กฎการอนุรักษ์ของสมการของไหลหนึ่งมิติในพิกัดลากรานเจียน

นางสาวชมพิศ แก้วมณี

มหาวิทยาลัยเทคโนโลยีสุรนารี

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

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CONSERVATION LAWS OF ONE-DIMENSIONAL EQUATIONS OF FLUIDS IN LAGRANGIAN COORDINATES

Chompit Kaewmanee

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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CONSERVATION LAWS OF ONE-DIMENSIONAL EQUATIONS OF FLUIDS IN LAGRANGIAN COORDINATES

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ชมพิศ แก้วมณี : กฎการอนุรักษ์ของสมการของไหลหนึ่งมิติในพิกัดลากรานเจียน (CONSERVATION LAWS OF ONE-DIMENSIONAL EQUATIONS OF FLUIDS IN LAGRANGIAN COORDINATES) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร. เซอร์เก เมเลชโก, 188 หน้า.

วิทยานิพนธ์นี้ได้ศึกษาสมการของไหลในพิกัดลากรานเจียน จากการสังเกตพบว่าสมการของไหลหนึ่งมิติที่มีความเฉื่อยภายในในพิกัดลากรานเจียนนั้น สามารถถูกเขียนได้ในรูปของสมการออยเลอร์-ลากรองจ์ที่มีฟังก์ชันลากรานเจียน และทำให้สามารถนำทฤษฎีบทของนออีเธอร์มาประยุกต์เพื่อสร้างกฎการอนุรักษ์ของสมการเหล่านั้น

วิทยานิพนธ์นี้ได้แสดงการศึกษาตัวแบบทางคณิตศาสตร์ 3 ชนิดได้แก่สมการพลศาสตร์แก๊ส สมการน้ำตื้นไฮเพอร์โบลิกและตัวแบบกรีน-นากห์ดี

ผลการศึกษาได้แสดงการจำแนกกลุ่มบริบูรณ์ของสมการพลศาสตร์แก๊สหนึ่งมิติในพิกัดลากรานเจียน จากนั้นประยุกต์ใช้ทฤษฎีบทของนออีเธอร์สร้างกฎการอนุรักษ์ในพิกัดลากรานเจียนสำหรับสมการน้ำตื้นไฮเพอร์โบลิกนั้นได้พบกฎการอนุรักษ์ใหม่ที่แตกต่างจากกฎการอนุรักษ์ในพิกัดออยเลอร์ และจากการใช้ทฤษฎีบทของนออีเธอร์ ทำให้ได้กฎการอนุรักษ์ใหม่ของสมการกรีน-นากห์ดีอีกด้วย

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา________

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CHOMPIT KAEWMANEE : CONSERVATION LAWS OF ONE-DIMENSIONAL EQUATIONS OF FLUIDS IN LAGRANGIAN COORDINATES. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D. 188 PP.

NOETHER'S THEOREM/GROUP CLASSIFICATION/HYPERBOLIC SHALLOW WATER EQUATIONS/GREEN-NAGHDI MODEL/GAS DYNAMICS EQUATIONS

The equations of fluids in Lagrangian coordinates are considered in this thesis. The observation that the equations of fluids with internal inertia in Lagrangian coordinates have the form of an Euler-Lagrange equation with a natural Lagrangian allows us to apply Noether's theorem for construction conservation laws for these equations.

In this thesis three types of these models are studied: the gas dynamics equations, the hyperbolic shallow water equations and the Green-Naghdi model.

As a result of this study, the complete group classification of one-dimensional gas dynamics equations in Lagrangian coordinates is obtained. Using Noether's theorem conservation laws in Lagrangian coordinates can be constructed. For the hyperbolic shallow water equations new conservation laws which have no analog in Eulerian coordinates are obtained. Finally, using Noether's theorem a new conservation law of the Green-Naghdi equations is found.

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CHAPTER I

INTRODUCTION

The theorem which concerns the physical results and conservation laws is Noether's theorem (Noether, 1918). In 1918, Emmy Noether established a fundamental theorem of physics which gives a connection between the symmetries of a physical system with a Lagrangian and the conservation laws for the associated Euler-Lagrange equations. The application of Noether's theorem depends on the following two conditions:

1. The differential equations (DEs) under consideration must be derived from a variational principle, i.e., they are Euler-Lagrange equations.

2. The symmetries must leave the variational integral invariant.

The latter implies that not every symmetry of the DEs can generate a conservation law through Noether's theorem. Therefore a suitable Lagrangian of the differential equations is needed for application of Noether's theorem. There are some differential equations which have no Lagrangian; that means the differential equations have no a variational principle. Some approaches were developed to overcome of the limitations of Noether's theorem. These developed methods use a formula which directly generates the conservation laws and does not require the existence of a Lagrangian. The most elementary method is the direct method. This method was first used by Laplace (1798) to derive the well-known Laplace vector of the two-body Kepler problem. The direct method is applicable to any differential equation with or without Lagrangian and the construction of conservation laws through the direct method is computationally more straightforward than Noether's theorem. Bluman, Cheviakov and Anco (2010) and Anco and Bluman (1997, 2002) derived a direct method to construct the conservation laws and applied it to systems of equations that do not admit a variational principle. Ibragimov (1985) and Bluman, Temuerchaolu, and Anco (2006) showed how to directly obtain a new conservation law from a known conservation law through the action of admitted symmetries, a contact transformation.

Ibragimov (2007a, 2007b, 2011) proved a new theorem for constructing conservation laws where the existence of a Lagrangian is not required. This theorem is based on the concept of the adjoint equation and he also proved that the adjoint equation admits all symmetries of the original equation which allows the use of Noether's theorem. He also applied his conservation law approach to the gas dynamics equations (Ibragimov, 2007b).

There is extensive literature developing methods to use Noether's theorem to derive conservation laws of differential-difference equations, see more details in Webb and Mace (2014), Webb and Zank (2007,2009), Webb (2015) and Ibragimov (2007b). One of the important model in continuum mechanics is the gas dynamics equations. The gas dynamics equations are defined by the well-known conservation laws: mass, momentum and energy conservation laws. The gas dynamics equations still attract attention of researchers to derive conservation laws by applying a variety of approaches combining with Noether's theorem.

Webb and Zank (2007) presented the role of the Lagrangian map for Lie symmetries in magnetohydrodynamics (MHD) and gas dynamics by converting the Eulerian Lie point symmetries of the Galilei group to Lagrange label space. They determined the conditions for the symmetries to be a variational symmetry of the action and Noether's theorem is used to obtained the corresponding conservation laws in Eulerian and Lagrangian form. Moreover, Webb and Zank (2009) investigated conservation laws associated with the scaling symmetries of the one-dimensional ideal gas dynamic equations and Sjöberg and Mahomed (2004) showed that new conservation laws of one-dimensional gas dynamics can be generated from non-local symmetries. Webb and Mace (2014) applied Noether's theorem to investigate conservation laws in magnetohydrodynamics (MHD) and gas dynamics by using Lagrange multipliers. In 2015, Webb (2015) applied the Lagrangian map to obtain the conservation laws of the gas dynamics equations and the Clebsch representation is used to transform the conservations laws into the simple form.

In this thesis we consider a class of dispersive models.

$$\begin{gathered}\dot\rho + \rho\,\mathrm{div}(u) = 0, \qquad \rho\dot u + \nabla p = 0, \qquad \dot S = 0,\\ p = \rho\tfrac{\delta W}{\delta\rho} - W = \rho\Big(\tfrac{\partial W}{\partial\rho} - \tfrac{\partial}{\partial t}\Big(\tfrac{\partial W}{\partial\dot\rho}\Big) - \mathrm{div}\Big(\tfrac{\partial W}{\partial\dot\rho}u\Big)\Big) - W,\end{gathered} \tag{1.1}$$

where t is time, ∇ is the gradient operator with respect to the space variables, ρ is the fluid density, u is the velocity field, p is the pressure, S is the entropy and $W(\rho,\dot\rho,S)$ is a given potential, the "dot" denotes the material time derivative: $\dot f = \frac{df}{dt} = f_t + u\nabla f$, and $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u.

The model (1.1) was derived by Gavrilyuk and Shugrin (1996) and Gavrilyuk and Teshukov (2001) using the Lagrangian

$$\mathcal{L} = \rho\frac{u^2}{2} - W(\rho,\dot\rho,S). \tag{1.2}$$

In this paper (Gavrilyuk and Teshukov, 2001), it was proven that these models include the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski, 1960; Kogarko, 1961, Wijngaarden, 1968) and the dispersive shallow water model (Green and Naghdi, 1976; Salmon,1998), where $W = W(\rho,\dot\rho)$. For the Green-Naghdi model,

$$W(\rho,\dot\rho) = \frac{1}{2}g\rho^2 - \frac{1}{6}\rho\dot\rho^2,$$

where g is the gravity and ρ is the height of a free surface. Here and below, “ $\dot{-}$ ” denotes the material time derivative. There has been an increasing number of studies of properties of the Green-Naghdi system in recent years. For the Iordanski-Kogarko-Wijngaarden model,

$$W(\rho, \dot{\rho}) = \rho(c_2\varepsilon(\rho_{20}) - 2\pi\rho_{10}R^3\dot{R}^2),$$

where ρ_{10} and ρ_{20} are the physical densities of the liquid and gas components, respectively, c_2 is the mass concentration of the gas component, R is the bubble radius and ε is the internal energy of a gas component. The physical density of a gas component ρ_{20} and the bubble radius R are related with the average density ρ by the formulae

$$\frac{4}{3}\pi nR^3 = \frac{1}{\rho} - \beta, \quad \rho_{20} = c_2\left(\frac{1}{\rho} - \beta\right)^{-1}.$$

Here the physical density of the liquid component ρ_{10}, the number of bubbles per unit mass n, and $\beta = (1 - c_2)\rho_{10}^{-1}$ are constant. Notice that if one assumes that the behavior of a gas component is not isentropic, then $\varepsilon_2 = \varepsilon(\rho_{20}, S)$ and the potential function is $W = W(\rho, \dot{\rho}, S)$.

The class of dispersive models (1.1) is an example of a medium whose behavior depends not only on thermodynamical variables but also on their derivatives with respect to space and time. In this particular case the potential function depends on the total derivative of the density, which reflects the dependence of the medium on its inertia.

The first model that this thesis is focused are the one-dimensional gas dynamics equations, in particular, the potential function is determined by the condition $W_{\dot{\rho}} = 0$,

$$\rho(u_t + uu_x) + p_x = 0, \qquad \rho_t + u\rho_x + \rho u_x = 0, \qquad S_t + uS_x = 0. \tag{1.3}$$

The second class of models which is analysed in the thesis are the shallow water equations which correspond to the potential $W = \gamma_1\rho^2 - \gamma\rho\dot{\rho}^2$. In particular, the potential for classical hyperbolic shallow water equations is determined by the condition $\gamma = 0$.

The main aim of the thesis is to derive the conservation laws for (1.1) in Lagrangian coordinates: the one-dimensional gas dynamics equations, the hyperbolic shallow water equations and the Green-Naghdi model, by using Noether's theorem. The Lagrangian map is applied in order to construct the Euler-Lagrange equations using a suitable Lagrangian. The group analysis method is used for finding the admitted Lie group of the Euler-Lagrange equations and the variational integral must be invariant under the action of this admitted symmetry. Finally, Noether's theorem is allowed to be applied for constructing conservation laws for these three models.

It is also worth mentioning that Webb and Zank (2007, 2009) derived conservation laws of the one-dimensional gas dynamics equations in Lagrangian coordinates by Noether's theorem. However they did not study all admitted generators, they only considered generators converted from the generators admitted by the gas dynamics equations in Eulerian coordinates.

The structure of this thesis is as follows. In Chapter II a review of Lie group analysis which is necessary for this study is provided. Noether's theorem, Noether's identities and variational principle concepts are given in Chapter III. Computation procedures including the three approaches, i.e. Shmyglevskii's approach, Ibragimov's approach and Lagrangian's approach, which satisfy the variational principle and the obtained Euler-Lagrange equation are performed in Chapter IV. Chapter V shows how one can apply group analysis to the Euler-Lagrange equation (in Lagrangian coordinate). The group classification of the Euler-Lagrange equation with respect to the arbitrary pressure function $P = P(X, \varphi_X)$ with the restrictions $P_X \neq 0$, and $P_{\varphi_X} < 0$ is presented. Noether's theorem is allowed to be applied for constructing conservation laws for the equations of fluids. The results of conservation laws of the gas dynamics equations and the hyperbolic shallow-water equations are shown in Chapter VI. Chapter VII provides the group classification of fluids with internal inertia (1.1), and conservation laws of the Green-Naghdi model in Lagrangian coordinates are presented there. A summary and discussion are summed up in the final Chapter VIII.

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CHAPTER II

GROUP ANALYSIS

Sophus Lie (1842-1899) was a Norwegian mathematician who applied the theory of continuous transformation groups to the theory of differential equations which then gave rise to the modern theory of the so-called *Lie groups.* He showed that the Lie groups of point transformations leaving invariant a differential equation, i.e., point symmetries of a differential equation, reduced to solving related linear systems of determining equations for its infinitesimal generators. He also showed that a point symmetry of a differential equation in the case of the nth-order ordinary differential equation would reduce its order to $n-1$, and in the case of a partial differential equation would find special solutions is called *invariant solutions.*

This chapter introduces basic background knowledge of Lie groups which is necessary for the later chapters. The mathematical tools of this method are provided in Ovsiannikov (1978) and Ibragimov (1985, 1994, 1999). Many examples and results with applications of this method are collected in the the Handbooks of Lie Group Analysis of differential equation (Ibragimov, 1994).

2.1 Local Lie group of transformations

Let V be an open set in $Z = R^N$, Δ be a symmetric interval in R^1. The invertible point transformations are presented as

$$\bar{z}^i = g^i(z; a), \tag{2.1}$$

where $i = 1, 2, ..., N$, $z \in V \subset Z$ and the parameter $a \in \Delta$.

For differential equations the variables z is separated into two parts, $z = (x, u) \in V \subset Z$, where $Z = R^n \times R^m$, $N = n + m$. Here, $x = (x_1, x_2, ..., x_n)$ is the n-tuples of the independent variables and $u = (u^1, u^2, ..., u^m)$ is the m-tuples of the dependent variables. By the notations given above, the invertible transformations of the equation (2.1) are represented as

$$\bar{x}_i = \varphi^i(x, u; a), \qquad \bar{u}^j = \psi^j(x, u; a) \tag{2.2}$$

where $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, $(x, u) \in V$.

2.1.1 Local one-parameter Lie group of transformations

Definition 1. A set of transformations of equation (2.1) is a local one-parameter Lie group G if it has the following properties:

(1) $g(z; 0) = z$ for all $z \in V$,

(2) $g(g(z; a), b) = g(z; a + b)$ for all $a, b, a + b \in \Delta, z \in V$,

(3) If $a \in \Delta$ and $g(z; a) = z$ for all $z \in V$, then $a = 0$,

(4) $g \in C^\infty(V, \Delta)$.

The group property is valid only locally, i.e., only for $|a|$ and $|b|$ sufficiently small. In group analysis, G is referred to as *a local one-parameter Lie group of transformations*. For brevity, it will be simply called *a Lie group* or *a group*.

Transformations (2.2) are called *point transformations*, and the group G is called *a group of point transformations*.

The representation of the functions $\varphi^i(x, u; a)$ and $\psi^j(x, u; a)$ are given as follows:

$$\bar{x}_i = \varphi^i(x, u; a) \approx x_i + \xi^i(x, u)a, \qquad \bar{u}^j = \psi^j(x, u; a) \approx u^j + \eta^j(x, u)a, \tag{2.3}$$

where

$$\xi^i(x, u) = \left.\frac{\partial \varphi^i(x, u; a)}{\partial a}\right|_{a=0}, \qquad \eta^j(x, u) = \left.\frac{\partial \psi^j(x, u; a)}{\partial a}\right|_{a=0}. \tag{2.4}$$

Consider the first-order differential operator

$$X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u^j}, \tag{2.5}$$

where i and j are repeated indexes which mean a summation of terms with respect to i from $i = 1$ to n and with respect to j from $j = 1$ to m, respectively. This operator X is called an *infinitesimal generator* or *a generator of the Lie group of transformations* (2.2), and the terms *infinitesimal operator*, *group operator*, *group generator*, and *Lie operator* can be used interchangeably. The functions ξ^i, η^j are called the coefficients of the generator.

For a given infinitesimal transformation (2.3), a corresponding group is completely determined by the Cauchy problem of a system of ordinary differential equations, called *Lie equations*:

$$\begin{aligned} \frac{d\varphi^i}{da} &= \xi^i(\varphi,\psi), \qquad \varphi^i\big|_{a=0} = x_i \\ \frac{d\psi^j}{da} &= \eta^j(\varphi,\psi), \qquad \psi^j\big|_{a=0} = u^j. \end{aligned} \tag{2.6}$$

There is a one-to-one correspondence between Lie groups of transformations and infinitesimal generators.

2.2 Prolongation of a Lie group

The space $Z = R^n \times R^m$ is prolonged by introducing the additional variables $p = (p^k_\alpha)$. Here $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index, and the notations $|\alpha| \equiv \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\alpha_{,i} \equiv (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_n)$ are used. The variable p^k_α plays the role of a derivative,

$$p^k_\alpha = \frac{\partial^{|\alpha|}u^k}{\partial x^\alpha} = \frac{\partial^{|\alpha|}u^k}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}.$$

The space J^l of the variables :

$$x = (x_i), \qquad u = (u^k), \qquad p = (p^k_\alpha),$$

for $i = 1,2,...,n;\ k = 1,2,...,m;\ |\alpha| \le l$ is called an l-th prolongation of the space Z. This space can be provided with a manifold structure. For convenience one agrees that $J^0 \equiv Z$.

Let the infinitesimal generator

$$X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u^j} \tag{2.7}$$

be an infinitesimal generator of a Lie group of transformations of equation (2.2).

Definition 2. The generator

$$X^l = X + \sum_{j,\alpha} \eta^j_\alpha \partial_{p^j_\alpha}, \qquad (j = 1,...,m, \quad |\alpha| \le l), \tag{2.8}$$

with the coefficients

$$\eta^j_{\tilde{\alpha},k} = D_k \eta^j_{\tilde{\alpha}} - \sum_i p^j_{\tilde{\alpha},i} D_k \xi^i, \qquad (|\tilde{\alpha}| \le l-1) \tag{2.9}$$

is called the l-th prolongation of the generator X. The operators

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha} p^j_{\alpha,k} \frac{\partial}{\partial p^j_\alpha}$$

are operators of the total derivatives with respect to x_k, $(k = 1,...,n)$.

A simple example for using the prolongation formula given by equation (2.8) is illustrated for $n = m = 1$. In this case, the generator X^1 includes a local Lie group of transformations in the space J^1 :

$$\bar{x} = \varphi(x,u\,;a), \qquad \bar{u} = \psi(x,u\,;a), \qquad \bar{p} = f(x,u,p\,;a), \tag{2.10}$$

with the generator

$$X^1 = \xi^x(x,u)\partial_x + \eta^u(x,u)\partial_u + \zeta^p(x,u,p)\partial_p\,, \tag{2.11}$$

where

$$\zeta^p = D_x(\eta^u) - pD_x(\xi^x), \qquad p = \frac{du}{dx}. \tag{2.12}$$

To derive the coefficients of the prolonged operator presented in (2.12), one formulates it through the following process.

Let a function $u = u_0(x)$ be given. Substituting it into the first part of equation (2.10), one obtains

$$\bar{x} = \varphi(x, u_0(x)\,; a).$$

Since $\varphi(x, u_0(x)\,; 0) = x$, then the Jacobian at $a = 0$ is

$$\frac{\partial \bar{x}}{\partial x}\Big|_{a=0} = \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}\frac{du_0}{dx}\Big)\Big|_{a=0} = 1.$$

By virtue of the inverse function theorem, in some neighborhood of $a = 0$, one can express x as a function of $\bar{x}$ and a,

$$x = \phi(\bar{x}, a). \tag{2.13}$$

Note that after substituting (2.13) into (2.10), one has the identity:

$$\bar{x} = \varphi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \tag{2.14}$$

The transformed function $u_a(\bar{x})$ is presented as follows

$$u_a(\bar{x}) = \psi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \tag{2.15}$$

Differentiating the function $u_a(\bar{x})$ with respect to $\bar{x}$, one finds :

$$\bar{u}_{\bar{x}} = \frac{\partial u_a}{\partial \bar{x}}(\bar{x}) = \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \psi}{\partial u}\frac{du_0}{dx}\frac{\partial \phi}{\partial \bar{x}} = \Big(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}u_0'\Big)\frac{\partial \phi}{\partial \bar{x}}.$$

Next, differentiating (2.14) with respect to $\bar{x}$, one gets

$$1 = \frac{\partial \varphi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \varphi}{\partial u}\frac{du_0}{dx}\frac{\partial \phi}{\partial \bar{x}} = \Big(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\Big)\frac{\partial \phi}{\partial \bar{x}}.$$

Since $\varphi(x, u_0(x); 0) = x$, then

$$\frac{\partial \varphi}{\partial x}(\phi(\bar{x}, 0), u_0(\phi(\bar{x}, 0)); 0) = 1$$

and

$$\frac{\partial \varphi}{\partial u}(\phi(\bar{x},0),u_0(\phi(\bar{x},0));0)=0,$$

one has $\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}u_0' \neq 0$ in some neighborhood of $a=0$. Thus,

$$\frac{\partial \phi}{\partial \bar{x}}=\Big(\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}u_0'\Big)^{-1},$$

and then

$$\bar{u}_{\bar{x}}=\Big(\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial u}u_0'\Big)\Big(\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial u}u_0'\Big)^{-1}=g(x,u_0,u_0';a). \tag{2.16}$$

The transformation as shown in equation (2.10) together with :

$$\bar{u}_{\bar{x}}=g(x,u,u';a), \qquad \text{and} \qquad \bar{p}=\frac{d\bar{u}}{d\bar{x}}$$

is called the prolongation of transformation (2.10). Now, one defines the coefficient ζ^p as follows:

$$\zeta^p(x,u,p)=\frac{\partial g(x,u,p;a)}{\partial a}\Big|_{a=0}, \qquad g\big|_{a=0}=p. \tag{2.17}$$

Therefore equation (2.16) is rewritten as

$$g(x,u,p;a)\Big(\frac{\partial \varphi(x,u;a)}{\partial x}+p\frac{\partial \varphi(x,u;a)}{\partial u}\Big)=\Big(\frac{\partial \psi(x,u;a)}{\partial x}+p\frac{\partial \psi(x,u;a)}{\partial u}\Big).$$

Differentiating the latter equation with respect to the group parameter a and then substituting $a=0$, one finds :

$$\Big(\frac{\partial g}{\partial a}\Big(\frac{\partial \varphi}{\partial x}+p\frac{\partial \varphi}{\partial u}\Big)+g\Big(\frac{\partial^2 \varphi}{\partial x \partial a}+p\frac{\partial^2 \varphi}{\partial u \partial a}\Big)\Big)\Big|_{a=0}=\Big(\frac{\partial^2 \psi}{\partial x \partial a}+p\frac{\partial^2 \psi}{\partial u \partial a}\Big)\Big|_{a=0}.$$

Since $\Big(\frac{\partial \varphi}{\partial x}+p\frac{\partial \varphi}{\partial u}\Big)\Big|_{a=0}=1$, the above equation can be solved for $\frac{\partial g}{\partial a}$ and after substituting it into equation (2.17), one obtains

$$\begin{aligned}\zeta^p(x,u,p)&=\Big(\frac{\partial^2 \psi}{\partial x \partial a}+p\frac{\partial^2 \psi}{\partial u \partial a}\Big)\Big|_{a=0}-g\big|_{a=0}\Big(\frac{\partial^2 \varphi}{\partial x \partial a}+p\frac{\partial^2 \varphi}{\partial u \partial a}\Big)\Big|_{a=0}\\&=\Big(\frac{\partial \eta^u}{\partial x}+p\frac{\partial \eta^u}{\partial u}\Big)-p\Big(\frac{\partial \xi^x}{\partial x}+p\frac{\partial \xi^x}{\partial u}\Big)\\&=D_x(\eta^u)-pD_x(\xi^x)\end{aligned}$$

where $\xi^x = \frac{\partial\varphi}{\partial a}\big|_{a=0}$, $\eta^u = \frac{\partial\psi}{\partial a}\big|_{a=0}$, $\zeta^p = \frac{\partial g}{\partial a}\big|_{a=0}$, and $D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + p_x\frac{\partial}{\partial p} + ...$

Therefore the coefficients of the first prolongation of the generator (2.7) can be derived by the above process and the the first prolongation of the generator (2.7) is

$$X^{(1)} = X + \zeta^p(x, u, p)\partial_p. \qquad (2.18)$$

Similarly, one can obtains the prolongation formulae for any order prolongation of an infinitesimal generator.

2.3 Admitted Lie group

Definition 3 (Admitted Lie group). A *symmetry group of a differential equation* is a group of transformations that converts every solution of the equation into another solution of the same equation. This equation is said to be *invariant* under the symmetry group.

The terms *a symmetry group*, *a group admitted by a differential equation*, and *an admitted group* are used interchangeably.

Consider a system of differential equations,

$$F^k(x, u, p) = 0, \qquad (k = 1, 2, ..., s). \qquad (2.19)$$

Here, x is the independent variable, u is the dependent variable, and p are arbitrary partial derivatives of u with respect to x.

Let $u = u_0(x)$ be a solution of system (2.19) and the transformations depending on a parameter a given by (2.10) belong to a group admitted by system (2.19). Therefore, by the definition of an admitted group,

$$\bar{u} = \psi(x, u_0(x); a)$$

is another solution of system (2.19), where $p_0(x)$ is the derivative of the function $u_0(x)$. Hence

$$F^k(\bar{x}, \bar{u}, \bar{p}) = 0, \qquad (k = 1, 2, ..., s). \tag{2.20}$$

whenever u satisfies system (2.19). Equation (2.19) is not changed (is invariant) under the Lie group of transformations as given in equation (2.2) or, in other words, the Lie group of transformations is admitted by equation (2.19).

Theorem 1. A system of differential equations (2.19) is not changed with respect to the Lie group of transformations (2.2) with the infinitesimal generator:

$$X = \xi^i(x, u)\partial_{x_i} + \eta^j(x, u)\partial_{u^j}$$

if and only if,

$$X^{(l)} F^k(x, u, p)\Big|_{(2.19)} = 0, \qquad (k = 1, 2, ..., s). \tag{2.21}$$

Equations (2.21) are called the *determining equations*.

These determining equations are linear homogeneous differential equations for the unknown $\xi^i(x, u)$ and $\eta^j(x, u)$. Any solution of the determining equations generates an infinitesimal generator of system (2.19). The set of transformations which is generated by one-parameter Lie groups corresponding to all admitted generators X is called the *Lie group admitted by system* (2.19) or one says that system (2.19) *admits the Lie group G*.

2.4 Equivalence Lie group

Consider a system of differential equations

$$F^k(x, u, p, \theta) = 0, \qquad (k = 1, 2, ..., s), \tag{2.22}$$

where x is the independent variable, u is the dependent variable, and $\theta = \theta(x, u)$ is an arbitrary element of system (2.22). Here $(x, u) \in V \subset R^{n+m}$ and $\theta : V \to R^t$.

A nondegenerate change of the dependent and independent variables, and the arbitrary element θ, which transforms system of differential equations (2.22) to a system of differential equations of the same class or same differential structure is called an *equivalence transformation.*

The problem of constructing a Lie group of equivalence transformations consists of generating a transformation of the space $R^{n+m+t}(x,u,\theta)$ that preserves the equations while only changing their representation of $\theta = \theta(x,u)$.

A Lie group of transformations of the space R^{n+m+t} depending on a one-parameter a is considered here. Assume that the transformations

$$\bar{x} = f^x(x,u,\theta;a), \qquad \bar{u} = f^u(x,u,\theta;a), \qquad \bar{\theta} = f^\theta(x,u,\theta;a), \tag{2.23}$$

compose a Lie group of equivalence transformations and the infinitesimal generator of this group (2.23) is

$$X^e = \xi^{x_i}\partial_{x_i} + \eta^{u^j}\partial_{u^j} + \zeta^{\theta^k}\partial_{\theta^k}, \tag{2.24}$$

with the coefficients

$$\begin{aligned} \xi^{x_i} &= \frac{\partial f^{x_i}(x,u,\theta;a)}{\partial a}\Big|_{a=0}, \qquad (i = 1,...,n) \\ \eta^{u^j} &= \frac{\partial f^{u^j}(x,u,\theta;a)}{\partial a}\Big|_{a=0}, \qquad (j = 1,...,m) \\ \zeta^{\theta^k} &= \frac{\partial f^{\theta^k}(x,u,\theta;a)}{\partial a}\Big|_{a=0}, \qquad (k = 1,...,t). \end{aligned} \tag{2.25}$$

The main point to construct a Lie group of equivalence transformations is to obtain that any solution $u = u_0(x)$ of system (2.22) with the function $\theta(x,u)$ is transformed by the transformation (2.23) into the solution $u = u_a(\bar{x})$ of a system of equations (2.22) with the same function F^k, but with another transformed function $\theta_a(\bar{x},\bar{u})$.

Consider the relations

$$\bar{x} = f^x(x,u,\theta(x,u);a), \qquad \bar{u} = f^u(x,u,\theta(x,u);a).$$

By virtue of the inverse function theorem, in some neighborhood of $a = 0$, one can express x and u as functions of $\bar{x}$, $\bar{u}$ and a :

$$x = g^x(\bar{x}, \bar{u}; a), \qquad u = g^u(\bar{x}, \bar{u}; a). \tag{2.26}$$

Substituting (2.26) into equation (2.23), the transformed function is defined as follows

$$\theta_a(\bar{x}, \bar{u}) = f^\theta(x, u, \theta(x, u); a),$$

where (x, u) of the latter equation has to be substituted by their expression (2.26). Because of the definition of the function $\theta_a(\bar{x}, \bar{u})$, there is the following identity with respect to x and u :

$$(\theta \circ (f^x, f^u))(x, u, \theta(x, u); a) = f^\theta(x, u, \theta(x, u); a).$$

As $u = u_0(x)$ is a given solution of equation (2.22), to obtain the transformed solution $T_a(u) = u_a(\bar{x})$, one considers the relation

$$\bar{x} = f^x(x, u_0(x), \theta(x, u_0(x)); a).$$

By virtue of the inverse function theorem, one finds $x = \psi^x(\bar{x}; a)$. Substituting $x = \psi^x(\bar{x}; a)$ into equation (2.23), one obtains the transformed function

$$u_a(\bar{x}) = f^u(x, u_0(x), \theta(x, u_0(x)); a).$$

As for the function θ_a, notice that there is an identity with respect to x, i.e.

$$(u_a \circ f^x)(x, u_0(x), \theta(x, u_0(x)); a) = f^u(x, u_0(x), \theta(x, u_0(x)); a). \tag{2.27}$$

A formula for transformations of partial derivative $\bar{p}_a = f^p(x, u, p, \theta, ...; a)$ can be obtained by differentiating equation (2.27) with respect to $\bar{x}$.

As the transformed function $u_a(\bar{x})$ is a solution of system of differential equations (2.22) with the transformed arbitrary element $\theta_a(\bar{x}, \bar{u})$, the system of differential equations

$$F^k(\bar{x}, u_a(\bar{x}), \bar{p}_a(\bar{x}), \theta_a(\bar{x}, u_a(\bar{x}))) = 0, \qquad (k = 1, 2, ..., s).$$

must be satisfied for an arbitrary $\bar{x}$. Because of the one-to-one correspondence between x and $\bar{x}$, one has

$$F^k(f^x(z(x);a), f^u(z(x);a), f^p(z_p(x);a), f^\theta(z(x);a)) = 0, \qquad (2.28)$$

where $z(x) = (x, u_0(x), \theta(x, u_0(x)))$, and $z_p(x) = (x, u_0(x), p_0(x), \theta(x, u_0(x)), ...)$.

Differentiating equation (2.28) with respect to the group parameter a, one obtains the determining equations

$$\tilde{X}^e F^k(x,u,p,\theta)\big|_{F^k=0} = 0, \qquad (k = 1, 2, ..., s). \qquad (2.29)$$

The sign $\big|_{F^k=0}$ means that the equations $\tilde{X}^e F^k(x,u,p,\theta)$ are considered on any solution $u_0(x)$ of equation (2.22). Here $\tilde{X}^e$ is the prolonged operator for the equivalence Lie group:

$$\tilde{X}^e = X^e + \zeta^{u^j_{x_i}} \partial_{u^j_{x_i}} + \zeta^{\theta^k_{x_i}} \partial_{\theta^k_{x_i}} + \zeta^{\theta^k_{u^j}} \partial_{\theta^k_{u^j}} + \dots , \qquad (2.30)$$

and the coefficients of the prolonged operator can be expressed as follows,

$$\begin{aligned}
\zeta^{u^j_{x_i}} &= D^e_{x_i}\zeta^{u^j} - u^j_{x_\beta} D^e_{x_i}\xi^{x_\beta}, \\
\zeta^{\theta^k_{x_i}} &= \tilde{D}^e_{x_i}\zeta^{\theta^k} - \theta^k_{x_\beta}\tilde{D}^e_{x_i}\xi^{x_\beta} - \theta^k_{u^j}\tilde{D}^e_{x_i}\zeta^{u^j}, \\
\zeta^{\theta^k_{u^j}} &= \tilde{D}^e_{u^j}\zeta^{\theta^k} - \theta^k_{x_i}\tilde{D}^e_{u^j}\xi^{x_i} - \theta^k_{u^\beta}\tilde{D}^e_{u_j}\zeta^{u^\beta},
\end{aligned} \qquad (2.31)$$

where

$$\begin{aligned}
D^e_{x_i} &= \partial_{x_i} + u^j_{x_i}\partial_{u^j} + (\theta^k_{x_i} + \theta^k_{u^j}u^j_{x_i})\partial_{\theta^k} + ..., \\
\tilde{D}^e_{x_i} &= \partial_{x_i} + \theta^k_{x_i}\partial_{\theta^k} + ..., \\
\tilde{D}^e_{u^j} &= \partial_{u^j} + \theta^k_{u^j}\partial_{\theta^k} + ...,
\end{aligned}$$

A solution of the determining equation (2.29) gives the coefficients of an infinitesimal generator X^e and after solving the Lie equation, one obtains the transformations as illustrated in equation (2.23). The set of transformations corresponding to this generator X^e, is called an *equivalence group*.

CHAPTER III

NOETHER'S THEOREM

In 1918, the German mathematician Emmy Noether formulated the correspondence between the symmetries of a variational principle and the conservation laws for the associated variational equations, i.e., she combined the methods of variational calculus with the theory of Lie groups to formulate a general approach for constructing conservation laws for Euler-Lagrange equations when their symmetries are known. It is commonly referred to as "Noether's theorem". The original proof of this theorem used calculus of variations, and an alternative proof were given in Ibragimov (1979).

Noether's theorem is applicable if the differential equations (DEs) under consideration satisfy a variational principle and the used symmetries leave the variational integral invariant. This implies that not every symmetry of a DE can generate a conservation law through Noether's theorem, and a suitable Lagrangian of the DE is needed.

Therefore, this Chapter will start with the basic idea of the variational principle; which an Euler-Lagrange equation can be derived. The variational derivative (or the Euler-Lagrange operator), Noether's identity, Noether's theorem and the conserved vectors formulae are also presented here.

3.1 Hamilton variational principle

Mechanics is the branch of physics studying motion. One of the aims of study in mechanics is trying to explain the World by means of the smallest possible number of universal laws and general principles. The most successful and fruitful attempts emanate from the idea that the observable events are extreme in their character, and this general principle found is called variational, i.e., they assert that certain parameters obtain their maximum or minimum values in realizable physical processes.

A variational principle was first formulated in mechanics by Pierre Mopertuis in 1744. His principle opened up a new idea of *the least action principle* which made the necessity of a technique to deal with the so-called *action functional* or *action integral*,

$$I(x(t)) = \int_{t_0}^{t_1} \mathcal{L}\big(x(t), x'(t), t\big)\, dt.$$

This technique was developed into a theory of dynamics by Euler, Lagrange, Jacobi and Hamilton; the developed techniques for their variational principle are reviewed by Berdichevsky (2009).

The most general formulation of mechanics through the principle of least action was explained by Hamilton. This theorem known as Hamilton's variational principle, states: the motion of the system from fixed time t_0 to t_1 is such that the *action integral*

$$I(q(t)) = \int_{t_0}^{t_1} \mathcal{L}\big(t, q(t), q'(t)\big)\, dt \tag{3.1}$$

is an extremum for the path $q(t)$ of motion. In other words, the variational of the action, δI, is zero for this path, i.e.

$$\delta I = 0. \tag{3.2}$$

Here t is time, $q = (q^1, ..., q^m)$ are coordinates and q' denotes the velocities of the particles of the system. The action is defined on the set of functions $q^\alpha = q^\alpha(t)$ such that the integral exists in an arbitrary interval of time $t_0 \leq t \leq t_1$.

To find the variational of the action functional, δI, one considers infinitesimally small variations, δq, of some function, q:

$$q \to \bar{q} = q + \delta q,$$

where $\delta q = \delta q(t)$ is an arbitrary function such that it is small everywhere in the interval $t_0 \le t \le t_1$ and vanishes at the boundary, i.e.,

$$\delta q(t_0) = 0 \qquad \text{and} \qquad \delta q(t_1) = 0.$$

The corresponding change (variation) in the function $\mathcal{L}\big(q, \dot{q}, t\big)$ is

$$\begin{aligned}
\delta I &= I(q + \delta q) - I(q) \\
&= \int_{t_0}^{t_1} \mathcal{L}\big(t, q + \delta q, \dot{q} + \delta \dot{q}\big) dt - \int_{t_0}^{t_1} \mathcal{L}\big(t, q, \dot{q}\big) dt \\
&= \int_{t_0}^{t_1} \Big(\mathcal{L}\big(t, q + \delta q, \dot{q} + \delta \dot{q}\big) - \mathcal{L}\big(t, q, \dot{q}\big)\Big) dt.
\end{aligned}$$

Applying multi-variables Taylor series, one has,

$$\mathcal{L}\big(t, q + \delta q, \dot{q} + \delta \dot{q}\big) = \mathcal{L}\big(t, q, \dot{q}\big) + \frac{\partial \mathcal{L}\big(t, q, \dot{q}\big)}{\partial q} \delta q + \frac{\partial \mathcal{L}\big(t, q, \dot{q}\big)}{\partial \dot{q}} \delta \dot{q} + O(\delta q^2).$$

Therefore the change in the action integral yields the linear principal part of δI (summation in $\alpha = 1, 2, ..., m$) :

$$\delta I = \int_{t_0}^{t_1} \big(\frac{\partial \mathcal{L}}{\partial q^\alpha} \delta q^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha\big) dt. \tag{3.3}$$

Since $\bar{q} = q + \delta q$, one of the explicit forms of the variations in the coordinate and velocity is

$$\begin{aligned}
\delta q &= \bar{q} - q, \\
\delta \dot{q} &= \frac{d\bar{q}}{dt} - \frac{dq}{dt} = \frac{d}{dt}\big(\bar{q} - q\big) = \frac{d}{dt} \delta q.
\end{aligned}$$

To put equation (3.3) into a suitable form for simplification, then this equation is rewritten as

$$\delta I = \int_{t_0}^{t_1} \big(\frac{\partial \mathcal{L}}{\partial q^\alpha} \delta q^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \frac{d\delta q^\alpha}{dt}\big) dt.$$

Integrating by-parts of the second term, one finds

$$\int_{t_0}^{t_1} \Big(\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \frac{d\delta q^\alpha}{dt}\Big) dt = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \delta q^\alpha \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \delta q^\alpha \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} dt.$$

Substituting this integral into equation (3.3), one has

$$\delta I = \int_{t_0}^{t_1} \Big(\frac{\partial \mathcal{L}}{\partial q^\alpha}\delta q^\alpha - \delta q^\alpha \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\Big)dt + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\delta q^\alpha\Big|_{t_0}^{t_1}$$
$$= \int_{t_0}^{t_1} \Big(\frac{\partial \mathcal{L}}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\Big)\delta q^\alpha dt + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\delta q^\alpha\Big|_{t=t_1} - \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\delta q^\alpha\Big|_{t=t_0},$$

and by the boundary condition $\delta q(t_0) = \delta q(t_1) = 0$:

$$\delta I = \int_{t_0}^{t_1} (\frac{\partial \mathcal{L}}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha})\delta q^\alpha dt.$$

The necessary condition for I to have an extremum is that $\delta I = 0$. Since for time interval $t_0 \leq t \leq t_1$ and function δq^α are arbitrary, this equation is satisfied if and only if

$$\frac{\partial \mathcal{L}}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} = 0, \qquad \alpha = 1, 2, ..., m. \tag{3.4}$$

Differential equations (3.4) are known as *Euler-Lagrange equation.* Thus the path $q = q(t)$ of a mechanical system with the Lagrangian $\mathcal{L}(t, q, \dot{q})$ solves the Euler-Lagrange equation.

One can treat a multi-dimensional problem with n independent variables $x = (x_1, x_2, ..., x_n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$ in a similar way.

Let $\mathcal{A}$ be the space of all differential functions of all finite orders, and $\mathcal{L} \in \mathcal{A}$, be a differential function of the sth order, $\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, u_{(3)}, ...,)$. Here, the notations $u_{(1)} = \{u_i^\alpha\} \equiv \{\frac{\partial u^\alpha(x)}{\partial x_i}\}, u_{(2)} = \{u_{i_1 i_2}^\alpha\} \equiv \{\frac{\partial^2 u^\alpha(x)}{\partial x_{i_1}\partial x_{i_2}}\}$, for $i_1 \leq i_2$, $u_{(3)} = \{u_{i_1 i_2 i_3}^\alpha\} \equiv \{\frac{\partial^3 u^\alpha(x)}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\}$, for $i_1 \leq i_2 \leq i_3, ...$ are the sets of first-order, second-order, third-order etc. partial derivatives where $\alpha = 1, 2, ..., m$; $i, i_1, i_2, ..., = 1, 2, ..., n$. Let $V \subset R^n$ be an arbitrary n-dimensional volume in the space of the independent variables x with the boundary ∂V.

An action integral, also termed *a variational integral* ,

$$I(u(x)) = \int_V \mathcal{L}(x, u, u_{(1)})dx \tag{3.5}$$

is defined on the set of functions $u = u(x)$ such that the action integral (3.5) exists. Here $\mathcal{L} = \mathcal{L}(x, u, u_{(1)})$ is a differential function of the first order.

The variation δI of the integral (3.5) is presented as

$$\delta I = \int_V \big(\mathcal{L}(x, u+a, u_{(1)} + a_{(1)}) - \mathcal{L}(x, u, u_{(1)})\big)dx.$$

Applying multi-variable Taylor series yields the linear terms, and one obtains:

$$\delta I = \int_V \big(\frac{\partial \mathcal{L}}{\partial u^\alpha} a^\alpha + \frac{\partial \mathcal{L}}{\partial u_i^\alpha} a_i^\alpha\big)dx.$$

Integrating by-parts of the second term and using the assumption that the functions $a^\alpha(x)$ vanish on the boundary, the above equation becomes, (see in Ibragimov (1999), Bluman, Cheviakov and Anco (2010)),

$$\delta I = \int_V \big(\frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\big(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\big)\big)a^\alpha dx.$$

A function $u = u(x)$ yields an extremum of the variational integral (3.5) if $\delta I = 0$ for any volume V and any $a = a(x)$ vanishing on the boundary. It then follows from the first expression that

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\big(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\big) = 0, \qquad \alpha = 1, 2, ..., m. \tag{3.6}$$

The operator $\dfrac{\delta \mathcal{L}}{\delta u^\alpha}$ is called the *Euler-Lagrange operator.*

Similarly, one can obtain Euler-Lagrange equations if $\mathcal{L}$ is a differential function of the second order, $\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)})$. The Euler-Lagrange equations then have the form

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\big(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\big) + D_{i_1}D_{i_2}\big(\frac{\partial \mathcal{L}}{\partial u_{i_1 i_2}^\alpha}\big) = 0, \qquad \alpha = 1, 2, ..., m.$$

In general, an Euler-Lagrange equation is in the following form

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \qquad \alpha = 1, 2, ..., m, \tag{3.7}$$

where the function $\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)})$ is a Lagrange function. The Euler-Lagrange operator is then defined by the formal sum

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} ... D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1...i_s}}, \tag{3.8}$$

where D_i is the total derivative with respect to x_i, i.e.,

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u^\alpha_{ii_1}\frac{\partial}{\partial u^\alpha_{i_1}} + u^\alpha_{ii_1i_2}\frac{\partial}{\partial u^\alpha_{i_1i_2}} + ... , \quad i = 1, 2, ..., n, \tag{3.9}$$

and for every s the summation is supposed over the repeated indices $i_1...i_s$ running from 1 to n.

3.2 Noether's theorem

Consider a one-parameter group G, of point transformations,

$$\bar{x}_i = \varphi^i(x, u; a), \qquad \bar{u}^\alpha = \psi^\alpha(x, u; a), \qquad i = 1, 2, ..., n; \quad \alpha = 1, 2, ..., m, \tag{3.10}$$

with its infinitesimal generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x_i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{3.11}$$

Definition 1. An action integral

$$\int_\Omega \mathcal{L}(x, u(x), u_{(1)}(x), ..., u_{(k)}(x))dx, \qquad \Omega \subset R^n \tag{3.12}$$

is said to be invariant under the action of the group G of transformations (3.10) if

$$\int_\Omega \mathcal{L}(x, u(x), u_{(1)}(x), ..., u_{(k)}(x))dx = \int_{\bar{\Omega}} \mathcal{L}(\bar{x}, \bar{u}(\bar{x}), \bar{u}_{(1)}(\bar{x}), ..., \bar{u}_{(k)}(\bar{x}))d\bar{x} \tag{3.13}$$

where function $u(x)$ is transformed into $\bar{u}(\bar{x})$, $\bar{\Omega} \subset R^n$ is the domain obtained from Ω by transformations (3.10).

Lemma 1. An action integral (3.12) is invariant under the group G of point transformations with infinitesimal operator (3.11) if and only if

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = 0. \tag{3.14}$$

Consider a Lie infinitesimal operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \dots \tag{3.15}$$

where the first, the second and the higher-order prolongations are

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\
\zeta_{i_1 i_2}^\alpha &= D_{i_2} D_{i_1}(\eta^\alpha) - u_j^\alpha D_{i_2} D_{i_1}(\xi^j) - u_{j i_1}^\alpha D_{i_2}(\xi^j), \\
\zeta_{i_1 \dots i_s}^\alpha &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \qquad s = 1, 2, \dots
\end{aligned}$$

The function $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ is called Lie's characteristic function. Consider the operator

$$\mathcal{N}^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{i i_1 i_2 \dots i_s}^\alpha}, \tag{3.16}$$

where the variational derivatives $\frac{\delta}{\delta u_i^\alpha}$ are obtained from (3.8) by replacing u^α by the corresponding derivative u_i^α, e.g.

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{j_s} \frac{\partial}{\partial u_{i j_1 \dots j_s}^\alpha}. \tag{3.17}$$

The operator $\mathcal{N}^i$ was introduced and called *Noether operator* in Ibragimov (1985). The operators (3.15), (3.16) and (3.17) are connected by the following theorem.

Theorem 1. **(Noether Identity)** The three operators (3.15), (3.16) and (3.17) satisfy the identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i \mathcal{N}^i. \tag{3.18}$$

Equation (3.18) is called *Noether identity* (see proof of this theorem in Ibragimov (1999)). Ibragimov (1979) used this identity for simplifying the proof of Noether's theorem.

Noether's theorem is a direct consequence of identities (3.18) and (3.14).

Theorem 2. **(Noether's theorem)** If the operator X given in (3.15) is admitted by the Euler-Lagrange equations (3.7) and satisfies the condition (3.14) of the invariance of the variational integral (an action integral), then the vector $C = (C^1, ..., C^n)$ defined by

$$C^i = \mathcal{N}^i(\mathcal{L}) \tag{3.19}$$

is a *conserved vector* for equation (3.7).

Consider the identity of equation (3.18). Applying this identity to $\mathcal{L}$, one obtains

$$X\mathcal{L} + D_i(\xi^i)\mathcal{L} = W^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i\big(\mathcal{N}^i(\mathcal{L})\big).$$

Taking into account equations (3.7) and (3.14), the vector with the components

$$C^i = \mathcal{N}^i(\mathcal{L}) \ , \qquad i = 1, \ldots, n.$$

satisfies the conservation equation

$$D_i\big(C^i\big)\Big|_{(3.7)} = 0. \tag{3.20}$$

Applying the operator (3.16) to $\mathcal{L}$, the vector field $C = (C^1, ..., C^n)$ in (3.19) can be expressed as

$$\begin{aligned} C^i = \xi^i \mathcal{L} + W^\alpha \Big[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\big(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\big) + D_j D_k\big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\big) - \ldots\Big] \\ + D_j\big(W^\alpha\big)\Big[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\big) + \ldots\Big] + D_j D_k\big(W^\alpha\big)\Big[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\Big]. \end{aligned} \tag{3.21}$$

If the invariance condition (3.14) is replaced by the divergence condition

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i(B^i), \qquad B^i \in \mathcal{A}, \tag{3.22}$$

then the fundamental identity (3.18) leads to the conservation law

$$D_i\big(C^i\big)\Big|_{(3.7)} = 0$$

where the conserved vector (3.19) is replaced by

$$C^i = \mathcal{N}^i(\mathcal{L}) - B^i, \qquad i = 1, ..., n. \tag{3.23}$$

The symmetry X satisfying condition (3.14) is called a *variational symmetry* while a symmetry satisfying condition (3.22) is called a *divergent symmetry.*

The second identity is

$$\frac{\delta}{\delta u^j}(X + D_i(\xi^i) - D_i(B^i)) = X(\frac{\delta}{\delta u^j}) + \frac{\delta}{\delta u^k}(\frac{\partial \eta^k}{\partial u^j} - \frac{\partial \xi^i}{\partial u^j}u_i^k + \delta_{kj}D_i\xi^i), \quad \text{for} \;\; j = 1, 2, ...m. \tag{3.24}$$

Applying this identity to the Lagrange function $\mathcal{L}$, one obtains

$$\frac{\delta}{\delta u^j}(X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) - D_i(B^i)) = X(\frac{\delta \mathcal{L}}{\delta u^j}) + \frac{\delta \mathcal{L}}{\delta u^k}(\frac{\partial \eta^k}{\partial u^j} - \frac{\partial \xi^i}{\partial u^j}u_i^k + \delta_{kj}D_i\xi^i).$$

If the symmetry X satisfies condition (3.22) and $\dfrac{\delta \mathcal{L}}{\delta u^k} = 0$, then the above equation is reduced to

$$X\left(\frac{\delta \mathcal{L}}{\delta u^j}\right)\Big|_{\frac{\delta \mathcal{L}}{\delta u^j}=0} = 0. \tag{3.25}$$

This latter equation shows that a variational (or divergent) symmetry is admitted by the Euler-Lagrange equation on the invariant manifold $\dfrac{\delta \mathcal{L}}{\delta u^j} = 0$. Here $\Big|\dfrac{\delta \mathcal{L}}{\delta u^j} = 0$ means that equations (3.25) are considered on the manifold $\dfrac{\delta \mathcal{L}}{\delta u^j} = 0$.

Remark. The identity of equation (3.24) becomes simpler by representing it in the case where a function $\mathcal{L}(x, u, p)$ does not depend on the second or higher derivatives. The coefficients $\xi^i = \xi^i(x, u)$ and $\eta^k = \eta^k(x, u)$ are also considered here. Identity (3.24) is valid in more general cases.

CHAPTER IV

EULER-LAGRANGE EQUATIONS

Recently, attention of scientist was attracted to the models with internal inertia (Gavrilyuk and Shugrin, 1996; Gavrilyuk and Teshukov, 2001)

$$\begin{gathered}\dot{\rho}+\rho\,\mathrm{div}(u)=0,\qquad \rho\dot{u}+\nabla p=0,\qquad \dot{S}=0,\\ p=\rho\frac{\delta W}{\delta\rho}-W=\rho\Big(\frac{\partial W}{\partial\rho}-\frac{\partial}{\partial t}\Big(\frac{\partial W}{\partial\dot{\rho}}\Big)-\mathrm{div}\Big(\frac{\partial W}{\partial\dot{\rho}}u\Big)\Big)-W,\end{gathered}\tag{4.1}$$

where, t is time, ∇ is the gradient operator with respect to the space variables, ρ is the fluid density, u is the velocity field, $W(\rho,\dot{\rho},S)$ is a given potential, the "dot" denotes the material time derivative: $\dot{f}=\frac{df}{dt}=f_t+u\nabla f$, and $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u.

The complete group classification of equations (4.1) has already been obtained by Siriwat, Kaewmanee, and Meleshko (2015) in the particular case where the potential function $W=W(\rho,\dot{\rho},S)$ satisfies condition $W_{\dot{\rho}}\neq 0$. Notice that the case $W_{\dot{\rho}}=0$ corresponds to the gas dynamic equations.

The starting objective of the present study was to construct conservation laws of one-dimensional equation of model (4.1) by applying Noether's theorem and using the complete group classification of Siriwat et al. (2015). Noether's theorem gives a procedure to find conservation laws for a system that admit a variational principle. When a given differential equations system admits a variational principle, then the extremum of its action integral yield the Euler-Lagrange equation. If one has a symmetry of the action integral, then one can obtain a conservation law through an explicit formula that involves the infinitesimal of the symmetry and the Lagrangian of the action integral.

To show that system (4.1) satisfies a variational principle, one requires a suitable Lagrangian. In this thesis, investigating for a suitable Lagrange function will be considered through three approaches, namely Shyglevskii's approach, Ibragimov's approach and Lagrangian's approach in Lagrangian coordinates (Newcomb, 1961). This chapter presents and discusses the concepts of each approach, in order to investigate the Euler-Lagrange equation.

To show the symmetries of the Euler-Lagrange equations leave the variational integral invariant needs 2 steps. The first step is to find an admitted Lie group. The second step is to show that the variational integral is invariant under the action of the admitted symmetries of the Euler-Lagrange equations such that the condition (3.14) holds.

4.1 Shmyglevskii's approach

This approach is named after Shmyglevskii (1980) even through the study of differential equations with variational principles was started earlier by Bateman (1929). Bateman derived various problems from variational principles including the hydrodynamical equations for non-viscous compressible fluid by using a variational principle form

$$\delta \int \mathcal{L} \text{ dtdxdydz} = 0,$$

in which the expression of the Lagrangian is

$$\mathcal{L} = \rho\big(\frac{u^2}{2} - U(\rho) + \dot{\varphi} + \eta\dot{\mu}\big)$$

where u is the velocity, ρ is the fluid density and U is the internal energy per mass and $\dot{f}$ is the substantial derivative : $\frac{\partial f}{\partial t} + u \cdot \nabla$.

Later on, Ito (1953) discovered that the internal energy U is not only a function of ρ but also of the entropy S. The parameters η and μ are conserved along the stream lines; such that the entropy is conserved in the reversible adiabatic process. Ito obtained a variational problem with two additive conditions, namely conservation of mass and entropy. The variational principle states that

$$\delta \int \rho\Big(\frac{u^2}{2} - U(\rho, S) + \dot{\varphi} + S\dot{\mu}\Big)\ \text{dtdxdydz} = 0\,, \tag{4.2}$$

where φ and μ play the roles of Lagrange's multipliers, S is the entropy and U is the internal energy depending on ρ and S. Varying variables ρ, u, φ, μ and S in such a manner that the variations vanish on the boundary of the region of integration, he obtained the Euler-Lagrange equations

$$\begin{aligned} u &= -\nabla\varphi - S\nabla\mu, \\ \frac{u^2}{2} - U - \rho U_\rho + \dot{\varphi} + S\dot{\mu} &= 0, \\ \dot{\mu} &= U_S, \\ \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) &= 0, \\ \frac{\partial(\rho S)}{\partial t} + \operatorname{div}(\rho S u) &= 0. \end{aligned} \tag{4.3}$$

Shmyglevskii (1980) presented the variational principle of gas dynamics. He considered the variational principle in the same manner as Ito, and then verified equations (4.3) with the thermodynamic equation

$$dU = TdS + \frac{p}{\rho^2}d\rho, \tag{4.4}$$

where p is a pressure, and he obtained the Euler-Lagrange equation

$$\dot{u} = -\frac{\nabla p}{\rho}. \tag{4.5}$$

Accordingly to Shmyglevskii's approach, the Lagrangian is

$$\mathcal{L} = \rho\Big(\frac{u^2}{2} + \dot{\varphi} + S\dot{\mu}\Big) - W(\rho, \dot{\rho}, S). \tag{4.6}$$

By the variational principle, the Euler-Lagrange equations are

$$u = -\nabla\varphi - S\nabla\mu + \rho^{-1}W_{\dot\rho}\nabla\rho, \tag{4.7a}$$

$$\frac{u^2}{2} + \dot\varphi + S\dot\mu = W_\rho - \frac{\partial W_{\dot\rho}}{\partial t} - \operatorname{div}(W_{\dot\rho}u), \tag{4.7b}$$

$$\dot\mu = \rho^{-1}W_S, \tag{4.7c}$$

$$\dot\rho + \rho\operatorname{div} u = 0, \tag{4.7d}$$

$$\frac{\partial(\rho S)}{\partial t} + \operatorname{div}(\rho S u) = 0. \tag{4.7e}$$

Excluding $\dot\rho$ from equation (4.7e), one has

$$\dot S = 0. \tag{4.8}$$

For the sake of simplicity, consider equation (4.7a) for the 1-dimension case,

$$u = \rho^{-1}W_{\dot\rho}\rho_x - \varphi_x - S\mu_x.$$

Differentiating this equation with respect to t and x, one gets

$$u_t = -\rho^{-2}\rho_t W_{\dot\rho}\rho_x + \rho^{-1}\Big(\frac{\partial W_{\dot\rho}}{\partial t}\Big)\rho_x + \rho^{-1}W_{\dot\rho}\rho_{xt} - \varphi_{xt} - S_t\mu_x - S\mu_{xt},$$

$$u_x = -\rho^{-2}\rho_x^2 W_{\dot\rho} + \rho^{-1}\Big(\frac{\partial W_{\dot\rho}}{\partial x}\Big)\rho_x + \rho^{-1}W_{\dot\rho}\rho_{xx} - \varphi_{xx} - S_x\mu_x - S\mu_{xx}.$$

One obtains that

$$\begin{aligned} u_t + uu_x &= -\rho^{-1}\Big(\rho_x\Big(\frac{\delta W}{\delta\rho}\Big) + \rho\frac{\partial}{\partial x}\Big(\frac{\delta W}{\delta\rho}\Big) - W_x\Big) \\ &= -\rho^{-1}\Big(\Big(\rho\frac{\delta W}{\delta\rho}\Big) - W\Big)_x \end{aligned}$$

where $W_x = \rho_x W_\rho + S_x W_S - W_{\dot\rho}(\rho u_x)_x$, $\quad \dfrac{\delta W}{\delta\rho} = W_\rho - \dfrac{\partial}{\partial t}W_{\dot\rho} - \operatorname{div}(W_{\dot\rho}u)$.

Introducing $p = \rho\dfrac{\delta W}{\delta\rho} - W$, one obtains

$$\rho\dot u + p_x = 0. \tag{4.9}$$

Equations (4.7d), (4.8) and (4.9) show that the one-dimensional equations of fluids with internal inertia

$$\begin{gathered} \dot\rho + \rho u_x = 0, \qquad \rho\dot u + p_x = 0, \qquad \dot S = 0, \\ p = \rho\frac{\delta W}{\delta\rho} - W = \rho\Big(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\Big(\frac{\partial W}{\partial\dot\rho}\Big) - \operatorname{div}\Big(\frac{\partial W}{\partial\dot\rho}u\Big)\Big) - W, \end{gathered} \tag{4.10}$$

can be derived from the Euler-Lagrange equation (4.7) corresponding to the Lagrangian (4.6) :

$$\mathcal{L} = \rho\big(\frac{u^2}{2} + \dot{\varphi} + S\dot{\mu}\big) - W(\rho, \dot{\rho}, S).$$

As an example, consider one of the models from Siriwat et al. (2015). Applying Noether's theorem to the model with the potential function

$$W = \rho^{-3}\dot{\rho}^2\eta, \tag{4.11}$$

equation (4.10) with (4.11) admit the generator

$$X = t\partial_t - u\partial_u. \tag{4.12}$$

One can check that this generator is a variational symmetry. Noether's theorem then gives the conservation laws

$$D_tC^1 + D_xC^2 = 0,$$

where

$$C^1 = t\rho\frac{u^2}{2} - t\rho^{-3}\eta\dot{\rho}^2 + \underline{\rho(\varphi + \eta\mu)},$$

$$C^2 = -t\rho\frac{u^3}{2} - tu\eta\rho^{-3}\dot{\rho}^2 - 4tu^2\eta\rho^{-3}\dot{\rho}\rho_x + 4tu^2\eta\rho^{-2}\rho_{tx} - 2tu^2\rho^{-2}\dot{\rho}\eta_x + 2tu^2\rho^{-2}\dot{\rho}\eta\eta_x$$
$$+2tu\rho^{-2}\dot{\rho}u_x\eta - 2tu^2\rho^{-2}u_x\eta\rho_x + 2tu\rho^{-2}\eta\rho_{tt} + 2tu\rho^{-2}\eta u_t\rho_x + 2tu^3\rho^{-3}\eta\rho_{xx} + \underline{\rho u(\varphi + \eta\mu)}.$$

The underlined terms still contain the unknown functions φ and μ which act in the roles of Lagrangian multipliers. This example shows that this approach can be applied to construct conservation laws. However, the Lagrangian multipliers φ and μ are not known, one therefore this approach is not suitable.

4.2 Ibragimov's approach

Ibragimov established the conservation law method in Ibragimov (2007a, 2011). He defined an adjoint equation for a non-linear differential equation and constructed a formal Lagrangian for an arbitrary equation considered together with its adjoint equation. It is proven that the adjoint equation inherits all symmetries of the original equation which means that application of Noether's theorem does not require existence of a classical Lagrangian. Ibragimov also applied his approach to construct conservation law for several equations such as fourth-order nonlinear partial differential equations, lubrication equations (Bruzon, Gandarias, and Ibragimov, 2007), gas dynamics equations (Ibragimov, 2007b) and Maxwell equations (Ibragimov, 2006).

Consider a system of sth-order differential equations

$$F_\alpha(x,u,u_{(1)},u_{(2)},\ldots,u_{(s)})=0\,,\qquad \alpha\quad=1,\ldots,m \tag{4.13}$$

with n independent variables $x=(x^1,x^2,\ldots,x^n)$ and m dependent variables $u=(u^1,u^2,\ldots,u^m)$. The adjoint system

$$F^*_\alpha(x,u,v,u_{(1)},v_{(1)},u_{(2)},v_{(2)},\ldots,u_{(s)},v_{(s)})\equiv\frac{\delta\mathcal{L}}{\delta u^\alpha}=0\qquad,\alpha\quad=1,\ldots,m \tag{4.14}$$

inherits the symmetries of the system (4.13), where $\mathcal{L}=v^\beta F_\beta(x,u,u_{(1)},\ldots,u_{(s)})$. Namely, if the system (4.13) admits a point transformation group with a generator

$$X=\xi^i(x,u)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{4.15}$$

then the adjoint system (4.14) also admits the operator (4.15). Then the quantities

$$C^i=v^\beta\big[\xi^i F_\beta+(\eta^\alpha-\xi^j u^\alpha_j)\frac{\partial F_\beta}{\partial u^\alpha_i}\big],\qquad i=1,\ldots,n, \tag{4.16}$$

furnish a conserved vector $C=(C^1,\ldots,C^n)$ for the system (4.13).

Applying Ibragimov's approach to equations (4.1), the formal Lagrangian is

$$\mathcal{L}=(R+\frac{u^2}{2})(\dot\rho+\rho u_x)+U(u_t+uu_x+\rho^{-1}p_x)+P\dot\eta\,, \tag{4.17}$$

where R, U, and P are Lagrangian multipliers or adjoint functions. The following functions

$$\begin{aligned}&U=\rho u, \qquad\qquad P=W_\eta-W_{\dot\rho\eta}(\dot\rho+\rho u_x),\\ &R=\rho^{-1}(p+W)+W_{\dot\rho\eta}\dot\eta\;=\;\frac{\delta W}{\delta\rho}+W_{\dot\rho\eta}\dot\eta\end{aligned} \tag{4.18}$$

satisfy adjoint system of equation (4.14).

Choosing the same example as in the first approach, the potential function and the admitted generator are

$$W = \rho^{-3}\dot{\rho}^2\eta \ , \qquad X = t\partial_t - u\partial_u.$$

Applying Noether's theorem, the conserved vectors are

$$C^1 = -tu\rho u_t - \rho u^2 - \frac{1}{2}tu^2\dot{\rho} + \frac{1}{2}tu^3\rho_x + 11t\eta\rho^{-4}\dot{\rho}^3 - 11tu\eta\rho^{-4}\dot{\rho}^2\rho_x + 8tu\eta\rho^{-3}\dot{\rho}\rho_{tx}$$
$$+ tu\eta_x\rho^{-3}\dot{\rho}^2,$$
$$C^2 = -\frac{3}{2}tu^2\rho u_t - \frac{3}{2}u^2\rho - \frac{1}{2}tu^3\dot{\rho} + \frac{1}{2}tu^4\rho_x - 5t\eta\rho^{-3}\dot{\rho}^2u_t + 23tu\eta\rho^{-4}\dot{\rho}^3 + 8tu\eta\rho^{-3}\dot{\rho}u_t\rho_x$$
$$- 47tu^2\eta\rho^{-4}\dot{\rho}^2\rho_x + 8tu^2\eta\rho^{-3}\rho_x^2u_t + 11u\eta\rho^{-3}\dot{\rho}^2 + 5tu^2\rho^{-3}\dot{\rho}^2\eta_x - 8tu^3\rho^{-3}\dot{\rho}\rho_x\eta_x$$
$$+ 8tu^2\eta\rho^{-3}\rho_x(\rho_{tt} + u\rho_{tx}) + 24tu^3\eta\rho^{-4}\dot{\rho}\rho_x^2.$$

This approach is not suitable because the equations for U, P, and R are more complicated than the original equations. For the multipliers U, P, and R given above is just an example solution which means that the Lagrangian can be found more and one can not say which one is the suitable Lagrangian and it also takes more computation on finding.

4.3 Lagrangian's approach

The Lagrangian map is applied here such that these Lagrangian coordinates satisfy the variational principles, and the Euler-Lagrange equations can be found.

4.3.1 Lagrangian map

Let $\mathcal{D}(t)$ be the position of a medium at the moment of time t. The particles are labeled by their initial position X in the reference space $\mathcal{D}(t_0)$, the motion of the continuum is defined as a diffeomorphism from $\mathcal{D}(t_0)$ into $\mathcal{D}(t)$,

$\varphi : \mathcal{D}(t_0) \to \mathcal{D}(t)$:

$$x = \varphi(X,t) \in \mathcal{D}(t).$$

Here, x is the material point (or particle) of continuous medium which is obtained as a result of movement of a fixed point $X \in \mathcal{D}(t_0)$. The set $\{x(t) \mid t \geq t_0\}$ is called the trajectory of the point $X \in \mathcal{D}(t_0)$.

The velocity u and the deformation gradient F are defined by

$$u = \frac{\partial \varphi(X,t)}{\partial t}, \qquad F = \frac{\partial x}{\partial X} = \frac{\partial \varphi(X,t)}{\partial X}.$$

Let f be a function of position x and time t, representing some physical property of the movement. There are two ways of describing the field f given on the moving continuous theorem. The first one is *Eulerian description*; it consists of giving value of the field f of x in the position $\mathcal{D}(t)$ at time t, i.e., it has a value $f(x,t)$. The second one is called *Lagrangian description.* This field is a function of each particle $X \in \mathcal{D}(t_0)$ at time t, writing $f(X,t)$. Coordinates (X,t) are called material or Lagrangian coordinates and (x,t) are called spatial or Eulerian coordinates.

To avoid possible confusion, we will use different notation for functions. The corresponding families of f in the Eulerian coordinates will be denoted by $\tilde{f}(x,t)$ and in the Lagrangian coordinates will be denoted by $\hat{f}(X,t)$. The functions $\tilde{f}(x,t)$ and $\hat{f}(X,t)$ are related by the identity

$$\hat{f}(X,t) = \tilde{f}(\varphi(X,t),t). \tag{4.19}$$

In this approach, the velocity u, the density ρ and the entropy S are defined as follows,

$$u = \frac{\partial \varphi(X,t)}{\partial t} = \varphi_t(X,t), \qquad \rho \det F = \rho_0(X), \qquad S = S_0(X), \tag{4.20}$$

where $\rho_0(X)$ is the reference density.

Therefore functions $u(x,t)$, $\rho(x,t)$, and $S(x,t)$ in Eulerian coordinates can be written through the relation (4.20) in Lagrangian coordinates as follow

$$u(\varphi(X,t),t)=\varphi_t(X,t), \qquad \rho(\varphi(X,t),t)\varphi_X(X,t)=\rho_0(X), \\ S(\varphi(X,t),t)=S_0(X). \tag{4.21}$$

4.3.2 Euler-Lagrange equations in Lagrangian coordinates

Newcomb (1962) was the first one who considered Lagrangian and Hamiltonian methods in gas dynamic equations which also can be found in Gavrilyuk (1996, 2001). The presentation of the Lagrangian of fluids containing gas bubbles (bubbly fluid) is

$$\mathcal{L}^E(t,x,\rho,u,S,\dot\rho)=\rho\frac{u^2}{2}-W(\rho,\dot\rho,S) \tag{4.22}$$

where $W=W(\rho,\dot\rho,S)$ is a given potential. Here $\mathcal{L}^E$ is presenting the Lagrange function in Eulerian coordinates.

Applying the relations (4.21) between Eulerian and Lagrangian coordinates, then the Lagrangian, $\mathcal{L}^L$, in Lagrangian coordinates is

$$\mathcal{L}^L(t,X,\varphi,\rho_0,\varphi_t,\varphi_X,\varphi_{tX},S_0)=\frac{\rho_0}{\varphi_X}\frac{\varphi_t^2}{2}-W\left(\frac{\rho_0}{\varphi_X},-\frac{\rho_0}{\varphi_X^2}\varphi_{tX},S_0(X)\right). \tag{4.23}$$

The present research considers the gas dynamics equations where the potential function is $W=W(\rho,S)$. Thus the studied Lagrangian is

$$\mathcal{L}(t,X,\varphi,\rho_0,\varphi_t,\varphi_X,S_0(X))=\rho_0\frac{\varphi_t^2}{2}-\varphi_X W\left(\frac{\rho_0}{\varphi_X},S_0(X)\right), \tag{4.24}$$

where $\mathcal{L}=\varphi_X\mathcal{L}^L$. The action integral is defined as

$$a=\int_{t_0}^{t_1}\int_{\mathcal{D}^E(t)}\mathcal{L}^E\,\mathrm{dx}\,\mathrm{dt}=\int_{t_0}^{t_1}\int_{\mathcal{D}^L(t_0)}\mathcal{L}\,\mathrm{dX}\,\mathrm{dt},$$

and the Euler-Lagrange equations related to the Lagrangian can be obtained by applying the variational principle, and it is in this following form:

$$\frac{\delta\,\mathcal{L}}{\delta\varphi}=0, \tag{4.25}$$

where

$$\frac{\delta}{\delta\varphi} = \frac{\partial}{\partial\varphi} - D_t\frac{\partial}{\partial\varphi_t} - D_X\frac{\partial}{\partial\varphi_X} + D_t^2\frac{\partial}{\partial\varphi_{tt}} + D_tD_X\frac{\partial}{\partial\varphi_{tX}} + D_X^2\frac{\partial}{\partial\varphi_{XX}} + \dots,$$

and D_t and D_X are total derivatives with respect to the Lagrangian coordinates. Moreover, the operators of total derivatives in Lagrangian and Eulerian coordinates are

$$D_X = \varphi_X D_x, \qquad D_t = \varphi_t D_x + D_{\tilde{t}}, \tag{4.26}$$

where, operators D_x, $D_{\tilde{t}}$ are total derivative in Eulerian coordinates.

Simplifying equation (4.25) for the Euler-Lagrange equation with the Lagrangian (4.24), one obtains

$$\rho_{0_X}W_{\rho\rho}\varphi_X\rho_0 + W_{\rho S}\varphi_X^2\rho_0 S_{0_X} - W_{\rho\rho}\varphi_{XX}\rho_0^2 - W_S\varphi_X^3 S_{0_X} + \varphi_{tt}\varphi_X^3\rho_0 = 0. \tag{4.27}$$

Because of the relations between variables in Eulerian and Lagrangian coordinates, one gets

$$\varphi_{tt} = \varphi_t u_x + u_{\tilde{t}}, \qquad \rho_X = \varphi_X\rho_x, \qquad S_{0X} = \varphi_X S_x, \qquad \rho_{0_X} = \rho_x\varphi_X^2 + \rho\varphi_{XX}.$$

Substituting them into (4.27), one has

$$\rho\big(u_{\tilde{t}} + uu_x\big) - S_xW_S + \rho S_xW_{\rho S} + \rho\rho_xW_{\rho\rho} = 0. \tag{4.28}$$

Introducing $p = \rho\frac{\delta W}{\delta\rho} - W = \rho W_\rho - W$, one gets

$$p_x = \rho_xW_\rho + \rho W_{\rho x} - W_x = \rho_xW_\rho + \rho\big[W_{\rho\rho}\rho_x + W_{\rho S}S_x\big] - \big[W_\rho\rho_x + W_SS_x\big]$$

$$= \rho\rho_xW_{\rho\rho} + \rho S_xW_{\rho S} - S_xW_S.$$

Finally, equation (4.28) becomes

$$\rho(u_{\tilde{t}} + uu_x) + p_x = 0. \tag{4.29}$$

The relation condition $\rho(\varphi(X,t),t)\varphi_X(X,t) = \rho_0(X)$, can be rewritten as the mass conservation law

$$\rho_{\tilde{t}} + u\rho_x + u_x\rho = 0. \tag{4.30}$$

Moreover, differentiating the relation condition $S(\varphi(X,t),t) = S_0(X)$, with respect to t, one finds

$$S_{\tilde{t}} + \varphi_t S_x = S_{\tilde{t}} + uS_x = 0. \tag{4.31}$$

Therefore (4.29), (4.30), and (4.31) can be obtained from the Euler-Lagrange equation (4.27) together with the relation condition (4.21). This means that the Euler-Lagrange equation in Lagrangian coordinates reduces to a gas dynamic equations in Eulerian coordinates.

Consider the Euler-Lagrange equation (4.27) :

$$\rho_{0_X} W_{\rho\rho}\varphi_X\rho_0 + \varphi_X^2 S_{0_X} W_{\rho S}\rho_0 - W_{\rho\rho}\varphi_{XX}\rho_0^2 - W_S\varphi_X^3 S_{0_X} + \varphi_{tt}\varphi_X^3\rho_0 = 0.$$

Without loss of generality one can assume that $\rho_0 = 1$. In fact, consider the change

$$\bar{X} = g(X), \quad \bar{\rho}_0(X) = \alpha(X)\rho_0(X), \quad \bar{t} = t, \quad \bar{\varphi} = \varphi, \quad \bar{S}_0 = S_0(X). \tag{4.32}$$

Let $X = h(\bar{X})$ be the inverse function of $g(X)$: $h(g(X)) = X$. The above transformation can be written as

$$\bar{\rho}_0(\bar{X}) = \alpha(h(\bar{X}))\rho_0(h(\bar{X}))$$

and one obtains the following conditions :

$$\bar{\rho}_{0_{\bar{X}}} = \alpha_X h'\rho_0(h(\bar{X})) + \alpha(h(\bar{X}))\rho_{0_X} h', \qquad \bar{S}_{0_{\bar{X}}} = S_{0X}h'.$$

The change in (4.32) maps a function $\varphi(X,t)$ to the function $\bar{\varphi}(\bar{X},\bar{t}) = \varphi(h(\bar{X}),\bar{t})$, and maps the potential function $W = W(\rho,S)$ to the function $\bar{W}(\bar{\rho},\bar{S}) =$

$W(\frac{\bar{\rho}_0(\bar{X})}{\bar{\varphi}_{\bar{X}}}, \bar{S}_0(\bar{X})) = W(\frac{\alpha(h(\bar{X}))\rho_0(h(\bar{X}))}{\varphi_X h'}, S_0(h(\bar{X})))$. If $h' = \alpha$, one finds these following relations,

$$\begin{gathered}\bar{\varphi}_{\bar{t}} = \varphi_t, \qquad \bar{\varphi}_{\bar{X}} = \varphi_X h', \qquad \bar{\varphi}_{\bar{t}\bar{t}} = \varphi_{tt}, \qquad \bar{\varphi}_{\bar{X}\bar{X}} = \varphi_X h'' + \varphi_{XX} h'^2, \\ \bar{W}_{\bar{\rho}\bar{\rho}} = W_{\rho\rho}, \qquad \bar{W}_{\bar{\rho}\bar{S}} = W_{\rho S} \qquad \bar{W}_{\bar{S}} = W_S.\end{gathered} \tag{4.33}$$

Substituting the Euler-Lagrange equation with all relations, one gets

$$\begin{aligned}
& -\bar{\rho}_{0_{\bar{X}}} \bar{W}_{\bar{\rho}\bar{\rho}} \bar{\varphi}_{\bar{X}} \bar{\rho}_0 - \bar{\varphi}^2_{\bar{X}} \bar{S}_{0_{\bar{X}}} \bar{W}_{\bar{\rho}\bar{S}} \bar{\rho}_0 + \bar{W}_{\bar{\rho}\bar{\rho}} \bar{\varphi}_{\bar{X}\bar{X}} \bar{\rho}_0^2 + \bar{W}_{\bar{S}} \bar{\varphi}^3_{\bar{X}} \bar{S}_{0_{\bar{X}}} - \bar{\varphi}_{\bar{t}\bar{t}} \bar{\varphi}^3_{\bar{X}} \bar{\rho}_0 \\
&= \frac{1}{h'^3}\Big(- \big(\alpha_X h' \rho_0 + \alpha \rho_{0_X} h'\big) W_{\rho\rho} \varphi_X h' \alpha \rho_0 - \varphi_X^2 h'^2 S_{0_X} h' W_{\rho S} \alpha \rho_0 \\
&\quad + W_{\rho\rho}\big(\varphi_X h'' + \varphi_{XX} h'^2\big) \alpha^2 \rho_0^2 + W_S \varphi_X^3 h'^3 S_{0_X} h' - \varphi_{tt} \varphi_X^3 h'^3 \alpha \rho_0 \Big) \\
&= -W_{\rho\rho} \varphi_X \frac{\alpha \alpha_X \rho_0^2}{h'} - W_{\rho\rho} \varphi_X \frac{\alpha^2 \rho_0 \rho_{0_X}}{h'} - W_{\rho S} \varphi_X^2 S_{0_X} \alpha \rho_0 \\
&\quad + W_{\rho\rho} \frac{\alpha^2 \rho_0^2 \varphi_X h''}{h'^3} + W_{\rho\rho} \varphi_{XX} \frac{\alpha^2 \rho_0^2}{h'} + W_S \varphi_X^3 S_{0_X} h' - \varphi_{tt} \varphi_X^3 \alpha \rho_0 \\
&= W_{\rho\rho} \varphi_X \rho_0^2 \Big(\frac{\alpha^2 h''}{h'^3} - \frac{\alpha \alpha_X}{h'}\Big) - W_{\rho\rho} \varphi_X \frac{\alpha^2 \rho_0 \rho_{0_X}}{h'} - W_{\rho S} \varphi_X^2 S_{0_X} \alpha \rho_0 \\
&\quad + W_{\rho\rho} \varphi_{XX} \frac{\alpha^2 \rho_0^2}{h'} + W_S \varphi_X^3 S_{0_X} h' - \varphi_{tt} \varphi_X^3 \alpha \rho_0.
\end{aligned}$$

As $h'(g(X)) = \alpha(X)$, the above equation becomes

$$-W_{\rho\rho} \varphi_X \rho_0 \rho_{0_X} - W_{\rho S} \varphi_X^2 S_{0_X} \rho_0 + W_{\rho\rho} \varphi_{XX} \rho_0^2 + W_S \varphi_X^3 S_{0_X} - \varphi_{tt} \varphi_X^3 \rho_0 = 0.$$

This means that (4.32) is an equivalence transformation. It does not change equation (4.27), it only changes the functions $\rho_0(X)$.

In the particular case of an isentropic solution, i.e. $S_0(X)$ is a constant, one can assume that $\rho_0(X)$ is a constant by the transformation with $\alpha(X)$ satisfying the condition

$$\rho_0(X)\alpha(X) = 1,$$

and one obtains that

$$\bar{\rho}(X,t) = \frac{1}{\bar{\varphi}_X(X,t)}.$$

Hence, the equation (4.27) transforms to

$$\varphi_{tt}\varphi_X^3 - W_{\rho\rho}\varphi_{XX} = 0, \qquad \text{where} \quad W = W(\frac{1}{\varphi_X}, S_0). \tag{4.34}$$

In the particular case where

$$\frac{1}{\varphi_X^3} W_{\rho\rho}(\frac{1}{\varphi_X}, S_0) = k^2 = \text{constant},$$

equation (4.34) becomes the wave equation, $\varphi_{tt} - k^2\varphi_{XX} = 0$.

Moreover for $W_{\rho\rho}(\frac{1}{\varphi_X}, S_0) = k^2\varphi_X^3$, it can be written in Eulerian coordinates as

$$W_{\rho\rho}(\rho, S_0) = k^2\rho^{-3}.$$

Since $p_\rho = \rho W_{\rho\rho}$, one obtains

$$p_\rho = k^2\rho^{-2}.$$

The pressure p in this case defines the Chaplygin gas.

From what was mentioned above, one can assume $\rho_0(X) = 1$, and $W(\rho, S_0(X)) = \tilde{W}(\rho, X)$, then equation (4.27) can be changed to

$$\varphi_X^2\tilde{W}_{\rho X} - \tilde{W}_{\rho\rho}\varphi_{XX} - \tilde{W}_X\varphi_X^3 + \varphi_{tt}\varphi_X^3 = 0. \tag{4.35}$$

From here on; the tilde ($\tilde{}$) symbol will be omitted. Since $p = \rho W_\rho - W$, and $W(\rho, X) = W(\frac{1}{\varphi_X}, X)$, one finds

$$W_{\rho\rho} = \frac{1}{\rho}p_\rho, \qquad W_{\rho X} = \frac{1}{\rho}\big(p_X + W_X\big).$$

Substituting the above relations into equation (4.35), it changes to

$$\varphi_{tt} + p_X + p_{\varphi_X}\varphi_{XX} = \varphi_{tt} + D_X p = 0. \tag{4.36}$$

Equation (4.36) takes the form of a one dimensional wave equation.

Consider the one dimensional wave equation presented in the following form

$$\varphi_{tt} = p_x, \qquad p = p(x, \varphi, \varphi_x).$$

There are several author who have studied the group properties of this type of equation. Ames, Lohner, and Adams (1981) demonstrated how a number of physical problems from gas dynamics, shallow water waves, dynamics of a finite non-linear string, elastic-plastic materials and electromagnetic transmission line satisfy the quasilinear wave equation $\varphi_{tt} = [f(\varphi)\varphi_x]_x$ for arbitrary $f \in C^2(R)$, $f > 0, f' \neq 0$. Ames et al. also provided the symmetry group which are presented in several cases of function $f(\varphi)$. Baikov and Gazizov (1989) and Suhubi and Bakkaloğlu (1991) considered arbitrary function $p = f(\varphi_x)$ and Vinokurov and Nurgalieva (1985) found the conservation law of this function p (see in Ibragimov (1993)). For the case of polytropic gas such that $p = -b(x)\varphi_x^\gamma$, $\gamma > 1$, the symmetry group was presented by Andreev, Kaptsov, Pukhnachov and Rodionov (1998). In 1987, Bluman and Kumei (1987) constructed a complete group classification of the wave equation with $p = -c(x)\varphi_x$, whereas Grimshaw, Pelinovsky and Pelinovsky (2010) showed the existence of traveling wave in the one-dimensional wave equation with a spatially-variable wave speed $c(x)$ and also provided the group of point transformations of this equation.

Notice that these authors studied the symmetry group just only in the particular case of function $p = -b(x)\varphi_x^\gamma$ when $\gamma > 1$ or $\gamma = 1$, (Chaplygin gas); in this present study, we will construct the group classification of the Euler-Lagrange equation which is reduced to one dimensional wave equation (4.36) for arbitrary function of $p = p\,(\varphi_x, x)$, i.e.

$$\varphi_{tt} + D_X p = 0, \qquad p = p\,(\varphi_x, x).$$

CHAPTER V

APPLICATION OF GROUP ANALYSIS TO THE EULER-LAGRANGE EQUATION

In this chapter the group analysis method is applied to construct the group classification of a gas dynamic equation in Lagrangian coordinates with respect to the arbitrary pressure function $P = P(X, \varphi_X)$ with the restrictions $P_X \neq 0$, and $P_{\varphi_X} < 0$. The studied equation is

$$\varphi_{tt} + D_X P = \varphi_{tt} + P_X + P_{\varphi_X}\varphi_{XX} = 0, \tag{5.1}$$

where D_X is the total derivative with respect to the Lagrangian coordinates.

Suppose the form of an infinitesimal generator is

$$X = \xi^t(t, X, \varphi)\partial_t + \xi^X(t, X, \varphi)\partial_X + \eta^\varphi(t, X, \varphi)\partial_\varphi, \tag{5.2}$$

with the unknown coefficients $\xi^t(t, X, \varphi)$, $\xi^X(t, X, \varphi)$, and $\eta^\varphi(t, X, \varphi)$. The determining equations will give conditions for solving for the coefficients of the generator X in order to obtain all possible generators which are admitted by equation (5.1). The way to study consists of the two following steps.

Step 1. Find the equivalence transformations of equation (5.1).

Step 2. Find the admitted Lie group which is admitted for all arbitrary elements and any specification of arbitrary elements.

5.1 Equivalence transformations of equation (5.1)

A transformation which transform equation (5.1) into an equation with the same differential structure is called an equivalence transformation. The algorithm presented by Meleshko, S.V (1996) has been applied to construct an equivalence Lie Group. This algorithm assumes dependence of all coefficients on all variables including the arbitrary elements. Here, the arbitrary element in equation (5.1) is the pressure function $P = P(X, \varphi_X)$ which depends on the independent variable and the derivative of the dependent variable and in order to simplify the equivalence Lie group, new dependent variables are introduced :

$$u = \varphi_t, \qquad \text{and} \qquad v = \varphi_x. \tag{5.3}$$

Hence $P = P(X, v)$ and equation (5.1) can rewrite as

$$u_t + P_X + P_v v_X = 0. \tag{5.4}$$

Moreover, a condition for the mixed derivatives

$$u_X - v_t = 0 \tag{5.5}$$

holds.

The independent variables are $x_1 = X$, $x_2 = t$, the dependent variables are $u_1 = u$, $u_2 = v$, $u_3 = \varphi$ and P is the arbitrary pressure function. An infinitesimal operator X^e of the equivalence Lie group is presented as follows,

$$\begin{aligned} X^e &= \xi^{x_i}\partial_{x_i} + \zeta^{u_j}\partial_{u_j} + \zeta^P\partial_P \\ &= \xi^X\partial_X + \xi^t\partial_t + \zeta^u\partial_u + \zeta^v\partial_v + \zeta^\varphi\partial_\varphi + \zeta^P\partial_P, \end{aligned}$$

with the coefficients

$$\xi^{x_i} = \xi^{x_i}(X, t, u, v, \varphi, P), \quad \zeta^{u_j} = \zeta^{u_j}(X, t, u, v, \varphi, P), \quad \zeta^P = \zeta^P(X, t, u, v, \varphi, P),$$

where $i = 1, 2$ and $j = 1, 2, 3$.

The prolonged operator is

$$\begin{aligned} \tilde{X}^e &= X^e + \zeta^{u_{j x_i}}\partial_{u_{j x_i}} + \zeta^{P_{x_i}}\partial_{P_{x_i}} + \zeta^{P_{u_j}}\partial_{P_{u_j}} \\ &= X^e + \zeta^{u_X}\partial_{u_X} + \zeta^{u_t}\partial_{u_t} + \zeta^{v_X}\partial_{v_X} + \zeta^{v_t}\partial_{v_t} + \zeta^{\varphi_X}\partial_{\varphi_X} + \zeta^{\varphi_t}\partial_{\varphi_t} \\ &+ \zeta^{P_X}\partial_{P_X} + \zeta^{P_t}\partial_{P_t} + \zeta^{P_u}\partial_{P_u} + \zeta^{P_v}\partial_{P_v} + \zeta^{P_\varphi}\partial_{P_\varphi}, \end{aligned}$$

where the coefficients of the prolonged operator are obtained by using the prolongation formulae

$$\zeta^{u_{j x_i}} = D^e_{x_i}\zeta^{u_j} - u_{j_{x_1}} D^e_{x_i}\xi^{x_1} - u_{j_{x_2}} D^e_{x_i}\xi^{x_2},$$
$$\zeta^{P_{x_i}} = \tilde{D}^e_{x_i}\zeta^P - P_{x_1}\tilde{D}^e_{x_i}\xi^{x_1} - P_{x_2}\tilde{D}^e_{x_i}\xi^{x_2} - P_{u_1}\tilde{D}^e_{x_i}\zeta^{u_1} - P_{u_2}\tilde{D}^e_{x_i}\zeta^{u_2} - P_{u_3}\tilde{D}^e_{x_i}\zeta^{u_3},$$
$$\zeta^{P_{u_j}} = \tilde{D}^e_{u_j}\zeta^P - P_{x_1}\tilde{D}^e_{u_j}\xi^{x_1} - P_{x_2}\tilde{D}^e_{u_j}\xi^{x_2} - P_{u_1}\tilde{D}^e_{u_j}\zeta^{u_1} - P_{u_2}\tilde{D}^e_{u_j}\zeta^{u_2} - P_{u_3}\tilde{D}^e_{u_j}\zeta^{u_3},$$

Here, the operators are :

$$D^e_{x_i} = \partial_{x_i} + u_{j_{x_i}}\partial_{u_j} + (P_{x_i} + u_{j_{x_i}} P_{u_j})\partial_P, \qquad \tilde{D}^e_{x_i} = \partial_{x_i} + P_{x_i}\partial_P, \qquad \tilde{D}^e_{u_j} = \partial_{u_j} + P_{u_j}\partial_P.$$

or their expression of the coefficients and operators are

$$\zeta^{u_X} = D^e_X\zeta^u - u_X D^e_X\xi^X - u_t D^e_X\xi^t, \qquad \zeta^{u_t} = D^e_t\zeta^u - u_X D^e_t\xi^X - u_t D^e_t\xi^t,$$
$$\zeta^{v_X} = D^e_X\zeta^v - v_X D^e_X\xi^X - v_t D^e_X\xi^t, \qquad \zeta^{v_t} = D^e_t\zeta^v - v_X D^e_t\xi^X - v_t D^e_t\xi^t,$$
$$\zeta^{\varphi_X} = D^e_X\zeta^\varphi - \varphi_X D^e_X\xi^X - \varphi_t D^e_X\xi^t, \qquad \zeta^{\varphi_t} = D^e_t\zeta^\varphi - \varphi_X D^e_t\xi^X - \varphi_t D^e_t\xi^t,$$
$$\zeta^{P_X} = \tilde{D}^e_X\zeta^P - P_X\tilde{D}^e_X\xi^X - P_t\tilde{D}^e_X\xi^t - P_u\tilde{D}^e_X\zeta^u - P_v\tilde{D}^e_X\zeta^v - P_\varphi\tilde{D}^e_X\zeta^\varphi,$$
$$\zeta^{P_t} = \tilde{D}^e_t\zeta^P - P_X\tilde{D}^e_t\xi^X - P_t\tilde{D}^e_t\xi^t - P_u\tilde{D}^e_t\zeta^u - P_v\tilde{D}^e_t\zeta^v - P_\varphi\tilde{D}^e_t\zeta^\varphi,$$
$$\zeta^{P_u} = \tilde{D}^e_u\zeta^P - P_X\tilde{D}^e_u\xi^X - P_t\tilde{D}^e_u\xi^t - P_u\tilde{D}^e_u\zeta^u - P_v\tilde{D}^e_u\zeta^v - P_\varphi\tilde{D}^e_u\zeta^\varphi,$$
$$\zeta^{P_v} = \tilde{D}^e_v\zeta^P - P_X\tilde{D}^e_v\xi^X - P_t\tilde{D}^e_v\xi^t - P_u\tilde{D}^e_v\zeta^u - P_v\tilde{D}^e_v\zeta^v - P_\varphi\tilde{D}^e_v\zeta^\varphi,$$
$$\zeta^{P_\varphi} = \tilde{D}^e_\varphi\zeta^P - P_X\tilde{D}^e_\varphi\xi^X - P_t\tilde{D}^e_\varphi\xi^t - P_u\tilde{D}^e_\varphi\zeta^u - P_v\tilde{D}^e_\varphi\zeta^v - P_\varphi\tilde{D}^e_\varphi\zeta^\varphi,$$
$$D^e_X = \partial_X + u_X\partial_u + v_X\partial_v + \varphi_X\partial_\varphi + (P_X + u_X P_u + v_X P_v + \varphi_X P_\varphi)\partial_P,$$
$$D^e_t = \partial_t + u_t\partial_u + v_t\partial_v + \varphi_t\partial_\varphi + (P_t + u_t P_u + v_t P_v + \varphi_t P_\varphi)\partial_P,$$
$$\tilde{D}^e_X = \partial_X + P_X\partial_P, \qquad \tilde{D}^e_t = \partial_t + P_t\partial_P,$$
$$\tilde{D}^e_u = \partial_u + P_u\partial_P, \qquad \tilde{D}^e_v = \partial_v + P_v\partial_P, \qquad \tilde{D}^e_\varphi = \partial_\varphi + P_\varphi\partial_P.$$

The conditions that $P = P(X, v)$ is an arbitrary function and does not depend on t , u , φ are

$$P_t = 0, \quad P_u = 0, \quad P_\varphi = 0. \tag{5.6}$$

The determining equations of the equivalence Lie group are

$$
\begin{gathered}
\left[\zeta^{u_t} + \zeta^{P_X} + v_X\zeta^{P_v} + P_v\zeta^{v_X}\right]\Big|_{(S)} = 0 \\
\left[\zeta^{u} - \zeta^{\varphi_t}\right]\Big|_{(S)} = 0, \qquad \left[\zeta^{v} - \zeta^{\varphi_X}\right]\Big|_{(S)} = 0, \qquad \left[\zeta^{u_X} - \zeta^{v_t}\right]\Big|_{(S)} = 0, \\
\zeta^{P_t}\Big|_{(S)} = 0, \qquad \zeta^{P_u}\Big|_{(S)} = 0, \qquad \zeta^{P_\varphi}\Big|_{(S)} = 0.
\end{gathered} \tag{5.7}
$$

After substituting ζ^u, ζ^{u_t}, ζ^{u_X}, ζ^v, ζ^{v_t}, ζ^{v_X}, ζ^{φ_t}, ζ^{φ_X}, ζ^{P_u}, ζ^{P_v}, ζ^{P_t}, ζ^{P_X}, ζ^{P_φ} and transition onto the manifold $(S) : u_t = -P_X - P_v v_X$, equation (5.7) can be split with respect to the variables u_X, v_X, v_t, P_X, P_v. The symbolic computer Reduce program was applied here. After solving the determining equations, the following basis of generators were obtained :

$$
\begin{gathered}
X_1^e = \partial_t, \qquad X_2^e = \partial_X, \qquad X_3^e = \partial_\varphi, \qquad X_4^e = \partial_P, \qquad X_5^e = t\partial_\varphi, \\
X_6^e = \varphi\partial_\varphi + P\partial_P, \qquad X_7^e = t\partial_t - 2P\partial_P, \qquad X_8^e = t^2\partial_\varphi - 2X\partial_P, \\
X_9^e = X\partial_X + P\partial_P,
\end{gathered}
$$

The following equivalence Lie group of transformations corresponding to these basis generators will be applied to simplify the function $P(X, \varphi_X)$ in the process of the group classification:

$$
\begin{aligned}
X_1^e &: \ \bar{t} = t + a, \quad \bar{X} = X, \quad \bar{\varphi} = \varphi, \quad \bar{P} = P\,; \\
X_2^e &: \ \bar{t} = t, \quad \bar{X} = X + a, \quad \bar{\varphi} = \varphi, \quad \bar{P} = P\,; \\
X_3^e &: \ \bar{t} = t, \quad \bar{X} = X, \quad \bar{\varphi} = \varphi + a, \quad \bar{P} = P\,; \\
X_4^e &: \ \bar{t} = t, \quad \bar{X} = X, \quad \bar{\varphi} = \varphi, \quad \bar{P} = P + a\,; \\
X_5^e &: \ \bar{t} = t, \quad \bar{X} = X, \quad \bar{\varphi} = \varphi + ta, \quad \bar{P} = P\,; \\
X_6^e &: \ \bar{t} = t, \quad \bar{X} = X, \quad \bar{\varphi} = \varphi e^a, \quad \bar{P} = Pe^a\,; \\
X_7^e &: \ \bar{t} = te^a, \quad \bar{X} = X, \quad \bar{\varphi} = \varphi, \quad \bar{P} = Pe^{-2a}\,; \\
X_8^e &: \ \bar{t} = t, \quad \bar{X} = X, \quad \bar{\varphi} = \varphi + t^2 a, \quad \bar{P} = P - 2aX\,; \\
X_9^e &: \ \bar{t} = t, \quad \bar{X} = Xe^a, \quad \bar{\varphi} = \varphi, \quad \bar{P} = Pe^a\,;
\end{aligned}
$$

Here, a is the group parameter. Moreover, a particular case of pressure function,

$$P(X,\varphi_X) = P_1(X)\big(\varphi_X + P_2(X)\big)^{\gamma} + P_3(X),$$

can be simplified by applying these equivalence Lie group of transformations

X^e_{10} : $\bar{t}=t,\quad \bar{X}=X,\quad \bar{\varphi}=\varphi e^a,\quad \bar{P}_1=P_1e^{-(\alpha-1)a},\quad \bar{P}_2=P_2e^a,\quad \bar{P}_3=P_3e^a$;

X^e_{11} : $\bar{t}=te^a,\quad \bar{X}=X,\quad \bar{\varphi}=\varphi,\quad \bar{P}_1=P_1e^{-2a},\quad \bar{P}_2=P_2,\quad \bar{P}_3=P_3e^{-2a}$;

X^e_{12} : $\bar{t}=t,\quad \bar{X}=Xe^a,\quad \bar{\varphi}=\varphi,\quad \bar{P}_1=P_1e^{(\alpha+1)a},\quad \bar{P}_2=P_2e^{-a},\quad \bar{P}_3=P_3e^a$;

X^e_{13} : $\bar{t}=t,\quad \bar{X}=X,\quad \bar{\varphi}=\varphi+t^2a,\quad \bar{P}_1=P_1,\quad \bar{P}_2=P_2,\quad \bar{P}_3=P_3-2aX$;

X^e_{14} : $\bar{t}=t,\quad \bar{X}=X,\quad ,\bar{\varphi}=\varphi,\quad \bar{P}_1=P_1,\quad \bar{P}_2=P_2,\quad \bar{P}_3=P_3+a$;

X^e_{15} : $\bar{t}=t,\ \bar{X}=X,\ \bar{\varphi}=\varphi+\zeta(X)a,\ \bar{P}_1=P_1,\ \bar{P}_2=P_2-\zeta'(X)a,\ \bar{P}_3=P_3$;

where the function $\zeta(X)$ and α are arbitrary.

5.2 Admitted Lie group of equation (5.1)

The infinitesimal generators of one-parameter Lie groups admitted by equation (5.1) are sought in the form:

$$X=\xi^t(t,X,\varphi)\partial_t+\xi^X(t,X,\varphi)\partial_X+\eta^{\varphi}(t,X,\varphi)\partial_{\varphi}. \tag{5.8}$$

The prolonged infinitesimal generator of (5.8) is

$$X^{(2)}=X+\eta^{\varphi_t}\partial_{\varphi_t}+\eta^{\varphi_X}\partial_{\varphi_X}+\eta^{\varphi_{tt}}\partial_{\varphi_{tt}}+\eta^{\varphi_{tX}}\partial_{\varphi_{tX}}+\eta^{\varphi_{XX}}\partial_{\varphi_{XX}} \tag{5.9}$$

with the coefficients

$$\begin{aligned}
\eta^{\varphi_t} &= \eta_t+\varphi_t\eta_\varphi-\varphi_t^2\xi^t_\varphi-\varphi_t\xi^t_t-\varphi_t\varphi_X\xi^X_\varphi-\varphi_X\xi^X_t,\\
\eta^{\varphi_X} &= \eta_X+\varphi_X\eta_\varphi-\varphi_t\varphi_X\xi^t_\varphi-\varphi_t\xi^t_X-\varphi_X^2\xi^X_\varphi-\varphi_X\xi^X_X,\\
\eta^{\varphi_{XX}} &= \eta_{XX}+2\varphi_X\eta_{\varphi X}+\varphi_X^2\eta_{\varphi\varphi}+\varphi_{XX}\eta_\varphi-2\varphi_t\varphi_X\xi^t_{\varphi X}-\varphi_t\varphi_X^2\xi^t_{\varphi\varphi}\\
&\quad -\varphi_t\varphi_{XX}\xi^t_\varphi-2\varphi_X\varphi_{tX}\xi^t_\varphi-\varphi_t\xi^t_{XX}-2\varphi_{tX}\xi^t_X-2\varphi_X^2\xi^X_{\varphi X}\\
&\quad -\varphi_X^3\xi^X_{\varphi\varphi}-3\varphi_X\varphi_{XX}\xi^X_\varphi-\varphi_X\xi^X_{XX}-2\varphi_{XX}\xi^X_X,
\end{aligned}$$

$$\eta^{\varphi_{tX}} = \eta_{tX} + \varphi_X\eta_{\varphi t} + \varphi_t\eta_{\varphi X} + \varphi_t\varphi_X\eta_{\varphi\varphi} + \varphi_{tX}\eta_\varphi - \varphi_t\varphi_X\xi^t_{\varphi t} - \varphi_t^2\xi^t_{\varphi X} - \varphi_X\varphi_t^2\xi^t_{\varphi\varphi}$$
$$- 2\varphi_t\varphi_{tX}\xi^t_\varphi - \varphi_X\varphi_{tt}\xi^t_\varphi - \varphi_t\xi^t_{tX} - \varphi_{tX}\xi^t_t - \varphi_{tt}\xi^t_X - \varphi_X^2\xi^X_{\varphi t} - \varphi_t\varphi_X\xi^X_{\varphi X}$$
$$- \varphi_t\varphi_X^2\xi^X_{\varphi\varphi} - \varphi_t\varphi_{XX}\xi^X_\varphi - 2\varphi_X\varphi_{tX}\xi^X_\varphi - \varphi_X\xi^X_{tX} - \varphi_{XX}\xi^X_t - \varphi_{tX}\xi^X_X,$$
$$\eta^{\varphi_{tt}} = \eta_{tt} + 2\varphi_t\eta_{\varphi t} + \varphi_t^2\eta_{\varphi\varphi} + \varphi_{tt}\eta_\varphi - 2\varphi_t^2\xi^t_{\varphi t} - \varphi_t^3\xi^t_{\varphi\varphi} - 3\varphi_t\varphi_{tt}\xi^t_\varphi - \varphi_t\xi^t_{tt} - 2\varphi_{tt}\xi^t_t$$
$$- 2\varphi_t\varphi_X\xi^X_{\varphi t} - \varphi_t^2\varphi_X\xi^X_{\varphi\varphi} - 2\varphi_t\varphi_{tX}\xi^X_\varphi - \varphi_{tt}\varphi_X\xi^X_\varphi - \varphi_X\xi^X_{tt} - 2\varphi_{tX}\xi^X_t.$$

The generator of (5.8) is admitted by equation (5.1), if and only if,

$$\left[X^{(2)}F(t, X, \varphi, \varphi_x, \varphi_{tt}, \varphi_{XX})\right]\Big|_{(S)} = 0.$$

The last equation becomes

$$\left[\eta^{\varphi_{tt}} + \xi^X P_{XX} + \eta^{\varphi_X} P_{\varphi_X X} + \varphi_{XX}\left(\xi^X P_{\varphi_X X} + \eta^{\varphi_X} P_{\varphi_X\varphi_X}\right) + \eta^{\varphi_{XX}} P_{\varphi_X}\right]\Big|_{(S)} = 0. \quad (5.10)$$

This equation is called the determining equation. Here (S) is the manifold defined by the relation $\varphi_{tt} = -P_X - P_{\varphi_X}\varphi_{XX}$. Substituting the coefficients $\eta^{\varphi_X}, \eta^{\varphi_{tt}}, \eta^{\varphi_{XX}}$ and the derivative $\varphi_{tt} = -P_X - P_{\varphi_X}\varphi_{XX}$, one obtains

$$\begin{aligned}
2\varphi_t\eta_{\varphi t} + 2\varphi_X\eta_{\varphi X}P_{\varphi_X} + \varphi_X^2\eta_{\varphi\varphi}P_{\varphi_X} + \varphi_t^2\eta_{\varphi\varphi} + \varphi_X\eta_\varphi P_{\varphi_X X}\\
+\varphi_X\varphi_{XX}\eta_\varphi P_{\varphi_X\varphi_X} + \varphi_{XX}\eta_\varphi P_{\varphi_X} + \varphi_{tt}\eta_\varphi + \eta_{tt} + \eta_{XX}P_{\varphi_X}\\
+\eta_X P_{\varphi_X X} + \varphi_{XX}\eta_X P_{\varphi_X\varphi_X} - 2\varphi_t^2\xi^t_{\varphi t} - 2\varphi_t\varphi_X\xi^t_{\varphi X}P_{\varphi_X} - \varphi_t\varphi_X^2\xi^t_{\varphi\varphi}P_{\varphi_X}\\
-\varphi_t^3\xi^t_{\varphi\varphi} - \varphi_t\varphi_X\xi^t_\varphi P_{\varphi_X X} - \varphi_t\varphi_X\varphi_{XX}\xi^t_\varphi P_{\varphi_X\varphi_X} - \varphi_t\varphi_{XX}\xi^t_\varphi P_{\varphi_X}\\
-2\varphi_{tX}\varphi_X\xi^t_\varphi P_{\varphi_X} - 3\varphi_t\varphi_{tt}\xi^t_\varphi - \varphi_t\xi^t_{tt} - 2\varphi_{tt}\xi^t_t - \varphi_t\xi^t_{XX}P_{\varphi_X} - \varphi_t\xi^t_X P_{\varphi_X X}\\
-\varphi_t\varphi_{XX}\xi^t_X P_{\varphi_X\varphi_X} - 2\varphi_{tX}\xi^t_X P_{\varphi_X} - 2\varphi_t\varphi_X\xi^X_{\varphi t} - 2\varphi_X^2\xi^X_{\varphi_X}P_{\varphi_X} - \varphi_X^3\xi^X_{\varphi\varphi}P_{\varphi_X}\\
-\varphi_t^2\varphi_X\xi^X_{\varphi\varphi} - \varphi_X^2\xi^X_\varphi P_{\varphi_X X} - \varphi_X^2\varphi_{XX}\xi^X_\varphi P_{\varphi_X\varphi_X} - 3\varphi_X\varphi_{XX}\xi^X_\varphi P_{\varphi_X}\\
-2\varphi_t\varphi_{tX}\xi^X_\varphi - \varphi_X\varphi_{tt}\xi^X_\varphi - \varphi_X\xi^X_{tt} - 2\varphi_{tX}\xi^X_t - \varphi_X\xi^X_{XX}P_{\varphi_X} - \varphi_X\xi^X_X P_{\varphi_X X}\\
-\varphi_X\varphi_{XX}\xi^X_X P_{\varphi_X\varphi_X} - 2\varphi_{XX}\xi^X_X P_{\varphi_X} + \varphi_{XX}\xi^X P_{\varphi_X X} + \xi^X P_{XX} = 0.
\end{aligned} \quad (5.11)$$

The equation (5.11) can be split with respect to the parametric derivatives φ_t , φ_{tX} , φ_{XX}. After splitting, one obtains these equations

$$2\varphi_X\eta_{\varphi X}P_{\varphi_X} + \varphi_X^2\eta_{\varphi\varphi}P_{\varphi_X} + \varphi_X\eta_\varphi P_{\varphi_X X} - \eta_\varphi P_X + \eta_{tt} + \eta_{XX}P_{\varphi_X}\eta_X P_{\varphi_X X}$$
$$+2\xi^t_t P_X - 2\varphi_X^2\xi^X_{\varphi X}P_{\varphi_X} - \varphi_X^3\xi^X_{\varphi\varphi}P_{\varphi_X} - \varphi_X^2\xi^X_\varphi P_{\varphi_X X} + \varphi_X\xi^X_\varphi P_X - \varphi_X\xi^X_{tt}$$
$$-\varphi_X\xi^X_{XX}P_{\varphi_X} - \varphi_X\xi^X_X P_{\varphi_X X} + \xi^X P_{XX} = 0, \qquad (5.12a)$$

$$2\varphi_X\xi^t_{\varphi X}P_{\varphi_X} + \varphi_X^2\xi^t_{\varphi\varphi}P_{\varphi_X} + \varphi_X\xi^t_\varphi P_{\varphi_X X} - 3\xi^t_\varphi P_X + \xi^t_{tt} - 2\eta_{\varphi t} + \xi^t_{XX}P_{\varphi_X}$$
$$+\xi^t_X P_{\varphi_X X} + 2\varphi_X\xi^X_{\varphi t} = 0, \qquad (5.12b)$$

$$\eta_{\varphi\varphi} - 2\xi^t_{\varphi t} - \varphi_X\xi^X_{\varphi\varphi} = 0, \qquad (5.12c)$$

$$\xi^t_{\varphi\varphi} = 0, \qquad (5.12d)$$

$$\varphi_X\xi^t_\varphi P_{\varphi_X} + \xi^t_X P_{\varphi_X} + \xi^X_t = 0, \qquad (5.12e)$$

$$\xi^X_\varphi = 0, \qquad (5.12f)$$

$$\varphi_X\eta_\varphi P_{\varphi_X\varphi_X} + \eta_X P_{\varphi_X\varphi_X} + 2\xi^t_t P_{\varphi_X} - \varphi_X^2\xi^X_\varphi P_{\varphi_X\varphi_X} - 2\varphi_X\xi^X_\varphi P_{\varphi_X}$$
$$-\varphi_X\xi^X_X P_{\varphi_X\varphi_X} - 2\xi^X_X P_{\varphi_X} + \xi^X P_{\varphi_X X} = 0, \qquad (5.12g)$$

$$\varphi_X\xi^t_\varphi P_{\varphi_X\varphi_X} - 2\xi^t_\varphi P_{\varphi_X} + \xi^t_X P_{\varphi_X\varphi_X} = 0. \qquad (5.12h)$$

As $P(X, \varphi_X)$ is an arbitrary function, one can split the above determining equations with respect to P_X, P_{φ_X}, $P_{\varphi_X X}$, $P_{\varphi_X\varphi_X}$. A solution for the determining equations is

$$\xi^t = k_1, \qquad \xi^X = 0, \qquad \eta = k_2 t + k_3.$$

The generator corresponding to these coefficients compose a basis of the kernel of admitted groups such that the kernel is admitted for all functions $P(X, \varphi_X)$ and they consist of the generators

$$X_1 = \partial_t, \qquad X_2 = \partial_\varphi, \qquad X_3 = t\partial_\varphi.$$

Extensions of the kernel depend on the value of the function $P(X, \varphi_X)$.

From equations (5.12c), (5.12d), and (5.12f), by a simple analysis, one finds

$$\xi^t(t, X, \varphi) = \varphi\xi^t_1(t, X) + \xi^t_0(t, X),$$

$$\eta(t,X,\varphi)=\varphi^2\xi^t_{1t}+\varphi\eta^1(t,X)+\eta^0(t,X).$$

Substituting and differentiating equations (5.12b) and (5.12e) with respect to φ, the relations

$$\xi^t_{1X}P_{\varphi_X}=0 \quad , \quad \xi^t_{1tt}=0$$

are obtained. As $P_{\varphi_X}<0$, then

$$\xi^t_1(t)=tk_1+k_2 \qquad \text{where} \quad k_1, \quad k_2 \text{ are constant.}$$

Substituting all above relations into (5.12e), then it becomes

$$\xi^X_t(t,X)=-P_{\varphi_X}(\xi^t_{0X}+\varphi_X k_1 t+\varphi_X k_2)$$

Differentiating this latter equation with respect to φ_X, one gets

$$P_{\varphi_X\varphi_X}\Big(\xi^t_{0X}+k_1t\varphi_X+k_2\varphi_X\Big)+P_{\varphi_X}\Big(k_1t+k_2\Big)=0, \tag{5.13}$$

and by taking linear combinations of equations (5.13) and (5.12h), one has

$$P_{\varphi_X}(k_1t+k_2)=0. \tag{5.14}$$

The latter equation implies that $k_1=0$, and $k_2=0$. Equation (5.12h) becomes

$$\xi^t_{0X}P_{\varphi_X\varphi_X}=0. \tag{5.15}$$

The study of this equation can be separated in 2 cases: $P_{\varphi_X\varphi_X}\neq 0$ and $P_{\varphi_X\varphi_X}=0$. If $P_{\varphi_X\varphi_X}=0$, then $P(X,\varphi_X)=a(X)\varphi_X+b(X)$. This type of pressure function is called the Chaplygin gas if $a(X)$ and $b(X)$ are constant and the group classification of this case has already been obtained by Bluman and Kumei (1986) and Grimshaw et al. (2010). Therefore in our study, we consider the case

$$P_{\varphi_X\varphi_X}\neq 0. \tag{5.16}$$

As $P_{\varphi_X\varphi_X} \neq 0$, one has from (5.15) that $\xi^t_{0X} = 0$ which means that $\xi^t_0 = \xi^t_0(t)$. Substituting this relation, then equation (5.12b) gives

$$\eta^1(t, X) = \frac{1}{2}\big(\xi^t_{0t} + \eta^{11}(X)\big).$$

Differentiating equations (5.12a) and (5.12g) with respect to φ, one has

$$\eta^{11} = k_3, \qquad \xi^t_0(t) = k_4t^2 + k_5t + k_6,$$

where k_3, k_4, k_5, and k_6 are constant. Moreover, one also derives equation (5.12g), one finds

$$\eta^0_X(t, X) = \frac{1}{2P_{\varphi_X\varphi_X}}\Big(P_{\varphi_X\varphi_X}\big(2\xi^X_X\varphi_X - k_3\varphi_X - 2k_4t\varphi_X - k_5\varphi_X\big) + P_{\varphi_X}\big(4\xi^X_X - 8k_4t - 4k_5\big) - 2\xi^X P_{\varphi_X X}\Big).$$

Differentiating this equation with respect to φ_X, one gets

$$\begin{aligned} &P_{\varphi_X\varphi_X\varphi_X}\big(- 4\xi^X_X P_{\varphi_X} + 2\xi^X P_{\varphi_X X} + 8k_4tP_{\varphi_X} + 4k_5P_{\varphi_X}\big) \\ &+P^2_{\varphi_X\varphi_X}\big(6\xi^X_X - k_3 - 10k_4t - 5k_5\big) - 2\xi^X P_{\varphi_X\varphi_X}P_{\varphi_X\varphi_X X} = 0. \end{aligned} \tag{5.17}$$

Substituting and differentiating equation (5.12a) with respect to t twice, one derives

$$\eta^0(t, X) = t^3\eta^{03}(X) + t^2\eta^{02}(X) + t\eta^{01}(X) + \eta^{00}(X),$$

where

$$\eta^{02}_X = 0, \qquad \eta^{03}_X = 0$$

such that $\eta^{02} = k_7$, and $\eta^{03} = k_8$ where k_7 and k_8 are constants,

$$\begin{aligned} \eta^{00}_X = \tfrac{1}{2P_{\varphi_X\varphi_X}}\Big(&- k_3\varphi_X P_{\varphi_X\varphi_X} + k_5\big(- \varphi_X P_{\varphi_X\varphi_X} - 4P_{\varphi_X}\big) \\ &+2\big(\xi^X_X\varphi_X P_{\varphi_X\varphi_X} + 2\xi^X_X P_{\varphi_X} - \xi^X P_{\varphi_X X}\big)\Big), \end{aligned} \tag{5.18}$$

$$\eta^{01}_X = \frac{k_4}{P_{\varphi_X\varphi_X}}\Big(- \varphi_X P_{\varphi_X\varphi_X} - 4P_{\varphi_X}\Big), \tag{5.19}$$

$$\begin{aligned} &k_5\big(4P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X} - 5P^2_{\varphi_X\varphi_X}\big) + 2\xi^X_X\big(- 2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X} + 3P^2_{\varphi_X\varphi_X}\big) \\ &+2\xi^X\big(P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X} - P_{\varphi_X\varphi_X X}P_{\varphi_X\varphi_X}\big) - k_3P^2_{\varphi_X\varphi_X} = 0, \end{aligned} \tag{5.20}$$

$$\frac{k_4}{P^2_{\varphi_X\varphi_X}}\Big(4P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X} - 5P^2_{\varphi_X\varphi_X}\Big) = 0. \tag{5.21}$$

Moreover, substituting all conditions into (5.12a) again, one gets these following conditions

$$k_8 = \frac{k_4}{6P^2_{\varphi_X\varphi_X}}\Big(8P_{\varphi_X}P_{\varphi_X X}P_{\varphi_X\varphi_X} - 4P^2_{\varphi_X}P_{\varphi_X\varphi_X X} - 3P^2_{\varphi_X\varphi_X}P_X\Big), \tag{5.22}$$

$$\begin{aligned}\tfrac{k_4}{6P^3_{\varphi_X\varphi_X}}\Big(&8P^2_{\varphi_X\varphi_X}\big(P_{\varphi_X}P_{\varphi_X XX} + P^2_{\varphi_X X}\big) + 8P^2_{\varphi_X}P^2_{\varphi_X\varphi_X X} - 3P^3_{\varphi_X\varphi_X}P_{XX}\\ &-4P_{\varphi_X}P_{\varphi_X\varphi_X}\big(4P_{\varphi_X X}P_{\varphi_X\varphi_X X} - P_{\varphi_X}P_{\varphi_X\varphi_X XX}\big)\Big) = 0,\end{aligned} \tag{5.23}$$

$$\begin{aligned}\tfrac{k_4}{6P^3_{\varphi_X\varphi_X}}\Big(&P_{\varphi_X\varphi_X}P_{\varphi_X X}\big(-8P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X} + 5P^2_{\varphi_X\varphi_X}\big)\\ &+P^2_{\varphi_X}\big(-4P_{\varphi_X\varphi_X}P_{\varphi_X\varphi_X\varphi_X X} + 8P_{\varphi_X\varphi_X X}P_{\varphi_X\varphi_X\varphi_X}\big)\Big) = 0.\end{aligned} \tag{5.24}$$

To simplify the calculation, let us introduce a new constant kk_3 which is $k_3 = kk_3 + k_5$ and by taking a linear combination of equation (5.17) and (5.20), then equation (5.17) becomes

$$\begin{aligned}&2k_5\big(2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X} - 3P^2_{\varphi_X\varphi_X}\big) + 2\xi^X_X\big(-2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X} + 3P^2_{\varphi_X\varphi_X}\big)\\ &-kk_3P^2_{\varphi_X\varphi_X} + 2\xi^X\big(P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X} - P_{\varphi_X\varphi_X X}P_{\varphi_X\varphi_X}\big)\Big) = 0.\end{aligned} \tag{5.25}$$

Since $P_{\varphi_X\varphi_X} \neq 0$, one gets

$$\begin{aligned}kk_3 = \tfrac{1}{P^2_{\varphi_X\varphi_X}}\Big(&2k_5\big(2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X} - 3P^2_{\varphi_X\varphi_X}\big) + 2\xi^X_X\big(-2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}\\ &+3P^2_{\varphi_X\varphi_X}\big) + 2\xi^X\big(P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X} - P_{\varphi_X\varphi_X X}P_{\varphi_X\varphi_X}\big)\Big).\end{aligned} \tag{5.26}$$

Differentiating equation (5.26) with respect to X and φ, one finds these two conditions

$$\begin{aligned}&4k_5\Big(P_{\varphi_X\varphi_X\varphi_X}\big(P_{\varphi_X X}P_{\varphi_X\varphi_X} - 2P_{\varphi_X\varphi_X X}P_{\varphi_X}\big) + P_{\varphi_X\varphi_X\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X}\Big)\\ &+2\xi^X_{XX}P_{\varphi_X\varphi_X}\Big(-2P_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X} + 3P^2_{\varphi_X\varphi_X}\Big) + 2\xi^X_X P_{\varphi_X\varphi_X X}\Big(4P_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X} - P^2_{\varphi_X\varphi_X}\Big)\\ &+2\xi^X P_{\varphi_X\varphi_X}\Big(P_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X XX} + P_{\varphi_X\varphi_X\varphi_X X}P_{\varphi_X X} - P_{\varphi_X\varphi_X XX}P_{\varphi_X\varphi_X} + P^2_{\varphi_X\varphi_X X}\Big)\\ &-2\xi^X_X P_{\varphi_X\varphi_X}\Big(P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X} - 2P_{\varphi_X\varphi_X\varphi_X X}P_{\varphi_X}\Big) - 4\xi^X P_{\varphi_X\varphi_X X}P_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X X} = 0,\end{aligned} \tag{5.27}$$

and

$$4k_5\Big(P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X\varphi_X}P_{\varphi_X\varphi_X}+P_{\varphi_X\varphi_X\varphi_X}P^2_{\varphi_X\varphi_X}-2P^2_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X}\Big)$$
$$+2\xi^X_X P_{\varphi_X\varphi_X}\Big(-2P_{\varphi_X\varphi_X\varphi_X\varphi_X}P_{\varphi_X}-2P_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X\varphi_X}\Big)$$
$$+2\xi^X P_{\varphi_X X}\Big(P_{\varphi_X\varphi_X\varphi_X\varphi_X}P_{\varphi_X\varphi_X}-2P^2_{\varphi_X\varphi_X\varphi_X}\Big)+8\xi^X_X P^2_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X}$$
$$+2\xi^X P_{\varphi_X\varphi_X}\Big(-P_{\varphi_X\varphi_X\varphi_X X}P_{\varphi_X\varphi_X}+2P_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X\varphi_X X}\Big)=0. \quad (5.28)$$

Substituting all relations, one finds ξ^X_{XX} from (5.12a)

$$\xi^X_{XX}=\xi^X_X\Big(-3P_{\varphi_X}P_{\varphi_X X}P_{\varphi_X\varphi_X}-2P_X P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}+2P^2_{\varphi_X}P_{\varphi_X\varphi_X X}+3P_X P^2_{\varphi_X\varphi_X}\Big)$$
$$+2k_5\Big(2P_{\varphi_X}P_{\varphi_X X}P_{\varphi_X\varphi_X}+P_X P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}-P^2_{\varphi_X}P_{\varphi_X\varphi_X X}-2P_X P^2_{\varphi_X\varphi_X}\Big)$$
$$+\xi^X\Big(P_{\varphi_X\varphi_X}(P_{\varphi_X}P_{\varphi_X XX}+P^2_{\varphi_X X})-P_{\varphi_X\varphi_X X}(P_{\varphi_X}P_{\varphi_X X}+P_X P_{\varphi_X\varphi_X})$$
$$+P_X P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X}-P^2_{\varphi_X\varphi_X}(P_{XX}+2k_7)\Big). \quad (5.29)$$

Differentiating equation (5.29) with respect to φ_X, one finds

$$\xi^X_X\Big(P_{\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X}(-2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}+6P^2_{\varphi_X\varphi_X})$$
$$+P^2_{\varphi_X\varphi_X}(-3P^2_{\varphi_X}P_{\varphi_X\varphi_X X}-6P_X P^2_{\varphi_X\varphi_X})$$
$$+P^2_{\varphi_X}P_{\varphi_X\varphi_X}(-2P_X P_{\varphi_X\varphi_X\varphi_X\varphi_X}+2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X X})$$
$$+2P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}(P_X P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}-P^2_{\varphi_X}P_{\varphi_X\varphi_X X}+5P_X P^2_{\varphi_X\varphi_X})\Big)$$
$$+2k_5\Big(P_{\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X}(P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}-4P^2_{\varphi_X\varphi_X})$$
$$+P^2_{\varphi_X\varphi_X}(2P^2_{\varphi_X}P_{\varphi_X\varphi_X X}+4P_X P^2_{\varphi_X\varphi_X})$$
$$+P^2_{\varphi_X}P_{\varphi_X\varphi_X}(P_X P_{\varphi_X\varphi_X\varphi_X\varphi_X}-P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X X})$$
$$+P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}(-P_X P_{\varphi_X}P_{\varphi_X\varphi_X\varphi_X}+P^2_{\varphi_X}P_{\varphi_X\varphi_X X}-3P_X P^2_{\varphi_X\varphi_X})\Big) \quad (5.30)$$
$$+\xi^X\Big(P_{\varphi_X}P_{\varphi_X\varphi_X}(-2P_{\varphi_X XX}P^2_{\varphi_X\varphi_X}+P^2_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X})$$
$$+P_{\varphi_X X}P^2_{\varphi_X\varphi_X}(-2P_X P_{\varphi_X\varphi_X\varphi_X}+2P_{\varphi_X}P_{\varphi_X\varphi_X X})$$
$$+P_{\varphi_X}P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X}(P_X P_{\varphi_X\varphi_X\varphi_X}+P_{\varphi_X}P_{\varphi_X\varphi_X X})$$
$$+P_{\varphi_X}P^2_{\varphi_X\varphi_X}(-P_{XX}P_{\varphi_X\varphi_X\varphi_X}+P_{\varphi_X}P_{\varphi_X\varphi_X XX})$$
$$+P_{\varphi_X\varphi_X}(-2P^2_{\varphi_X X}P^2_{\varphi_X\varphi_X}+P_X P_{\varphi_X}P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X\varphi_X}$$
$$-P^2_{\varphi_X}P_{\varphi_X X}P_{\varphi_X\varphi_X\varphi_X X}-P^2_{\varphi_X}P^2_{\varphi_X\varphi_X X}$$

$$+2P_X P_{\varphi_X\varphi_X X} P^2_{\varphi_X\varphi_X} + 2P_{XX} P^3_{\varphi_X\varphi_X}\big)\Big)$$

$$+2k_7 P^2_{\varphi_X\varphi_X}\big(- P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X} + 2P^2_{\varphi_X\varphi_X}\big) = 0$$

Taking a linear combination of equations (5.27) and (5.30), one finds

$$\begin{aligned}k_7 = \tfrac{1}{2P^4_{\varphi_X\varphi_X}}\Big(\xi^X_X\Big(P_{\varphi_X} P_{\varphi_X\varphi_X}\big(- 3P_{\varphi_X X} P^2_{\varphi_X\varphi_X} + 4P_X P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X\varphi_X}\big)\\ +P_X P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X}\big(8P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X} + 2P^2_{\varphi_X\varphi_X}\big) + P^2_{\varphi_X\varphi_X}\big(2P^2_{\varphi_X} P_{\varphi_X\varphi_X X} + 3P_X P^2_{\varphi_X\varphi_X}\big)\Big)\\ +2k_5\Big(P_{\varphi_X} P_{\varphi_X\varphi_X}\big(2P_{\varphi_X X} P^2_{\varphi_X\varphi_X} - 2P_X P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X\varphi_X}\big)\\ +P_X P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X}\big(4P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X} - P^2_{\varphi_X\varphi_X}\big) + P^2_{\varphi_X\varphi_X}\big(- P^2_{\varphi_X} P_{\varphi_X\varphi_X X} - 2P_X P^2_{\varphi_X\varphi_X}\big)\Big)\\ +\xi^X\Big(P^3_{\varphi_X\varphi_X}\big(P_{\varphi_X} P_{\varphi_X XX} + P^2_{\varphi_X X}\big) + P_X P_{\varphi_X} P_{\varphi_X X}\big(P_{\varphi_X\varphi_X} P_{\varphi_X\varphi_X\varphi_X\varphi_X} + 4P^2_{\varphi_X\varphi_X\varphi_X}\big)\\ +P_{\varphi_X X} P^2_{\varphi_X\varphi_X}\big(P_X P_{\varphi_X\varphi_X\varphi_X} - P_{\varphi_X} P_{\varphi_X\varphi_X X}\big) + P^3_{\varphi_X\varphi_X}\big(- P_X P_{\varphi_X\varphi_X X} - P_{XX} P_{\varphi_X\varphi_X}\big)\\ +P_X P_{\varphi_X} P_{\varphi_X\varphi_X}\big(2P_{\varphi_X\varphi_X} P_{\varphi_X\varphi_X\varphi_X X} - 4P_{\varphi_X\varphi_X X} P_{\varphi_X\varphi_X\varphi_X}\big)\Big)\Big)\end{aligned} \tag{5.31}$$

Let us introduce a new function

$$\mu_1 = \frac{P_{\varphi_X} P_{\varphi_X\varphi_X\varphi_X}}{P^2_{\varphi_X\varphi_X}}. \tag{5.32}$$

After substituting function $\mu_1(X, \varphi_X)$ into all above conditions, the following equations are the latest form, and they must be analyzed to find the solution of the determining equation.

Equation (5.21) becomes

$$k_4\big(4\mu_1 - 5\big) = 0. \tag{5.33}$$

Equation (5.23) becomes

$$\begin{aligned}k_4\Big(P^2_{\varphi_X\varphi_X}\big(8P_{\varphi_X} P_{\varphi_X\varphi_X X} + 8P^2_{\varphi_X X} - 3P_{XX} P_{\varphi_X\varphi_X}\big) + 8P^2_{\varphi_X} P^2_{\varphi_X\varphi_X X}\\ +P_{\varphi_X\varphi_X}\big(- 16P_{\varphi_X} P_{\varphi_X X} P_{\varphi_X\varphi_X X} - 4P^2_{\varphi_X} P_{\varphi_X\varphi_X XX}\big)\Big) = 0.\end{aligned} \tag{5.34}$$

Equation (5.24) becomes

$$\begin{aligned}k_4\Big(P_{\varphi_X\varphi_X}\big(- 4\mu_{1X} P_{\varphi_X} + 4\mu_1 P_{\varphi_X X} - 5P_{\varphi_X X}\big)\\ +2P_{\varphi_X\varphi_X X} P_{\varphi_X}\big(- 4\mu_1 + 5\big)\Big) = 0.\end{aligned} \tag{5.35}$$

Equation (5.27) becomes

$$\begin{aligned}
&\xi^X_X P_{\varphi_X} P_{\varphi_X\varphi_X}\Big(P_{\varphi_X\varphi_X}\big(4\mu_{1\varphi_X}\mu_1 P_X - 6\mu_{1\varphi_X}P_X - 2\mu_{1X}P_{\varphi_X} + \mu_1 P_{\varphi_X X}\big) \\
&-P_{\varphi_X}P_{\varphi_X\varphi_X X}\Big) + \xi^X\Big(P^2_{\varphi_X\varphi_X}\big(-2\mu_{1\varphi_X}\mu_1 P_X P_{\varphi_X X} + 3\mu_{1\varphi_X}P_X P_{\varphi_X X} \\
&-\mu_1 P^2_{\varphi_X X} + \mu_{1X}P_{\varphi_X}P_{\varphi_X X} + 2\mu_{1X}\mu_1 P_X P_{\varphi_X\varphi_X} - 3\mu_{1X}P_X P_{\varphi_X\varphi_X} \\
&+\mu_1 P_{\varphi_X}P_{\varphi_X XX}\big) - P^2_{\varphi_X}\big(P_{\varphi_X\varphi_X}P_{\varphi_X\varphi_X XX} - P^2_{\varphi_X\varphi_X X}\big)\Big) \\
&+2k_5 P_{\varphi_X}P^2_{\varphi_X\varphi_X}\Big(\mu_{1\varphi_X}P_X\big(-2\mu_1+3\big) + \mu_{1X}P_{\varphi_X}\Big) = 0.
\end{aligned} \tag{5.36}$$

Equation (5.28) becomes

$$2\xi^X_X\mu_{1\varphi_X}P_{\varphi_X} - 2k_5\mu_{1\varphi_X}P_{\varphi_X} + \xi^X\big(-\mu_{1\varphi_X}P_{\varphi_X X} + \mu_{1X}P_{\varphi_X\varphi_X}\big) = 0. \tag{5.37}$$

Equation (5.30) becomes

$$\begin{aligned}
&\xi^X_X P_{\varphi_X}P_{\varphi_X\varphi_X}\Big(\mu_{1\varphi_X}P_X P_{\varphi_X\varphi_X}\big(-4\mu_1+6\big) + P_{\varphi_X\varphi_X}\big(2\mu_{1X}P_{\varphi_X} - \mu_1 P_{\varphi_X X}\big) \\
&+P_{\varphi_X}P_{\varphi_X\varphi_X X}\Big) + \xi^X\Big(P^2_{\varphi_X\varphi_X}\big(2\mu_{1\varphi_X}\mu_1 P_X P_{\varphi_X X} - 3\mu_{1\varphi_X}P_X P_{\varphi_X X} \\
&-\mu_{1X}P_{\varphi_X}P_{\varphi_X X} - 2\mu_{1X}\mu_1 P_X P_{\varphi_X\varphi_X} + 3\mu_{1X}P_X P_{\varphi_X\varphi_X} \\
&-\mu_1 P_{\varphi_X}P_{\varphi_X XX} + \mu_1 P^2_{\varphi_X X}\big) + P^2_{\varphi_X}\big(P_{\varphi_X\varphi_X}P_{\varphi_X\varphi_X XX} - P^2_{\varphi_X\varphi_X X}\big)\Big) \\
&+2k_5 P_{\varphi_X}P^2_{\varphi_X\varphi_X}\Big(\mu_{1\varphi_X}P_X\big(2\mu_1-3\big) - \mu_{1X}P_{\varphi_X}\Big) = 0.
\end{aligned} \tag{5.38}$$

Equation (5.22) becomes

$$k_8 = \frac{k_4}{6P^2_{\varphi_X\varphi_X}}\Big(8P_{\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X} - 4P_{\varphi_X\varphi_X X}P^2_{\varphi_X} - 3P^2_{\varphi_X\varphi_X}P_X\Big). \tag{5.39}$$

Equation (5.26) becomes

$$\begin{aligned}
kk3 = \tfrac{1}{P_{\varphi_X\varphi_X}P_{\varphi_X}}\Big(&2\xi^X_X P_{\varphi_X\varphi_X}P_{\varphi_X}(-2\mu_1+3) + 2k_5 P_{\varphi_X\varphi_X}P_{\varphi_X}(2\mu_1-3) \\
&+2\xi^X\big(P_{\varphi_X X}P_{\varphi_X\varphi_X}\mu_1 - P_{\varphi_X\varphi_X X}P_{\varphi_X}\big)\Big).
\end{aligned} \tag{5.40}$$

Equation (5.31) becomes

$$\begin{aligned}
k_7 = \tfrac{1}{2P^2_{\varphi_X\varphi_X}P_{\varphi_X}}\Big(&\xi^X_X P_{\varphi_X}\big(4\mu_{1\varphi_X}P_{\varphi_X\varphi_X}P_{\varphi_X}P_X - 3P_{\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X} \\
&+2P_{\varphi_X\varphi_X X}P^2_{\varphi_X} - 2P^2_{\varphi_X\varphi_X}P_X\mu_1 + 3P^2_{\varphi_X\varphi_X}P_X\big) \\
&+2k_5 P_{\varphi_X}\big(-2\mu_{1\varphi_X}P_{\varphi_X\varphi_X}P_{\varphi_X}P_X + 2P_{\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X} - P_{\varphi_X\varphi_X X}P^2_{\varphi_X}
\end{aligned} \tag{5.41}$$

$$+P^2_{\varphi_X\varphi_X}P_X\mu_1 - 2P^2_{\varphi_X\varphi_X}P_X\big) + \xi^X\big(- 2\mu_{1_{\varphi_X}}P_{\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X}P_X$$
$$+2\mu_{1X}P^2_{\varphi_X\varphi_X}P_{\varphi_X}P_X + P_{\varphi_X XX}P_{\varphi_X\varphi_X}P^2_{\varphi_X}$$
$$+P^2_{\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X} - P_{\varphi_X X}P_{\varphi_X\varphi_X X}P^2_{\varphi_X}$$
$$+P_{\varphi_X X}P^2_{\varphi_X\varphi_X}P_X\mu_1 - P_{\varphi_X\varphi_X X}P_{\varphi_X\varphi_X}P_{\varphi_X}P_X - P^2_{\varphi_X\varphi_X}P_{\varphi_X}P_{XX}\big)\Big).$$

Differentiating equation (5.41) with respect to φ_X, one finds

$$\begin{aligned}
&\xi^X_X P_X P_{\varphi_X\varphi_X}\Big(P_{\varphi_X}\big(4\mu_{1_{\varphi_X\varphi_X}}P_X + 4\mu_{1_{\varphi_X}}P_{\varphi_X X}P_{\varphi_X\varphi_X X}\big) \\
&+\mu_{1_{\varphi_X}}P_X P_{\varphi_X\varphi_X}\big(- 4\mu_1 + 2\big)P_{\varphi_X\varphi_X}\big(2\mu_{1X}P_{\varphi_X} - \mu_1 P_{\varphi_X X}\big)\Big) \\
&+\xi^X\Big(P_X P_{\varphi_X}P_{\varphi_X\varphi_X}\big(2\mu_{1_{\varphi_X X}}P_{\varphi_X\varphi_X} - 2\mu_{1_{\varphi_X\varphi_X}}P_{\varphi_X X}\big) \\
&+\mu_{1_{\varphi_X}}P_{\varphi_X X}P_{\varphi_X\varphi_X}\big(- 2P_{\varphi_X X}P_{\varphi_X} + 2\mu_1 P_X P_{\varphi_X\varphi_X} + P_X P_{\varphi_X\varphi_X}\big) \\
&+P^2_{\varphi_X}\big(P_{\varphi_X\varphi_X}P_{\varphi_X\varphi_X XX} - P^2_{\varphi_X\varphi_X X}\big) \\
&+P_{\varphi_X}P_{\varphi_X\varphi_X}\big(- 2\mu_{1_{\varphi_X}}P_X P_{\varphi_X\varphi_X X} + \mu_{1X}P_{\varphi_X X}P_{\varphi_X\varphi_X}\big) \\
&+P^2_{\varphi_X\varphi_X}\big(- \mu_{1X}P_X P_{\varphi_X\varphi_X} - \mu_1 P_{\varphi_X}P_{\varphi_X XX} + \mu_1 P^2_{\varphi_X X}\big)\Big) \\
&+2k_5 P_{\varphi_X}P_{\varphi_X\varphi_X}\Big(P_{\varphi_X}\big(- 2\mu_{1_{\varphi_X\varphi_X}}P_X - 2\mu_{1_{\varphi_X}}P_{\varphi_X X} - \mu_{1X}P_{\varphi_X\varphi_X}\big) \\
&+\mu_{1_{\varphi_X}}P_X P_{\varphi_X\varphi_X}\big(2\mu_1 - 1\big)\Big) = 0.
\end{aligned} \tag{5.42}$$

Further study of the determining equations is separated in several cases as indicated in Figures 5.1-5.4. Details of solving the determining equations and finding the function $P(X, \varphi_X)$ of all these cases are given in Appendix A, and results of this analysis are summarized in Tables 5.1-5.2. Here details of the study of one of the branches are presented. We start analyzing the branch choosing $\mu_{1_{\varphi_X}} \neq 0$.

5.3 Case $\mu_{1_{\varphi_X}} \neq 0$

By equation (5.37), assuming $\mu_{1_{\varphi_X}} \neq 0$,and since $P_{\varphi_X} \neq 0$, then ξ^X_X can be found:

$$\xi^X_X = \frac{1}{2\mu_{1_{\varphi_X}}P_{\varphi_X}}\Big(2k_5\mu_{1_{\varphi_X}}P_{\varphi_X} + \xi^X\big(\mu_{1_{\varphi_X}}P_{\varphi_X X} - \mu_{1X}P_{\varphi_X\varphi_X}\big)\Big). \tag{5.43}$$

Next, consider equation (5.33), differentiating it with respect to φ_X, one has

$$k_4 = 0$$

and

$$\begin{gathered} 2k_5\mu_{1_{\varphi_X}} P_{\varphi_X}\Big(\mu_{1_{\varphi_X}} P_{\varphi_X X} - \mu_{1X} P_{\varphi_X\varphi_X}\Big) \\ +\xi^X\Big(2\mu_{1_{\varphi_X X}}\mu_{1X} P_{\varphi_X} P_{\varphi_X\varphi_X} + 2\mu_{1_{\varphi_X}}^2 P_{\varphi_X} P_{\varphi_X XX} - \mu_{1_{\varphi_X}}^2 P_{\varphi_X X}^2 \\ -2\mu_{1_{\varphi_X}}\mu_{1XX} P_{\varphi_X} P_{\varphi_X\varphi_X} - 2\mu_{1_{\varphi_X}}\mu_{1X} P_{\varphi_X} P_{\varphi_X\varphi_X X} + \mu_{1X}^2 P_{\varphi_X\varphi_X}^2\Big) = 0. \end{gathered} \tag{5.44}$$

Differentiating equation (5.43) with respect to φ_X, one has

$$\xi^X\Big(\frac{\mu_{1_{\varphi_X}} P_{\varphi_X X} - \mu_{1X} P_{\varphi_X\varphi_X}}{2\mu_{1_{\varphi_X}} P_{\varphi_X}}\Big)_{\varphi_X} = 0. \tag{5.45}$$

Let $\Delta = \Big(\frac{\mu_{1_{\varphi_X}} P_{\varphi_X X} - \mu_{1X} P_{\varphi_X\varphi_X}}{2\mu_{1_{\varphi_X}} P_{\varphi_X}}\Big)_{\varphi_X}$. Consider equation (5.45), in order to analyze the transformations, we will consider two cases: $\Delta \neq 0$ and $\Delta = 0$.

5.3.1 Case $\Delta \neq 0$

Considering equation (5.45), when $\Delta \neq 0$, then $\xi^X = 0$. In this case one finds $k_5 = 0$ and a solution for the determining equations is

$$\xi^t = k_6, \qquad \xi^X = 0, \qquad \eta = t\eta^{01} + \eta^{00}, \qquad \eta^{00}_X = 0, \qquad \eta^{01}_X = 0.$$

This case has no extension of the kernels of admitted Lie algebras.

5.3.2 Case $\Delta = 0$

Considering equation (5.45), in this case $\Big(\frac{\mu_{1_{\varphi_X}} P_{\varphi_X X} - \mu_{1X} P_{\varphi_X\varphi_X}}{2\mu_{1_{\varphi_X}} P_{\varphi_X}}\Big)_{\varphi_X}$=0, one can introduce a function $\mu_2 = \mu_2(X)$ such that

$$\mu_2 = \frac{\mu_{1_{\varphi_X}} P_{\varphi_X X} - \mu_{1X} P_{\varphi_X\varphi_X}}{2\mu_{1_{\varphi_X}} P_{\varphi_X}}. \tag{5.46}$$

From the latter equation one has

$$\mu_{1X} = \frac{\mu_{1_{\varphi_X}}\Big(P_{\varphi_X X} - 2\mu_2 P_{\varphi_X}\Big)}{P_{\varphi_X\varphi_X}}. \tag{5.47}$$

Equation (5.44) reduces to the equation

$$k_5\mu_2 + \xi^X\Big(\mu_{2X} + \mu_2^2\Big) = 0. \tag{5.48}$$

Differentiating equation (5.48) with respect to X, one gets

$$k_5\Big(2\mu_{2X} + \mu_2^2\Big) + \xi^X\Big(\mu_{2XX} + 3\mu_{2X}\mu_2 + \mu_2^3\Big) = 0. \tag{5.49}$$

Equations (5.48) and (5.49) are algebraic linear homogeneous equations with respect to k_5 and ξ^X with the determinant $\mu_2\mu_{2XX} - 2\mu_{2X}^2$. If this determinant is not equal to zero, then $k_5 = 0$ and $\xi^X = 0$. In this case there is no extension of the kernel of admitted Lie algebras. Hence, one has to assume that

$$\mu_2\mu_{2XX} - 2\mu_{2X}^2 = 0.$$

The general solution of this equation is ${\mu_2}^1(X) = 0$, and ${\mu_2}^2(X) = \frac{1}{k_1X+k_2}$, where k_1 and k_2 are constants such that $k_1^2 + k_2^2 \neq 0$.

Case $\mu_2(X) \neq 0$

Substituting $\mu_2(X) = \frac{1}{k_1X+k_2}$ into equation (5.43) and (5.48), they become

$$\xi^X_X = k_5 + \frac{\xi^X}{k_1X + k_2},$$

and

$$k_5\Big(k_1X + k_2\Big) + \xi^X\Big(-k_1 + 1\Big) = 0. \tag{5.50}$$

- **Case I** $k_1 = 1$

Substituting $k_1 = 1$ into equation (5.50), one gets $k_5 = 0$, then

$$\xi^X = k_9\big(X + k_2\big). \tag{5.51}$$

Equation (5.40) also becomes

$$\begin{aligned} kk_3 \quad &= \frac{k_9}{P_{\varphi_X}P_{\varphi_X\varphi_X}}\Big(-2P_{\varphi_X}P_{\varphi_X\varphi_XX}(X + k_2) \\ +2\mu_1 P_{\varphi_X\varphi_X}&\big(P_{\varphi_XX}(X + k_2) - 2P_{\varphi_X}\big) + 6P_{\varphi_X}P_{\varphi_X\varphi_X}\Big). \end{aligned} \tag{5.52}$$

Let $g = g(X, \varphi_X)$ be such that

$$g = \tfrac{1}{P_{\varphi_X} P_{\varphi_X\varphi_X}}\Big(-2P_{\varphi_X} P_{\varphi_X\varphi_X X}(X+k_2) \\ +2\mu_1 P_{\varphi_X\varphi_X}\big(P_{\varphi_X X}(X+k_2) - 2P_{\varphi_X}\big) + 6P_{\varphi_X} P_{\varphi_X\varphi_X}\Big),$$

one can rewrite equation (5.52) as

$$kk_3 = k_9 g. \tag{5.53}$$

Differentiating equation (5.53) with respect to X and φ_X, one has

$$k_9 g_X = 0, \qquad k_9 g_{\varphi_X} = 0.$$

If $g_X^2 + g_{\varphi_X}^2 \neq 0$, one has $k_9 = 0$. In this case there is no extension of the kernel of admitted Lie algebras. Hence, one has to assume that g is constant.

As kk_3 is constant, say $g = kk_3$, the latter can be rewritten as

$$P_{\varphi_X\varphi_X X} = \frac{1}{2P_{\varphi_X}(X+k_2)}\Big(2\mu_1 P_{\varphi_X\varphi_X}\big(P_{\varphi_X X}(X+k_2) - 2P_{\varphi_X}\big) + P_{\varphi_X\varphi_X\varphi_X}(6-g)\Big).$$

A similar study applies to equation (5.41), which can be also rewritten in the new form

$$k_7 = k_9 ko_4,$$

or

$$\begin{aligned} P_{\varphi_X XX} = \frac{1}{2P_{\varphi_X}(X+k_2)^2}\Big(&-2P_{\varphi_X X}^2(X+k_2)^2 - k_2 g P_{\varphi_X} P_{\varphi_X X} \\ &+2\mu_1\big(P_{\varphi_X X}^2(X+k_2)^2 - 4P_{\varphi_X} P_{\varphi_X X}(X+k_2) + 4P_{\varphi_X}^2\big) \\ &+P_{\varphi_X} P_{\varphi_X X}(12k_2 - gX + 12X) + 2P_{XX} P_{\varphi_X\varphi_X}(X+k_2)^2 \\ &-P_{\varphi_X\varphi_X}(X+k_2)\big(gP_X - 4ko_4\big) + 2P_{\varphi_X}^2(g-6)\Big), \end{aligned} \tag{5.54}$$

where ko_4 is constant.

Substituting all these relations into equation (5.18), it becomes

$$\eta_X^{00} = \frac{k_9}{2P_{\varphi_X\varphi_X}}\Big(-2k_2 P_{\varphi_X X} - 2X P_{\varphi_X X} - \varphi_X P_{\varphi_X\varphi_X}(g-2) + 4P_{\varphi_X}\Big). \tag{5.55}$$

Since η^{00} only depends on X and for existence of extension of the kernel of admitted Lie algebras one obtains that

$$\eta^{00}_X = f(X)k_9,$$

equation (5.55) provides that

$$P_{\varphi_X X} = \frac{1}{2(X+k_2)}\Big(P_{\varphi_X\varphi_X}(-g\varphi_X + 2\varphi_X - 2f(X)) + 4P_{\varphi_X}\Big) \tag{5.56}$$

where $f(X)$ is a function of X only. Substituting (5.56) into (5.54), one derives

$$\begin{aligned} &\varphi_X^2 P_{\varphi_X\varphi_X} g(4-g) - 4\varphi_X P_{\varphi_X\varphi_X}\big(\varphi_X + gf(X)\big) \\ &+4f(X)P_{\varphi_X\varphi_X}\big(2\varphi_X - f(X)\big) + 4\varphi_X P_{\varphi_X}(g-2) + 8f(X)P_{\varphi_X} \\ &+4P_{XX}(X+k_2)^2 - (X+k_2)\big(2gP_X - 8ko_4 - 4f(X)P_{\varphi_X}\big) = 0. \end{aligned} \tag{5.57}$$

Notice that $(X+k_2)^2 \neq 0$, one from equation (5.57), one can find

$$\begin{aligned} P_{XX} = \frac{1}{4(X+k_2)}\Big(&\varphi_X^2 P_{\varphi_X\varphi_X} g(g-4) + 4\varphi_X P_{\varphi_X\varphi_X}\big(\varphi_X + gf(X)\big) \\ &-4f(X)P_{\varphi_X\varphi_X}\big(2\varphi_X - f(X)\big) - 4\varphi_X P_{\varphi_X}(g-2) - 8f(X)P_{\varphi_X} \\ &+(X+k_2)\big(2gP_X - 8ko_4 - 4f(X)P_{\varphi_X}\big)\Big). \end{aligned} \tag{5.58}$$

Finally, the solution of the determining equations is

$$k_4 = 0, \qquad k_5 = 0, \qquad k_8 = 0, \qquad kk3 = k_9 g, \qquad k_7 = k_9 ko_4,$$

$$\eta^{00}_X = f(X)k_9, \qquad \eta^{01}_X = 0, \qquad \xi^t = k_6, \qquad \xi^X = k_9(X+k_2),$$

$$\eta = \frac{2k_9 ko_4 t^2 + 2\eta^{01}t + 2\eta^{00} + k_9 g\varphi}{2},$$

for $k_{10} = \eta^{01}$ then, the generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_9X^3$$

with the basis of generators

$$\begin{aligned} &X^1 = \partial_t, \qquad X^2 = t\partial_\varphi \\ &X^3 = (X+k_2)\partial_X + \Big(ko_4 t^2 + \int f(X)dX + \frac{g\varphi}{2}\Big)\partial_\varphi. \end{aligned} \tag{5.59}$$

Here the function $P(X, \varphi_X)$ satisfies the two equations (5.56) and (5.58).

Finding : Pressure function

Equation (5.56) can be rewritten as

$$2(X+k_2)P_{\varphi_X X} + P_{\varphi_X \varphi_X}\big(\varphi_X(g-2) + 2f(X)\big) = 4P_{\varphi_X}.$$

Thus, the general solution of equation (5.56) is

$$P_{\varphi_X} = \phi(\tilde{Z})(X+k_2)^2 \tag{5.60}$$

where $\tilde{Z} = \varphi_X(X+k_2)^{-\alpha} - \int f(X)(X+k_2)^{-\alpha-1}dX$.

Integrating equation (5.60) with respect to φ_X we obtain the pressure function,

$$P(X, \varphi_X) = \tilde{\phi}(\tilde{Z})(X+k_2)^{\alpha+2} + h(X) \tag{5.61}$$

where $\phi(\tilde{Z})$ is such that $\phi(\tilde{Z}) = \tilde{\phi}(\tilde{Z})'$. Substituting this function into (5.58), one derives

$$2(X+k_2)h_{XX} - gh_X + 4ko_4 = 0.$$

To find the integral $-\int f(X)(X+k_2)^{-\alpha-1}dX$, let us introduce the function

$$C(X) = -\int f(X)(X+k_2)^{-\alpha-1}dX.$$

Then $C'(X) = -f(X)(X+k_2)^{-\alpha-1}$ or $f(X) = -(X+k_2)^{\alpha+1}C'(X)$. Consider

$$\int f(X)dX = -\int (X+k_2)^{\alpha+1}C'(X)dX.$$

Integrating by-parts, one has

$$\begin{aligned}\int f(X)dX &= -\big[C(X)(X+k_2)^{\alpha+1} - \int (\alpha+1)(X+k_2)^{\alpha}C(X)dX\big]\\ &= -C(X)(X+k_2)^{\alpha+1} + (\alpha+1)\int C(X)(X+k_2)^{\alpha}dX\\ &= -C(X)(X+k_2)^{\alpha+1} + (\alpha+1)\tilde{f}(X)\end{aligned}$$

where $\tilde{f}(X) = \int C(X)(X+k_2)^{\alpha} dX$ or $C(X) = (X+k_2)^{-\alpha}\tilde{f}'(X)$.

Therefore

$$-\int f(X)(X+k_2)^{-\alpha-1}dX = (X+k_2)^{-\alpha}\tilde{f}'(X),$$

$$\int f(X)dX = -(X+k_2)\tilde{f}'(X) + (\alpha+1)\tilde{f}(X),$$

and hence, $\eta^{00} = k_9\Big(-(X+k_2)\tilde{f}'(X) + (\alpha+1)\tilde{f}(X)\Big)$.

Then generator in equation (5.59) can be rewritten as

$$\begin{aligned} &X^1 = \partial_t, \qquad X^2 = t\partial_\varphi \\ &X^3 = (X+k_2)\partial_X + \Big(\beta t^2 - (X+k_2)\tilde{f}'(X) + (\alpha+1)\big(\tilde{f}(X)+\varphi\big)\Big)\partial_\varphi \end{aligned} \tag{5.62}$$

where $\beta = ko_4$ and $2(\alpha+1) = g$. The pressure function (5.61) can be written as

$$P(X,\varphi_X) = \phi(Z)(X+k_2)^{\alpha+2} + h(X) \tag{5.63}$$

where $Z = (X+k_2)^{-\alpha}\big(\varphi_X + \tilde{f}'(X)\big)$ and $(X+k_2)h_{XX} - (\alpha+1)h_X + 2\beta = 0$. By virtue of the equivalence transformations corresponding to the generators X_2^e, X_{15}^e, it can be assumed that $k_2 = 0$ and $\tilde{f}(X) = 0$. The generator X^3 in equation (5.62) is changed to

$$X^3 = X\partial_X + \Big(\beta t^2 + (\alpha+1)\varphi\Big)\partial_\varphi.$$

Later on the equivalence transformation corresponding to the operator X_8^e will be applied and this transformation allows one to simplify that $\beta = 0$. For $\alpha \neq -1$, the extension of the kernel and the related pressure function are

$$X_4 = X\partial_X + (\alpha+1)\varphi\partial_\varphi, \qquad P(X,\varphi_X) = \phi(Z)X^{\alpha+2} + h(X) \tag{5.64}$$

where $Z = X^{-\alpha}\varphi_X$ and $Xh_{XX} - (\alpha+1)h_X = 0$. The result of this case is presented in Table 5.1 as the model M_1.

Further study of the determining equations of the other branches are presented in Appendix A.

5.4 Results of the group classification

The result of the group classification of equation (5.1) is summarized in Tables 5.1-5.2. The first column presents the number of the extension, forms of the function $P(X,\varphi_X)$ are presented in the second column, and the extensions of the kernel of admitted Lie algebra can be found in the third column. The restrictions on functions and constants are given in the fourth column.

Table 5.1 Group classification of the equation $\varphi_{tt} + D_X P = 0$.

No.	$P(X,\varphi_X)$	Extensions	Remarks
M_1	$\Phi(Z)X^{\alpha+2\gamma} + h(X)$	$(\gamma-1)t\partial_t - X\partial_X$	$\gamma \neq 0,\ 1,$
	$Z = X^{-\alpha}\varphi_X$	$-(\alpha+1)\varphi\partial_\varphi$	
	$Xh''(X) - (\alpha+2\gamma-1)h'(X) = 0$		
M_2	$\Phi(Z)e^{(2\beta-\alpha)X} + h(X)$	$\beta t\partial_t - \partial_X + \alpha\varphi\partial_\varphi$	$\alpha \neq 0$
	$Z = e^{\alpha X}\varphi_X$		
	$h''(X) - \big(2\beta-\alpha\big)h'(X) = 0$		
M_3	$\Phi(Z)X^{\alpha} + h(X)$	$t\partial_t + X\partial_X$	$\alpha \neq -1$
	$Z = X^{-\alpha}\varphi_X$	$+\big(\alpha+1\big)\varphi\partial_\varphi$	
	$Xh''(X) - (\alpha-1)h'(X) = 0$		
M_4	$\Phi(\varphi_X) + \beta X + \gamma X^2$	$\partial_X - \gamma t^2\partial_\varphi$	$\gamma,\ \beta \neq 0$
M_5	$\Phi(\varphi_X)$	$t\partial_t + X\partial_X + \varphi\partial_\varphi,\quad \partial_X$	
M_6	$e^{\beta X}\big(\varphi_X^{\gamma} + \frac{k_2}{\alpha^2}e^{(\alpha-\beta)X}\big)$	$\big(\beta+\alpha(\gamma-1)\big)t\partial_t-$	$\alpha,\ \beta \neq 0$
		$2\gamma\partial_X + 2(\beta-\alpha)\varphi\partial_\varphi$	$\gamma \neq 0,\ 1$
			$\alpha - \beta \neq 0$
M_7	$e^{\beta X}\varphi_X^{\gamma}$	$(\gamma-1)\partial_X - \beta\varphi\partial_\varphi,$	$\beta \neq 0$
		$(\gamma-1)t\partial_t - 2\varphi\partial_\varphi$	$\gamma \neq 0,\ 1$

Table 5.2 Group classification of the equation $\varphi_{tt} + D_X P = 0$ (continued).

No.	$P(x,\varphi_x)$	Extensions	Remarks
M_8	$b(X)\varphi_X^\gamma + k_1 b^{m+1}(X)$	$\big(\gamma(1+m+2l)-m\big)t\partial_t+$	$\gamma \neq 0,\, 1$
	$b^l(X) = \frac{\beta}{X}$	$(2\gamma l)X\partial_X + 2(\gamma l - m)\varphi\partial_\varphi$	$m \neq \gamma l$
			$m \neq -1,$
			$l,\, \beta \neq 0,$
M_9	$b(X)\varphi_X^\gamma$	$l(\gamma-1)X\partial_X + \big(l(\gamma+1)+1\big)\varphi\partial_\varphi,$	$\gamma \neq 0,\, 1$
	$b^l(X) = \frac{\beta}{X}$	$(\gamma-1)t\partial_t - 2\varphi\partial_\varphi$	$l,\, \beta \neq 0$
M_{10}	$b(X)\varphi_X^\gamma + k_1 X^2$	$(2l-1)t\partial_t + 2l\gamma X\partial_X+$	$\gamma \neq 0,\, 1$
	$b^l(X) = \beta X$	$2\big(2l-1+l\gamma\big)\varphi\partial_\varphi$	$l,\, \beta,\, k_1 \neq 0$
M_{11}	$e^{\beta X}\varphi_X^\gamma + k_1 X^2$	$-\beta t\partial_t + 2\gamma\partial_X - 2\beta\varphi\partial_\varphi$	$\gamma \neq 0,\, 1$
			$\beta,\, k_1 \neq 0$
M_{12}	$k_1 X\varphi_X^\gamma + k_2 X^{\alpha+1}$	$\big(\gamma(1-\alpha)+\alpha\big)t\partial_t+$	$\gamma \neq 0,\, 1$
		$2\gamma X\partial_X + 2(\gamma+\alpha)\varphi\partial_\varphi$	$k_1,\, k_2 \neq 0$
			$\alpha \neq -1,\, 0$
			$\gamma + \alpha \neq 0$
M_{13}	$\beta\varphi_X^\gamma + k_1 X^2$	$\partial_X - k_1 t^2\partial_\varphi,$	$\gamma \neq 0,\, 1,\, -2$
		$t\partial_t + \gamma X\partial_X + (\gamma+2)\varphi\partial_\varphi$	$k_1,\, \beta \neq 0$
M_{14}	$\beta\varphi_X^{-3} + k_1 X^2$	$\partial_X - k_1 t^2\partial_\varphi,$	$\beta,\, k_1 \neq 0$
		$t\partial_t - 3X\partial_X - \varphi\partial_\varphi$	
M_{15}	$\beta\varphi_X^\gamma$	$(\gamma-1)t\partial_t - 2\varphi\partial_\varphi,$	$\beta \neq 0$
		$(\gamma-1)X\partial_X + (\gamma+1)\varphi\partial_\varphi$	$\gamma \neq 0,\, 1$
M_{16}	$\beta\varphi_X^{-3}$	$\partial_X, \quad 2X\partial_X + \varphi\partial_\varphi,$	$\beta \neq 0$
		$t^2\partial_t + t\varphi\partial_\varphi, \quad 2t\partial_t + \varphi\partial_\varphi$	
M_{17}	$b(X)\varphi_X^{-3}$	$t^2\partial_t + t\varphi\partial_\varphi,$	

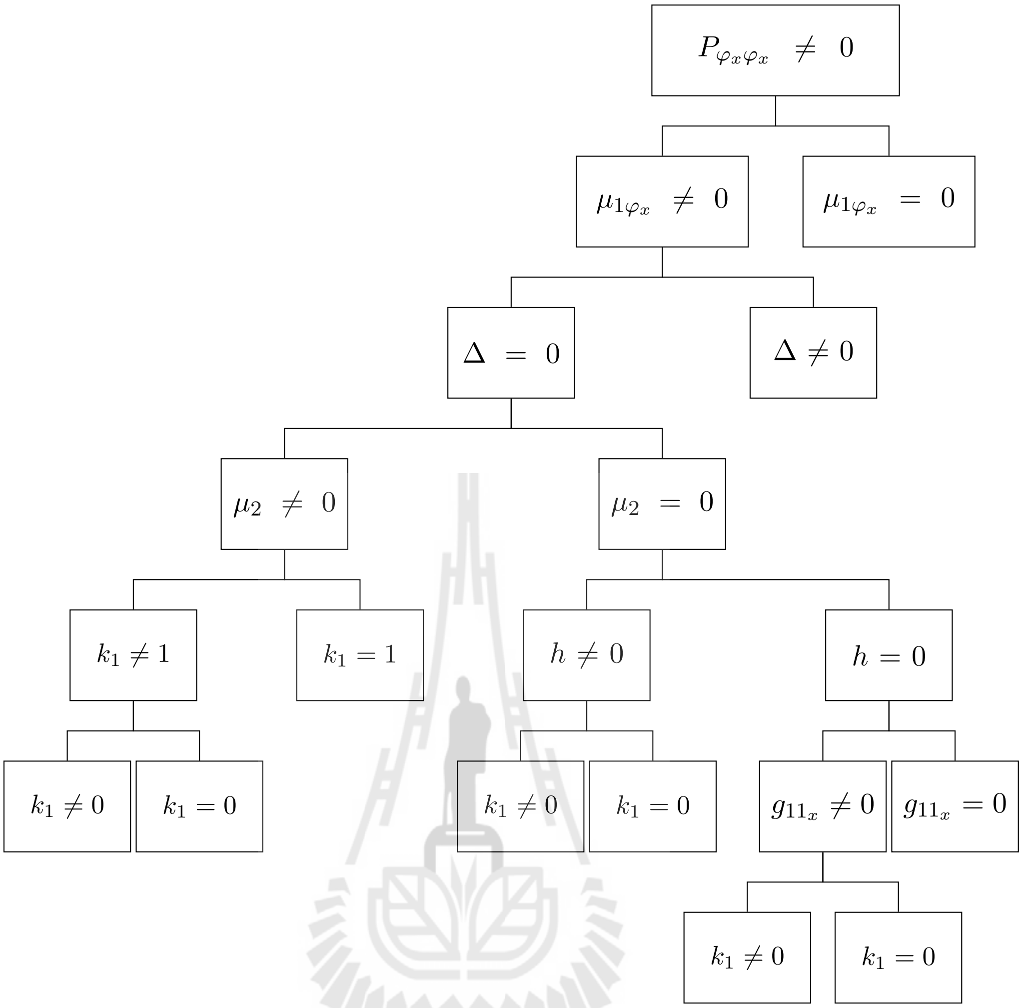

Figure 5.1 Tree diagram of $\mu_{1\varphi_x} \neq 0$.

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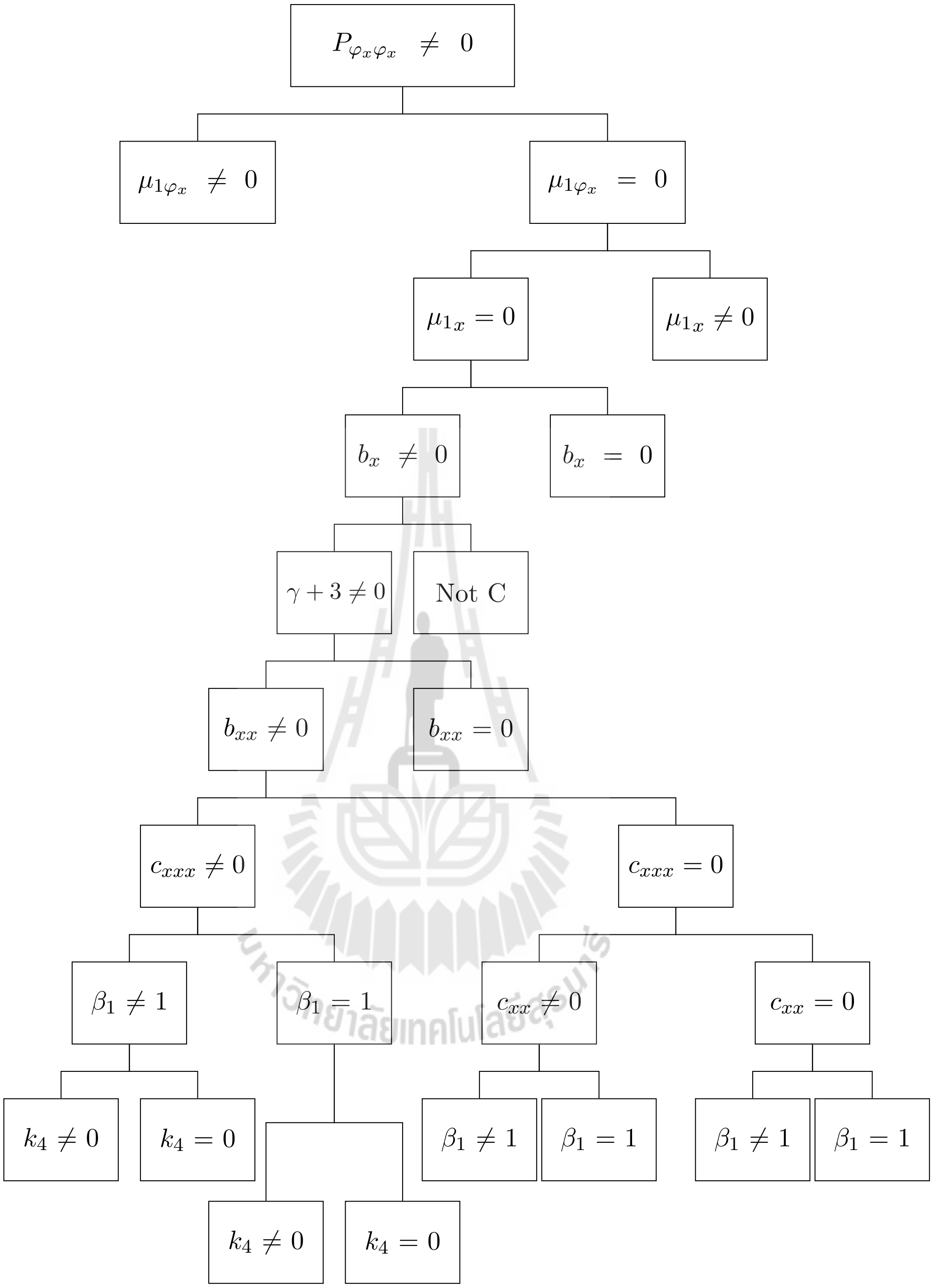

Figure 5.2 Tree diagram of $\mu_{1\varphi_x} = 0$.

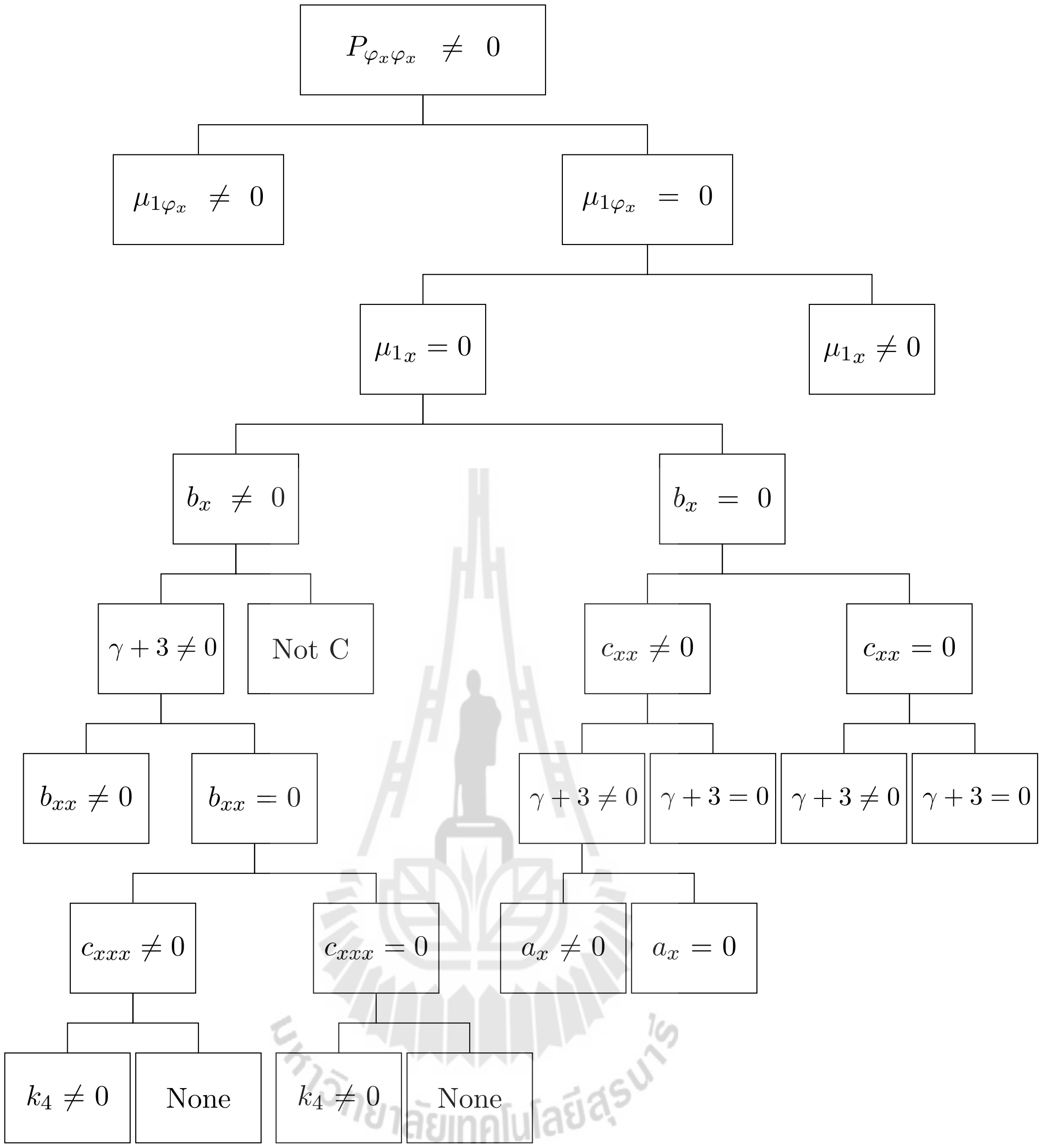

Figure 5.3 Tree diagram of $\mu_{1\varphi_x} = 0$ with $\mu_{1x} = 0$.

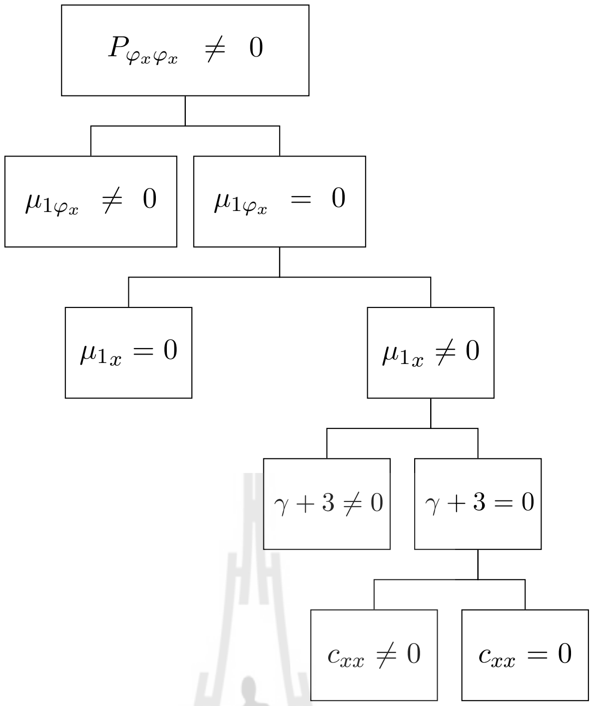

Figure 5.4 Tree diagram of $\mu_{1\varphi_x} = 0$ with $\mu_{1x} \neq 0$.

CHAPTER VI

CONSERVATION LAWS

6.1 Constructing Lagrangians

In this chapter we apply Noether's theorem to construct conservation laws of the gas dynamics equations in Lagrangian coordinates corresponding to the pressure function $P(X,\varphi_X)$ presented in Tables 5.1- 5.2. The Lagrangian of the gas dynamics equations in Lagrangian coordinates is

$$\mathcal{L}(X,\varphi_t,\varphi_X) = \frac{\varphi_t^2}{2} - \varphi_X W(X,\varphi_X), \tag{6.1}$$

where X is the Lagrangian mass coordinate .

The function $W(X,\varphi_X)$ and the pressure function $P(X,\varphi_X)$ are related as follows. Let

$$\tilde{W}(X,\rho) = W(X,\rho^{-1}), \qquad p(X,\rho) = P(X,\rho^{-1}).$$

The relation is

$$p = \rho\tilde{W}_\rho - \tilde{W}, \tag{6.2}$$

and $\rho = \varphi_X^{-1}$.

Consider the function $\bar{W}(X,\rho) = \tilde{W}(X,\rho) + \rho\, g(X)$. Then

$$\rho\bar{W}_\rho - \bar{W} = \rho\tilde{W}_\rho - \tilde{W}.$$

This means that the function $\tilde{W}(X,\rho)$ can be found up to the term $\rho\, g(X)$. This term will be omitted in the further study.

In order to construct the Lagrangian, one has to analyze a non-homogeneous equation (6.2) for the potential function $\bar{W}(X,\rho)$. This equation can be solved by the method of variation of parameters, in which one assumes that

$$\tilde{W}(X,\rho) = \rho f(X,\rho).$$

Differentiating the latest equation with respect to ρ and substituting it into equation (6.2) yields:

$$\frac{\partial f}{\partial \rho} = \frac{1}{\rho^2} p.$$

Hence

$$f(X,\rho) = \int \frac{1}{\rho^2}\, p \, d\rho + g(X).$$

Therefore, the general solution of the non-homogeneous equation (6.2) is

$$\tilde{W}(X,\rho) = \rho \int \frac{1}{\rho^2}\, p(X,\rho)\, d\rho. \tag{6.3}$$

Consider the pressure function $P(X,\varphi_X)$ of the model M_1 which is presented in Table 5.1. The Lagrangian can be constructed by applying equation (6.3) and the process is analyzed as follows

$$\begin{aligned}\tilde{W}(X,\rho) &= \rho \int \frac{1}{\rho^2}\, p(X,\rho) d\rho \\ &= \rho \int \frac{1}{\rho^2}\Big(\Phi(Z)X^{\alpha+2\gamma} + h(X)\Big) d\rho \\ &= \rho \int \frac{1}{\rho^2}\Phi(Z)X^{\alpha+2\gamma} d\rho - h(X)\end{aligned}$$

where $Z = X^{-\alpha}\rho^{-1}$.

In the first integral term, one applies the change $u = \frac{X^{-\alpha}}{\rho}$. Then

$$\begin{aligned}\int \frac{1}{\rho^2}\Phi(Z)X^{\alpha+2\gamma} d\rho &= \int \frac{1}{\rho^2}\Phi(\frac{X^{-\alpha}}{\rho})X^{\alpha+2\gamma} d\rho \\ &= -\int \Phi(u) X^{2(\alpha+\gamma)} du \\ &= -X^{2(\alpha+\gamma)}\tilde{\Phi}(u),\end{aligned}$$

where $\tilde{\Phi}(u) = \int \Phi(u) du$ or $\tilde{\Phi}'(u) = \Phi(u)$. Therefore the potential function $W(X,\varphi_X)$ relating to the pressure function in M_1 is

$$W(X,\varphi_X) = -\varphi_X^{-1}\tilde{\Phi}(Z)X^{2(\alpha+\gamma)} - h(X),$$

where $Z = X^{-\alpha}\varphi_X$.

The potential functions of the other models can be solved in similar way. The results are summarized in Tables 6.1-6.2. The first column gives the number of models, the second column presents the pressure functions $P(X,\varphi_X)$, and the potential functions $W(X,\varphi_X)$ are shown in the third column. The restrictions for constants are in the fourth column.

Table 6.1 The potential functions of the equation $\varphi_{tt} + D_X P = 0$.

No.	$P(X,\varphi_X)$	$W(X,\varphi_X)$	Remarks
M_1	$\Phi(Z)X^{\alpha+2\gamma} + h(X)$	$-\varphi_X{}^{-1}\Phi(Z)X^{2(\alpha+\gamma)} - h(X)$	$\gamma \neq 0,\ 1,$
	$Z = X^{-\alpha}\varphi_X$		$\alpha \neq -1$
	$Xh''(X) = (\alpha + 2\gamma - 1)h'(X)$		
M_2	$\Phi(Z)e^{(2\beta-\alpha)X} + h(X)$	$-\varphi_X{}^{-1}e^{2(\beta-\alpha)X}\Phi(Z) - h(X)$	$\alpha \neq 0$
	$Z = e^{\alpha X}\varphi_X$		
	$h''(X) = (2\beta - \alpha)h'(X)$		
M_3	$\Phi(Z)X^{\alpha} + h(X)$	$-\varphi_X{}^{-1}\Phi(Z)X^{2\alpha} - h(X)$	$\alpha \neq -1$
	$Z = X^{-\alpha}\varphi_X$		
	$Xh''(X) = (\alpha - 1)h'(X)$		
M_4	$\Phi(\varphi_X) + \beta X + \gamma X^2$	$-\varphi_X{}^{-1}\Phi(\varphi_X) - \left(\beta X + \gamma X^2\right)$	$\alpha,\ \beta \neq 0$
M_5	$\Phi(\varphi_X)$	$-\varphi_X{}^{-1}\Phi(\varphi_X)$	
M_6	$e^{\beta X}\left(\varphi_X^{\gamma} + \frac{k_2}{\alpha^2}e^{(\alpha-\beta)X}\right)$	$-\varphi_X{}^{-1}\ln(\varphi_X)e^{\beta X} - \frac{k_2}{\alpha^2}e^{\alpha X}$	$\gamma = -1$
		$-\frac{1}{(\gamma+1)}\varphi_X^{\gamma}e^{\beta X} - \frac{k_2}{\alpha^2}e^{\alpha X}$	$\gamma \neq -1$
M_7	$e^{\beta X}\varphi_X^{\gamma}$	$-\varphi_X{}^{-1}\ln(\varphi_X)e^{\beta X}$	$\gamma = -1$
		$-\frac{1}{(\gamma+1)}\varphi_X^{\gamma}e^{\beta X}$	$\gamma \neq -1$
M_8	$b(X)\varphi_X^{\gamma} + k_1 b^{m+1}(X)$	$-\varphi_X{}^{-1}\ln(\varphi_X)b(X) - k_1 b^{m+1}(X)$	$\gamma = -1$
	$b^{l}(X) = \frac{\beta}{X}$	$-\frac{1}{(\gamma+1)}\varphi_X^{\gamma}b(X) - k_1 b^{m+1}(X)$	$\gamma \neq -1$

Table 6.2 The potential functions of the equation $\varphi_{tt} + D_X P = 0$ (continued).

No.	$P(X, \varphi_X)$	$W(X, \varphi_X)$	Remarks
M_9	$b(X)\varphi_X^\gamma$	$-\varphi_X{}^{-1}\ln(\varphi_X)b(X)$	$\gamma = -1$
	$b^l(X) = \frac{\beta}{X}$	$-\frac{1}{(\gamma+1)}\varphi_X^\gamma b(X)$	$\gamma \neq -1$
M_{10}	$b(X)\varphi_X^\gamma + k_1X^2$	$-\varphi_X{}^{-1}\ln(\varphi_X)b(X) - k_1X^2$	$\gamma = -1$
	$b^l(X) = \beta X$	$-\frac{1}{(\gamma+1)}\varphi_X^\gamma b(X) - k_1X^2$	$\gamma \neq -1$
M_{11}	$e^{\beta X}\varphi_X^\gamma + k_1X^2$	$-\varphi_X{}^{-1}\ln(\varphi_X)e^{\beta X} - k_1X^2$	$\gamma = -1$
		$-\frac{1}{(\gamma+1)}\varphi_X^\gamma e^{\beta X} - k_1X^2$	$\gamma \neq -1$
M_{12}	$k_1X\varphi_X^\gamma + k_2X^{\alpha+1}$	$-k_1X\varphi_X{}^{-1}\ln(\varphi_X) - k_2X^{\alpha+1}$	$\gamma = -1$
		$-\frac{k_1}{(\gamma+1)}X\varphi_X^\gamma - k_2X^{\alpha+1}$	$\gamma \neq -1$
M_{13}	$\beta\varphi_X^\gamma + k_1X^2$	$-\beta\varphi_X{}^{-1}\ln(\varphi_X) - k_1X^2$	$\gamma = -1$
		$-\frac{\beta}{(\gamma+1)}\varphi_X^\gamma - k_1X^2$	$\gamma \neq -1$
M_{14}	$\beta\varphi_X^{-3} + k_1X^2$	$\frac{\beta}{2}\varphi_X^{-3} - k_1X^2$	
M_{15}	$\beta\varphi_X^\gamma$	$-\beta\varphi_X{}^{-1}\ln(\varphi_X)$	$\gamma = -1$
		$-\frac{\beta}{(\gamma+1)}\varphi_X^\gamma$	$\gamma \neq -1$
M_{16}	$\beta\varphi_X^{-3}$	$\frac{\beta}{2}\varphi_X^{-3}$	
M_{17}	$b(X)\varphi_X^{-3}$	$\frac{1}{2}b(X)\varphi_X^{-3}$	

6.2 Conservation laws of equation (5.1)

Noether's theorem is applied to derive conservation laws

$$D_tC^t + D_XC^X = 0.$$

Using the kernel of the admitted Lie algebras $X_1 = \partial_t$, $X_2 = \partial_\varphi$, $X_3 = t\partial_\varphi$, one finds the conserved vectors which are already known as conservation laws of energy, momentum and center of mass, respectively. It is very worthy to our study to construct the conservation laws for the use of extensions of the kernel which they have not yet been studied and found. Details of the study are given in Appendix

B and results are summarized in Tables 6.3-6.8. The following details perform the study of Model 1.

The extension of the kernel of admitted Lie algebras in M_1 is given by the generator

$$X_4 = (\gamma - 1)t\partial_t - X\partial_X - (\alpha + 1)\varphi\partial_\varphi.$$

Determining equation for vector B^i is

$$Y\mathcal{L} + \mathcal{L}\big(D_t\xi^t + D_X\xi^X\big) = D_tB^1 + D_XB^2 \tag{6.4}$$

where Y is the extension of the generator X_4, $B^1 = B^1(t, X, \varphi, \varphi_t, \varphi_X)$, and $B^2 = B^2(t, X, \varphi, \varphi_t, \varphi_X)$. Equation (6.4) has to be satisfied for any function $\varphi(t, X)$.

Substituting the Lagrangian $\mathcal{L}$ into equation (6.4), one obtains

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t \\ -\varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X \\ -h_X X\varphi_X - (2\alpha+\gamma+2)X^{2\alpha+2\gamma}\Phi(Z) \\ -\alpha h\varphi_X - \Big(\frac{2\alpha+\gamma+2}{2}\Big)\varphi_t^2 + (\gamma-2)h\varphi_X = 0. \end{aligned} \tag{6.5}$$

Splitting equation (6.5) with respect to φ_{tt}, φ_{tX}, φ_{XX}, one gets

$$B^1_{\varphi_t} = 0, \qquad B^2_{\varphi_X} = 0, \qquad B^1_{\varphi_X} + B^2_{\varphi_t} = 0, \tag{6.6}$$

$$\begin{aligned} 2\varphi_t B^1_\varphi + 2B^1_t + 2\varphi_X B^2_\varphi + 2h_X X\varphi_X + (2\alpha+\gamma+2)X^{2\alpha+2\gamma}\Phi(Z) \\ +2\alpha h\varphi_X + \big(2\alpha+\gamma+2\big)\varphi_t^2 - 2(\gamma-2)h\varphi_X = 0. \end{aligned} \tag{6.7}$$

Solving equation (6.6), one finds

$$B^1 = -\varphi_X h_1 + h_3, \qquad B^2 = \varphi_t h_1 + h_2,$$

where $h_i = h_i(t, X, \varphi)$. Substituting B^1 and B^2 into equation (6.7), and splitting it again with respect to φ_X and φ_t, one obtains the condition $\gamma = -2\alpha - 2$. Solving the latter equations, one derives

$$\begin{aligned} B^1 &= -t\varphi_X\big(Xh'(X) + 3\alpha h(X) + 4h(X)\big), \\ B^2 &= t\varphi_t\big(Xh'(X) + 3\alpha h(X) + 4h(X)\big). \end{aligned} \tag{6.8}$$

The symmetry is divergent. Using Noether's theorem, the conserved vectors are

$$\begin{aligned} C^t &= -tX\varphi_X h'(X) + (\alpha+1)\Big(\varphi\varphi_t - t\varphi_X h(X)\Big) - (\alpha + \frac{3}{2})t\varphi_t^2 \\ &\quad - X\varphi_t\varphi_X + (2\alpha+3)tX^{-2\alpha-4}\Phi(Z), \\ C^X &= tX\varphi_t h'(X) + (\alpha+1)\big(\varphi + t\varphi_t\big)h(X) + \frac{1}{2}X\varphi_t^2 + X^{-2\alpha-3}\Phi(Z) \\ &\quad + \Big((\alpha+1)\varphi - (2\alpha+3)t\varphi_t - X\varphi_X\Big)X^{-3\alpha-4}\Phi'(Z). \end{aligned} \tag{6.9}$$

Further details of the study on constructing the conservation laws of other extensions of the kernel can be found in Appendix B.

6.3 Results of conservation laws

The conserved vectors in Lagrangian coordinates of the gas dynamic equations are summarized in Tables 6.3-6.8. The first column gives the number of the model. The second column presents the conserved vectors C^t and C^X, and the restriction of conditions can be found in the third column.

Table 6.3 The conserved vectors in Lagrangian coordinates.

No.	C^t and C^X	Remarks
M_1	$C^t = -tX\varphi_X h'(X) + (\alpha+1)\big(\varphi\varphi_t - t\varphi_X h(X)\big) - (\alpha+\frac{3}{2})t\varphi_t^2 - X\varphi_t\varphi_X + (2\alpha+3)tX^{-2\alpha-4}\Phi(Z),$	$\gamma = -2\alpha - 2$
	$C^X = tX\varphi_t h'(X) + (\alpha+1)\big(\varphi + t\varphi_t\big)h(X) + \big((\alpha+1)\varphi - (2\alpha+3)t\varphi_t - X\varphi_X\big)X^{-3\alpha-4}\Phi'(Z) + +\frac{1}{2}X\varphi_t^2$	
	$+X^{-2\alpha-3}\Phi(Z).$	
M_2	$C^t = -t\varphi_X h'(X) - 2\alpha t e^{2\alpha X}\Phi(Z) - \alpha\varphi\varphi_t + \alpha t\varphi_X h(X) + \alpha t\varphi_t^2 - \varphi_t\varphi_X,$	$\beta = 2\alpha$
	$C^X = t\varphi_t h'(X) - \alpha\varphi h(X) - \alpha t\varphi_t h(X) + \frac{1}{2}\varphi_t^2 + e^{2\alpha X}\Phi(Z) + \big(-\alpha\varphi + 2\alpha t\varphi_t - \varphi_X\big)e^{3\alpha X}\Phi'(Z)$	
M_3	$C^t = tX\varphi_X h'(X) + \frac{1}{2}t\varphi_t^2 + X\varphi_t\varphi_X - \frac{t}{X^2}\Phi(Z),$	$\alpha = -1$
	$C^X = -tX\varphi_t h'(X) - \frac{1}{2}X\varphi_t^2 - \frac{1}{X}\Phi(Z) + \big(\frac{t}{X}\varphi_t + \varphi_X\big)\Phi'(Z).$	
M_4	$C^t = \varphi_t\varphi_X + \gamma t^2\varphi_t + 2\gamma tX\varphi_X,$	
	$C^X = \beta\varphi + \beta\gamma t^2 X - \frac{1}{2}\varphi_t^2 - 2\gamma tX\varphi_t + \gamma^2 t^2 X^2 - \Phi(Z) + \big(\varphi_X + \gamma t^2\big)\Phi'(Z).$	
M_5	$X_2 = \partial_X$	
	$C^t = \varphi_t\varphi_X,$	
	$C^X = \frac{-1}{2}\varphi_t^2 - \Phi(Z) + \varphi_X\Phi'(Z).$	

Table 6.4 The conserved vectors in Lagrangian coordinates (continued).

No.	C^t and C^X	Remarks
M_6	$C^t = \frac{2}{3\alpha}\big(2\alpha^2 t\ln\varphi_X e^{\frac{2\alpha X}{3}} + 2k_2 t\varphi_X e^{\alpha X} + \alpha^2\varphi\varphi_t - \alpha^2 t\varphi_t^2 + 3\alpha\varphi_t\varphi_X\big),$	$\gamma=-1, \quad \beta=\frac{2\alpha}{3}$
	$C^X = \frac{1}{3\alpha}\varphi_X^{-1}\big(\alpha e^{\frac{2\alpha X}{3}}\big(-6\varphi_X\ln\varphi_X + 2\alpha\varphi - 4\alpha t\varphi_t + 3\varphi_X\big) + 2k_2 e^{\alpha X}\big(\varphi\varphi_X - 2t\varphi_t\varphi_X\big) - 3\alpha\varphi_t^2\varphi_X\big).$	
	$C^t = -4\alpha t e^{2\alpha X}, \quad C^X = 2e^{2\alpha X}$	$\gamma=-1, \quad \beta=2\alpha$
	$C^t = \frac{2\gamma}{3\alpha}\big(\big(\frac{-2\alpha^2}{\gamma+1}\big)te^{\frac{\alpha X(\gamma+3)}{3}}\varphi_X^{\gamma+1} - 2k_2 t\varphi_X e^{\alpha X} - \alpha^2\varphi\varphi_t + \alpha^2 t\varphi_t^2 - 3\alpha\varphi_t\varphi_X\big),$	$\gamma\neq-1, \quad \beta=\frac{\alpha(\gamma+3)}{3}$
	$C^X = \frac{\gamma}{3\alpha}\big(\big(\frac{-6\alpha\gamma}{\gamma+1}\big)e^{\frac{\alpha X(\gamma+3)}{3}}\varphi_X^{\gamma+1} + \big(2\alpha^2 e^{\frac{\alpha X(\gamma+3)}{3}}\varphi_X^{\gamma} + 2k_2 e^{\alpha X}\big)\big(2t\varphi_t - \varphi\big) + 3\alpha\varphi_t^2\big).$	
M_7	$X_2 = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$	
	$C^t = 2\varphi_X^{-2}\big(-te^{\beta X} + \varphi\varphi_t\varphi_X^2 - t\varphi_t^2\varphi_X^2\big), \quad C^X = 2e^{\beta X}\big(\varphi - 2t\varphi_t\big)\varphi_X^{-3}.$	$\gamma\neq-1, \quad \gamma=-3$
M_8	$C^t = 2(1-l)\big(k_1 l^2\beta^2 tX^2\varphi_X b^{\frac{3}{2}}(X) + t\ln\varphi_X b(X) - \frac{t}{2}\varphi_t^2\big) + (4l-1)tb(X)+,$	$\gamma=-1, \quad m=\frac{4l-1}{-2}$
	$(1-2l)\varphi\varphi_t - 2lX\varphi_t\varphi_X$	
	$C^X = k_1 l^2\beta^2 X^2 b^{\frac{3}{2}}(X)\big((1-2l)\varphi - 2(1-l)t\varphi_t\big) + (1-2l)\varphi_X^{-1}\varphi b(X) - 2(1-l)\varphi_X^{-1}t\varphi_t b(X)+$	
	$2lXb(X)\big(\ln\varphi_X - 1\big) + lX\varphi_t^2.$	

Table 6.5 The conserved vectors in Lagrangian coordinates (continued).

No.	C^t and C^X	Remarks
M_9	$X_1 = l(\gamma-1)X\partial_X + \big(l(\gamma+1)+1\big)\varphi\partial_\varphi,$	
	$C^t = -\varphi\varphi_t - 2X\varphi_t\varphi_X - \frac{3t}{\beta X},$	$\gamma = -1, \quad l = 1$
	$C^X = \frac{-2}{\beta}\ln\varphi_X + X\varphi_t^2 + \frac{1}{\beta}(\varphi X^{-1}\varphi_X^{-1} + 2).$	
	$C^t = \big(\frac{1-\gamma}{3\gamma+1}\big)\big(\varphi\varphi_t + 2X\varphi_t\varphi_X\big),$	$\gamma \neq -1, \quad l = \frac{-2}{3\gamma+1}$
	$C^X = \frac{(3\gamma+1)^{-3(\gamma+1)/2}}{\gamma+1}(2\beta X)^{(3\gamma+1)/2}\big((1-\gamma^2)\varphi\varphi_X^{\gamma} + 2\gamma(1-\gamma)X\varphi_X^{\gamma+1}\big) + \big(\frac{\gamma-1}{3\gamma+1}\big)X\varphi_t^2.$	
	$X_2 = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi,$	
	$C^t = 2\varphi_X^{-2}\big(-tb(X) + \varphi\varphi_t\varphi_X^2 - t\varphi_t^2\varphi_X^2\big),$	$\gamma \neq -1, \quad \gamma = -3$
	$C^X = 2b(X)\big(\varphi - 2t\varphi_t\big)\varphi_X^{-3}.$	
M_{10}	$C^t = \frac{\gamma}{4(\gamma+1)^4}\Big(\big(\frac{\beta}{2(\gamma+1)}\big)^{2\gamma}\beta^2 tX^{2\gamma+2}\varphi_X^{\gamma+1} + \big(\gamma^2(\gamma+3)+3\gamma+1\big)\big(4\varphi\varphi_t - 2t\varphi_t^2 + 4X\varphi_t\varphi_X + 4k_1 tX^2\varphi_X\big)\Big),$	$\gamma \neq -1, \quad l = \frac{1}{2(\gamma+1)}$
	$C^X = \frac{\gamma X}{4(\gamma+1)^4}\Big(\big(\frac{\beta}{2(\gamma+1)}\big)^{2\gamma}(\gamma+1)\beta^2X^{2\gamma+1}\varphi_X^{\gamma}\big(\varphi - t\varphi_t\big) + \big(\frac{\beta}{2(\gamma+1)}\big)^{2\gamma}\gamma\beta^2X^{2\gamma+2}\varphi_X^{\gamma+1}$ $+\big(\gamma^2(\gamma+3)+3\gamma+1\big)\big(4k_1X\varphi - 4k_1tX\varphi_t - 2\varphi_t^2\big)\Big).$	

Table 6.6 The conserved vectors in Lagrangian coordinates (continued).

No.	C^t and C^X	Remarks
M_{12}	$C^t = -4k_2 tX^{7/2}\varphi_X - 4k_1 tX \ln\varphi_X - 3\varphi\varphi_t + 2t\varphi_t^2 - 2X\varphi_t\varphi_X + 5k_1 tX,$	$\gamma = -1, \quad \alpha = \frac{5}{2}$
	$C^X = -3k_2X^{7/2}\varphi + 4k_2 tX^{7/2}\varphi_t + 2k_1X^2\ln\varphi_X + k_1X\varphi_X^{-1}(-3\varphi + 4t\varphi_t) + X\varphi_t^2 - 2k_1X^2$	
	$C^t = \left(\frac{-2k_2\gamma(3\gamma-1)}{(\gamma+1)(\gamma+3)}\right)tX\varphi_X^{\gamma+1} - \left(\frac{2k_2\gamma(3\gamma-1)}{\gamma+3}\right)tX^{\frac{3-4\gamma}{\gamma+3}}\varphi_X - \left(\frac{2\gamma(\gamma-2)}{\gamma+3}\right)\varphi\varphi_t + \left(\frac{\gamma(3\gamma-1)}{\gamma+3}\right)t\varphi_t^2 + 2\gamma X\varphi_t\varphi_X,$	$\gamma \neq -1, \quad \alpha = \frac{-5\gamma}{\gamma+3}$
	$C^X = \left(\frac{-2k_1\gamma(\gamma-2)}{\gamma+3}\right)X\varphi\varphi_X^{\gamma} + \left(\frac{2k_1\gamma(3\gamma-1)}{\gamma+3}\right)tX\varphi_t\varphi_X^{\gamma} + \left(\frac{2k_1\gamma^2}{\gamma+1}\right)X^2\varphi_X^{\gamma+1} - \left(\frac{2k_2\gamma(\gamma-2)}{\gamma+3}\right)X^{\frac{3-4\gamma}{\gamma+3}}\varphi +$ $\left(\frac{2k_2\gamma(3\gamma-1)}{\gamma+3}\right)tX^{\frac{3-4\gamma}{\gamma+3}}\varphi_t$	
M_{13}	$X_1 = t\partial_t + \gamma X\partial_X + (\gamma+2)\varphi\partial_\varphi$	
	$C^t = -\beta t\ln\varphi_X + 2\beta t - \varphi\varphi_t + \frac{1}{2}t\varphi_t^2 - X\varphi_t\varphi_X - k_1 tX^2\varphi_X$	$\gamma = -1$
	$C^X = \beta X\ln\varphi_X - \beta(\varphi - t\varphi_t)\varphi_X^{-1} - \beta X - k_1X^2\varphi + \frac{1}{2}X\varphi_t^2 + k_1 tX^2\varphi_t.$	
	$X_2 = \partial_X - k_1t^2\partial_\varphi$	
	$C^t = \varphi_t\varphi_X + k_1t^2\varphi_t + 2k_1tX\varphi_X,$	$\gamma = -1$
	$C^X = -\beta\ln\varphi_X + \beta + k_1\beta t^2\varphi_X^{-1} - \frac{1}{2}\varphi_t^2 - 2k_1tX\varphi_t + k_1^2t^2X^2.$	
	$C^t = \varphi_t\varphi_X + k_1t^2\varphi_t + 2k_1tX\varphi_X,$	$\gamma = -1$
	$C^X = \frac{\beta\gamma}{\gamma+1}\varphi_X^{\gamma+1} + \beta k_1t^2\varphi_X^{\gamma} - \frac{1}{2}\varphi_t^2 - 2k_1tX\varphi_t + k_1^2t^2X^2.$	

Table 6.7 The conserved vectors in Lagrangian coordinates (continued).

No.	C^t and C^X	Remarks
M_{14}	$X_1 = \partial_X - k_1 t^2 \partial_\varphi$	
	$C^t = \varphi_t \varphi_X + k_1 t^2 \varphi_t + 2k_1 t X \varphi_X,$	
	$C^X = \frac{3}{2}\beta \varphi_X^{-2} + \beta k_1 t^2 \varphi_X^{-3} - \frac{1}{2}\varphi_t^2 - 2k_1 t X \varphi_t +$	
	$k_1^2 t^2 X^2.$	
M_{15}	$X_1 = (\gamma - 1) t \partial_t - 2\varphi \partial_\varphi,$	
	$C^t = 2\beta t \ln \varphi_X - 2\beta t + 2\varphi \varphi_t - t\varphi_t^2 + 2X \varphi_t \varphi_X,$	$\gamma = -1$
	$C^X = -2\beta X \ln \varphi_X + 2\beta\big(\varphi \varphi_X^{-1} - t\varphi_t \varphi_X^{-1}\big) - X\varphi_t^2.$	
	$C^t = 2\big(- \beta t \varphi_X^{-2} + \varphi \varphi_t - t\varphi_t^2\big),$	$\gamma \neq -1, \quad \gamma = -3$
	$C^X = 2\beta \varphi_X^{-3}\big(\varphi - 2t\varphi_t\big).$	
	$X_2 = (\gamma - 1) X \partial_X + (\gamma + 1)\varphi \partial_\varphi$	
	$C^t = -2\beta t, \quad C^X = 2\beta X.$	$\gamma = -1$
	$C^t = \frac{2}{3}\big(- \varphi \varphi_t - 2X \varphi_t \varphi_X\big),$	$\gamma \neq -1, \quad \gamma = -1/3$
	$C^X = \frac{2}{3}\big(X\varphi_t^2 - \beta \varphi \varphi_X^{-1/3} + \beta X \varphi_X^{2/3}\big).$	
	$X_3 = \partial_X$	
	$C^t = \varphi_t \varphi_X, \qquad C^X = -\beta \ln \varphi_X + \beta - \frac{1}{2}\varphi_t^2.$	$\gamma = -1$
	$C^t = \varphi_t \varphi_X, \qquad C^X = \big(\frac{\beta\gamma}{\gamma+1}\big)\varphi_X^{\gamma+1} - \frac{1}{2}\varphi_t^2.$	$\gamma \neq -1$
M_{16}	$X_1 = \partial_X$	
	$C^t = \varphi_t \varphi_X, \qquad C^X = \frac{3\beta}{2}\varphi_X^{-2} - \frac{1}{2}\varphi_t^2.$	
	$X_3 = t^2 \partial_t + t\varphi \partial_\varphi,$	
	$C^t = \frac{1}{2}\beta t^2 \varphi_X^{-2} + \frac{1}{2}\varphi_t^2 - t\varphi \varphi_t + \frac{1}{2}t^2 \varphi_t^2,$	
	$C^X = \beta t(-\varphi + t\varphi_t)\varphi_X^{-3}.$	

Table 6.8 The conserved vectors in Lagrangian coordinates (continued).

No.	C^t and C^X	Remarks
M_{16}	$X_4 = 2t\partial_t + \varphi\partial_\varphi$	
	$C^t = \beta t\varphi_X^{-2} - \varphi\varphi_t + t\varphi_t^2,$	
	$C^X = \beta\varphi_X^{-3}(-\varphi + 2t\varphi_t).$	
M_{17}	$C^t = \frac{1}{2}b(X)t^2\varphi_X^{-2} + \frac{1}{2}\varphi^2 - t\varphi\varphi_t + \frac{1}{2}t^2\varphi_t^2,$	
	$C^X = tb(X)(-\varphi + t\varphi_t)\varphi_X^{-3}$	

The following section provides conservation laws of the hyperbolic shallow water equations. This models belong to the particular class model (1.1) considered by Gavrilyuk and Teshukov (2001).

6.4 Hyperbolic shallow water equations

The one-dimensional hyperbolic shallow-water equations are

$$h_t + uh_x + hu_x = 0, \quad u_t + uu_x + gh_x = 0,$$

where u is the velocity of the fluid and h is the location of the free surface. Here $g = 2\gamma_1$. It is well-known that exchanging the depth h by ρ (density of a gas), these equations describe one-dimensional isentropic gas flow

$$\rho_t + u\rho_x + \rho u_x = 0, \qquad u_t + uu_x + \frac{1}{\rho}p_x = 0 \tag{6.10}$$

with the pressure

$$p = \gamma_1\rho^2. \tag{6.11}$$

The admitted Lie algebra of equations (6.10) with (6.11) is infinite-dimensional and defined by the generators (Szatmari and Bihlo, 2014 ; Chirkunov and Pikmullina , 2014 ; Chirkunov, Dobrokhotov, Medvedec, and Minenkov, 2014)

$$Y_1 = t\partial_t + x\partial_x, \;\; Y_2 = t\partial_x + \partial_u, \;\; Y_3 = x\partial_x + u\partial_u + 2\rho\partial_\rho,$$

$$Y_4 = 2(x - 3tu)\partial_t + 3t(2\rho - u^2)\partial_x + (u^2 + 4\rho)\partial_u + 4\rho u\partial_\rho,$$

$$Y_h = f(u,\rho)\partial_t + g(u,\rho)\partial_x,$$

where

$$g_u - uf_u + \rho f_\rho = 0, \ \ g_\rho - uf_\rho + f_u = 0.$$

Choosing the function $W(\rho)$ such that

$$\gamma_1\rho^2 = \rho W_\rho - W \qquad \text{or} \qquad W = \gamma_1\rho^2,$$

equations (6.10) are equivalent to the Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta\varphi} = 0 \tag{6.12}$$

with the Lagrangian

$$\mathcal{L} = \rho_0(\frac{1}{2}\varphi_t^2 - \gamma_1\rho_0\varphi_\xi^{-1}).$$

The Euler-Lagrange equation (6.12) is

$$\varphi_\xi\varphi_{tt} + 2\gamma_1(\rho_0\varphi_\xi^{-1})_\xi = 0. \tag{6.13}$$

Here

$$\frac{\delta}{\delta\varphi} = \frac{\partial}{\partial\varphi} - D_t\frac{\partial}{\partial\varphi_t} - D_\xi\frac{\partial}{\partial\varphi_\xi} + D_t^2\frac{\partial}{\partial\varphi_{tt}} + D_tD_\xi\frac{\partial}{\partial\varphi_{t\xi}} + D_\xi^2\frac{\partial}{\partial\varphi_{\xi\xi}} + ... \tag{6.14}$$

is the variational derivative. Because of the equivalence transformation

$$\xi = \alpha(x_0) \qquad \text{where} \quad \alpha'(x_0) = \rho_0(x_0), \qquad \text{one can assume} \quad \rho_0 = 1.$$

In our further study we will consider the Euler-Lagrange equation (6.13) in reduced Lagrangian coordinates:

$$\varphi_\xi^3\varphi_{tt} - 2\gamma_1\varphi_{\xi\xi} = 0. \tag{6.15}$$

Calculations show that the Lie group admitted by equation (6.15) consists of the transformations corresponding to the generators

$$X_1 = \partial_t, \ \ X_2 = \partial_\xi, \ \ X_3 = \partial_\varphi, \ \ X_4 = t\partial_\varphi,$$

$$X_5 = t\partial_t + 4\xi\partial_\xi + 2\varphi\partial_\varphi, \;\; X_6 = \varphi\partial_\varphi + 3\xi\partial_\xi.$$

Functions T^1 and T^2 are called densities of a conservation law if

$$\left(D_t T^1 + D_\xi T^2\right)_{|S} = 0, \tag{6.16}$$

where (S) is a system of studied equations, $|S$ means that equation (6.16) is identically satisfied for any solution of the system of equations (S).

Assume that

$$T^1 = T^1(t, \xi, \varphi, \varphi_t, \varphi_\xi), \;\; T^2 = T^2(t, \xi, \varphi, \varphi_t, \varphi_\xi).$$

Substituting the latter representation of the densities into equation (6.16), excluding φ_{tt} found from equation (6.15), and splitting it with respect to $\varphi_{t\xi}$ and $\varphi_{\xi\xi}$, one obtains the overdetermined system of equations:

$$T^1_\varphi \varphi_t + T^1_t + T^2_\varphi \varphi_\xi + T^2_\xi = 0,$$

$$T^1_{\varphi_\xi} + T^2_{\varphi_t} = 0, \;\; 2\gamma_1 T^1_{\varphi_t} + \varphi_\xi^3 T^2_{\varphi_\xi} = 0.$$

The general solution of this system is

$$(T^1, T^2) = c_1(T^1_1, T^2_1) + c_2(T^1_2, T^2_2) + c_3(T^1_3, T^2_3) + (\tilde{P}, \tilde{Q}),$$

where c_i, $(i = 1, 2, 3)$ are constant,

$$T^1_1 = t\varphi_t - \varphi, \;\; T^2_1 = t\gamma_1\varphi_\xi^{-2}$$

$$T^1_2 = \varphi_t(5t\varphi_t - 2\xi\varphi_\xi - 6\varphi) + 10\gamma_1 t\varphi_\xi^{-1},$$

$$T^2_2 = \varphi_\xi^{-2}(2\gamma_1(5t\varphi_t - 2\xi\varphi_\xi - 3\varphi) + \xi\varphi_t^2,$$

$$T^1_3 = 6\gamma_1\varphi_\xi^{-1}\left(40t\varphi_t + \xi\varphi_\xi(10\ln(\varphi_\xi) + 3) - 15\varphi\right) + 5\varphi_t^2(8t\varphi_t - 3\xi\varphi_\xi - 9\varphi),$$

$$T^2_3 = 160\gamma_1^2 t\varphi_\xi^{-3} + 30\gamma_1\varphi_t\varphi_\xi^{-2}(4t\varphi_t - 2\xi\varphi_\xi - 3\varphi) + 5\xi\varphi_t^3,$$

and the functions $\tilde{P}(\varphi_t, \varphi_\xi)$ and $\tilde{Q}(\varphi_t, \varphi_\xi)$ satisfy the conditions

$$\tilde{P}_{\varphi_\xi} + \tilde{Q}_{\varphi_t} = 0, \;\; \varphi_\xi^3\tilde{Q}_{\varphi_\xi} + 2\gamma_1\tilde{P}_{\varphi_t} = 0. \tag{6.17}$$

Notice that excluding $\tilde{Q}(\varphi_t,\varphi_\xi)$ from the latter equations, one derives that the function $\tilde{P}(\varphi_t,\varphi_\xi)$ has to satisfy

$$2\gamma_1\tilde{P}_{\varphi_t\varphi_t}-\varphi_\xi^3\tilde{P}_{\varphi_\xi\varphi_\xi}=0. \tag{6.18}$$

The conservation laws related with the densities (T_1^1,T_1^2) and $(\tilde{P},\tilde{Q})$ are known in the theory of the gas dynamics equations. The conservation law corresponding to (T_1^1,T_1^2) is the center of mass conservation law (Ibragimov, 1985). The conservation laws related with the densities $(\tilde{P},\tilde{Q})$ are as follows. It is well known (Whitham, 1974) that the hyperbolic shallow water equations (6.10) have an infinite number of conservation laws in Eulerian coordinates:

$$D_tP+D_xQ=0, \tag{6.19}$$

where the functions $P(u,\rho)$ and $Q(u,\rho)$ satisfy the equations

$$Q_u=uP_u+\rho P_\rho,\ \ Q_\rho=2\gamma_1P_u+uP_\rho. \tag{6.20}$$

As densities of conservation laws in Lagrangian coordinates and Eulerian coordinates are related by the formulae

$$P=\rho\tilde{P},\ \ Q=\rho u\tilde{P}+\tilde{Q},$$

one also obtains an infinite number of conservation laws in Lagrangian coordinates with

$$\tilde{P}(\varphi_t,\varphi_\xi)=\varphi_\xi P(\varphi_t,\varphi_\xi^{-1}),\ \ \tilde{Q}(\varphi_t,\varphi_\xi)=Q(\varphi_t,\varphi_\xi^{-1})-\varphi_tP(\varphi_t,\varphi_\xi^{-1}).$$

Equations (6.20) become (6.17).

6.4.1 Applications of symmetries for deriving conservation laws.

Direct checking shows that the symmetries X_1, X_2 and X_3 satisfy (3.22) with $(B^1,B^2)=0$, and the symmetry X_4 is a divergent symmetry with the vector $(B^1,B^2)=(\varphi,0)$. The symmetries X_5, X_6 and X_c do not satisfy equation (3.22).

Using the generators X_1, X_2, X_3 and X_4, Noether's theorem allows one to derive conservation laws

$$D_t T^1 + D_\xi T^2 = 0,$$

where the densities of the conservation laws T^1, T^2 are presented in Table 6.9.

Table 6.9 Conservation laws of the hyperbolic shallow-water equations.

	T^1	T^2	Remark
X_1	$\varphi_t^2 + 2\gamma_1\varphi_\xi^{-1}$	$2\gamma_1\varphi_t\varphi_\xi^{-2}$	energy
X_2	$\varphi_t\varphi_\xi$	$2\gamma_1\varphi_\xi^{-1} - \varphi_t^2/2$	(Whitham, 1974)
X_3	φ_t	$\gamma_1\varphi_\xi^{-2}$	momentum
X_4	$t\varphi_t - \varphi$	$\gamma_1 t\varphi_\xi^{-2}$	center of mass

Remark. The system of modified one-dimensional shallow-water equations studied in Szatmari and Bihlo (2014),

$$\rho_t + u\rho_x + \rho u_x = 0, \quad u_t + uu_x + g\left(1 + \frac{H}{\rho}\right)\rho_x = 0,$$

where H is constant, can be rewritten in form (1.1) with the potential function

$$W = \gamma_1\rho(\rho + 2H\ln\rho).$$

The Euler-Lagrange equation is

$$\varphi_\xi^3\varphi_{tt} - 2\gamma_1\varphi_{\xi\xi}(1 + H\varphi_\xi^2) = 0.$$

Remark. The one-dimensional shallow-water equations with arbitrary bottom

$$\eta_t + ((\eta + H)u)_x = 0, \quad u_t + uu_x + g\eta_x = 0, \tag{6.21}$$

where $H = H(x)$ can be changed, by setting

$$\eta = \rho - H,$$

to

$$\rho_t + u\rho_x + \rho u_x = 0, \quad u_t + uu_x + 2\gamma_1\rho_x = 2\gamma_1 H'. \qquad (6.22)$$

Group analysis of equations (6.21) is given in Aksenov and Druzhkov (2016). The potential function W for equations (6.22) can be chosen as follows

$$W = \gamma_1\rho(\rho - 2H(x)),$$

and the Euler-Lagrange equation is

$$\varphi_\xi^3\varphi_{tt} - 2\gamma_1\varphi_{\xi\xi}(1 + H'(\varphi)\varphi_\xi^3) = 0.$$

CHAPTER VII

FLUIDS WITH INTERNAL INERTIA

This chapter is focused on the group classification of a class of dispersive models (Gavrilyuk and Teshukov, 2001)

$$\begin{array}{c} \dot{\rho} + \rho\, div(u) = 0, \quad \rho\dot{u} + \nabla p = 0, \quad \dot{S} = 0, \\ p = \rho\frac{\delta W}{\delta\rho} - W = \rho(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}(\frac{\partial W}{\partial\dot{\rho}}) - div(\frac{\partial W}{\partial\dot{\rho}}u)) - W, \end{array} \tag{7.1}$$

where t is time, ∇ is the gradient operator with respect to space variables, ρ is the fluid density, u is the velocity field, $W(\rho, \dot{\rho}, S)$ is a given potential, "dot" denotes the material time derivative: $\dot{f} = \frac{df}{dt} = f_t + u\nabla f$ and $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u. The method used in Siriwat and Meleshko (2012) (as well in Hematulin, Meleshko and Gavrilyuk (2007); Siriwat and Meleshko (2008)) followed the classical approach developed in Ovsiannikov (1978) for the gas dynamics equations. In contrast to the gas dynamics equations, this method becomes very complicated and cumbersome for the group classification of equations (7.1) with $W_S \neq 0$. In Siriwat and Meleshko (2012), a complete group classification of the one-dimensional equations (7.1) for a particular case where the function $W = W(\rho, \dot{\rho}, S)$ satisfies the condition $W_{S\dot{\rho}\dot{\rho}} = 0$ was performed. It is worth to notice that the used approach did not take into account the algebraic properties of the admitted Lie group. On the other hand the knowledge of algebraic structure of admitted Lie groups allow essentially simplify the group classification.

7.1 Some results of Siriwat and Meleshko (2012)

For the sake of completeness it is necessary to review here some results of Siriwat and Meleshko (2012).

The basis of generators of the equivalence Lie group consists of the generators

$$X_1^e = \partial_x,\ X_2^e = \partial_t,\ X_3^e = t\partial_x + \partial_u,\ X_4^e = t\partial_t + x\partial_x,$$
$$X_5^e = t\partial_t + 2\rho\partial_\rho - u\partial_u,\ X_6^e = \partial_W,\ X_7^e = -u\partial_u + \rho\partial_\rho - W\partial_W + t\partial_t,$$
$$X_8^e = \rho\varphi(S)\partial_W,\ X_9^e = \dot{\rho}g(\rho,S)\partial_W,\ X_{10}^e = h(S)\partial_S,$$

where the functions $g(\rho,S),\varphi(S)$ and $h(S)$ are arbitrary. Here only the essential part of the operators X_i^e, $(i = 5, 6, ..., 10)$ is written.

Since the equivalence transformations corresponding to the operators X_5^e , X_6^e, X_7^e, X_8^e,X_9^e and X_{10}^e are applied for simplifying the function W in the process of the group classification, let us present these transformations. Because the function W depends on ρ, $\dot{\rho}$ and S only, the transformations of these variables are presented:

$$\begin{array}{lllll}
X_5^e: & \rho' = \rho e^{2a}, & \dot{\rho}' = \dot{\rho}e^a, & S' = S, & W' = W; \\
X_6^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S, & W' = W + a; \\
X_7^e: & \rho' = \rho e^a, & \dot{\rho}' = \dot{\rho}, & S' = S, & W' = We^{-a}; \\
X_8^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S, & W' = \rho\varphi(S)a + W; \\
X_9^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S, & W' = \dot{\rho}h(\rho,S)a + W \\
X_{10}^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = q(S,a), & W' = W;
\end{array}$$

Here a is the group parameter. The group classification is performed up to this set of equivalence transformations.

The kernel of admitted Lie algebras is determined for all functions $W(\rho,\dot{\rho},S)$ and it consists of the generators

$$X_1 = \partial_x,\ \ X_2 = \partial_t,\ \ X_3 = t\partial_x + \partial_u.$$

Table 7.1 Functions $W(\rho, \dot{\rho}, S)$ such that equations (7.1) admit projective transformations.

	$W(\rho, \dot{\rho}, S)$	Extensions	Remarks
M_1	$q_0\rho^{-3}\dot{\rho}^2 + \rho^3 S$	$X_p, \quad X_4 - 2S\partial_S, \quad X_5 - X_6$	
M_2	$\rho^{-3}\dot{\rho}^2 S + q_1\rho^3 S^k$	$X_p, \quad X_5 - X_6, \quad X_6 - (k+1)X_4 + 2S\partial_S$	$q_1 \neq 0$
M_3	$\rho^{-3}\dot{\rho}^2 S + \rho^3\mu(S)$	$X_p, \quad X_5 - X_6$	$\mu' \neq q_1 S^k$
M_4	$\rho^{-3}\dot{\rho}^2 S$	$X_p, \quad X_4, \quad X_5 - X_6, \quad X_6 + 2S\partial_S$	

Extensions of the kernel depend on the value of the function $W(\rho, \dot{\rho}, S)$. They can only be operators of the form

$$k_p X_p + k_4 X_4 + k_5 X_5 + k_6 X_6 + \zeta\partial_S,$$

where $\zeta = \zeta(S)$ and

$$X_4 = t\partial_t - u\partial_u - \dot{\rho}\partial_{\dot{\rho}}, \;\; X_5 = x\partial_x + 2t\partial_t - u\partial_u - 2\dot{\rho}\partial_{\dot{\rho}}, \;\; X_6 = \rho\partial_\rho + \dot{\rho}\partial_{\dot{\rho}},$$

$$X_p = tx\partial_x + t^2\partial_t + (x - ut)\partial_u - t\rho\partial_\rho - (\rho + 3t\dot{\rho})\partial_{\dot{\rho}},$$

Since the function $W(\rho, \dot{\rho}, S)$ depends on $\dot{\rho}$, the term with $\partial_{\dot{\rho}}$ is also presented in the generators.

In Siriwat and Meleshko (2012), it is shown that if the function $W(\rho, \dot{\rho}, S)$ is not equivalent to one of the functions presented in Table 7.1, then $k_p = 0$.

7.1.1 Group classification of equations (7.1) with $k_p = 0$

In the present study we focus on the case where $k_p = 0$. In this case one can reduce the determining equations*. to the equation

$$k_6\rho g_\rho + \dot{\rho} g_{\dot{\rho}}(k_6 - k_4 - 2k_5) + \zeta g_S = g(2k_5 - k_6) + \dot{\rho}^{-2}(\rho\varphi + c), \qquad (7.2)$$

*Equations (2)–(7) of Siriwat and Meleshko (2012) where $g = (\dot{\rho}^{-1}W)_{\dot{\rho}}$, the constant c and the function $\varphi(S)$ are arbitrary and obtained during the integration. Relations between the constants k_4, k_5, k_6 and $\zeta(S)$ depend on the function $W(\rho, \dot{\rho}, S)$.

Notice that the study given in Siriwat and Meleshko (2012) analyzes the case where $W_{\dot{\rho}\dot{\rho}S} = 2g_S + \dot{\rho}g_{\dot{\rho}S} = 0$. Application of an algebraic approach allows us to omit this restriction.

7.1.2 Algebraic properties of admitted Lie algebras

The commutator table of the Lie algebra $L_6 = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ is

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	X_1	0
X_2	0	0	$-X_1$	$-X_2$	$-X_2$	0
X_3	0	X_1	0	X_3	$2X_3$	0
X_4	0	X_2	$-X_3$	0	0	0
X_5	X_1	$-X_3$	$-2X_3$	0	0	0
X_6	0	0	0	0	0	0

The Lie algebra $\{X_1, X_2, X_3\}$ is a kernel of admitted Lie algebras, the Lie algebra $\{X_4, X_5, X_6\}$ is an Abelian subalgebra. The generator X_ζ belongs to the center for any function $\zeta(S)$. Since the Lie algebra $\{X_1, X_2, X_3\}$ composes the kernel of admitted Lie algebras, then the basis generators of an admitted Lie algebra related with the generators X_4, X_5, X_6 and X_ζ can be chosen in the form

$$\beta X_6 + qX_5 + \gamma X_4 + X_\zeta. \tag{7.3}$$

The latter generators also compose a Lie algebra.

Notice that if $\zeta \neq 0$ for one of the basis generators, then for this generator[†] one can assume that $\zeta = 1$.

[†]Only for a single basis generator: for other basis generators the function $\zeta = \zeta(S)$.

7.1.3 Strategy of further study

In the approach used in Siriwat and Meleshko (2012) it was tried to find the coefficients β, q, γ and ζ of the basis generators simultaneously with the function $W(\rho, \dot{\rho}, S)$ by solving the determining equations. This led to a complicated and cumbersome study.

It is well known that the set of admitted generators composes a Lie algebra (Ovsiannikov, 1978): the property to compose a Lie algebra is automatically satisfied for solutions of the determining equations.

The idea of the algebraic approach used in the present paper is to separate the study of group classification into two steps. In the first step one makes a preliminary study of possible coefficients of the basis generators using the requirement of admitted generators to compose a Lie algebra. In the second step one substitutes these coefficients of each basis generator of the Lie algebra into the determining equation (7.2). Solving the obtained system of equations, the function $W(\rho, \dot{\rho}, S)$ and additional restrictions for the coefficients of the basis generators are obtained.

Here we have to notice that the function $\varphi(S)$ and the constant c can be different for each basis generator.

Let us also notice that if one can choose basis generators such that two of them have the form

$$\zeta_1(S)\partial_S, \;\; \zeta_2(S)\partial_S, \tag{7.4}$$

then this case is reduced to $W_S = 0$. Indeed, since the generators (7.4) are basis generators, then $\zeta_i \neq 0$ and $\zeta_1\zeta_2' - \zeta_1'\zeta_2 \neq 0$. By virtue of the equivalence transformation related with X_{10}^e, one can assume that $\zeta_1 = 1$ and $\zeta_2' \neq 0$. Substituting the coefficients of the generators (7.4) into (7.2) one obtains the equations

$$g_S = \dot{\rho}^{-2}(\rho\varphi_1 + c_1), \quad \rho(\varphi_2 - \zeta_2\varphi_1) + c_2 - \zeta_2 c_1 = 0. \tag{7.5}$$

Splitting the second equation with respect to ρ, and then with respect to S, one finds that

$$\varphi_2 = \zeta_2 \varphi_1, \;\; c_1 = 0, \;\; c_2 = 0.$$

Integration of the first equation (7.5) gives

$$g = \rho \dot{\rho}^{-2} \psi + \widetilde{f},$$

where $\psi'(S) = \varphi_1(S)$ and $\widetilde{f} = \widetilde{f}(\rho, \dot{\rho})$. Hence,

$$W(\rho, \dot{\rho}, S) = \rho \psi(S) + f(\rho, \dot{\rho}) + \dot{\rho} h(\rho, S).$$

where $h(\rho, S)$ is an arbitrary function of the integration, and $\widetilde{f}(\rho, \dot{\rho}) = f_{\dot{\rho}}(\rho, \dot{\rho})$. Using the equivalence transformations corresponding to X_8^e and X_9^e, one can assume that $\psi = 0$ and $h = 0$, which means that $W_S = 0$.

In the preliminary study of Lie algebras of dimension more than 1, it is sufficient for our goals to use classifications of two- and three dimensional Lie algebras. These classifications are well-known[‡]. For the sake of completeness they are presented in Appendix C.

Further study depends on the dimension of a Lie algebra composed by the generators of the form (7.3).

7.2 Results of the group classification of equations (7.1)

The result of the group classification of equations (7.1) with $W_S \neq 0$ is summarized in Tables 7.2-7.4. The representation of the function $W(\rho, \dot{\rho}, S)$ is simplified by equivalence transformations.

The first column in Tables 7.2-7.4 presents the number of the extension, forms of the function $W(\rho, \dot{\rho}, S)$ are given in the second column, extensions of the

[‡]See for example in Ibragimov (1996)

kernel of admitted Lie algebras are in the third column, restrictions for constants are in the fourth column. Details of the study are presented in Appendix D.

Table 7.2 Group classification of a class of dispersive models (7.1).

No.	$W(\rho,\dot\rho,S)$	Extensions	Remarks
M_1	$q_0\rho^{-3}\dot\rho^2+\rho^3 S$	$X_p,\quad X_4-2S\partial_S,\quad X_5-X_6$	
M_2	$\rho^{-3}\dot\rho^2 S+q_1\rho^3S^k$	$X_p,\quad X_5-X_6,$ $X_6-(k+1)X_4+2S\partial_S$	$q_1\neq 0$
M_3	$\rho^{-3}\dot\rho^2S+\rho^3\mu(S)$	$X_p,\quad X_5-X_6$	$\mu'\neq q_1S^k$, $\mu\neq 0$
M_4	$\rho^{-3}\dot\rho^2S$	$X_p,\quad X_4,\quad X_5-X_6,$ $X_6+2S\partial_S$	
M_5	$\rho^\alpha\phi(\dot\rho\rho^\beta,S)$	$-(\alpha+\beta)X_4+(\beta+\frac{\alpha+1}{2})X_5$ $+X_6$	$\alpha(\alpha-1)\neq 0$
M_6	$\phi(\dot\rho\rho^{-\gamma},S)-q_0\ln(\rho)$	$2\gamma(X_4-X_5)+X_5+2X_6$	
M_7	$\rho\phi(\dot\rho\rho^\alpha,S)+\rho\ln(\rho)\psi(S)$	$-(\alpha+1)(X_4-X_5)+X_6$	
M_8	$\dot\rho\ln(\dot\rho)\phi(\rho,S)$	X_5	
M_9	$\dot\rho^\alpha\phi(\rho,S)$	$X_4+\frac{2-\alpha}{2(\alpha-1)}X_5$	$\alpha(\alpha-1)\neq 0$
M_{10}	$\phi(\rho,S)+\ln(\dot\rho)(q_0+\rho\psi(S))$	$-X_4+X_5$	$q_0\psi\neq 0$
M_{11}	$e^{\alpha S}\phi(\rho e^{-S},\dot\rho e^{\beta S})$	$-(\alpha+\beta)X_4+(\beta+\frac{\alpha+1}{2})X_5$ $+X_6+\partial_S$	$\alpha\neq 0$
M_{12}	$\phi(\rho e^{-S},\dot\rho e^{-\gamma S})+q_0S$	$2\gamma(X_4-X_5)+X_5$ $+2(X_6+\partial_S)$	
M_{13}	$e^{\alpha S}\phi(\rho,\dot\rho e^{(1-\frac{\alpha}{2})S})$	$(-\alpha-2)X_4+2X_5+2\partial_S$	$\alpha\neq 0$
M_{14}	$\phi(\rho,\dot\rho e^S)+q_0S$	$-X_4+X_5+\partial_S$	
M_{15}	$e^{-2S}\phi(\rho,\dot\rho e^S)$	$X_4+\partial_S$	
M_{16}	$h(\rho)\dot\rho^{-2}+\alpha S$	$\partial_S,\quad -3X_4+2X_5+2S\partial_S$	$h\alpha\neq 0$

Table 7.3 Group classification of a class of dispersive models (7.1) (continued).

No.	$W(\rho,\dot\rho,S)$	Extensions	Remarks
M_{17}	$\rho^{2q_2-2\alpha-1}e^{-2(\beta-q_1)S}\phi\left(\dot\rho\rho^{\alpha}e^{\beta S}\right)$	$(\beta-2q_1)X_4+q_1X_5+\partial_S,$	$(\alpha-q_2+1)(2(\alpha-q_2)+1)\neq 0$
		$(\alpha+1-2q_2)X_4+q_2X_5+X_6$	
M_{18}	$\rho e^{\alpha S}\left(q_0\ln(\rho)+\phi\left(\dot\rho\rho^{\lambda}e^{\beta S}\right)\right)$	-$(\alpha+\beta)X_4+(\frac{\alpha}{2}+\beta)X_5+\partial_S,$	
		$-(\lambda+1)(X_4-X_5)+X_6$	
M_{19}	$q_0\ln(\rho)+\alpha S+\phi\left(\dot\rho\rho^{q_2-1/2}e^{q_1S}\right)$	$\gamma_1(X_4-X_5)+\partial_S,$	
		$2\gamma_2(X_4-X_5)+X_5+2X_6$	
M_{20}	$e^{-2(\beta-q_1)S}\phi\left(\dot\rho\rho^{\alpha}e^{\beta S}\right).$	$(\beta-2q_1)X_4+q_1X_5+\partial_S,$	$\beta-q_1\neq 0$
		$2\gamma_2(X_4-X_5)+X_5+2X_6$	
M_{21}	$\dot\rho^{\alpha}e^{\beta S}\phi\left(\rho e^{-\beta_1S}\right)$	$(\beta-(1-\alpha)\beta_1)X_4+(\alpha-2)\beta_1X_6+(\alpha-2)\partial_S,$	$(\alpha-1)(\alpha-2)\neq 0$
		$(2-2\alpha)X_4+(\alpha-2)X_5$	
M_{22}	$\dot\rho\ln(\dot\rho)e^{-\gamma_1S}\phi\left(\rho e^{-\beta_1S}\right)$	$\gamma_1X_4+\beta_1X_6+\partial_S,\ \ X_5$	
M_{23}	$\dot\rho^2e^{\alpha S}\phi\left(\rho e^{\beta S}\right)$	$(\alpha-\beta)X_5-2\beta X_6+2\partial_S,\ \ X_4$	
M_{24}	$\rho^{-\gamma_2}\dot\rho\ln(\dot\rho)\phi\left(S\right)$	$X_5,\ \ \gamma_2X_4+X_6$	

Table 7.4 Group classification of a class of dispersive models (7.1) (continued).

No.	$W(\rho,\dot\rho,S)$	Extensions	Remarks
M_{25}	$\rho^{\frac{2q_2-1}{2q_1+1}}\dot\rho^{\frac{2(q_1+1)}{2q_1+1}}\phi(S)$	$X_4+q_1X_5,\ \ q_2X_5+X_6$	$(1+2q_1)(1+q_1)\neq 0,$
M_{26}	$\rho\left(\ln(\rho)\psi(S)+\ln(\dot\rho)\phi(S)\right)$	$X_4-X_5,\ \ X_6$	
M_{27}	$q_0\ln(\rho)+\alpha\ln(\dot\rho)+S$	$X_4-X_5,\ \ X_5+2X_6$	
M_{28}	$q_0\ln(\dot\rho)+\alpha\ln(\rho)+S$	$\partial_S,\ \ X_4+2X_6,\ \ -X_4+X_5$	
M_{29}	$\rho e^{-S}\left(\ln(\dot\rho)+q_0\ln(\rho)\right)$	$X_4+2\partial_S,\ \ X_6,\ \ -X_4+X_5$	
M_{30}	$q_0\rho^{-q_2}\dot\rho\ln(\dot\rho)e^{-q_1S}$	$q_1X_4+\partial_S,\ \ q_2X_4+X_6,\ \ X_5$	
M_{31}	$\rho^{\alpha}\dot\rho^{\beta}e^{\lambda S}$	$-\lambda X_4+(2-\beta)\partial_S,\ \ (1-(\beta+\alpha))X_4+(2-\beta)X_6$	$\frac{2\beta-2}{(2-\beta)^2}\neq 0$
		$(2\beta-2)X_4+(2-\beta)X_5$	
M_{32}	$q_0\dot\rho^2\rho^{\alpha}e^{2\beta S}$	$\beta X_5+\partial_S,\ \ (\alpha+1)X_5+2X_6,\ \ X_4$	
M_{33}	$q_0\rho^{-\gamma_3}\dot\rho\ln(\dot\rho)-S$	$\partial_S,\ \ q_2X_5+S\partial_S,\ \ \gamma_3X_4+q_3X_5+X_6$	$\alpha q_0\neq 0$
M_{34}	$q_0\rho^{\alpha}\dot\rho^{\beta}\ln(\dot\rho)-S$	$\partial_S,\ \ (1-\beta)X_4+\beta\beta_2X_6+\beta S\partial_S,$	$\beta(1-\beta)\neq 0$
		$-\alpha X_4+\beta X_5$	
M_{35}	$q_0\rho\ln(\dot\rho/\rho)-\alpha S$	$\partial_S,\ \ q_2X_5+X_6+S\partial_S,\ \ X_4$	$\gamma_3+2\neq 0$

7.3 Green-Naghdi models

The models (7.1) were derived by Gavrilyuk and Shugrin (1996) and Gavrilyuk and Teshukov (2001) using the Lagrangian

$$\mathcal{L} = \rho\frac{u^2}{2} - W\left(\rho, \dot{\rho}\right). \tag{7.6}$$

The Green-Naghdi model corresponds to the potential $W = \gamma_1\rho^2 - \gamma\rho\dot{\rho}^2$. In particular, the potential for classical hyperbolic shallow water equations is determined by the condition $\gamma = 0$.

The Green-Naghdi system is used to model highly nonlinear weakly dispersive waves propagating at the surface of a shallow layer of a perfect fluid. In Eulerian coordinates these equations are

$$\begin{gathered} \rho_t + u\rho_x + \rho u_x = 0, \\ \rho(u_t + uu_x + 2\gamma_1\rho_x) = 2\gamma\left(\rho^3(u_{xt} + uu_{xx} - u_x^2)\right)_x, \end{gathered} \tag{7.7}$$

where ρ is the water depth, u is the horizontal velocity, g is the gravity and ε is the ratio of the vertical length scale to the horizontal length scale. For $\varepsilon = 0$ equations (7.7) become the classical hyperbolic shallow water equations corresponding to hydrostatic pressure distribution as considered in the previous section. Here $g = 2\gamma_1$ and $\gamma = \varepsilon^2/6$ are introduced for convenience.

7.4 Conservation laws of Green-Naghdi models

The admitted Lie algebra of the Green-Naghdi equations is four-dimensional and determined by the generators (Bagderina and Chupakhin, 2005)

$$Y_1 = \partial_t,\;\; Y_2 = \partial_x,\;\; Y_3 = t\partial_x + \partial_u,\;\; Y_4 = t\partial_t + 2x\partial_x + u\partial_u + 2\rho\partial_\rho.$$

System (7.7) has four associated conservation laws (Gavrilyuk, Kalisch, and Khorsand, 2015):

$$D_t{}^gT_i^t + D_x{}^gT_i^x = 0,\;\; (i = 0, 1, 2, 3), \tag{7.8}$$

where

$$\begin{aligned}
{}^{g}T_0^t &= \rho, \qquad {}^{g}T_0^x = \rho u, \\
{}^{g}T_1^t &= \rho u, \\
{}^{g}T_1^x &= 2\gamma\rho^3(u_x^2 - uu_{xx} - u_{tx}) + \rho(u^2 + \gamma_1\rho) \\
{}^{g}T_2^t &= \tfrac{1}{2}\rho(2\gamma_1\rho + u^2 + 2\gamma\rho^2 u_x^2), \\
{}^{g}T_2^x &= \tfrac{1}{2}\rho u(2\gamma\rho^2(3u_x^2 - 2uu_{xx} - 2u_{tx}) + u^2 + 4\gamma_1\rho) \\
{}^{g}T_3^t &= u - \tfrac{2\gamma}{\rho}(\rho^3 u_x)_x, \\
{}^{g}T_3^x &= \tfrac{1}{2}u^2 + \gamma\rho(-2\rho uu_{xx} - 3\rho u_x^2 - 6\rho_x uu_x) + 2\gamma_1\rho,
\end{aligned}$$

which describe the conservation of mass ($i = 0$), momentum ($i = 1$), and energy ($i = 2$) due to the surface wave motion. The fourth conservation law ($i = 3$) can be interpreted in terms of a concrete kinematic quantity related to the evolution of the tangent velocity at a free surface (Gavrilyuk et al., 2015).

Here we also note that if

$$D_t T^t + D_x T^x = 0,$$

then

$$D_t(T^t + D_x f) + D_x(T^x - D_t f) = 0,$$

for any function f.

7.4.1 The Green-Naghdi equations in Lagrangian coordinates

One can check that choosing the Lagrangian

$$\mathcal{L} = \rho_0 \frac{\varphi_t^2}{2} + \varphi_\xi^{-4}\rho_0^2\left(\gamma\rho_0\varphi_{t\xi}^2 - \gamma_1\varphi_\xi^3\right),$$

the Green-Naghdi equations are equivalent to the Euler-Lagrange equation (3.7)

$$\frac{\delta\mathcal{L}}{\delta\varphi} = 0, \tag{7.9}$$

where $u=\varphi_t$, $\rho=\rho_0(\xi)\varphi_\xi^{-1}$, $\dot{\rho}=-\rho_0(\xi)\varphi_\xi^{-2}\varphi_{t\xi}$ and $\dfrac{\delta}{\delta\varphi}$ is the variational derivative (6.14). The Euler-Lagrange equation (7.9) in reduced coordinates has the form:

$$\begin{gathered}2\gamma\left(\varphi_\xi^2\varphi_{tt\xi\xi}-4\varphi_\xi\varphi_{\xi\xi}\varphi_{tt\xi}-4\varphi_\xi\varphi_{t\xi}\varphi_{t\xi\xi}+10\varphi_{t\xi}^2\varphi_{\xi\xi}\right)\\+\varphi_\xi^3\left(2\gamma_1\varphi_{\xi\xi}-\varphi_\xi^3\varphi_{tt}\right)=0.\end{gathered}\tag{7.10}$$

7.4.2 Conservation Laws of Green-Naghdi Model in Lagrangian Coordinates

Calculations show that the symmetries X_1, X_2 and X_3 are variational.

Applying Noether's theorem*, one finds the following conservation laws. For the generator $X_1=\partial_t$:

$$\begin{aligned}T^1&=\frac{1}{2}\varphi_t^2-\frac{\gamma\varphi_t(\varphi_\xi\varphi_{t\xi\xi}-4\varphi_{t\xi}\varphi_{\xi\xi})}{\varphi_\xi^5}+\frac{\gamma_1}{\varphi_\xi},\\T^2&=\frac{\gamma_1\varphi_t}{\varphi_\xi^2}-\frac{\gamma(\varphi_t\varphi_{tt\xi}-\varphi_{tt}\varphi_{t\xi})}{\varphi_\xi^4}.\end{aligned}\tag{7.11}$$

For the generator $X_2=\partial_\xi$:

$$\begin{aligned}T^1&=\varphi_t\varphi_\xi-\frac{\gamma(\varphi_\xi\varphi_{t\xi\xi}-5\varphi_{t\xi}\varphi_{\xi\xi})}{\varphi_\xi^4},\\T^2&=-\frac{\varphi_t^2}{2}-\frac{\gamma\varphi_{tt\xi}}{\varphi_\xi^3}+\frac{2\gamma_1}{\varphi_\xi}.\end{aligned}\tag{7.12}$$

For the generator $X_3=\partial_\varphi$:

$$\begin{aligned}T^1&=-\varphi_t+\frac{\gamma(\varphi_\xi\varphi_{t\xi\xi}-4\varphi_{t\xi}\varphi_{\xi\xi})}{\varphi_\xi^5},\\T^2&=\frac{\gamma\varphi_{tt\xi}}{\varphi_\xi^4}-\frac{\gamma_1}{\varphi_\xi^2}.\end{aligned}\tag{7.13}$$

The symmetry $X_4=t\partial_\varphi$ is divergent, with

$$(B_1,B_2)=(\varphi,0),$$

*Because of the presence of mixed derivatives in the Lagrangian, for using Noether's theorem one has to rewrite the Lagrangian in a symmetric form Ibragimov (2014).

and provides the conservation law:

$$T^1 = -t\varphi_t + \frac{\gamma t(\varphi_\xi\varphi_{t\xi\xi} - 4\varphi_{t\xi}\varphi_{\xi\xi})}{\varphi_\xi^5},$$
$$T^2 = \frac{\gamma(t\varphi_{tt\xi} - \varphi_{t\xi})}{\varphi_\xi^4} - \frac{\gamma_1 t}{\varphi_\xi^2}. \tag{7.14}$$

The generator $X_5 = t\partial_t + 4\xi\partial_\xi + 2\varphi\partial_\varphi$ is not divergent, hence, does not provide a conservation law.

7.4.3 Relations between conservation laws in Lagrangian and Eulerian coordinates

The operators of total derivatives in Lagrangian and Eulerian coordinates are related as follows

$$D_\xi = \varphi_\xi D_x,$$
$$D_t = \varphi_t D_x + D_{\tilde{t}}, \tag{7.15}$$

where ($\tilde{}$) is used in order to distinguish time in Eulerian coordinates from time in Lagrangian coordinates. Because the variables $\rho(t,x)$ and $u(t,x)$ are considered in Eulerian coordinates, omitting $\tilde{}$ in further study is not misleading.

Let T^1 and T^2 be the conserved vector in Lagrangian coordinates:

$$D_t T^1 + D_\xi T^2 = 0.$$

By the definition of velocity $u = \varphi_t$ and density $\rho = \varphi_\xi^{-1}$, one has that

$$u_x = \varphi_{t\xi}\varphi_\xi^{-1},$$

and

$$\begin{aligned} D_tT^1 + D_\xi T^2 &= D_t(\varphi_\xi \rho T^1) + D_\xi T^2 \\ &= \varphi_{t\xi}(\rho T^1) + \varphi_\xi D_t(\rho T^1) + D_\xi T^2 \\ &= \varphi_{t\xi}\varphi_\xi^{-1} T^1 + \varphi_\xi \Big(u D_x(\rho T^1) + D_{\tilde{t}}(\rho T^1)\Big) + \varphi_\xi D_x T^2 \\ &= \varphi_{t\xi}\varphi_\xi^{-1} T^1 + \varphi_\xi \Big(D_x(\rho u T^1) - u_x(\rho T^1) + D_{\tilde{t}}(\rho T^1)\Big) + \varphi_\xi D_x T^2 \\ &= \left(\varphi_{t\xi}\varphi_\xi^{-1} - u_x\right) T^1 + \varphi_\xi \Big(D_x(\rho u T^1 + T^2) + D_{\tilde{t}}(\rho T^1)\Big) \\ &= \varphi_\xi \Big(D_x(\rho u T^1 + T^2) + D_{\tilde{t}}(\rho T^1)\Big). \end{aligned}$$

Thus, the conserved vector in Eulerian coordinates is

$$T^t = \rho T^1, \quad T^x = \rho u T^1 + T^2. \tag{7.16}$$

In order to derive representations of the obtained conservation laws in Eulerian coordinates one can use the following relations:

$$\varphi_t = u, \;\; \varphi_\xi = \rho^{-1}, \;\; \varphi_{t\xi} = \rho^{-1}u_x, \;\; \varphi_{tt} = uu_x + u_t, \;\; \varphi_{\xi\xi} = -\rho_x\rho^{-3},$$

$$\varphi_{t\xi\xi} = \rho^{-2}\big(u_{xx} - u_x\rho_x\rho^{-1}\big), \;\; \varphi_{tt\xi} = \rho^{-1}\big(uu_{xx} + u_x^2 + u_{xt}\big),$$

$$\varphi_{ttt} = u_{tt} + u^2u_{xx} + 2uu_{tx} + uu_x^2 + u_xu_t, \;\; \varphi_{\xi\xi\xi} = \rho^{-4}\big(3\rho_x^2\rho^{-1} - \rho_{xx}\big),$$

$$\varphi_{tt\xi\xi} = \rho^{-3}\left(\rho\left(u_{txx} + uu_{xxx} + 3u_xu_{xx}\right) - uu_{xx} - u_x^2 - u_{xt}\right).$$

Therefore, the corresponding generators in Eulerian coordinates become as follows.

For the generator $X_1 = \partial_t$:

$$\begin{aligned} T^t &= \frac{1}{2}\rho u^2 - 3\gamma uu_x\rho_x\rho^2 - \gamma uu_{xx}\rho^3 + \gamma_1\rho^2 = {}^gT_2^t - \gamma(\rho^3uu_x)_x, \\ T^x &= u\Big(\frac{1}{2}\rho u^2 - 3\gamma uu_x\rho_x\rho^2 - 2\gamma uu_{xx}\rho^3 + 2\gamma_1\rho^2\Big) + \gamma\rho^3\big(-uu_{tx} + uu_x^2 + u_xu_t\big). \end{aligned} \tag{7.17}$$

By virtue of the equivalence transformation the last term in T^t can be moved to T^x.

For the generator $X_2 = \partial_\xi$:

$$\begin{aligned} T^t &= u - \gamma(u_{xx}\rho^2 + 4\rho\rho_xu_x) = {}^gT_3^t + \gamma(\rho^2u_x)_x, \\ T^x &= \frac{u^2}{2} - 2\gamma uu_{xx}\rho^2 - \gamma u_{tx}\rho^2 - 4\gamma\rho\rho_xuu_x + 2\gamma_1\rho. \end{aligned} \tag{7.18}$$

The term $\gamma(\rho^2 u_x)_x$ in T^t can be moved to the coefficient T^x. Hence, this conservation law is equivalent to $(7.8)_{|i=3}$.

For the generator $X_3 = \partial_\varphi$:

$$
\begin{aligned}
T^t &= -\rho u + \gamma(\rho^3 u_x)_x = -\,{}^g T_1^t + \gamma(\rho^3 u_x)_x, \\
T^x &= \rho\Big(2\gamma u u_{xx}\rho^2 + \gamma u_{tx}\rho^2 + 3\gamma\rho\rho_x u u_x - \gamma_1\rho - u^2\Big).
\end{aligned}
\tag{7.19}
$$

This conservation law is also equivalent to the conservation law to $(7.8)_{|i=1}$.

For the generator $X_4 = t\partial_\varphi$:

$$
\begin{aligned}
T_4^t &= \rho(x - tu + \gamma\rho t(\rho u_{xx} + 3\rho_x u_x)) = \rho(x - tu) + \gamma(t\rho^3 u_x)_x, \\
T_4^x &= \rho(u(x - ut) + \gamma\rho^2 t(2uu_{xx} + u_{tx} + u_x^2) - \gamma\rho^2 u_x + 3\gamma\rho t\rho_x u u_x - \gamma_1\rho t).
\end{aligned}
\tag{7.20}
$$

For the gas dynamics equations ($\gamma = 0$) this conservation was obtained in Ibragimov (1985) and it is called the center of mass conservation law. For the Green-Naghdi equations ($\gamma \neq 0$) we also call it by the same name.

CHAPTER VIII

CONCLUSIONS

The equations of fluids in Lagrangian coordinates are considered in this thesis. With a natural Lagrangian, the equations of fluids in Lagrangian coordinates have the form of an Euler-Lagrange equation and Noether's theorem is allowed to be applied for constructing conservation laws. Three types of these models are studied: the gas dynamics equations, the hyperbolic shallow water equations and the Green-Naghdi model.

For the one-dimensional gas dynamics equations the complete group classification in Lagrangian coordinates with respect to the pressure function $P(X, \varphi_X)$ with the restrictions $P_{\varphi_X \varphi_X} \neq 0$ and $P_X \neq 0$ is obtained. The kernel of admitted Lie algebras is determined for all function $P(X, \varphi_X)$. Extensions of the kernel depend on the value of the function $P(X, \varphi_X)$. These extensions of the kernel are found by solving the conditions given by the determining equations. The group classification separates this model into 17 different classes presented in Table 1.

Using Noether's theorem the kernel of admitted Lie algebra X_1, X_2, X_3, for an arbitrary potential function $W(\rho, S)$ gives rise to the well-known conservation laws; the energy, the momentum, and the center of mass, respectively. For the extensions, first we needed to find the potential function corresponding to the function $P(X, \varphi_X)$, and then Noether's theorem was applied for deriving conservation laws. The results of the study of constructing conservation laws of the one-dimensional gas dynamics equations are presented in Table 3. The hyperbolic shallow water equations is a particular case of the one-dimensional isentropic gas flow with the pressure $p = \gamma_1 \rho^2$. Using Noether's theorem to derive conservation laws we obtained new conservation laws which have no analog in Eulerian coordinates. The derivation of the conservation laws of these models are performed in Chapter VI.

The group classification of one-dimensional nonisentropic equations of fluids with internal inertia are obtained in the particular case where the potential function $W = W(\rho, \dot{\rho}, S)$ satisfies the condition $W_{\dot{\rho}} \neq 0$, and is performed in Chapter VII. The Green-Naghdi model corresponds to the potential $W = \gamma_1 \rho^2 - \gamma \rho \dot{\rho}^2$. Using Noether's theorem a new conservation law in Lagrangian coordinates of the Green-Naghdi equations is found.

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APPENDICES

APPENDIX A

APPLICATION OF GROUP ANALYSIS TO EULER-LAGRANGE EQUATION

A.1 Case $\mu_{1_{\varphi_X}} \neq 0$

- **Case II** $k_1 \neq 1$

Consider equation (5.50) when $k_1 \neq 1$. One has

$$\xi^X = \frac{k_5\big(k_1X + k_2\big)}{k_1 - 1}. \tag{A.1}$$

As $\xi^X \neq 0$, it leads to $k_5 \neq 0$. Substituting all relations into equation (5.40), it becomes

$$\begin{aligned} kk_3 \quad &= \frac{k_5}{(k_1-1)P_{\varphi_X}P_{\varphi_X\varphi_X}}\Big(-2P_{\varphi_X}P_{\varphi_X\varphi_X X}(k_1X+k_2) \\ &+6P_{\varphi_X}P_{\varphi_X\varphi_X} + 2\mu_1 P_{\varphi_X\varphi_X}\big(P_{\varphi_X X}(k_1X+k_2) - 2P_{\varphi_X}\big)\Big). \end{aligned} \tag{A.2}$$

Let $g_1 = g_1(X, \varphi_X)$ such that

$$\begin{aligned} g_1 = \tfrac{1}{(k_1-1)P_{\varphi_X}P_{\varphi_X\varphi_X}}\Big(&-2P_{\varphi_X}P_{\varphi_X\varphi_X X}(k_1X+k_2) + 6P_{\varphi_X}P_{\varphi_X\varphi_X} \\ &+2\mu_1 P_{\varphi_X\varphi_X}\big(P_{\varphi_X X}(k_1X+k_2) - 2P_{\varphi_X}\big)\Big), \end{aligned}$$

one can rewrite equation (A.2) as

$$kk_3 = k_5 g_1. \tag{A.3}$$

Differentiating equation (A.3) with respect to X and φ_X, one has

$$k_5 g_{1_X} = 0, \qquad k_5 g_{1_{\varphi_X}} = 0.$$

Since $k_5 \neq 0$, hence, g_1 is constant. As $kk3$ is constant, say $g_1 = kk3$, the latter can be written as

$$P_{\varphi_X\varphi_X X} = \frac{1}{2P_{\varphi_X}(k_1X+k_2)}\Big(2\mu_1 P_{\varphi_X\varphi_X}\big(P_{\varphi_X X}(k_1X+k_2)-2P_{\varphi_X}\big) + P_{\varphi_X}P_{\varphi_X\varphi_X}((1-k_1)g_1+6)\Big).$$

Substituting all relations into equation (5.18), it becomes

$$\eta_X^{00} = \frac{k_5}{2(k_1-1)P_{\varphi_X\varphi_X}}\Big(-2P_{\varphi_X X}(k_1X+k_2) - g_1(k_1-1)\varphi_X P_{\varphi_X\varphi_X} + 2\varphi_X P_{\varphi_X\varphi_X} + 4P_{\varphi_X}\Big), \quad \text{(A.4)}$$

Since η^{00} only depends on X and for existence of extension of the kernel of admitted Lie algebra, one obtains that

$$\eta_X^{00} = k_5 f_1(X).$$

Equation (A.4) provides that

$$P_{\varphi_X X} = \frac{1}{2(k_1X+k_2)}\Big(g_1(1-k_1)\varphi_X P_{\varphi_X\varphi_X} + 2P_{\varphi_X\varphi_X}\big(\varphi_X - k_1 f_1(X) + 2f_1(X)\big) + 4P_{\varphi_X}\Big) \quad \text{(A.5)}$$

where $f_1(X)$ is a function of X only.

A similar study can be performed for equation (5.41), which can be also written in the new form

$$k_7 = k_5 ko_4,$$

or

$$\begin{aligned} P_{XX} = \frac{1}{4(k_1X+k_2)^2}\bigg(&4f_1'(X)P_{\varphi_X}(k_1X+k_2)(1-k_1) \\ &+P_{\varphi_X\varphi_X}\big(\varphi_X^2\big(k_1g_1(k_1g_1-2g_1-4)+(g_1+2)^2\big) \\ &+4\varphi_X f_1(X)\big(k_1g_1(k_1-2)-2k_1+g_1+2\big)+4f_1^2(X)(k_1+1)^2\big) \\ &+2\big(-2g_1(k_1-1)\varphi_X P_{\varphi_X}+4P_{\varphi_X}\big(\varphi_X - f_1(X)(k_1-1)\big) \\ &+P_X k_1(g_1-2)\big(k_1X+k_2-X-1\big)+4ko_4(k_1X+k_2)(1-k_1)\big)\bigg), \end{aligned} \quad \text{(A.6)}$$

where ko_4 is constant.

Finally, a solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = k_5 g_1, \qquad k_7 = k_5 ko_4, \qquad \eta_X^{00} = f_1(X)k_5,$$
$$\eta_X^{01} = 0, \qquad \xi^t = k_5 t + k_6, \qquad \xi^X = \frac{k_5(k_1 X + k_2)}{k_1 - 1},$$
$$\eta = \frac{2k_5 ko_4 t^2 + k_5\varphi(g_1+2) + 2k_{10}t + 2\eta^{00}}{2}.$$

For $k_{10} = \eta^{01}$, the generator corresponding to these coefficients is

$$X = k_6 X_1 + k_{10} X_2 + k_5 X_3$$

with the basis of generators

$$X^1 = \partial_t, \qquad X^2 = t\partial_\varphi$$
$$X^3 = t\partial_t + \frac{k_1 X + k_2}{k_1 - 1}\partial_X + \Big(ko_4 t^2 + \int f_1(X)dx + \frac{(g_1+2)\varphi}{2}\Big)\partial_\varphi. \qquad (A.7)$$

Here the function $P(X, \varphi_X)$ satisfies (A.5) and (A.6).

Finding : Pressure function

Rewrite equation (A.5)

$$2(k_1 X + k_2)P_{\varphi_X X} - P_{\varphi_X\varphi_X}\Big(\varphi_X\big(2 + g_1(1-k_1)\big) + 2f_1(X)(1-k_1)\Big) = 4P_{\varphi_X}.$$

To simplify the extensions of kernel in this case, one has to separate into two cases: $k_1 \neq 0$ and $k_1 = 0$.

- $k_1 \neq 0$

The general solution of equation (A.5) for $k_1 \neq 0$ is

$$P_{\varphi_X} = \phi(\tilde{Z})(X + \frac{k_2}{k_1})^{\frac{2}{k_1}}. \qquad (A.8)$$

where $\tilde{Z} = \varphi_X(X + \frac{k_2}{k_1})^{-\alpha} - \frac{(k_1-1)}{k_1}\int f_1(X)(X + \frac{k_2}{k_1})^{-\alpha-1}dX$. Integrate equation (A.8) with respect to φ_X to obtain a pressure function,

$$P(X, \varphi_X) = \tilde{\phi}(\tilde{Z})(X + \gamma k_2)^{\alpha+2\gamma} + h(X) \qquad (A.9)$$

where $\phi(\tilde{Z}) = \tilde{\phi}(\tilde{Z})'$, $\tilde{Z} = \varphi_X(X+\gamma k_2)^{-\alpha} - (1-\gamma)\int f_1(X)(X+\gamma k_2)^{-\alpha-1}dX$ and $\gamma = \frac{1}{k_1}$. Substituting this function into (A.6), one derives

$$(X+\gamma k_2)h_{XX} = h_X(\alpha+2\gamma-1) + 2k_4(\gamma-1).$$

To find the integral $-(1-\gamma)\int f_1(X)(X+\gamma k_2)^{-\alpha-1}dX$, let us introduce the function

$$C(X) = -(1-\gamma)\int f_1(X)(X+\gamma k_2)^{-\alpha-1}dX,$$

then $C'(X) = -(1-\gamma)f_1(X)(X+\gamma k_2)^{-\alpha-1}$ or $f_1(X) = \frac{-1}{1-\gamma}(X+\gamma k_2)^{\alpha+1}C'(X)$. Consider

$$\int f_1(X)dX = \frac{-1}{1-\gamma}\int (X+\gamma k_2)^{\alpha+1}C'(X)dX.$$

Integrating by-parts, one has

$$\begin{aligned}
\int f_1(X)dX &= \frac{-1}{1-\gamma}\Big(C(X)(X+\gamma k_2)^{\alpha+1} - \int(\alpha+1)(X+\gamma k_2)^{\alpha}C(X)dX\Big)\\
&= \frac{1}{1-\gamma}\Big(-C(X)(X+\gamma k_2)^{\alpha+1} + (\alpha+1)\int C(X)(X+\gamma k_2)^{\alpha}dX\Big)\\
&= \frac{1}{1-\gamma}\Big(-C(X)(X+\gamma k_2)^{\alpha+1} + (\alpha+1)\tilde{f}_1(X)\Big)
\end{aligned}$$

where $\tilde{f}_1(X) = \int C(X)(X+\gamma k_2)^{\alpha}dX$ or $C(X) = (X+\gamma k_2)^{-\alpha}\tilde{f}_1{}'(X)$. Therefore

$$\begin{aligned}
-(1-\gamma)\int f_1(X)(X+\gamma k_2)^{-\alpha-1}dX &= (X+\gamma k_2)^{-\alpha}\tilde{f}_1{}'(X),\\
\int f_1(X)dX &= \frac{1}{1-\gamma}\big(-(X+\gamma k_2)\tilde{f}'(X) + (\alpha+1)\tilde{f}_1(X)\big),
\end{aligned}$$

and hence $\eta^{00} = \frac{k_5}{1-\gamma}\big(-X\tilde{f}_1{}'(X) + (\alpha+1)\tilde{f}_1(X)\big)$. The generators of equation (A.7) become

$$\begin{aligned}
X^1 &= \partial_t, \qquad X^2 = t\partial_\varphi\\
X^3 &= t\partial_t + \frac{-(X+\gamma k_2)}{\gamma-1}\partial_X\\
&\quad + \frac{1}{\gamma-1}\Bigg(\beta(\gamma-1)t^2 + (X+\gamma k_2)\tilde{f}_1{}'(X) - (\alpha+1)\big(\tilde{f}_1(X)+\varphi\big)\Bigg)\partial_\varphi
\end{aligned} \tag{A.10}$$

where $\beta = ko_4$ and $\frac{g_1+2}{2} = \frac{\alpha+1}{1-\gamma}$.

The pressure function in equation (A.9) can also be written as

$$P(X,\varphi_X) = \phi(Z)X^{\alpha+2\gamma} + h(X) \tag{A.11}$$

where $Z = X^{-\alpha}\big(\varphi_X + \tilde{f}_1'(X)\big)$ and $Xh_{XX} = (\alpha+2\gamma-1)h_X + (2\beta(\gamma-1))$.

By virtue of the equivalence transformations corresponding to the generator X_2^e, X_{15}^e, it can be assumed that $k_2 = 0$, $\tilde{f}_1(X) = 0$. The generator X^3 in equation (A.10) is changed to

$$X^3 = t\partial_t + \frac{-X}{\gamma-1}\partial_X + \frac{1}{\gamma-1}\big(\beta(\gamma-1)t^2 - (\alpha+1)\varphi\big)\partial_\varphi.$$

Later on the equivalence transformation corresponding to the operator X_8^e is applied and this transformation allows one to simplify to $\beta = 0$. For $\alpha \neq -1$ and $\gamma \neq 0,\ 1$, the extensions of the kernel and the related pressure function are

$$\begin{gathered} X_4 = (\gamma-1)t\partial_t - X\partial_X - (\alpha+1)\varphi\partial_\varphi, \\ P(X,\varphi_X) = \phi(Z)X^{\alpha+2\gamma} + h(X) \end{gathered} \tag{A.12}$$

where $Z = X^{-\alpha}\varphi_X$ and $Xh_{XX} = (\alpha+2\gamma-1)h_X$.

Notice that the solution of (5.64) is a particular case of (A.12) when $\gamma = 1$. Thus the general form of the solution of these two cases is presented in Table 5.1 as model M_1.

- $k_1 = 0$

Substitute $k_1 = 0$ into equation (A.5). The general solution of this equation is

$$P(X,\varphi_X) = \tilde{\phi}(\tilde{Z})e^{(2\beta-\alpha)X} + h(X) \tag{A.13}$$

where $\beta \neq 0$, $\tilde{\phi}(\tilde{Z}) = \int \phi(\tilde{Z})d\tilde{Z}$, and $\tilde{Z} = \varphi_X e^{\alpha X} + \beta \int f_1(X)e^{\alpha X}dX$.

Substituting this pressure function into (A.6), one derives

$$h_{XX} = h_X\big(2\beta - \alpha\big) + 2\beta\gamma.$$

Using a similar study as in the previous case to simplify $\beta \int f_1(X)e^{\alpha X}dX$, one gets

$$\beta \int f_1(X)e^{\alpha X}dX = e^{\alpha X}\tilde{f}_1{}'(X).$$

and

$$\int f_1(X)dX = \frac{1}{\beta}\big(\tilde{f}_1{}'(X) + \alpha\tilde{f}_1(X)\big).$$

Hence, $\eta^{00} = \frac{k_5}{\beta}\big(\tilde{f}_1{}'(X) + \alpha\tilde{f}_1(X)\big)$. The generators in equation (A.7) become

$$\begin{aligned} &X^1 = \partial_t, \qquad\qquad X^2 = t\partial_\varphi \\ &X^3 = \beta t\partial_t - \partial_X + \big(\beta\gamma t^2 + \tilde{f}_1{}'(X) + \alpha\big(\tilde{f}_1(X) + \varphi\big)\big)\partial_\varphi. \end{aligned} \tag{A.14}$$

The pressure function in equation (A.13) is written as

$$P(X, \varphi_X) = \phi(Z)e^{(2\beta-\alpha)X} + h(X) \tag{A.15}$$

where $Z = e^{\alpha X}\big(\varphi_X + f_1'(X)\big)$ and $h_{XX} = h_X\big(2\beta - \alpha\big) + 2\beta\gamma$.

By virtue of the equivalence transformations corresponding to the generator X^e_{15} and X^e_8, it can be assumed that $\tilde{f}_1(X) = 0$, and $\gamma = 0$.

The extension of the kernel and the related pressure function are

$$X_5 = \beta t\partial_t - \partial_X + \alpha\varphi\partial_\varphi, \qquad P(X, \varphi_X) = \phi(Z)e^{(2\beta-\alpha)X} + h(X) \tag{A.16}$$

where $Z = e^{\alpha X}\varphi_X$, $h_{XX} = h_X\big(2\beta - \alpha\big)$ where β, $\alpha \neq 0$. The result of this case is labelled as M_2 in Table 5.1.

Case $\mu_2(X) = 0$

Substituting all conditions and $\mu_2(X) = 0$ into equation (5.43), it becomes $\xi^X_X = k_5$. Solving the latter equation, one gets

$$\xi^X = k_5X + k_9. \tag{A.17}$$

Remark. : As $\xi^X \neq 0$, then $k_5X + k_9 \neq 0$. Equation (5.40) becomes

$$kk3 = \frac{2(k_5X + k_9)}{P_{\varphi_X}P_{\varphi_X\varphi_X}}\big(P_{\varphi_X X}P_{\varphi_X\varphi_X}\mu_1 - P_{\varphi_X\varphi_X X}P_{\varphi_X}\big). \tag{A.18}$$

Let $g_2 = g_2(X, \varphi_X)$ such that

$$g_2 = \frac{P_{\varphi_X}}{P_{\varphi_X\varphi_X}}\big(P_{\varphi_X X} P_{\varphi_X\varphi_X}\mu_1 - P_{\varphi_X\varphi_X X} P_{\varphi_X}\big), \tag{A.19}$$

one can rewrite equation (A.18) as

$$kk_3 \ = \frac{2(k_5 X + k_9) g_2}{P_{\varphi_X}^2}. \tag{A.20}$$

By equation (A.19), one finds

$$P_{\varphi_X\varphi_X X} = \frac{P_{\varphi_X\varphi_X}\Big(P_{\varphi_X X} P_{\varphi_X}\mu_1 - g_2\Big)}{P_{\varphi_X}^2}.$$

Differentiating this equation with respect to φ_X and comparing the result with $(P_{\varphi_X\varphi_X\varphi_X})_X$, one derives

$$-g_{2_{\varphi_X}} P_{\varphi_X} + 2P_{\varphi_X\varphi_X} g_2 = 0.$$

Solving this latter equation, one gets two solutions :

$$g_2(X, \varphi_X) = h(X) P_{\varphi_X}^2, \qquad g_2(X, \varphi_X) = 0.$$

Substitution all relation into equation (5.36), it becomes

$$k_5\big(X h_X + h\big) + k_9 h_X = 0. \tag{A.21}$$

Differentiate equation (A.21) with respect to X, one gives

$$k_5\big(X h_{XX} + 2h_X\big) + k_9 h_{XX} = 0. \tag{A.22}$$

Equations (A.21) and (A.22) are algebraic linear homogeneous equations with respect to k_5 and k_9 with determinant $hh_{XX} - 2h_X^2$. If this determinant is not equal to zero, then $k_5 = 0$ and $k_9 = 0$. In this case there is no extension of the kernel. Hence, one has to assume that

$$hh_{XX} - 2h_X^2 = 0.$$

The general solution of this equation is $h^1(X) = 0$, and $h^2(X) = \frac{1}{k_1X+k_2}$ where k_1 and k_2 are constant such that $k_1^2 + k_2^2 \neq 0$.

Case I $h(X) \neq 0$

Substituting $h(X) = \frac{1}{k_1X+k_2}$ into equation (A.21), one gets

$$k_5k_2 - k_9k_1 = 0. \tag{A.23}$$

To analyze the extensions of the kernel, one has to split into 2 cases: $k_1 \neq 0$ and $k_1 = 0$.

- $k_1 \neq 0$

As $k_1 \neq 0$, from equation (A.23), one can find

$$k_9 = \frac{k_2}{k_1}k_5,$$

and

$$\xi^X = k_5X + \frac{k_2}{k_1}k_5 = \frac{k_5\big(k_1X + k_2\big)}{k_1}.$$

Substituting all conditions into equation (5.41) and performing a study of this equation similar to the previous cases, it can be rewritten in new form

$$k_7 = k_5ko_3$$

or

$$\begin{aligned} P_{\varphi_X XX} = \frac{1}{P_{\varphi_X}(k_1X+k_2)^2}\Big(&P_{\varphi_X X}^2\big(\mu_1X^2k_1^2 - X^2k_1^2 + 2\mu_1Xk_1k_2 \\ &-2Xk_1k_2 + \mu_1k_2^2 - k_2^2\big) + P_{XX}P_{\varphi_X\varphi_X}(k_1X+k_2)^2 \\ &+P_{\varphi_X}P_{\varphi_X X}\big(-Xk_1^2 - k_1k_2 - Xk_1 - k_2\big) \\ &+P_XP_{\varphi_X\varphi_X}\big(Xk_1^2 + k_1k_2 - Xk_1 + k_2\big) \\ &+P_{\varphi_X\varphi_X}\big(2Xk_1^2ko_3 + 2k_1k_2ko_3\big)\Big) \end{aligned} \tag{A.24}$$

where ko_3 is constant. Substituting all these relations into equation (5.18), it becomes

$$\eta_X^{00} = \frac{-k_5}{k_1P_{\varphi_X\varphi_X}}\Big(P_{\varphi_X X}\big(k_1X + k_2\big) + \varphi_XP_{\varphi_X\varphi_X}\Big). \tag{A.25}$$

Since η^{00} only depends on X and for the existence of extension of kernel of admitted Lie algebras, one obtains that

$$\eta^{00}_X = f_2(X)k_5.$$

Equation (A.25) provides that

$$P_{\varphi_X X} = \frac{-P_{\varphi_X\varphi_X}\big(\varphi_X + k_1 f_2(X)\big)}{k_1 X + k_2}, \tag{A.26}$$

where $f_2(X)$ is a function of X only. Substituting (A.26) into (A.24), one derives

$$\begin{aligned} P_{XX} = &\frac{1}{(k_1X+k_2)^2}\Big((k_1X+k_2)\big(-f_{2X}(X)k_1P_{\varphi_X} - k_1P_X \\ &+P_X - 2k_1ko_3\big) + P_{\varphi_X\varphi_X}\big(\varphi_X^2 + 2k_1f_2(X)\varphi_X + k_1^2{f_2}^2(X)\big)\Big). \end{aligned} \tag{A.27}$$

Finally, a solution of the determining equations is

$$\begin{aligned} &k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{2}{k_1}k_5, \qquad k_7 = ko_3k_5, \qquad k_9 = \frac{k_2}{k_1}k_5, \\ &\eta^{00}_X = g_2(X)k_5, \qquad \eta^{01}_X = 0, \qquad \xi^t = k_5t + k_6, \qquad \xi^X = \frac{k_5(k_1X+k_2)}{k_1}, \\ &\eta = \frac{k_5k_1ko_3t^2 + k_5(k_1+1)\varphi + \eta^{01}k_1t + \eta^{00}k_1}{k_1}. \end{aligned}$$

For $k_{10} = \eta^{01}$, then the generator corresponding to these coefficients is

$$X = k_5X_1 + k_6X_2 + k_{10}X_3$$

where

$$\begin{aligned} &X_1 = \partial_t, \qquad\qquad X_2 = t\partial_\varphi \\ &X_3 = t\partial_t + (\frac{k_1X+k_2}{k_1})\partial_X + \Big(ko_3t^2 + \int f_2(X)dx + (\frac{k_1+1}{k_1})\varphi\Big)\partial_\varphi. \end{aligned} \tag{A.28}$$

Here the function $P(X, \varphi_X)$ satisfies two equations (A.27) and (A.26).

Finding : Pressure function

Solving (A.26), one gets a general solution

$$P(X, \varphi_X) = \tilde{\phi}(\tilde{Z})(X + \alpha k_2)^\alpha + h(X) \tag{A.29}$$

where $\tilde{\phi}(\tilde{Z}) = \int \phi(\tilde{Z})d\tilde{Z}$, $\tilde{Z} = \varphi_X(X+\alpha k_2)^{-\alpha} - \int f_2(X)(X+\alpha k_2)^{-\alpha-1}dX$. Substituting this equation into (A.27), one derives

$$(X+\alpha k_2)h_{XX} = h_X(\alpha-1) - 2ko_3.$$

A similar study as in previous cases, gives

$$-\int f_2(X)(X+\alpha k_2)^{-\alpha-1}dX = (X+\alpha k_2)^{-\alpha}\tilde{f}_2{}'(X),$$

and

$$\int f_2(X)dX = -(X+\alpha k_2)\tilde{f}_2{}'(X) + (\alpha+1)\tilde{f}_2(X)$$

and $\eta^{00} = k_5\Big(-X\tilde{f}_2{}'(X) + (\alpha+1)\tilde{f}_2(X)\Big)$; therefore the generator in equation (A.28) can be written as

$$\begin{aligned} &X^1 = \partial_t, \qquad X^2 = t\partial_\varphi \\ &X^3 = t\partial_t + (X+\alpha k_2)\partial_X + \Big(\beta t^2 + (\alpha+1)\big(\varphi + \tilde{f}_2(X)\big) \\ &\quad - (X+\alpha k_2)\tilde{f}_2{}'(X)\Big)\partial_\varphi \end{aligned} \tag{A.30}$$

where $\beta = ko_3$. The pressure function in equation (A.29) becomes

$$P(X,\varphi_X) = \phi(Z)(X+\alpha k_2)^\alpha + h(X) \tag{A.31}$$

where $Z = (X+\alpha k_2)^{-\alpha}\big(\varphi_X + f_2{}'(X)\big)$ and $(X+\alpha k_2)h_{XX} = h_X(\alpha-1) - 2\beta$. By virtue of the equivalence transformations corresponding to the generators X_2^e and X_{15}^e, it can be assumed that $k_2 = 0$, $\tilde{f}_2(X) = 0$. The equivalence transformation corresponding to the operator X_8^e is also used and this transformation allows one to simplify that $\beta = 0$.

For the condition $\alpha \neq -1$, the extensions of kernel and the pressure function are

$$\begin{aligned} &X_6 = t\partial_t + X\partial_X + (\alpha+1)\varphi\partial_\varphi, \qquad P(X,\varphi_X) = \phi(Z)X^\alpha + h(X) \\ &\text{where} \quad Z = X^{-\alpha}\varphi_X \quad \text{and} \quad Xh_{XX} = h_X(\alpha-1). \end{aligned} \tag{A.32}$$

The result of this case is presented in Table 5.1 as the model M_3.

- $k_1 = 0$

Substituting $k_1 = 0$ into equation (A.23), one gets

$$k_5 k_2 = 0.$$

Assume $k_2 \neq 0$ then these relations

$$k_5 = 0, \qquad kk3 = \frac{2}{k_2} k_9, \qquad \xi^X = k_9 \qquad \text{where} \quad k_9 \neq 0$$

are obtained. Substituting all conditions into (5.41) and with a similar study as in the previous cases for this equation, it can be rewritten in the new form

$$k_7 = ko_3 k_9$$

or

$$P_{\varphi_X XX} = \frac{1}{k_2 P_{\varphi_X}} \Big(P_{\varphi_X X}^2 k_2 (\mu_1 - 1) - P_{\varphi_X} P_{\varphi_X X} + P_{\varphi_X \varphi_X} \big(P_{XX} k_2 - P_X + 2 k_2 ko_3 \big) \Big) \tag{A.33}$$

where ko_3 is constant. Substituting all relations into (5.18), it becomes

$$\eta_X^{00} = \frac{-k_9}{k_2 P_{\varphi_X \varphi_X}} \Big(k_2 P_{\varphi_X X} + \varphi_X P_{\varphi_X \varphi_X} \Big). \tag{A.34}$$

A similar study for (A.34) as performed in the previous cases gives

$$\eta_X^{00} = f_3(X) k_9$$

and provides that

$$P_{\varphi_X X} = \frac{-P_{\varphi_X \varphi_X} \big(\varphi_X + k_2 f_3(X) \big)}{k_2}, \tag{A.35}$$

where $f_3(X)$ is a function of X only. Substituting (A.35) into (A.33), one derives

$$P_{XXX} = \frac{1}{k_2^3} \Big(- f_{3XX}(X) k_2^3 P_{\varphi_X} + 3 k_2^2 f_{3X}(X) P_{\varphi_X \varphi_X} \big(\varphi_X + k_2 f_3(X) \big) - \varphi_X^2 P_{\varphi_X \varphi_X \varphi_X} \big(\varphi_X + 3 k_2 f_3(X) \big) - P_{\varphi_X \varphi_X \varphi_X} k_2^2 {f_3}^2(X) \big(3 \varphi_X + k_2 f_3(X) \big) - \varphi_X P_{\varphi_X \varphi_X} \big(\varphi_X + 2 k_2 f_3(X) \big) - P_{\varphi_X \varphi_X} k_2^2 \big({f_3}^2(X) - 1 \big) \Big),$$

and

$$P_{XX} = \frac{1}{k_2^2}\Big(- f_{3X}(X)k_2^2 P_{\varphi_X} + \varphi_X^2 P_{\varphi_X\varphi_X} + k_2\big(P_X - 2k_2ko_3\big) \\ + k_2 f_3(X) P_{\varphi_X\varphi_X}\big(2\varphi_X + k_2 f_3(X)\big)\Big). \tag{A.36}$$

Finally, a solution of the determining equations is

$$k_4 = 0, \qquad k_5 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{2}{k_2}k_9, \qquad k_7 = ko_3k_9,$$

$$\eta_X^{00} = f_3(X)k_9, \qquad \eta_X^{01} = 0, \qquad \xi^t = k_6, \qquad \xi^X = k_9,$$
$$\eta = \frac{k_9k_2ko_3t^2 + k_9\varphi + \eta^{01}k_2t + \eta^{00}k_2}{k_2}$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6X_1 + k_{10}X_2 + k_9X_3$$

with

$$X^1 = \partial_t, \qquad X^2 = t\partial_\varphi$$
$$X^3 = \partial_X + \Big(ko_3t^2 + \int f_3(X)dx + (\frac{\varphi}{k_2})\Big)\partial_\varphi. \tag{A.37}$$

Here the function $P(X, \varphi_X)$ satisfies the two equations (A.35) and (A.36).

Finding : Pressure function

Solving equation (A.35), one derives the general solution

$$P(X, \varphi_X) = \tilde{\phi}(\tilde{Z})e^{\alpha X} + h(X) \tag{A.38}$$

where $\tilde{\phi}(\tilde{Z}) = \int \phi(\tilde{Z})d\tilde{Z}$, $\tilde{Z} = \varphi_X e^{-\alpha X} - \int f_3(X)e^{-\alpha X}dX$. Substituting this equation into (A.36), one finds

$$h_{XX} = \alpha h_X - 2ko_3.$$

Performing a similar study as in the previous cases for finding $-\int f_3(X)e^{-\alpha X}dX$, one gets

$$-\int f_3(X)e^{-\alpha X}dX = e^{-\alpha X}\tilde{f}_3{}'(X)$$

and

$$\int f_3(X)dX = -\tilde{f}_3{}'(X) + \alpha\tilde{f}_3(X).$$

Hence, $\eta^{00} = k_9\big(-\tilde{f}_3{}'(X) + \alpha\tilde{f}_3(X)\big)$ and the generators in (A.37) become

$$\begin{aligned} &X^1 = \partial_t, \qquad\qquad X^2 = t\partial_\varphi \\ &X^3 = \partial_X + \Big(\beta t^2 - f_3(X)'(X) + \alpha\big(f_3(X) + \varphi\big)\Big)\partial_\varphi \end{aligned} \tag{A.39}$$

where $\beta = ko_3$ and the pressure function in equation (A.38) can also be written as

$$P(X, \varphi_X) = \phi(Z)e^{\alpha X} + h(X) \tag{A.40}$$

where $Z = e^{-\alpha X}\big(\varphi_X + f_3'(X)\big)$ and $h_{XX} = \alpha h_X - 2\beta$.

By virtue of the equivalence transformations corresponding to the generators X^e_{15} and X^e_8 allows us to assume that $\tilde{f}_3(X) = 0$ and $\beta = 0$.

Therefore when $\alpha \neq 0$, the extension of the kernel and the pressure function are

$$X = \partial_X + \alpha\varphi\partial_\varphi, \qquad P(X, \varphi_X) = \phi(Z)e^{\alpha X} + h(X) \tag{A.41}$$

where $Z = e^{-\alpha X}\varphi_X$ and $h_{XX} = \alpha h_X$.

Consider equation (A.16) and (A.41), one notices that (A.41) is a particular case of (A.16) when $\beta = 0$ with $-\alpha = \tilde{\alpha}$. Thus the general form of these two cases with the related pressure function is given as model M_2 in Table 5.1.

Case II $\qquad h(X) = 0$

Substituting $h(X) = 0$ into equation (A.20), one finds

$$kk3 = 0,$$

and

$$\eta^{00}_X = \frac{-P_{\varphi_X X}}{P_{\varphi_X\varphi_X}}\Big(ko_5 X + ko_9\Big). \tag{A.42}$$

Performing a similar study as in previous cases for (A.42), one obtains

$$\eta_X^{00} = f_4(X)\big(ko_5X + ko_9\big),$$

and provides that

$$P_{\varphi_X X} = -f_4(X)P_{\varphi_X\varphi_X} \tag{A.43}$$

where $f_4(X)$ is a function of X only. Integrating equation (A.43) with respect to φ_X, one obtains

$$P_X = -f_4(X)P_{\varphi_X} + g_{11}(X). \tag{A.44}$$

Substituting all relations, one finds

$$k_7 = \frac{-k_5\big(g_{11X}X + g_{11}\big) - g_{11X}k_9}{2}. \tag{A.45}$$

Differentiating equation (A.45) with respect to X, one gets

$$k_5\big(-g_{11XX}X - 2g_{11X}\big) - g_{11XX}k_9 = 0. \tag{A.46}$$

Differentiating equation (A.46) with respect to X, one gets

$$k_5\big(-g_{11XXX}X - 3g_{11XX}\big) - g_{11XXX}k_9 = 0. \tag{A.47}$$

Equations (A.46) and (A.47) are algebraic linear homogeneous equations with respect to k_5 and k_9 with the determinant $2g_{11XXX}g_{11X} - 3{g_{11XX}}^2$. If this determinant is not equal to zero, then $k_5 = 0$ and $k_9 = 0$. In this case there is no extension of the kernel. Hence, one has to assume that

$$2g_{11XXX}g_{11X} - 3{g_{11XX}}^2 = 0.$$

The general solution of this equation is $g_{11X}^1(X) = 0$ and $g_{11X}^2(X) = \frac{4}{\big(k_1X+k_2\big)^2}$. where k_1 and k_2 are constant such that $k_1^2 + k_2^2 \neq 0$.

Case II.1 $g_{11X}(X) \neq 0$

Substituting $g_{11X}(X) = \frac{4}{(k_1X+k_2)^2}$, into (A.46), it becomes

$$-4k_5k_2 + 4k_9k_1 = 0. \quad \text{(A.48)}$$

One has to study in 2 cases : $k_1 \neq 0$ and $k_1 = 0$.

- $k_1 \neq 0$

As $k_1 \neq 0$, equation (A.48) and (A.45) give

$$k_9 = \frac{k_2}{k_1}k_5, \qquad k_7 = \frac{-ko_3}{2}k_5.$$

Finally, a solution of the determining equations is

$$k_4 = 0, \quad k_8 = 0, \quad kk3 = 0, \quad k_7 = \frac{-ko_3}{2}k_5, \quad k_9 = \frac{k_2}{k_1}k_5,$$
$$\eta^{00}_X = (\frac{k_1X + k_2}{k_1})g_4(X)k_5, \quad \eta^{01}_X = 0 \quad \xi^t = k_5t + k_6$$
$$\xi^X = \frac{k_5(k_1X + k_2)}{k_1}, \quad \eta = \frac{-k_5ko_3t^2 + 2k_5\varphi + 2\eta^{01}t + 2\eta^{00}}{2}$$

with $k_{10} = \eta^{01}$, then the generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_5X^3$$

with

$$\begin{aligned} X^1 &= \partial_t, \qquad X^2 = t\partial_\varphi \\ X^3 &= t\partial_t + (\frac{k_1X + k_2}{k_1})\partial_X \\ &\quad + \Big(-\frac{ko_3}{2}t^2 + \varphi + (\frac{k_1X + k_2}{k_1})\int f_4(X)dx\Big)\partial_\varphi \end{aligned} \quad \text{(A.49)}$$

Here the function $P(X, \varphi_X)$ satisfies these two equations (A.43) and (A.44).

Finding : Pressure function

Solving equation (A.43), one gets the general solution

$$P(X, \varphi_X) = \tilde{\phi}(\tilde{Z}) + h(X) \quad \text{(A.50)}$$

where $\tilde{\phi}(\tilde{Z}) = \int \phi(\tilde{Z})d\tilde{Z}$, $\tilde{Z} = \varphi_X - \int g_4(X)dX$. Substituting this function into (A.44), one finds

$$k_1{}^2 h(X) = k_1{}^2(ko_3 X + ko_4) - 4\ln(k_1 X + k_2).$$

Consider $\int f_4(X)dX = -\tilde{g}'(X)$, one obtains

$$\int X f_4(X)dX = -X\tilde{g}'(X) + \tilde{g}(X),$$

and

$$\begin{aligned}\eta^{00} &= k_5 \int X f_4(X)dX + k_5 \frac{k_2}{ko_1} \int f_4(X)dX \\ &= k_5\big(-X\tilde{g}'(X) + \tilde{g}(X)\big) + k_5 \frac{k_2}{k_1}\big(-\tilde{g}'(X)\big) \\ &= k_5\big(-X\tilde{g}'(X) + \tilde{g}(X) - \alpha\beta\tilde{g}'(X)\big)\end{aligned}$$

where $\alpha = \frac{1}{k_1}$ $(\alpha \neq 0)$, and $\beta = k_2$.

Hence the generator in equation (A.49) can be written as

$$\begin{aligned} &X^1 = \partial_t, \qquad X^2 = t\partial_\varphi \\ &X^3 = t\partial_t + (X + \alpha\beta)\partial_X + \Big(-\gamma t^2 + \varphi + \big(-X\tilde{g}'(X) + \tilde{g}(X) \\ &\quad - \alpha\beta\tilde{g}'(X)\big)\Big)\partial_\varphi \end{aligned} \tag{A.51}$$

where $\gamma = 2ko_3$. Equation (A.50) can also be written as

$$P(X, \varphi_X) = \phi(Z) + h(X) \tag{A.52}$$

where $Z = \varphi_X + \tilde{g}'(X)$ and $h(X) = 2\big(\gamma X - 2\alpha^2 \ln\big(X + \alpha\beta\big)\big)$ $(\alpha \neq 0)$.

By virtue of the equivalence transformations corresponding to the generators X^e_{15}, X^e_8, and X^e_2, one can assume that $\tilde{g}(X) = 0$, $\gamma = 0$, and $\beta = 0$.

Therefore the extensions of kernel and the pressure function are

$$X = t\partial_t + X\partial_X + \varphi\partial_\varphi, \qquad P(X, \varphi_X) = \phi(Z) + h(X) \tag{A.53}$$

where $Z = \varphi_X$, $h(X) = -4\alpha^2 \ln X$ $(\alpha \neq 0)$.

Notice that the result in this case is a particular form of the result in (A.32) when placing $\alpha = 0$. Therefore the general form of these two cases is presented as model M_3 in the Table 5.1.

- $k_1 = 0$

Substituting $k_1 = 0$ into equation (A.48) and (A.45), and supposing that $k_2 \neq 0$, one obtains

$$k_5 = 0, \qquad k_7 = \frac{-2}{{k_2}^2} k_9.$$

Finally, the solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = 0, \qquad k_5 = 0, \qquad k_7 = \frac{-2}{{k_2}^2} k_9,$$

$$\eta_X^{00} = f_4(X) k_9, \qquad \eta_X^{01} = 0 \qquad \xi^t = k_6 \qquad \xi^X = k_9,$$

$$\eta = \frac{-2k_9 t^2 + {k_2}^2 \eta^{01} t + {k_2}^2 \eta^{00}}{{k_2}^2}$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6 X^1 + k_{10} X^2 + k_9 X^3$$

with

$$X^1 = \partial_t, \qquad X^2 = t\partial_\varphi, \qquad X^3 = \partial_X + \Big(\frac{-2t^2}{{k_2}^2} + \int f_4(X) dx\Big)\partial_\varphi. \quad \text{(A.54)}$$

Here the function $P(X, \varphi_X)$ satisfies the two equations (A.43) and (A.44).

Finding : Pressure function

Solving (A.43), one derives the general solution

$$P(X, \varphi_X) = \phi(Z) + h(X) \quad \text{(A.55)}$$

where $Z = \varphi_X - \int f_4(X) dX$. Substituting this equation into (A.44), one finds

$$h(X) = ko_3 X + \frac{2X^2}{{k_2}^2}.$$

Consider $\int f_4(X)dX = \tilde{g}(X)$, then

$$\eta^{00} = k_9 \int f_4(X)dX = k_9\tilde{g}(X),$$

and the generator of equation (A.54) can be written as

$$X^1 = \partial_t, \qquad X^2 = t\partial_\varphi, \qquad X^3 = \partial_X + \Big(-\gamma t^2 + \tilde{g}(X)\Big)\partial_\varphi \tag{A.56}$$

where $\gamma = \frac{2}{{ko_2}^2}$ $(\gamma \neq 0)$. For $\beta = ko_3$, the equation (A.55) can also be written as

$$\begin{aligned} P(X, \varphi_X) &= \phi(Z) + \beta X + \gamma X^2 \\ Z &= \varphi_X - \tilde{g}(X). \end{aligned} \tag{A.57}$$

By virtue of the equivalence transformations corresponding to the generator X^e_{15} one can assume that $\tilde{g}(X) = 0$.

The extensions of the kernel and its related pressure function are

$$X_7 = \partial_X - \gamma t^2\partial_\varphi, \qquad P(X, \varphi_X) = \phi(Z) + \beta X + \gamma X^2 \tag{A.58}$$

where $Z = \varphi_X$ with $\gamma \neq 0$. In the Table 5.1, this is model M_4.

Case II.2 $g_{11X}(X) = 0$

This case $g_{11X}(X) = 0$ means that $g_{11}(X) = ko_{11}$, for some constant ko_{11}. Substituting $g_{11}(X) = ko_{11}$ into (A.45), one finds

$$k_7 = \frac{-k_1}{2}k_5.$$

Finally, a solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = 0, \qquad k_7 = \frac{-k_1}{2}k_5, \qquad \eta^{00}_X = g_4(X)\big(k_5X + k_9\big),$$

$$\eta^{01}_X = 0, \qquad \xi^t = k_5t + k_6, \qquad \xi^X = k_5X + k_9, \qquad \eta = \frac{-k_5k_1t^2 + 2k_5\varphi + 2t\eta^{01} + 2\eta^{00}}{2}$$

with $k_{10} = \eta^{01}$, then the generator corresponding to these coefficients is

$$X = k_5 X^1 + k_9 X^2 + k_6 X^3 + k_{10} X^4$$

with

$$\begin{aligned} &X^1 = \partial_t, \qquad\qquad X^2 = t\partial_\varphi \\ &X^3 = t\partial_t + X\partial_X + \Big(\frac{-k_1}{2}t^2 + \varphi + \int X f_4(X)dX\Big)\partial_\varphi \\ &X^4 = \partial_X + \int f_4(X)dX\partial_\varphi. \end{aligned} \tag{A.59}$$

Here the function $P(X, \varphi_X)$ satisfies the two equations (A.43) and (A.44).

Finding : Pressure function

Solving (A.43), one derives the general solution

$$P(X, \varphi_X) = \phi(Z) + h(X) \tag{A.60}$$

where $Z = \varphi_X - \int f_4(X)dX$. Substituting this equation into (A.44), one finds

$$h(X) = k_1 X + k_2.$$

Consider $C(X) = -\int f_4(X)dX$; one obtains $\int X f_4(X)dX = -X\tilde{g}'(X) + \tilde{g}(X)$ where $\tilde{g}(X) = \int C(X)dX$, which gives the generator in equation (A.59) of form

$$\begin{aligned} &X^1 = \partial_t, \qquad\qquad X^2 = t\partial_\varphi \\ &X^3 = t\partial_t + X\partial_X + \Big(\frac{-\beta}{2}t^2 + \varphi - X\tilde{g}'(X) + \tilde{g}(X)\Big)\partial_\varphi \\ &X^4 = \partial_X - \tilde{g}'(X)\partial_\varphi \end{aligned} \tag{A.61}$$

where $\beta = ko_1$. By virtue of the equivalence transformation corresponding to the generators X_{15}^e and X_8^e, it can be assumed that $\tilde{g}(X) = 0$, and $\beta = 0$. In this case, there are two extensions of the kernel which are related to the following pressure function

$$X_8 = t\partial_t + X\partial_X + \varphi\partial_\varphi, \qquad X_9 = \partial_X, \qquad P(X, \varphi_X) = \phi(Z) \tag{A.62}$$

where $Z = \varphi_X$. In the Table 5.1, this is model M_5.

A.2 Case $\mu_{1_{\varphi_X}} = 0$

Consider equation (5.37)

$$2\xi^X_X \mu_{1_{\varphi_X}} P_{\varphi_X} - 2k_5 \mu_{1_{\varphi_X}} P_{\varphi_X} + \xi^X\big(- \mu_{1_{\varphi_X}} P_{\varphi_X X} + \mu_{1X} P_{\varphi_X \varphi_X}\big) = 0.$$

Substituting $\mu_{1_{\varphi_X}} = 0$ into this equation, it becomes

$$\xi^X \mu_{1X} P_{\varphi_X \varphi_X} = 0. \tag{A.63}$$

As $P_{\varphi_X \varphi_X} \neq 0$, then the latter equation holds when assuming in 2 cases: $\mu_{1X} \neq 0$ and $\mu_{1X} = 0$.

A.2.1 Case $\mu_{1X} = 0$

From equation (A.63), as $P_{\varphi_X \varphi_X} \neq 0$, and assuming $\xi^X \neq 0$ then $\mu_{1X} = 0$ that is

$$\mu_1 = k_1, \qquad \text{where } k_1 \text{ is a constant.}$$

Finding : Pressure function

Construct the pressure function by solving equation (5.32)

$$\frac{P_{\varphi_X \varphi_X \varphi_X} P_{\varphi_X}}{{P_{\varphi_X \varphi_X}}^2} = \mu_1 = k_1.$$

Since $\left(\frac{P_{\varphi_X}}{P_{\varphi_X \varphi_X}}\right)_{\varphi_X} = 1 - \frac{P_{\varphi_X} P_{\varphi_X \varphi_X \varphi_X}}{P^2_{\varphi_X \varphi_X}}$, then $\left(\frac{P_{\varphi_X}}{P_{\varphi_X \varphi_X}}\right)_{\varphi_X} = 1 - \mu_1$. Integrating the latter equation with respect to φ_X, one gets

$$\frac{P_{\varphi_X}}{P_{\varphi_X \varphi_X}} = (1 - \mu_1)\varphi_X + a$$
$$\frac{P_{\varphi_X \varphi_X}}{P_{\varphi_X}} = \frac{1}{(1 - \mu_1)\big(\varphi_X + \tilde{a}(X)\big)}, \qquad \tilde{a} = \frac{a}{1 - \mu_1}$$

where $\mu_1 \neq 1$. The pressure function is

$$P(X, \varphi_X) = \tilde{b}(X)\big(\varphi_X + \tilde{a}(X)\big)^{\frac{1}{1-\mu_1}+1} + c(X), \qquad \tilde{b}(X) = \frac{b(X)}{\frac{1}{1-\mu_1} + 1}.$$

Let $\gamma = \frac{1}{1-\mu_1} + 1$. As $P_{\varphi_X} \neq 0$, then $\gamma \neq 0$. The general form of the pressure function can be written as

$$P(X, \varphi_X) = b(X)\big(\varphi_X + a(X)\big)^{\gamma} + c(X), \qquad \gamma \neq 0\,,\,1. \tag{A.64}$$

Moreover one finds $P_{\varphi_X\varphi_X} = \gamma(\gamma-1)b(X)(a(X)+\varphi_X)^{\gamma-2}$, as $P_{\varphi_X\varphi_X} \neq 0$, then this condition

$$\gamma(\gamma-1)b(X) \neq 0$$

is obtained. Substituting the pressure function, P into equation (5.36), it becomes

$$\frac{-2\xi^X_X b_X b + 2\xi^X\big(-b_{XX}b + b_X^2\big)}{b^2(\gamma-1)} = 0. \tag{A.65}$$

From this equation, one can study in 2 cases: $b_X \neq 0$ and $b_X = 0$.

Case $b_X \neq 0$

Assuming $b_X \neq 0$, then (A.65) becomes

$$\xi^X_X = \frac{\xi^X\big(-b_{XX}b + b_X^2\big)}{bb_X}. \tag{A.66}$$

Integrating equation (A.66) with respect to X, one obtains

$$\xi^X = k_{11}\frac{b}{b_X} \tag{A.67}$$

with the relations $k_{11} \neq 0$ and $b \neq 0$ (as $\xi^X \neq 0$). Substituting all conditions into (5.33), one finds

$$k_4\gamma(\gamma+3)(\gamma-1)b(X)(a(X)+\varphi_X)^{\gamma} = 0.$$

Once consider $\gamma + 3 \neq 0$, and as $\gamma(\gamma-1)b(X) \neq 0$, one gets

$$k_4 = 0.$$

Substituting all conditions into equation (5.42), it becomes

$$\begin{aligned} &2k_5 b_X^2 c_{XX}\gamma(-\gamma+1) + k_{11}\Big((a+\varphi_X)^{\gamma} b_{XXX} b_X b(\gamma-1) \\ &-2(a+\varphi_X)^{\gamma} b_{XX}^2 b(\gamma-1) + (a+\varphi_X)^{\gamma} b_{XX} b_X^2(\gamma-1) - 2b_{XX}c_{XX}b(\gamma-1) \\ &+b_X^2 c_{XX}(\gamma-1) - b_X c_{XXX} b(\gamma-1)^2\Big) = 0. \end{aligned} \tag{A.68}$$

Differentiating equation (A.68) with respect to φ;

$$k_{11}(a+\varphi_X)^{\gamma}\gamma\Big(b_{XXX}b_Xb-2b_{XX}^2b+b_{XX}b_X^2\Big)=0.$$

Since $k_{11}\neq 0$, and $b_X\neq 0$, one derives

$$b_{XXX}=\frac{b_{XX}\big(2b_{XX}b-b_X^2\big)}{bb_X}. \tag{A.69}$$

Finding the extension of the kernel in this case, one has to consider 2 cases : $b_{XX}\neq 0$ and $b_{XX}=0$.

Case I $b_{XX}\neq 0$

Solving equation (A.69), the general solution is

$$b_X=b^{\beta_1}ko_2,$$

where β_1 is constant and since $b_X\neq 0$ then $ko_2\neq 0$. Therefore equation (A.68) changes to

$$-2k_5b^{\beta_1}\gamma ko_2c_{XX}+k_{11}\Big(-c_{XXX}b(\gamma-1)-b^{\beta_1}c_{XX}ko_2(2\beta_1-1)\Big)=0. \tag{A.70}$$

Differentiating equation (A.70) with respect to X, one gets

$$\begin{aligned}&-2k_5\gamma ko_2b^{\beta_1}\big(c_{XXX}b+\beta ko_2b^{\beta_1}c_{XX}\big)+k_{11}\big(-c_{XXXX}b^2(\gamma-1)\\&-ko_2b^{\beta_1+1}c_{XXX}(\gamma-2\beta_1-2)-ko_2^2\beta_1b^{2\beta_1}c_{XXX}(2\beta_1-1)\big)=0.\end{aligned} \tag{A.71}$$

Equations (A.70) and (A.71) are algebraic linear homogeneous equations with respect to k_5 and k_{11} with the determinant $\gamma(\gamma-1)ko_2b^{\beta_1+1}(X)\big(c_{XXXX}c_{XX}-c_{XXX}^2-ko_2(\beta_1-1)b^{\beta_1-1}c_{XXX}c_{XX}\big)$. If this determinant is not equal to zero, then $k_5=0$ and $k_{11}=0$ (which contradicts our condition $k_{11}\neq 0$). Hence, one has to assume that

$$\gamma(\gamma-1)ko_2b^{\beta_1+1}(X)\big(c_{XXXX}c_{XX}-c_{XXX}^2-ko_2(\beta_1-1)b^{\beta_1-1}c_{XXX}c_{XX}\big)=0.$$

Since $\gamma(\gamma-1)b(X) \neq 0$ and $ko_2 \neq 0$, it yields

$$c_{XXXX}c_{XX} - c_{XXX}^2 - ko_2(\beta_1 - 1)b^{\beta_1 - 1}c_{XXX}c_{XX} = 0.$$

To solve the latter equation, one has to determine 2 cases: $c_{XXX} \neq 0$ and $c_{XXX} = 0$.

Case I.1 $c_{XXX} \neq 0$

Let us consider

$$\frac{c_{XXXX}}{c_{XXX}} - \frac{c_{XXX}}{c_{XX}} - ko_2(\beta_1 - 1)b^{\beta_1 - 1} = 0. \qquad \text{(A.72)}$$

To complete the analysis, one has to consider 2 cases: $\beta_1 = 1$, and $\beta_1 \neq 1$.

Case I.1a $\beta_1 = 1$

Substituting $\beta_1 = 1$ in to (A.72), one can find the general solution of equation (A.72) as

$$c(X) = \frac{ko_4}{ko_3^2}\,e^{ko_3 X} + ko_5 X + ko_6$$

where ko_3, ko_4, ko_5, ko_6 are constant and $ko_3 \neq 0$. Next, we will consider the value of ko_4 which is $ko_4 = 0$ and $ko_4 \neq 0$.

- $ko_4 \neq 0$

Substituting $c(X)$ into (A.70), one derives

$$k_5 = \frac{k_{11}\big(-ko_2(\gamma-1)^2 - ko_3(\gamma-1)^3\big)}{2\gamma ko_2(\gamma-1)^2}$$

and

$$\eta_X^{00} = \frac{k_{11}\big(-a_X\gamma - ako_2 + ko_3\big)}{\gamma ko_2}. \qquad \text{(A.73)}$$

Performing a similar study as in previous cases for equation (A.73), one can rewrite it as

$$\eta_X^{00} = k_{11}f_5(X)$$

where $f_5(X)$ is a function of X only. Equation (A.73) provides that

$$a'(X) = \frac{-a(X)ko_2 + a(X)ko_3 - \gamma ko_2 g_5(X)}{\gamma}.$$

Finally, a solution of the determining equations is

$$\begin{aligned}
k_4 &= 0, \qquad k_8 = 0, \qquad kk3 = 0, \qquad k_5 = \frac{k_{11}\big(-ko_2 - ko_3(\gamma-1)\big)}{2\gamma ko_2},\\
k_7 &= \frac{k_{11}ko_3ko_5}{2ko_2}, \qquad \eta^{00}_X = g_5(X)k_{11}, \qquad \eta^{01}_X = 0.\\
\xi^t &= \frac{k_{11}t\big(-ko_2 - ko_3(\gamma-1)\big) + 2k_6\gamma ko_2}{2\gamma ko_2}, \qquad \xi^X = \frac{k_{11}}{ko_2},\\
\eta &= \frac{k_{11}ko_3ko_5\gamma t^2 + 2k_{11}\varphi\big(-ko_2 + ko_3\big) + 2t\gamma ko_2\eta^{01} + 2\gamma ko_2\eta^{00}}{2\gamma ko_2}
\end{aligned}$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3$$

with

$$\begin{aligned}
X^1 &= \partial_t, \qquad\qquad X^2 = t\partial_\varphi\\
X^3 &= \frac{t\big(-ko_2 - ko_3(\gamma-1)\big)}{2\gamma ko_2}\partial_t + \frac{1}{ko_2}\partial_X\\
&\quad + \Big(\frac{ko_3ko_5}{2ko_2}t^2 + \frac{-ko_2+ko_3}{\gamma ko_2}\varphi + \int g_5(X)dX\Big)\partial_\varphi.
\end{aligned} \tag{A.74}$$

Letting $ko_2 = \beta$, $ko_3 = \alpha$, $ko_6 = 0$, and $\int g_5(X)dX = \tilde{g}(X)$ then the basis generator X^3 in equation (A.74) can be written as

$$X^3 = \big(\frac{-1}{2\gamma} - \frac{\alpha}{2\beta} + \frac{\alpha}{2\beta\gamma}\big)t\partial_t + \frac{1}{\beta}\partial_X + \Big(\frac{\alpha}{2\beta}k_3t^2 + \big(\frac{-1}{\gamma} + \frac{\alpha}{\beta\gamma}\big)\varphi + \tilde{g}(X)\Big)\partial_\varphi, \tag{A.75}$$

and the pressure function is

$$P(X,\varphi_X) = b(X)(a(X)+\varphi_X)^\gamma + c(X)$$

$$\begin{aligned}
a'(X) &= \frac{a(X)}{\gamma}\big(\alpha-\beta\big) - \beta\tilde{g}'(X), \qquad b(X) = k_1\,e^{\beta X},\\
c(X) &= \frac{k_2}{\alpha^2}\,e^{\alpha X} + k_3X
\end{aligned} \tag{A.76}$$

where $k_1 \neq 0, \quad \alpha \neq 0, \quad \beta \neq 0, \quad$ and $\quad \gamma \neq 0,\ 1.$

By virtue of the equivalence transformation corresponding to the generators X^e_{15} and X^e_8, it can be assumed that $a(X) = 0$ and $k_3 = 0$.

For $\alpha - \beta \neq 0$ the extension of the kernel and the pressure function are

$$\begin{aligned} X_{10} &= \big(\beta + \alpha(\gamma - 1)\big)t\partial_t - 2\gamma\partial_X + 2\big(\beta - \alpha\big)\varphi\partial_\varphi \\ P(X, \varphi_X) &= k_1 e^{\beta X}\left(\varphi_X^\gamma + \frac{k_2}{k_1\alpha^2}e^{(\alpha-\beta)X}\right) \end{aligned} \tag{A.77}$$

The result of this case is presented in Table 5.1 as model M_6.

- $ko_4 = 0$

Substituting $ko_4 = 0$, then

$$\eta_X^{00} = \frac{k_{11}\big(-a_X(\gamma - 1) - ako_2\big) - 2ko_2k_5a}{(\gamma - 1)ko_2}$$

and it can be written as

$$\eta_X^{00} = \frac{g_6(X)k_{11} - 2ako_2k_5}{ko_2(\gamma - 1)}$$

where $g_6(X)$ is a function of X only. This equation provides

$$a'(X) = \frac{-(ko_2a(X) + g_6(X))}{\gamma - 1}.$$

Finally, a solution of the determining equations is

$$\begin{aligned} &k_4 = 0, \qquad k_8 = 0, \qquad kk3 = 0, \qquad k_7 = \frac{-ko_5}{2(\gamma - 1)}\big(k_{11} + 2\gamma k_5\big), \\ &\eta_X^{00} = \frac{g_6(X)k_{11} - 2ako_2k_5}{ko_2(\gamma - 1)}, \qquad \eta_X^{01} = 0, \qquad \xi^t = k_5t + k_6, \qquad \xi^X = \frac{k_{11}}{ko_2}, \\ &\eta = \frac{-k_{11}ko_5t^2}{2(\gamma - 1)} - \frac{k_{11}\varphi}{\gamma - 1} - \frac{\gamma ko_5k_5t^2}{\gamma - 1} - \frac{2k_5\varphi}{\gamma - 1} + t\eta^{01} + \eta^{00} \end{aligned}$$

with $k_{10} = \eta^{01}$. The generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3 + k_5X^4.$$

Letting $ko_2 = \beta$, $ko_3 = \alpha$, $ko_6 = 0$, $\tilde{\gamma} = \gamma - 1$, $\int g_6(X)dX = \tilde{g}(X)$ and $\int a(X)dX = \tilde{a}(X)$, the basis generators can be written as

$$\begin{aligned} X^1 &= \partial_t, \qquad\qquad X^2 = t\partial_\varphi \\ X^3 &= \partial_X + \frac{1}{2\tilde{\gamma}}\Big(-\beta k_2 t^2 - 2\big(\beta\varphi - \tilde{g}(X)\big)\Big)\partial_\varphi \\ X^4 &= t\partial_t + \frac{1}{\tilde{\gamma}}\Big(-k_2\big(\tilde{\gamma}+1\big)t^2 - 2\big(\tilde{a}+\varphi\big)\Big)\partial_\varphi \end{aligned} \tag{A.78}$$

and the pressure function with $a(X)$, $b(X)$, and $c(X)$ becomes

$$\begin{aligned} P(X,\varphi_X) &= b(X)(\tilde{a}'(X)+\varphi_X)^{\tilde{\gamma}+1} + c(X) \\ \tilde{a}''(X) &= \frac{-\big(\beta\tilde{a}'(X)+\tilde{g}'(X)\big)}{\tilde{\gamma}}, \qquad b(X) = k_1 e^{\beta X}, \\ c(X) &= k_2 X + k_3 \end{aligned} \tag{A.79}$$

where $k_1 \neq 0$ $\beta \neq 0$, and $\tilde{\gamma} \neq 0$ -1.

By virtue of the equivalence transformation corresponding to the operator X_{15}^e, X_2^e, and X_8^e, it can be assumed $\tilde{a}(X) = 0$, $\tilde{g}'(X) = 0$, $k_3 = 0$, and $k_2 = 0$.

For letting $\gamma = \tilde{\gamma} + 1$, the extensions of kernel of admitted Lie group and the pressure function are

$$X_{11} = (\gamma-1)\partial_X - \beta\varphi\partial_\varphi, \qquad X_{12} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi \tag{A.80}$$

$$P(X,\varphi_X) = k_1 e^{\beta X}\varphi_X^\gamma$$

where $\beta \neq 0$, and $\gamma \neq 0$, 1. The result of this case is presented in Table 5.1 as model M_7.

Case I.1b $\beta_1 \neq 1$

Solving equation (A.72), one obtains the solution

$$c(X) = ko_4 b^{c_1} + ko_5 X + ko_6$$

where c_1 is constant and $c_1 \neq 0$. To analyze the solutions of this equation, one has to split into 2 cases : $ko_4 \neq 0$ and $ko_4 = 0$.

- $ko_4 \neq 0$

For $ko_4 \neq 0$, one obtains

$$\eta_X^{00} = k_{11}\Big(\frac{-a_X b^{1-\beta_1}}{ko_2} + \frac{a(X)\big(c_1-1\big)}{\gamma}\Big),$$

Performing a similar study for this equation as previously, it can be rewritten as

$$\eta_X^{00} = k_{11}g_7(X),$$

where $g_7(X)$ is a function of X only. This equation also provides

$$a'(X) = \frac{ko_2 b^{\beta_1-1}}{\gamma}\Big(a(X)\big(c_1-1\big) - \gamma g_7(X)\Big).$$

Finally, a solution of the determining equations is

$$\begin{aligned}
&k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{k_{11}\big(c_1\gamma + c_1 - 1\big)}{\gamma}, \qquad k_7 = \frac{k_{11}ko_5}{2}\big(\beta_1 + c_1 - 1\big),\\
&\eta_X^{00} = g_7(X)k_{11}, \qquad \eta_X^{01} = 0, \qquad \xi^X = \frac{k_{11}}{ko_2}b^{1-\beta_1},\\
&\xi^t = \frac{k_{11}t}{2\gamma}\Big(-2\gamma(\beta_1-1) + c_1(1-\gamma) - 1\Big) + k_6,\\
&\eta = k_{11}\Big(\frac{ko_5}{2}\big(\beta_1 + c_1 - 1\big)t^2 + \frac{\varphi}{\gamma}\big(-\gamma\beta_1 + \gamma + c_1 - 1\big)\Big) + t\eta^{01} + \eta^{00}
\end{aligned}$$

with $k_{10} = \eta^{01}$. The generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3.$$

Letting $ko_2 = \beta$, $c_1 - 1 = m$, $\beta_1 - 1 = l$, $ko_4 = k_1$, $ko_5 = k_2$, $ko_6 = k_3$, and $\int g_7(X)dX = \tilde{g}(X)$, finally, the basis generators are

$$\begin{aligned}
&X^1 = \partial_t, \qquad X^2 = t\partial_\varphi\\
&X^3 = \big(-l + \frac{m}{2\gamma} - \frac{(m+1)}{2}\big)t\partial_t + \frac{b^{-l}}{\beta}\partial_X + \Big(\frac{k_2}{2}t^2\big(l+m+1\big)\\
&\quad + \big(-l + \frac{m}{\gamma}\big)\varphi + \tilde{g}(X)\Big)\partial_\varphi
\end{aligned} \tag{A.81}$$

and the pressure function is

$$P(X, \varphi_X) = b(X)(a(X) + \varphi_X)^\gamma + c(X)$$

$$a'(X) = \frac{\beta b^l}{\gamma}\Big(ma(X) - \gamma\tilde{g}'(X)\Big), \qquad b'(X) = \beta b^{l+1},$$
$$c(X) = k_1 b^{m+1} + k_2 X + k_3 \tag{A.82}$$

where $\beta \neq 0$, $\gamma \neq 0\,,\,1$, $l \neq 0$, and $m \neq -1$.

By virtue of the equivalence transformations corresponding to the operators X_{15}^e, X_2^e and X_8^e one can assume $a(X) = 0$, $k_2 = 0$, and $k_3 = 0$.

For $m \neq \gamma l$, therefore, the extensions of the kernel and the pressure function are

$$X_{13} = \big(\gamma(1 + m + 2l) - m\big)t\partial_t + (2\gamma l)X\partial_X + 2(\gamma l - m)\varphi\partial_\varphi$$
$$P(X, \varphi_X) = b(X)\big(\varphi_X^\gamma + k_1 b^m(X)\big), \qquad b^l(X) = \frac{-1}{l\beta X}$$

where l, β, $\neq 0$, $\gamma \neq 0$, 1, and $m \neq -1$. In Table 1, this is model M_8.

- $ko_4 = 0$

Substituting $ko_4 = 0$, one obtains

$$\eta_X^{00} = \frac{k_{11}\Big(-a_X b(X)(\gamma - 1) - a(X)b^{\beta_1}ko_2\big(2\beta_1 - 1\big)\Big) - 2k_5 a(X)b^{\beta_1}ko_2}{b^{\beta_1}ko_2\big(\gamma - 1\big)}.$$

Performing a similar study for this equation as previously, one can rewrite it as

$$\eta_X^{00} = g_8(X)k_{11} - \frac{2a(X)}{\gamma - 1}k_5,$$

and it provides

$$a'(X) = \frac{ko_2 b^{\beta_1 - 1}(X)}{\gamma - 1}\Big(a(X)\big(1 - 2\beta_1\big) + \big(1 - \gamma\big)g_8(X)\Big)$$

where $g_8(X)$ is function of X only. Finally, a solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{2k_{11}\big(-\gamma^2(\beta_1-1)-\gamma+\beta_1\big)+2k_5\big(1-\gamma^2\big)}{\big(\gamma-1\big)^2},$$

$$k_7 = \frac{ko_5k_{11}\big(-\gamma^4(\beta_1-1)+\gamma^3(2\beta_1-3)+\gamma(3\gamma-2\beta_1)-\gamma+\beta_1\big)-2ko_5\gamma k_5\big(\gamma-1\big)^3}{2\big(\gamma-1\big)^4},$$

$$\eta^{00}_X = g_8(X)k_{11} - \frac{2a(X)}{\gamma-1}k_5, \qquad \eta^{01}_X = 0, \qquad \xi^t = k_5t + k_6, \qquad \xi^X = \frac{b^{1-\beta_1}}{ko_2}k_{11},$$

$$\eta = k_{11}\big(-\gamma^2\beta_1+\gamma^2-\gamma+\beta_1\big)\big(ko_5t^2+2\varphi\big)+2k_5\big(1-\gamma\big)\Big(\gamma ko_5t^2+2\varphi\Big)$$
$$+\,t\eta^{01}+\eta^{00}$$

with $k_{10} = \eta^{01}$. The generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3 + k_5X^4.$$

Letting $ko_2 = \beta$, $\beta_1 - 1 = l$, $\gamma - 1 = \tilde{\gamma}$, $ko_5 = k_1$, $ko_6 = k_2$, $\int g_8(X)dX = \tilde{g}(X)$, and $\int a(X)dX = \tilde{a}(X)$, finally, the basis generators and the pressure are

$$X_1 = \partial_t, \qquad X_2 = t\partial_\varphi$$
$$X_3 = \frac{b^{-l}}{\beta}\partial_X + \frac{1}{2\gamma}\Big(-\big(k_1t^2+2\varphi\big)\big(\gamma l+2l+1\big)+2\gamma\tilde{g}(X)\Big)\partial_\varphi \tag{A.83}$$
$$X_4 = t\partial_t + \frac{1}{\gamma}\Big(-k_1t^2\big(\gamma+1\big)-2\big(\tilde{a}(X)+\varphi\big)\Big)\partial_\varphi$$

$$P(X,\varphi_X) = b(X)(\tilde{a}'(X)+\varphi_X)^{\tilde{\gamma}+1}+c(X)$$
$$\tilde{a}''(X) = \frac{\beta b^l}{\gamma}\Big(-(2l+1)\tilde{a}'(X)-\gamma\tilde{g}'(X)\Big), \tag{A.84}$$
$$b'(X) = \beta b^{l+1}, \qquad c(X) = k_1X + k_2$$

where $\beta \neq 0$, $\tilde{\gamma} \neq 0, -1$, and $l \neq 0$.

By virtue of the equivalence transformations corresponding to the operators X^e_{15}, X^e_2 and X^e_8, one can assume that $\tilde{a}(X) = 0$, $k_1 = 0$ and $k_2 = 0$.

For $\tilde{\gamma} + 1 = \gamma$, the extensions of the kernel and the pressure function are

$$X_{14} = l(\gamma-1)X\partial_X + \big(l(\gamma+1)+1\big)\varphi\partial_\varphi,$$
$$X_{15} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi, \tag{A.85}$$
$$P(X,\varphi_X) = b(X)\varphi^\gamma_X, \qquad \text{where} \quad b^{-l}(X) = -l\beta X,$$

where $l,\ \beta,\ \neq 0$, and $\gamma \neq 0,\ 1$. In Table 5.1, this is model M_9.

Case I.2 $\quad c_{XXX} = 0$

The general solution of this equation is

$$c(X) = c_1X^2 + c_2X + c_3 \qquad c_1, c_2, c_3 \quad \text{are constant.}$$

Substituting $c(X)$ into equation (A.68), one obtains

$$\frac{c_1}{\gamma - 1}\Big(k_{11}\big(1 - 2\beta_1\big) - 2\gamma k_5\Big) = 0.$$

We will consider 2 cases : $c_{XX} \neq 0$ and $c_{XX} = 0$.

Case I.2a $\quad c_{XX} \neq 0$

For $c_{XX} \neq 0$ that is $c_1 \neq 0$, such that one finds

$$k_5 = \frac{1 - 2\beta_1}{2\gamma}k_{11}.$$

Substituting all relations then

$$\eta_X^{00} = k_{11}\Big(\frac{-a_Xb^{1-\beta_1}}{ko_2} - \frac{a(X)\big(2\beta_1 - 1\big)}{\gamma}\Big).$$

Performing a similar study for this equation as previously, it can be written as

$$\eta_X^{00} = g_9(X)k_{11}$$

and it provides

$$a'(X) = \frac{ko_2b^{\beta_1-1}(X)}{\gamma}\Big(a(X)\big(1 - 2\beta_1\big) - \gamma g_9(X)\Big)$$

where $g_9(X)$ is a function of X only.

- $\beta_1 \neq 1$

For $\beta_1 \neq 1$, there are relations of $a(X)$, $b(X)$, $c(x)$ as follows :

$$a'(X) = \frac{ko_2}{\gamma} b^{\beta_1 - 1}\Big(a(X)\big(1 - 2\beta_1\big) - \gamma g_9(X)\Big)$$

$$b'(X) = b^{\beta_1}(X) ko_2 \qquad c(X) = c_1 X^2 + c_2 X + c_3$$

and a solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad k_5 = \frac{k_{11}\big(1 - 2\beta_1\big)}{2\gamma}, \qquad \eta_X^{01} = 0$$

$$k_7 = k_{11}\Big(\frac{1}{2}\big(1 - \beta_1\big)\big(2c_1 X + c_2\big) - \frac{c_1 b^{1-\beta_1}}{ko_2}\Big), \qquad \eta_X^{00} = g_9(X) k_{11},$$

$$\xi^t = \Big(\frac{-2\beta_1 + 1}{2\gamma}\Big) t k_{11} + k_6, \qquad \xi^X = \Big(\frac{b^{1-\beta_1}}{ko_2}\Big) k_{11},$$

$$\eta = k_{11} t^2 \Big(\big(\frac{1 - \beta_1}{2}\big)\big(2c_1 X + c_2\big) - \big(\frac{c_1}{ko_2}\big) b^{1-\beta_1}\Big) + k_{11}\big(\frac{\varphi}{\gamma}\big)\Big(\gamma\big(1 - \beta_1\big) - 2\beta_1 + 1\Big)$$
$$+ t\eta^{01} + \eta^{00}$$

with $k_{10} = \eta^{01}$. The generator corresponding to these coefficients is

$$X = k_6 X^1 + k_{10} X^2 + k_{11} X^3.$$

Letting $ko_2 = \beta$, $1 - \beta_1 = l$, $c_1 = k_1$, $c_2 = k_2$, $c_3 = k_3$, and $\int g_9(X) dX = \tilde{g}(X)$ or $g_9(X) = \tilde{g}'(X)$, finally, the basis generators and the pressure function can be written as

$$X^1 = \partial_t, \qquad X^2 = t\partial_\varphi$$

$$X^3 = \big(\frac{2l - 1}{2\gamma}\big) t\partial_t + \frac{b^l}{\beta}\partial_X + \Big(\frac{t^2}{2\beta}\big(2l\beta k_1 X + l\beta k_2 - 2k_1 b^l\big) + \frac{\varphi}{\gamma}\big(l\gamma + 2l - 1\big) + \tilde{g}(X)\Big)\partial_\varphi \tag{A.86}$$

and

$$\begin{aligned} P(X, \varphi_X) &= b(X)(a(X) + \varphi_X)^\gamma + c(X) \\ a'(X) &= \frac{\beta b^{-l}}{\gamma}\Big(a(X)\big(2l - 1\big) - \gamma \tilde{g}'(X)\Big), \\ b'(X) &= \beta b^{1-l}, \qquad c(X) = k_1 X^2 + k_2 X + k_3 \end{aligned} \tag{A.87}$$

where $\beta \neq 0$, $\gamma \neq 0, 1$, $l \neq 0$ and $k_1 \neq 0$.

By virtue of the equivalence transformations corresponding to the operators X_{15}^e, X_2^e and X_8^e, one can assume $a(X) = 0$, $k_2 = 0$, and $k_3 = 0$.

Therefore the extensions of the kernel and the pressure function are

$$X_{16} = (2l-1)t\partial_t + (2\gamma l)X\partial_X + \big(2(2l-1)+2l\gamma\big)\varphi\partial_\varphi$$

$$P(X,\varphi_X) = b(X)\varphi_X^\gamma + k_1X^2 \qquad \text{where} \quad b^l(X) = l\beta X\,,$$

$l,\ \beta\ \neq 0,\ \ \gamma \neq 0,\ 1$ and $k_1 \neq 0$. In Table 5.2, this is model M_{10}.

- $\beta_1 = 1$

Substituting $\beta_1 = 1$, and solving equation $b_X = bko_2$, one gets

$$b(X) = c_4e^{ko_2X}.$$

The condition for $a(X)$, such that $a'(X) = \frac{-ko_2}{\gamma}\big(a(X)+\gamma g_9(X)\big)$ is obtained. Finally, a solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{-k_{11}}{\gamma}, \qquad k_5 = \frac{-k_{11}}{2\gamma},$$
$$k_7 = \frac{-k_{11}c_1}{ko_2}, \qquad \eta_X^{00} = g_9(X)k_{11}, \qquad \eta_X^{01} = 0,$$
$$\xi^t = \frac{-k_{11}t + 2\gamma k_6}{2\gamma}, \qquad \xi^X = \frac{k_{11}}{ko_2}, \qquad \eta = -k_{11}\big(\frac{c_2t^2}{ko_2} + \frac{\varphi}{\gamma}\big) + t\eta^{01} + \eta^{00}$$

with $k_{10} = \eta^{01}$. The generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3.$$

Letting $ko_2 = \beta$, $c_1 = k_1$, $c_2 = k_2$, $c_3 = k_3$, $c_4 = k_4$, and $\int g_9(X)dX = \tilde{g}(X)$ or $g_9(X) = \tilde{g}'(X)$, then the basis generators and the pressure function can be written as

$$X^1 = \partial_t, \qquad\qquad X^2 = t\partial_\varphi$$
$$X^3 = \frac{-t}{2\gamma}\partial_t + \frac{1}{\beta}\partial_X + \Big(-\big(\frac{k_2t^2}{\beta} + \frac{\varphi}{\gamma}\big) + \tilde{g}(X)\Big)\partial_\varphi \tag{A.88}$$

and

$$P(X,\varphi_X) = b(X)(a(X)+\varphi_X)^\gamma + c(X),$$
$$a'(X) = \frac{-\beta}{\gamma}\Big(a(X) + \gamma\tilde{g}'(X)\Big), \tag{A.89}$$
$$b(X) = k_4e^{\beta X}, \qquad c(X) = k_1X^2 + k_2X + k_3$$

where $\beta \neq 0$, $\gamma \neq 0\,,\,1$, $k_1 \neq 0$ and $k_4 \neq 0$.

By virtue of the equivalence transformations corresponding to the operators X^e_{15}, X^e_2 and X^e_8, one can assume $a(X) = 0$, $k_2 = 0$, and $k_3 = 0$.

Therefore the extensions of the kernel and the pressure function are

$$X_{17} = -\beta t\partial_t + 2\gamma\partial_X - 2\beta\varphi\partial_\varphi, \qquad P(X,\varphi_X) = e^{\beta X}\varphi_X^\gamma + k_1 X^2 \tag{A.90}$$

where $\beta \neq 0$, $\gamma \neq 0$, 1 and $k_1 \neq 0$. In Table 5.2, this is model M_{11}.

Case I.2b $c_{XX} = 0$

This case is considering $c_{XX} = 0$; substituting $c_1 = 0$, one gets

$$\eta^{00}_X = k_{11}\Big(\frac{-b^{1-\beta_1}}{ko_2}a_X - (\frac{2\beta_1 - 1}{\gamma - 1})a(X)\Big) - \frac{2a(X)k_5}{\gamma - 1}.$$

Performing a similar study for this equation as previously, one can rewrite it as

$$\eta^{00}_X = g_{10}(X)k_{11} - \frac{2a(X)}{\gamma - 1}k_5,$$

and it provides

$$a'(X) = \frac{ko_2 b^{\beta_1 - 1}}{\gamma - 1}\Big(a(X)\big(1 - 2\beta_1\big) + \big(1 - \gamma\big)g_{10}(X)\Big).$$

where $g_{10}(X)$ is function of X only. To find conditions for function $b(X)$, one has to consider 2 cases : $\beta_1 = 1$ and $\beta_1 \neq 1$.

- $\beta_1 = 1$

For $\beta_1 = 1$, one obtains the conditions for $a(X)$, $b(X)$ and $c(X)$ as follows :

$$a'(X) = (\frac{ko_2}{1 - \gamma})a(X) - ko_2 g_{10}(X), \qquad b(X) = c_4 e^{ko_2 X}, \qquad c(X) = c_2 X + c_3.$$

A solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{2k_{11}(1-\gamma) + 2k_5(1-\gamma^2)}{(\gamma-1)^2},$$

$$k_7 = \frac{-k_{11}c_2(\gamma-1)^3 - 2k_5\gamma c_2(\gamma-1)^3}{2(\gamma-1)^4}, \qquad \eta_X^{00} = g_{10}(X)k_{11} - \frac{2a(X)}{\gamma-1}k_5$$

$$\eta_X^{01} = 0 \qquad \xi^t = k_5 t + k_6, \qquad \xi^X = \frac{k_{11}}{ko_2},$$

$$\eta = \frac{k_{11}(1-\gamma)\left(t^2c_2 + 2\varphi\right) + k_5(1-\gamma)\left(2t^2\gamma c_2 + 4\varphi\right)}{2(\gamma-1)^2} + t\eta^{01} + \eta^{00}$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3 + k_5X^4.$$

Letting $ko_2 = \beta$, $1-\gamma = \tilde{\gamma}$, $c_2 = k_1$, $c_3 = k_2$, $c_4 = k_3$, $\int g_{10}(X)dX = \tilde{g}(X)$ and $\int a(X)dX = \tilde{a}(X)$ then the basis generators and the pressure function become

$$\begin{aligned} X^1 &= \partial_t, \qquad X^2 = t\partial_\varphi \\ X^3 &= \frac{1}{\beta}\partial_X + (\tilde{g}(X) + \frac{k_1t^2}{2\gamma} + \frac{\varphi}{\gamma})\partial_\varphi \\ X^4 &= t\partial_t + \frac{1}{\gamma}(2\tilde{a}(X) - \tilde{\gamma}k_1t^2 + k_1t^2 + 2\varphi)\partial_\varphi \end{aligned} \tag{A.91}$$

and

$$\begin{gathered} P(X,\varphi_X) = b(X)(\tilde{a}'(X) + \varphi_X)^{1-\tilde{\gamma}} + c(X) \\ \tilde{a}''(X) = \frac{\beta}{\tilde{\gamma}}\left(\tilde{a}'(X) - \tilde{\gamma}\tilde{g}'(X)\right), \qquad b(X) = k_3e^{\beta X}, \qquad c(X) = k_1X + k_2 \end{gathered} \tag{A.92}$$

where $\beta \neq 0$, $\tilde{\gamma} \neq 0, 1$, and $k_3 \neq 0$.

By virtue of the equivalence transformations corresponding to the operators X_{15}^e, X_2^e and X_8^e, one can assume $\tilde{a}'(X) = 0$, $k_1 = 0$, and $k_2 = 0$.

For $1 - \tilde{\gamma} = \gamma$, therefore, the extensions of the kernel and the pressure function are

$$\begin{gathered} X_1 = (1-\gamma)\partial_X + \beta\varphi\partial_\varphi, \qquad X_2 = (1-\gamma)t\partial_t + 2\varphi\partial_\varphi \\ P(X,\varphi_X) = e^{\beta X}\varphi_X^\gamma, \end{gathered} \tag{A.93}$$

where $\beta \neq 0$, and $\gamma \neq 0, 1$. This case is equivalent to the generator of equation (A.80).

- $\beta_1 \neq 1$

For $\beta_1 \neq 1$, then the condition for $a(X)$, $b(X)$ and $b(X)$ are given as

$$a'(X) = \frac{ko_2 b^{\beta_1 - 1}}{\gamma - 1}\Big(a(X)\big(1 - 2\beta_1\big) + (1-\gamma)g_{10}(X)\Big),$$

$$b'(X) = ko_2 b^{\beta_1}, \qquad c(X) = c_2 X + c_3$$

and a solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{2k_{11}\big(-\gamma^2(\beta-1) - \gamma + \beta\big) + 2k_5(1-\gamma^2)}{\big(\gamma-1\big)^2},$$

$$k_7 = \frac{k_{11}c_2\big(-\gamma^4(\beta-1) + \gamma^3(2\beta-3) + 3\gamma^2 - 2\gamma\beta - \gamma + \beta\big) - 2k_5\gamma c_2(\gamma-1)^3}{2\big(\gamma-1\big)^4},$$

$$\eta_X^{00} = g_{10}(X)k_{11} - \frac{2a(X)}{\gamma-1}k_5, \qquad \eta_X^{01} = 0 \qquad \xi^t = k_5 t + k_6 \qquad \xi^X = \frac{k_{11}b^{1-\beta_1}}{ko_2}$$

$$\eta = k_{11}\Big(\frac{c_2t^2 + 2\varphi}{2(\gamma-1)}\Big)\Big(-\beta(1+\gamma)+\gamma\Big) - \frac{k_5}{\gamma-1}\Big(c_2\gamma t^2 + 2\varphi\Big) + t\eta^{01} + \eta^{00},$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3 + k_5X^4.$$

Letting $ko_2 = \beta$, $\gamma - 1 = \tilde{\gamma}$, $\beta_1 - 1 = l$, $c_2 = k_1$, $c_3 = k_2$, $\int g_{10}(X)dX = \tilde{g}(X)$ and $\int a(X)dX = \tilde{a}(X)$, then the basis generators and the pressure function can be written as

$$X^1 = \partial_t, \qquad X^2 = t\partial_\varphi$$

$$X^3 = \frac{b^{-l}}{\beta}\partial_X + \frac{1}{2\tilde{\gamma}}\Big(\big(k_1t^2 + 2\varphi\big)\big(-1 - 2l - \tilde{\gamma}l\big) + 2\tilde{\gamma}\tilde{g}(X)\Big)\partial_\varphi \tag{A.94}$$

$$X^4 = t\partial_t - \frac{1}{\tilde{\gamma}}\Big(\big(\tilde{\gamma}+1\big)k_1t^2 + 2\big(\varphi + \tilde{a}(X)\big)\Big)\partial_\varphi$$

and

$$P(X, \varphi_X) = b(X)(\tilde{a}'(X) + \varphi_X)^{\tilde{\gamma}+1} + c(X),$$

$$\tilde{a}''(X) = \frac{\beta b^l}{\tilde{\gamma}}\Big(-\tilde{a}'(X)\big(2l+1\big) - \tilde{\gamma}\tilde{g}'(X)\Big), \qquad b'(X) = \beta b^{l+1}(X), \tag{A.95}$$

$$c(X) = k_1X + k_2$$

where $\beta \neq 0$, $\tilde{\gamma} \neq 0\,, -1$, and $l \neq 0$.

By virtue of the equivalence transformations corresponding to the operators X_{15}^e, X_2^e and X_8^e, one can assume $\tilde{a}(X) = 0$, $k_1 = 0$, and $k_2 = 0$.

For $\tilde{\gamma} + 1 = \gamma$, therefore, the extensions of the kernel and the pressure function are

$$X_1 = l(\gamma - 1)X\partial_X + \big(l(\gamma+1)+1\big)\varphi\partial_\varphi, \qquad X_2 = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$$
$$P(X, \varphi_X) = b(X)\varphi_X^\gamma \qquad \text{where} \quad b^{-l}(X) = -l\beta X, \tag{A.96}$$

l, $\beta \neq 0$, and $\gamma \neq 0,\ 1$. This case is equivalent to the generator in equation (A.85).

Case II $\quad b_{XX} = 0$

This case is considering $b_{XX} = 0$; solving this equation gives

$$b(X) = ko_2 X + ko_3, \qquad ko_2, ko_3 \quad \text{are constant.}$$

After substituting $b(X) = ko_2 X + ko_3$ into equation (A.68), it becomes

$$k_{11}\Big(c_{XXX}\big(ko_2X + ko_3\big)\big(1-\gamma\big) + c_{XX}ko_2\Big) - 2k_5 c_{XX}\gamma ko_2 = 0. \tag{A.97}$$

Differentiating equation (A.97) with respect to X, then

$$k_{11}\Big(c_{XXXX}\big(ko_2X + ko_3\big)\big(1-\gamma\big) - c_{XXX}ko_2(\gamma-2)\Big) - 2k_5 c_{XXX}\gamma ko_2 = 0. \tag{A.98}$$

Equations (A.97) and (A.98) are algebraic linear homogeneous equations with respect to k_5 and k_{11} with the determinant

$$\gamma(\gamma-1)ko_2\Big(c_{XXXX}c_{XX}\big(ko_2X + ko_3\big) - c_{XXX}^2\big(ko_2X + ko_3\big) + ko_2 c_{XXX}c_{XX}\Big).$$

If this determinant is not equal to zero, then $k_5 = 0$ and $k_{11} = 0$ (which contradicts our condition $k_{11} \neq 0$). Hence, one has to assume that

$$c_{XXXX}c_{XX}\big(ko_2X + ko_3\big) - c_{XXX}^2\big(ko_2X + ko_3\big) + ko_2 c_{XXX}c_{XX} = 0. \tag{A.99}$$

There are 2 assumptions to determine solutions of this equation: $c_{XXX} \neq 0$ and $c_{XXX} = 0$.

Case II.1 $\quad c_{XXX} \neq 0$

From equation (A.99), assuming $c_{XXX} \neq 0$ one obtains

$$c_{XXXX}c_{XX}\big(ko_2X + ko_3\big) - c_{XXX}^2\big(ko_2X + ko_3\big) + ko_2c_{XXX}c_{XX} = 0.$$

Integrating this equation with respect to X, one solution is obtained as

$$c(X) = ko_5\Big(X + \frac{ko_3}{ko_2}\Big)^{ko_4} + ko_6X + ko_7, \qquad ko_4, ko_5, ko_6, ko_7 \quad \text{are constant.}$$

- $ko_4 \neq 0$, and $ko_5 \neq 0$

  Substituting $c(X)$, one derives

$$k_5 = \frac{k_{11}\Big(- \gamma^3 ko_4 + 2\gamma^3 + 3\gamma^2 ko_4 - 5\gamma^2 - 3\gamma ko_4 + 4\gamma + ko_4 - 1\Big)}{2\gamma(\gamma - 1)^2}$$

and

$$\eta_X^{00} = \frac{k_{11}\Big(- a_X\gamma ko_2X - a_X\gamma ko_3 + a(X)ko_2ko_4 - a(X)ko_2\Big)}{\gamma ko_2}.$$

Moreover η_X^{00} can be written as

$$\eta_X^{00} = g_{11}(X)k_{11}, \qquad g_{11}(X) \quad \text{is function of X only,}$$

and it provides

$$a'(X) = \frac{ko_2\Big(a(X)ko_4 - a(X) - \gamma g_{11}(X)\Big)}{\gamma\big(ko_2X + ko_3\big)}.$$

A solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{k_{11}\big(\gamma ko_4 + ko_4 - 1\big)}{\gamma},$$

$$k_5 = \frac{k_{11}\big(- \gamma ko_4 + 2\gamma + ko_4 - 1\big)}{2\gamma}, \qquad k_7 = \frac{k_{11}ko_6\big(ko_4 - 1\big)}{2},$$

$$\eta_X^{00} = g_{11}(X)k_{11}, \qquad \eta_X^{01} = 0.$$

$$\xi^t = \Big(\frac{-\gamma ko_4 + 2\gamma + ko_4 - 1}{2\gamma}\Big)k_{11}t + k_6, \qquad \xi^X = \Big(\frac{ko_2 X + ko_3}{ko_2}\Big)k_{11},$$
$$\eta = k_{11}\Big(\frac{ko_6(ko_4-1)}{2}t^2 + \frac{\varphi}{\gamma}\big(\gamma + ko_4 - 1\big)\Big) + t\eta^{01} + \eta^{00}$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6 X^1 + k_{10} X^2 + k_{11} X^3.$$

Letting $ko_2 = k_1$, $ko_5 = k_2$, $ko_6 = k_3$, $ko_4 - 1 = \alpha$, $\frac{ko_3}{ko_2} = \beta$, and $\int g_{11}(X)dX = \tilde{g}(X)$, then the basis generators and the pressure function can be written as

$$\begin{aligned} X^1 &= \partial_t, \qquad\qquad X^2 = t\partial_\varphi \\ X^3 &= \frac{1}{2\gamma}\Big(\gamma\big(1-\alpha\big) + \alpha\Big)t\partial_t + \big(X+\beta\big)\partial_X \\ &\quad + \Big(\frac{k_3\alpha}{2}t^2 + \varphi\big(1 + \frac{\alpha}{\gamma}\big) + \tilde{g}(X)\Big)\partial_\varphi, \end{aligned} \tag{A.100}$$

and

$$\begin{aligned} &P(X,\varphi_X) = b(X)(a(X)+\varphi_X)^\gamma + c(X), \qquad a'(X) = \frac{\alpha a(X) - \gamma\tilde{g}'(X)}{\gamma\big(X+\beta\big)}, \\ &b(X) = k_1\big(X+\beta\big), \qquad c(X) = k_2\big(X+\beta\big)^{\alpha+1} + k_3 X, \end{aligned} \tag{A.101}$$

where $k_1 \neq 0$, $k_2 \neq 0$, $\alpha \neq 0, -1$, and $\gamma \neq 0, 1$.

By virtue of the equivalence transformations corresponding to the operators X^e_{15}, X^e_2 and X^e_8 one can assume $a(X) = 0$, $k_3 = 0$ and $\beta = 0$.

For $\gamma + \alpha \neq 0$, therefore, the extensions of the kernel and the pressure function are

$$\begin{aligned} X_{18} &= \big(\gamma(1-\alpha) + \alpha\big)t\partial_t + 2\gamma X\partial_X + 2(\gamma+\alpha)\varphi\partial_\varphi \\ P(X,\varphi_X) &= X\Big(k_1\varphi_X^\gamma + k_2 X^\alpha\Big) \end{aligned} \tag{A.102}$$

where $\alpha \neq -1,\ 0$, $\gamma \neq 0,\ 1$, $\gamma + \alpha \neq 0$, k_1, $k_2 \neq 0$. In Table 1, the result of this case is presented as model M_{12}.

Remark. Either $ko_4 = 0$ or $ko_5 = 0$ lead a contradiction to the condition $c_{XXX} \neq 0$.

Case II.2 $c_{XXX} = 0$

The general solution of $c_{XXX} = 0$ is

$$c(X) = ko_4 X^2 + ko_5 X + ko_6,$$

where ko_4, ko_5, and ko_6 are constant. Substituting $c(X)$ into equation (A.97), one gets

$$ko_4\big(\gamma - 1\big)^2\big(k_{11} - 2k_5\gamma\big) = 0.$$

- $ko_4 \neq 0$

Since $\gamma \neq 0\,,1$ and assuming $ko_4 \neq 0$, then

$$k_5 = \frac{k_{11}}{2\gamma}.$$

Substituting all relations, η_X^{00} is

$$\eta_X^{00} = \frac{k_{11}\Big(- a_X\gamma\big(ko_2X + ko_3\big) + ko_2a(X)\Big)}{\gamma ko_2}$$

which can be rewritten in the following form,

$$\eta_X^{00} = g_{12}(X)k_{11}.$$

This equation also provides

$$a'(X) = \frac{ko_2\Big(a(X) - \gamma g_{12}(X)\Big)}{\gamma\big(ko_2X + ko_3\big)}$$

where $g_{12}(X)$ is function of X only. Finally, a solution of the determining equation is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = (\frac{2\gamma+1}{\gamma})k_{11}, \qquad k_5 = \frac{k_{11}}{2\gamma},$$

$$k_7 = (\frac{ko_2ko_5 - 2ko_3ko_4}{2ko_2})k_{11}, \qquad \eta_X^{00} = g_{12}(X)k_{11}, \qquad \eta_X^{01} = 0,$$

$$\xi^t = \frac{k_{11}t}{2\gamma} + k_6, \qquad \xi^X = \Big(\frac{ko_2X + ko_3}{ko_2}\Big)k_{11},$$

$$\eta = k_{11}\Big(\frac{t^2}{2ko_2}\big(ko_2ko_5 - 2ko_3ko_4\big) + \frac{\varphi}{\gamma}\big(\gamma+1\big)\Big) + t\eta^{01} + \eta^{00}$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6 X^1 + k_{10} X^2 + k_{11} X^3.$$

Letting $ko_2 = k_1$, $ko_4 = k_2$, $ko_5 = k_3$, $ko_6 = k_4$, $\frac{ko_3}{ko_2} = \beta$, and $\int g_{12}(X)dX = \tilde{g}(X)$, then the basis generator can be written as

$$\begin{aligned} X^1 &= \partial_t, \qquad X^2 = t\partial_\varphi \\ X^3 &= \frac{t}{2\gamma}\partial_t + (X+\beta)\partial_X + \Big(t^2\big(\frac{k_3}{2} - k_2\beta\big) + \varphi\big(1+\frac{1}{\gamma}\big) + \tilde{g}(X)\Big)\partial_\varphi, \end{aligned} \tag{A.103}$$

and the pressure function with the conditions for $a(X)$, $b(X)$, and $c(X)$ is

$$\begin{aligned} P(X,\varphi_X) &= b(X)(a(X)+\varphi_X)^\gamma + c(X) \qquad a'(X) = \frac{a(X) - \gamma\tilde{g}'(X)}{\gamma\big(X+\beta\big)}, \\ b(X) &= k_1\big(X+\beta\big), \qquad c(X) = k_2X^2 + k_3X + k_4 \end{aligned} \tag{A.104}$$

where $k_1 \neq 0$, $k_2 \neq 0$, and $\gamma \neq 0, 1$.

By virtue of the equivalence transformations corresponding to the operator X^e_{15}, X^e_2 and X^e_8 one can assume $a(X) = 0$, $k_3 = 0$, $k_4 = 0$ and $\beta = 0$.

Therefore the extensions of the kernel and the related pressure function are

$$X_1 = t\partial_t + 2\gamma X\partial_X + 2(\gamma+1)\varphi\partial_\varphi, \qquad P(X,\varphi_X) = X\Big(k_1\varphi_X^\gamma + k_2X\Big) \tag{A.105}$$

where $\gamma \neq 0,\ 1,\ -1$, and $k_1,\ k_2 \neq 0$. This case is a particular case of equation (A.102) when $\alpha = 1$. Thus the general form of these two equations (A.102) and (A.105) is given in Table 5.2 as a model M_{12}.

- $ko_4 = 0$

If $ko_4 = 0$, there exists extensions of the kernel and its corresponding pressure function as follows

$$X_3 = t\partial_t + 2\gamma X\partial_X + 2(\gamma+1)\varphi\partial_\varphi, \qquad P(X,\varphi_X) = k_1X\varphi_X^\gamma \tag{A.106}$$

where $\gamma \neq 0,\ 1,\ -1$, and $k_1 \neq 0$. This case is given in Table 5.2 as a model M_{12a}.

Case $b_X = 0$

Substituting $P(X, \varphi_X) = b(X)\big(\varphi_X + a(X)\big)^{\gamma} + c(X)$, $\mu_{1_{\varphi_X}} = 0$, and $b_X = 0$ or $b(X) = ko_2$, where $ko_2 \neq 0$, one obtains the relation

$$\frac{2\xi^X_X c_{XX} - 2k_5 c_{XX}\gamma + \xi^X c_{XXX}\big(1-\gamma\big)}{2(\gamma-1)} = 0. \tag{A.107}$$

Construct ξ^X_X by assuming $c_{XX} \neq 0$.

Case I $c_{XX} \neq 0$, $c_{XXX} = 0$

Assuming $c_{XX} \neq 0$, one can find ξ^X_X,

$$\xi^X_X = \frac{2k_5 c_{XX}\gamma + \xi^X c_{XXX}\big(\gamma-1\big)}{2c_{XX}},$$

such that

$$\xi^X = k_5\gamma X + k_9$$

and

$$c(X) = ko_3 X^2 + ko_4 X + ko_5, \qquad \text{where} \quad ko_3 \neq 0.$$

Substituting all relations into equation (5.34), it becomes

$$\begin{aligned} &ko_2\gamma\big(a(X)+\varphi_X\big)^{\gamma}\Big(a(X)a_{XX}\big(\gamma+3\big) + \varphi_X a_{XX}\big(\gamma+3\big) \\ &+a_X^2\gamma(\gamma+2) - 3a_X^2\Big) - 6ko_3(\gamma-1)\Big(a(X)+\varphi_X\Big)^2 = 0. \end{aligned} \tag{A.108}$$

Differentiating equation (A.108) with respect to φ_X, then

$$ko_2 k_4\big(a(X)+\varphi_X\big)^{\gamma-3}\Big(\frac{\gamma(\gamma+3)}{6}\Big)\Big(a_{XX}\big(a(X)+\varphi_X\big) + a_X^2\big(\gamma-2\big)\Big) = 0 \tag{A.109}$$

is obtained. To consider $\gamma + 3 \neq 0$, one will consider two cases as follows :

Case I.1 $a_{XX}\big(a(X)+\varphi_X\big) + a_X^2\big(\gamma-2\big) \neq 0$

In this case, one obtains $k_4{=}0$ and after substitution one can find all solutions of the determining equations as follows :

$$k_8 = 0, \qquad kk3 = 2k_5\big(\gamma+1\big), \qquad k_7 = -ko_3k_{11} + \frac{ko_4}{2}\gamma k_5$$
$$\eta_X^{00} = k_5\Big(-a_X\gamma X + 2a(X)\Big) - a_Xk_{11}, \qquad \eta_X^{01} = 0$$
$$\xi^t = k_5t + k_6, \qquad \xi^X = \gamma Xk_5 + k_{11},$$
$$\eta = k_{11}\Big(-ko_3t^2\Big) + k_5\Big(\frac{ko_4\gamma}{2}t^2 + \varphi\big(\gamma+2\big)\Big) + t\eta^{01} + \eta^{00}$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_{11}X^3 + k_5X^4.$$

Letting $ko_2 = \beta$, $ko_3 = k_1$, $ko_4 = k_2$, $ko_5 = k_3$, and $\int a(X)dX = \tilde{a}(X)$, the basis generators and the pressure function can be written as

$$\begin{aligned} &X^1 = \partial_t, \qquad X^2 = t\partial_\varphi, \qquad X^3 = \partial_X + \Big(-k_1t^2 - \tilde{a}'(X)\Big)\partial_\varphi \\ &X^4 = t\partial_t + \gamma X\partial_X + \Big(\frac{k_2\gamma}{2}t^2 + \varphi\big(\gamma+2\big) - \gamma\big(X\tilde{a}'(X) - \tilde{a}(X)\big) \\ &\quad + 2\tilde{a}(X)\Big)\partial_\varphi, \end{aligned} \tag{A.110}$$

and

$$P(X,\varphi_X) = \beta(\tilde{a}'(X) + \varphi_X)^\gamma + k_1X^2 + k_2X + k_3 \tag{A.111}$$

where $\beta \neq 0$, $\gamma \neq 0\,, 1,\, -3$, and $k_1 \neq 0$.

By virtue of the equivalence transformations corresponding to the operators X_{15}^e, X_2^e and X_8^e, one can assume $\tilde{a}'(X) = 0$, $k_2 = 0$, and $k_3 = 0$.

For $\gamma \neq -2$, therefore, the extensions of the kernel and the pressure function are

$$\begin{aligned} &X_{19} = t\partial_t + \gamma X\partial_X + \big(\gamma+2\big)\varphi\partial_\varphi, \qquad X_{20} = \partial_X - k_1t^2\partial_\varphi, \\ &P(X,\varphi_X) = \beta\varphi_X^\gamma + k_1X^2 \end{aligned} \tag{A.112}$$

where $\beta \neq 0$, $\gamma \neq 0$, 1, -2, -3, and $k_1 \neq 0$. The result of this case is given in Table 5.2 as model M_{13}.

Case I.2 $a_{XX}\big(a(X)+\varphi_X\big)+a_X^2\big(\gamma-2\big)=0$

Solving equation $a_{XX}\big(a(X)+\varphi_X\big)+a_X^2\big(\gamma-2\big)=0$, one gets 2 general solutions, i.e., $a_X\neq 0$ and $a_X=0$.

Case $a_X\neq 0$

The solution of equation $a_{XX}\big(a(X)+\varphi_X\big)+a_X^2\big(\gamma-2\big)=0$ is

$$a_X=ko_6\big(a(X)+\varphi_X\big)^{2-\gamma},\qquad ko_6 \quad \text{is a constant.}$$

Substituting and considering $\gamma+3\neq 0$, and applying the equivalence transformation corresponding to the generator X_{15}^e, and this transformation allows to assume $a(X)=0$. Then

$$\eta_X^{00}=k_5\Big(-ko_6\gamma X\varphi_X^{2-\gamma}\Big)-ko_6\varphi_X^{2-\gamma}k_{11}$$

and one obtains $\eta^{00}=k_5\Big(-ko_6\frac{X^2}{2}\gamma\varphi_X^{2-\gamma}\Big)-ko_6X\varphi_X^{2-\gamma}k_{11}$. Moreover, a solution of the determining equations is

$$k_4=0,\qquad k_8=0,\qquad kk3=2k_5(\gamma+1),\qquad k_7=-ko_3k_{11}+\frac{ko_4}{2}\gamma k_5$$
$$\eta_X^{00}=k_5\Big(2a(X)-ko_6\gamma X\big(a(X)+\varphi_X\big)^{2-\gamma}\Big)-ko_6\big(a(X)+\varphi_X\big)^{2-\gamma}k_{11},\qquad \eta_X^{01}=0,$$
$$\xi^t=k_5t+k_6,\qquad \xi^X=\gamma Xk_5+k_{11},$$
$$\eta=k_{11}\Big(-ko_3t^2\Big)+k_5\Big(\frac{ko_4\gamma}{2}t^2+\varphi\big(\gamma+2\big)\Big)+t\eta^{01}+\eta^{00}$$

with $k_{10}=\eta^{01}$; then the generator corresponding to these coefficients is

$$X=k_6X^1+k_{10}X^2+k_{11}X^3+k_5X^4.$$

Letting $ko_2=\beta$, $ko_3=k_1$, $ko_4=k_2$, $ko_5=k_3$, $ko_6=0$, finally, the basis generator and the pressure function can be written as

$$\begin{aligned}&X^1=\partial_t,\qquad X^2=t\partial_\varphi\qquad X^3=\partial_X-k_1t^2\partial_\varphi\\&X^4=t\partial_t+\gamma X\partial_X+\Big(\frac{k_2\gamma}{2}t^2+\varphi\big(\gamma+2\big)\Big)\partial_\varphi\end{aligned}\tag{A.113}$$

and

$$P(X,\varphi_X) = \beta\varphi_X^{\gamma} + k_1X^2 + k_2X + k_3 \tag{A.114}$$

where $\beta \neq 0$, $\gamma \neq 0\,,1$ and $k_1 \neq 0$.

By virtue of the equivalence transformation corresponding to the generators X_2^e and X_8^e, one can assume $k_2 = 0$ and $k_3 = 0$.

For $\gamma \neq -2$, therefore the extensions of the kernel and the pressure function are

$$\begin{gathered} X_1 = \partial_X - k_1t^2\partial_\varphi, \qquad X_2 = t\partial_t + \gamma X\partial_X + (\gamma+2)\varphi\partial_\varphi \\ P(X,\varphi_X) = \beta\varphi_X^{\gamma} + k_1X^2 \end{gathered} \tag{A.115}$$

where $\gamma \neq 0,\, 1,\, -2,\, -3,$ and $k_1 \neq 0$. This case is equivalent to the generator in equation (A.112)

Case $a_X = 0$

If $a_X = 0$, then $a(X) = ko_6$ and a solution of the determining equations is given as follows :

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = 2k_5(\gamma+1), \qquad k_7 = -ko_3k_{11} + \frac{ko_4}{2}\gamma k_5$$

$$\eta_X^{00} = 2ko_6k_5, \qquad \eta_X^{01} = 0, \qquad \xi^t = k_5t + k_6, \qquad \xi^X = \gamma Xk_5 + k_{11},$$

$$\eta = k_{11}\Big(-ko_3t^2\Big) + k_5\Big(\frac{ko_4\gamma}{2}t^2 + \varphi(\gamma+2)\Big) + t\eta^{01} + \eta^{00}.$$

with $k_{10} = \eta^{01}$, then the generator corresponding to these coefficients is

$$X = k_6X_1 + k_{10}X_2 + k_{11}X_3 + k_5X_4.$$

Letting $ko_2 = \beta$, $ko_3 = k_1$, $ko_4 = k_2$, $ko_5 = k_3$, $ko_6 = k_4$, then the basis generator and the pressure function can be written as

$$\begin{gathered} X_1 = \partial_t, \qquad X_2 = t\partial_\varphi \qquad X_3 = \partial_X - k_1t^2\partial_\varphi \\ X_4 = t\partial_t + \gamma X\partial_X + \Big(\frac{k_2\gamma}{2}t^2 + \varphi(\gamma+2) + 2k_4X\Big)\partial_\varphi \end{gathered} \tag{A.116}$$

and

$$P(X, \varphi_X) = \beta(\varphi_X + k_4)^\gamma + k_1 X^2 + k_2 X + k_3 \tag{A.117}$$

where $\beta \neq 0$, $\gamma \neq 0\,,1$ and $k_1 \neq 0$.

By virtue of the equivalence transformation corresponding to the generators X^e_{15}, X^e_2 and X^e_8, one can assume $k_2 = 0$, $k_3 = 0$, and $k_4 = 0$.

For $\gamma \neq -2$, therefore the extensions of the kernel and the pressure function are

$$\begin{gathered} X_1 = \partial_X - k_1 t^2 \partial_\varphi, \qquad X_1 = t\partial_t + \gamma X \partial_X + \big(\gamma + 2\big)\varphi\partial_\varphi \\ P(X, \varphi_X) = \beta\varphi_X^\gamma + k_1 X^2 \end{gathered} \tag{A.118}$$

where $\beta \neq 0$, $\gamma \neq 0, 1, -2, -3,$ and $k_1 \neq 0$. This case is equivalent to the generator in equation (A.112).

- $\gamma + 3 = 0$

Substituting $\gamma = -3$, there exists a solution of the determining equations as

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = -4k_5, \qquad k_7 = -ko_3 k_{11} - \frac{3ko_4}{2} k_5$$

$$\eta^{00}_X = -a_X k_{11} + k_5\big(3Xa_X + 2a(X)\big), \qquad \eta^{01}_X = 0, \qquad \xi^t = k_5 t + k_6,$$

$$\xi^X = -3Xk_5 + k_{11}, \qquad \eta = -k_{11}ko_3 t^2 - \frac{3ko_4}{2} t^2 k_5 - \varphi k_5 + t\eta^{01} + \eta^{00}.$$

with $k_{10} = \eta^{01}$; then the generator corresponding to these coefficients is

$$X = k_6 X_1 + k_{10} X_2 + k_{11} X_3 + k_5 X_4.$$

The pressure function is given by

$$P(X, \varphi_X) = ko_2(a(X) + \varphi_X)^\gamma + ko_3 X^2 + ko_4 X + ko_5.$$

Letting $ko_2 = \beta$, $ko_3 = k_1$, $ko_4 = k_2$, $ko_5 = k_3$ and $\int a(X)dX = \tilde{a}(X)$ or $a(X) = \tilde{a}'(X)$, finally, the basis generators and the pressure function can be written as

$$X_1 = \partial_t, \qquad X_2 = t\partial_\varphi \qquad X_3 = \partial_X - \big(k_1 t^2 + \tilde{a}'(X)\big)\partial_\varphi$$
$$X_4 = t\partial_t - 3X\partial_X + \Big(\frac{-3k_2}{2}t^2 - \varphi + 3X\tilde{a}'(X) - \tilde{a}(X)\Big)\partial_\varphi \tag{A.119}$$

and

$$P(X, \varphi_X) = \beta(\varphi_X + \tilde{a}'(X))^{-3} + k_1 X^2 + k_2 X + k_3 \tag{A.120}$$

where $\beta \neq 0$, and $k_1 \neq 0$.

By applying the equivalence transformations corresponding to the generators X^e_{15}, X^e_2 and X^e_8, one can assume $k_2 = 0$, $k_3 = 0$, and $\tilde{a}'(X) = 0$.

Then the extensions of the kernel and the pressure function are

$$X_{21} = \partial_X - k_1 t^2 \partial_\varphi, \qquad X_{22} = t\partial_t - 3X\partial_X - \varphi\partial_\varphi \tag{A.121}$$

$$P(X, \varphi_X) = \beta\varphi_X^{-3} + k_1 X^2$$

where $\beta \neq 0$,and $k_1 \neq 0$. This case is presented as a model M_{14} in Table 5.2.

Case II $c_{XX} = 0$

The general solution of $c_{XX} = 0$ is $c(X) = ko_3 X + ko_4$. Substituting all conditions into (5.33), one gets

$$k_4 ko_2 \gamma\big(\gamma + 3\big)\big(\gamma - 1\big)\big(a(X) + \varphi_X\big)^\gamma = 0.$$

Case II.1 $\gamma + 3 \neq 0$

Since $\gamma\big(\gamma - 1\big) \neq 0$, $\big(a(X) + \varphi_X\big)^\gamma \neq 0$, and $ko_2 \neq 0$, then $k_4 = 0$. Substituting all relations into (5.29), it becomes $\xi^{XX} = 0$, that is

$\xi^X = k_9 X + k_{11}$, k_9, k_{11} is constant (and not equal to zero).

A solution of the determining equations is

$$k_4 = 0, \qquad k_8 = 0, \qquad kk3 = \frac{-2k_5\big(\gamma^2-1\big)+2k_9\big(\gamma^2-1\big)}{\big(\gamma-1\big)^2},$$

$$k_7 = \frac{-2k_5\gamma ko_3\big(\gamma-1\big)^3 + k_9 ko_3\big(\gamma^4-2\gamma^3+2\gamma-1\big)}{2\big(\gamma-1\big)^4},$$

$$\eta_X^{00} = -k_{11}a_X + k_5\big(\frac{-2a(X)}{\gamma-1}\big) + k_9\Big(-a_X X + \frac{2a(X)}{\gamma-1}\Big), \qquad \eta_X^{01} = 0.$$

$$\xi^t = k_5 t + k_6, \qquad \xi^X = k_9 X + k_{11},$$

$$\eta = \frac{-k_5}{\gamma-1}\Big(\gamma ko_3 t^2 + 2\varphi\Big) + \frac{k_9\big(\gamma+1\big)}{2\big(\gamma-1\big)}\Big(ko_3 t^2 + 2\varphi\Big) + t\eta^{01} + \eta^{00}$$

with $k_{10} = \eta^{01}$, then the generator corresponding to these coefficients is

$$X = k_6X^1 + k_{10}X^2 + k_5X^3 + k_9X^4 + k_{11}X^5.$$

The pressure function is

$$P(X,\varphi_X) = ko_2(a(X)+\varphi_X)^\gamma + ko_3X + ko_4.$$

Letting $ko_2 = \beta$, $ko_3 = k_1$, $ko_4 = k_2$, $\gamma - 1 = \tilde{\gamma}(X)$ and $\int a(X)dX = \tilde{a}(X)$, finally, the basis generators and the pressure function can be written as

$$\begin{aligned} X^1 &= \partial_t, \qquad X^2 = t\partial_\varphi \\ X^3 &= t\partial_t - \frac{1}{\tilde{\gamma}}\Big(2\big(\varphi+\tilde{a}(X)\big) + k_1t^2\big(\tilde{\gamma}+1\big)\Big)\partial_\varphi \\ X^4 &= X\partial_X + \Big(\frac{\tilde{\gamma}+2}{2\tilde{\gamma}}\Big)\Big(2\big(\varphi+\tilde{a}(X)\big) + k_1t^2\Big)\partial_\varphi \\ X^5 &= \partial_X - \tilde{a}'(X)\partial_\varphi \end{aligned} \tag{A.122}$$

and

$$P(X,\varphi_X) = \beta(\varphi_X + \tilde{a}'(X))^{\tilde{\gamma}+1} + k_1X + k_2 \tag{A.123}$$

where $\beta \neq 0$, and $k_1 \neq 0$ and $\tilde{\gamma} \neq 0\,,\,-1$.

By applying the equivalence transformations corresponding to the generators X_{15}^e, X_2^e and X_8^e, one can assume $k_1 = 0$, $k_2 = 0$, and $\tilde{a}(X) = 0$.

Therefore, for $\tilde{\gamma}+1=\gamma$, the extensions of the kernel and the pressure function are

$$X_{23}=(\gamma-1)t\partial_t-2\varphi\partial_\varphi, \qquad X_{24}=(\gamma-1)X\partial_X+(\gamma+1)\varphi\partial_\varphi,$$
$$X_{25}=\partial_X, \qquad \text{and} \qquad P(X,\varphi_X)=\beta\varphi_X^\gamma, \tag{A.124}$$

where $\beta\neq 0$, and $\gamma\neq 0,\ 1$. In Table 5.2, this case is presented as a model M_{15}

Case II.2 $\gamma+3=0$

Substituting $\gamma=-3$ one gets a solution of the determining equations

$$k_8=\frac{-ko_3k_4}{2}, \qquad kk3=-k_5+k_9, \qquad k_7=\frac{-3ko_3}{4}k_5+\frac{ko_3}{4}k_9$$
$$\eta_X^{00}=-a_Xk_{11}+\frac{a(X)}{2}k_5+\frac{k_9}{2}\Big(-2Xa_X-a(X)\Big), \qquad \eta_X^{01}=a(X)k_4.$$
$$\xi^t=k_4t^2+k_5t+k_6, \qquad \xi^X=k_9X+k_{11},$$
$$\eta=k_4\Big(\frac{-ko_3t^3}{2}+t\varphi\Big)+k_5\Big(\frac{-3ko_3t^2}{4}+\frac{\varphi}{2}\Big)+k_9\Big(\frac{ko_3t^2}{4}+\frac{\varphi}{2}\Big)+t\eta^{01}+\eta^{00}$$

and the generator corresponding to these coefficients is

$$X=k_5X^1+k_6X^2+k_{11}X^3+k_9X^4+k_4X^5.$$

The pressure function is given as

$$P(X,\varphi_X)=ko_2(a(X)+\varphi_X)^{-3}+ko_3X+ko_4.$$

Letting $ko_2=\beta$, $ko_3=k_1$, $ko_4=k_2$, and $\int a(X)dX=\tilde{a}(X)$, then the basis generators and the pressure function can be written as

$$X^1=\partial_t, \qquad X^2=\partial_X-\tilde{a}'(X)\partial_\varphi,$$
$$X^3=X\partial_X+\Big(-X\tilde{a}'(X)+\frac{a(X)+\varphi}{2}+\frac{k_1t^2}{4}\Big)\partial_\varphi,$$
$$X^4=t^2\partial_t+\Big(a(X)+\varphi-\frac{k_1t^2}{2}\Big)t\partial_\varphi, \tag{A.125}$$
$$X^5=t\partial_t+\Big(\frac{a(X)+\varphi}{2}-\frac{3k_1t^2}{4}\Big)\partial_\varphi,$$

and

$$P(X,\varphi_X)=\beta(\varphi_X+\tilde{a}'(X))^{-3}+k_1X+k_2 \tag{A.126}$$

where $\beta \neq 0$. By applying the equivalence transformations corresponding to the generators X^e_{15}, X^e_6 and X^e_8, one can assume $k_1 = 0$, $k_2 = 0$, and $\tilde{a}'(X) = 0$.

Therefore the extensions of the kernel and the pressure function are

$$\begin{aligned} X_{26} &= \partial_X, & X_{27} &= 2X\partial_X + \varphi\partial_\varphi, \\ X_{28} &= t^2\partial_t + t\varphi\partial_\varphi, & X_{29} &= 2t\partial_t + \varphi\partial_\varphi, \\ & P(X,\varphi_X) = \beta\varphi_X^{-3} \end{aligned} \tag{A.127}$$

where $\beta \neq 0$. In Table 5.2, this case is presented as a model M_{16}.

A.2.2 Case $\mu_{1X} \neq 0$

As $\mu_{1X} \neq 0$, by equation (A.63), one assumes $\xi^X = 0$. Substituting $\mu_{1_{\varphi_X}} = 0$ and $\xi^X = 0$ into equation (5.36), it becomes

$$4k_5\mu_{1X} = 0. \tag{A.128}$$

As $\mu_{1X} \neq 0$, one gets

$$k_5 = 0.$$

Construct the pressure function by solving equation (5.32)

$$\frac{P_{\varphi_X\varphi_X\varphi_X}P_{\varphi_X}}{{P_{\varphi_X\varphi_X}}^2} = \mu_1(X),$$

the general form of pressure function is

$$P(X,\varphi_X) = b(X)\big(\varphi_X + a(X)\big)^\gamma + c(X), \qquad \gamma \neq 1. \tag{A.129}$$

For $P(X,\varphi_X) = b(X)\big(\varphi_X + a(X)\big)^\gamma + c(X)$, since $P_{\varphi_X\varphi_X} \neq 0$, then one obtains condition

$$\gamma(\gamma-1)b(X) \neq 0.$$

Moreover, equation (5.33) becomes

$$\gamma(\gamma-1)(\gamma+3)b(X)\big(a(X)+\varphi_X\big)^\gamma k_4 = 0.$$

Case I $\gamma + 3 \neq 0$

Assuming $\gamma + 3 \neq 0$ and since $\gamma(\gamma - 1)b(X) \neq 0$, then $k_4 = 0$ and a solution of determining equations is

$$kk3 = 0 \qquad k_5 = 0, \qquad k_7 = 0, \qquad k_8 = 0 \qquad \eta^{00}_X = 0, \qquad \eta^{01}_X = 0$$

$$\xi^t = k_6, \qquad \xi^X = 0, \qquad \eta = t\eta^{01} + \eta^{00}$$

with $k_9 = \eta^{00}$ and $k_{10} = \eta^{01}$. Then the generator corresponding to these coefficients is

$$X = k_6 X_1 + k_9 X_2 + k_{10} X_3.$$

In this case there exists no any extension of the kernel.

Case II $\gamma + 3 = 0$

Substituting $\gamma = -3$, $\mu_{1\varphi_X} = 0$, $\xi^X = 0$, and $P(X, \varphi_X)$, then equation (5.34) is changed to

$$k_4 c_{XX} = 0.$$

For $c_{XX} \neq 0$

Assuming $c_{XX} \neq 0$ then $k_4 = 0$ and a solution of the determining equations is

$$kk3 = 0 \qquad k_5 = 0, \qquad k_7 = 0, \qquad k_8 = 0, \qquad \eta^{00}_X = 0, \qquad \eta^{01}_X = 0.$$

$$\xi^t = k_6, \qquad \xi^X = 0, \qquad \eta = t\eta^{01} + \eta^{00}$$

This case does not have any generator extension.

For $c_{XX} = 0$

The general solution of $c_{XX} = 0$ is $c(X) = ko_1 X + ko_2$. Substituting all relations, one obtains a solution of the determining equations as

$$kk3 = 0 \qquad k_5 = 0, \qquad k_7 = 0, \qquad k_8 = \frac{-ko_1 k_4}{2},$$

$$\eta^{00}_X = 0, \qquad \eta^{01}_X = a(X)k_4 \qquad \xi^t = k_4 t^2 + k_6, \qquad \xi^X = 0,$$

$$\eta = \frac{-ko_1 t^3}{2} k_4 + t\varphi k_4 + t\eta^{01} + \eta^{00}.$$

with $k_{10} = \eta^{00}$; then the generator corresponding to these coefficients is

$$X = k_6 X_1 + k_{10} X_2 + k_4 X_3.$$

Letting $ko_1 = k_1$, $ko_2 = k_2$, and $\int a(X)dX = \tilde{a}(X)$, then the basis generators and the pressure function are presented as

$$X_1 = \partial_t, \qquad X_2 = \partial_\varphi, \qquad X_3 = t^2\partial_t + \Big(\frac{-k_1 t^2}{2} + \varphi + \tilde{a}(X)\Big)t\partial_\varphi \quad \text{(A.130)}$$

and

$$P(X, \varphi_X) = b(X)(\varphi_X + \tilde{a}'(X))^{-3} + k_1 X + k_2. \quad \text{(A.131)}$$

By virtue of the equivalence transformations corresponding to the generators X_{15}^e, X_2^e, X_8^e, one can assume that $\tilde{a}(X) = 0$, $k_1 = 0$, and $k_2 = 0$.

Therefore the extensions of the kernel and the pressure function are

$$X_{30} = t^2\partial_t + t\varphi\partial_\varphi, \qquad P(X, \varphi_X) = b(X){\varphi_X}^{-3}. \quad \text{(A.132)}$$

The result of this case is presented in Table 5.2 as the model M_{17}.

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APPENDIX B

CONSERVATION LAWS

Details of constructing the conservation laws of the gas dynamic equations for all extensions of the kernel of the admitted Lie algebras are presented here.

The extension of the kernel in M_2 is given by the generator

$$X_5 = \beta t\partial_t - \partial_X + \alpha\varphi\partial_\varphi.$$

Substituting the Lagrangian into equation (6.4), one obtains

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t \\ -\varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} - \varphi_{XX} B^2_{\varphi_X} - B^2_X \\ -\varphi_X h'(X) + (\alpha - \frac{\beta}{2})\varphi_t^2 + (\alpha+\beta)\varphi_X h(X) \\ +e^{2(\beta-\alpha)X}\Phi(Z)\big(2\alpha - \beta\big) = 0. \end{aligned} \tag{B.1}$$

Solving equation (B.1) for B^i, one finds the condition $\beta = 2\alpha$, and it satisfies

$$B^1 = -t\varphi_X\big(h'(X) - 3\alpha h(X))\big), \qquad B^2 = t\varphi_t\big(h'(X) - 3\alpha h(X)\big).$$

This symmetry is divergent. Using Noether's theorem, the conserved vectors are

$$\begin{aligned} C^t &= -t\varphi_X h'(X) - 2\alpha t e^{2\alpha X}\Phi(Z) - \alpha\varphi\varphi_t + \alpha t\varphi_X h(X) + \alpha t\varphi_t^2 - \varphi_t\varphi_X, \\ C^X &= t\varphi_t h'(X) - \alpha\varphi h(X) - \alpha t\varphi_t h(X) + \frac{1}{2}\varphi_t^2 + e^{2\alpha X}\Phi(Z) \\ &\quad + \Big(-\alpha\varphi + 2\alpha t\varphi_t - \varphi_X\Big) e^{3\alpha X}\Phi'(Z). \end{aligned}$$

The extension of the kernel in M_3 is given by the generator

$$X_6 = t\partial_t + X\partial_X + \big(\alpha+1\big)\varphi\partial_\varphi.$$

Substituting the Lagrangian into equation (6.4), one obtains

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} - \varphi_{XX} B^2_{\varphi_X} - B^2_X \\ +X\varphi_X h'(X) + 2(\alpha+1)X^{2\alpha}\Phi(Z) + (\alpha+1)\varphi_t^2 + (\alpha+2)\varphi_X h(X) = 0. \end{aligned} \tag{B.2}$$

Solving equation (B.2) for B^i, one finds the condition $\alpha = -1$, which satisfies

$$B^1 = t\varphi_X\big(h'(X) + h(X))\big), \qquad B^2 = -t\varphi_t\big(h'(X) + h(X)\big).$$

The symmetry is divergent. Using Noether's theorem, the conserved vectors are

$$\begin{aligned} C^t &= tX\varphi_X h'(X) + \frac{1}{2}t\varphi_t^2 + X\varphi_t\varphi_X - \frac{t}{X^2}\Phi(Z), \\ C^X &= -tX\varphi_t h'(X) - \frac{1}{2}X\varphi_t^2 - \frac{1}{X}\Phi(Z) + \big(\frac{t}{X}\varphi_t + \varphi_X\big)\Phi'(Z). \end{aligned}$$

The extension of the kernel in M_4 is given by the generator

$$X_7 = \partial_X - \gamma t^2 \partial_\varphi.$$

Substituting the Lagrangian into equation (6.4), one finds

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} \\ -\varphi_{XX}B^2_{\varphi_X} - B^2_X + \beta\varphi_X - 2\gamma t\varphi_t + 2\gamma X\varphi_X = 0. \end{aligned} \tag{B.3}$$

Solving equation (B.3) then vectors B^i are derived

$$B^1 = 2\gamma tX\varphi_X, \qquad B^2 = -2\gamma tX\varphi_t + \beta\varphi.$$

This symmetry is divergent. Using Noether's theorem, then the conserved vectors are

$$\begin{aligned} C^t &= \varphi_t\varphi_X + \gamma t^2\varphi_t + 2\gamma tX\varphi_X, \\ C^X &= \beta\varphi + \beta\gamma t^2 X - \frac{1}{2}\varphi_t^2 - 2\gamma tX\varphi_t + \gamma^2t^2X^2 - \Phi(Z) \\ &\quad + \big(\varphi_X + \gamma t^2\big)\Phi'(Z). \end{aligned}$$

The extension of the kernel in M_5 is given by the generator

$$X_8 = \partial_X - \gamma t^2 \partial_\varphi.$$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} \\ -\varphi_{XX}B^2_{\varphi_X} - B^2_X + \varphi_t^2 + 2\Phi(Z) = 0. \end{aligned} \tag{B.4}$$

In this case one can not find vector Bi; therefore this case does not provide conservation laws.

Next consider the extension of the kernel in M_5, which is given by the generator

$$X_9 = \partial_X.$$

The determining equation for vector B^i is

$$\varphi_t B^1_\varphi + \varphi_{tt} B^1_{\varphi_t} + \varphi_{tX} B^1_{\varphi_X} + B^1_t + \varphi_X B^2_\varphi + \varphi_{tX} B^2_{\varphi_t} + \varphi_{XX} B^2_{\varphi_X} + B^2_X = 0. \quad \text{(B.5)}$$

Solving this equation, one finds $B^1 = 0$ and $B^2 = 0$. This symmetry is called a variational symmetry and the conservation laws are

$$C^t = \varphi_t \varphi_X, \qquad C^X = \frac{-1}{2}\varphi_t^2 - \Phi(\varphi_X) + \varphi_X \Phi'(Z).$$

The extension of the kernel in M_6 is given by the generator

$$X_{10} = \big(\beta + \alpha(\gamma - 1)\big)t\partial_t - 2\gamma\partial_X + 2(\beta - \alpha)\varphi\partial_\varphi.$$

There are 2 cases to be considered, which are $\gamma = -1$ and $\gamma \neq -1$.

Case 1. $\gamma = -1, \quad W(X, \varphi_X) = -\varphi_X^{-1}\ln(\varphi_X)e^{\beta X} - \frac{k_2}{\alpha^2}e^{\alpha X}.$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned} &-\varphi_t B^1_\varphi + \big(\frac{k_2}{2}e^{\alpha X} + \frac{e^{\beta X}}{\varphi_X^2}(\beta\varphi_X - \varphi_{XX})\big)B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t \\ &-\varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X - \frac{k_2}{\alpha^2}(2\alpha - 3\beta)e^{\alpha X}\varphi_X \\ &-\big((2\alpha - 3\beta)\ln\varphi_X + 2(\alpha - \beta)\big)e^{\beta X} - \frac{(2\alpha - 3\beta)}{2}\varphi_t^2 = 0. \end{aligned} \quad \text{(B.6)}$$

Solving the above equation for a particular condition $\beta = \frac{2\alpha}{3}$, one finds $B^1 = 0$ and $B^2 = -e^{\frac{2\alpha X}{3}}$. The conserved vectors are

$$\begin{aligned} C^t &= \frac{2}{3\alpha}\Big(2\alpha^2 t\ln\varphi_X e^{\frac{2\alpha X}{3}} + 2k_2 t\varphi_X e^{\alpha X} + \alpha^2\varphi\varphi_t - \alpha^2 t\varphi_t^2 + 3\alpha\varphi_t\varphi_X\Big), \\ C^X &= \frac{1}{3\alpha\varphi_X}\Big(\alpha e^{\frac{2\alpha X}{3}}\big(-6\varphi_X\ln\varphi_X + 2\alpha\varphi - 4\alpha t\varphi_t + 3\varphi_X\big) \\ &\quad + 2k_2 e^{\alpha X}\big(\varphi\varphi_X - 2t\varphi_t\varphi_X\big) - 3\alpha\varphi_t^2\varphi_X\Big). \end{aligned}$$

Moreover, for another condition $\beta = 2\alpha$, one obtains

$$B^1 = 2(-2\alpha t e^{2\alpha X} + \alpha\varphi\varphi_t - \varphi_t\varphi_X),$$
$$B^2 = \frac{1}{\alpha\varphi_X}\Big((2\alpha\varphi_X \ln\varphi_X + 2\alpha^2\varphi)e^{2\alpha X} + 2k_2\varphi\varphi_X e^{\alpha X} + \alpha\varphi_t^2\varphi_X\Big).$$

Using Noether's theorem, the conserved vectors are

$$C^t = -4\alpha t e^{2\alpha X}, \qquad C^X = 2e^{2\alpha X}.$$

Case 2. $\gamma \neq -1, \quad W(X,\varphi_X) = -\frac{\varphi_X^\gamma}{(\gamma+1)}e^{\beta X} - \frac{k_2}{\alpha^2}e^{\alpha X}$

Here, the determining equation for vector B^i is

$$\begin{aligned}
-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t \\
-\varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X \\
+(\frac{3\beta - \alpha(\gamma+3) + 2\beta(\alpha-\gamma)}{\gamma+1})e^{\beta X}\varphi_X^{\gamma+1} \\
+k_2(\frac{3\beta-\alpha(\gamma+3)}{\alpha^2})e^{\alpha X}\varphi_X + \big(3\beta - \alpha(\gamma+3)\big)\frac{\varphi_t^2}{2} = 0.
\end{aligned} \tag{B.7}$$

Solving this equation for a particular case $\beta = \frac{\alpha(\gamma+3)}{3}$, one gets a variational symmetry such that $B^1 = 0$ and $B^2 = 0$. Using Noether's theorem, the conserved vectors are

$$C^t = \frac{2\gamma}{3\alpha}\big((\frac{-2\alpha^2}{\gamma+1})te^{\frac{\alpha X(\gamma+3)}{3}}\varphi_X^{\gamma+1} - 2k_2 t\varphi_X e^{\alpha X} - \alpha^2\varphi\varphi_t + \alpha^2 t\varphi_t^2 - 3\alpha\varphi_t\varphi_X\big),$$
$$C^X = \frac{\gamma}{3\alpha}\big((\frac{-6\alpha\gamma}{\gamma+1})e^{\frac{\alpha X(\gamma+3)}{3}}\varphi_X^{\gamma+1} + \big(2\alpha^2 e^{\frac{\alpha X(\gamma+3)}{3}}\varphi_X^\gamma + 2k_2 e^{\alpha X}\big)\big(2t\varphi_t - \varphi\big)$$
$$+ 3\alpha\varphi_t^2\big).$$

The extensions of the kernel in M_7 are given by the generators

$$X_{11} = (\gamma-1)\partial_X - \beta\varphi\partial_\varphi, \qquad X_{12} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi.$$

In this case we will consider 2 cases, $\gamma = -1$ and $\gamma \neq -1$.

Case 1. $\gamma = -1, \quad W(X,\varphi_X) = \varphi_X^{-1}\ln(\varphi_X)e^{\beta X}$

I. $X_{11} = (\gamma-1)\partial_X - \beta\varphi\partial_\varphi$,

Substituting the Lagrangian into equation (6.4), one gets

$$-\varphi_t B^1_\varphi + \big(\frac{\beta}{\varphi_X} - \frac{\varphi_{XX}}{\varphi_X^2}\big)e^{\beta X}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t}$$
$$-\varphi_{XX}B^2_{\varphi_X} - B^2_X - \beta\big(2\ln\varphi_X + 1\big)e^{\beta X} - \beta\varphi_t^2 = 0. \tag{B.8}$$

Solving equation (B.8), one finds X_{11} is not divergent. Hence, it does not provide a conservation law.

II. $X_{12} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$,

The determining equation for vector B^i is

$$-\varphi_t B^1_\varphi + \big(\frac{\beta}{\varphi_X} - \frac{\varphi_{XX}}{\varphi_X^2}\big)e^{\beta X}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t}$$
$$-\varphi_{XX}B^2_{\varphi_X} - B^2_X - 2\big(\ln\varphi_X + 1\big)e^{\beta X} - \varphi_t^2 = 0. \tag{B.9}$$

This extension generator is not divergent either.

Case 2. $\gamma \neq -1, \quad W(X,\varphi_X) = -\frac{\varphi_X^\gamma}{(\gamma+1)}e^{\beta X}$

I. $X_{11} = (\gamma-1)\partial_X - \beta\varphi\partial_\varphi$,

The determining equation is

$$-\varphi_t B^1_\varphi + \big(\beta\varphi_X^\gamma + \gamma\varphi_X^{\gamma-1}\varphi_{XX}\big)e^{\beta X}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi$$
$$-\varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X - \frac{2\beta}{\gamma+1}\varphi_X^{\gamma+1}e^{\beta X} - \beta\varphi_t^2 = 0. \tag{B.10}$$

In this case one cannot find B^i, thus a conservation law cannot be constructed.

II. $X_{12} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$,

The determining equation is

$$-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t}$$
$$-\varphi_{XX}B^2_{\varphi_X} - B^2_X - \big(\frac{\gamma+3}{\gamma+1}\big)\varphi_X^{\gamma+1}e^{\beta X} - \big(\frac{\gamma+3}{2}\big)\varphi_t^2 = 0. \tag{B.11}$$

For the condition $\gamma = -3$, there exists a variational symmetry and the conserved vectors are

$$C^t = \frac{2}{\varphi_X^2}\Big(-te^{\beta X} + \varphi\varphi_t\varphi_X^2 - t\varphi_t^2\varphi_X^2\Big), \qquad C^X = \frac{2e^{\beta X}\Big(\varphi - 2t\varphi_t\Big)}{\varphi_X^3}.$$

The extension of the kernel in M_8 is given by the generator

$$X_{13} = \big(\gamma(1+m+2l)-m\big)t\partial_t + (2\gamma l)X\partial_X + 2(\gamma l - m)\varphi\partial_\varphi$$

There are 2 cases of the Lagrangian to be considered : $\gamma = -1$ and $\gamma \neq -1$.

Case 1. $\gamma = -1, \quad W(X,\varphi_X) = -\varphi_X^{-1}\ln(\varphi_X)b(X) - c(X)$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} \\ -B^2_X - k_1(4l+2m-1)\varphi_X b^{m+1}(x) - (4l+2m-1)\ln\varphi_X b(X) \\ -2mb(X) - (2l+m)\varphi_t^2 + \frac{1}{2}\varphi_t^2 = 0. \end{aligned} \tag{B.12}$$

Solving the above equation, one finds the condition $m = \frac{4l-1}{-2}$, which is satisfied when $B^1 = t(4l-1)b(X)$ and $B^2 = 0$. Using Noether's theorem, one derives the conserved vectors as

$$\begin{aligned} C^t &= 2(1-l)k_1 l^2\beta^2 tX^2\varphi_X b^{\frac{3}{2}}(X) + 2(1-l)t\ln\varphi_X b(X) + (4l-1)tb(X) \\ &\quad + (1-2l)\varphi\varphi_t - (1-l)t\varphi_t^2 - 2lX\varphi_t\varphi_X, \\ C^X &= k_1 l^2\beta^2 X^2 b^{\frac{3}{2}}(X)\Big((1-2l)\varphi - 2(1-l)t\varphi_t\Big) \\ &\quad + \frac{(1-2l)}{\varphi_X}\varphi b(X) - \frac{2(1-l)}{\varphi_X}t\varphi_t b(X) + 2lXb(X)\big(\ln\varphi_X - 1\big) + lX\varphi_t^2. \end{aligned}$$

Case 2. $\gamma \neq -1, \quad W(X,\varphi_X) = -\frac{\varphi_X^\gamma}{(\gamma+1)}b(X) - c(X)$

The determining equation for the vectors B^i is

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X \\ + \frac{1}{\gamma+1}\Big(k_1\big(\gamma^2(4l-m-1) + 4\gamma l - 4\gamma m - \gamma - 3m\big)\varphi_X b^{m+1}(X) \\ + \big(\gamma^2(4l-m-1) + 4\gamma l - 4\gamma m - \gamma - 3m\big)\frac{\varphi_t^2}{2} \\ + (4\gamma l - \gamma m - \gamma - 3m)\varphi_X^{\gamma+1}b(X)\Big) = 0. \end{aligned} \tag{B.13}$$

In this case, the symmetry is not divergent, hence, it can not provide a conservation law.

The extensions of the kernel in M_9 are given by the generators

$$X_{14} = l(\gamma-1)X\partial_X + \big(l(\gamma+1)+1\big)\varphi\partial_\varphi, \qquad X_{15} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi.$$

There are 2 cases of the Lagrangian to be considered : $\gamma=-1$ and $\gamma\neq-1$,

Case 1. $\gamma=-1$, $W(X,\varphi_X) = -\varphi_X^{-1}\ln(\varphi_X)b(X)$

I. $X_{14} = l(\gamma-1)X\partial_X + \big(l(\gamma+1)+1\big)\varphi\partial_\varphi$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned}-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X}\\ -B^2_X - 2(l-1)\ln\varphi_X b(X) + (2l+1)b(X) - (l-1)\varphi_t^2 = 0.\end{aligned} \tag{B.14}$$

For the particular condition $l=1$, one obtains $B^1 = \frac{-3t}{\beta X}$ and $B^2=0$. The conserved vectors are

$$C^t = -\varphi\varphi_t - 2X\varphi_t\varphi_X - \frac{3t}{\beta X}, \qquad C^X = \frac{-2}{\beta}\ln\varphi_X + X\varphi_t^2 + \frac{1}{\beta}\big(\varphi X^{-1}\varphi_X^{-1}+2\big).$$

II. $X_{15} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$

The determining equation is

$$\begin{aligned}-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X}\\ -B^2_X - 2\big(\ln\varphi_X + 1\big)b(X) - \varphi_t^2 = 0.\end{aligned} \tag{B.15}$$

In this case the symmetry is not divergent, therefore conservation laws cannot be obtained.

Case 2. $\gamma\neq-1$, $W(X,\varphi_X) = -\frac{\varphi_X^\gamma}{(\gamma+1)}b(X)$

I. $X_{14} = l(\gamma-1)X\partial_X + \big(l(\gamma+1)+1\big)\varphi\partial_\varphi$

The determining equation is

$$\begin{aligned}-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X}\\ -B^2_X + \Big(\frac{3\gamma l + l + 2}{\gamma+1}\Big)\varphi_X^{\gamma+1}b(X) + \big(\frac{3\gamma l + l + 2}{2}\big)\varphi_t^2 = 0.\end{aligned} \tag{B.16}$$

The symmetry satisfying this case is a variational symmetry and the conserved vectors are found as

$$C^t = \left(\frac{1-\gamma}{3\gamma+1}\right)\big(\varphi\varphi_t + 2X\varphi_t\varphi_X\big),$$
$$C^X = \frac{(3\gamma+1)^{-3(\gamma+1)/2}}{\gamma+1}(2\beta X)^{(3\gamma+1)/2}\big((1-\gamma^2)\varphi\varphi_X^{\gamma} + 2\gamma(1-\gamma)X\varphi_X^{\gamma+1}\big) + \left(\frac{\gamma-1}{3\gamma+1}\right)X\varphi_t^2.$$

II. $X_{15} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$

The determining equation is

$$-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X - \left(\frac{\gamma+3}{\gamma+1}\right)\varphi_X^{\gamma+1}b(X) - \left(\frac{\gamma+3}{2}\right)\varphi_t^2 = 0. \tag{B.17}$$

For the particular condition $\gamma = -3$, there exists a variational symmetry such that $B^1 = 0$ and $B^2 = 0$, and the conserved vectors are

$$C^t = 2\varphi_X^{-2}\big(-tb(X) + \varphi\varphi_t\varphi_X^2 - t\varphi_t^2\varphi_X^2\big), \qquad C^X = 2b(X)\big(\varphi - 2t\varphi_t\big)\varphi_X^{-3}.$$

The extension of the kernel in M_{10} is given by the generator

$$X_{16} = (2l-1)t\partial_t + 2l\gamma X\partial_X + 2\big(2l-1+l\gamma\big)\varphi\partial_\varphi.$$

This model has 2 cases of the Lagrangian, when $\gamma = -1$ and $\gamma \neq -1$.

Case 1. $\gamma = -1$, $W(X,\varphi_X) = -\varphi_X^{-1}\ln(\varphi_X)b(X) - k_1X^2$

Substituting the Lagrangian into equation (6.4), one gets

$$-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X - 3\ln\varphi_X b(X) + 2(2l-1)b(X) - 3k_1X^2\varphi_X - \frac{3}{2}\varphi_t^2 = 0. \tag{B.18}$$

The symmetry is not divergent, then it can not provide a conservation law.

Case 2. $\gamma \neq -1$, $W(X,\varphi_X) = -\frac{\varphi_X^{\gamma}}{(\gamma+1)}b(X) - k_1X^2$

The determining equation for vector B^i is

$$-\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} - \varphi_{XX} B^2_{\varphi_X} - B^2_X$$
$$+3\big(2l(\gamma+1)-1\big)\big((\frac{1}{\gamma+1})\varphi_X^{\gamma+1} b(X) + \frac{1}{2}\varphi_t^2 + k_1 X^2 \varphi_X\big) = 0. \quad \text{(B.19)}$$

For a particular condition $l = \frac{1}{2(\gamma+1)}$, one obtains $B^i = 0$. This symmetry is variational and the conserved vectors are

$$C^t = \frac{\gamma}{4(\gamma+1)^4}\big((\frac{\beta}{2(\gamma+1)})^{2\gamma}\beta^2 t X^{2\gamma+2}\varphi_X^{\gamma+1}$$
$$+ \big(\gamma^2(\gamma+3)+3\gamma+1\big)\big(4\varphi\varphi_t - 2t\varphi_t^2 + 4X\varphi_t\varphi_X + 4k_1 t X^2 \varphi_X\big)\big),$$
$$C^X = \frac{\gamma X}{4(\gamma+1)^4}\big((\frac{\beta}{2(\gamma+1)})^{2\gamma}(\gamma+1)\beta^2 X^{2\gamma+1}\varphi_X^{\gamma}\big(\varphi - t\varphi_t\big) \quad \text{(B.20)}$$
$$+ (\frac{\beta}{2(\gamma+1)})^{2\gamma}\gamma\beta^2 X^{2\gamma+2}\varphi_X^{\gamma+1}$$
$$+ \big(\gamma^2(\gamma+3)+3\gamma+1\big)\big(4k_1 X\varphi - 4k_1 t X\varphi_t - 2\varphi_t^2\big)\big).$$

The extension of the kernel in M_{11} is given by the generator

$$X_{17} = -\beta t\partial_t + 2\gamma\partial_X - 2\beta\varphi\partial_\varphi.$$

This model has 2 cases of the Lagrangian, when $\gamma = -1$ and $\gamma \neq -1$.

Case 1. $\gamma = -1$, $W(X, \varphi_X) = -\varphi_X^{-1}\ln(\varphi_X)e^{\beta X} - k_1 X^2$

Substituting the Lagrangian into equation (6.4), one gets

$$-\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} - \varphi_{XX} B^2_{\varphi_X}$$
$$-B^2_X - \beta\big(3\ln\varphi_X + 2\big)e^{\beta X} - k_1\big(3\beta X + 4\big)X\varphi_X - \frac{3\beta}{2}\varphi_t^2 = 0. \quad \text{(B.21)}$$

In this case the symmetry is not divergent, hence, it can not provide a conservation law.

Case 2. $\gamma \neq -1$, $W(X, \varphi_X) = -\frac{\varphi_X^\gamma}{(\gamma+1)}e^{\beta X} - k_1 X^2$

The determining equation is

$$-\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} - \varphi_{XX} B^2_{\varphi_X} - B^2_X$$
$$-(\frac{3\beta}{\gamma+1})\varphi_X^{\gamma+1}e^{\beta X} - \frac{3\beta}{2}\varphi_t^2 - 3k_1\beta X^2\varphi_X + 4k_1\gamma X\varphi_X = 0. \quad \text{(B.22)}$$

In this case the symmetry is not divergent, hence, it can not provide a conservation law.

The extension of the kernel in M_{12} is given by the generator

$$X_{18} = \big(\gamma(1-\alpha)+\alpha\big)t\partial_t + 2\gamma X\partial_X + 2(\gamma+\alpha)\varphi\partial_\varphi.$$

This model has 2 cases of the Lagrangian when $\gamma=-1$ and $\gamma\neq -1$.

Case 1. $\gamma=-1,\quad W(X,\varphi_X) = -k_1X\varphi_X^{-1}\ln(\varphi_X) - k_2X^{\alpha+1}$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned}-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X \\ +(2\alpha-5)\big(k_2\varphi_X X^{\alpha+1} + k_1X\ln\varphi_X + \frac{1}{2}\varphi_t^2\big) + 2\alpha k_1 X = 0.\end{aligned} \tag{B.23}$$

For the condition $\alpha=\frac{5}{2}$ one finds $B^1 = 5k_1tX$ and $B^2=0$. The conserved vectors are

$$\begin{aligned}C^t &= -4k_2tX^{7/2}\varphi_X - 4k_1tX\ln\varphi_X - 3\varphi\varphi_t + 2t\varphi_t^2 - 2X\varphi_t\varphi_X + 5k_1tX,\\ C^X &= -3k_2X^{7/2}\varphi + 4k_2tX^{7/2}\varphi_t + 2k_1X^2\ln\varphi_X - \frac{3k_1X\varphi}{\varphi_X}\\ &\quad + X\varphi_t^2 + \frac{4k_1tX\varphi_t}{\varphi_X} - 2k_1X^2.\end{aligned}$$

Case 2. $\gamma\neq -1,\quad W(X,\varphi_X) = -k_1X\frac{\varphi_X^\gamma}{(\gamma+1)} - k_2X^{\alpha+1}$

The determining equation for vector B^i is

$$\begin{aligned}-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X \\ +(\alpha\gamma+3\alpha+5\gamma)\Big(\big(\frac{k_1}{\gamma+1}\big)X\varphi_X^{\gamma+1} + k_2X^{\alpha+1}\varphi_X + \frac{1}{2}\varphi_t^2\Big) = 0.\end{aligned} \tag{B.24}$$

The symmetry is a variational symmetry for the condition $\alpha=\frac{-5\gamma}{\gamma+3}$. The conserved vectors are given as follow

$$\begin{aligned}C^t &= \big(\frac{-2k_2\gamma(3\gamma-1)}{(\gamma+1)(\gamma+3)}\big)tX\varphi_X^{\gamma+1} - \big(\frac{2k_2\gamma(3\gamma-1)}{\gamma+3}\big)tX^{\frac{3-4\gamma}{\gamma+3}}\varphi_X\\ &\quad - \big(\frac{2\gamma(\gamma-2)}{\gamma+3}\big)\varphi\varphi_t + \big(\frac{\gamma(3\gamma-1)}{\gamma+3}\big)t\varphi_t^2 + 2\gamma X\varphi_t\varphi_X,\\ C^X &= \big(\frac{-2k_1\gamma(\gamma-2)}{\gamma+3}\big)X\varphi\varphi_X^\gamma + \big(\frac{2k_1\gamma(3\gamma-1)}{\gamma+3}\big)tX\varphi_t\varphi_X^\gamma\\ &\quad + \big(\frac{2k_1\gamma^2}{\gamma+1}\big)X^2\varphi_X^{\gamma+1} - \big(\frac{2k_2\gamma(\gamma-2)}{\gamma+3}\big)X^{\frac{3-4\gamma}{\gamma+3}}\varphi + \big(\frac{2k_2\gamma(3\gamma-1)}{\gamma+3}\big)tX^{\frac{3-4\gamma}{\gamma+3}}\varphi_t.\end{aligned}$$

The extensions of the kernel in M_{13} are given by the generators

$$X_{19} = t\partial_t + \gamma X\partial_X + (\gamma+2)\varphi\partial_\varphi, \qquad X_{20} = \partial_X - k_1 t^2\partial_\varphi.$$

This model has 2 cases of the Lagrangian, when $\gamma = -1$ and $\gamma \neq -1$.

Case 1. $\gamma = -1, \quad W(X,\varphi_X) = -\beta\varphi_X^{-1}\ln(\varphi_X) - k_1X^2$

I. $X_{19} = t\partial_t + \gamma X\partial_X + (\gamma+2)\varphi\partial_\varphi$

The determining equation for vector B^i is

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi \\ -\varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X + 2\beta = 0. \end{aligned} \tag{B.25}$$

Solving the above equation for B^i, one obtains a divergent symmetry. Using Noether's theorem, the conserved vector are

$$\begin{aligned} C^t &= -\beta t\ln\varphi_X + 2\beta t - \varphi\varphi_t + \frac{1}{2}t\varphi_t^2 - X\varphi_t\varphi_X - k_1tX^2\varphi_X, \\ C^X &= \beta X\ln\varphi_X - \beta\varphi_X^{-1}\big(\varphi - t\varphi_t\big) - \beta X - k_1X^2\varphi + \frac{1}{2}X\varphi_t^2 + k_1tX^2\varphi_t. \end{aligned}$$

II. $X_{20} = \partial_X - k_1t^2\partial_\varphi$

The determining equation is

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} \\ -\varphi_{XX}B^2_{\varphi_X} - B^2_X - 2k_1\big(t\varphi_t - X\varphi_X\big) = 0. \end{aligned} \tag{B.26}$$

Solving this equation, one finds

$$B^1 = 2k_1tX\varphi_X, \qquad B^2 = -2k_1tX\varphi_t.$$

The conserved vectors are obtained as

$$\begin{aligned} C^t &= \varphi_t\varphi_X + k_1t^2\varphi_t + 2k_1tX\varphi_X, \\ C^X &= -\beta\ln\varphi_X + \beta + k_1\beta t^2\varphi_X^{-1} - \frac{1}{2}\varphi_t^2 - 2k_1tX\varphi_t + k_1^2t^2X^2. \end{aligned}$$

Case 2. $\gamma \neq -1, \quad W(X,\varphi_X) = -\frac{\beta\varphi_X^\gamma}{(\gamma+1)} - k_1X^2$

I. $X_{19} = t\partial_t + \gamma X\partial_X + (\gamma+2)\varphi\partial_\varphi$

The determining equation is

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} \\ -B^2_X + 3\beta\varphi_X^{\gamma+1} + \frac{3(\gamma+1)}{2}\varphi_t^2 + 3(\gamma+1)k_1X^2\varphi_X = 0. \end{aligned} \tag{B.27}$$

In this case the symmetry is not divergent, therefore conserved vectors can not be constructed.

II. $X_{20} = \partial_X - k_1t^2\partial_\varphi$

The determining equation is

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi \\ -\varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X - 2k_1\big(t\varphi_t - X\varphi_X\big) = 0. \end{aligned} \tag{B.28}$$

Solving the above equation, one obtains a divergent symmetry such that $B^1 = 2k_1tX\varphi_X$ and $B^2 = -2k_1tX\varphi_t$ and the conserved vectors

$$\begin{aligned} C^t &= \varphi_t\varphi_X + k_1t^2\varphi_t + 2k_1tX\varphi_X, \\ C^X &= \frac{\beta\gamma}{\gamma+1}\varphi_X^{\gamma+1} + \beta k_1t^2\varphi_X^{\gamma} - \frac{1}{2}\varphi_t^2 - 2k_1tX\varphi_t + k_1^2t^2X^2. \end{aligned}$$

The extensions of the kernel in M_{14} are given by 2 generators. The first generator is

$$X_{21} = \partial_X - k_1t^2\partial_\varphi,$$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t} \\ -\varphi_{XX}B^2_{\varphi_X} - B^2_X - 2k_1(t\varphi_t - X\varphi_X) = 0. \end{aligned} \tag{B.29}$$

Solving this equation, one finds a divergent symmetry where $B^1 = 2k_1tX\varphi_X$ and $B^2 = -2k_1tX\varphi_t$. Using Noether's theorem, one finds

$$\begin{aligned} C^t &= \varphi_t\varphi_X + k_1t^2\varphi_t + 2k_1tX\varphi_X, \\ C^X &= \frac{3}{2}\beta\varphi_X^{-2} + \beta k_1t^2\varphi_X^{-3} - \frac{1}{2}\varphi_t^2 - 2k_1tX\varphi_t + k_1^2t^2X^2. \end{aligned}$$

For the second generator

$$X_{22} = t\partial_t - 3X\partial_X - \varphi\partial_\varphi,$$

the determining equation of the symmetry of equation is

$$\begin{aligned} -\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} \\ -\varphi_{XX} B^2_{\varphi_X} - B^2_X + \frac{3\beta}{\varphi_X^2} - 3\varphi_t^2 - 6k_1 X^2 \varphi_X = 0. \end{aligned} \tag{B.30}$$

This generator is not divergent, therefore a conserved vector does not exist.

The extensions of the kernel in M_{15} are given by the generators

$$X_{23} = (\gamma - 1)t\partial_t - 2\varphi\partial_\varphi, \qquad X_{24} = (\gamma - 1)X\partial_X + (\gamma + 1)\varphi\partial_\varphi, \qquad X_{25} = \partial_X.$$

This model has 2 cases of the Lagrangian, when $\gamma = -1$ and $\gamma \neq -1$.

Case 1. $\gamma = -1, \quad W(X, \varphi_X) = -\beta\varphi_X^{-1} \ln(\varphi_X)$

I. $X_{23} = (\gamma - 1)t\partial_t - 2\varphi\partial_\varphi$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned} -\varphi_t B^1_\varphi - \frac{\beta\varphi_{XX}}{\varphi_X^2} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi \\ -\varphi_{tX} B^2_{\varphi_t} - \varphi_{XX} B^2_{\varphi_X} - B^2_X - 2\beta \ln \varphi_X - 2\beta - \varphi_t^2 = 0. \end{aligned} \tag{B.31}$$

Solving the above equation, one obtains a divergent symmetry such that

$$B^1 = 2\big(-\beta t + X\varphi_t\varphi_X\big), \qquad B^2 = X\big(-2\beta \ln \varphi_X - \varphi_t^2\big).$$

Using Noether's theorem, the conserved vectors are

$$C^t = 2\beta t \ln \varphi_X - 2\beta t + 2\varphi\varphi_t - t\varphi_t^2 + 2X\varphi_t\varphi_X,$$

$$C^X = -2\beta X \ln \varphi_X + 2\beta\varphi_X^{-1}\big(\varphi - t\varphi_t\big) - X\varphi_t^2.$$

II. $X_{24} = (\gamma - 1)X\partial_X + (\gamma + 1)\varphi\partial_\varphi$

The determining equation for vector B^i is

$$\begin{aligned} -\varphi_t B^1_\varphi - \frac{\beta\varphi_{XX}}{\varphi_X^2} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi \\ -\varphi_{tX} B^2_{\varphi_t} - \varphi_{XX} B^2_{\varphi_X} - B^2_X - 2\beta \ln \varphi_X + 2\beta - \varphi_t^2 = 0. \end{aligned} \tag{B.32}$$

The symmetry is divergent, as one finds

$$B^1 = 2\big(-\beta t + X\varphi_t\varphi_X\big), \qquad B^2 = X\big(-2\beta\ln\varphi_X - \varphi_t^2 + 4\beta\big).$$

The conserved vectors are

$$C^t = -2\beta t, \qquad C^X = 2\beta X.$$

III. $X_{25} = \partial_X$

The determining equation is

$$\varphi_t B^1_\varphi + \varphi_{tt} B^1_{\varphi_t} + \varphi_{tX} B^1_{\varphi_X} + B^1_t + \varphi_X B^2_\varphi + \varphi_{tX} B^2_{\varphi_t} + \varphi_{XX} B^2_{\varphi_X} + B^2_X = 0. \quad \text{(B.33)}$$

This symmetry is a variational symmetry where $B^i = 0$ and the conserved vectors are

$$C^t = \varphi_t\varphi_X, \qquad C^X = -\beta\ln\varphi_X + \beta - \frac{1}{2}\varphi_t^2.$$

Case 2. $\gamma \neq -1, \quad W(X, \varphi_X) = -\frac{\beta\varphi_X^\gamma}{(\gamma+1)}$

I. $X_{23} = (\gamma-1)t\partial_t - 2\varphi\partial_\varphi$

The determining equation is

$$\begin{aligned} &-\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} \\ &-\varphi_{XX} B^2_{\varphi_X} - B^2_X - \frac{\beta(\gamma+3)}{\gamma+1}\varphi_X^{\gamma+1} - \frac{(\gamma+3)}{2}\varphi_t^2 = 0. \end{aligned} \quad \text{(B.34)}$$

For the condition $\gamma = -3$, one obtains a variational symmetry and the conserved vectors are given as follows,

$$C^t = 2\Big(-\beta t\varphi_X^{-2} + \varphi\varphi_t - t\varphi_t^2\Big), \qquad C^X = 2\beta\varphi_X^{-3}\big(\varphi - 2t\varphi_t\big).$$

II. $X_{24} = (\gamma-1)X\partial_X + (\gamma+1)\varphi\partial_\varphi$

The determining equation is

$$\begin{aligned} &-\varphi_t B^1_\varphi - \varphi_{tt} B^1_{\varphi_t} - \varphi_{tX} B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX} B^2_{\varphi_t} \\ &-\varphi_{XX} B^2_{\varphi_X} - B^2_X + \Big(\frac{\beta(3\gamma+1)}{\gamma+1}\Big)\varphi_X^{\gamma+1} + \Big(\frac{3\gamma+1}{2}\Big)\varphi_t^2 = 0. \end{aligned} \quad \text{(B.35)}$$

Solving this equation, for the condition $\gamma = -\frac{1}{3}$, one finds a variational symmetry and the conserved vectors are

$$C^t = \frac{2}{3}\Big(- \varphi\varphi_t - 2X\varphi_t\varphi_X\Big), \qquad C^X = \frac{2}{3}\Big(X\varphi_t^2 - \beta\varphi\varphi_X^{-1/3} + \beta X\varphi_X^{2/3}\Big).$$

III. $X_{25} = \partial_X$

The determining equation is

$$\begin{aligned}-\varphi_t B^1_\varphi - \varphi_{tt}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi - \varphi_{tX}B^2_{\varphi_t}\\ -\varphi_{XX}B^2_{\varphi_X} - B^2_X = 0.\end{aligned} \tag{B.36}$$

This symmetry is a variational symmetry and the conserved vectors are

$$C^t = \varphi_t\varphi_X, \qquad C^X = \Big(\frac{\beta\gamma}{\gamma+1}\Big)\varphi_X^{\gamma+1} - \frac{1}{2}\varphi_t^2.$$

The extensions of the kernel in M_{16} are given by the generators

$$X_{26} = \partial_X, \qquad X_{27} = 2X\partial_X + \varphi\partial_\varphi, \qquad X_{28} = t^2\partial_t + t\varphi\partial_\varphi \qquad X_{29} = 2t\partial_t + \varphi\partial_\varphi.$$

I. $X_{26} = \partial_X$

Substituting the Lagrangian into equation (6.4), one gets

$$\varphi_t B^1_\varphi + \varphi_{tt}B^1_{\varphi_t} + \varphi_{tX}B^1_{\varphi_X} + B^1_t + \varphi_X B^2_\varphi + \varphi_{tX}B^2_{\varphi_t} + \varphi_{XX}B^2_{\varphi_X} + B^2_X = 0. \tag{B.37}$$

This symmetry is a variational symmetry and the conserved vectors are

$$C^t = \varphi_t\varphi_X, \qquad C^X = \frac{3}{2}\beta\varphi_X^{-2} - \frac{1}{2}\varphi_t^2.$$

II. $X_{27} = 2X\partial_X + \varphi\partial_\varphi$

The determining equation for vector B^i is

$$\begin{aligned}-\varphi_t B^1_\varphi - \frac{\beta\varphi_{XX}}{\varphi_X^2}B^1_{\varphi_t} - \varphi_{tX}B^1_{\varphi_X} - B^1_t - \varphi_X B^2_\varphi\\ -\varphi_{tX}B^2_{\varphi_t} - \varphi_{XX}B^2_{\varphi_X} - B^2_X - \frac{2\beta}{\varphi_X^2} + 2\varphi_t^2 = 0.\end{aligned} \tag{B.38}$$

In this case, the symmetry is not divergent, hence, it can not provide a conservation law.

III. $X_{28} = t^2\partial_t + t\varphi\partial_\varphi$

The determining equation for vector B^i is

$$\begin{aligned}&\varphi_t B^1_\varphi + \varphi_{tt} B^1_{\varphi_t} + \varphi_{tX} B^1_{\varphi_X} + B^1_t + \varphi_X B^2_\varphi \\ &+\varphi_{tX} B^2_{\varphi_t} + \varphi_{XX} B^2_{\varphi_X} + B^2_X - \varphi\varphi_t = 0.\end{aligned} \tag{B.39}$$

The symmetry is divergent where $B^1 = \frac{\varphi^2}{2}$, $B^2 = 0$ and the conserved vectors are

$$C^t = \frac{1}{2}\Big(\beta t^2\varphi_X^{-2} + \varphi_t^2 - t\varphi\varphi_t + t^2\varphi_t^2\Big), \qquad C^X = \beta t(-\varphi + t\varphi_t)\varphi_X^{-3}.$$

IV. $X_{29} = 2t\partial_t + \varphi\partial_\varphi$

The determining equation for vector B^i is

$$\varphi_t B^1_\varphi + \varphi_{tt} B^1_{\varphi_t} + \varphi_{tX} B^1_{\varphi_X} + B^1_t + \varphi_X B^2_\varphi + \varphi_{tX} B^2_{\varphi_t} + \varphi_{XX} B^2_{\varphi_X} + B^2_X = 0. \tag{B.40}$$

Solving this equation, one finds a variational symmetry and the conserved vectors are

$$C^t = \beta t\varphi_X^{-2} - \varphi\varphi_t + t\varphi_t^2, \qquad C^X = \beta(-\varphi + 2t\varphi_t)\varphi_X^{-3}.$$

The extension of the kernel in M_{17} is given by the generator

$$X_{30} = t^2\partial_t + t\varphi\partial_\varphi.$$

Substituting the Lagrangian into equation (6.4), one gets

$$\begin{aligned}&\varphi_t B^1_\varphi + \varphi_{tt} B^1_{\varphi_t} + \varphi_{tX} B^1_{\varphi_X} + B^1_t + \varphi_X B^2_\varphi \\ &+\varphi_{tX} B^2_{\varphi_t} + \varphi_{XX} B^2_{\varphi_X} + B^2_X - \varphi\varphi_t = 0.\end{aligned} \tag{B.41}$$

Solving this equation, one obtains $B^1 = \frac{\varphi^2}{2}$, and $B^2 = 0$ and the conserved vectors are

$$C^t = \frac{1}{2}b(X)t^2\varphi_X^{-2} + \frac{1}{2}\varphi^2 - t\varphi\varphi_t + \frac{1}{2}t^2\varphi_t^2, \qquad C^X = tb(X)(-\varphi + t\varphi_t)\varphi_X^{-3}.$$

APPENDIX C

THE CLASSIFICATIONS OF TWO- AND THREE-DIMENSIONAL LIE ALGEBRAS

In an appropriate basis, every two-dimensional Lie algebra will have a commutator table as one of the following two forms:

$$L(2,1):\left|\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & 0 & 0 \\ e_2 & & 0 \end{array}\right|,\qquad L(2,2):\left|\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & 0 & e_1 \\ e_2 & & 0 \end{array}\right|.$$

Here e_i are the basis vectors of a Lie algebra.

All three-dimensional Lie algebras are exhausted by the list:

$$L(3,1):\left|\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & 0 \\ e_2 & & 0 & e_1 \\ e_3 & & & 0 \end{array}\right|,\qquad L(3,2,p):\left|\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & & 0 & p\,e_2 \\ e_3 & & & 0 \end{array}\right|,\qquad 0<|p|\le 1,$$

$$L(3,3):\left|\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & & 0 & e_1+e_2 \\ e_3 & & & 0 \end{array}\right|,\qquad L(3,4,p):\left|\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & p\,e_1-e_2 \\ e_2 & & 0 & e_1+p\,e_2 \\ e_3 & & & 0 \end{array}\right|,\qquad p\ge 0,$$

$$L(3,5):\left|\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & 2e_2 \\ e_2 & & 0 & e_3 \\ e_3 & & & 0 \end{array}\right|,\qquad L(3,6):\left|\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_3 & -e_2 \\ e_2 & & 0 & e_1 \\ e_3 & & & 0 \end{array}\right|,$$

$$L(3,7): \begin{array}{|c|ccc|} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_1 & 0 \\ e_2 & & 0 & 0 \\ e_3 & & & 0 \end{array}, \qquad L(3,0): \begin{array}{|c|ccc|} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & 0 \\ e_2 & & 0 & 0 \\ e_3 & & & 0 \end{array}.$$

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APPENDIX D

THE GROUP CLASSIFICATION OF EQUATION (7.1)

D.1 One-dimensional Lie algebras

The function $\zeta(S)$ in the basis generator (7.3) of a one-dimensional Lie algebra can be reduced to one of the two cases: either $\zeta = 0$ or $\zeta = 1$. This can be done by virtue of the equivalence transformation related with the generator X_{10}^e. The set of possible basis generators of one-dimensional Lie algebras is exhausted by the following generators

$$\zeta = 0 \ : \ X_6 + qX_5 + \gamma X_4, \quad X_5 + \gamma X_4, \quad X_4$$

$$\zeta = 1 \ : \ X_6 + qX_5 + \gamma X_4 + \partial_S, \quad X_5 + \gamma X_4 + \partial_S, \quad X_4 + \partial_S$$

In the second step one has to substitute the coefficients of each generator into the determining equation (7.2) and solve it with respect to the function $W(\rho, \dot{\rho}, S)$. Here we present the calculations of the case where $X_6 + qX_5 + \gamma X_4$. The study of the other cases is similar and is summarized in Tables 7.2-7.5.

Substituting

$$k_4 = \gamma, \qquad k_5 = q, \qquad k_6 = 1, \qquad \zeta = 0$$

into equation (7.2), one obtains

$$\rho g_\rho + \dot{\rho} g_{\dot{\rho}}(1 - \gamma - 2q) = g(2q - 1) + \dot{\rho}^{-2}(\rho\varphi + k). \tag{D.1}$$

The characteristic system of equations is

$$\frac{d\rho}{1} = \frac{d\dot{\rho}}{\dot{\rho}(1 - \gamma - 2q)} = \frac{dg}{g(2q - 1) + \dot{\rho}^{-2}(\rho\varphi + k)}.$$

Invariants of the characteristic system of equations depend on the vanishing of the expression

$$(\gamma+q)((\gamma+q)-1/2).$$

If $(\gamma+q)((\gamma+q)-1/2)\neq 0$, then the solution of (D.1) is

$$g(\rho,\dot\rho,S)=\rho^{2q-1}\widetilde{\phi}(\dot\rho\rho^{\gamma+2q-1},S)+\dot\rho^{-2}\left(\frac{\rho\varphi(S)}{2(\gamma+q)}+\frac{k}{2(\gamma+q)-1}\right),$$

where the function $\widetilde{\phi}$ is an arbitrary function. Integrating the function $g(\rho,\dot\rho,S)$, one finds

$$W(\rho,\dot\rho,S)=\rho^{1-2(\gamma+q)}\phi(\dot\rho\rho^{\gamma+2q-1},S)-\frac{\rho\varphi(S)}{2(\gamma+q)}+\frac{k}{2(\gamma+q)-1}+\dot\rho h(\rho,S),$$

where $h(\rho,S)$ is an arbitrary function of the integration. Using the equivalence transformations corresponding to X_6^e, X_8^e and X_9^e, one gets that the system of equations (7.1) with the function

$$W(\rho,\dot\rho,S)=\rho^{1-2(\gamma+q)}\phi(\dot\rho\rho^{\gamma+2q-1},S)$$

admits the generator

$$X_6+qX_5+\gamma X_4,\quad (\gamma+q)((\gamma+q)-1/2)\neq 0.$$

Similarly, one finds that for the function

$$W(\rho,\dot\rho,S)=\phi(\dot\rho\rho^{q-1/2},S)-q_0\ln(\rho),$$

the extension of the kernel of the admitted Lie algebra is defined by the generator

$$X_6+qX_5+(-q+1/2)X_4,$$

and for the function

$$W(\rho,\dot\rho,S)=\rho\phi(\dot\rho\rho^{q-1},S)+\rho\ln(\rho)\psi(S)$$

the admitted generator is

$$X_6+q(X_5-X_4).$$

Here q_0 and $\psi(S)$ are arbitrary.

D.2 Two-dimensional Lie algebras

Since for the basis generators

$$X = \beta_1 X_6 + q_1 X_5 + \gamma_1 X_4 + X_{\zeta_1}, \qquad Y = \beta_2 X_6 + q_2 X_5 + \gamma_2 X_4 + X_{\zeta_2},$$

their commutator is

$$[X, Y] = [X_{\zeta_1}, X_{\zeta_2}],$$

substituting the commutator into the equation

$$[X, Y] = pX,$$

one has

$$[X_{\zeta_1}, X_{\zeta_2}] = p\left(\beta_1 X_6 + q_1 X_5 + \gamma_1 X_4 + X_{\zeta_1}\right),$$

where $p = 0$ or $p = 1$. From these conditions one finds that

$$\zeta_2'\zeta_1 - \zeta_1'\zeta_2 = p\zeta_1, \tag{D.2}$$

and

$$p\beta_1 = 0, \qquad pq_1 = 0, \qquad p\gamma_1 = 0. \tag{D.3}$$

Let us consider the case where $p = 1$. For this case one finds that the basis of the Lie algebra consists of the generators

$$X = X_{\zeta_1}, \qquad Y = \beta_2 X_6 + q_2 X_5 + \gamma_2 X_4 + X_{\zeta_2},$$

where $\zeta_1 \neq 0$. By virtue of equivalence transformations, one can assume that $\zeta_1 = 1$. The general solution of equation (D.2) is

$$\zeta_2 = S + c_0,$$

where the constant c_0 can be assumed to be zero. Thus, in the case $p = 1$ the Lie algebras have the form:

$$\{\partial_S, \beta X_6 + qX_5 + \gamma X_4 + S\partial_S\}. \tag{D.4}$$

The set of such Lie algebras is exhausted by the following list

1. $\partial_S,\ \ X_6 + qX_5 + \gamma X_4 + \partial_S$ 2. $\partial_S,\ \ X_5 + \gamma X_4 + S\partial_S$

3. $\partial_S,\ \ X_4 + S\partial_S$ 4. $\partial_S,\ \ S\partial_S$.

Let us consider the case $p = 0$. For this case equation (D.2) becomes

$$\zeta_2'\zeta_1 - \zeta_1'\zeta_2 = 0. \tag{D.5}$$

Notice that if $\zeta_1^2 + \zeta_2^2 \neq 0$, then one can assume that $\zeta_1 = 1$. In this case equation (D.5) gives that $\zeta_2 = k\zeta_1$. Hence, one also can assume that $\zeta_2 = 0$. Thus, Lie algebras in this case have the following forms

$$\{\beta_1 X_6 + q_1 X_5 + \gamma_1 X_4 + \partial_S, \qquad \beta_2 X_6 + q_2 X_5 + \gamma_2 X_4\}, \tag{D.6}$$

$$\{\beta_1 X_6 + q_1 X_5 + \gamma_1 X_4, \qquad \beta_2 X_6 + q_2 X_5 + \gamma_2 X_4\}. \tag{D.7}$$

The set of all possible Lie algebras of the forms (D.6) and (D.7) is exhausted by the list

$$\begin{aligned}
\zeta_1 = 1\ &:\ q_1X_5 + \gamma_1X_4 + \partial_S,\ \ X_6 + q_2X_5 + \gamma_2X_4,\\
&:\ \beta_1X_6 + \gamma_1X_4 + \partial_S,\ \ X_5 + \gamma_2X_4,\\
&:\ \beta_1X_6 + q_1X_5 + \partial_S,\ \ X_4,\\
\zeta_1 = 0\ &:\ q_1X_5 + \gamma_1X_4,\ \ X_6 + q_2X_5 + \gamma_2X_4,\\
&:\ X_4,\ \ X_5.
\end{aligned}$$

Similar to the one-dimensional Lie algebras, further obtaining of the function $W(\rho, \dot{\rho}, S)$ consists of solving the determining equations (7.2) where the coefficients are defined by the obtained Lie algebras. Results of these studies are summarized in Tables 7.2-7.5.

D.3 Three-dimensional Lie algebras

Let the basis generators of a three-dimensional Lie algebra be

$$X = \widetilde{X} + X_{\zeta_1}, \qquad Y = \widetilde{Y} + X_{\zeta_2}, \qquad Z = \widetilde{Z} + X_{\zeta_3},$$

where

$$\widetilde{X} = \beta_1 X_6 + q_1 X_5 + \gamma_1 X_4, \quad \widetilde{Y} = \beta_2 X_6 + q_2 X_5 + \gamma_2 X_4, \quad \widetilde{Z} = \beta_3 X_6 + q_3 X_5 + \gamma_3 X_4.$$

Notice that

$$[\widetilde{X}, \widetilde{Y}] = 0, \qquad [\widetilde{X}, \widetilde{Z}] = 0, \qquad [\widetilde{Y}, \widetilde{Z}] = 0. \tag{D.8}$$

Let us first study the Abelian Lie algebra. In this case one has

$$[X, Y] = 0, \qquad [X, Z] = 0, \qquad [Y, Z] = 0,$$

which means that

$$\zeta_1\zeta_2' - \zeta_1'\zeta_2 = 0, \quad \zeta_1\zeta_3' - \zeta_1'\zeta_3 = 0, \quad \zeta_2\zeta_3' - \zeta_2'\zeta_3 = 0. \tag{D.9}$$

If $\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 0$, then the basis of this Lie algebra is

$$X_4, \qquad X_5, \qquad X_6.$$

This case is reduced to $W_{\dot\rho} = 0$.

If, for example, $\zeta_1 \neq 0$, then one can assume that $\zeta_1 = 1$ and, hence,

$$\zeta_2 = c_1, \qquad \zeta_3 = c_2,$$

where c_1 and c_2 are constant. Without loss of generality one can assume that $c_1 = 0$ and $c_2 = 0$. Hence, the list of all possible Abelian three-dimensional Lie algebras consists of the following Lie algebras

$$\begin{aligned}
\zeta_1 = 1 \ &: \ q_1 X_4 + \partial_S, \quad X_6 + q_2 X_4, \quad X_5 + q_3 X_4,\\
&: \ \beta X_5 + \partial_S, \quad X_6 + q_1 X_5, \quad X_4,\\
\zeta_1 = 0 \ &: \ X_6 + \partial_S, \quad X_5, \quad X_4,\\
&: \ \partial_S, \quad X_5, \quad X_4.
\end{aligned}$$

Let us study three-dimensional non-Abelian Lie algebras. In the case $L(3,1)$ one has

$$[X, Y] = 0, \qquad [X, Z] = 0, \qquad [Y, Z] = X,$$

which means that

$$\zeta_1\zeta_2' - \zeta_1'\zeta_2 = 0, \quad \zeta_1\zeta_3' - \zeta_1'\zeta_3 = 0, \quad \zeta_2\zeta_3' - \zeta_2'\zeta_3 = \zeta_1. \tag{D.10}$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = 0, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{X}.$$

Because of (D.8), one has that $\widetilde{X} = 0$. Hence, $\zeta_1 \neq 0$, and one can assume that $\zeta_1 = 1$. Equations (D.10) become contradictory.

In the case $L(3, 2, p)$, $(0 < |p| \leq 1)$ one obtains

$$[X, Y] = 0, \qquad [X, Z] = X, \qquad [Y, Z] = pY,$$

which means that

$$\zeta_1\zeta_2' - \zeta_1'\zeta_2 = 0, \quad \zeta_1\zeta_3' - \zeta_1'\zeta_3 = \zeta_1, \quad \zeta_2\zeta_3' - \zeta_2'\zeta_3 = p\zeta_2. \tag{D.11}$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \qquad [\widetilde{X}, \widetilde{Z}] = \widetilde{X}, \qquad [\widetilde{Y}, \widetilde{Z}] = p\widetilde{Y}.$$

Because of (D.8), one has that $\widetilde{X} = 0$ and $\widetilde{Y} = 0$. Hence, $\zeta_1\zeta_2 \neq 0$, and one can assume that $\zeta_1 = 1$. Equations (D.11) give that ζ_2 is constant, which contradicts the property that X, Y and Z are basis generators of the Lie algebra.

Similar contradictions are obtained for $L(3, 3)$ and $L(3, 4, p)$. Indeed, for $L(3, 3)$ one has

$$[X, Y] = 0, \qquad [X, Z] = X, \qquad [Y, Z] = X + Y,$$

or

$$\zeta_1\zeta_2' - \zeta_1'\zeta_2 = 0, \quad \zeta_1\zeta_3' - \zeta_1'\zeta_3 = \zeta_1, \quad \zeta_2\zeta_3' - \zeta_2'\zeta_3 = \zeta_1 + \zeta_2. \tag{D.12}$$

and

$$[\widetilde{X}, \widetilde{Y}] = 0, \quad [\widetilde{X}, \widetilde{Z}] = \widetilde{X}, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{X} + \widetilde{Y}.$$

Hence, $\widetilde{X}=0$ and $\widetilde{Y}=0$. Similar for $L(3,4,p)$:

$$[X,Y]=0, \qquad [X,Z]=pX-Y, \qquad [Y,Z]=X+pY, \qquad (p\geq 0),$$

which means that

$$\zeta_1\zeta_2'-\zeta_1'\zeta_2=0, \quad \zeta_1\zeta_3'-\zeta_1'\zeta_3=p\zeta_1-\zeta_2, \quad \zeta_2\zeta_3'-\zeta_2'\zeta_3=\zeta_1+p\zeta_2 \qquad \text{(D.13)}$$

and

$$[\widetilde{X},\widetilde{Y}]=0, \quad [\widetilde{X},\widetilde{Z}]=p\widetilde{X}-\widetilde{Y}, \quad [\widetilde{Y},\widetilde{Z}]=\widetilde{X}+p\widetilde{Y}.$$

Because of (D.8), one also has that $\widetilde{X}=0$ and $\widetilde{Y}=0$. Hence, in both these cases $\zeta_1\zeta_2\neq 0$, and one can assume that $\zeta_1=1$. Equations (D.12) and (D.13) give that ζ_2 is constant, which contradicts the property that X, Y and Z compose a basis of the Lie algebra.

Let us consider the algebra $L(3,5)$. In this case one has

$$[X,Y]=X, \qquad [X,Z]=2Y, \qquad [Y,Z]=Z,$$

which means that

$$\zeta_1\zeta_2'-\zeta_1'\zeta_2=\zeta_1, \quad \zeta_1\zeta_3'-\zeta_1'\zeta_3=2\zeta_2, \quad \zeta_2\zeta_3'-\zeta_2'\zeta_3=\zeta_3 \qquad \text{(D.14)}$$

and

$$[\widetilde{X},\widetilde{Y}]=\widetilde{X}, \quad [\widetilde{X},\widetilde{Z}]=2\widetilde{Y}, \quad [\widetilde{Y},\widetilde{Z}]=\widetilde{Z}.$$

Because of (D.8), one has that $\widetilde{X}=0$, $\widetilde{Y}=0$ and $\widetilde{Z}=0$. Hence, $\zeta_1\zeta_2\zeta_3\neq 0$, and, for example, $\zeta_1=1$. The general solution of equations (D.14) is

$$\zeta_2=S+c_1, \qquad \zeta_3=S^2+2c_1S+c_1^2.$$

Thus, the basis generators are

$$X=\partial_S, \qquad Y=S\partial_S, \qquad Z=S^2\partial_S.$$

As noticed in the previous section in this case $W_S=0$.

For the Lie algebra $L(3,6)$:

$$[X,Y]=Z, \qquad [X,Z]=-Y, \qquad [Y,Z]=X,$$

which mean that

$$\zeta_1\zeta_2'-\zeta_1'\zeta_2=\zeta_3, \quad \zeta_1\zeta_3'-\zeta_1'\zeta_3=-\zeta_2, \quad \zeta_2\zeta_3'-\zeta_2'\zeta_3=\zeta_1, \tag{D.15}$$

and

$$[\widetilde{X},\widetilde{Y}]=\widetilde{Z}, \quad [\widetilde{X},\widetilde{Z}]=-\widetilde{Y}, \quad [\widetilde{Y},\widetilde{Z}]=\widetilde{X}.$$

Because of (D.8), one has that $\widetilde{X}=0$, $\widetilde{Y}=0$ and $\widetilde{Z}=0$. Hence, $\zeta_1\zeta_2\zeta_3\neq 0$, and assuming that $\zeta_1=1$, one obtains the general solution of the first two equations of (D.15)

$$\zeta_2=c_1\sin(S)+c_2\cos(S), \qquad \zeta_3=c_1\cos(S)-c_2\sin(S)$$

and the contradiction $c_1^2+c_2^2+1=0$.

Let us study the Lie algebra $L(3,-1)$:

$$[X,Y]=X, \qquad [X,Z]=0, \qquad [Y,Z]=0,$$

which mean that

$$\zeta_1\zeta_2'-\zeta_1'\zeta_2=\zeta_1, \quad \zeta_1\zeta_3'-\zeta_1'\zeta_3=0, \quad \zeta_2\zeta_3'-\zeta_2'\zeta_3=0, \tag{D.16}$$

and

$$[\widetilde{X},\widetilde{Y}]=\widetilde{X}, \quad [\widetilde{X},\widetilde{Z}]=0, \quad [\widetilde{Y},\widetilde{Z}]=0.$$

Hence, $\widetilde{X}=0$ and one can assume that $\zeta_1=1$. Solving equations (D.16), one finds that the basis generators have the form

$$X=\partial_S, \quad Y=S\partial_S+\beta_2X_6+q_2X_5+\gamma_2X_4, \quad Z=\beta_3X_6+q_3X_5+\gamma_3X_4.$$

Thus, one only needs to study Lie algebras with the following basis generators

$$X=\partial_S, \qquad Y=S\partial_S, \qquad Z=S^2\partial_S, \tag{D.17}$$

and

$$X = \partial_S, \quad Y = S\partial_S + \beta_2 X_6 + q_2 X_5 + \gamma_2 X_4, \quad Z = \beta_3 X_6 + q_3 X_5 + \gamma_3 X_4.$$

According to the remark in the previous section for the case (D.17) one can assume that $W_S = 0$. Hence, for the non-Abelian three-dimensional Lie algebras the list of possible Lie algebras is

1. $\partial_S, \ \ S\partial_S + q_2 X_5 + \gamma_2 X_4, \ \ X + q_3 X_5 + \gamma_3 X_4,$
2. $\partial_S, \ \ S\partial_S + \beta_2 X_6 + \gamma_2 X_4, \ \ X_5 + \gamma_3 X_4,$
3. $\partial_S, \ \ S\partial_S + \beta_2 X_6 + q_2 X_5, \ \ X_4.$

D.4 Lie algebras of dimension greater than 3

If the dimension of the Lie algebra is greater or equal to 4, then one can either choose the basis generators such that two of the generators have the form (7.4) or the admitted Lie algebra is four-dimensional and the basis generators can be chosen such as

$$X_4 + \zeta_1(S)\partial_S, \ \ X_5 + \zeta_2(S)\partial_S, \ \ X_6 + \zeta_3(S)\partial_S, \ \ \partial_S. \tag{D.18}$$

Substituting the coefficients of the generators (D.18) into (7.2) one obtains reduction to the case either where $W_S = 0$ or $W_{\dot\rho} = 0$.

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