Group Classification of Three–wave Equations in Nonlinear optics

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Abstract

First step of Lie group analysis application to the nonlinear optics three–wave equations is a group classification: the finding of an admissible group and equivalence group, construction of an optimal system of subalgebras. We completed these operations. One of the interesting result is that nontrivial equivalence group allows to simplify these equations.

1. Introduction

The study of the light evolution in quadratic nonlinear media are strongly stimulated by many physical fascinating phenomena: self–focusing and self–trapping of light beams, soliton formation, optical–vortex solitons, parametric frequency conversation, pulse compression. Their straight applications are very important in laser related technology: long-distance optical net, all-optical switching and signal processing, laser power systems [1]. Analytical investigations of the light evolution are usually restricted by particular representations of solutions: for example, one dimensional three waves interactions and two-dimensional

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but two waves interaction process. It has been shown that one (1+1) and two (2+1) dimensional solitary waves involving quadratic nonlinearities can be formed under variety of experimental conditions and combination of equation's parameters (for example, [2–4]). Usually in real situations with arbitrary boundary conditions the equations describing the process were integrated numerically. However in this time the element of generality is lost and despite on achieved progress in the development of numerical methods, the calculation of complex flows is remained hard to come by a problem.
We would like to apply the power of the classical group analysis to an investigation of system of differential equations describing three–wave interaction in approach of slowly varying complex amplitudes. In this report we made the first step of application of group analysis: we found admissible group, equivalence group and we constructed self–normalized optimal system of subalgebras.

2. Governing equations

We consider two beams of fundamental frequency and different polarizations that propagate through the quadratic nonlinear media and generate a third beam of second harmonic frequency. In the slowly varying envelope approximation the beam evolution of three–wave interaction is described by the reduced system of equations

$$M_1A_1 = i\sigma_1 A_3 A_2^* e^{i\Delta kz}, M_2A_2 = i\sigma_2 A_3 A_1^* e^{i\Delta kz}, \qquad (2.1)$$
$$M_3A_3 = i\sigma_3 A_1 A_2 e^{-i\Delta kz}.$$

Here

$$A_1 = u_1 + iu_2,\ A_2 = u_3 + iu_4,\ A_3 = u_5 + iu_6;$$

are complex–valued amplitudes, $A_{1,2}$ are amplitudes of two fundamental harmonic fields of different polarizations and A_3 is an amplitude of the second–harmonic field,

$$M_j = \frac{\partial}{\partial z} + \beta_j \frac{\partial}{\partial x} + \frac{i}{2k_j}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{k_j}{\omega}\frac{\partial}{\partial t}, \quad (j = 1, 2, 3),$$

z is coordinate along the propagation direction, (x, y) is the transverse coordinate, t is time, $k_{1,2}$ are the linear wavenumbers at fundamental frequencies, k_3 is the linear wavenumber at second–harmonic frequency, $\Delta k = k_3 - (k_1 + k_2)$ is a wavevector–mismatch, a symbol $*$ denotes the complex conjugation, β_j are walkoff angles of the fundamental and second harmonics, ω is a light frequency, σ_j are nonlinear coupling coefficients, $\Delta k = k_3 - (k_1 + k_2)$. We consider a case of exact phase–matched condition: $\Delta k = k_3 - (k_1 + k_2) = 0$.

3. The equivalence transformations

The first step of group classification is the seeking of group equivalence. Here we explain new approach to the construction of group equivalence *
Almost all systems of differential equations have arbitrary elements: arbitrary functions or arbitrary constants. The notion of the arbitrary element is related to the fact that a lot of particular problems of mathematical physics contain a set of experimentally determined parameters or functions. These parameters and functions play role of an arbitrary elements. For example, for the gas dynamic equations such arbitrary element is the state equation. In our equations such parameters are $k_1, k_2, \omega, \beta_1, \beta_2, \beta_3, \sigma_1, \sigma_2, \sigma_3$.
The nondegenerate change of dependent and independent variables, which transfers a system of differential equations of given class to the system of equations of the same class is called the transformation of equivalence for the given class of equations. Search and determination of these transformations are the basic stage in group classification.
Let us consider a system of partial differential equations with arbitrary elements $\phi = (\phi^1, \dots, \phi^r)$:

$$F_k(x, u, p, \phi) = 0, (k = 1, 2, \dots, s). \tag{3.1}$$

Here $x = (x_1, \dots, x_n) \in R^n$ are independent variables, $u = (u^1, \dots, u^m) \in R^m$ are dependent variables, p are derivatives

*This approach was appeared as a result of work one of coauthor on the SUBMODELS program [5] (participants: L.V.Ovsiannikov, S.V.Khabirov, A.A.Talyshev, A.P.Chupahin, A.A.Cherevko, S.V.Golovin).

of dependent variables u with respect to independent x until some order r. For the sake of simplicity, below we consider the first order systems ($r = 1$). Without the loss of generality we take that in the system the arbitrary elements depend only from dependent and independent variables. The system (3.1) is called the system with arbitrary elements and determines the class of differential equations which definite representative is determined by setting the arbitrary elements $\phi(x, u)$.
The problem of the seeking of the equivalent transformations consists of in the construction of such a transformation of space $R^{n+m+r}(x, u, \phi)$ that preserves only the equations that change their representative $\phi = \phi(x, u)$. For the purpose we are looking for an one–parametric group of transformations of the space R^{n+m+r} with generator †.

$$X^e = \xi^x \partial_x + \zeta^u \partial_u + \zeta^\phi \partial_\phi.$$

Here the coordinates are:

$$\xi^i = \xi^i(x, u, \phi), \quad \zeta^{u^j} = \zeta^{u^j}(x, u, \phi), \quad \zeta^{\phi^k} = \zeta^{\phi^k}(x, u, \phi)$$

$$(i = 1, \ldots, n; j = 1, \ldots, m; k = 1, \ldots, r),$$

instead earlier [7] case $\xi^i = \xi^i(x, u, \phi)$, $\zeta^{u^j} = \zeta^{u^j}(x, u, \phi)$, $(i = 1, \ldots, n; j = 1, \ldots, m$. Since there are the derivatives p in (3.1), it is necessary to determine how they are transformed or how the coordinate of the expended operator

$$\bar{X}^e = X^e + \zeta^{u_x} \partial_{u_x} + \zeta^{\phi_x} \partial_{\phi_x} + \zeta^{\phi_u} \partial_{\phi_u} + \ldots,$$

to the generator X^e are determined.
Because the functions $\phi(x, u)$ and $u(x)$ act in different spaces, and hence, the coordinates of the extended operator $\bar{X}^e$, connected with them are defined by different formulas

$$\zeta^{u_\lambda} = D^e_\lambda \zeta^u - u_x D^e_\lambda \xi^x, \;\; D^e_\lambda = \partial_\lambda + u_\lambda \partial_u + (\phi_u u_\lambda + \phi_\lambda) \partial_\phi$$

Here λ takes the values x_i. The coordinates of the extended operator, related to the arbitrary elements, are defined by the formulas

$$\zeta^{\phi_\lambda} = \tilde{D}^e_\lambda \zeta^\phi - \phi_x \tilde{D}^e_\lambda \xi^x - \phi_u \tilde{D}^e_\lambda \zeta^u, \;\; \tilde{D}^e_\lambda = \partial_\lambda + \phi_\lambda \partial_\phi \quad (\lambda = u^j, x_i)$$

†Detail description on this approach is in [6]

To construct an equivalence group we need to obtain the group admissible by it with the extended operator $\bar{X}^e$ constructed above. Also here we must take into account possible special previously known properties of the arbitrary elements (for example, $\phi_{x_i} = 0$).
Thus, the finding of equivalence group is executed with the help of usual algorithm of a finding of admissible group of continuous transformations, but with a more general kind of coordinates of operator X^e. The assumption of dependence of all its coordinates on arbitrary elements, in general case, expand the equivalence group in comparison with earlier used algorithm [6].

4. Equivalence transformations of (2.1)

As a result of integration of determining equations for the group equivalence we have got that group equivalence of (2.1) is 18 parametric group. Part of generators in which arbitrary elements can be transformed consists of nine transformations:

$$X_1^e = x\partial_x + y\partial_y + z\partial_z - \sum_{\alpha=1}^{6} u_\alpha \partial_{u_\alpha} - k_1\partial_{k_1} - k_2\partial_{k_2},$$

$$X_2^e = t\partial_t - \omega\partial_\omega,\ X_3^e = \omega t\partial_x + \sum_{\alpha=1}^{3} k_\alpha \partial_{\beta_\alpha},\ X_4^e = z\partial_x + \sum_{\alpha=1}^{3} \partial_{\beta_\alpha},$$

$$X_5^e = \frac{(k_1 x - (\beta_3 - \beta_2)\omega t - (k_3\beta_2 - k_2\beta_3)z)}{k_1}(u_4\partial_{u_3} - u_3\partial_{u_4} + +u_6\partial_{u_5} - u_5\partial_{u_6}) - \frac{1}{k_2}\partial_{\beta_2} - \frac{1}{k_3}\partial_{\beta_3},$$

$$X_6^e = 2\sigma_1\partial_{\sigma_1} - (u_3\partial_{u_3} + u_4\partial_{u_4} + u_5\partial_{u_5} + u_6\partial_{u_6})$$

$$X_7^e = 2\sigma_2\partial_{\sigma_2} - (u_1\partial_{u_1} + u_2\partial_{u_2} + u_5\partial_{u_5} + u_6\partial_{u_6})$$

$$X_8^e = 2\sigma_3\partial_{\sigma_3} - (u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3} + u_4\partial_{u_4})$$

$$X_9^e = x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2\sum_{\alpha=1}^{6} u_\alpha\partial_{u_\alpha} - \sum_{\alpha=1}^{3} \beta_\alpha\partial_{\beta_\alpha}.$$

Due to the structure of generators X_3^e, X_4^e, X_5^e it allows to account

$$\beta_1 = \beta_2 = \beta_3 = 0.$$

Then generators of another part are

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = \partial_z,\ X_4 = x\partial_y - y\partial_x,$$

$$X_6 = xX_{10} + z\partial_x,\ X_7 = yX_{10} + z\partial_y,\ X_8 = \partial_t,$$

$$X_9 = (k_1 - k_2)zX_{10} + ((k_1 + k_2)z - 2\omega t)X_{11}$$

$$X_{10} = k_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + k_2(u_4\partial_{u_3} - u_3\partial_{u_4}) + (k_1+k_2)(u_6\partial_{u_5} - u_5\partial_{u_6}),$$

$$X_{11} = k_1(u_2\partial_{u_1} - u_1\partial_{u_2}) - k_2(u_4\partial_{u_3} - u_3\partial_{u_4}) + (k_1-k_2)(u_6\partial_{u_5} - u_5\partial_{u_6}),$$

Remark. In original calculations the coefficients of generators X_4, X_6 depend of $\beta_1, \beta_2, \beta_3$ and they have more complex form.

Remark. In the experiments it is very important to have $\beta_1^2 + \beta_2^2 + \beta_3^2 \neq 0$. But in this case admissible group is very cumbersome. Group equivalence gives us the possibility to account that the coefficients $\beta_j = 0$ $(j = 1, 2, 3)$. This fact is very important with physical point of view because it allows to understand the reason of existing possible solitary wave at the walk-off effect.

5. The admissible group

By virtue of equivalence group we account that $\beta_1 = \beta_2 = \beta_3 = 0$. After direct integration of determining equations we get that an admissible algebra

$$L_{11} = \{X_1, X_2, \dots, X_{11}\},$$

where

$$X_5 = x\partial_x + y\partial_y + 2z\partial_z + 2t\partial_t - 2\sum_{\alpha=1}^{6} u_\alpha\partial_{u_\alpha}.$$

Generator X_5 can be obtained from X_9^e by assuming of $\beta_1 = \beta_2 = \beta_3 = 0$. The table of commutators is

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
X_1	0	0	0	$-X_2$	X_1	X_{10}	0	0	0
X_2		0	0	X_1	X_2	0	X_{10}	0	0
X_3			0	0	$2X_3$	X_1	X_2	0	$\alpha X_{10} + \beta X_{11}$
X_4				0	0	X_7	$-X_6$	0	0
X_5					0	X_6	X_7	$-2X_8$	$2X_9$
X_6						0	0	0	0
X_7							0	0	0
X_8								0	$-\omega X_{11}$
X_9									0

Here $\alpha = (k_1 - k_1)/2$, $\beta = (k_1 + k_1)/2$. Two generators X_{10} and X_{11} compose the center of algebra. Inner automorphisms are constructed immediately with help of table of commutators

$$A_1: \ x_1' = x_1 + a_1 x_5,\ x_2' = x_2 - a_1 x_4,\ x_{10}' = x_{10} + a_1 x_6;$$

$$A_2: \ x_1' = x_1 + a_2 x_4,\ x_2' = x_2 + a_2 x_5,\ x_{10}' = x_{10} + a_2 x_7;$$

$$A_3: \ x_1' = x_1 + a_3 x_6,\ x_2' = x_2 + a_3 x_7,\ x_3' = x_3 + 2a_3 x_5,$$
$$x_{10}' = x_{10} + \alpha a_3 x_9,\ x_{11}' = x_{11} + \beta a_3 x_9;$$

$$A_4: \ x_1' = x_1 \cos(a_4) - x_2 \sin(a_4),\ x_2' = x_1 \sin(a_4) + x_2 \cos(a_4),$$
$$x_6' = x_6 \cos(a_4) - x_7 \sin(a_4),\ x_7' = x_6 \sin(a_4) + x_7 \cos(a_4),$$

$$A_5: \ x_1' = a_5^{-1} x_1,\ x_2' = a_5^{-1} x_2,\ x_3' = a_5^{-2} x_3,\ x_6' = a_5 x_6,$$
$$x_7' = a_5 x_7,\ x_8' = a_5^{-2} x_8,\ x_9' = a_5^2 x_9;$$

$$A_6: \ x_1' = x_1 - a_6 x_3,\ x_6' = x_6 - a_6 x_5,$$
$$x_7' = x_7 - a_6 x_4,\ x_{10}' = x_{10} - a_6 x_1 + \frac{a_6^2}{2} x_3;$$

$$A_7: \ x_2' = x_2 - a_7 x_3,\ x_6' = x_6 + a_7 x_4,$$
$$x_7' = x_7 - a_7 x_5,\ x_{10}' = x_{10} - a_7 x_2 + \frac{a_7^2}{2} x_3;$$

$$A_8: \ x_8' = x_8 + 2a_8 x_5,\ x_{11}' = x_{11} - \omega a_8 x_9;$$

$$A_9: \ x_{10}' = x_{10} - \alpha a_9 x_3,$$
$$x_{11}' = x_{11} - a_9(\omega x_8 - \beta x_3),\ x_9' = x_9 - 2a_9 x_5.$$

Also we have one involution

$$E: \ x_1' = -x_1,\ x_2' = -x_2,\ x_6' = -x_6,\ x_7' = -x_7.$$

We constructed the self–normalized optimal system of subalgebras of algebra L_{11} with the using a method which is being developed by L.V.Ovsiannikov in program SUBMODELS [5]. By two–steps algorithm [5] for the construction of optimal system of subalgebras on the first step we constructed optimal system of $L_9 = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9\}$ which consists of generators of L_{11} without center $\{X_{10}, X_{11}\}$. Because $L_9 = J^1 \oplus N^1$, where $J^1 = \{X_1, X_2, X_3\}$ is ideal and $N^1 = \{X_4, X_5, X_6, X_7, X_8, X_9\}$ is subalgebra then we can construct optimal system for subalgebra $N^1 = J^2 \oplus N^2$ with ideal $J^2 = \{X_6, X_7\}$ and subalgebra $N^2 = \{X_4, X_5, X_8, X_9\}$. Because optimal system is very cumbersome then we can not write it here.

This research was partially financial supported by Russian Fund for Fundamental Researches (96–01–01780) and National Research Council of Thailand (2308).

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