [h_{P'} - h_P]^2 \, d\mu = \sum_{i \in I} \int_{A_i} [h_{P'} - h_P]^2 \, d\mu = \sum_{i \in I} \int_{A_i} \left[h_{P'}^2 - h_P^2\right] d\mu = \int_A \left[h_{P'}^2 - h_P^2\right] d\mu, \quad (5.15)$$

and hence,

$$\int_A h_P^2 \, d\mu \leq \int_A h_{P'}^2 \, d\mu, \quad (5.16)$$

for any refinement $\mathscr{P}'$ of $\mathscr{P}$.

Step 3: We now construct a sequence $\{\mathscr{R}_n\}$ of partitions so that the corresponding sequence $\{h_{R_n}\}$ converges a.e.

For this, let us set $K := \mu(\Omega) < \infty$. Now if $\mathscr{P}$ is any partition of Ω, then as $0 \leq h_P \leq 1$ we have

$$\int h_P^2 \leq K$$

and hence,

$$C := \sup\left\{ \int h_P^2 \, d\mu \ : \ \mathscr{P} \text{ is a partition of } \Omega \right\} \leq K.$$

By a characterization of the supremum, for each $\varepsilon = \frac{1}{4^n K}$ there exists a partition $\mathscr{P}_n$ of Ω with

$$C - \frac{1}{4^n K} < \int h_{P_n}^2 \, d\mu \leq C. \quad (5.17)$$

Now if we modify these partitions inductively by setting

$$\mathscr{R}_1 := \mathscr{P}_1, \qquad \mathscr{R}_2 := \mathscr{R}_1 \wedge \mathscr{P}_2, \qquad \ldots \qquad \mathscr{R}_n := \mathscr{R}_{n-1} \wedge \mathscr{P}_n,$$

for $n \geq 2$, then $\{\mathscr{R}_n\}$ will be a sequence of partitions so that each $\mathscr{R}_n$ refines both, $\mathscr{R}_{n-1}$ as well as $\mathscr{P}_n$. Hence by (5.16) and (5.17),

$$C - \frac{1}{4^n K} < \int h_{P_n}^2 \, d\mu \underset{(5.16)}{\leq} \int h_{R_n}^2 \, d\mu \underset{(5.16)}{\leq} \int h_{R_{n+1}}^2 \, d\mu \leq C \quad (5.18)$$

for all n. We are now ready to show that $\lim_{n\to\infty} h_{R_n}(\omega)$ exists a.e.

In fact, for all $n \in \mathbb{N}$, we have

$$\int \left[h_{R_{n+1}} - h_{R_n}\right]^2 d\mu \underset{(5.15)}{=} \int \left[h_{R_{n+1}}^2 - h_{R_n}^2\right] d\mu \underset{(5.18)}{\leq} \frac{1}{4^n K}. \quad (5.19)$$

Now since $\|\cdot\|_1 \leq \|\cdot\|_2 \sqrt{K}$ (see Exercise 3.3), we have

$$\int \left|h_{R_{n+1}} - h_{R_n}\right| d\mu = \left\|h_{R_{n+1}} - h_{R_n}\right\|_1 \leq \left\|h_{R_{n+1}} - h_{R_n}\right\|_2 \sqrt{K} \underset{(5.19)}{\leq} \frac{1}{4^{n/2}\sqrt{K}} \sqrt{K} = \frac{1}{2^n},$$

and hence,

$$\sum_{n=1}^{\infty} \int \left|h_{R_{n+1}} - h_{R_n}\right| d\mu \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus, by the Beppo-Levi Theorem, the telescoping series

$$\sum_{n=1}^{\infty}\left[h_{R_{n+1}}-h_{R_n}\right]$$

converges a.e. to an integrable function f, and

$$\int f\,d\mu=\sum_{n=1}^{\infty}\int\left[h_{R_{n+1}}-h_{R_n}\right]d\mu.$$

Set

$$h=h_{R_1}+f=h_{R_1}+\lim_{N\to\infty}\sum_{n=1}^{N-1}\left[h_{R_{n+1}}-h_{R_n}\right]=\lim_{N\to\infty}h_{R_n},$$

in the sense of a.e. convergence.

Step 4: Finally, we verify that h has the desired properties. We first note that as $0\le h_{R_n}\le 1$ for all n, then

$$0\le h\le 1 \text{ a.e.}$$

Modifying h on a null set, we may thus assume that

$$0\le h\le 1.$$

To verify (5.9), let $A\in\mathscr{F}$ be given. First set

$$\mathscr{S}_0=\{A,A^c\},$$

a partition of Ω, and for each $n\in\mathbb{N}$, set

$$\mathscr{S}_n=\mathscr{S}_0\wedge\mathscr{R}_n.$$

Then $\mathscr{S}_n$ is a refinement of $\mathscr{S}_0$ as well as $\mathscr{R}_n$, and hence by (5.18),

$$C-\frac{1}{4^nK}<\int h_{R_n}^2\,d\mu\underset{(5.16)}{\le}\int h_{S_n}^2\,d\mu\le C.$$

Arguing as in (5.19), then for each n,

$$\int\left[h_{S_n}-h_{R_n}\right]^2d\mu\underset{(5.15)}{=}\int\left[h_{S_n}^2-h_{R_n}^2\right]d\mu\underset{(5.18)}{\le}\frac{1}{4^nK},$$

so that

$$\int|h_{S_n}-h_{R_n}|\,d\mu=\|h_{S_n}-h_{R_n}\|_1\le\|h_{S_n}-h_{R_n}\|_2\sqrt{K}\le\frac{1}{2^n}\to 0\quad\text{as } n\to\infty. \tag{5.20}$$

Therefore,

$$\nu(A)\underset{(5.10)}{=}\int_A h_{S_n}\,d\mu=\int_A\left[h_{S_n}-h_{R_n}\right]d\mu+\int_A h_{R_n}\,d\mu.$$

Now by (5.20), the first integral on the right-hand side goes to zero as $n\to\infty$. As for the second integral, since $h_{R_n}\to h$ a.e. and $|h_{R_n}|\le 1\in L^1(\Omega,\mathscr{F},\mu)$, the Dominated Convergence Theorem implies that the second integral tends to $\int h\,d\mu$ as $n\to\infty$. Thus

$$\nu(A)=\int_A h\,d\mu,$$

which completes the proof. ■

The assumption that $\mu(\Omega)<\infty$ may be weakened to μ is σ-finite:

Corollary 5.2.5 Let $(\Omega, \mathscr{F}, \mu)$ be a *σ-finite* measure space, and ν a measure on $(\Omega, \mathscr{F})$ with $\nu \leq \mu$. Then there exists an $\mathscr{F}$-measurable function $h : \Omega \to [0,1]$ satisfying

$$\nu(E) = \int_E h\,d\mu \qquad \forall E \in \mathscr{F}.$$

Proof. By σ-finiteness, there exists a sequence $\{A_n\} \subseteq \mathscr{F}$ of measurable sets satisfying

1. $\mu(A_n) < \infty$ for all n, and
2. $\Omega = \bigcup_{n=1}^{\infty} A_n$.

Applying Theorem 1.1.1, we may assume further that these sets are mutually disjoint.

Now we can apply the above Theorem to each measure space $(A_n, \mathscr{F}_n, \mu)$, where

$$\mathscr{F}_n = \{E \in \mathscr{F} \,|\, E \subseteq A_n\} = \{E \cap A_n : E \in \mathscr{F}\}$$

(see Exercise 2.8) to obtain $\mathscr{F}_n$-measurable functions

$$h_n : A_n \to [0,1]$$

satisfying

$$\nu(E_n) = \int_{E_n} h_n\,d\mu \qquad \forall E_n \in \mathscr{F}_n.$$

Since the collection $\{A_n\}$ forms a partition of Ω, we can "glue" the functions h_n together and define $h : \Omega \to [0,1]$ by

$$h(\omega) = h_n(\omega) \qquad \text{where} \quad \omega \in A_n.$$

Then

1. obviously, $0 \leq h \leq 1$,
2. h is $\mathscr{F}$-measurable. In fact, for each $a \in \mathbb{R}$ we have since each h_n is $\mathscr{F}_n$-measurable,

$$\{\omega \in \Omega \,|\, h(\omega) < a\} = \bigcup_{n=1}^{\infty} \underbrace{\{\omega \in A_n \,|\, h_n(\omega) < a\}}_{\in \mathscr{F}_n \subseteq \mathscr{F}} \in \mathscr{F}.$$

3. Given $E \in \mathscr{F}$, set $E_n = E \cap A_n \in \mathscr{F}_n$. Since

$$E = E \cap \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (E \cap A_n) = \bigcup_{n=1}^{\infty} E_n,$$

a disjoint union, and $h = h_n$ on A_n, then by Corollary 2.7.6,

$$\nu(E) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} \int_{E_n} h_n\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} h\,d\mu \underset{\text{Cor 2.7.6}}{=} \int_E h\,d\mu.$$

Thus the assertion holds. ■

We are now ready to prove the main result of this section.

Theorem 5.2.6 *(Radon-Nikodym Theorem).* Let μ and ν be σ-finite measures on a measurable space $(\Omega, \mathscr{F})$ with $\nu \ll \mu$. Then there exists an $\mathscr{F}$-measurable function $h \geq 0$ such that

$$\nu(E) = \int_E h\,d\mu \qquad \forall E \in \mathscr{F}.$$

Furthermore, h is essentially unique. That is, if $\tilde{h}$ is another $\mathscr{F}$-measurable function with the property that $\nu(E) = \int_E \tilde{h}\,d\mu\ \forall E \in \mathscr{F}$, then $h = \tilde{h}$ μ-a.e.

Proof. 1. Existence. Set $\varphi = \mu + \nu$. Then clearly,

(i) $0 \leq \mu, \nu \leq \varphi$,

(ii) φ is a σ-finite measure on $(\Omega, \mathscr{F})$ (see Exercise 1.6).

By the previous Corollary, there exist $\mathscr{F}$-measurable functions $h_\mu, h_\nu : \Omega \to [0,1]$ with

$$\mu(E) = \int_E h_\mu\,d\varphi \qquad \text{and} \qquad \nu(E) = \int_E h_\nu\,d\varphi \qquad \forall E \in \mathscr{F}.$$

Set

$$F = \{\omega \in \Omega \,|\, h_\mu(\omega) > 0\} \in \mathscr{F} \qquad \text{so that} \qquad F^c = \{\omega \in \Omega \,|\, h_\mu(\omega) = 0\} \in \mathscr{F}.$$

Then

$$\mu(F^c) = \int_{F^c} h_\mu\,d\varphi = 0$$

and hence, as $\nu \ll \mu$, then $\nu(F^c) = 0$ as well. Now set

$$h(\omega) = \begin{cases} \frac{h_\nu(\omega)}{h_\mu(\omega)} & \text{if } \omega \in F \\ 0 & \text{if } \omega \in F^c. \end{cases} \tag{5.21}$$

Then

(i) clearly, $h \geq 0$,

(ii) since $h = \dfrac{h_\nu}{h_\mu + \mathbf{1}_{F^c}} \mathbf{1}_F$, then h is $\mathscr{F}$-measurable,

(iii) for each $E \in \mathscr{F}$, since $\nu(F^c) = \mu(F^c) = 0$, we have

$$\nu(E) = \nu(E \cap F) + \nu(E \cap F^c) = \nu(E \cap F) + 0 = \int_{E \cap F} h_\nu\,d\varphi = \int_{E \cap F} h h_\mu\,d\varphi$$
$$\underset{(5.7)}{=} \int_{E \cap F} h\,d\mu + 0 = \int_{E \cap F} h\,d\mu + \int_{E \cap F^c} h\,d\mu = \int_E h\,d\mu.$$

2. Essential Uniqueness. Suppose that $\tilde{h} : \Omega \to [0, \infty]$ is another $\mathscr{F}$-measurable function with

$$\nu(E) = \int_E \tilde{h}\,d\mu \qquad \forall E \in \mathscr{F}.$$

Set

$$N = \left\{\omega \in \Omega \mid h(\omega) > \tilde{h}(\omega)\right\} \in \mathscr{F}.$$

We claim: $\mu(N) = 0$. For suppose to the contrary that $\mu(N) > 0$. Then by Theorem 2.6.6,

$$\int_N \left[h - \tilde{h}\right] d\mu > 0. \tag{5.22}$$

Note that the difference of the two functions is defined on N, as $\tilde{h}$ is finite valued on N.

For each $n \in \mathbb{N}$, set

$$N_n := \{\omega \in N \,|\, \tilde{h}(\omega) \le n\},$$

so that

$$N_n \in \mathscr{F}, \qquad \{N_n\} \uparrow \qquad \text{and} \qquad N = \bigcup_{n=1}^{\infty} N_n.$$

Next we modify these sets so they have finite measure. In fact, since μ is σ-finite, there exists $\{A_n\}_{n=1}^{\infty} \uparrow \subseteq \mathscr{F}$, with

$$\mu(A_n) < \infty \quad \forall n \qquad \text{and} \qquad \Omega = \bigcup_{n=1}^{\infty} A_n.$$

We set

$$B_n = N_n \cap A_n \in \mathscr{F}, \qquad \text{so that} \qquad \{B_n\} \uparrow \qquad \text{and also} \qquad N = \bigcup_{n=1}^{\infty} B_n.$$

We now have

$$0 < \int \left[h - \tilde{h}\right] d\mu \underset{\text{Cor 2.7.5}}{=} \lim_{n \to \infty} \int_{B_n} \left[h - \tilde{h}\right] d\mu,$$

from which we conclude that

$$\int_{B_n} \left[h - \tilde{h}\right] d\mu > 0$$

for sufficiently large n, and consequently

$$\nu(B_n) = \int_{B_n} h \, d\mu = \int_{B_n} \left[h - \tilde{h}\right] d\mu + \int_{B_n} \tilde{h} \, d\mu > 0 + \nu(B_n) = \nu(B_n),$$

since $\mu(B_n) < \infty$, which is impossible. Thus, the claim follows.

Now by symmetry,

$$\tilde{N} = \left\{\omega \in \Omega \mid \tilde{h}(\omega) > h(\omega)\right\}$$

is also a null set. Hence, $h = \tilde{h}$ a.e. ■

R The σ-finiteness condition in this Theorem cannot be dropped. For example, consider the measurable space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with the Lebesgue measure λ and the counting measure μ_c which is not σ-finite. By Example 5.4, $\lambda \ll \mu_c$. Now suppose, there exists h as in the Theorem. Let $\{x\}$ be any singleton. We obtain

$$0 = \lambda(\{x\}) = \int_{\{x\}} h \, d\mu_c = \int h \mathbf{1}_{\{x\}} \, d\mu_c = h(x)\mu_c(\{x\}) = h(x)$$

so that $h(x) = 0$ for all $x \in \mathbb{R}$. But then for all $E \in \mathscr{B}(\mathbb{R})$,

$$\lambda(E) = \int_E h \, d\mu_c = \int_E 0 \, d\mu_c = 0,$$

which contradicts the fact that $\lambda \neq 0$.

(R) The function h is called the *Radon-Nikodym* derivative of μ with respect to ν, and denoted by

$$h = \frac{d\mu}{d\nu}.$$

Exercise 5.8 Let $\nu \ll \mu$ and $\mu \ll \varphi$ be three measures on $(\Omega, \mathscr{F})$.

1. Prove the *chain rule*:

$$\frac{d\varphi}{d\nu} = \frac{d\varphi}{d\mu} \cdot \frac{d\mu}{d\nu}$$

in the sense of equality a.e.

2. Show: If $\nu \approx \mu \quad \Leftrightarrow \quad 0 < \frac{d\mu}{d\nu} < \infty$ a.e. ■

5.3 From Premeasure to Measure

Since algebras of sets need not be closed under countable unions, the concept of measure does not apply to them. In this section we will introduce the corresponding concept for algebras, the premeasures, and show how the results on the outer measures of Section 4.3 can be applied to extend a premeasure on an algebra $\mathscr{A}$ to a measure on the σ-algebra generated by $\mathscr{A}$. While these results will be needed for the construction of product measures in the Section 5.4 below, they are interesting in their own right.

We begin with a construction of algebras.

Definition 5.3.1 Let Ω be an arbitrary set, and $\mathscr{E}$ a nonempty collection of subsets of Ω. Then $\mathscr{E}$ is called an *elementary family* on Ω provided the following hold:

(E1) Whenever $A, B \in \mathscr{E}$, then $A \cap B \in \mathscr{E}$.

(E2) For all $A \in \mathscr{E}$, the complement A^c is the finite disjoint union of elements of $\mathscr{E}$.

■ **Example 5.6** Let

$$\mathscr{E} = \{(a,b] \cap \mathbb{R} \mid -\infty \leq a \leq b \leq \infty\} = \{(a,b] \mid -\infty \leq a \leq b < \infty\} \cup \{(a,\infty) \mid -\infty \leq a < \infty\}.$$

Then $\mathscr{E}$ is an elementary family on $\mathbb{R}$. ■

■ **Example 5.7** Given measurable spaces $(X, \mathscr{E})$ and $(Y, \mathscr{F})$, we set

$$\mathscr{E}_o = \{A \times B \mid A \in \mathscr{E},\ B \in \mathscr{F}\} \subseteq \mathscr{P}(X \times Y).$$

$\mathscr{E}_o$ is called the set of *measurable rectangles* on $X \times Y$.

We observe that $\mathscr{E}_o$ is an elementary family. In fact, if $A_1 \times B_1, A_2 \times B_2 \in \mathscr{E}_o$, then

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \cap (B_1 \cap B_2) \in \mathscr{E}_o$$

since $\mathscr{E}$ and $\mathscr{F}$ are σ-algebras. Furthermore, for each $A \times B \in \mathscr{E}_o$ we have

$$(A \times B)^c = (A^c \times Y) \cup (X \times B^c) = (A^c \times B) \cup (A^c \times B^c) \cup (A \times B^c)$$

and the right-hand side is a finite disjoint union of members of $\mathscr{E}_o$. Thus, (E1) and (E2) hold. ■

Proposition 5.3.1 Given an elementary family $\mathscr{E}$ on Ω, let

$$\mathscr{A} = \left\{ E = \bigcup_{k=1}^{n} A_k \mid n \in \mathbb{N},\, A_k \in \mathscr{E},\, A_j \cap A_k = \emptyset \; (j \neq k) \right\}$$

denote the collection of all finite *disjoint* unions of members of $\mathscr{E}$. Then $\mathscr{A}$ is an algebra on Ω.

Proof. 1. First let $A, B \in \mathscr{A}$, say $A = \bigcup_{k=1}^{n} A_k$ and $B = \bigcup_{j=1}^{m} B_j$, where the families $\{A_k\}_{k=1}^{n} \subseteq \mathscr{E}$ and $\{B_j\}_{j=1}^{m} \subseteq \mathscr{E}$ are each disjoint. Then

$$A \cap B = \left[\bigcup_{k=1}^{n} A_k\right] \cap \left[\bigcup_{j=1}^{m} B_j\right] = \bigcup_{k=1}^{n} \bigcup_{j=1}^{m} [A_k \cap B_j]$$

is, by (E1), a finite disjoint union of elements in $\mathscr{E}$, and hence $A \cap B \in \mathscr{A}$. It now follows by induction that $\mathscr{A}$ is closed under finite intersections.

2. Note also that

$$A^c = \left[\bigcup_{k=1}^{m} A_k\right]^c = \bigcap_{k=1}^{m} A_k^c.$$

Now by assumption (E2), each A_k^c is a member of $\mathscr{A}$. It follows from part 1. that $A^c \in \mathscr{A}$.

Finally, applying Exercise 1.1 we conclude that $\mathscr{A}$ is an algebra. ■

We are now ready to introduce the concept of premeasure on an algebra $\mathscr{A}$.

Definition 5.3.2 Let Ω be a set and $\mathscr{A}$ an algebra on Ω. A set function

$$\rho : \mathscr{A} \to [0, \infty]$$

satisfying

(PM1) $\rho(\emptyset) = 0$,

(PM2) Whenever $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ is a countable collection of *pairwise disjoints* sets such that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$, then

$$\rho\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \rho(A_n), \qquad \text{("}\sigma\text{-additivity")}$$

is called a *premeasure* on $\mathscr{A}$.

Clearly, this definition coincides with that of a measure when $\mathscr{A}$ is a σ-algebra. Applying exactly the same arguments as used for measures in Section 1.4, it is easy to see that a premeasure is (finitely) additive and hence monotone. Furthermore, the notions of *finite* and σ-*finite* premeasures are defined in exactly the same way as they are for measures.

Note that by Exercise 4.2, every premeasure ρ on $\mathscr{A}$ defines an outer measure μ^* on Ω by

$$\mu^*(E) := \inf\left\{ \sum_{n=1}^{\infty} \rho(A_n) \,:\, A_n \in \mathscr{A},\, E \subseteq \bigcup_{n=1}^{\infty} A_n \right\} \qquad (E \subseteq \Omega). \tag{5.23}$$

The next theorem tells us that μ^* is an extension of ρ to a measure on $\sigma(\mathscr{A})$.

Theorem 5.3.2 Let $\mathscr{A}$, ρ and μ^* be as above, and let $\mathscr{F}_\mu$ denote the σ-algebra of μ^*-measurable sets as in Carathéodory's Theorem. Then

1. $\mu^*(E) = \rho(E)$ for all $E \in \mathscr{A}$.
2. $\sigma(\mathscr{A}) \subseteq \mathscr{F}_\mu$.
3. If ν is another extension of ρ to a measure on $\sigma(\mathscr{A})$, then $\nu \leq \mu^*$ on $\sigma(\mathscr{A})$.
4. If ρ is σ-finite, then μ^* is the *unique* extension of ρ to a measure on $\sigma(\mathscr{A})$.

Proof. 1. Let $E \in \mathscr{A}$ be given. Choosing $A_1 = E$ and $A_n = \emptyset$ for $n \geq 2$, then clearly,

$$E \subseteq \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \rho(A_n) = \rho(A_1) = \rho(E),$$

so that by definition of μ^*,

$$\mu^*(E) \leq \rho(E). \tag{5.24}$$

Conversely, let $\{A_n\}$ be any collection in $\mathscr{A}$ with $E \subseteq \bigcup_{n=1}^{\infty} A_n$. Applying Theorem 1.1.1, there exists a disjoint collection $\{B_n\}$ in $\mathscr{A}$ so that

$$B_n \subseteq A_n \quad \text{for all } n \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Set $\tilde{B}_n = E \cap B_n$ for all n. Then $\{\tilde{B}_n\}$ is a disjoint collection of elements of $\mathscr{A}$ with union

$$\bigcup_{n=1}^{\infty} \tilde{B}_n = \bigcup_{n=1}^{\infty} [E \cap B_n] = E \cap \left[\bigcup_{n=1}^{\infty} B_n\right] = E \cap \left[\bigcup_{n=1}^{\infty} A_n\right] = E,$$

and since ρ is a premeasure on $\mathscr{A}$ then by monotonicity,

$$\rho(E) = \sum_{n=1}^{\infty} \rho(\tilde{B}_n) \leq \sum_{n=1}^{\infty} \rho(A_n).$$

Now as $\{A_n\}$ was an arbitrary countable covering of E by elements of $\mathscr{A}$, it follows from (5.23) that

$$\rho(E) \leq \mu^*(E).$$

Together with (5.24), equality follows.

2. Since $\mathscr{F}_\mu$ is a σ-algebra, it suffices to show that $\mathscr{A} \subseteq \mathscr{F}_\mu$. To this end, given $E \in \mathscr{A}$, let $A \subseteq \Omega$ be arbitrary. By definition of $\mu^*(A)$, given $\varepsilon > 0$, there exists a countable collection $\{A_n\} \subseteq \mathscr{A}$ with

$$A \subseteq \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \rho(A_n) \leq \mu^*(A) + \varepsilon.$$

Applying Theorem 1.1.1 and monotonicity of ρ, we may assume that the sets A_n are disjoint. Since ρ is a premeasure, then

$$\begin{aligned}
\mu^*(A) + \varepsilon &\geq \sum_{n=1}^{\infty} \rho(A_n) = \sum_{n=1}^{\infty} \rho\Big([A_n \cap E] \cup [A_n \cap E^c]\Big) = \sum_{n=1}^{\infty} \Big[\rho(A_n \cap E) + \rho(A_n \cap E^c)\Big] \\
&= \sum_{n=1}^{\infty} \rho(A_n \cap E) + \sum_{n=1}^{\infty} \rho(A_n \cap E^c) \underset{\text{def. of } \mu^*}{\geq} \mu^*\Big(\bigcup_{n=1}^{\infty} [A_n \cap E]\Big) + \mu^*\Big(\bigcup_{n=1}^{\infty} [A_n \cap E^c]\Big) \\
&= \mu^*\left(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap E\right) + \mu^*\left(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap E^c\right) = \mu^*(A \cap E) + \mu^*(A \cap E^c).
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary then

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

and as $A \subseteq \Omega$ was arbitrary, it follows that $E \in \mathscr{F}_\mu$. This shows that $\mathscr{A} \subseteq \mathscr{F}_\mu$.

3. Let ν be another extension of ρ to a measure on $\sigma(\mathscr{A})$. If $E \in \sigma(\mathscr{A})$ and if $\{A_n\}_{n=1}^\infty$ is a covering of E by elements of $\mathscr{A}$, then by σ-subadditivity of ν,

$$\nu(E) \leq \sum_{n=1}^\infty \nu(A_n) \underset{\mu=\rho \text{ on } \mathscr{A}}{=} \sum_{n=1}^\infty \rho(A_n)$$

so that by (5.23),

$$\nu(E) \leq \mu^*(E). \tag{5.25}$$

Thus, $\nu \leq \mu^*$.

For the last part of the proof, we will need:

Claim: Let $E \in \sigma(\mathscr{A})$ with $\mu^*(E) < \infty$. Than $\nu(E) = \mu^*(E)$.

In fact by definition of of μ^*, given $\varepsilon > 0$ there exists a countable covering $\{A_n\} \subseteq \mathscr{A}$ of E with

$$\sum_{n=1}^\infty \rho(A_n) < \mu^*(E) + \varepsilon. \tag{5.26}$$

As in the proof of part 2. we may assume that the sets A_n are mutually disjoint. Then (5.26) still holds, so setting $A = \bigcup_{n=1}^\infty A_n$ we have by σ-additivity of μ^* and ν,

$$\nu(A) = \sum_{n=1}^\infty \nu(A_n) = \sum_{n=1}^\infty \rho(A_n) = \sum_{n=1}^\infty \mu^*(A_n) = \mu^*(A) < \mu^*(E) + \varepsilon < \infty.$$

Since $E \subseteq A$ and $\nu(A) = \mu^*(A) < \infty$ then

$$\begin{aligned} \mu^*(E) &\leq \mu^*(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu^*(A \setminus E) \\ &= \nu(E) + \mu^*(A) - \mu^*(E) < \nu(E) + \varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows that $\mu^*(E) \leq \nu(E)$. Together with the reverse inequality (5.25), the claim follows.

4. Finally, suppose that ρ is σ-finite. Then there exists a countable collection $\{A_n\}_{n=1}^\infty$ of elements of $\mathscr{A}$ with $\mu^*(A_n) = \rho(A_n) < \infty$ for all n and $\Omega = \bigcup_{n=1}^\infty A_n$. We may again assume that the sets A_n are disjoint. Then for all all $E \in \sigma(\mathscr{A})$ it follows by σ-additivity and the claim that

$$\mu^*(E) = \mu^*\left(\bigcup_{n=1}^\infty [A_n \cap E]\right) = \sum_{n=1}^\infty \mu^*(A_n \cap E) \underset{\text{claim}}{=} \sum_{n=1}^\infty \nu(A_n \cap E) = \nu\left(\bigcup_{n=1}^\infty [A_n \cap E]\right) = \nu(E).$$

This shows that $\mu = \nu$ and completes the proof. ■

R Observe that the main statement of this theorem is similar to Proposition A in Section 4.2. There the starting point was the collection $\mathscr{I}_6$ of half-open intervals which is not yet an algebra, and the generalized length L_F of such intervals. Since the topological methods used in the proof of Proposition A are not available in general, we had to work with premeasures instead.

Exercise 5.9 (The condition that ρ be σ-finite can not be removed in the above Theorem.) Let $\mathscr{A}$ be the collection of *finite unions* of sets of the form $(a,b] \cap \mathbb{Q}$, where $-\infty \leq a \leq b \leq \infty$.

1. Show that $\mathscr{A}$ is an algebra on $\mathbb{Q}$.
2. Show that $\sigma(\mathscr{A}) = \mathscr{P}(\mathbb{Q})$.
3. Show that $\rho(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$ is a premeasure on $\mathscr{A}$.
4. Find μ^* and find another extension ν of ρ to $\mathscr{P}(\mathbb{Q})$ which is different from μ^*.

■

5.4 Product Measures

The product of two σ-algebras

Recall the concept of product topology: Given two topological spaces X and Y, the product topology is the weakest topology on the Cartesian product $X \times Y$ containing all "open rectangles" $A \times B$, where A and B are open subsets of X, respectively Y.

Given two measurable spaces $(X, \mathscr{E})$ and $(Y, \mathscr{F})$, we introduce a σ-algebra onto the Cartesian product $X \times Y$ in a similar way:

Definition 5.4.1 Let $(X, \mathscr{E})$ and $(Y, \mathscr{F})$ be measurable spaces.

1. A set of the form $A \times B$ with $A \in \mathscr{E}$ and $B \in \mathscr{F}$ is called a *measurable rectangle* on $X \times Y$. (see also Example 5.7.)
2. The σ-algebra on $X \times Y$ generated by the collection of measurable rectangles is called the *product σ-algebra*, and denoted by $\mathscr{E} \otimes \mathscr{F}$. That is,

$$\mathscr{E} \otimes \mathscr{F} = \sigma\big(\{A \times B \,|\, A \in \mathscr{E},\, B \in \mathscr{F}\}\big).$$

(R) In a similar way, if $(X_1, \mathscr{F}_1), (X_2, \mathscr{F}_2), \ldots, (X_n, \mathscr{F}_n)$ is a finite collection of measurable spaces, then the product σ-algebra on $X_1 \times X_2 \cdots \times X_n$ is defined by

$$\mathscr{F}_1 \otimes \mathscr{F}_2 \otimes \cdots \otimes \mathscr{F}_n := \sigma\Big(\{A_1 \times A_2 \times \cdots \times A_n \,|\, A_i \in \mathscr{F}_i,\, i = 1 \ldots n\}\Big).$$

■ **Example 5.8** Consider the measurable spaces $\big(\mathbb{R}, \mathscr{B}(\mathbb{R})\big)$ and $\big(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2)\big)$.

<u>Claim:</u> $\mathscr{B}(\mathbb{R}^2) = \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$.

In fact, recall from Corollary 2.4.2 that

$$\mathscr{B}(\mathbb{R}^2) = \sigma\Big(\big\{(r_1, s_2) \times (r_2, s_2) \,\big|\, r_i < s_i,\ r_i, s_i \in \mathbb{Q},\ i = 1,2\big\}\Big).$$

Now since such open squares $(r_1, s_2) \times (r_2, s_2)$ are simply measurable rectangles in $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$ and the latter is a σ-algebra, it follows that

$$\mathscr{B}(\mathbb{R}^2) \subseteq \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}).$$

To show the reverse inclusion, first let

$$\mathscr{E}_o := \big\{A \times \mathbb{R} \,|\, A \in \mathscr{B}(\mathbb{R})\big\}.$$

It is easy to see that $\mathscr{E}_o$ is a σ-algebra on $\mathbb{R}\times\mathbb{R}$, because $\mathscr{B}(\mathbb{R})$ itself is a σ-algebra. Furthermore, the map

$$\Phi : A \in \mathscr{B}(\mathbb{R}) \mapsto A\times\mathbb{R} \in \mathscr{E}_o$$

clearly is a bijection of σ-algebras preserving unions, intersections and complements. That is, we may identify $\mathscr{B}(\mathbb{R})$ and $\mathscr{E}_o$ as σ-algebras through the mapping Φ. Now since $\mathscr{B}(\mathbb{R})$ is generated by the open subsets U of $\mathbb{R}$, then

$$\mathscr{E}_o = \sigma\Big(\big\{U\times\mathbb{R}\,\big|\,U\subseteq\mathbb{R}\text{ is open}\big\}\Big).$$

But the sets $U\times\mathbb{R}$ are open subsets of $\mathbb{R}^2$, and hence

$$\mathscr{E}_o \subseteq \sigma\Big(\big\{V \mid V\subseteq\mathbb{R}^2\text{ is open}\big\}\Big) = \mathscr{B}(\mathbb{R}^2).$$

By a symmetric argument, if

$$\mathscr{F}_o := \big\{\mathbb{R}\times B \,|\, B\in\mathscr{B}(\mathbb{R})\big\}$$

then $\mathscr{F}_o \subseteq \mathscr{B}(\mathbb{R}^2)$.

Now let $A\times B$ be a measurable rectangle, with $A,B\in\mathscr{B}(\mathbb{R})$. Since $A\times\mathbb{R}\in\mathscr{E}_o\subseteq\mathscr{B}(\mathbb{R}^2)$ and $\mathbb{R}\times B\in\mathscr{F}_o\subseteq\mathscr{B}(\mathbb{R}^2)$, it follows that

$$A\times B = (A\times\mathbb{R})\cap(\mathbb{R}\times B)\in\mathscr{B}(\mathbb{R}^2).$$

Therefore,

$$\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}) = \sigma\big(\{A\times B\,\big|\,A,B\in\mathscr{B}(\mathbb{R})\}\big)\subseteq\mathscr{B}(\mathbb{R}^2)$$

which proves the reverse inclusion, and hence the assertion. ■

(R) In a similar way one shows that

$$\mathscr{B}(\mathbb{R}^n) = \mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R})\otimes\cdots\otimes\mathscr{B}(\mathbb{R})$$

and also

$$\mathscr{B}(\mathbb{R}^{m+n}) = \mathscr{B}(\mathbb{R}^m)\otimes\mathscr{B}(\mathbb{R}^n)$$

for all positive integers m and n.

However, if $\mathscr{M}(\mathbb{R}^n)$ denotes the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$, then

$$\mathscr{M}(\mathbb{R}^n) \supsetneq \mathscr{M}(\mathbb{R})\otimes\mathscr{M}(\mathbb{R})\otimes\mathscr{M}(\mathbb{R})\otimes\cdots\otimes\mathscr{M}(\mathbb{R}).$$

(See the discussion in the second part of this section.)

By the definition of the product σ-algebra $\mathscr{E}\otimes\mathscr{F}$, one can obtain $\mathscr{E}\otimes\mathscr{F}$-measurable functions $h : X\times Y\to\mathbb{R}^*$ as products of functions of a single variable:

■ **Example 5.9** Let $(X,\mathscr{E})$ and $(Y,\mathscr{F})$ be measurable spaces, let $f : X\to\mathbb{R}^*$ be $\mathscr{E}$-measurable and $g : Y\to\mathbb{R}^*$ be $\mathscr{F}$-measurable.

Claim: The function $h : X\times Y\to\mathbb{R}^*$ given by

$$h(x,y) = f(x)g(y)$$

is $\mathscr{E}\otimes\mathscr{F}$-measurable.

To see this, we first extend f and g to functions $\hat{f},\hat{g}: X\times Y$ which are constant with respect to the second variable by

$$\hat{f}(x,y) = f(x) \qquad \text{and} \qquad \hat{g}(x,y) = g(y).$$

Let us show that $\hat{f}$ is $\mathscr{E}\otimes\mathscr{F}$-measurable. In fact, for each $a\in\mathbb{R}$ we have

$$\begin{aligned}\{(x,y)\in X\times Y : \hat{f}(x,y) < a\} &= \{(x,y)\in X\times Y : f(x) < a\}\\ &= \underbrace{\{x\in X : f(x) < a\}}_{\in\mathscr{E} \text{ by Thm. 1.5.4}}\times Y,\end{aligned}$$

a measurable rectangle in $\mathscr{E}\otimes\mathscr{F}$. Applying Theorem 1.5.4 again, it follows that $\hat{f}$ is $\mathscr{E}\otimes F$-measurable.

By a symmetric argument, $\hat{g}$ is also $\mathscr{E}\otimes\mathscr{F}$-measurable. Hence,

$$h(x,y) = f(x)g(y) = \hat{f}(x,y)\hat{g}(x,y)$$

is the product of two $\mathscr{E}\otimes\mathscr{F}$-measurable functions, and thus is also measurable. ∎

We now discuss some relationship between $\mathscr{E}\otimes\mathscr{F}$-measurable sets (respectively functions), and $\mathscr{E}$- and $\mathscr{F}$-measurable sets (respectively functions).

Definition 5.4.2 (Sections of sets) Let $E\subseteq X\times Y$.

1. Given an element $x\in X$ we set

$$E_x := \{y\in Y : (x,y)\in E\}\subseteq Y \qquad \text{("}x\text{-section").}$$

2. Similarly, given $y\in Y$ we set

$$E^y := \{x\in X : (x,y)\in E\}\subseteq X \qquad \text{("}y\text{-section").}$$

Definition 5.4.3 (Sections of functions) Let $f(x,y)$ be a function defined on $X\times Y$.

1. Given an element $x\in X$ we define a function f_x on Y by

$$f_x(y) := f(x,y) \quad \forall y\in Y \qquad \text{("}x\text{-section").}$$

2. Similarly, given an element $y\in Y$ we define a function f^y on X by

$$f^y(x) := f(x,y) \quad \forall x\in X \qquad \text{("}y\text{-section").}$$

■ **Example 5.10** Given $E\in\mathscr{E}\otimes\mathscr{F}$, let us set $f=\mathbf{1}_E$. Then for each fixed $x\in X$

$$\operatorname{range}(f_x)\subseteq\operatorname{range}(f) = \{0,1\},$$

and for all $y\in Y$,

$$(\mathbf{1}_E)_x(y)=1 \quad\Leftrightarrow\quad \mathbf{1}_E(x,y)=1 \quad\Leftrightarrow\quad (x,y)\in E \quad\Leftrightarrow\quad y\in E_x \quad\Leftrightarrow\quad \mathbf{1}_{E_x}(y)=1$$

which shows that $(\mathbf{1}_E)_x = \mathbf{1}_{E_x}$. In a similar way, $(\mathbf{1}_E)^y = \mathbf{1}_{E^y}$ for each $y\in Y$. ∎

Proposition 5.4.1 Let $(X,\mathscr{E})$ and $(Y,\mathscr{F})$ be measurable spaces.

1. Let $E \in \mathscr{E}\otimes\mathscr{F}$. Then for each $x \in X$ and $y \in Y$, we have $E_x \in \mathscr{F}$ and $E^y \in \mathscr{E}$.
2. Let $f : X\times Y \to \mathbb{R}^*$ be $\mathscr{E}\otimes\mathscr{F}$-measurable. Then for each $x \in X$ the function f_x is $\mathscr{F}$-measurable and for each $y \in Y$ the function f^y is $\mathscr{E}$-measurable.

Proof. 1. Given any $x \in X$ we set

$$\mathscr{G}_x = \left\{ E \subseteq X\times Y \;\middle|\; E_x \in \mathscr{F} \right\}.$$

Claim: $\mathscr{G}_x$ is a σ-algebra on $X\times Y$ containing $\mathscr{E}\otimes\mathscr{F}$.

In fact, observe that for each $E \subset X\times Y$ we have

$$y \in (E^c)_x \quad\Leftrightarrow\quad (x,y)\in E^c \quad\Leftrightarrow\quad (x,y)\notin E \quad\Leftrightarrow\quad y \notin E_x \quad\Leftrightarrow\quad y \in (E_x)^c.$$

That is,

$$(E^c)_x = (E_x)^c. \tag{5.27}$$

Similarly, if $\{E_n\}_{n=1}^{\infty}$ is a collection of subsets of $X\times Y$, then

$$y \in \left[\bigcup_{n=1}^{\infty} E_n\right]_x \;\Leftrightarrow\; (x,y)\in \bigcup_{n=1}^{\infty} E_n \;\Leftrightarrow\; \exists n,\ (x,y)\in E_n \;\Leftrightarrow\; \exists n,\ y\in (E_n)_x \;\Leftrightarrow\; y \in \bigcup_{n=1}^{\infty}(E_n)_x,$$

which shows that

$$\left[\bigcup_{n=1}^{\infty} E_n\right]_x = \bigcup_{n=1}^{\infty}(E_n)_x. \tag{5.28}$$

Now since $\mathscr{F}$ is a σ-algebra, it follows from (5.27) and (5.28) that $(E^c)_x \in \mathscr{F}$ and $[\bigcup_{n=1}^{\infty} E_n]_x \in \mathscr{F}$ for all $E, E_n \in \mathscr{G}_x$, which shows that $\mathscr{G}_x$ is indeed a σ-algebra. Now clearly, all measurable rectangles of the form $E = A\times B$ generating $\mathscr{E}\otimes\mathscr{F}$ are elements of $\mathscr{G}_x$, since either $E_x = B$ (if $x\in A$) or $E_x = \emptyset$ (if $x \notin A$). This shows that $\mathscr{E}\otimes\mathscr{F} \subseteq \mathscr{G}_x$ and proves the claim. In particular $E_x \in \mathscr{F}$ for all $E \in \mathscr{E}\otimes\mathscr{F}$.

By a symmetric argument, given any $y \in Y$ we have that $E^y \in \mathscr{E}$ for all $E \in \mathscr{E}\otimes\mathscr{F}$, and thus 1. follows.

2. Next let $\tilde{x} \in X$ be arbitrary, but fixed. Note that for all $a \in \mathbb{R}$,

$$\left\{ y\in Y \;\middle|\; f_{\tilde{x}}(y) > a \right\} = \left\{ y \in Y \;\middle|\; f(\tilde{x},y) > a \right\} = E_{\tilde{x}}$$

where

$$E = \left\{ (x,y)\in X\times Y \;\middle|\; f(x,y) > a \right\} \in \mathscr{E}\times\mathscr{F}$$

since f is $\mathscr{E}\otimes\mathscr{F}$ measurable. Then by part 1., $E_x \in \mathscr{F}$ no matter what a, and it follows that $f_{\tilde{x}}$ is $\mathscr{F}$-measurable.

By a symmetric argument, $f^{\tilde{y}}$ is $\mathscr{E}$-measurable for all $\tilde{y} \in Y$. Thus, the proof is complete. ■

(R) Clearly, the results of Example 5.9 and Proposition 5.4.1 can be extended to complex-valued functions, by simply splitting functions into their real and imaginary parts.

■ **Example 5.11** Let $f:\mathbb{R}^2\to\mathbb{R}^*$ be a Borel-measurable function. Since $\mathscr{B}(\mathbb{R}^2)=\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R})$, then by the above proposition, the functions

$$f^y : x\in\mathbb{R}\mapsto f(x,y) \qquad \text{and} \qquad f_x : y\in\mathbb{R}\mapsto f(x,y)$$

are Borel-measurable for all x and y in $\mathbb{R}$. ■

The Product of Two Measures

We are now ready to introduce product measures. For this purpose, let $(X,\mathscr{E},\mu)$ and $(Y,\mathscr{F},\nu)$ be two measure spaces. By Example 5.7, the collection of measurable rectangles

$$\mathscr{E}_o=\{E=A\times B \mid A\in\mathscr{E},\ B\in\mathscr{F}\}$$

is an elementary family, so that the collection of finite disjoint unions of measurable rectangles,

$$\mathscr{A}=\left\{E=\bigcup_{k=1}^{n}E_k \mid n\in\mathbb{N},\ E_k\in\mathscr{E}_o,\ E_j\cap E_k=\emptyset\ (j\neq k)\right\}$$

is an algebra on $X\times Y$. Clearly, $\sigma(\mathscr{A})=\sigma(\mathscr{E}_o)=\mathscr{E}\otimes\mathscr{F}$.

Given a measurable rectangle $A\times B\in\mathscr{E}_o$, we set

$$\rho(A\times B)=\mu(A)\nu(B)\in[0,\infty]$$

and hope to be able to extend ρ to a measure on $\mathscr{E}\otimes\mathscr{F}$.

Proposition 5.4.2 ρ extends to a premeasure on $\mathscr{A}$ by

$$\rho(E)=\sum_{k=1}^{n}\mu(A_k)\nu(B_k) \tag{5.29}$$

where $E=\bigcup_{k=1}^{n}(A_k\times B_k)\in\mathscr{A}$, a disjoint union of measurable rectangles.

Proof. We first must show that ρ in (5.29) is well defined. For this, let $E\in\mathscr{A}$ be represented as two different disjoint unions of measurable rectangles,

$$E=\bigcup_{k=1}^{n}(A_k\times B_k)=\bigcup_{j=1}^{m}(\tilde{A}_j\times\tilde{B}_j). \tag{5.30}$$

We need to show that

$$\sum_{k=1}^{n}\mu(A_k)\nu(B_k)=\sum_{j=1}^{m}\mu(\tilde{A}_j)\nu(\tilde{B}_j).$$

Since

$$\begin{aligned}E=E\cap E&=\left[\bigcup_{k=1}^{n}(A_k\times B_k)\right]\bigcap\left[\bigcup_{j=1}^{m}(\tilde{A}_j\times\tilde{B}_j)\right]\\&=\bigcup_{k=1}^{n}\bigcup_{j=1}^{m}\left[(A_k\times B_k)\cap(\tilde{A}_j\times\tilde{B}_j)\right]=\bigcup_{k=1}^{n}\bigcup_{j=1}^{m}\left[(A_k\cap\tilde{A}_j)\times(B_k\cap\tilde{B}_j)\right]\end{aligned}$$

where all unions are disjoint, it suffices, by symmetry, to show that

$$\sum_{k=1}^{n} \mu(A_k)\nu(B_k) = \sum_{k=1}^{n}\sum_{j=1}^{m} \mu(A_k \cap \tilde{A}_j)\nu(B_k \cap \tilde{B}_j).$$

Note that for each k, $A_k \times B_k$ is the disjoint unions of measurable rectangles,

$$A_k \times B_k = (A_k \times B_k) \cap E = \bigcup_{j=1}^{m} \left[(A_k \cap \tilde{A}_j) \times (B_k \cap \tilde{B}_j)\right],$$

hence it suffices to show that

$$\mu(A)\nu(B) = \sum_{j=1}^{m} \mu(\tilde{A}_j)\nu(\tilde{B}_j) \tag{5.31}$$

whenever $A \times B$ is a measurable rectangle expressed as a disjoint union of measurable rectangles,

$$A \times B = \bigcup_{j=1}^{m} (\tilde{A}_j \times \tilde{B}_j). \tag{5.32}$$

Observe that

$$\mathbf{1}_{A\times B}(x,y) = \mathbf{1}_A(x)\mathbf{1}_B(y)$$

and since the union in (5.32) is disjoint, then

$$\mathbf{1}_A(x)\mathbf{1}_B(y) = \mathbf{1}_{A\times B}(x,y) = \sum_{j=1}^{m} \mathbf{1}_{\tilde{A}_j\times\tilde{B}_j}(x,y) = \sum_{j=1}^{m} \mathbf{1}_{\tilde{A}_j}(x)\mathbf{1}_{\tilde{B}_j}(y). \tag{5.33}$$

Integrating over x, then for all $y \in Y$ by linearity of the integral,

$$\mu(A)\mathbf{1}_B(y) = \int_X \mathbf{1}_A(x)\mathbf{1}_B(y)\,d\mu(x) = \sum_{j=1}^{m}\int_X \mathbf{1}_{\tilde{A}_j}(x)\mathbf{1}_{\tilde{B}_j}(y)\,d\mu(x) = \sum_{j=1}^{m} \mu(\tilde{A}_j)\mathbf{1}_{\tilde{B}_j}(y). \tag{5.34}$$

Integrating further over y,

$$\mu(A)\nu(B) = \int_Y \mu(A)\mathbf{1}_B(y)\,d\nu(y) = \sum_{j=1}^{m} \mu(\tilde{A}_j)\int_Y \mathbf{1}_{\tilde{B}_j}(y)\,d\nu(y) = \sum_{j=1}^{m} \mu(\tilde{A}_j)\nu(\tilde{B}_j), \tag{5.35}$$

which shows that (5.31) holds. Thus, ρ is well defined.

Next we need to show that ρ is a premeasure. Clearly

$$\rho(\emptyset) = \rho(\emptyset \times \emptyset) = \mu(\emptyset)\nu(\emptyset) = 0.$$

To prove σ-additivity, let $E \in \mathscr{A}$ be expressed as a disjoint *countable* union of members of $\mathscr{A}$,

$$E = \bigcup_{N=1}^{\infty} E_N, \qquad E_N \in \mathscr{A}. \tag{5.36}$$

We recycle the arguments of the first part of this proof with further refinements. Since $E \in \mathscr{A}$ then it is a disjoint union of the form

$$E = \bigcup_{k=1}^{n} (A_k \times B_k), \qquad A_k \times B_k \in \mathscr{E}_o.$$

On the other hand, as each $E_N \in \mathscr{E}_o$ is a finite disjoint union of measurable rectangles and the union in (5.36) disjoint, then E can be represented as a disjoint union of the form

$$E = \bigcup_{N=1}^{\infty} \bigcup_{j \in S_N} (\tilde{A}_j \times \tilde{B}_j)$$

where $\{S_N\}_{N=1}^{\infty}$ is a partition of $\mathbb{N}$ by finite subsets and

$$E_N = \bigcup_{j \in S_N} (\tilde{A}_j \times \tilde{B}_j) \qquad \text{for all } N.$$

That is,

$$E = \bigcup_{k=1}^{n} (A_k \times B_k) = \bigcup_{j=1}^{\infty} (\tilde{A}_j \times \tilde{B}_j).$$

The arguments of the first part of the proof carry over without modification, except that now $m = \infty$, while equations (5.34) and (5.35) require the application of Corollary 2.7.4. It thus follows that

$$\rho(E) = \sum_{k=1}^{n} \mu(A_k)\nu(B_k) = \sum_{j=1}^{\infty} \mu(\tilde{A}_j)\nu(\tilde{B}_j) = \sum_{N=1}^{\infty} \sum_{j \in S_N} \mu(\tilde{A}_j)\nu(\tilde{B}_j) = \sum_{N=1}^{\infty} \rho(E_N),$$

which completes the proof. ■

We note that if μ and ν are σ-finite, then ρ is also σ-finite. To see this let $\{A_k\}_{k=1}^{\infty} \subseteq \mathscr{E}$ and $\{B_j\}_{j=1}^{\infty} \subseteq \mathscr{F}$ be sets of finite measure, with $X = \bigcup_{k=1}^{\infty} A_k$ and $Y = \bigcup_{j=1}^{\infty} B_j$. Then

$$X \times Y = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} (A_k \times B_j) \qquad \text{with} \qquad \rho(A_k \times B_j) = \mu(A_k)\nu(B_j) < \infty \qquad \text{for all } k, j.$$

Applying Theorem 5.3.2 to the current setup we immediately obtain:

Theorem 5.4.3 Let $(X, \mathscr{E}, \mu)$ and $(Y, \mathscr{F}, \nu)$ be measure spaces. There exists a measure on $(X \times Y, \mathscr{E} \otimes \mathscr{F})$ denoted by $\mu \times \nu$ and satisfying:

1. for all $A \in \mathscr{E}$, $B \in \mathscr{F}$,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B), \tag{5.37}$$

2. if π is another measure on $(X \times Y, \mathscr{E} \otimes \mathscr{F})$ satisfying

$$\pi(A \times B) = \mu(A)\nu(B) \qquad \text{for all } A \in \mathscr{E}, B \in \mathscr{F}, \tag{5.38}$$

then $\pi \leq \mu \times \nu$,

3. If μ and ν are σ-finite, then $\pi = \mu \times \nu$ is the *unique* measure on $(X \times Y, \mathscr{E} \otimes \mathscr{F})$ satisfying (5.38).

(R) We call the measure space $(X \times Y, \mathscr{E} \otimes \mathscr{F}, \mu \times \nu)$ the *product measure space* of $(X, \mathscr{E}, \mu)$ and $(Y, \mathscr{F}, \nu)$, and $\mu \times \nu$ the *product measure*.

R The measure space $(X \times Y, \mathscr{E} \otimes \mathscr{F}, \mu \times \nu)$ need not be complete, even when $(X, \mathscr{E}, \mu)$ and $(Y, \mathscr{F}, \nu)$ are both complete.

In fact, suppose that $\mathscr{E} \neq \mathscr{P}(X)$ and that $\mathscr{F}$ contains a non-empty ν-null set B. Then we can pick $A \subset X$ with $A \notin \mathscr{E}$, let $E = A \times B$ and $N = X \times B$. Thus, $E \subset N$ with $(\mu \times \nu)(N) = \mu(X)\nu(B) = 0$, while by Proposition 5.4.1, $E \notin \mathscr{E} \otimes \mathscr{F}$.

However, this is not a problem, since $\mu \times \nu$ is simply the restriction of the complete measure μ^* on $\mathscr{F}_\mu$ of Theorem 5.3.2 to $\mathscr{E} \otimes \mathscr{F}$.

Observe that by Theorem 5.3.2 as well as the construction of $\mathscr{A}$ and ρ, the product measure is given by

$$(\mu \times \nu)(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) \ : \ A_n \in \mathscr{E},\ B_n \in \mathscr{F},\ E \subseteq \bigcup_{n=1}^{\infty} (A_n \times B_n) \right\}$$

for all $E \in \mathscr{E} \otimes \mathscr{F}$. It turns out that, at least for σ-finite measure spaces, there is a simpler way of computing this measure by means of iterated integrals. First, however, some remarks on measurability.

From here on, let $(X, \mathscr{E}, \mu)$ and $(Y, \mathscr{F}, \nu)$ be σ-*finite* measure spaces. Observe that if $E = A \times B$ is a measurable rectangle, then

$$E_x = \begin{cases} \emptyset & \text{if } x \notin A \\ B & \text{if } x \in A \end{cases} \qquad \text{and} \qquad E^y = \begin{cases} \emptyset & \text{if } y \notin B \\ A & \text{if } y \in B, \end{cases}$$

so that $\nu(E_x) = \nu(B)\mathbf{1}_A(x)$ for all $x \in X$, and $\mu(E^y) = \mu(A)\mathbf{1}_B(y)$ for all $y \in Y$. It follows that the function $x \in X \mapsto \nu(E_x)$ is $\mathscr{E}$-measurable and the function $y \in Y \mapsto \mu(E^y)$ is $\mathscr{F}$-measurable, and

$$(\mu \times \nu)(E) = \mu(A)\nu(B) = \int_X \nu(B)\mathbf{1}_A(x)\, d\mu(x) = \int_X \nu(E_x)\, d\mu(x) \tag{5.39}$$

while also

$$(\mu \times \nu)(E) = \mu(A)\nu(B) = \int_Y \mu(A)\mathbf{1}_B(y)\, d\nu(y) = \int_Y \mu(E^y)\, d\nu(y). \tag{5.40}$$

We thus expect the measure of an arbitrary set $E \in \mathscr{E} \otimes \mathscr{F}$ to be of the form

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y),$$

which is what we will need to prove now.

Observe that $\nu(E_x)$ and $\mu(E^y)$ are well-defined for all $E \in \mathscr{E} \otimes \mathscr{F}$ by Proposition 5.4.1. Thus, for the above two integrals to make sense, we first must make sure that the maps $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are both measurable for general E. This requires the introduction of yet another class of sets.

Definition 5.4.4 Let Ω be a set. A collection $\mathscr{C} \subseteq \mathscr{P}(\Omega)$ is called a *monotone family*, provided that for any countable family $\{A_k\}_{k=1}^{\infty} \subseteq \mathscr{C}$ we have:

(MC1) if $\{A_k\} \uparrow$ then $\bigcup_{k=1}^{\infty} A_k \in \mathscr{C}$,

(MC2) if $\{A_k\} \downarrow$ then $\bigcap_{k=1}^{\infty} A_k \in \mathscr{C}$.

(That is, $\mathscr{C}$ is closed under countable increasing unions and countable decreasing intersections.)

Note that every σ-algebra on Ω is a monotone class. In addition, if $\mathscr{C}$ is both an algebra and a monotone class, then $\mathscr{C}$ is also a σ-algebra, since for every countable collection $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{C}$ we have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}\left[\bigcup_{k=1}^{n} A_k\right] \in \mathscr{C} \qquad \text{as} \qquad \left\{\bigcup_{k=1}^{n} A_k\right\}\uparrow .$$

Furthermore (similar to what has been shown for σ-algebras), if $\mathscr{K}$ is a collection of subsets of Ω, and $\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda}$ is the collection of *all* monotone classes containing $\mathscr{K}$, then

$$\mathscr{C} = \bigcap_{\lambda\in\Lambda} \mathscr{C}_\lambda$$

is the smallest monotone class containing $\mathscr{K}$, called the *monotone class generated by* $\mathscr{K}$.

Lemma 5.4.4 *(Monotone Class Lemma)* Let $\mathscr{A}$ be an algebra of sets on Ω. Then the monotone class $\mathscr{C}$ generated by $\mathscr{A}$ coincides with the σ-algebra $\sigma(\mathscr{A})$ generated by $\mathscr{A}$.

Proof. Since every σ-algebra is also a monotone class, then clearly, $\mathscr{C} \subseteq \sigma(\mathscr{A})$. To show that $\sigma(\mathscr{A}) \subseteq \mathscr{C}$, it suffices to show that $\mathscr{C}$ is a σ-algebra.

For each $A \in \mathscr{C}$, we set

$$\mathscr{C}(A) = \left\{B \in \mathscr{C} \;\middle|\; A\backslash B,\ B\backslash A,\ A\cap B \in \mathscr{C}\right\}. \tag{5.41}$$

Claim: $\mathscr{C} = \mathscr{C}(A)$. For this end,we note that the collection $\mathscr{C}(A)$ has the following properties:

(i) Since $\emptyset \in \mathscr{A}$ and $\mathscr{A} \subseteq \mathscr{C}$, then $\emptyset, A \in \mathscr{C}(A)$.

(ii) By definition (5.41), clearly $B \in \mathscr{C}(A) \quad\Leftrightarrow\quad A \in \mathscr{C}(B)$.

(iii) $\mathscr{C}(A)$ is a monotone class. In fact, let $\{B_k\}_{k=1}^{\infty} \subseteq \mathscr{C}(A)$. Then

$$A\backslash\left[\bigcup_{k=1}^{\infty} B_k\right] = \bigcap_{k=1}^{\infty} A\backslash B_k, \quad \left[\bigcup_{k=1}^{\infty} B_k\right]\backslash A = \bigcup_{k=1}^{\infty} B_k\backslash A, \quad A\cap\left[\bigcup_{k=1}^{\infty} B_k\right] = \bigcup_{k=1}^{\infty}(A\cap B_k).$$

and by assumption, $A\backslash B_k, B_k\backslash A, A\cap B_k \in \mathscr{C}$ for all k. So if $\{B_k\}\uparrow$, then $\{A\backslash B_k\}\downarrow$, $\{B_k\backslash A\}\uparrow$ and $\{A\cap B_k\}\uparrow$, and as $\mathscr{C}$ is a monotone class, then the three above sets are elements of $\mathscr{C}$, and thus $\bigcup_{k=1}^{\infty} B_k \in \mathscr{C}(A)$ also. Similarly,

$$A\backslash\left[\bigcap_{k=1}^{\infty} B_k\right] = \bigcup_{k=1}^{\infty} A\backslash B_k, \quad \left[\bigcap_{k=1}^{\infty} B_k\right]\backslash A = \bigcap_{k=1}^{\infty} B_k\backslash A, \quad A\cap\left[\bigcap_{k=1}^{\infty} B_k\right] = \bigcap_{k=1}^{\infty}(A\cap B_k).$$

So if $\{B_k\}\downarrow$, then $\{A\backslash B_k\}\uparrow$, $\{B_k\backslash A\}\downarrow$ and $\{A\cap B_k\}\downarrow$, and as $\mathscr{C}$ is a monotone class, then the three above sets are elements of $\mathscr{C}$, and thus $\bigcap_{k=1}^{\infty} B_k \in \mathscr{C}(A)$ also.

(iv) For each $A \in \mathscr{A}$, we have $\mathscr{C}(A) = \mathscr{C}$. In fact by definition, $\mathscr{C}(A) \subseteq \mathscr{C}$. Conversely, note that if $B \in \mathscr{A}$ then also $B \in \mathscr{C}(A)$ since $A, B \in \mathscr{A}$ and $\mathscr{A}$ is an algebra. Thus, $\mathscr{A} \subseteq \mathscr{C}(A)$. Now as $\mathscr{C}$ is the smallest monotone class containing $\mathscr{A}$, it follows that $\mathscr{C} \subseteq \mathscr{C}(A)$.

(v) It now follows from (iv) and (ii) that for every $B \in \mathscr{C}$ and $A \in \mathscr{A}$ we have $A \in \mathscr{C}(B)$, which shows that $\mathscr{A} \subseteq \mathscr{C}(B)$ for all $B \in \mathscr{C}$. Now again, as $\mathscr{C}$ is the smallest monotone class containing $\mathscr{A}$, it follows that $\mathscr{C} \subseteq \mathscr{C}(B)$, that is, $\mathscr{C} = \mathscr{C}(B)$ for all $B \in \mathscr{C}$. Thus the claim is proved.

Now by the claim we have that $A\backslash B, A\cap B \in \mathscr{C}$ for all $A, B \in \mathscr{C}$. Now since $\Omega \in \mathscr{A} \subseteq \mathscr{C}$, (5.41) and Exercise 1.1 show that $\mathscr{C}$ is an algebra on Ω. On the other hand, $\mathscr{C}$ is also a monotone class by (iii), and hence it is a σ-algebra. Thus the proof is complete. ■

The monotone class property enables us to apply convergence theorems for integrals in the following two proofs.

Proposition 5.4.5 Let $(X,\mathscr{E},\mu)$ and $(Y,\mathscr{F},\nu)$ be *σ-finite* measure spaces. Then for each $E \in \mathscr{E}\otimes\mathscr{F}$,

(1) the function $f : x \mapsto \nu(E_x)$ is $\mathscr{E}$-measurable, and
(2) the function $g : y \mapsto \mu(E^y)$ is $\mathscr{F}$-measurable.

Proof. By symmetry, it suffices to proof assertion (1). We assume first that ν is a finite measure.
Recall that

$$\mathscr{A} = \left\{ E = \bigcup_{k=1}^{n} E_k \;\middle|\; n \in \mathbb{N},\ E_k = A_k \times B_k,\ A_k \in \mathscr{E},\ B_k \in \mathscr{F},\ E_k \cap E_j = \emptyset\ (k \neq j) \right\}$$

is an algebra on $X \times Y$, and $\mathscr{E}\otimes\mathscr{F} = \sigma(\mathscr{A})$. Furthermore, the discussion preceding (5.39) shows that assertion (1) holds for measurable rectangles $A \times B$. Now it is easy to see that every $E \in \mathscr{A}$ can expressed as a union of measurable rectangles of the form

$$E = \bigcup_{j=1}^{m} (A_j \times B_j) \tag{5.42}$$

where the sets B_j are mutually disjoint (but the sets A_j need not be disjoint). In addition, since for each $x \in X$,

$$E_x = \bigcup_{j=1}^{m} (A_j \times B_j)_x, \tag{5.43}$$

and $(A_j \times B_j)_x$ is either empty or equals B_j, then the union in (5.43) is disjoint, so that by additivity of the measure,

$$\nu(E_x) = \sum_{j=1}^{m} \nu\left((A_j \times B_j)_x\right).$$

Thus, $x \to \nu(E_x)$ is the finite sum of measurable functions, and hence is also measurable. This shows that assertion (1) also holds for the elements of $\mathscr{A}$. So if we set

$$\mathscr{C} = \{E \in \mathscr{E}\otimes\mathscr{F} \mid (1) \text{ holds}\},$$

then $\mathscr{A} \subseteq \mathscr{C}$.

Claim: $\mathscr{C}$ is a monotone class. In fact, let $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{C}$ be given.

If $\{E_n\}\uparrow$, we set $E = \bigcup_{n=1}^{\infty} E_n$. Then for each $x \in X$, $\{(E_n)_x\}\uparrow$ and also $E_x = \bigcup_{n=1}^{\infty}(E_n)_x$, so if we set

$$f_n(x) = \nu\big((E_n)_x\big) \qquad \text{and} \qquad f(x) = \nu(E_x), \tag{5.44}$$

then by monotonicity of the measure, $\{f_n(x)\}\uparrow$ and hence by Theorem 1.4.2, $f_n(x) \to f(x)$ for all $x \in X$. Now each f_n is $\mathscr{E}$-measurable since (1) holds for E_n, and hence f is $\mathscr{E}$-measurable as well by Theorem 1.5.7.

On the other hand, if $\{E_n\}\downarrow$, we set $E = \bigcap_{n=1}^{\infty} E_n$. Now for each $x \in X$, $\{(E_n)_x\}\downarrow$ and also $E_x = \bigcap_{n=1}^{\infty}(E_n)_x$, so if we define f_n and f as in (5.44), then $\{f_n(x)\}\downarrow$ so that by monotonicity and finiteness of the measure, again $f_n(x) \to f(x)$ for all $x \in X$. Since assertion (1) holds for each set

E_n then each f_n is $\mathscr{E}$-measurable, so that again, f is $\mathscr{E}$-measurable as well. Thus, $E \in \mathscr{C}$ again and the claim is proved.

Since $\mathscr{A} \subseteq \mathscr{C} \subseteq \mathscr{E} \otimes \mathscr{F} = \sigma(\mathscr{A})$, the Monotone Class Lemma yields that $\mathscr{C} = \mathscr{E} \otimes \mathscr{F}$, thus assertion (1) holds for all $E \in \mathscr{E} \otimes \mathscr{F}$.

In the general σ-finite case, there exists a collection $\{Y_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ with $Y = \bigcup_{n=1}^{\infty} Y_n$ and $\nu(Y_n) < \infty$ for all n. Replacing each Y_n with $\bigcup_{k=1}^{n} Y_k$ we may assume that $\{Y_n\} \uparrow$. For each n we now set

$$\nu_n(B) = \nu(B \cap Y_n) \quad \text{for } B \in \mathscr{F}.$$

Then ν_n is a finite measure on $\mathscr{F}$ which coincides with ν on measurable subsets of Y_n. Now given $E \in \mathscr{E} \otimes \mathscr{F}$, we set

$$E_n = E \cap (X \times Y_n) \qquad (n \in \mathbb{N})$$

so that $\{E_n\} \uparrow$ and also $E = \bigcup_{n=1}^{\infty} E_n$. Since each ν_n is a finite measure and $(E_n)_x \subseteq Y_n$, we obtain by the above that

$$x \mapsto \nu_n\big((E_n)_x\big) = \nu\big((E_n)_x\big)$$

is $\mathscr{E}$-measurable for all n. Now since $E = \bigcup_{n=1}^{\infty} E_n$, we may repeat the arguments of $\{E_n\} \uparrow$ above (which are valid for arbitrary ν) to conclude that $f(x) = \nu(E_x)$ is $\mathscr{E}$-measurable. ■

Theorem 5.4.6 Let $(X, \mathscr{E}, \mu)$ and $(Y, \mathscr{F}, \nu)$ be σ-*finite* measure spaces. Then for all $E \in \mathscr{E} \otimes \mathscr{F}$,

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y).$$

Proof. Again, by symmetry it suffices to show that

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu(x). \tag{5.45}$$

Suppose first that μ and ν are finite measures. We let $\mathscr{A}$ be again the algebra generated by the measurable rectangles, but now set

$$\mathscr{C} = \{\, E \in \mathscr{E} \otimes \mathscr{F} \mid (5.45) \text{ holds } \}.$$

By (5.39), $\mathscr{C}$ contains all measurable rectangles $E = A \times B$. Thus if $E \in \mathscr{A}$ is again expressed as in (5.42),

$$E = \bigcup_{j=1}^{m} E_j, \qquad \text{with the sets } (E_j)_x \text{ disjoint}$$

then by additivity of measures and integrals,

$$\begin{aligned}(\mu \times \nu)(E) &= \sum_{j=1}^{m} (\mu \times \nu)(E_j) = \sum_{j=1}^{m} \int_X \nu\left((E_j)_x\right) d\mu(x) \\ &= \int_X \left[\sum_{j=1}^{m} \nu\big((E_j)_x\big)\right] d\mu(x) = \int_X \nu\Big(\bigcup_{j=1}^{m} (E_j)_x\Big)\, d\mu(x) = \int_X \nu\left(E_x\right) d\mu(x),\end{aligned}$$

which shows that $\mathscr{A} \subseteq \mathscr{C}$.

Claim: $\mathscr{C}$ is a monotone class. In fact, let $\{E_n\}_{n=1}^{\infty} \subseteq \mathscr{C}$ be given.

If $\{E_n\} \uparrow$, we set $E = \bigcup_{n=1}^{\infty} E_n$. Keeping the notation of (5.44) we may apply Theorem 1.4.2 together with the Monotone Convergence Theorem to obtain

$$\begin{aligned}(\mu \times \nu)(E) &= \lim_{n\to\infty} (\mu \times \nu)(E_n) = \lim_{n\to\infty} \int_X \nu\big((E_n)_x\big)\, d\mu(x) \\ &= \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x) = \int_X \nu(E_x)\, d\mu(x)\end{aligned} \tag{5.46}$$

which shows that $E \in \mathscr{C}$.

On the other hand, if $\{E_n\} \downarrow$, we set $E = \bigcap_{n=1}^{\infty} E_n$. We keep again the notation of (5.44). Since ν is a finite measure, then $f_1(x)$ is a bounded function, and as μ is also finite, then $f_1(x)$ is integrable over X. Now since $0 \le f_n \le f_1$ we can apply the Dominated Convergence Theorem to obtain identity (5.46). Thus, $E \in \mathscr{C}$ again, and the claim follows.

Now since $\mathscr{A} \subseteq \mathscr{C} \subseteq \mathscr{E} \otimes \mathscr{F} = \sigma(\mathscr{A})$, the Monotone Class Lemma yields that $\mathscr{C} = \mathscr{E} \otimes \mathscr{F}$, thus the assertion of the theorem holds.

Next let μ and ν be σ-finite. Then there exist collections $\{X_n\}_{n=1}^{\infty} \subseteq \mathscr{E}$ and $\{Y_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ which we may assume to be increasing, with $X = \bigcup_{n=1}^{\infty} X_n$, $Y = \bigcup_{n=1}^{\infty} Y_n$ and $\mu(X_n) < \infty$, $\nu(Y_n) < \infty$ for all n. For each n we now set

$$\mu_n(A) = \mu(A \cap X_n) \quad (A \in \mathscr{E}), \qquad \nu_n(B) = \nu(B \cap Y_n) \quad (B \in \mathscr{F})$$

and

$$(\mu \times \nu)_n(E) = (\mu \times \nu)\,(E \cap (X_n \times Y_n)) \quad (E \in \mathscr{E} \otimes \mathscr{F}).$$

These are all finite measures on the respective spaces and furthermore, by definition of the product measure, it is not difficult to see that

$$(\mu \times \nu)_n = \mu_n \times \nu_n.$$

Now given $E \in \mathscr{E} \otimes \mathscr{F}$, we set

$$E_n = E \cap (X_n \times Y_n) \qquad (n \in \mathbb{N})$$

so that $E_n \subseteq X_n \times Y_n$, $\{E_n\} \uparrow$ and also $E = \bigcup_{n=1}^{\infty} E_n$. By the case of finite measures we obtain that for all n,

$$\begin{aligned}(\mu \times \nu)(E_n) &= (\mu_n \times \nu_n)(E_n) = \int_X \nu_n\big((E_n)_x\big)\, d\mu_n(x) \\ &= \int_X \nu\big((E_n)_x\big)\, d\mu_n(x) = \int_X \nu\big((E_n)_x\big)\, d\mu(x)\end{aligned}$$

where we have used the fact that $E_n \subseteq X_n \times Y_n$, so that in particular, $(E_n)_x \subseteq Y_n$ and $(E_n)_x = \emptyset$ for $x \notin X_n$. Applying the same arguments as in (5.46) then (5.45) follows. ■

The next two theorems show that an integral over a product space can be expressed as an iterated integral, just as we are used to from elementary calculus. The first theorem deals with functions in the class $\mathscr{L}^+$, and the second with functions in L^1.

Theorem 5.4.7 *(Tonelli)* Let $(X,\mathscr{E},\mu)$ and $(Y,\mathscr{F},\nu)$ be *σ-finite* measure spaces and let $f \in \mathscr{L}^+(X \times Y, \mathscr{E} \otimes \mathscr{F})$. Then the functions

$$g(x) = \int_Y f_x(y)\,d\nu(y) = \int_Y f(x,y)\,d\nu(y)$$
$$h(y) = \int_X f^y(x)\,d\mu(x) = \int_X f(x,y)\,d\mu(x)$$

are in $\mathscr{L}^+(X)$, respectively $\mathscr{L}^+(Y)$, and the double integral can be written as an iterated integral,

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu) = \int_X \underbrace{\int_Y f(x,y)\,d\nu(y)}_{=g(x)}\,d\mu(x) = \int_Y \underbrace{\int_X f(x,y)\,d\mu(x)}_{=h(y)}\,d\nu(y). \quad (5.47)$$

Proof. By symmetry, it suffices to show that g is measurable, and that

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu) = \int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x). \quad (5.48)$$

Clearly, $g,h \geq 0$. The remainder of the proof is straightforward; we simply move through the steps involved in the definition of the integral.

First let f be an indicator function, $f = \mathbf{1}_E$ for some $E \in \mathscr{E}\otimes\mathscr{F}$. Then by Example 5.10 and Proposition 5.4.5,

$$g(x) = \int_Y (\mathbf{1}_E)_x(y)\,d\nu(y) = \int_Y \mathbf{1}_{E_x}(y)\,d\nu(y) = \nu(E_x)$$

is $\mathscr{E}$-measurable, and then by Theorem 5.4.6,

$$\begin{aligned}\int_{X\times Y} \mathbf{1}_E(x,y)\,d(\mu\times\nu) &= (\mu\times\nu)(E) = \int_X \nu(E_x)\,d\mu(x)\\ &= \int_X\int_Y \mathbf{1}_{E_x}(y)\,d\nu(y)\,d\mu(x) = \int_X\int_Y \mathbf{1}_E(x,y)\,d\nu(y)\,d\mu(x).\end{aligned} \quad (5.49)$$

Next let $f = \sum_{k=1}^n c_k\mathbf{1}_{E_k} \in \mathscr{S}^+(X\times Y,\mathscr{E}\otimes\mathscr{F})$. Then $f_x = \sum_{k=1}^n c_k\,(\mathbf{1}_{E_k})_x$, so that by linearity of the integral,

$$g(x) = \int_Y f_x(y)\,d\nu(y) = \sum_{k=1}^n c_k g_k(x) \quad \text{where} \quad g_k(x) = \int_Y (\mathbf{1}_{E_k})_x(y)\,d\nu(y).$$

Since each $g_k(x)$ is $\mathscr{E}$-measurable, it follows from Theorem 1.5.6 that g is also $\mathscr{E}$-measurable, and then again by linearity of all integrals together with (5.49),

$$\begin{aligned}\int_{X\times Y} f\,d(\mu\times\nu) &= \sum_{k=1}^n c_k \int_{X\times Y}\mathbf{1}_{E_k}\,d(\mu\times\nu) = \sum_{k=1}^n c_k\int_X\int_Y \mathbf{1}_{E_k}\,d\nu\,d\mu\\ &= \int_X\int_Y\left[\sum_{k=1}^n c_k\mathbf{1}_{E_k}\right]d\nu\,d\mu = \int_X\int_Y f\,d\nu\,d\mu,\end{aligned}$$

that is, (5.48) holds.

Finally, let $f \in \mathscr{L}^+(X \times Y, \mathscr{E} \otimes \mathscr{F})$ be arbitrary. By the Structure Theorem for Measurable Functions, there exists as sequence $\{\varphi_n\} \uparrow$ in $\mathscr{S}^+$ so that $\varphi_n \to f$ pointwise. Then for each $x \in X$, $(\varphi_n)_x \in \mathscr{S}^+$, $\{(\varphi_n)_x\} \uparrow$ and $(\varphi_n)_x \to f_x$ pointwise. By the Monotone Convergence Theorem,

$$g(x) = \int_Y f_x(y)\, d\nu(y) = \lim_{n\to\infty} g_n(x) \quad \text{where} \quad g_n(x) = \int_Y (\varphi_n)_x(y)\, d\nu(y).$$

Since each $g_n(x)$ is $\mathscr{E}$-measurable, it follows from Theorem 1.5.7 that g is also $\mathscr{E}$-measurable. Furthermore, since (5.48) holds for each φ_n, since $\{\int_Y (\varphi_n)_x d\nu\} \uparrow$ and applying the Monotone convergence Theorem several times, then

$$\begin{aligned}\int_{X\times Y} f\, d(\mu\times\nu) &= \lim_{n\to\infty} \int_{X\times Y} \varphi_n\, d(\mu\times\nu) = \lim_{n\to\infty} \int_X \int_Y \varphi_n\, d\nu\, d\mu \\ &= \int_X \lim_{n\to\infty} \left[\int_Y \varphi_n\, d\nu\right] d\mu = \int_X \int_Y \left[\lim_{n\to\infty} \varphi_n\right] d\nu\, d\mu = \int_X \int_Y f\, d\nu\, d\mu,\end{aligned}$$

so that (5.48) holds. ■

R We note that when f is integrable, then $g(x)$ and $h(y)$ must be finite valued a.e., that is, f_x and f^y will be integrable a.e.

Theorem 5.4.8 *(Fubini)* Let $(X, \mathscr{E}, \mu)$ and $(Y, \mathscr{F}, \nu)$ be σ-*finite* measure spaces and let $f \in L^1(X \times Y, \mathscr{E} \otimes \mathscr{F}, \mu \times \nu)$. Then

$$f_x \in L^1(Y, \mathscr{F}, \nu) \quad \text{a.e. } x \qquad \text{and} \qquad f^y \in L^1(X, \mathscr{E}, \mu) \quad \text{a.e. } y,$$

so that the functions

$$g(x) = \int_Y f(x,y)\, d\nu(y) \qquad \text{and} \qquad h(y) = \int_X f(x,y)\, d\mu(x)$$

are are defined a.e. Furthermore,

$$g(x) \in L^1(X, \mathscr{E}, \mu) \qquad \text{and} \qquad h(y) \in L^1(Y, \mathscr{F}, \nu),$$

and

$$\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \int_X \underbrace{\int_Y f(x,y)\, d\nu(y)}_{=g(x)}\, d\mu(x) = \int_Y \underbrace{\int_X f(x,y)\, d\mu(x)}_{=h(y)}\, d\nu(y). \quad (5.50)$$

Proof. Again, by symmetry we only need to prove half of the assertions.

First let $f : X \times Y \to \mathbb{R}^*$. Since f is integrable, then so are f^+ and f^-, and hence by the Remark following Tonelli's Theorem, the functions $(f^+)_x = (f_x)^+$ are integrable a.e. x, as are the functions $(f^-)_x = (f_x)^-$. This shows that $f_x = (f_x)^+ - (f_x)^- \in L^1(Y, \mathscr{F}, \nu)$ a.e. x.

As usual, modifying g on a null set N (to be precise, we modify f on the null set $N \times Y$) we may assume that the above holds everywhere, so that $g(x)$ is defined everywhere. By definition of the integral,

$$g^+(x) = \int_Y f^+(x,y)\, d\nu(y) \qquad \text{and} \qquad g^-(x) = \int_Y f^-(x,y)\, d\nu(y),$$

so that by linearity of the integral and Tonelli's Theorem,

$$\int_X |g|\,d\mu = \int_X (g^+ + g^-)\,d\mu = \int_X \int_Y (f^+ + f^-)\,d\nu\,d\mu$$
$$= \int_X \int_Y |f|\,d\nu\,d\mu = \int_{X\times Y} |f|\,d(\mu\times\nu) < \infty,$$

since f is assumed to be integrable. This shows that g is integrable, and then we obtain by definition of the integral as well as Tonelli's Theorem that

$$\int_X g\,d\mu = \int_X g^+\,d\mu - \int_X g^-\,d\mu = \int_X \int_Y f^+\,d\nu\,d\mu - \int_X \int_Y f^-\,d\nu\,d\mu$$
$$= \int_{X\times Y} f^+\,d(\nu\times\mu) - \int_{X\times Y} f^-\,d(\nu\times\mu) = \int_{X\times Y} f\,d(\mu\times\nu),$$

so that the left-hand equality of (5.50) holds.

Now when f is complex valued and integrable, we simply apply the above to its real and imaginary parts. The details are easy and left to the reader. ■

Exercise 5.10 Complete the proof of Fubini's Theorem for complex valued functions. ■

(R) Given a $\mathscr{E}\otimes\mathscr{F}$-measurable function $f: X\times Y \to \mathbb{K}$, one usually first applies Tonelli's theorem to $|f|$ in order to check whether $f \in L^1(X\times Y)$. Then one can use Fubini's theorem to express the double integral of f as an iterated integral as in (5.50). The next example illustrates this idea.

■ **Example 5.12** Let $(X,\mathscr{E},\mu)$ and $(Y,\mathscr{F},\nu)$ be σ-*finite* measure spaces, $f\in L^1(X)$ and $g \in L^1(Y)$. Set $h(x,y) := f(x)g(y)$. Then by Example 5.9, h is $\mathscr{E}\otimes\mathscr{F}$-measurable.
<u>Claim:</u> h is integrable, and

$$\int_{X\times Y} h(x,y)\,d(\mu\times\nu) = \left[\int_X f(x)\,d\mu\right]\left[\int_Y g(y)\,d\nu\right].$$

In fact, we have

$$\int_{X\times Y} |h(x,y)|\,d(\nu\times\mu) = \int_X\int_Y |h(x,y)|\,d\nu\,d\mu \qquad \text{(by Tonelli)}$$
$$= \int_X\int_Y |f(x)|\,|g(y)|\,d\nu\,d\mu = \int_X |f(x)|\left[\int_Y |g(y)|\,d\nu\right]d\mu$$
$$= \int_X |f(x)|\,\|g\|_1\,d\mu = \|g\|_1 \int_X |f(x)|\,d\mu = \|g\|_1\,\|f\|_1 < \infty.$$

Hence, $h \in L^1(X\times Y)$ and we can apply Fubini's theorem to repeat essentially the same computations,

$$\int_{X\times Y} h(x,y)\,d(\nu\times\mu) = \int_X\int_Y h(x,y)\,d\nu\,d\mu \qquad \text{(by Fubini)}$$
$$= \int_X\int_Y f(x)\,g(y)\,d\nu\,d\mu$$
$$= \int_X f(x)\left[\int_Y g(y)\,d\nu\right]d\mu = \left[\int_X f(x)\,d\mu\right]\left[\int_Y g(y)\,d\nu\right].$$

This proves the claim. ■

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