EXACT SOLUTIONS OF

THE POPULATION BALANCE EQUATION

Fubiao Lin

A Thesis Submitted in Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy in Applied Mathematics

Suranaree University of Technology

Academic Year 2015

ผลเฉลยแม่นตรงของสมการสมดุลประชากร

นายฝูเปียว หลิน

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์
มหาวิทยาลัยเทคโนโลยีสุรนารี
ปีการศึกษา 2558

EXACT SOLUTIONS OF THE POPULATION BALANCE EQUATION

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

Thesis Examining Committee

(Asst. Prof. Dr. Eckart Schulz)

Chairperson

(Prof. Dr. Sergey V. Meleshko)

Member (Thesis Advisor)

(Prof. Dr. Adrian E. Flood)

Member

(Asst. Prof. Dr. Prapart Pue-on)

Member

(Asst. Prof. Dr. Jessada Tanthanuch)

Member

(Prof. Dr. Sukit Limpijumnong)

Vice Rector for Academic Affairs and Innovation

(Prof. Dr. Santi Maensiri)

Dean of Institute of Science

ฝูเปียว หลิน : ผลเฉลยแม่นตรงของสมการสมดุลประชากร (EXACT SOLUTIONS OF THE POPULATION BALANCE EQUATION) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร.เซอร์เก เมเลชโก, 113 หน้า.

สมการสมดุลประชากรได้ถูกใช้เพื่อสร้างตัวแบบของกระบวนการต่าง ๆ อย่างหลากหลาย ตัวอย่างเช่น การเกิดพอลิเมอร์ การตกผลึก การก่อเกิดเมฆ และ พลศาสตร์ของเซลล์ แต่ทั้งนี้ยังขาดผลเฉลยเชิงวิเคราะห์ซึ่งมีจำเป็นมากกว่าจะมีเพียงผลเฉลยที่ได้จากกลวิธีเชิงตัวเลข จุดมุ่งหมายของวิทยานิพนธ์นี้ คือ การหาผลเฉลยเชิงวิเคราะห์ของสมการสมดุลประชากร โดยใช้วิธีการวิเคราะห์กลุ่มที่ได้พัฒนาขึ้นและวิธีโมเมนต์

วิทยานิพนธ์นี้ได้ศึกษาตัวแบบสมดุลประชากร 3 แบบ สัมประสิทธิ์ของตัวก่อกำเนิดกลุ่มที่ปรากฏในสมการกำหนดถูกแทนให้อยู่ในรูปอนุกรมเทย์เลอร์ และสามารถหาผลเฉลยของสมการกำหนดดังกล่าวได้ทั้งหมด ทั้งนี้ยังสามารถหาระบบเหมาะที่สุดของพีชคณิตย่อย ผลเฉลยยืนยง และสมการลดรูปทั้งหมด สำหรับผลเฉลยเชิงตัวเลขของสมการลดรูปที่ได้จากการศึกษา ถูกสร้างโดยใช้วิธีฮอมอโทปีเพอร์เทอร์เบชัน นอกจากนี้ยังได้นำวิธีการจำแนกกลุ่มขั้นต้นมาใช้หาสมการกำหนดอีกด้วย

ในวิทยานิพนธ์นี้ ได้ใช้แนวคิดกลุ่มปรับมาตรา หาผลเฉลยทางกายภาพเชิงวิเคราะห์ชัดแจ้งของสมการการจับก้อนสโมลอูชอฟสกี้ที่มีเคอร์เนลเป็นค่าคงตัว

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา Fubiao Lin

ปีการศึกษา 2558 ลายมือชื่ออาจารย์ที่ปรึกษา________

ลายมือชื่ออาจารย์ที่ปรึกษาร่วม________

FUBIAO LIN : EXACT SOLUTIONS OF THE POPULATION BALANCE EQUATION. THESIS ADVISOR : PROF. SERGEY V. MELESHKO, Ph.D. 113 PP.

POPULATION BALANCE EQUATION/LIE GROUP METHOD/METHOD OF MOMENTS/GROUP CLASSIFICATION/EXACT SOLUTION

Population balance equations (PBEs) have been used to model a wide range of processes including polymerization, crystallization, cloud formation and cell dynamics, but the lack of analytical solutions necessitates the use of numerical techniques. The goal of this dissertation is to find analytic solutions of the PBEs by using the developed group analysis method and the method of moments.

Three models of the PBEs were investigated in this dissertation. The coefficients of the group generators in the determining equations are represented by the Taylor series and the determining equations are successfully solved. The optimal system of subalgebras, invariant solutions and all the reduced equations are obtained, and the numerical solutions for the reduced equations are determined by the homotopy perturbation method. Furthermore, finding the determining equations by use of the preliminary group classification is also considered.

In particular, an explicit analytical physical solution of the Smoluchowski coagulation equation with constant kernel is given by using the scaling group approach.

School of Mathematics

Academic Year 2015

Student's Signature Fubiao Lin

Advisor's Signature

Co-advisor's Signature

ACKNOWLEDGEMENTS

It took me approximately three years to finish the curriculum and dissertation. Of course, without the help, encouragement and support of my Professors it would have become more difficult to complete the research, and therefore my advisors must be most sincerely thanked. I profoundly appreciate Prof. Dr. Sergey Meleshko and Prof. Dr. Adrian Flood, my supervisors whose patient help, great support, valuable comments and useful suggestions made this dissertation get off the ground and come to a smooth close. The success of my dissertation is partly due to the inspiration provided by Prof. Sergey and Adrian's ample good papers and books; their scientific fruits have a long history and have attracted attention from the applied mathematics and chemical engineering communities all over the world.

When I wrote this graduation dissertation, I met a lot of difficulties and troubles; however my advisors gave me invaluable suggestions and comments, constant encouragement and guidance. This dissertation could not have reached its present form without their consistent and illuminating instruction. Moreover, Prof. Sergey and Adrian taught me a lot, including how to use the software Reduce, how to improve my English level, and how to submit academic articles, which is helpful and important for me to write the graduation dissertation. During the past three years your scholarship and profound knowledge have deeply influenced me. You also have given me assistance with my English and much enlightenment on how to be a good teacher. The completion of this dissertation has largely relied on your guidance and hard work. In addition, you gave me a lot of support in life and provided me with many opportunities that will benefit the development of my future.

Second, I would like to express my heartfelt gratitude to Asst. Prof. Dr. Eckart Schulz, who led me into the world of advanced and abundant knowledge. Also, I am greatly indebted to the professors, teachers and students at the School of Mathematics: Asst. Prof. Dr. Jessada Tanthanuch, students Fae, Toey, Jear, Pom, and so on, who have instructed and helped me a lot in the past three years.

Third, my thanks go to my beloved family for their love and great confidence in me throughout all these years. My indebtedness also goes to my parents who are the supporters of my life and study, and who have worked hard to let me have a happy life and healthy body.

Fourth, thanks to my fellow classmates who have shared happiness and sadness with me during the difficult course of the thesis. Thanks to the people who spent their precious time in reading this dissertation. Thanks to all other professors, teachers, friends and classmates and so on, who taught and helped me. Thank you very much.

Fifth, I thank the laboratory of the School of Mathematics and the library of SUT, which provided good environments for my study.

Sixth, I am warmly grateful to my former university, Suranaree University of Technology, Thailand, which gave me much knowledge in the precious three years. Because there is no end to study, I shall continue it after graduation. As for me, it's a new starting point. On the other hand, I gratefully acknowledge financial support from Guizhou University of Finance and Economics, China.

Eventually, I thank the anonymous teachers and staff of the Center for International Affairs for their warm help and transportation that helped obtain visa extensions and the like. Thanks ever so much.

Fubiao Lin

CONTENTS

CONTENTS (Continued)

CONTENTS (Continued)

CONTENTS (Continued)

CONTENTS (Continued)

CONTENTS (Continued)

Page

LIST OF TABLES

Table **Page**

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In this chapter, the background and applications of the population balance equations (PBEs) are introduced, and the PBEs that have been investigated here are also presented.

1.1 Population Balance Equation

Ramkrishna (2000) has presented a good introduction to the population balance concept and an explanation on the PBE as: "The PBE is an equation in the number density and may be regarded as representing a number balance on particles of a particular state. The equation is often coupled with conservation equation for entities in the particles environmental (or continuous) phase." In particular, a PBE (e.g., Randolph and Larson, 1971; 1988; Ramkrishna, 2000) is an equation of continuity for discrete entities (particles, crystals, emulsion droplets, bubbles, and people, among others) having a particular characteristic, which could be a measure of size (diameter or volume of a particle), age, chemical activity, or other characteristic.

The problem with PBEs modeling particulate processes (e.g., Randolph and Larson, 1971; 1988; Ramkrishna, 2000) is that they typically involve both a partial differential equation (the population balance), an integral equation (the equation that represents the time-dependent mass of crystal as a function of the crystal population density), and an algebraic equation (the calculation of the growth rate from the concentration of the solute, which is dependent on the total mass of the crystals). This system of equations is particularly difficult to solve, particularly since the integral term is a non-local operator.

PBEs are essential to scientists and engineers of widely varying disciplines. They are of interest to physicists (astrophysicists, high-energy physicists, geophysicists, meteorologists), chemists (colloidal chemists, statistical mechanicians), biophysicists concerned with populations of cells of various kinds, and food scientists dealing with preparations of emulsions or sterilization of food. All of these fields have an indispensable need for PBEs (e.g., Ramkrishna, 2000).

In physics, specifically non-equilibrium statistical mechanics, the Boltzmann equation describes the statistical behavior of a thermodynamic system not in thermodynamic equilibrium. It was devised by Ludwig Boltzmann in 1872. The famous Boltzmann equation of chemical physics was perhaps the first PBE and is now more than a century old. However the current uses of the PBEs may be regarded as relatively new in light of the variety of applications in which engineers have more recently put it to use (e.g., Ramkrishna, 2000). The trend is, of course, associated with the realization that the methodology of population balances is indispensable for a rational treatment of dispersed phase processes in engineering.

Numerous phenomena in the chemical engineering sciences are described by systems involving PBEs (e.g., Lakshmikantham and Rama Mohana Rao, 1995; Ramkrishna, 2000), and the need arose to determine the functional dependencies between the variables involved. PBEs have been used to model a wide range of chemical and physical processes, including polymerization, solution crystallization, cloud formation, and cell dynamics, which are best described by PBEs (Ramkrishna, 2000). In our use of it, the population consists of particles or crystals within a crystallizer, or a particular subsection of a crystallizer, and the particle characteristic of most significance is the particle size based on a particular linear dimension

(Hulburt and Katz, 1964; Randolph and Larson, 1964).

Although the application of population balances has been growing enormously in recent times (e.g., Hulburt and Katz, 1964; Randolph and Larson, 1964; 1988; Ramkrishna, 2000) it is typically impossible to obtain solutions to these equations for all but the most simple of realistic industrial systems (for instance steady-state crystallizers with a fully mixed suspension, plug flow crystallizers in the absence of agglomeration or breakage, etc.).

The major difficulty is not the ability to produce a fundamental, analytical model of the processes, but the ability to solve these models and the difficulty to compute solutions except by numerical methods. An accurate numerical solution of the PBE can be challenging since population density can extend many orders of magnitude in size and time, and the rate of change in the distribution can be very sharp at some points. This has led to the development of many algorithms (e.g., Randolph and Larson, 1971; 1988; Ramkrishna, 2000) for solving the PBE, such as variations of the method of moments (e.g., McCoy, 2002), the method of characteristics (e.g., Briesen, 2006), the method of weighted residuals or orthogonal collocation (e.g., Rawlings, Witkowski, and Eaton, 1992), Monte Carlo simulation (e.g., Smith and Matsoukas, 1998), the spectral methods, the finite element method, the finite volume method, the method of classes and so on. In more recent years, Ramkrishna (2000) has made a major contribution to the needed unification of theory and computational techniques of population balances modeling of particulate systems.

Despite advances in the numerical solution techniques, clearly it would be preferable to have analytic solutions that can describe a range of conditions with a single set of solutions.

1.2 Preliminary Stage of Studied Population Balance Equations

Three particular forms of the population balance are presented in the next two sections. In this section, the most important aspects of the crystallization and granulation processes models are discussed.

The particulate nature of solids is characterised by a number of properties, such as size, shape, liquid and gas content, porosity, composition and age. These properties are denoted as internal coordinates, whereas the Euclidian coordinates, such as the position in rectangular coordinates (x, y, z), cylindrical coordinates (r, φ, z), and spherical coordinates (r, θ, φ) used to specify the locations of (the centroid of) particles describing continuous motion through space are defined as the external coordinates. Usually when dealing with nucleation and growth problems, the internal coordinate is a characteristic length, whereas when dealing with aggregation and breakage, the internal coordinate is the particle volume. For instance, Hulbert and Katz (1964) developed a population balance model for aggregation using volume as the internal coordinate.

The most important property for the characterisation of particles is particle size. Randolph and Larson (1988) pointed out that: "As no two particles will be exactly the same size, the material must be characterised by the distribution of sizes or particle-size distribution." If only the size (such as particle length or volume) is of interest, a single-variable distribution function is sufficient to characterise the particulate system. But if additional properties are also important, multivariable distributions of the number density as functions of several internal properties and locations must be developed.

For instance, in a well-mixed system having only one particle characteristic of interest, the characteristic lengths of the population of particles in the system, the population balance has two independent variables, the particle size, and the time from the initiation of the crystallization. The dependent variable is the population density of crystals of a particular size at the time, which is a probability distribution defined for all non-negative values of size and time. In the following sections three models using the PBE are considered.

1.3 The Homogeneous Population Balance Equations Investigated in the Study

The one-dimensional homogeneous PBEs (Ramkrishna, 2000) used for batch crystallization units are given by

$$\bar{G}(\bar{c}) = k_g \left(\frac{\bar{c} - c_{sat}}{c_{sat}}\right)^g, \quad \bar{B}_0(\bar{c}) = k_b \left(\frac{\bar{c} - c_{sat}}{c_{sat}}\right)^b, \tag{1.1}$$

$$\frac{\partial \bar{f}}{\partial \bar{t}} + \bar{G}(\bar{c})\frac{\partial \bar{f}}{\partial \bar{L}} = 0, \tag{1.2}$$

$$\frac{\partial \bar{c}}{\partial \bar{t}} = -3\rho_{\bar{c}} k_v \bar{G}(\bar{c}) \int_0^\infty \bar{f}\bar{L}^2 \, d\bar{L}, \tag{1.3}$$

where $\bar{t}$ denotes the time, $\bar{c}$ is the solution concentration, $\bar{L}$ is an internal coordinate, the characteristic length of the particle (it can also represent age, composition, or other characteristics of an entity in a distribution depending on the system being modelled, although this may alter the mass balance equation), $\bar{f}(\bar{L}, \bar{t})$ is the probability distribution representing the number concentration of particles of a particular size, $\bar{L}$, at the time $\bar{t}$ (this is commonly known as the population density), $\bar{G}(\bar{c})$ is the crystal growth rate, and $\bar{B}_0(\bar{c})$ is the rate of nucleation of particles of zero size. For a seeded batch crystallizer the dominant mechanism of crystal birth is secondary nucleation, and the nucleation kinetics are typically in power-law form (Nyvlt, Sohnel, Matuchova, and Broul, 1985; William, 2001).

Constants k_b, b, k_g and g are the kinetic parameters, $\rho_{\bar{c}}$ is the crystal density, and c_{sat} is the concentration of the saturated solution. The constant k_v depends on geometry of crystals, in particular, $k_v = 1$ for crystals with the shape of a cube. The integro-differential equation (1.3) is a mass balance which gives the solution concentration.

1.4 The Nonhomogeneous Population Balance Equations Investigated in the Study

Hulburt and Katz (1964) developed the PBE for aggregation using volume as the internal coordinate. The PBE for a well-mixed batch system of constant volume is given by Randolph and Larson (1971). One-dimensional population balance models for both batch and continuous systems as special cases of the generalized population balance model are more frequently applied to industrial granulation processes than its generalized format. For a well-mixed batch system with only one internal coordinate $\bar{x}$ (particle size, volume), the general PBE (Ramkrishna, 2000; Cameron et al., 2005) for a system having aggregation and breakage, and also crystal growth is reduced to the form

$$\begin{aligned}\frac{\partial \bar{f}(\bar{x},\bar{t})}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}[G\bar{f}(\bar{x},\bar{t})] = \frac{1}{2}\int_0^{\bar{x}} K(\bar{x}-\bar{y},\bar{y})\bar{f}(\bar{x}-\bar{y},\bar{t})\bar{f}(\bar{y},\bar{t})\,d\bar{y} \\ -\bar{f}(\bar{x},\bar{t})\int_0^{\infty} K(\bar{x},\bar{y})\bar{f}(\bar{y},\bar{t})\,d\bar{y}, \qquad (1.4)\end{aligned}$$

where $\bar{f}$ is the one-dimensional population density function, representing the particle number distribution in property space, location space and time space. The second term on the right hand side of equation (1.4) accounts for the formation of a cluster of volume $\bar{x}$ resulting from the merger of two clusters with respective volumes $\bar{y}$ and $\bar{x}-\bar{y}, \bar{y} \in (0,\infty)$, i.e., the coagulation of smaller particles to produce one of volume $\bar{x}$. The third term on the right hand side of equation (1.4) describes the loss of the cluster of volume $\bar{x}$ by coagulation with other clusters, i.e., removal of particles of volume $\bar{x}$ as they in turn coagulate to produce larger particles.

The PBE (Ramkrishna, 2000) of continuous systems involving aggregation, breakage, and growth processes with one internal coordinate $\bar{x}$ (such as particle volume or mass) and one external coordinate $\bar{y}$ (i.e., Euclidian coordinate, such as rectangular coordinates, cylindrical coordinates and spherical coordinates used to specify the locations of particles) is given by

$$\frac{\partial \bar{f}(\bar{x},\bar{y},\bar{t})}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}[G\bar{f}(\bar{x},\bar{y},\bar{t})] + \frac{\partial}{\partial \bar{y}}[Z\bar{f}(\bar{x},\bar{y},\bar{t})] = \frac{1}{2}\int_0^{\bar{x}} K(\bar{x}-\bar{z},\bar{z})\bar{f}(\bar{x}-\bar{z},\bar{y},\bar{t})\bar{f}(\bar{z},\bar{y},\bar{t})\,d\bar{z} - \bar{f}(\bar{x},\bar{y},\bar{t})\int_0^{\infty} K(\bar{x},\bar{z})\bar{f}(\bar{z},\bar{y},\bar{t})\,d\bar{z}, \quad (1.5)$$

where $\bar{f}$ is the one-dimensional population density function. The first term on the right hand side of equation (1.5) corresponds to the birth of particles with volume $\bar{x}$ due to the aggregation of two smaller particles with volumes $\bar{x}-\bar{z}$ and $\bar{z}$. Similarly, the second term describes the consumption of particles having volume $\bar{x}$ due to the aggregation with particles of any other volume $\bar{z}$. The spatial velocity is defined as the rate of change of position on the $\bar{y}$-axis with respect to time $\bar{t}$, that is

$$Z = \frac{d\bar{y}}{d\bar{t}}\ .$$

For instance, when Stokes' law (e.g., Batchelor, 1967) is valid (using the terminal velocity of dilute free settling particles in the laminar flow regime), the spatial velocity function can be modeled by

$$Z = z_0\bar{x}^p, \quad p > 0, \quad (1.6)$$

where z_0 is constant.

1.4.1 Crystal Growth Rate

The function G represents a crystal growth rate, having units of mass/time or volume/time. Crystal growth is a multiple step process. The analysis of crystal growth via mass transfer controlled growth is very well known, and there are several theoretical models for integration controlled crystal growth, which are promising, but probably not yet complete. The growth rate G can be a function of the crystal size (volume), position, time and other variables. For instance, the general arguments for equations (1.4) and (1.5) can be made about the crystal growth rate such that, respectively

$$G = G(\bar{x}, \bar{t}), \quad G = G(\bar{x}, \bar{y}, \bar{t}).$$

The growth rate G is typically a highly nonlinear function of its arguments. However it is becoming increasingly recognized that real size dependent growth does not occur to any significant level, with any apparent size dependent growth really being due to a growth rate dispersion mechanism, and as such the growth rate may be considered independent of particle size.

In this dissertation, the growth rate function G is not related to aggregation, but to the growth of clusters by mass transfer of solute from solution to the cluster.

1.4.2 Coagulation Kernel

The aggregation kernel $K(\bar{x}, \bar{z})$ describes the merging rate (kinetics) of two particles of volumes $\bar{x}$ and $\bar{z}$; that is the merging of clusters of volume $\bar{x}$ and volume $\bar{z}$ to produce clusters of volume $\bar{x} + \bar{z}$ which occurs at a rate determined by a symmetric nonnegative coagulation kernel $K(\bar{x}, \bar{z})$, i.e.,

$$K(\bar{x}, \bar{z}) = K(\bar{z}, \bar{x}) \geq 0. \tag{1.7}$$

Different kernels, such as the Brownian motion kernel and the aerosol kernel, may arise in many applications. By far the largest application area is physical chemistry; other fields of application include astronomy, biological entities and bubbles warms. Smit, Hounslow, and Paterson (1994) gave examples of kernels used in physical chemistry. A wider class of important and interesting coagulation kernels $K(\bar{x}, \bar{z})$ arising in applications are homogeneous (Fournier and Laurençot, 2006); that is, there exists some exponent γ such that

$$K(\alpha\bar{x}, \alpha\bar{z}) = \alpha^{\gamma} K(\bar{x}, \bar{z}), \tag{1.8}$$

for every $\alpha, \bar{x}, \bar{z} > 0$. For instance, it has long been recognized that three particular and important kernels $K(\bar{x}, \bar{z})$ are mathematically tractable:

$$K(\bar{x}, \bar{z}) = k_0, \quad k_1(\bar{x} + \bar{z}), \quad k_2\bar{x}\bar{z}, \tag{1.9}$$

where k_0, k_1 and k_2 are positive constants. In addition, notice that for equation (1.8) it is not difficult to show that

$$\bar{x}K_{\bar{x}}(\bar{x}, \bar{z}) = \gamma K(\bar{x}, \bar{z}) - \bar{z}K_{\bar{z}}(\bar{x}, \bar{z}). \tag{1.10}$$

Hence, the general solution of equation (1.10) is given by

$$K(\bar{x}, \bar{z}) = \bar{z}^{\gamma} H\left(\frac{\bar{x}}{\bar{z}}\right), \tag{1.11}$$

where H is an arbitrary function of a single variable.

CHAPTER II

THE METHOD OF MOMENTS AND SOME OTHER TECHNIQUES

In this chapter, the basic concepts of moments and the method of moments are reviewed. In addition, the ideas for finding symmetries of integro-differential equations by using the method of moments are introduced. The homotopy perturbation method and dimensional analysis are also presented in this chapter.

2.1 A Brief Introduction to Moments

Moments are mathematical formulations that allow us to calculate various properties of the distribution. The general idea of consideration of such a system goes back to the pioneering paper (Maxwell, 1867) where the Boltzmann equation was studied by using the power moments defined on a solution of the Boltzmann equation. They are also well-known from statistics (see, e.g., Dette and Studden, 1997): the mean and variance of a statistical distribution are moments of the distribution. Solving the moment form of the population balance (see, e.g., Flood, 2002; 2009) is a very popular method of modeling the evolution of particle size in many particulate systems. In many cases knowledge of the moments of the particle size distribution is sufficient to characterize the system for engineering purposes. The moments of the distribution may be defined in a generic way by:

$$M_j = \int_{-\infty}^{\infty} (L - L_M)^j f(L)\, dL.$$

This equation represents the jth moment of the function $f(L)$ measured around the point L_M, where L_M may be chosen arbitrarily. For instance, there are two values commonly chosen for the arbitrary parameter L_M, namely zero ($L_M = 0$) and the mean of the distribution ($L_M = \bar{L}_M$).

One of the independent variables of the particle size distribution, length, cannot accept a negative value. In this context the lower integration limit will be taken as zero. In this particular case, moments around the original are described by

$$M_j = \int_0^\infty L^j f(L)\, dL.$$

The values of the moments around the origin have particular significance. The zeroth moment around the origin ($j = 0$) represents the total number of particles in the system. The first moment around the origin represents the total linear length in the system; therefore, the first moment divided by the zeroth moment is the number average particle size in the system. Similarly, the second moment around the origin represents the total value of L^2 in the system, which is related to the total surface area in the system divided by the surface area shape factor. The average surface area in the system is the total surface area (calculated from the second moment around the origin) divided by the zeroth moment. The most significant moment about the mean is the second moment which is the variance of the distribution.

2.2 The Method of Moments

The method of moments is a simple deterministic method widely applicable in modeling various particulate processes (Randolph and Larson, 1988). This method is easy to use and computationally inexpensive. For instance, the method of moments is a simple, yet powerful, modeling approach to predict the kinetic profiles of average properties in polymerization systems (Mastan and Zhu, 2015).

Solving the moment form of the PBE is a very popular method of modeling the evolution of particle size in many particulate systems (Ramkrishna, 2000). One of the major advantages of the method of moments is that it simplifies a partial differential equation involving $n + 1$ independent variables into a series of differential equations of n independent variables. Another major advantage of the method of moments is the simplicity of its derivation. The versatility of the method of moments is highlighted by presenting case studies such as those in this thesis. It is our hope that this work can assist interested researchers in utilizing the method of moments for their kinetic modeling.

2.2.1 Applying the Method of Moments to Population Balance Equations

The method of moments represents the distribution by its moments (Randolph and Larson, 1988), reducing the dimensionality of the PBEs.

Practical needs can often be fulfilled by calculating the moments of the population density function. The calculation of such moments can usually be accomplished by directly taking moments of the PBE producing a set of moment equations.

Under certain conditions, the moment equations are closed, that is, the differential equations for the lower order moments do not depend on values for the higher-order moments. If there are only two independent variables, for instance particle size and time, this results in a series of ordinary differential equations that can be solved very efficiently and accurately using ordinary differential equation solvers.

2.2.2 Finding Symmetries of Integro-differential Equations by Using the Method of Moments

The use of the method of moments for finding symmetries of integro-differential equations is based on the idea of the studying an infinite system of partial differential equations (Taranov, 1976; Bunimovich and Krasnoslobodtsev, 1982; 1983).

The method for obtaining symmetries consists of the following steps (Taranov, 1976). A finite subsystem of m moment equations is chosen. Applying the classical group analysis method developed for partial differential equations to the chosen subsystem, one finds the admitted Lie group (algebra) of this subsystem. Expanding the subsystem and letting $m \to \infty$, the intersection of all calculated Lie groups is carried out. The final step consists of returning the obtained symmetries for the moment representation to the symmetries of the original integro-differential equations.

The first application of this method was done in (Taranov, 1976) for the system of the Vlasov-Maxwell collisionless plasma equations, and later this method was applied in (Bunimovich and Krasnoslobodtsev, 1982; 1983) for the models of the Boltzmann equation.

There are some problems in applications of this method. One of them is that, for some equations, the construction of the moment system is impossible. Another problem with the moment system is the infinite number of equations that are involved. However, it is worth noting that among the indirect methods of studying symmetries of integro-differential equations, the method of moments is the most universal one, despite of the substantial restrictions of its applications.

2.3 The Homotopy Perturbation Method

The homotopy perturbation technique was firstly proposed by He (1999) based on the introduction of a homotopy and an artificial parameter for the solution of algebraic equations and ordinary differential equations. To describe this method, we consider the following nonlinear differential equation

$$A(u) - f(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega \tag{2.1}$$

with the boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad \mathbf{r} \in \Gamma,$$

where A is a differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can, generally speaking, be divided into two parts L and N, where L is a linear operator and N is a nonlinear operator. Therefore, equation (2.1) can be rewritten as

$$L(u) + N(u) - f(\mathbf{r}) = 0.$$

By means of the homotopy technique (Liao, 1995; 1997), one can construct a homotopy $v(\mathbf{r}, p) : \Omega \times [0, 1] \to R$:

$$\begin{aligned} H(v,p) &= (1-p)[L(v) - L(u_0)] + p[A(v) - f(\mathbf{r})] \\ &= L(v) - (1-p)L(u_0) + p[N(v) - f(\mathbf{r})] = 0, \ p \in [0,1], \ \mathbf{r} \in \Omega, \end{aligned} \tag{2.2}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation solution of equation (2.1) which satisfies the boundary conditions. Therefore, by means of equation (2.2) we have

$$H(v, 0) = L(v) - L(u_0) = 0, \quad H(v, 1) = A(v) - f(\mathbf{r}) = 0,$$

the changing process of p from zero to unity is just that of $v(\mathbf{r}, p)$ from $u_0(\mathbf{r})$ to $u(\mathbf{r})$. In topology, this is called deformation, $L(v) - L(u_0)$ and $A(v) - f(\mathbf{r})$ are called homotopic. He (1999) used the imbedding parameter p as a "small parameter", and assumed that the solution of equation (2.2) can be written as a power series in p, i.e.,

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + p^4 v_4 + \cdots .$$

By setting $p = 1$, the following approximation solution of (2.1) is obtained,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + v_4 + \cdots . \tag{2.3}$$

The coupling of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation methods. On the other hand, the proposed technique can take full advantage of the traditional perturbation techniques.

The series (2.3) is convergent for most cases, however, the convergence rate depends on the nonlinear operator $A(v)$. The derivative of $N(v)$ with respect to v must be small, because the parameter p may be relatively large, i.e., $p \to 1$. The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one, in order that the series converges. Some examples and studies of this method can also be found in (He, 2000; 2003).

The proposed method does not require small parameters in equations, so the limitations of the traditional perturbation methods can be eliminated. In addition, the initial approximation can be freely selected with possible unknown constants; the approximations obtained by this method are valid not only for small parameters, but also for very large parameters. Moreover, this method can be applicable to nonlinear partial differential equations.

2.4 Dimensional Analysis

It is essential to apply the technique of dimensional analysis to deal with the complicated constant coefficients of original integro-differential equations in this dissertation.

Dimensional analysis (e.g., Bridgman, 1922; Sedov, 1993) is a well-known methodology in physics, chemistry and other traditional engineering and science areas, such as fluid dynamics, heat flow, electromagnetics, astronomy and economics. Szirtes (2007) regards dimensional analysis as an art, his intention here is to remind us that the practitioner of dimensional analysis is offered a refreshing scope for personal initiative in the approach to any given problem.

Dimensional analysis (e.g., Langhaar, 1951; Szirtes, 2007) is often indispensable in treating problems that would otherwise prove intractable. One of its most attractive features is the manner in which it may be applied across a broad range of disciplines–including engineering, physics, biometry, physiology, economics, astronomy, and even social sciences. In particular, in science and engineering, dimensional analysis is the analysis of the relationships between different physical quantities by identifying their fundamental dimensions (such as length, mass, time, and electric charge) and units of measure (such as miles vs. kilometers, or pounds vs. kilograms vs. grams) and tracking these dimensions as calculations or comparisons are performed.

The concept of a scaling group is closely related with the dimensional analysis (Ovsiannikov, 1978), the basis for dimensional analysis is a scaling group. One of the modelling stages of a problem in continuum mechanics is the dimensional analysis of the quantities of the variables involved. This analysis allows forming representations of solutions, which are called self-similar solutions. When constructing exact solutions by the dimensional analysis theory one can use the theory of invariant solutions with respect to the scaling group.

CHAPTER III

GROUP ANALYSIS METHOD

In mathematical physics there are various methods for constructing exact solutions. One of them is based on symmetries of a given equation: the group analysis method (e.g., Ovsiannikov, 1978; Olver, 1993; Ibragimov, 1999; Meleshko, 2005; Grigoriev et al., 2010).

In order to investigate exact solutions of PBEs, the background and difficulty of application of classical Lie group analysis method are reviewed. In addition, some results and applications of the development of the group analysis method to integro-differential equations are also presented in this chapter.

3.1 Classical Group Analysis Method and Difficulties of Its Applications

Before investigation of exact solutions of the PBEs in later chapters, it is essential to know about the history of development and contributions of the classical Lie group analysis method.

3.1.1 Background of the Classical Lie Group Method

A Norwegian mathematician, Sophus Lie, largely created a systematic theory of groups of continuous transformations (Lie, 1871) which are now universally known as Lie groups. A knowledgeable and detailed review of this subject may be found in the historical remarks by Bourbaki (1896), and the references therein.

"Lie groups have had a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically-based science. The application of Lie's continuous symmetry groups include such diverse fields as algebraic topology, differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics and so on (Olver, 1993)." The main idea of original classical Lie groups is to search for the symmetries of differential equations (Lie, 1871). The classical Lie group theory provides an effective tool for studying symmetry groups for systems of differential equations (see, e.g., Lie, 1888; 1896).

From the viewpoint of theoretical treatment, every differential equation can be solved using the Lie group method. However, in practical applications, the most important obstacle for applications of the Lie group method is the fact that it requires huge amounts of analytical calculations that may hardly be performed by pencil and paper for almost any nontrivial example, resulting in limitations of research and other considerations. "Therefore, the Lie group method have been sleeping for nearly half a century, until Birkhoff (1950) called attention to the unexploited applications of Lie groups to the differential equations of fluid mechanics. Subsequently, Ovsiannikov (1978) and his school began a systematic program of successfully applying these methods to a wide range of physically important problems (Olver, 1993)."

In recent years, with the rapid development of science and technology, in particular, the update of computers and softwares, Lie group theory has been widely applied and intensively studied in physics, engineering and natural science. For instance, many applications of Lie group theory can be found in a wide range of literature (see, e.g., Ovsiannikov, 1978; Olver, 1993; Ibragimov, 1999). A collection of results by using this approach can be also found in the Handbooks of Lie group analysis (Ibragimov, 1994; 1995; 1996), and the references therein.

Due to the crucial role of various kinds of transformations in Lie's theory of differential equations, a major part of his work is devoted to establishing a systematic theory of groups of continuous transformations. In summary, Olver (1993) has given an appraisal for Lie's work: "It is impossible to overestimate the importance of Lie's contribution to modern science and mathematics".

3.1.2 Difficulty of Applying the Classical Lie Group Method to Integro-differential Equations

The group analysis method was developed especially for differential equations. The difficulty of applying the classical Lie group method to integro-differential equations will be presented in this subsection.

The principal aim of the group analysis method is to obtain exact solutions and admitted symmetries. The classical group analysis theory provides a universal tool for calculating symmetry groups for systems of differential equations. However, the classical group analysis method cannot be directly applied to integro-differential equations. The major obstacle is that the frames (see, e.g., Ovsiannikov, 1978; Olver,1993; Ibragimov, 1999) of these equations are not locally defined in the space of differential functions. In other words, the main difficulty arises from the integral (nonlocal) terms present in integro-differential equations. In consequence, the crucial idea of splitting determining equations into over-determined systems, commonly used in classical group analysis, fails.

3.2 Development of the Lie Group Method to Integro-Differential Equations

An approach to overcome the difficulty that the classical Lie group method cannot be directly applied to integro-differential equations was proposed by Grigoriev and Meleshko (1986; 1987). In recent years, some applications of Lie group analysis for integro-differential equations were developed (see, e.g., Meleshko, 2005; Grigoriev et al., 2010), and the references therein.

3.2.1 Definition of a Lie Group Admitted by Integro-differential Equation

By means of the work of Meleshko (2005) and Grigoriev et al. (2010), the definition of a Lie group admitted by an integro-differential equation is reviewed. Let us consider an abstract system of integro-differential equation:

$$\Phi(x, u) = 0, \tag{3.1}$$

where u is the vector of the dependent variables, and x is the vector of the independent variables. Assume that a one-parameter Lie group $G^1(X)$ of transformations

$$\bar{x} = f^x(x, u; a), \quad \bar{u} = f^u(x, u; a) \tag{3.2}$$

has the group generator

$$X = \xi^x(x, u)\partial_x + \xi^u(x, u)\partial_u.$$

The one-parameter Lie group $G^1(X)$ of transformations (3.2) is a symmetry group admitted by integro-differential equation (3.1) if $G^1(X)$ satisfies the following equation,

$$(\bar{X}\Phi)(x, u_0(x)) = 0, \tag{3.3}$$

for any solution $u_0(x)$ of integro-differential equation (3.1). Equations (3.3) are called the determining equations. The practical construction of the determining equation for integro-differential equation (3.1) is performed by using the canonical Lie-Bäcklund operator $\bar{X}$

$$\bar{X} = \bar{\eta}^j \partial_{u^j}, \qquad \bar{\eta}^j = \xi^{u^j}(x,u) - \xi^{x_i}(x,u) u^j_{x_i},$$

where the action of the derivative $\bar{\eta}^j \partial_{u^j}$ on the integral terms should be understood in terms of Fréchet derivatives. Most importantly, we should note that the determining equation (3.3) has to be satisfied for arbitrary solutions of the original integro-differential equation (3.1).

Obtaining the determining equations for integro-differential equations, like for differential equations, is not difficult. The crucial difficulty is to find the general solutions of the determining equations.

For a system of differential equations (without nonlocal terms), the approach of constructing the determining equations (3.3) coincides with the determining equations constructed by usual way after some simplifications. These simplifications are related with a consideration of the infinitesimal generator X as Lie-Bäcklund operator (Ibragimov, 1983; Meleshko, 2005; Grigoriev et al., 2010).

3.3 Methods of Finding Symmetries of Integro-differential Equation

There are some different known approaches for calculating symmetry groups of integro-differential equations. Loosely speaking, these approaches can be separated in two large groups: indirect and direct methods.

3.3.1 Indirect Methods

The first indirect approach which is mentioned here is the method of moments, where for each differential equation of the system, containing a finite number of terms, classical group analysis (for differential equations) is applied. Then the process of expansion is carried out. The ideas of the method of moments for obtaining symmetries of integro-differential equations can be found in (Taranov, 1976). In addition, some statements have also been presented in Chapter II.

The second indirect method is to find a representation of an admitted Lie group or a solution on the basis of priori assumption. This approach supposes a priori choice of a form of symmetries or a solution on the basis of some assumptions about the representation. A well-known BKW-solution of the Boltzmann equation was found in this way. For the Boltzmann equation this approach was applied in (Bobylev, 1975; Grigoriev and Meleshko, 1986; Bobylev and Ibragimov, 1989). The main problem in this approach is to discover a representation of the admitted Lie group or solution.

In the third indirect method, the original integro-differential equations are transformed into differential equations. After that the classical algorithm of group analysis is applied to the differential equations. The results of the works (Krook and Wu, 1977; Nonnenmacher, 1984) were obtained by this approach.

The last indirect method developed in (Chetverikov and Kudryavtsev, 1995a; 1995b) consists in reducing the original integro-differential equation to a system of boundary differential equations. This approach is based on Vinogradov's idea to use the notion of a covering (Krasilshchik and Vinogradov, 1984). Introducing nonlocal variables, one needs to eliminate integrals and transform the original integro-differential equation to a system of boundary (or functional) differential equations. As an example of such a transition one can consider the one-dimensional nonlinear Gammerstein integral equation of the second kind:

$$u(x) = \int_a^b K(x, s, u(s))\, ds, \tag{3.4}$$

where the kernel $K(x, s, u)$ is a given function and $x \in [a, b], s \in [a, b]$. The equivalent system of boundary differential equations is introduced as follows

$$v_s(x, s) = K(x, s, u(s)), \quad v(x, a) = 0, \quad u(x) = v(x, b).$$

The latter system involves two independent variables x and s, two dependent variables u and v, and the restrictions of the dependent variable v to the boundary conditions $v(x, a) = 0$ and $v(x, b) = u(x)$. The new dependent variable v is nonlocal because it depends on all values of a solution $u(x)$ on the interval $[a, b]$. For this reason one calls the derived system a covering of equation (3.4). In particular, some applications of this approach to the Smoluchowski coagulation equation were given in (Chetverikov and Kudryavtsev, 1995a; 1995b).

It is worth noting that no heuristic expedient permits a complete answer to the question: what is the widest Lie group of transformations, admitted by a given nonlocal equation. In the first three approaches listed above one needs to prove that the constructed admitted Lie group is a complete group, because, in general case, either loss of some invariant properties or obtaining new ones is possible. For instance, the one-parameter semigroup of integral transformations (Bobylev, 1975) is transferred by the Fourier transformation to the group of point transformations.

3.3.2 Direct Methods

The main advantage of the direct method is that it is able to answer the question about completeness of an admitted Lie group. For the completeness of description group properties of equations with nonlocal operators it is necessary to use a successive approach of group analysis: constructing determining equations and finding their general solutions. According to the algorithm (Meleshko, 2005; Grigoriev et al., 2010), for using the direct method of finding a symmetry group of integro-differential equation one needs to solve a determining equation. In the following, a summary of the methods and ideas which are applied for solving the determining equation (3.3) is presented.

The direct method of finding symmetries were developed in (Grigoriev and Meleshko, 1986; 1987) and applied to finding symmetries of the kinetic Boltzmann equation, the equations of motion of a viscoelastic medium and the Vlasov-Maxwell equations of plasma theory. The method of obtaining the determining equations for integro-differential equations is similar to, and not more difficult than, the way used for differential equations. But one has to notice that the determining equations of integro-differential equations are still integro-differential.

The main obstacle of this approach is in finding the general solutions of the determining equations. For partial differential equations the main method for solving and simplifying determining equations is by their splitting, for which a huge amount of computations need to be accomplished with the aid of computers and professional softwares. Moreover, the techniques for investigating symmetries of a system of differential equations have been improved in recent years with the rapid development and update of computers and softwares. It should be noted that contrary to partial differential equations, approaches of solving determining equations of integro-differential equations depend on the studied equations: there is no a general method for solving determining equations of integro-differential equations.

As mentioned, the main efforts for finding a Lie group admitted by an integro-differential equation are devoted to solving determining equations. By means of the definition of a Lie group admitted by an integro-differential equation, some important approaches and ideas for simplifying and solving determining equations are the following.

Since the determining equation (3.3) has to be satisfied for any solution of the original equation (3.1), the arbitrariness of the solution $u_0(x)$ plays a vital role in the process of solving the determining equations. The important circumstance is the knowledge of properties of solutions of the original integro-differential equation. For instance, one of these properties is the existence of a solution of the Cauchy problem (see, e.g., Lakshmikantham and Rama Mohana Rao, 1995). The most important is that the arbitrariness of the initial conditions in the Cauchy problem allows one to split the determining equations. It is worth to note that the particular class of solutions of the original integro-differential equation may allow us to find the general solutions of the determining equations.

In order to find solutions of the determining equation (3.3) one can expand coordinates of the group generator into formal power series and equate coefficients of various powers. This approach for studying group properties of integro-differential equations was proposed in (Grigoriev and Meleshko, 1986; 1987) and later was applied in (Grigoriev and Meleshko, 1995; 1998; Meleshko, 2005; Grigoriev et al., 2010; Suriyawichitseranee, Grigoriev and Meleshko, 2015).

3.4 Scaling Group

One particular approach for finding exact solutions of integro-differential equation is related with a scaling group. In this section, the definition of a scaling group and its properties are provided (Ovsiannikov, 1978; Meleshko, 2005).

Let $(x_1, x_2, \cdots, x_n)$ and $(u_1, u_2, \cdots, u_m)$ be the independent and dependent variables respectively, and $a = (a_1, \cdots, a_r)$. A transformation

$$h_a : R^{n+m} \to R^{n+m}$$

of the form

$$\bar{x}_i = x_i \prod_{\alpha=1}^{r} (a_\alpha)^{\lambda_\alpha^i}, \quad i = 1, \cdots, n, \quad \bar{u}_k = u_k \prod_{\alpha=1}^{r} (a_\alpha)^{\mu_\alpha^k}, \quad k = 1, \cdots, m$$

is called a scaling group H^r of transformations of the space $R^{n+m}(x, u)$. The variables a_α $(\alpha = 1, \cdots, r)$ are called its parameters. It is natural to require

$$\text{rank} \begin{pmatrix} \lambda_1^1 & \cdots & \lambda_1^n & \mu_1^1 & \cdots & \mu_1^m \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_r^1 & \cdots & \lambda_r^n & \mu_r^1 & \cdots & \mu_r^m \end{pmatrix} = r.$$

Otherwise, introducing new scaling parameters, it is possible to reduce the number r.

The infinitesimal operator of the scaling group H^r is defined by the linear differential operator

$$\zeta_j \partial = \sum_{i=1}^{n} \lambda_j^i x_i \frac{\partial}{\partial x_i} + \sum_{l=1}^{m} \mu_j^l u_l \frac{\partial}{\partial u_l}.$$

CHAPTER IV

EXACT SOLUTIONS OF HOMOGENEOUS POPULATION BALANCE EQUATIONS (1.1)-(1.3)

In this chapter, the homogeneous PBEs (1.1)-(1.3) are studied by using the method of moments and Lie group method, and the exact solutions are also presented.

4.1 Dimensional Analysis

For dimensional analysis, let us consider the following transformations

$$\bar{t} = t_0 t, \quad \bar{L} = L_0 L, \quad \bar{f} = f_0 f, \quad \frac{\bar{c} - c_{sat}}{c_{sat}} = \tilde{c}c, \tag{4.1}$$

where t_0, L_0, f_0 and $\tilde{c}$ are positive real numbers. Applying the transformations (4.1) to equations (1.1)-(1.3), one obtains the following equations

$$\bar{G}(c_{sat}(1 + \tilde{c}c)) = k_g \tilde{c}^g c^g, \qquad \bar{B}_0(c_{sat}(1 + \tilde{c}c)) = k_b \tilde{c}^b c^b,$$

$$\frac{\partial f}{\partial t} + \frac{t_0 k_g \tilde{c}^g}{L_0} c^g \frac{\partial f}{\partial L} = 0, \quad \frac{\partial c}{\partial t} = -\frac{3\rho_{\bar{c}} k_v f_0 L_0^3 t_0 k_g \tilde{c}^g}{c_{sat}\tilde{c}} c^g \int_0^\infty f L^2 \, dL.$$

Letting that

$$L_0 = t_0 k_g \tilde{c}^g, \quad f_0 = \frac{c_{sat}}{3\rho_{\bar{c}} k_v t_0^4 k_g^4 \tilde{c}^{4g-1}},$$

and denoting

$$G(c) = k_g \tilde{c}^g c^g, \quad B_0(c) = k_b \tilde{c}^b c^b, \tag{4.2}$$

equations (1.2) and (1.3) become

$$\frac{\partial f}{\partial t} + c^g \frac{\partial f}{\partial L} = 0, \tag{4.3}$$

$$\frac{\partial c}{\partial t} = -c^g \int_0^\infty f L^2 \, dL. \tag{4.4}$$

4.2 Search for Solutions of the Form $f = f(L,t), c = c(t)$

4.2.1 The Methods of Moments

The moments for equations (4.3) and (4.4) are defined as

$$M_m = \int_0^\infty L^m f(L,t)\, dL, \quad m = 0, 1, 2, \cdots .$$

Multiplying (4.3) with L^m and integrating it with respect to L, one obtains a system of ordinary differential equations for the moments. Since the population density must vanish for (arbitrarily large sizes) infinite sized particles (Ramkrishna, 2000), and because of equation (4.2), we have for $m = 0$

$$\frac{dM_0}{dt} = c^g f(0,t) = c^g \frac{B_0(c)}{G(c)} = kc^b, \tag{4.5}$$

where $k = \frac{k_b \tilde{c}^b}{k_g \tilde{c}^g} > 0$. For $m = 1, 2$ integration by parts leads to the equations

$$\frac{dM_m}{dt} = mc^g M_{m-1}, \quad m = 1, 2.$$

Thus we derive the Cauchy problem which consists of the equations

$$\frac{dM_0}{dt} = kc^b, \quad \frac{dM_1}{dt} = c^g M_0, \quad \frac{dM_2}{dt} = 2c^g M_1, \quad \frac{dc}{dt} = -c^g M_2, \tag{4.6}$$

and the initial conditions

$$c(L,0) = c_0, \quad M_0(0) = \int_0^\infty f(L,0)\, dL, \tag{4.7}$$

$$M_1(0) = \int_0^\infty f(L,0) L \, dL, \quad M_2(0) = \int_0^\infty f(L,0) L^2 \, dL. \tag{4.8}$$

Assume that $M_2 \neq 0$, we derive from equations (4.6) the relations

$$\frac{dM_0}{dc} = -k\frac{c^{b-g}}{M_2}, \quad \frac{dM_1}{dc} = -\frac{M_0}{M_2}, \quad \frac{dM_2}{dc} = -2\frac{M_1}{M_2}. \tag{4.9}$$

4.2.2 Admitted Scaling Group

According to the algorithm (Meleshko, 2005; Grigoriev et al., 2010), a scaling group H^1 is defined by

$$M_0 = \tilde{M}_0 e^{k_0 a}, \quad M_1 = \tilde{M}_1 e^{k_1 a}, \quad M_2 = \tilde{M}_2 e^{k_2 a}, \quad c = \tilde{c} e^{k_c a},$$

where a is a group parameter. Under this change, equations (4.9) become

$$\frac{d\tilde{M}_0}{d\tilde{c}} = -k\frac{\tilde{c}^{b-g}}{\tilde{M}_2} e^{(k_c(b-g)-k_2-k_0+k_c)a}, \quad \frac{d\tilde{M}_1}{d\tilde{c}} = -\frac{\tilde{M}_0}{\tilde{M}_2} e^{(k_0-k_2-k_1+k_c)a},$$
$$\frac{d\tilde{M}_2}{d\tilde{c}} = -2\frac{\tilde{M}_1}{\tilde{M}_2} e^{(k_1-2k_2+k_c)a}.$$

An admitted scaling group requires that the manifold, assigned by equations (4.9), is invariant. This condition leads to the equations

$$k_0 = \frac{1}{4}(3(b-g)+1)k_c, \quad k_1 = \frac{1}{2}(b-g+1)k_c, \quad k_2 = \frac{1}{4}(b-g+3)k_c. \quad (4.10)$$

The infinitesimal generator of the group H^1 is

$$X = 4c\frac{\partial}{\partial c} + (3(b-g)+1)M_0\frac{\partial}{\partial M_0} + 2(b-g+1)M_1\frac{\partial}{\partial M_1} + (b-g+3)M_2\frac{\partial}{\partial M_2}.$$

A solution invariant with respect to the scaling group H^1 has the representation

$$M_0 = \alpha c^{\frac{k_0}{k_c}}, \quad M_1 = \beta c^{\frac{k_1}{k_c}}, \quad M_2 = \gamma c^{\frac{k_2}{k_c}}, \quad (4.11)$$

where the coefficients k_0, k_1, k_2 and k_c are related by (4.10). By virtue of the initial conditions (4.7) and (4.8) we also have

$$\alpha c_0^{\frac{k_0}{k_c}} = M_0(0), \quad \beta c_0^{\frac{k_1}{k_c}} = M_1(0), \quad \gamma c_0^{\frac{k_2}{k_c}} = M_2(0). \quad (4.12)$$

For finding dependence on time t of the moments we have to solve the last equation of (4.6), which becomes

$$c^{-\frac{1}{4}(b+3g+3)}dc = -\gamma dt. \quad (4.13)$$

Notice that since $M_i \geq 0$ $(i = 0, 1, 2)$, it is required that $\alpha \geq 0, \beta \geq 0$ and $\gamma \geq 0$. Substituting (4.11) into equations (4.9), and using the conditions $\alpha \geq 0, \beta \geq 0$ and $\gamma \geq 0$, we find that

$$\alpha = p^3(g-b-1)(g-b-3)\sqrt{2}, \quad \beta = p^2(g-b-3), \quad \gamma = 2p\sqrt{2},$$

and

$$g-b-3 > 0,$$

where

$$p = \sqrt[4]{\frac{k}{(g-b-1)(g-b-3)\left(3(g-b)-1\right)}}.$$

Integration of equation (4.13) depends on the value of $b + 3(g+1)$.

Assume that $b + 3g \neq 1$, the solution of equation (4.13) is

$$c(t) = (\mu\gamma t + c_0^{-\mu})^{-\frac{1}{\mu}},$$

where $\mu = \frac{1}{4}(b + 3g - 1)$. The general solution of equation (4.3) is

$$f(L,t) = F(\varphi(L,t)),$$

where F is an arbitrary positive function of a single independent variable, and $\varphi(L,t)$ is a first integral of equation (4.3)

$$\varphi(L,t) = L - \frac{1}{\gamma(\mu - g)}(\mu\gamma t + c_0^{-\mu})^{1-\frac{g}{\mu}}.$$

In particular, for $t = 0$ one derives that

$$f(L,0) = F\left(L - \frac{4}{\gamma(b-g-1)}c_0^{\frac{g-b+1}{4}}\right).$$

Since the function $f(L,0)$ has to satisfy the conditions in (4.12), the function F has to satisfy the relations

$$\alpha c_0^{\frac{k_0}{k_c}} = \int_0^\infty F\left(L - \frac{4}{\gamma(b-g-1)}c_0^{\frac{g-b+1}{4}}\right) dL,$$

$$\beta c_0^{\frac{k_1}{k_c}} = \int_0^\infty F\left(L - \frac{4}{\gamma(b-g-1)}c_0^{\frac{g-b+1}{4}}\right) L\, dL,$$

$$\gamma c_0^{\frac{k_2}{k_c}} = \int_0^\infty F\left(L - \frac{4}{\gamma(b-g-1)}c_0^{\frac{g-b+1}{4}}\right) L^2\, dL.$$

Letting $b+3g=1$, equations (4.13) becomes

$$\frac{dc}{c} = -\gamma dt. \tag{4.14}$$

Because the general solution of (4.14) is

$$c(t) = c_0 e^{-\gamma t},$$

the first integral of equation (4.3) is $L + \frac{c_0^g}{\gamma g} e^{-\gamma g t} = C$, where C is constant. Thus the general solution of equation (4.3) is

$$f(L,t) = F\left(L + \frac{c_0^g}{\gamma g} e^{-\gamma g t}\right),$$

where F is an arbitrary positive function of a single variable. By virtue of the initial conditions (4.12), this function has to satisfy the conditions

$$\alpha c_0^{\frac{k_0}{k_c}} = \int_0^\infty F\left(L + \frac{c_0^g}{\gamma g}\right) dL, \quad \beta c_0^{\frac{k_1}{k_c}} = \int_0^\infty F\left(L + \frac{c_0^g}{\gamma g}\right) L\, dL,$$

$$\gamma c_0^{\frac{k_2}{k_c}} = \int_0^\infty F\left(L + \frac{c_0^g}{\gamma g}\right) L^2\, dL.$$

Remark 1: The solutions presented above are exact solutions of equations (4.3) and (4.4). In real industrial applications using the PBE, the kinetic parameters b and g are assumed to satisfy the following inequalities

$$0 \leq b \leq 6, \quad 1 \leq g \leq 2.$$

For such b and g the found solutions contradict the requirements that $\alpha \geq 0, \beta \geq 0$ and $\gamma \geq 0$.

4.3 Application of Scaling Group

Since the application of the group analysis method to integro-differential equations is complicated, usually a particular class of transformations is considered.

In this section we apply a scaling group of transformations for constructing exact solutions of equations (4.3) and (4.4).

First of all we note that in the previous sections we considered the case where it is assumed that $c_L = 0$. This property is usually considered in formulation of a model, whereas equation (4.4) only requires that

$$\frac{\partial}{\partial L}(c^{-g}c_t) = 0.$$

Notice that the first prolongation of the generator (Ovsiannikov, 1978)

$$X = k_1 L\frac{\partial}{\partial L} + k_2 t\frac{\partial}{\partial t} + k_3 f\frac{\partial}{\partial f} + k_4 c\frac{\partial}{\partial c}$$

is

$$\underset{1}{X} = X + (k_4 - k_1)c_L\frac{\partial}{\partial c_L} + (k_4 - k_2)c_t\frac{\partial}{\partial c_t}.$$

Applying $\underset{1}{X}$ to the derivative c_L, one obtains $\underset{1}{X}\, c_L = (k_4 - k_1)c_L$. According to the definition of the invariance of the value c_L, one has $\underset{1}{X}\, c_L = 0$, which leads to the condition $(k_4 - k_1)c_L = 0$, which means that either $k_4 - k_1 = 0$ or one assumes that $c_L = 0$.

A one-parameter scaling group H^1 of transformations of the space $R^{2+2}(L, t, f, c)$ is defined as follows

$$\bar{L} = La^{\lambda_1}, \quad \bar{t} = ta^{\lambda_2}, \quad \bar{f} = fa^{\mu_1}, \quad \bar{c} = ca^{\mu_2},$$

where $a > 0$ is a group parameter*. The infinitesimal operator of the group H^1 is

$$X = \lambda_1 L\frac{\partial}{\partial L} + \lambda_2 t\frac{\partial}{\partial t} + \mu_1 f\frac{\partial}{\partial f} + \mu_2 c\frac{\partial}{\partial c}.$$

Since equation (4.3) is a partial differential equation, application of the classical group analysis method gives that equation (4.3) is invariant with respect to the scaling group H^1 if

$$\mu_2 = \frac{1}{g}(\lambda_1 - \lambda_2).$$

*For a Lie group one assumes that $a = e^b$, where b is a Lie group parameter.

For obtaining conditions of invariance of equation (4.4), it is recalled here how a solution is transformed under the scaling group H^1

$$\bar{f}(\bar{L},\bar{t}) = a^{\mu_1} f(a^{-\lambda_1}\bar{L}, a^{-\lambda_2}\bar{t}), \quad \bar{c}(\bar{L},\bar{t}) = a^{\mu_2} c(a^{-\lambda_1}\bar{L}, a^{-\lambda_2}\bar{t}).$$

This leads to the equation

$$\begin{aligned} \frac{\partial \bar{c}}{\partial \bar{t}} + \bar{c}^g \int_0^\infty \bar{f}\bar{L}^2\, d\bar{L} &= a^{\mu_2-\lambda_2} c_t + c^g a^{g\mu_2+\mu_1+3\lambda_1} \int_0^\infty f L^2\, dL \\ &= c^g (a^{g\mu_2+\mu_1+3\lambda_1} - a^{\mu_2-\lambda_2}) \int_0^\infty f L^2\, dL = 0, \end{aligned}$$

where we used the property that $c(L,t)$ is a solution of equation (4.4). The latter equation gives

$$(1-g)\mu_2 - \mu_1 = 3\lambda_1 + \lambda_2.$$

Looking for invariants of the admitted scaling group in the form

$$J_k(L,t,f,c) = L^{\theta_1^k} t^{\theta_2^k} f^{\sigma_1^k} c^{\sigma_2^k}, \ k = 1,2,3,$$

we derive

$$\lambda_1\theta_1^k + \lambda_2\theta_2^k + ((\frac{1}{g} - 4)\lambda_1 - \frac{1}{g}\lambda_2)\sigma_1^k + \frac{1}{g}(\lambda_1 - \lambda_2)\sigma_2^k = 0, \quad k = 1,2,3.$$

Further study of invariants depends on the values of λ_1 and λ_2.

4.3.1 Case $\lambda_1 = 0, \ \lambda_2 \neq 0$

In this case the invariants can be chosen as follows

$$J_1 = L, \quad J_2 = t^{\frac{1}{g}} f, \quad J_3 = t^{\frac{1}{g}} c$$

and an invariant solution has the representation

$$f(L,t) = t^{-\frac{1}{g}}\phi_1(L), \quad c(L,t) = t^{-\frac{1}{g}}\phi_2(L).$$

Substituting the representation of the invariant solution into equations (4.3) and (4.4), we have

$$\phi_2^g\phi_1' - \frac{1}{g}\phi_1 = 0, \quad \int_0^\infty \phi_1 L^2\, dL - \frac{1}{g}\phi_2^{1-g} = 0. \tag{4.15}$$

If one assumes that $g \neq 1$, then from the second equation of (4.15) one has that ϕ_2 is constant, say C_2. The general solution of the first equation of (4.15) is

$$\phi_1(L) = C_1 e^{\frac{1}{gC_2^g}L},$$

where $C_1 > 0$ is constant. This leads to a divergent integral in the second equation of (4.15). Hence one has to assume that $g = 1$. For $g = 1$ the function ϕ_1 is an arbitrary function satisfying the second condition of (4.15). The function ϕ_2 is defined by the first equation of (4.15), i.e.,

$$\int_0^\infty \phi_1 L^2\, dL = 1, \qquad \phi_2 = \frac{\phi_1}{\phi_1'}.$$

4.3.2 Case $\lambda_1 \neq 0, \ \lambda_2 \neq 0$

Assuming that $\lambda_2 \neq 0$, one can choose the following invariants of the admitted scaling group

$$J_1 = t^\alpha L, \quad J_2 = t^{\frac{1}{g}+(\frac{1}{g}-4)\alpha} f, \quad J_3 = t^{\frac{1}{g}(1+\alpha)} c,$$

where $\alpha = -\frac{\lambda_1}{\lambda_2} \neq 0$. A representation of self-similar solutions is

$$f(L,t) = t^{(4-\frac{1}{g})\alpha - \frac{1}{g}}\phi_1(x), \quad c(L,t) = t^{-\frac{1}{g}(1+\alpha)}\phi_2(x), \tag{4.16}$$

where $x = t^\alpha L$, ϕ_1 and ϕ_2 are nonnegative continuously differentiable functions. Substituting the representation of the invariant solutions into equations (4.3) and (4.4), one derives

$$\beta\phi_1 + \phi_1'(\alpha x + \phi_2^g) = 0, \tag{4.17}$$

$$\gamma\phi_2(x) + \alpha x\phi_2' + \phi_2^g \int_0^\infty x^2\phi_1(x)\, dx = 0, \tag{4.18}$$

where

$$\beta = (4 - \frac{1}{g})\alpha - \frac{1}{g}, \quad \gamma = -\frac{1}{g}(1+\alpha).$$

4.3.3 Solutions with $c_L = 0$

The condition $c_L = 0$ implies that $\phi_2(x) = C_2 > 0$, where C_2 is constant. The general solution of equation (4.17) is

$$\phi_1(x) = C_1|\alpha x + C_2^g|^{-\frac{\beta}{\alpha}},$$

where $C_1 > 0$ is also constant. Equation (4.18) is rewritten as

$$\int_0^\infty x^2|\alpha x + C_2^g|^{-n}\,dx = -\frac{\gamma}{C_1 C_2^{g-1}}, \tag{4.19}$$

where $n = \frac{\beta}{\alpha}$. For convergence of the integral in (4.19) one has to require that $\alpha > 0,\ \ n-2 > 1$, which leads to the condition

$$g > 1 + \frac{1}{\alpha}.$$

Because of this condition equation (4.19) becomes

$$6C_1 C_2^{g(4-n)-1} = -\gamma\alpha^3(n-1)(n-2)(n-3). \tag{4.20}$$

Finally, the invariant solution is

$$f(L,t) = C_1 t^\beta(\alpha t^\alpha L + C_2^g)^{-n}, \qquad c(t) = C_2 t^{-\frac{1}{g}(1+\alpha)},$$

where $\alpha > \frac{1}{g-1}$, and the constants $C_1 > 0,\ C_2 > 0$ are related by the condition (4.20).

4.3.4 Solutions without the Assumption that $c_L = 0$

For $c_L \neq 0$ one has to assume that $\phi_2'(x) \neq 0$. Denoting

$$0 < I = \int_0^\infty x^2\phi_1(x)\,dx < \infty,$$

the general solution of equation (4.18) for $g \neq 1$ is given by the formula

$$\phi_2(x) = \left(\frac{C\gamma}{x^m - CI}\right)^{\frac{1}{g-1}}, \qquad (4.21)$$

where $\gamma = -\frac{1}{g}(1+\alpha)$, $m = \frac{\gamma(g-1)}{\alpha}$ and C is a constant of the integration such that $\frac{C\gamma}{x^m - CI} > 0$ for any $x > 0$. Substituting (4.21) into (4.17), one obtains

$$\phi_1(x) = e^{\psi(x)}, \quad \psi(x) = -\int \frac{\beta(x^m - CI)^n}{\alpha x(x^m - CI)^n + (C\gamma)^n} dx,$$

where $n = \frac{g}{g-1}$. Thus, we obtained the following invariant solution of equations (4.3) and (4.4)

$$f(L,t) = t^{(4-\frac{1}{g})\alpha - \frac{1}{g}} e^{\psi(t^\alpha L)}, \quad c(L,t) = t^\gamma \left(\frac{C\gamma}{t^{\alpha m}L^m - CI}\right)^{\frac{1}{g-1}},$$

where the constant I is found from the equation

$$0 < I = \int_0^\infty x^2 e^{-\int \frac{\beta(x^m - CI)^n}{\alpha x(x^m - CI)^n + (C\gamma)^n} dx} dx < \infty.$$

Similarly, we can obtain an invariant solution where $g = 1$. In this case for $\alpha \neq 0$, the general solution of equations (4.17), (4.18) is

$$\phi_1(x) = C_1 \mid \alpha x^{\frac{I+\gamma+\alpha}{\alpha}} + C_2 \mid^{-\frac{\beta}{I+\gamma+\alpha}}, \quad \phi_2(x) = C_2 x^{-\frac{I+\gamma}{\alpha}}, \qquad (4.22)$$

where $C_2 > 0$ and $C_1 > 0$ are constant. Substituting (4.22) into (4.16) respectively, one finds the invariant solutions of equations (4.3) and (4.4)

$$f(L,t) = C_1 t^\beta \mid \alpha t^{I+\gamma+\alpha} L^{\frac{I+\gamma+\alpha}{\alpha}} + C_2 \mid^{-\frac{\beta}{I+\gamma+\alpha}}, \quad c(L,t) = C_2 t^{-I} L^{-\frac{I+\gamma}{\alpha}},$$

where as in the previous case the constant I has to be found from the nonlinear functional equation

$$\Phi(I) = I - C_1 \int_0^\infty x^2 \mid \alpha x^{\frac{I+\gamma+\alpha}{\alpha}} + C_2 \mid^{-\frac{\beta}{I+\gamma+\alpha}} dx = 0,$$

$$\beta\phi_1 + \phi_1'(\alpha x + \phi_2) = 0, \quad \gamma\phi_2(x) + \alpha x \phi_2' + \phi_2 \int_0^\infty x^2 \phi_1(x)\, dx = 0,$$

where $\beta = 3\alpha - 1,\ \gamma = -(1+\alpha)$.

Remark 2: One can notice that releasing the physical condition that $\phi_2(x) > 0$, one can obtain many exact solutions of equations (4.3) and (4.4) of the form (4.16) in the case $g = 1$ and $\alpha = 0$. In fact, in this case $\beta = \gamma = -1$, and equations (4.17) and (4.18) become

$$\phi_1'\phi_2 - \phi_1 = 0, \quad \int_0^\infty x^2\phi_1(x)\,dx = 1. \tag{4.23}$$

Choosing the function $\phi_1(x)$ such that $\phi_1' \neq 0$ and the integral $\int_0^\infty x^2\phi_1(x)\,dx = 1$, the function ϕ_2 becomes $\phi_2 = \frac{\phi_1}{\phi_1'}$. In particular,

$$\phi_1(x) = \frac{p^3}{2}e^{-px}, \;\; \phi_2(x) = -\frac{1}{p}$$

is a solution of equations (4.23), where p is constant.

4.4 Admitted Lie Group of Equations (4.3) and (4.4)

In this section the recently developed group analysis for finding admitted Lie groups of integro-differential equations is applied (Meleshko, 2005; Grigoriev et al., 2010). Since we consider solutions where $c = c(t)$, equations (4.3) and (4.4) are supplemented by the auxiliary equation

$$c_L = 0. \tag{4.24}$$

Let a one-parameter Lie group of transformations admitted by equations (4.3), (4.4) and (4.24) have the infinitesimal generator

$$X = \xi^L\frac{\partial}{\partial L} + \xi^t\frac{\partial}{\partial t} + \eta^f\frac{\partial}{\partial f} + \eta^c\frac{\partial}{\partial c}.$$

According to the algorithm (Meleshko, 2005; Grigoriev et al., 2010), the determining equations for (4.3), (4.4) and (4.24) are

$$(D_t\bar{\eta}^f + gc^{g-1}\bar{\eta}^c f_L + c^g D_L\bar{\eta}^f)\,|_{(S)} = 0, \tag{4.25}$$

$$(D_L\bar{\eta}^c)\mid_{(S)}=0, \tag{4.26}$$

$$\left(D_t\bar{\eta}^c + gc^{g-1}\bar{\eta}^c\int_0^\infty fL^2\,dL + c^g\int_0^\infty L^2\bar{\eta}^f\,dL\right)\Big|_{(S)} = 0, \tag{4.27}$$

where

$$\bar{\eta}^f = \eta^f - \xi^L f_L - \xi^t f_t, \quad \bar{\eta}^c = \eta^c - \xi^t c_t,$$

D_L and D_t are the total derivatives with respect to the independent variables L and t, respectively. The subscript $|_{(S)}$ means that the expression is satisfied for any solution of system (4.3), (4.4) and (4.24). In particular, this allows us to derive

$$f_t = -c^g f_L, \quad f_{tL} = -c^g f_{LL}, \quad f_{tt} = -gc^{g-1}c_t f_L + c^{2g} f_{LL}, \tag{4.28}$$

$$c_{tt} = gc^{2g-1}\left(\int_0^\infty L^2 f(L,t)\,dL\right)^2 - 2c^{2g}\int_0^\infty Lf(L,t)\,dL. \tag{4.29}$$

In deriving equation (4.29) we used integration by parts and the property that population density must vanish for infinite sized particles. Notice also that

$$f(t,L) = 0, \;\; c(t) = c_0 \tag{4.30}$$

is a solution of equations (4.3) and (4.4).

In the case of partial differential equations, their determining equations are usually split with respect to parametric derivatives. The set of parametric derivatives is defined on the base of the arbitrariness of initial data of a Cauchy problem having a solution. The complete set of solutions of the determining equations (4.25), (4.26) and (4.27) is also sought under the assumption that there exists a solution of the Cauchy problem

$$f(t_0,L) = f_0(L), \;\; c(t_0) = c_0$$

for arbitrary constant c_0 and sufficiently smooth function $f_0(L)$. This assumption allows us to consider f_L and c_t as parametric variables in the determining equations (4.25), (4.26) and (4.27). In fact, let us take the form of $f_0(L)$ as follows

$$f_0(L) = e^{-L}(a_0 + a_1 L), \tag{4.31}$$

where a_0 and a_1 are arbitrary constants. Using equations (4.4) and (4.31), we find that

$$f_L(t_0, L) = e^{-L}(a_1 - a_0 - a_1 L), \quad c_t(t_0) = -c_0^g(2a_0 + 6a_1).$$

By virtue of the arbitrariness of the constants a_0 and a_1, and since the Jacobian is given by

$$\left.\frac{\partial(f_L, c_t)}{\partial(a_0, a_1)}\right|_{t=t_0} = 2c_0^g(4 - L)e^{-L},$$

we can consider $f_L(t_0, L)$ and $c_t(t_0)$ as parametrical variables. This allows us to split the determining equations (4.25), (4.26) and (4.27) with respect to f_L and c_t.

Consider equation (4.26). It can be rewritten as follows

$$\eta^c_L + \eta^c_f f_L - (\xi^t_L + \xi^t_f f_L)c_t = 0,$$

splitting this equation with respect to f_L and c_t, one obtains

$$\eta^c_L = 0, \quad \eta^c_f = 0, \quad \xi^t_L = 0, \quad \xi^t_f = 0. \tag{4.32}$$

Substituting (4.28) into (4.25), one has

$$f_L(c^g\xi^t_t + gc^{g-1}\eta^c - \xi^L_t - \xi^L_L c^g) + f_L c_t(\xi^t_c c^g - \xi^L_c) + \eta^f_c c_t + \eta^f_t + c^g\eta^f_L = 0.$$

Splitting this equation with respect to f_L and c_t, we get

$$\xi^L_c = c^g\xi^t_c, \tag{4.33}$$

$$\xi^L_t + \xi^L_L c^g = c^g\xi^t_t + gc^{g-1}\eta^c, \tag{4.34}$$

$$\eta^f_c = 0, \quad \eta^f_t + c^g\eta^f_L = 0. \tag{4.35}$$

The general solution of equations (4.32) and (4.35) is

$$\eta^f = \eta^f(f), \;\; \eta^c = \eta^c(t, c), \;\; \xi^t = \xi^t(t, c). \tag{4.36}$$

Substituting them into equation (4.27) and simplifying, one has

$$\eta_t^c + (\xi_t^L + \xi_L^L c^g - c^g \eta_c^c) \int_0^\infty f L^2 \, dL - c^{2g} \xi_c^t (\int_0^\infty f L^2 \, dL)^2$$
$$+2c^{2g} \xi^t \int_0^\infty f L \, dL + c^g \int_0^\infty L^2 \eta^f(f) \, dL + c^g \int_0^\infty L^2 (\xi^t c^g - \xi^L) f_L \, dL = 0. \quad (4.37)$$

Using (4.32), the property that the population density must vanish for infinite sized particles (Ramkrishna, 2000), and applying integration by parts, the last term in equation (4.37) can be represented in the form

$$c^g \int_0^\infty L^2 (\xi^t c^g - \xi^L) f_L \, dL$$
$$= -2c^{2g} \xi^t \int_0^\infty f L \, dL + 2c^g \int_0^\infty f L \xi^L \, dL + c^g \int_0^\infty f L^2 \xi_L^L \, dL.$$

Hence, equation (4.37) becomes

$$\eta_t^c + (\xi_t^L + \xi_L^L c^g - c^g \eta_c^c) \int_0^\infty f L^2 \, dL - c^{2g} \xi_c^t (\int_0^\infty f L^2 \, dL)^2$$
$$+c^g \int_0^\infty L^2 \eta^f(f) \, dL + 2c^g \int_0^\infty f L \xi^L \, dL + c^g \int_0^\infty f L^2 \xi_L^L \, dL = 0. \quad (4.38)$$

The determining equations have to be satisfied for any solution of equations (4.3) and (4.4), in particular, for the solution (4.30). Substituting (4.30) into (4.38), we derive

$$\eta_t^c + \eta^f(0) c^g \int_0^\infty L^2 \, dL = 0. \quad (4.39)$$

Because of the divergence of the integral $\int_0^\infty L^2 \, dL = \infty$ in (4.39), we conclude that

$$\eta^f(0) = 0, \quad \eta_t^c = 0,$$

which by virtue of (4.32) gives that

$$\eta^c = \eta^c(c). \quad (4.40)$$

Using equations (4.33), (4.34), (4.36) and (4.40), we derive that

$$\xi^L = C_1 L + \mu(f) + h(t, c),$$

where C_1 is constant, μ and h are arbitrary continuously differentiable functions and

$$h_c = c^g \xi^t_c.$$

Equation (4.38) becomes

$$(\eta^c_c - \xi^t_t - gc^{-1}\eta^c)c_t - \xi^t_c c_t^2 - 3C_1 c_t + 2c^g h \int_0^\infty fL\,dL$$
$$+c^g \int_0^\infty L^2 \eta^f(f)\,dL + 2c^g \int_0^\infty fL\mu(f)\,dL = 0. \qquad (4.41)$$

For further steps in solving the determining equation (4.41) we assume that the functions μ and η^f are represented by Taylor series

$$\mu = \sum_{k=0}^{\infty} q_k f^k, \quad \eta^f = \sum_{k=1}^{\infty} p_k f^k.$$

The coefficients of these series can be found using a particular class of solutions (Meleshko, 2005; Grigoriev et al., 2010) of (4.3) and (4.4). Consider the class of solutions which is defined by the initial conditions

$$f(t_0, L) = ae^{-bL}, \ \ c(t_0) = c_0, \quad a > 0, \ b > 0,$$

at a given but arbitrary time $t = t_0$. Hence, by virtue of (4.4) we obtain

$$c_t(t_0) = -2c_0^g b^{-3} a.$$

Notice also that

$$\int_0^\infty Le^{-(k+1)bL}\,dL = \frac{b^{-2}}{(k+1)^2}, \qquad \int_0^\infty L^2 e^{-kbL}\,dL = \frac{2b^{-3}}{k^3}.$$

Considering the determining equation (4.41) at the time $t = t_0$, we have

$$b^3 \left(\xi^t_t(t_0, c_0) + gc_0^{-1}\eta^c(c_0) - \eta^c_c(c_0) + 3C_1 + \sum_{k=1}^{\infty} \frac{p_k a^{k-1}}{k^3} \right)$$
$$+b^4 \left(h(t_0, c_0) + \sum_{k=0}^{\infty} \frac{q_k a^k}{(k+1)^2} \right) - 2\xi^t_c(t_0, c_0)c_0^g a = 0. \qquad (4.42)$$

Using the arbitrariness of the value b, the equation (4.42) can be split into a series of equations by equating to zero the coefficients with b:

$$\xi^t_t(t_0,c_0)+gc_0^{-1}\eta^c(c_0)-\eta^c_c(c_0)+3C_1+\sum_{k=1}^{\infty}\frac{p_k a^{k-1}}{k^3}=0,$$

$$\xi^t_c(t_0,c_0)=0,\qquad h(t_0,c_0)+\sum_{k=0}^{\infty}\frac{q_k a^k}{(k+1)^2}=0.$$

Because of the arbitrariness of the value a, the latter equations give

$$h(t_0,c_0)=-q_0,\quad q_k=0,\quad (k=1,2,\ldots).$$

$$p_1=\eta^c_c(c_0)-\xi^t_t(t_0,c_0)-gc_0^{-1}\eta^c(c_0)-3C_1,\quad p_k=0,\quad (k=2,3,\ldots).$$

Thus, we obtain

$$h+\mu=0,\qquad \xi^t_c=0,$$

and

$$\xi^L=C_1L,\quad \eta^f=p_1 f.$$

Substituting ξ^L into (4.34), and using the property that $\eta^c=\eta^c(c)$, we derive that

$$\xi^t=C_2t+C_3,\qquad \eta^c=\frac{1}{g}(C_1-C_2)c,\qquad p_1=\frac{1}{g}\left((1-4g)C_1-C_2\right),$$

where C_2 and C_3 are arbitrary constants.

Therefore, the infinitesimal generators admitted by equations (4.3) and (4.4) form the three-dimensional Lie algebra L_3 spanned by the generators

$$X_1=L\frac{\partial}{\partial L}+(\frac{1}{g}-4)f\frac{\partial}{\partial f}+\frac{1}{g}c\frac{\partial}{\partial c},\ X_2=t\frac{\partial}{\partial t}-\frac{1}{g}f\frac{\partial}{\partial f}-\frac{1}{g}c\frac{\partial}{\partial c},\ X_3=\frac{\partial}{\partial t}.\quad (4.43)$$

The transformations of solutions corresponding to these generators are

$$X_1:\qquad \bar{c}(\bar{t})=c(\bar{t})e^{\frac{1}{g}a},\quad \bar{f}(\bar{L},\bar{t})=f(e^{-a}\bar{L},\bar{t})e^{(\frac{1}{g}-4)a},$$

$$X_2:\qquad \bar{c}(\bar{t})=c(\bar{t}e^{-a})e^{-\frac{1}{g}a},\quad \bar{f}(\bar{L},\bar{t})=f(\bar{L},\bar{t}e^{-a})e^{-\frac{1}{g}a},$$

$$X_3:\qquad \bar{c}(\bar{t})=c(\bar{t}-a),\quad \bar{f}(\bar{L},\bar{t})=f(\bar{L},\bar{t}-a).$$

Notice that since the Lie group is admitted, these transformations map any solution of equations (4.3) and (4.4) into a solution.

Table 4.1 Commutators of Lie algebra $L_3 = \{X_1, X_2, X_3\}$ for generator (4.43).

	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	$-X_3$
X_3	0	X_3	0

4.5 Classification of Subalgebras of Lie Algebra

For constructing all invariant solutions related with the Lie algebra $L_3 = \{X_1, X_2, X_3\}$ one needs to have an optimal system of subalgebras. Applying Table 4.1 of commutators one obtains the following inner automorphisms,

$$A_1: \quad \hat{x}^3 = x^3 e^{a_1}; \qquad A_2: \quad \hat{x}^3 = x^3 - x^2 a_2,$$

where only the transformed coordinates are presented. The optimal system of subalgebras of the Lie algebra L_3 consists of the subalgebra

$$\{X_1, X_2, X_3\}, \quad \{X_1, X_2\}, \quad \{X_1, X_3\}, \quad \{X_2 + \alpha X_1, X_3\},$$
$$\{X_1 + \beta X_2\}, \quad \{X_1 \pm X_3\}, \quad \{X_2\}, \quad \{X_3\}.$$

4.6 Invariant Solutions of Equation (4.3) and (4.4)

The invariant solutions corresponding to the subalgebras

$$\{X_1 + \beta X_2\}, \quad \{X_2\}$$

are discussed in the previous section. For the subgroup corresponding to the subalgebras $\{X_1, X_2, X_3\}$ and $\{X_1, X_2\}$ there is no invariant solution. The invariant solution for subalgebra $\{X_1, X_3\}$ is

$$c = C_1 L^{\frac{1}{g}}, \quad f = C_2 L^{\frac{1}{g} - 4},$$

where $C_1 \geq 0, C_2 \geq 0$, and $C_1C_2 = 0$. The invariant solution corresponding to the subalgebra $\{X_2 + \alpha X_1, X_3\}$ has the representation

$$c = C_1 L^{\frac{\alpha-1}{g\alpha}}, \quad f = C_2 L^{\frac{\alpha-1}{g\alpha}-4},$$

where $C_1 \geq 0, C_2 \geq 0$, and $C_1C_2 = 0$.

The invariant solutions with respect to the subgroup with the generator X_3 have the representation

$$c = 0, \quad f = C; \qquad c = C, \quad f = 0,$$

where C is constant.

Invariants of subgroup with the generator $X_1 \pm X_3$ are $\ln L \mp t, cL^{-\frac{1}{g}}, fL^{4-\frac{1}{g}}$. The representation of invariant solutions is

$$c = L^{\frac{1}{g}}\phi(x), \quad f = L^{\frac{1}{g}-4}\varphi(x), \quad x = \ln L \mp t.$$

Substituting them into equations (4.3) and (4.4), one obtains the reduced equations

$$\varphi'(x)(\phi^g(x) \mp 1) = 0, \quad \int_{-\infty}^{\infty} e^{(\frac{1}{g}-1)x}\varphi(x)\,dx \mp e^{(\frac{1}{g}-1)x}\phi^{-g}(x)\phi'(x) = 0.$$

CHAPTER V

A NOTE ON SMOLUCHOWSKI COAGULATION EQUATION

The application of the Lie group method as developed for integro-differential equations is complicated (see, e.g., Meleshko, 2005; Grigoriev et al., 2010), due to the fact that solving the determining equations is difficult. Even more, the way of solving determining equations depends on the integro-differential equations studied. Conversely, for simplicity, the study of new integro-differential equations is started by considering the self-similar solutions. In this chapter, an explicit analytical solution of the Smoluchowski coagulation equation is presented by using a scaling group.

5.1 Smoluchowski Coagulation Equation

The study of aggregation started from the work of Smoluchowski (1917), who first defined the birth and death rates for a discrete system composed of interacting monomers. This equation can be rewritten for a continuous system in terms of the particle volume, i.e., assume that $G = 0$, equation (1.4) becomes Smoluchowski coagulation equation which describes the time evolution of the size distribution of particles coagulating by two-body collisions. This equation was first applied to small, suspended particles which collide and coagulate by virtue of their Brownian motion and has subsequently been applied to interacting polymers and to other physical systems. Even much more important, it is widely used for modelling growth in many fields of science. Examples include planetesimal accumulation, mergers in dense clusters of stars, coalescence of interstellar dust grains and galaxy mergers in astrophysics, aerosol coalescence in atmospheric physics, colloids and polymerization and gelation (see, e.g., Smoluchowski, 1917; Schumann, 1940; Scott, 1968; Ramkrishna, 2000), and the references therein.

Assuming that $G = 0$ and $K = k_0$, equation (1.4) can be reduced to

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}k_0 x \int_0^1 f(x(1-s),t)f(xs,t)\,ds - k_0 f(x,t)\int_0^\infty f(x,t)\,dx. \tag{5.1}$$

Simplification can be effected by introducing two new variables, a particular implicit solution of equation (5.1) was given by Schumann (1940), i.e.,

$$f(x,t) = \frac{N_0^2}{V(1+pt)^2} exp\left(-\frac{N_0 x}{V(1+pt)}\right).$$

The initial distribution corresponding to this solution is

$$f(x,0) = \frac{N_0^2}{V} exp\left(-\frac{N_0 x}{V}\right),$$

where N_0 is initial number of nuclear drops, $p = \frac{1}{2}N_0 k_0$, V is volume of condensed water per unit volume of space, i.e., the first moment $V = \int_0^\infty x f(x,t)\,dx$.

5.2 Admitted Scaling Group

According to the algorithm (Ovsiannikov, 1978; Meleshko, 2005), a scaling group H^1 of transformations of the space $R^{2+1}(x,t,f)$ with one parameter admitted by equation (5.1) can be defined as

$$\bar{x} = xa^{\lambda_1}, \quad \bar{t} = ta^{\lambda_2}, \quad \bar{f} = fa^{\mu}, \tag{5.2}$$

where a is a parameter. The corresponding infinitesimal operator of the group H^1 is provided by

$$X = \lambda_1 x\frac{\partial}{\partial x} + \lambda_2 t\frac{\partial}{\partial t} + \mu f\frac{\partial}{\partial f}. \tag{5.3}$$

The transformations (5.2) leave the manifold, assigned by equation (5.1) invariant, one obtains

$$\frac{\partial \bar{f}(\bar{x},\bar{t})}{\partial \bar{t}} = \frac{1}{2}k_0\bar{x}\int_0^1 \bar{f}(\bar{x}(1-s),\bar{t})\bar{f}(\bar{x}s,\bar{t})\,ds - k_0\bar{f}(\bar{x},\bar{t})\int_0^\infty \bar{f}(\bar{x},\bar{t})\,d\bar{x}. \quad (5.4)$$

Substituting (5.2) into (5.4) and taking note of $f(x,t)$ is solution of (5.1), one has

$$(a^{\mu-\lambda_2} - a^{2\mu+\lambda_1})\frac{\partial f(x,t)}{\partial t} = 0,$$

which shows that

$$\mu = -\lambda_2 - \lambda_1,$$

substituting this into (5.3), the admitted operator is presented by

$$X = \lambda_1 x\frac{\partial}{\partial x} + \lambda_2 t\frac{\partial}{\partial t} - (\lambda_1+\lambda_2)f\frac{\partial}{\partial f}. \quad (5.5)$$

5.3 An Explicit Analytic Solution of Equation (5.1) with Kernel $k_0 = 2$

Assuming that $\lambda_2 \neq 0$, and setting $\alpha = \frac{\lambda_1}{\lambda_2}$, by means of operator (5.5) one obtains

$$X = \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - (1+\alpha)f\frac{\partial}{\partial f}.$$

The invariants of this generator are $xt^{-\alpha}$, $t^{1+\alpha}f$. Thus, assume that the self-similar solutions of (5.1) have the representation

$$f(x,t) = t^{-1-\alpha}\varphi(z), \quad z = xt^{-\alpha}, \quad (5.6)$$

where the function φ satisfies the equation

$$\alpha z\varphi'(z) + \frac{k_0}{2}z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds + \left(1+\alpha-k_0\int_0^\infty \varphi(z)\,dz\right)\varphi(z) = 0. \quad (5.7)$$

In particular, let us take a kernel of physical interest, $k_0 = 2$ and $\alpha = 1$. Then equation (5.7) becomes

$$z\varphi'(z) + z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds + \left(2 - 2\int_0^\infty \varphi(z)\,dz\right)\varphi(z) = 0. \tag{5.8}$$

Since it is not difficult to check that $\varphi(z) = e^{-z}$ is a solution of equation (5.8), hence, by means of (5.6) we derive that

$$f(x,t) = t^{-2}e^{-xt^{-1}}$$

is a physical solution of Smoluchowski coagulation equation (5.1) with $k_0 = 2$. Thus, the total number of clusters $N(t)$ and total mass of clusters $M(t)$ at time t are presented by

$$N(t) = \int_0^\infty f(x,t)\,dx = \frac{1}{t}, \quad M(t) = \int_0^\infty xf(x,t)\,dx = 1. \tag{5.9}$$

By virtue of (5.9) it is easy to check that $N(t)$ is a non-increasing function with respect to time t. It is well-known that $M(t)$ might not remain constant throughout time evolution for some coagulation kernel $K(x,y)$. However, in light of (5.9) it is shown that $M(t)$ still remains constant throughout time evolution for the Smoluchowski coagulation equation with kernel $k_0 = 2$. Thus, the first moment, i.e., the total volume, is a constant. In addition, notice that

$$\lim_{x\to\infty} t^{-2}e^{-xt^{-1}} = 0, \qquad \lim_{t\to\infty} t^{-2}e^{-xt^{-1}} = 0, \tag{5.10}$$

which shows that population density $f(x,t) = t^{-2}e^{-xt^{-1}}$ must vanish for (arbitrarily large sizes) infinite sized particles (Ramkrishna, 2000). Moreover, the second identity of (5.10) shows that the population density $f(x,t) = t^{-2}e^{-xt^{-1}}$ is an asymptotically stable solution of (5.1) with kernel $k_0 = 2$.

5.4 An Explicit Analytic Solution of Equation (5.1) with Arbitrary Kernel k_0

In this section, an explicit analytic solution of equation (5.1) with arbitrary constant coagulation kernel k_0 is presented. Note that $f(x,t) = t^{-2}e^{-xt^{-1}}$ is a solution of Smoluchowski equation (5.1) with $k_0 = 2$. The main idea, from this point, assume that

$$f(x,t) = \alpha t^{-2} e^{-\beta x t^{-1}}$$

is a solution of the Smoluchowski equation (5.1) with arbitrary kernel k_0 again, where α and β are positive constants. Hence, by simplification and calculation one has

$$(\alpha k_0 - 2\beta)(2t - \beta x) = 0.$$

The latter equation leads to $\alpha = 2\frac{\beta}{k_0}$, $\beta > 0$. Therefore, an explicit analytic solution of equation (5.1) with arbitrary constant coagulation kernel k_0 is given by

$$f(x,t) = 2\frac{\beta}{k_0} t^{-2} e^{-\beta x t^{-1}}, \quad \beta > 0.$$

The result just obtained is not a general solution, but is a particular solution corresponding to the initial condition presented by

$$f(x,t_0) = 2\frac{\beta}{k_0} t_0^{-2} e^{-\beta x t_0^{-1}}, \quad \beta > 0, \quad t_0 > 0.$$

The explicit analytic solutions are useful for testing numerical programs and checking the accuracy of numerical solutions of PBEs. However, analytic solutions of PBEs are seldom available, the challenging question thus arises what explicit analytic solutions would be obtained for other initial conditions. This has motivated us to develop specialized approaches for solving the PBEs.

CHAPTER VI

EXACT SOLUTIONS OF NONHOMOGENEOUS POPULATION BALANCE EQUATION (1.4)

In this chapter, exact solutions of equation (1.4) with constant growth rate and kernels (1.9) are presented by using the developed Lie group method (Meleshko, 2005; Grigoriev et al., 2010). Furthermore, the symmetry group of equation (1.4) with constant growth rate and homogenous kernel (1.8) is also provided.

6.1 Dimensional Analysis

In order to study the dimensionless equation (1.4) by using the group analysis method, the technique of dimensional analysis will be used to deal with the complicated constant coefficients of equation (1.4).

For dimensional analysis, let us consider the following transformations

$$\bar{x} = x_0 x, \quad \bar{t} = t_0 t, \quad \bar{f}(\bar{x}, \bar{t}) = f_0 f(x,t), \tag{6.1}$$

where x_0, t_0, f_0 are positive real numbers.

In this section, equation (1.4) with $G = G_0$ and kernels (1.9) is studied, where G_0 is constant. Notice that equation (1.4) can be rewritten as follows

$$\frac{\partial \bar{f}(\bar{x},\bar{t})}{\partial \bar{t}} + G_0 \frac{\partial \bar{f}(\bar{x},\bar{t})}{\partial \bar{x}} = \frac{1}{2}\bar{x}\int_0^1 K(\bar{x}(1-s), \bar{x}s)\bar{f}(\bar{x}(1-s),\bar{t})\bar{f}(\bar{x}s,\bar{t})\,ds \\ -\bar{f}(\bar{x},\bar{t})\int_0^\infty K(\bar{x},\bar{y})\bar{f}(\bar{y},\bar{t})\,d\bar{y}. \tag{6.2}$$

6.1.1 Constant Kernel

For constant kernel $K = k_0$, applying the transformations (6.1) to equation (6.2), one obtains the following equation

$$\begin{aligned}\frac{\partial f(x,t)}{\partial t} + \frac{G_0 t_0}{x_0}\frac{\partial f(x,t)}{\partial x} = {} & \frac{1}{2}k_0 f_0 t_0 x_0 x \int_0^1 f(x(1-s),t) f(xs,t)\, ds \\ & -k_0 f_0 t_0 x_0 f(x,t) \int_0^\infty f(x,t)\, dx. \qquad (6.3)\end{aligned}$$

If we set

$$x_0 = G_0 t_0, \quad f_0 = \frac{1}{k_0 G_0 t_0^2},$$

equation (6.3) can be written in the form

$$\begin{aligned}\frac{\partial f(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x} = {} & \frac{1}{2}x \int_0^1 f(x(1-s),t) f(xs,t)\, ds \\ & -f(x,t) \int_0^\infty f(x,t)\, dx. \qquad (6.4)\end{aligned}$$

6.1.2 Sum Kernel

Similarly, for sum kernel $K(\bar{x}, \bar{y}) = k_1(\bar{x} + \bar{y})$, if one sets

$$x_0 = G_0 t_0, \quad f_0 = \frac{1}{k_1 G_0^2 t_0^3},$$

equation (6.2) is written in the form

$$\begin{aligned}\frac{\partial f(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x} = {} & \frac{1}{2}x^2 \int_0^1 f(x(1-s),t) f(xs,t)\, ds \\ & -x f(x,t) \int_0^\infty f(x,t)\, dx - f(x,t) \int_0^\infty x f(x,t)\, dx. \qquad (6.5)\end{aligned}$$

6.1.3 Product Kernel

In a similar way, for product kernel $K(\bar{x}, \bar{y}) = k_2 \bar{x}\bar{y}$, if one sets

$$x_0 = G_0 t_0, \quad f_0 = \frac{1}{k_2 G_0^3 t_0^4},$$

equation (6.2) is reduced to

$$\frac{\partial f(x,t)}{\partial t}+\frac{\partial f(x,t)}{\partial x}=\frac{1}{2}x^3\int_0^1 s(1-s)f(x(1-s),t)f(xs,t)\,ds$$
$$-xf(x,t)\int_0^\infty xf(x,t)\,dx. \qquad (6.6)$$

6.2 Admitted Lie Group

The classical group analysis cannot be applied to equations (6.4)-(6.6). One needs to use the method developed for equations with non-local terms (Meleshko, 2005; Grigoriev et al., 2010) instead. A generator of the admitted Lie group is sought in the form

$$X=\xi^x(x,t,f)\frac{\partial}{\partial x}+\xi^t(x,t,f)\frac{\partial}{\partial t}+\eta^f(x,t,f)\frac{\partial}{\partial f}.$$

According to the algorithm (Meleshko, 2005; Grigoriev et al., 2010), the determining equation for equation (6.4) is given by

$$\Big(D_t\varphi+D_x\varphi-x\int_0^1\varphi(xs,t)f(x(1-s),t)\,ds$$
$$+\varphi\int_0^\infty f\,dx+f\int_0^\infty\varphi\,dx\Big)\Big|_{(S)}=0, \qquad (6.7)$$

where D_x, D_t denote the total derivatives with respect to the independent variables x, t, respectively, the subscript $|_{(S)}$ means that the expression is satisfied for any solution of equation (6.4), and the function $\varphi(x,t)$ is presented by

$$\varphi(x,t)=\eta^f(x,t,f(x,t))-\xi^x(x,t,f(x,t))f_x(x,t)-\xi^t(x,t,f(x,t))f_t(x,t).$$

Differentiating equation (6.4) with respect to x and t, respectively, we notice that

$$\int_0^1 f_t(x(1-s),t)f(xs,t)\,ds=\int_0^1 f_t(xs,t)f(x(1-s),t)\,ds,$$
$$\int_0^1(1-s)f_x(x(1-s),t)f(xs,t)\,ds=\int_0^1 sf_x(xs,t)f(x(1-s),t)\,ds,$$

and thus we arrive at the following

$$f_{tx}(x,t) + f_{xx}(x,t) = \frac{1}{2}\int_0^1 f(xs,t)f(x(1-s),t)\,ds$$
$$+x\int_0^1 sf_x(xs,t)f(x(1-s),t)\,ds - f_x(x,t)\int_0^\infty f(x,t)\,dx,$$

$$f_{tt}(x,t) + f_{xt}(x,t) = x\int_0^1 f_t(xs,t)f(x(1-s),t)\,ds$$
$$-f_t(x,t)\int_0^\infty f(x,t)\,dx - f(x,t)\int_0^\infty f_t(x,t)\,dx.$$

The approach for constructing the general solution of the determining equation (6.7) is as follows. We analyze the determining equation on the subset of solutions of equation (6.4) determined by the initial conditions

$$f(x,t_0) = ae^{-bx}, \quad a>0, \quad b>0, \tag{6.8}$$

at the given (arbitrary) time $t = t_0$, we consider the resulting equation at an arbitrary initial time t_0. Accordingly, the determining equation (6.7) is written in terms of the following functions:

$$\hat{\xi}^x(x,t) = \xi^x(x,t,ae^{-bx}), \quad \hat{\xi}^t(x,t) = \xi^t(x,t,ae^{-bx}), \quad \hat{\eta}^f(x,t) = \eta^f(x,t,ae^{-bx}),$$
$$\hat{\xi}^x_t(x,t) = \xi^x_t(x,t,ae^{-bx}), \quad \hat{\xi}^x_x(x,t) = \xi^x_x(x,t,ae^{-bx}), \quad \hat{\xi}^x_f(x,t) = \xi^x_f(x,t,ae^{-bx}), \cdots.$$

For solving the determining equation (6.7), we use the solutions corresponding to initial data (6.8) by varying the parameters a and b. It is worth noting that the particular class of solutions allows us to find the general solution of the determining equation (6.7).

We proceed now to the calculations, it is assumed that the coefficients of the infinitesimal generator X can be presented by the formal Taylor series with respect to f:

$$\xi^x(x,t,f) = \sum_{j=0}^{\infty} q_j(x,t)f^j, \quad \xi^t(x,t,f) = \sum_{j=0}^{\infty} r_j(x,t)f^j, \quad \eta^f(x,t,f) = \sum_{j=0}^{\infty} p_j(x,t)f^j.$$

Calculating the derivatives of function $f(x,t)$ at the time t, one has

$$f_x(x,t) = -abe^{-bx}, \quad f_{xx}(x,t) = ab^2e^{-bx},$$
$$f_t(x,t) = abe^{-bx} + (\frac{1}{2}x - b^{-1})a^2e^{-bx}, \quad f_{tx}(x,t) + f_{xx}(x,t) = (\frac{3}{2} - \frac{1}{2}bx)a^2e^{-bx},$$
$$f_{tt}(x,t) + f_{tx}(x,t) = (bx-2)a^2e^{-bx} + (\frac{1}{4}x^2 - \frac{3}{2}b^{-1}x + \frac{3}{2}b^{-2})a^3e^{-bx}.$$

Then the determining equation (6.7) becomes

$$\begin{aligned} \hat{\eta}^f_x + \hat{\eta}^f_t + \hat{\eta}^f_f(f_x+f_t) - f_x(\hat{\xi}^x_x + \hat{\xi}^x_t) - f_t(\hat{\xi}^t_x + \hat{\xi}^t_t) \\ -\hat{\xi}^x_f(f_x^2 + f_xf_t) - \hat{\xi}^t_f(f_t^2 + f_xf_t) - \hat{\xi}^t(f_{tx} + f_{tt}) - \hat{\xi}^x(f_{xt} + f_{xx}) \\ -x\int_0^1 \hat{\varphi}(xs,t)f(x(1-s),t)\,ds + \hat{\varphi}\int_0^\infty f(x,t)\,dx + f\int_0^\infty \hat{\varphi}(x,t)\,dx = 0. \end{aligned} \quad (6.9)$$

For analyzing equation (6.9), we use representations of the functions p_i, q_i, r_i in the formal Taylor series:

$$p_i(x,t) = \sum_{j=0}^\infty p_{ij}(t)x^j, \quad q_i(x,t) = \sum_{j=0}^\infty q_{ij}(t)x^j, \quad r_i(x,t) = \sum_{j=0}^\infty r_{ij}(t)x^j. \quad (6.10)$$

Using the arbitrariness of the value a and equating to zero the coefficients with respect to $a^k (k = 0,1,2,3,\cdots)$, the determining equation can be split into a series of equations. In Section 6.7 it is shown that we ultimately arrive at the following general solution of the determining equation (6.7)

$$\xi^x(x,t,f) = c_1x, \quad \xi^t(x,t,f) = c_1t + c_2, \quad \eta^f(x,t,f) = -2c_1f.$$

It contains two arbitrary constants c_1 and c_2. Hence, the infinitesimal symmetries of equation (6.4) form the two-dimensional Lie algebra L_2 spanned by the following operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 2f\frac{\partial}{\partial f}. \quad (6.11)$$

In a similar way, one can find the infinitesimal symmetries of equations (6.5) and (6.6). They are, respectively

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 3f\frac{\partial}{\partial f},$$

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 4f\frac{\partial}{\partial f}.$$

6.3 Invariant Solutions of Equations (6.4)-(6.6)

For constructing all invariant solutions related with the Lie algebra $L_2 = \{X_1, X_2\}$ one needs to have an optimal system of subalgebras of Lie algebra L_2. The optimal system of Lie algebra L_2 consists of the subalgebras

$$\{X_1, X_2\}, \quad \{X_1\}, \quad \{X_2\}. \tag{6.12}$$

For subalgebra $\{X_1, X_2\}$ there is only one invariant $x^n f (n = 2, 3, 4)$ corresponding to equation (6.4)-(6.6), respectively. Notice that the invariant solution for this subgroup is $f(x,t) = cx^{-n} (n = 2, 3, 4)$, where c is constant, and the integral is divergent. Hence, an invariant solution can not be found for this subalgebra.

6.3.1 Invariant Solutions of Equation (6.4)

Invariants of the subgroup with the generator X_1 are x, f. The representation of the invariant solutions is

$$f(x,t) = \varphi(x).$$

Substituting this into equation (6.4), one obtains the reduced equation

$$\varphi'(x) - \frac{1}{2}x\int_0^1 \varphi(x(1-s))\varphi(xs)\, ds + \varphi(x)\int_0^\infty \varphi(x)\, dx = 0.$$

Invariants of the subgroup with the generator X_2 are $xt^{-1}, t^2 f$. The representation of the invariant solutions is

$$f(x,t) = t^{-2}\varphi(z), \quad z = xt^{-1}.$$

Substituting this into equation (6.4), one obtains the reduced equation

$$(1-z)\varphi'(z) - 2\varphi(z) - \frac{1}{2}z\int_0^1 \varphi(z(1-s))\varphi(zs)\, ds + \varphi(z)\int_0^\infty \varphi(z)\, dz = 0.$$

6.3.2 Invariant Solutions of Equation (6.5)

Invariants of the subgroup with the generator X_1 are x, f. The representation of the invariant solutions is

$$f(x,t) = \varphi(x).$$

Substituting this into equation (6.5), one finds the reduced equation

$$\varphi'(x) - \frac{1}{2}x^2 \int_0^1 \varphi(x(1-s))\varphi(xs)\,ds$$
$$+x\varphi(x)\int_0^\infty \varphi(x)\,dx + \varphi(x)\int_0^\infty x\varphi(x)\,dx = 0.$$

Invariants of the subgroup with the generator X_2 are $xt^{-1}, t^3 f$. The representation of the invariant solutions is presented by

$$f(x,t) = t^{-3}\varphi(z), \quad z = xt^{-1}.$$

Substituting this into equation (6.5), one finds the reduced equation

$$(1-z)\varphi'(z) - 3\varphi(z) - \frac{1}{2}z^2 \int_0^1 \varphi(z(1-s))\varphi(zs)\,ds$$
$$+z\varphi(z)\int_0^\infty \varphi(z)\,dz + \varphi(z)\int_0^\infty z\varphi(z)\,dz = 0.$$

6.3.3 Invariant Solutions of Equation (6.6)

Invariants of the subgroup with the generator X_1 are x, f. The representation of the invariant solutions is

$$f(x,t) = \varphi(x).$$

Substituting this into equation (6.6), one obtains the reduced equation

$$\varphi'(x) - \frac{1}{2}x^3 \int_0^1 s(1-s)\varphi(x(1-s))\varphi(xs)\,ds + x\varphi(x)\int_0^\infty x\varphi(x)\,dx = 0.$$

Invariants of the subgroup with the generator X_2 are $xt^{-1}, t^4 f$. The representation of the invariant solutions is given by

$$f(x,t) = t^{-4}\varphi(z), \quad z = xt^{-1}.$$

Substituting this into equation (6.6), we derive the reduced equation

$$(1-z)\varphi'(z) - 4\varphi(z) - \frac{1}{2}z^3 \int_0^1 s(1-s)\varphi(z(1-s))\varphi(zs)\, ds + z\varphi(z)\int_0^\infty z\varphi(z)\, dz = 0. \quad (6.13)$$

6.4 Reduced Equation and Admitted Lie Group

6.4.1 Reduced Equation (1.5)

If $Z =$ constant, equation (1.5) can be reduced to the form of equation (1.4), where $\bar{y}$ is a parameter. In fact, let us consider the invertible transformations

$$\bar{\tau} = \bar{y} + (1-Z)\bar{t}, \quad \tilde{y} = \bar{y} - Z\bar{t}.$$

Then equation (1.5) becomes

$$\frac{\partial g(\bar{x},\tilde{y},\bar{\tau})}{\partial \bar{\tau}} + \frac{\partial}{\partial \bar{x}}[Gg(\bar{x},\tilde{y},\bar{\tau})] = \frac{1}{2}\int_0^{\bar{x}} K(\bar{x}-\bar{z},\bar{z})g(\bar{x}-\bar{z},\tilde{y},\bar{\tau})g(\bar{z},\tilde{y},\bar{\tau})\, d\bar{z} - g(\bar{x},\tilde{y},\bar{\tau})\int_0^\infty K(\bar{x},\bar{z})g(\bar{z},\tilde{y},\bar{\tau})\, d\bar{z},$$

where $g(\bar{x},\tilde{y},\bar{\tau}) = \bar{f}(\bar{x}, Z\bar{\tau} + (1-Z)\tilde{y}, \bar{\tau} - \tilde{y})$.

If $G = 0$, this equation becomes the classical Smoluchowski coagulation equation. If $G = G_0$, where G_0 is constant, and with kernels (1.9), it will be reduced to our previous studied case.

6.4.2 Admitted Lie Group of Equation (6.2) with Homogenous Kernel (1.8)

For the general homogenous kernel (1.8) and constant growth rate G, the dimensionless equation (6.2) can be written in the form

$$\frac{\partial f(x,t)}{\partial t}+\frac{\partial f(x,t)}{\partial x}=\frac{1}{2}x\int_0^1 K(x(1-s),xs)f(x(1-s),t)f(xs,t)\,ds \\ -f(x,t)\int_0^\infty K(x,y)f(y,t)\,dy. \quad (6.14)$$

Equation (6.14) admits the following infinitesimal operators

$$X_1=\frac{\partial}{\partial t},\quad X_2=x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-(2+\gamma)f\frac{\partial}{\partial f}. \quad (6.15)$$

In fact, substituting the coefficients of operator X_2 of (6.15) into the determining equation of (6.14), by calculation and simplification the remaining terms are denoted by the following equation

$$\frac{\gamma}{2}x\int_0^1 K(x(1-s),xs)f(x(1-s),t)f(xs,t)\,ds \\ -\frac{1}{2}x^2\int_0^1 (K(x(1-s),xs))'_x f(x(1-s),t)f(xs,t)\,ds+xf\int_0^\infty K_x(x,y)f(y,t)\,dy \\ -f\int_0^\infty yK(x,y)f_x(y,t)\,dy-(1+\gamma)f\int_0^\infty K(x,y)f(y,t)\,dy=0. \quad (6.16)$$

Note that since (1.8), i.e., $K(x(1-s),xs)=x^\gamma K((1-s),s)$, or using (1.11), one can verify that the first term is equal to the second term on the left-hand side of (6.16). Hence, equation (6.16) is reduced to

$$\int_0^\infty xK_x(x,y)f(y,t)\,dy-\int_0^\infty yK(x,y)f_x(y,t)\,dy \\ -(1+\gamma)\int_0^\infty K(x,y)f(y,t)\,dy=0. \quad (6.17)$$

Using identity (1.10), equation (6.17) can be rewritten as follows

$$\int_0^\infty yf(y,t)K_y(x,y)\,dy+\int_0^\infty yK(x,y)f_x(y,t)\,dy+\int_0^\infty K(x,y)f(y,t)\,dy=0.$$

Using symmetric property of (1.7), the population density f must vanish for infinite sized particles (Ramkrishna, 2000) and applying integration by parts to the second term on the left-hand side of above equation, leads to the determining equation (6.16) holds.

In a similar way, for arbitrary kernel (1.7) (condition (1.8) is unnecessary), it is trivial to check that the coefficient of the first generator of (6.15) is a solution of the determining equation of (6.14).

Hence, the group of point Lie transformations admitted by eqution (6.14) with kernel (1.8) is specified by the set of infinitesimal operators (6.15).

6.5 Numerical Solutions

In this section, the homotopy perturbation method (He, 1999; 2000; 2003) will be used to numerically solve the reduced integro-differential equation.

6.5.1 Verifying Accuracy of the Homotopy Perturbation Method

First we need to verify the accuracy and efficiency of homotopy perturbation technique. In particular, let us consider equation (1.4), assume that $G = 0, K = 2$, we obtain Smoluchowski coagulation equation

$$\frac{\partial f(x,t)}{\partial t} = x\int_0^1 f(x(1-s),t)f(xs,t)\,ds - 2f(x,t)\int_0^\infty f(x,t)\,dx, \qquad (6.18)$$

with the following initial condition

$$f(x,1) = f_0(x). \qquad (6.19)$$

Applying the scaling group (Meleshko, 2005; Grigoriev et al., 2010) to equation (6.18), an explicit physical analytical solution was given in Chapter V, i.e.,

$$f(x,t) = t^{-2}e^{-xt^{-1}}, \quad f(x,1) = e^{-x}, \quad t > 0, \quad x \geq 0. \tag{6.20}$$

By virtue of equation (6.18), one can construct a homotopy (He, 1999; 2000; 2003) which satisfies

$$\frac{\partial \psi}{\partial t} - \frac{\varphi_0}{\partial t} + p\frac{\varphi_0}{\partial t} - px\int_0^1 \psi(x(1-s))\psi(xs)\,ds + 2p\psi(x)\int_0^\infty \psi(x)\,dx = 0, \tag{6.21}$$

where $0 \leq p \leq 1$ is an embedding parameter, φ_0 is an initial approximation which satisfies the initial condition (6.19). Suppose the solution of equation (6.21) has the form

$$\psi(x,t) = \sum_{j=0}^{\infty} p^j \psi_j(x,t). \tag{6.22}$$

Substituting (6.22) into (6.21) and equating terms with the identical powers of p^k $(k = 0, 1, \cdots)$, and assume that the initial approximation is

$$\varphi_0(x,1) = e^{-x}, \quad x \geq 0.$$

By calculation and collection we obtain

$$\begin{aligned}
\psi_0(x,t) &= e^{-x}, \\
\psi_1(x,t) &= (x-2)(t-1)e^{-x}, \\
\psi_2(x,t) &= \frac{1}{2}(x^2 - 6x + 6)(t-1)^2 e^{-x}, \\
\psi_3(x,t) &= \frac{1}{6}(x^3 - 12x^2 + 36x - 24)(t-1)^3 e^{-x}, \\
\psi_4(x,t) &= \frac{1}{24}(x^4 - 20x^3 + 120x^2 - 240x + 120)(t-1)^4 e^{-x}, \\
\psi_5(x,t) &= \frac{1}{120}(x^5 - 30x^4 + 300x^3 - 1200x^2 + 1800x - 720)(t-1)^5 e^{-x}, \\
&\cdots
\end{aligned}$$

and so on. Taking the domain $\Omega = [0, 50] \times [1, 1.5]$ to carry out experiments, we compare the exact solution (6.20) with the k-th order approximate solutions

Table 6.1 Errors comparison of numerical solution to equation (6.18), $h_1 = h_2 = \frac{1}{100}$.

i	$\parallel f - \tilde{f}_i \parallel_2$	RE_i
1	7.8786	0.2081
2	3.9845	0.1052
3	2.0225	0.0534
4	1.0273	0.0271
5	0.5217	0.0138

$\tilde{f}_k(x,t)(k = 1, \cdots, 5)$, i.e.,

$$\tilde{f}_k(x,t) = \sum_{j=0}^{k} \psi_j(x,t), \quad k = 1, \cdots, 5.$$

Setting

$$\parallel f - \tilde{f}_k \parallel_2^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(f(x_i, t_j) - \tilde{f}_k(x_i, t_j) \Big)^2, \quad k = 1, \cdots, 5,$$

$$\parallel f \parallel_2^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (f(x_i, t_j))^2, \quad RE_k = \frac{\parallel f - \tilde{f}_k \parallel_2}{\parallel f \parallel_2}, \quad k = 1, \cdots, 5,$$

where $x_i = (i-1)h_1,\ t_j = 1 + (j-1)h_2,\ n = \frac{50}{h_1} + 1,\ m = \frac{1}{2h_2} + 1$, and RE_k $(k = 1, \cdots, 5)$ denotes the relative errors. The numerical results are listed in Table 6.1.

It can be seen from Table 6.1 that the numerical results obtained show high accuracy of the method as compared with the exact solution. The solution obtained by this method is valid for not only nonlinear ordinary differential equations but also nonlinear partial differential equations (He, 1999). The method gives convergent successive approximations and handles linear and nonlinear problems in a similar manner (He, 2000; 2003).

6.5.2 Numerical Solution of Equation (6.13)

In this subsection, the numerical analytical solution of equation (6.13) is determined by the homotopy perturbation method, the corresponding figures are also provided by using Matlab.

Let us consider equation (6.13) with the initial condition

$$\varphi(0) = a, \quad a \geq 0.$$

One can construct a homotopy (He, 1999; 2000; 2003) which satisfies

$$\psi' - 4\psi + (p-1)(\varphi_0' - 4\varphi_0) - pz\psi' - \frac{p}{2}z^3 \int_0^1 s(1-s)\psi(z(1-s))\psi(zs)\,ds + pz\psi(z)\int_0^\infty z\psi(z)\,dz = 0, \tag{6.23}$$

where $p \in [0,1]$ is an embedding parameter, φ_0 is an initial approximate solution which satisfies the initial condition. In addition, notice that the integral $\int_0^\infty z\varphi_0(z)\,dz$ needs to be convergent, hence we choose

$$\varphi_0(z) = ae^{-z}. \tag{6.24}$$

Assume that solution of (6.23) has to be of the form

$$\psi(z) = \sum_{j=0}^{\infty} p^j \psi_j(z). \tag{6.25}$$

Substituting (6.24) and (6.25) into (6.23) and equating the terms with identical powers of p^k $(k = 0, 1, \cdots)$, for each $\psi_k(z)$ $(k = 0, 1, \cdots)$, one obtains a corresponding first-order linear ordinary differential equation. Solving these equations using Matlab and taking into account the integral $\int_0^\infty z\psi_k(z)\,dz$ $(k = 1, 2, \cdots)$ needing to be convergent, one sets integration constant equal to 0. By calculation and collection one finds

$$
\begin{aligned}
\psi_0(z) &= ae^{-z}, \\
\psi_1(z) &= -\frac{a}{7500}e^{-z}[125az^3 + 75az^2 - (1470a + 1500)z - 294a + 7200], \\
\psi_2(z) &= \frac{a}{11250000}e^{-z}[1250a^2z^6 + 2625a^2z^5 - (71625a^2 + 75000a)z^4 \\
&- (94500a^2 - 390000a)z^3 + (384300a^2 + 1170000a + 450000)z^2 \\
&+ (177120a^2 - 3304800a - 2430000)z + 35424a^2 - 660960a - 486000], \\
&\cdots
\end{aligned}
$$

and so on, other components can be obtained by using Matlab. A few terms approximation to solution of equation (6.13) can be obtained by setting $p = 1$ in (6.25); that is one can derive the k-th order approximate solution of equation (6.6) as follows

$$\tilde{f}_k(x,t) = t^{-4}\sum_{j=0}^{k}\psi_j(xt^{-1}), \quad k = 1, 2, \cdots,$$

notice that $\psi_0(0) = a$, thus the initial condition is reduced to

$$\sum_{j=1}^{k}\psi_j(0) = 0, \quad k = 1, 2, \cdots.$$

For simplicity we take $k = 5$, and notice the above initial condition and by using Matlab to find an approximate positive real root $a \approx 16.948273$. At last the figures of $\tilde{f}_5$ and $x\tilde{f}_5$ are presented in Figure 6.1.

6.6 Preliminary Group Classification of Equation (6.4) with Source

There are studies where the PBE is considered with a source term $S(x, t, f)$

$$Lf = S, \tag{6.26}$$

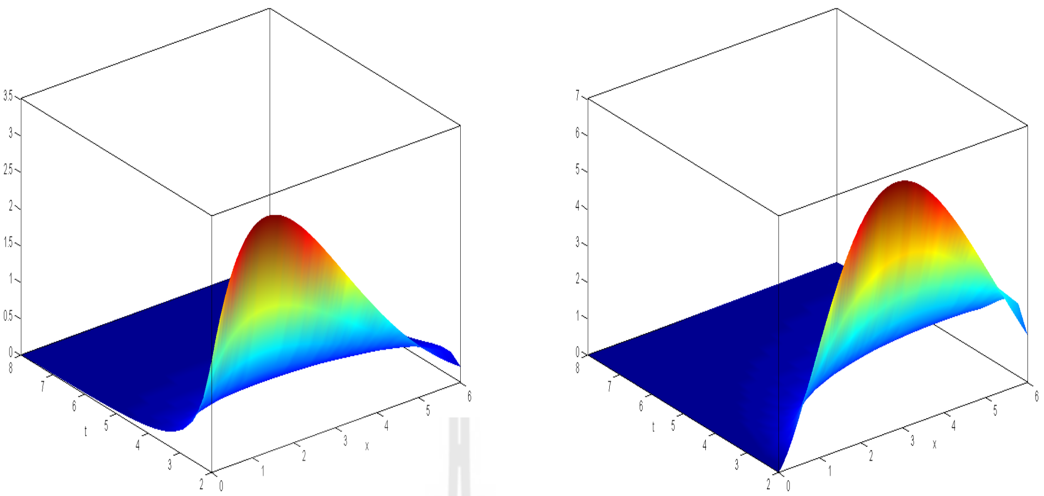

Figure 6.1 The left figure is $\tilde{f}_5$, the right figure is $x\tilde{f}_5$.

where

$$Lf = \frac{\partial f(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x} - \frac{1}{2}x\int_0^1 f(x(1-s),t)f(xs,t)\,ds + f(x,t)\int_0^\infty f(x,t)\,dx.$$

In particular, the authors of (Kumar and Ramkrishna, 1997) considered the source function dependent on x. The source function means the rate of nucleation particles of volume x.

6.6.1 Preliminary Group Classification

In this subsection the nonhomogeneous integro-differential equation (6.26) is considered. Solving the determining equation corresponding to (6.26) where $S \neq 0$ is a complicated task. However, for the case $S = 0$ the general solution of the determining equation was found. This allows us to apply the method of preliminary group classification (Akhatov, Gazizov, and Ibragimov, 1989) in equation (6.26) by the result attained for $S = 0$.

The infinitesimal symmetries of $Lf = 0$ form the two-dimensional Lie algebra $L_2 = \{X_1, X_2\}$ spanned by operators (6.11). Assume that the admitted generator of equation (6.26) is

$$X = c_1X_1 + c_2X_2.$$

The corresponding canonical Lie-Bäcklund operators are

$$\bar{X}_1 = -f_t\frac{\partial}{\partial f}, \quad \bar{X}_2 = -(2f + xf_x + tf_t)\frac{\partial}{\partial f},$$

prolonging and applying them to Lf, one obtains

$$\tilde{X}_1(Lf) = -D_t(Lf), \quad \tilde{X}_2(Lf) = -3(Lf) - xD_x(Lf) - tD_t(Lf), \qquad (6.27)$$

where $\tilde{X}_i (i = 1, 2)$ are prolongation of $\bar{X}_i (i = 1, 2)$, respectively. For deriving the second equation of (6.27) one has to use integration by parts and the property that population density must vanish for infinite sized particles (Ramkrishna, 2000), i.e.,

$$\int_0^\infty xf_x(x,t)\,dx = -\int_0^\infty f(x,t)\,dx.$$

Hence, one obtains that

$$\tilde{X}(Lf) = -c_1D_t(Lf) - c_2(3(Lf) + xD_x(Lf) + tD_t(Lf))$$

and the determining equation

$$\tilde{X}(Lf - S)\Big|_{(6.26)} = 0$$

becomes

$$c_2xS_x + (c_2t + c_1)S_t - 2c_2fS_f = -3c_2S.$$

Using the known optimal system of subalgebras (6.12) of the Lie algebra L_2 one classifies equation (6.26) with respect to the function $S(x,t,f)$. The results of the preliminary group classification are listed in Table 6.2.

Remark. In the particular case, where $g = c$ is constant, the function $S = cx^{-3}$ approximates a nucleation rate of infinite sized particles.

Table 6.2 Preliminary group classification for equation (6.26).

Subalgebra	Function S
$\{X_1\}$	$S = \psi(x, f)$
$\{X_2\}$	$S = t^{-3}\psi(xt^{-1}, ft^2)$
$\{X_1, X_2\}$	$S = x^{-3}g(x^2 f)$

6.6.2 Invariant Solutions

To find invariant solutions, one uses a subalgebra of the Lie algebra $\{X_1, X_2\}$. Representations of invariant solutions are given in Subsection 6.3.1. The reduced equations have the forms

$$\varphi'(x) - \frac{1}{2}x\int_0^1 \varphi(x(1-s))\varphi(xs)\,ds + \varphi(x)\int_0^\infty \varphi(s)\,ds = \psi(x, \varphi(x)),$$

for $f(x,t) = \varphi(x)$ and

$$(1-z)\varphi'(z) - 2\varphi(z) - \frac{1}{2}z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds + \varphi(z)\int_0^\infty \varphi(s)\,ds = \psi(z, \varphi(z)),$$

for $f(x,t) = t^{-2}\varphi(z), \quad z = xt^{-1}$.

In the case of $S = x^{-3}g(x^2 f)$, invariant solutions have the same forms as above where function ψ has the form

$$\psi(z_1, z_2) = z_1^{-3} g(z_1^2 z_2).$$

6.7 Solving the Determining Equation of (6.7)

If $k = 0$, then equation (6.9) yields

$$\frac{\partial p_0}{\partial x} + \frac{\partial p_0}{\partial t} = 0.$$

For $k = 1$, equation (6.9) becomes

$$(\frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial t} - x\int_0^1 p_0(xs,t)\,ds + \int_0^\infty p_0(x,t)\,dx)be^{-bx} + (\frac{\partial q_0}{\partial x} + \frac{\partial q_0}{\partial t} - \frac{\partial r_0}{\partial x} - \frac{\partial r_0}{\partial t})b^2e^{-bx} + p_0 = 0.$$

Splitting the latter equation with respect to b, one has

$$p_0 = 0, \quad \frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial t} = 0, \quad \frac{\partial q_0}{\partial x} + \frac{\partial q_0}{\partial t} - \frac{\partial r_0}{\partial x} - \frac{\partial r_0}{\partial t} = 0. \tag{6.28}$$

If $k = 2$, then equation (6.9) reduces to the condition

$$(\frac{\partial p_2}{\partial x} + \frac{\partial p_2}{\partial t})be^{-bx} + (\frac{\partial q_1}{\partial x} + \frac{\partial q_1}{\partial t} - \frac{\partial r_1}{\partial x} - \frac{\partial r_1}{\partial t})b^2e^{-bx} + (\frac{\partial r_0}{\partial x} + \frac{\partial r_0}{\partial t})(1 - \frac{1}{2}bx) + \frac{1}{2}p_1bx + (r_0 - \frac{1}{2}q_0)(b - b^2x) - x\int_0^1 [bp_1(xs,t) + b^2(q_0(xs,t) - r_0(xs,t))]\,ds + \int_0^\infty [bp_1(x,t)e^{-bx} + b^2(q_0(x,t) - r_0(x,t))e^{-bx}]\,dx = 0. \tag{6.29}$$

Applying (6.10) to the last integral in equation (6.29) it can be rewritten

$$(\frac{\partial p_2}{\partial x} + \frac{\partial p_2}{\partial t})be^{-bx} + (\frac{\partial q_1}{\partial x} + \frac{\partial q_1}{\partial t} - \frac{\partial r_1}{\partial x} - \frac{\partial r_1}{\partial t})b^2e^{-bx} + (\frac{\partial r_0}{\partial x} + \frac{\partial r_0}{\partial t})(1 - \frac{1}{2}bx) + \frac{1}{2}p_1bx + (r_0 - \frac{1}{2}q_0)(b - b^2x) - x\int_0^1 [bp_1(xs,t) + b^2(q_0(xs,t) - r_0(xs,t))]\,ds + (q_{00} - r_{00})b + p_{10} + q_{01} - r_{01} + \sum_{j=1}^\infty [p_{1j} + (j+1)(q_{0j+1} - r_{0j+1})]j!b^{-j} = 0. \tag{6.30}$$

Equating the coefficients with respect to b^0 and $b^{-j} (j \geq 1)$ in (6.30) to zero, we also used here the expansion of the exponent e^{-bx}, one has

$$\frac{\partial r_0}{\partial x} + \frac{\partial r_0}{\partial t} = r_{01} - p_{10} - q_{01}, \quad p_{1j} = (j+1)(r_{0j+1} - q_{0j+1}), \quad j \geq 1. \tag{6.31}$$

Hence, equation (6.30) is reduced to

$$(\frac{\partial p_2}{\partial x} + \frac{\partial p_2}{\partial t})e^{-bx} + (\frac{\partial q_1}{\partial x} + \frac{\partial q_1}{\partial t} - \frac{\partial r_1}{\partial x} - \frac{\partial r_1}{\partial t})be^{-bx} + \frac{1}{2}(p_1 + p_{10} + q_{01} - r_{01})x + (r_0 - \frac{1}{2}q_0)(1 - bx) + q_{00} - r_{00} - x\int_0^1 [p_1(xs,t) + b(q_0(xs,t) - r_0(xs,t))]\,ds = 0.$$

Splitting this equation with respect to b, one obtains

$$\frac{\partial p_2}{\partial x}+\frac{\partial p_2}{\partial t}=0,\qquad \frac{\partial q_1}{\partial x}+\frac{\partial q_1}{\partial t}-\frac{\partial r_1}{\partial x}-\frac{\partial r_1}{\partial t}=0, \tag{6.32}$$

$$\frac{1}{2}q_0-r_0=\int_0^1 (q_0(xs,t)-r_0(xs,t))\,ds, \tag{6.33}$$

$$\frac{1}{2}(p_1+p_{10}+q_{01}-r_{01})x+q_{00}-r_{00}+r_0-\frac{1}{2}q_0-x\int_0^1 p_1(xs,t)\,ds=0. \tag{6.34}$$

Applying (6.10) to equation (6.33), one obtains

$$\sum_{j=0}^{\infty}\frac{1}{j+1}[(j-1)q_{0j}-2jr_{0j}]x^j=0,$$

where equating the coefficients of $x^j(j=0,1,\cdots)$ to zero, one obtains

$$q_{00}=0,\quad r_{01}=0,\quad q_{0j}=\frac{2j}{j-1}r_{0j},\quad j\geq 2. \tag{6.35}$$

Equation (6.35) and the second equation of (6.31) lead to

$$p_{1j}=-\frac{(j+2)(j+1)}{j}r_{0j+1},\quad j\geq 1. \tag{6.36}$$

A similar analysis of equation (6.34) gives that

$$r_{02}=\frac{1}{2}q_{02},\qquad p_{1j}=\frac{2(j+1)}{j(j-1)}r_{0j+1},\quad j\geq 2. \tag{6.37}$$

By a summary the results obtained in (6.28)-(6.37), one has

$$p_1=-2c_1,\quad r_0=c_1t+c_2,\quad q_0=c_1x, \tag{6.38}$$

where c_1 and c_2 are constant.

If $k = 3 + n (n \geq 1)$, then equation (6.9) becomes

$$
\begin{aligned}
&(\frac{\partial p_{n+3}}{\partial x} + \frac{\partial p_{n+3}}{\partial t})b^2 e^{-(n+2)bx} + (\frac{\partial q_{n+2}}{\partial x} + \frac{\partial q_{n+2}}{\partial t} - \frac{\partial r_{n+2}}{\partial x} - \frac{\partial r_{n+2}}{\partial t})b^3 e^{-(n+2)bx} \\
&+(\frac{n+2}{2}b^2 x - (n+1)b)p_{n+2}e^{-(n+1)bx} + (b - \frac{1}{2}b^2 x)(\frac{\partial r_{n+1}}{\partial x} + \frac{\partial r_{n+1}}{\partial t})e^{-(n+1)bx} \\
&+(\frac{n+2}{2}bx - n - \frac{3}{2})b^2 q_{n+1}e^{-(n+1)bx} + (n + 2 - \frac{n+3}{2}bx)r_{n+1}b^2 e^{-(n+1)bx} \\
&+[(n+1)bx - \frac{n+1}{4}b^2x^2 - (n+\frac{1}{2})]r_n e^{-nbx} - x\int_0^1 [b^2 p_{n+2}(xs,t)e^{-(n+1)bxs} \\
&+r_n(xs,t)e^{-nbxs}(b - \frac{1}{2}b^2 xs) + b^3(q_{n+1}(xs,t) - r_{n+1}(xs,t))e^{-(n+1)bxs}]\,ds \\
&+\int_0^\infty [b^2 p_{n+2}(x,t)e^{-(n+2)bx} + (b - \frac{1}{2}b^2 x)r_n(x,t)e^{-(n+1)bx} \\
&+b^3(q_{n+1}(x,t) - r_{n+1}(x,t))e^{-(n+2)bx}]\,dx = 0. \quad (6.39)
\end{aligned}
$$

Substituting (6.10) into the last integral of equation (6.39), one obtains

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\frac{(-x)^i}{i!}\Big[(\frac{\partial p_{n+3}}{\partial x} + \frac{\partial p_{n+3}}{\partial t} + b(\frac{\partial q_{n+2}}{\partial x} + \frac{\partial q_{n+2}}{\partial t} - \frac{\partial r_{n+2}}{\partial x} - \frac{\partial r_{n+2}}{\partial t}))(n+2)^i b^{i+2} \\
&+p_{n+2}(\frac{n+2}{2}b^2x - (n+1)b)(n+1)^i b^i + (\frac{\partial r_{n+1}}{\partial x} + \frac{\partial r_{n+1}}{\partial t})(b - \frac{1}{2}b^2 x)(n+1)^i b^i \\
&+q_{n+1}(\frac{n+2}{2}bx - n - \frac{3}{2})(n+1)^i b^{i+2} + r_{n+1}(n + 2 - \frac{n+3}{2}bx)(n+1)^i b^{i+2} \\
&+r_n((n+1)bx - \frac{n+1}{4}b^2x^2 - (n+\frac{1}{2}))n^i b^i - x\int_0^1 (p_{n+2}(xs,t)(n+1)^i s^i b^{i+2} \\
&+r_n(xs,t)(b - \frac{1}{2}b^2 xs)n^i s^i b^i + (q_{n+1}(xs,t) - r_{n+1}(xs,t))(n+1)^i s^i b^{i+3})\,ds\Big] \\
&+\sum_{i=0}^{\infty}\left[\frac{i!p_{(n+2)i}}{(n+2)^{i+1}b^{i-1}} + \frac{i!r_{ni}(2n-i+1)}{2(n+1)^{i+2}b^i} + \frac{i!(q_{(n+1)i} - r_{(n+1)i})}{(n+2)^{i+1}b^{i-2}}\right] = 0. \quad (6.40)
\end{aligned}
$$

Equating the coefficients of b^0 on the left-hand side of (6.40) to zero, one has

$$
\begin{aligned}
r_n(x,t) = \frac{2}{2n+1}\Big[&\frac{p_{(n+2)1}(t)}{(n+2)^2} + \frac{r_{n0}(t)}{n+1} - \frac{r_{n0}(t)}{2(n+1)^2} \\
&+\frac{2(q_{(n+1)2}(t) - r_{(n+1)2}(t))}{(n+2)^3}\Big], \quad n \geq 1,
\end{aligned}
$$

which means that

$$
r_n(x,t) = r_{n0}(t), \quad r_{ni}(t) = 0, \quad i \geq 1, \quad n \geq 1,
$$

where

$$r_{n0} = \frac{(n+1)^2}{n(n+2)^3}\left(p_{(n+2)1} + \frac{2q_{(n+1)2}}{n+2}\right), \quad n \geq 1. \tag{6.41}$$

Equating the coefficients of b and $b^{-i}(i \geq 1)$ in (6.40) to zero, one has

$$p_{n+2} = \frac{n(2n+3)}{2(n+1)}r_{n0}x + \frac{r'_{(n+1)0}}{n+1} + \frac{p_{(n+2)0}}{(n+1)(n+2)} + \frac{q_{(n+1)1}}{(n+1)(n+2)^2},$$

$$q_{(n+1)i+1} = -\frac{n+2}{i+1}p_{(n+2)i}, \quad i \geq 2, \quad n \geq 1.$$

Thus, we have

$$p_{(n+2)0} = \frac{1}{(n+2)(n^2+3n+1)}[(n+2)^2 r'_{(n+1)0} + q_{(n+1)1}],$$
$$p_{(n+2)1} = \frac{(2n+3)n}{2(n+1)}r_{n0}, \quad p_{(n+2)i} = 0, \quad q_{(n+1)j} = 0, \; i \geq 2, \quad j \geq 3, \quad n \geq 1. \tag{6.42}$$

Notice that (6.41) becomes

$$q_{(n+1)2} = \frac{n(n+2)(2n^2+4n+1)}{4(n+1)}r_{n0}, \quad n \geq 1. \tag{6.43}$$

Equating the coefficients of b^2 in (6.40) to zero, one has

$$\frac{n(n+2)(2n+3)(2n^2+2n-1)}{2(n+1)}r_{n0}x^2$$
$$-2[2p'_{(n+3)1} + \frac{n+1}{n^2+3n+1}r'_{(n+1)0} - \frac{(n+1)(2n^2+7n+2)}{n^2+3n+1}q_{(n+1)1}]x$$
$$-4p_{(n+3)1} - 4p'_{(n+3)0} + \frac{2(2n^2+7n+4)}{n+2}q_{(n+1)0} - \frac{4(n^2+4n+3)}{n+2}r_{(n+1)0} = 0.$$

Splitting with respect to x in the latter equation, and taking note of equations (6.42) and (6.43), one obtains

$$r_n = 0, \quad p_{n+2} = 0, \quad q_{n+1} = 0, \quad n \geq 1. \tag{6.44}$$

If $k = 3$, using the second equation of (6.38) and (6.44), then equation (6.9) becomes

$$[(bx-1)p_2 + (b^2x - \frac{3}{2}b)q_1]e^{-bx} - x\int_0^1 (bp_2(xs,t) + b^2q_1(xs,t))e^{-bxs}\,ds$$
$$+\int_0^\infty (bp_2(x,t) + b^2q_1(x,t))e^{-2bx}\,dx = 0. \tag{6.45}$$

Using (6.10), equation (6.45) is rewritten

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i[p_2(bx-1)b^i+q_1(b^2x-\frac{3}{2}b)b^i+\frac{1}{2}q_{10}b+\frac{1}{2}p_{20}+\frac{1}{4}q_{11}$$
$$-x\int_0^1(p_2(xs,t)+q_1(xs,t)b)b^{i+1}s^i\,ds]+\sum_{i=1}^{\infty}\frac{i!}{2^{i+1}}(\frac{i+1}{2}q_{1i+1}+p_{2i})b^{-i}=0. \quad (6.46)$$

Equating the coefficients of b^0 and $b^{-i}(i\geq 1)$ on the left-hand side of (6.46) to zero, one has

$$p_2(x,t)=\frac{1}{2}p_{20}(t)+\frac{1}{4}q_{11}(t), \qquad q_{1i+1}=-\frac{2}{i+1}p_{2i}, \quad i\geq 1, \qquad (6.47)$$

The first equation in (6.47) gives that

$$p_{2i}=0, \quad i\geq 1, \quad p_2=p_{20}=\frac{1}{2}q_{11}.$$

The second equation in (6.47) leads to

$$q_1=q_{10}+q_{11}x.$$

Hence, equation (6.45) is reduced to

$$e^{-bx}(b^2q_{11}x^2+b^2q_{10}x-\frac{1}{2}bq_{10}+q_{11})-\frac{1}{2}bq_{10}-q_{11}=0.$$

Splitting with respect to b in the latter equation leads to

$$p_2=0, \quad q_1=0.$$

In summary, the general solution of the determining equation (6.7) is

$$\xi^x(x,t,f)=c_1x, \quad \xi^t(x,t,f)=c_1t+c_2, \quad \eta^f(x,t,f)=-2c_1f.$$

CHAPTER VII

EXACT SOLUTIONS OF NONHOMOGENEOUS POPULATION BALANCE EQUATION (1.5)

In this chapter, exact solutions of nonhomogeneous PBE (1.5) with constant growth rate and constant kernel are presented by using the developed group analysis method (Meleshko, 2005; Grigoriev et al., 2010). Moreover, the symmetry group of nonhomogeneous PBE (1.5) with constant growth rate and homogeneous kernel (1.8) is also provided.

Assume that $G = G_0$ is constant and $K = k_0$, equation (1.5) with spatial velocity function (1.6) can be rewritten as follows

$$\frac{\partial \bar{f}(\bar{x},\bar{y},\bar{t})}{\partial \bar{t}} + G_0 \frac{\partial \bar{f}(\bar{x},\bar{y},\bar{t})}{\partial \bar{x}} + z_0 \bar{x}^p \frac{\partial \bar{f}(\bar{x},\bar{y},\bar{t})}{\partial \bar{y}} =$$

$$\frac{1}{2} k_0 \bar{x} \int_0^1 \bar{f}(\bar{x}(1-s),\bar{y},\bar{t}) \bar{f}(\bar{x}s,\bar{y},\bar{t})\, ds - k_0 \bar{f}(\bar{x},\bar{y},\bar{t}) \int_0^\infty \bar{f}(\bar{z},\bar{y},\bar{t})\, d\bar{z}. \tag{7.1}$$

7.1 Dimensional Analysis

For dimensional analysis, let us consider the transformations

$$\bar{x} = x_0 x, \quad \bar{y} = y_0 y, \quad \bar{t} = t_0 t, \quad \bar{f}(\bar{x},\bar{y},\bar{t}) = f_0 f(x,y,t), \tag{7.2}$$

where x_0, y_0, t_0, f_0 are positive real numbers.

In the following, the dimensionless equation (7.1) will be obtained by transformations (7.2).

7.1.1 Zero Growth Rate

In case where the growth rate $G = 0$, applying (7.2) to equation (7.1), one has

$$\frac{\partial f(x,y,t)}{\partial t} + \frac{z_0 x_0^p t_0}{y_0} x^p \frac{\partial f(x,y,t)}{\partial y} = \frac{1}{2} k_0 f_0 t_0 x_0 x \int_0^1 f(x(1-s),y,t) f(xs,y,t)\, ds$$
$$-k_0 f_0 t_0 x_0 f(x,y,t) \int_0^\infty f(x,y,t)\, dx.$$

Letting

$$y_0 = z_0 x_0^p t_0, \quad f_0 = \frac{1}{k_0 x_0 t_0},$$

one obtains the equation in dimensionless form,

$$\frac{\partial f(x,y,t)}{\partial t} + x^p \frac{\partial f(x,y,t)}{\partial y} = \frac{1}{2} x \int_0^1 f(x(1-s),y,t) f(xs,y,t)\, ds$$
$$-f(x,y,t) \int_0^\infty f(x,y,t)\, dx. \qquad (7.3)$$

7.1.2 Constant Growth Rate

In case where the growth rate $G = G_0$ is constant, similarly, applying (7.2) to equation (7.1) and letting

$$x_0 = t_0 G_0, \quad y_0 = z_0 t_0^{p+1} G_0^p, \quad f_0 = \frac{1}{k_0 G_0 t_0^2},$$

one has the dimensionless equation

$$\frac{\partial f(x,y,t)}{\partial t} + \frac{\partial f(x,y,t)}{\partial x} + x^p \frac{\partial f(x,y,t)}{\partial y} = \frac{1}{2} x \int_0^1 f(x(1-s),y,t) f(xs,y,t)\, ds$$
$$-f(x,y,t) \int_0^\infty f(x,y,t)\, dx. \quad (7.4)$$

7.2 Admitted Lie Group

The classical group analysis method cannot be directly applied to equations (7.3) and (7.4). One needs to use the method developed for equations with non-local terms (Grigoriev and Meleshko, 1986; Meleshko, 2005; Grigoriev et al., 2010) instead. A generator of the admitted Lie group is sought in the form

$$X = \xi^x(x,y,t,f)\frac{\partial}{\partial x} + \xi^y(x,y,t,f)\frac{\partial}{\partial y} + \xi^t(x,y,t,f)\frac{\partial}{\partial t} + \eta^f(x,y,t,f)\frac{\partial}{\partial f}. \quad (7.5)$$

According to the algorithm (Meleshko, 2005; Grigoriev et al., 2010), the determining equation of (7.3) is presented by

$$\Big(D_t\zeta^f + x^p D_y\zeta^f - x\int_0^1 \zeta^f(xs,y,t)f(x(1-s),y,t)\,ds$$
$$+\zeta^f\int_0^\infty f(x,y,t)\,dx + f\int_0^\infty \zeta^f(x,y,t)\,dx\Big)\Big|_{(S)} = 0, \quad (7.6)$$

where D_y and D_t denote the total derivatives with respect to the independent variables y and t, respectively. The subscript $|_{(S)}$ means that the expression is satisfied for arbitrary solution $f(x,y,t)$ of system (7.3), and the function ζ^f is given by

$$\zeta^f(x,y,t) = \eta^f(x,y,t,f(x,y,t)) - \xi^x(x,y,t,f(x,y,t))f_x(x,y,t)$$
$$-\xi^y(x,y,t,f(x,y,t))f_y(x,y,t) - \xi^t(x,y,t,f(x,y,t))f_t(x,y,t). \quad (7.7)$$

Differentiating equation (7.3) with respect to x, y and t, one obtains

$$f_{tx}(x,y,t) + x^p f_{yx}(x,y,t) = -px^{p-1}f_y(x,y,t) + \frac{1}{2}\int_0^1 f(x(1-s),y,t)f(xs,y,t)\,ds$$
$$+x\int_0^1 sf_x(xs,y,t)f(x(1-s),y,t)\,ds - f_x(x,y,t)\int_0^\infty f(x,y,t)\,dx,$$

$$f_{tt}(x,y,t) + x^p f_{yt}(x,y,t) = x\int_0^1 f_t(xs,y,t)f(x(1-s),y,t)\,ds$$
$$-f_t(x,y,t)\int_0^\infty f(x,y,t)\,dx - f(x,y,t)\int_0^\infty f_t(x,y,t)\,dx,$$

$$f_{ty}(x,y,t) + x^p f_{yy}(x,y,t) = x\int_0^1 f_y(xs,y,t)f(x(1-s),y,t)\,ds$$
$$-f_y(x,y,t)\int_0^\infty f(x,y,t)\,dx - f(x,y,t)\int_0^\infty f_y(x,y,t)\,dx.$$

Here the following relations were used

$$\int_0^1 (1-s) f_x(x(1-s), y, t) f(xs, y, t)\, ds = \int_0^1 f(x(1-s), y, t) s f_x(xs, y, t)\, ds,$$
$$\int_0^1 f_t(x(1-s), y, t) f(xs, y, t)\, ds = \int_0^1 f(x(1-s), y, t) f_t(xs, y, t)\, ds,$$
$$\int_0^1 f_y(x(1-s), y, t) f(xs, y, t)\, ds = \int_0^1 f(x(1-s), y, t) f_y(xs, y, t)\, ds.$$

The main difficulty in finding an admitted Lie group of integro-differential equation (7.3) consists of solving the determining equation (7.6). Here one needs to notice that the determining equation (7.6) has to be satisfied for arbitrary solutions of the original equation (7.3). To solve the determining equation (7.6) we need to study the existence of solutions of Cauchy problem (7.3).

The different fields of applications of the Lie group method to partial differential equations have not caught special attention to the existence of solutions of Cauchy problem, but this problem plays an important role in solving the determining equations: it allows splitting them. There is also no special study of the Cauchy problem focusing on partial differential equations because either the studied system of equations is a geometrical approach or a Cauchy–Kovalevskaya type is applied to find an admitted Lie group.

For the system of integro-differential equation (7.3) we suppose that there exists a solution of the Cauchy problem of equation (7.3) with the following initial conditions:

$$f(x, y, t_0) = ae^{-bx-cy}, \quad a > 0, \quad b > 0, \quad \forall c \in R. \tag{7.8}$$

The approach for constructing the general solution of the determining equation (7.6) is as follows. We analyze the determining equation on the subset of solutions of equation (7.3) determined by the initial conditions (7.8) at given (arbitrary) time $t = t_0$; we consider the resulting equation at an arbitrary initial time

t_0. Accordingly, the determining equation (7.6) is written in terms of the following functions:

$$\hat{\xi}^x(x,y,t) = \xi^x(x,y,t,ae^{-bx-cy}), \quad \hat{\xi}^t(x,y,t) = \xi^t(x,y,t,ae^{-bx-cy}),$$
$$\hat{\eta}^f(x,y,t) = \eta^f(x,y,t,ae^{-bx-cy}), \quad \hat{\xi}^x_t(x,y,t) = \xi^x_t(x,y,t,ae^{-bx-cy}),$$
$$\hat{\xi}^x_x(x,y,t) = \xi^x_x(x,y,t,ae^{-bx-cy}), \quad \hat{\xi}^x_f(x,y,t) = \xi^x_f(x,y,t,ae^{-bx-cy}), \cdots .$$

For solving the determining equation (7.6), we use the solutions corresponding to initial data (7.8) by varying the parameters a, b and c. It is worth noting that the particular class of solutions allows us to find the general solution of the determining equation (7.6).

We proceed now to the calculations. It is assumed that the coefficients of an infinitesimal generator X can be presented by the formal Taylor series with respect to f:

$$\xi^x(x,y,t,f) = \sum_{j=0}^{\infty} q_j(x,y,t)f^j, \quad \xi^y(x,y,t,f) = \sum_{j=0}^{\infty} h_j(x,y,t)f^j,$$
$$\xi^t(x,y,t,f) = \sum_{j=0}^{\infty} r_j(x,y,t)f^j, \quad \eta^f(x,y,t,f) = \sum_{j=0}^{\infty} p_j(x,y,t)f^j. \tag{7.9}$$

Calculating the derivatives of function $f(x,y,t)$ at the time $t = t_0$, one obtains

$$f_x(x,y,t) = -abe^{-bx-cy}, \; f_y(x,y,t) = -ace^{-bx-cy}, \; f_{yy}(x,y,t) = ac^2e^{-bx-cy},$$
$$f_t + x^p f_y = (\frac{1}{2}x - b^{-1})a^2e^{-bx-2cy}, \quad f_{ty} + x^p f_{yy} = a^2ce^{-bx-2cy}(2b^{-1} - x),$$
$$f_{tx} + x^p f_{yx} = acpx^{p-1}e^{-bx-cy} + (\frac{3}{2} - \frac{1}{2}bx)a^2e^{-bx-2cy},$$
$$f_{tt} + x^p f_{yt} = a^3e^{-bx-3cy}(\frac{1}{4}x^2 - \frac{3}{2}b^{-1}x + \frac{3}{2}b^{-2})$$
$$+a^2e^{-bx-2cy}[\frac{x^{p+1}}{p+1}c - b^{-1}cx^p - b^{-p-1}c\Gamma(p+1)],$$

where the Gamma function Γ is given by

$$\Gamma(p+1) = \int_0^{\infty} x^p e^{-x} dx, \quad p > 0.$$

Hence, the determining equation (7.6) becomes

$$\hat{\eta}^f_t + x^p\hat{\eta}^f_y + \hat{\eta}^f_f(f_t + x^p f_y) - f_x(\hat{\xi}^x_t + x^p\hat{\xi}^x_y) - f_x(f_t + x^p f_y)\hat{\xi}^x_f$$
$$-f_y(\hat{\xi}^y_t + x^p\hat{\xi}^y_y) - f_y(f_t + x^p f_y)\hat{\xi}^y_f - f_t(\hat{\xi}^t_t + x^p\hat{\xi}^t_y) - f_t(f_t + x^p f_y)\hat{\xi}^t_f$$
$$-(f_{tx} + x^p f_{xy})\hat{\xi}^x - (f_{ty} + x^p f_{yy})\hat{\xi}^y - (f_{tt} + x^p f_{ty})\hat{\xi}^t$$
$$-x\int_0^1 \hat{\zeta}^f(xs,y,t)f(x(1-s),y,t)\,ds + \hat{\zeta}^f\int_0^\infty f\,dx + f\int_0^\infty \hat{\zeta}^f\,dx = 0. \quad (7.10)$$

For studying equation (7.10), we use representations of the functions q_j, h_j, r_j, p_j in the formal Taylor series:

$$q_j(x,y,t) = \sum_{i=0}^\infty q_{ji}(y,t)x^i, \qquad h_j(x,y,t) = \sum_{i=0}^\infty h_{ji}(y,t)x^i,$$
$$p_j(x,y,t) = \sum_{i=0}^\infty p_{ji}(y,t)x^i, \qquad r_j(x,y,t) = \sum_{i=0}^\infty r_{ji}(y,t)x^i. \quad (7.11)$$

Applying the arbitrariness of the value a, and equating to zero the coefficients with respect to $a^k (k = 0,1,2,3,\cdots)$, the determining equation can be split into a series of equations. In Section 7.7 it is shown that we finally find the following general solution of the determining equation (7.10),

$$\xi^x = c_2x, \quad \xi^t = -(c_1+c_2)t + c_3, \quad \xi^y = ((p-1)c_2 - c_1)y + c_4, \quad \eta^f = c_1f.$$

It contains four arbitrary constants c_1, c_2, c_3 and c_4. Thus, the infinitesimal symmetries of equation (7.3) form the four-dimensional Lie algebra L_4 spanned by the operators

$$X_1 = y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f}, \qquad X_2 = x\frac{\partial}{\partial x} + (p-1)y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t},$$
$$X_3 = \frac{\partial}{\partial t}, \qquad X_4 = \frac{\partial}{\partial y}. \quad (7.12)$$

Similarly, equation (7.4) admits the three-dimensional Lie algebra L_3 spanned by the operators

$$X_1 = x\frac{\partial}{\partial x} + (p+1)y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - 2f\frac{\partial}{\partial f}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial y}. \quad (7.13)$$

7.3 Admitted Lie Group of Equation (1.5) with Homogeneous Kernel (1.8) and Spatial Velocity (1.6)

In this section, the infinitesimal symmetries of equation (1.5) with the homogeneous coagulation kernel (1.8) and spatial velocity (1.6) for zero growth rate and constant growth rate are found.

7.3.1 Zero Growth Rate

For homogeneous kernel (1.8), consider the dimensionless equation (1.5) with (1.6) and growth rate $G = 0$, one derives equation

$$\frac{\partial f(x,y,t)}{\partial t} + x^p \frac{\partial f(x,y,t)}{\partial y} = \frac{1}{2} x \int_0^1 K(x(1-s), xs) f(x(1-s), y, t) f(xs, y, t)\, ds$$
$$- f(x,y,t) \int_0^\infty K(x,z) f(z,y,t)\, dz, \; p > 0. \quad (7.14)$$

In the previous section, it the case where $K = k_0$ was studied. Similar analysis of the cases where $K = k_1(x+z)$ and $K = k_2 xz$ allows one to guess that equation (7.14) admits the generators

$$Y_1 = y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f}, \quad Y_2 = x\frac{\partial}{\partial x} + (p-\gamma-1)y\frac{\partial}{\partial y} - (1+\gamma)t\frac{\partial}{\partial t}, \quad p > 0,$$

$$Y_3 = \frac{\partial}{\partial t}, \qquad Y_4 = \frac{\partial}{\partial y}. \quad (7.15)$$

Here we check directly using the algorithm (Meleshko, 2005; Grigoriev et al., 2010) that these generators are admitted by equation (7.14) with a homogeneous kernel (1.8). The determining equation for (7.14) is presented by

$$\Big(D_t\zeta^f + x^p D_y \zeta^f - x\int_0^1 K(x(1-s), xs)\zeta^f(xs, y, t) f(x(1-s), y, t)\, ds$$
$$+ \zeta^f \int_0^\infty K(x,z) f(z,y,t)\, dz + f\int_0^\infty K(x,z)\zeta^f(z,y,t)\, dz\Big)\Big|_{(S)} = 0, \quad (7.16)$$

where the subscript $|_{(S)}$ means that the expression is satisfied for any solution of equation (7.14). In addition, we have also used the property

$$\begin{aligned}&\int_0^1 K(x(1-s),xs)\zeta^f(xs,y,t)f(x(1-s),y,t)\,ds\\=&\int_0^1 K(x(1-s),xs)\zeta^f(x(1-s),y,t)f(xs,y,t)\,ds.\end{aligned}$$

which follows from symmetric kernel (1.7).

Differentiating equation (7.14) with respect to y,t and x, respectively, we derive

$$\begin{aligned}f_{ty}+x^p f_{yy}=&\,x\int_0^1 K(x(1-s),xs)f(x(1-s),y,t)f_y(xs,y,t)\,ds\\&-f_y\int_0^\infty K(x,z)f(z,y,t)\,dz-f\int_0^\infty K(x,z)f_y(z,y,t)\,dz,\end{aligned}$$

$$\begin{aligned}f_{tt}+x^p f_{yt}=&\,x\int_0^1 K(x(1-s),xs)f(x(1-s),y,t)f_t(xs,y,t)\,ds\\&-f_t\int_0^\infty K(x,z)f(z,y,t)\,dz-f\int_0^\infty K(x,z)f_t(z,y,t)\,dz,\end{aligned}$$

$$\begin{aligned}f_{tx}+x^p f_{yx}=&-px^{p-1}f_y+\frac{1}{2}\int_0^1 K(x(1-s),xs)f(x(1-s),y,t)f(xs,y,t)\,ds\\&+\frac{1}{2}x\int_0^1 (K(x(1-s),xs))'_x f(x(1-s),y,t)f(xs,y,t)\,ds\\&+x\int_0^1 sK(x(1-s),xs)f(x(1-s),y,t)f_x(xs,y,t)\,ds\\&-f_x\int_0^\infty K(x,z)f(z,y,t)\,dz-f\int_0^\infty K_x(x,z)f(z,y,t)\,dz,\end{aligned}$$

where we use symmetric kernel (1.7) and

$$\begin{aligned}&\int_0^1 K(x(1-s),xs)f(x(1-s),y,t)f_y(xs,y,t)\,ds\\=&\int_0^1 K(x(1-s),xs)f_y(x(1-s),y,t)f(xs,y,t)\,ds,\end{aligned}$$

$$\begin{aligned}&\int_0^1 K(x(1-s),xs)f(x(1-s),y,t)f_t(xs,y,t)\,ds\\=&\int_0^1 K(x(1-s),xs)f_t(x(1-s),y,t)f(xs,y,t)\,ds,\end{aligned}$$

$$\int_0^1 sK(x(1-s),xs)f(x(1-s),y,t)f_x(xs,y,t)\,ds$$
$$=\int_0^1 (1-s)K(x(1-s),xs)f_x(x(1-s),y,t)f(xs,y,t)\,ds.$$

Let us check that the generator Y_2 is admitted by equation (7.14). In fact, for generator Y_2, one has

$$\xi^x = x, \quad \xi^t = -(1+\gamma)t, \quad \xi^y = (p-\gamma-1)y, \quad \eta^f = 0,$$

$$\zeta^f(x,y,t) = (1+\gamma-p)yf_y(x,y,t) - xf_x(x,y,t) + (1+\gamma)tf_t(x,y,t).$$

Substituting them into the determining equation (7.16), by calculation and simplification one derives that

$$\begin{aligned}\frac{\gamma}{2}x\int_0^1 K(x(1-s),xs)f(x(1-s),y,t)f(xs,y,t)\,ds\\ -\frac{1}{2}x^2\int_0^1 (K(x(1-s),xs))'_x f(x(1-s),y,t)f(xs,y,t)\,ds\\ -(1+\gamma)f\int_0^\infty K(x,z)f(z,y,t)\,dz - f\int_0^\infty K(x,z)zf_x(z,y,t)\,dz\\ +xf\int_0^\infty K_x(x,z)f(z,y,t)\,dz = 0. \qquad (7.17)\end{aligned}$$

Notice that because of homogeneous kernel (1.8), one obtains

$$K(x(1-s),xs) = x^\gamma K(1-s,s).$$

Hence, (7.17) becomes

$$\begin{aligned}\int_0^\infty xK_x(x,z)f(z,y,t)\,dz - (1+\gamma)\int_0^\infty K(x,z)f(z,y,t)\,dz\\ -\int_0^\infty K(x,z)zf_x(z,y,t)\,dz = 0. \qquad (7.18)\end{aligned}$$

Substituting identity (1.10) into (7.18), one obtains the equation

$$\begin{aligned}\int_0^\infty zK(x,z)f_x(z,y,t)\,dz + \int_0^\infty K(x,z)f(z,y,t)\,dz\\ +\int_0^\infty zK_z(x,z)f(z,y,t)\,dz = 0. \qquad (7.19)\end{aligned}$$

Integration by parts to the third integral of (7.19) and using the property that population density must vanish for infinite sized particles (Ramkrishna, 2000), one has

$$\int_0^\infty zK_z(x,z)f(z,y,t)\,dz = -\int_0^\infty zK(x,z)f_z(z,y,t)\,dz - \int_0^\infty K(x,z)f(z,y,t)\,dz.$$

Thus, the generator Y_2 is admitted by equation (7.14). Similarly, one can show that the other generators of (7.15) are also admitted by equation (7.14).

7.3.2 Constant Growth Rate

For homogeneous kernel (1.8), considering the dimensionless equation (1.5) with (1.6) and constant growth rate G, one derives the equation

$$\begin{aligned} \frac{\partial f(x,y,t)}{\partial t} + \frac{\partial f(x,y,t)}{\partial x} + x^p \frac{\partial f(x,y,t)}{\partial y} &= \\ \frac{1}{2}x\int_0^1 K(x(1-s),xs)f(x(1-s),y,t)f(xs,y,t)\,ds & \\ -f(x,y,t)\int_0^\infty K(x,z)f(z,y,t)\,dz, \qquad p>0 & \end{aligned} \tag{7.20}$$

which admits the generators

$$X_1 = x\frac{\partial}{\partial x} + (p+1)y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - (2+\gamma)f\frac{\partial}{\partial f}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \frac{\partial}{\partial y}.$$

The proof is similar to the previous case.

7.4 Optimal System of Subalgebras

7.4.1 Optimal System of Subalgebras of Lie Algebra L_4

For constructing all invariant solutions of equation (7.3) related with the Lie algebra $L_4 = \{X_1, X_2, X_3, X_4\}$ one needs to have an optimal system of one

Table 7.1 Commutators of Lie algebra $L_4 = \{X_1, X_2, X_3, X_4\}$ for generators (7.12).

	X_1	X_2	X_3	X_4
X_1	0	0	$-X_3$	$-X_4$
X_2	0	0	X_3	$(1-p)X_4$
X_3	X_3	$-X_3$	0	0
X_4	X_4	$(p-1)X_4$	0	0

and two-dimensional subalgebras. Using Table 7.1 of commutators for generators (7.12) the inner automorphisms are presented by

$$A_1: \quad \hat{x}^3 = x^3 e^{a_1}, \;\; \hat{x}^4 = x^4 e^{a_1}, \qquad A_2: \quad \hat{x}^3 = x^3 e^{-a_2}, \;\; \hat{x}^4 = x^4 e^{(p-1)a_2},$$

$$A_3: \quad \hat{x}^3 = (x^2 - x^1)a_3 + x^3, \qquad A_4: \quad \hat{x}^4 = [(1-p)x^2 - x^1]a_4 + x^4,$$

where $a_i (i = 1, 2, 3, 4)$ are parameters of automorphisms $A_i (i = 1, 2, 3, 4)$, respectively, and only the transformed coordinates are presented. The optimal system of two-dimensional subalgebras of L_4 consists of the subalgebras

$$\{X_1, X_2\}, \quad \{X_1 + \alpha X_2, X_4\}, \quad \{X_1 + X_2 + X_3, X_4\},$$

$$\{X_1 + X_2 - X_3, X_4\}, \quad \{X_1 + \alpha X_2, X_3\}, 1 - p \neq \tfrac{1}{\alpha}, \quad \{X_1, X_3 + \beta X_4\},$$

$$\{X_2, X_4\}, \quad \{X_2, X_3\}, \quad \{X_2 - X_4, X_3\}, \quad \{X_2 + X_4, X_3\}, \quad \{X_3, X_4\}.$$

The optimal system of one-dimensional subalgebras of L_4 consists of the subalgebras

$$\{X_2 \pm X_4\}, \; p = 1, \quad \{X_2\}, \; p \neq 1, \quad \{X_4\}, \quad \{X_3 + \alpha X_4\},$$

$$\{X_1 + X_2 \pm X_3\}, \quad \{X_1 + \alpha X_2\}, \; (1-p)\alpha \neq 1, \; \alpha \neq 1.$$

Table 7.2 Commutators of Lie algebra $L_3 = \{X_1, X_2, X_3\}$ for generators (7.13).

	X_1	X_2	X_3
X_1	0	$-X_2$	$-(p+1)X_3$
X_2	X_2	0	0
X_3	$(p+1)X_3$	0	0

7.4.2 Optimal System of Subalgebras of Lie Algebra L_3

In a similar way, for Lie algebra $L_3 = \{X_1, X_2, X_3\}$, the commutators for generators (7.13) are presented in Table 7.2. The inner automorphisms are given by

$$A_1: \ \hat{x}^2 = x^2 e^{a_1}, \ \hat{x}^3 = x^3 e^{(p+1)a_1}; \ \ A_2: \ \hat{x}^2 = x^2 - x^1 a_2; \ \ A_3: \ \hat{x}^3 = x^3 - (p+1)x^1 a_3,$$

where only the transformed coordinates are presented, a_1, a_2, a_3 are parameters of automorphisms A_1, A_2, A_3, respectively. The optimal system of subalgebras of L_3 consists of the subalgebras

$$\{X_1, X_2, X_3\}, \ \ \{X_2, X_3\}, \ \ \{X_1, X_3\}, \ \ \{X_1, X_2\}, \ \ \{X_2 + \alpha X_3\}, \ \ \{X_3\}, \ \ \{X_1\}.$$

7.5 Invariant Solutions of Equation (7.3)

7.5.1 Invariant Solutions with $\{X_1, X_2\}$

The invariants of the subalgebra $\{X_1, X_2\}$ are $xtf, x^{-p}t^{-1}y$, the representation of the invariant solution is

$$f = x^{-1}t^{-1}\varphi(z), \ z = x^{-p}t^{-1}y,$$

substituting this into equation (7.3), one obtains the reduced equation

$$2p(1-z)\varphi'(z) + 2\left[\int_0^\infty z^{-1}\varphi(z)\,dz - p\right]\varphi(z) - p\int_0^1 \varphi(z(1-s)^{-p})\varphi(zs^{-p})\,ds = 0.$$

7.5.2 Invariant Solutions with $\{X_1 + \alpha X_2, X_4\}, \alpha \neq 1$

The invariants of the subalgebra $\{X_1 + \alpha X_2, X_4\}$ are $x^{\alpha-1}t^{\alpha}, t^{\frac{1}{1-\alpha}}f, \alpha \neq 1$, the representation of the invariant solution is

$$f = t^{\frac{1}{\alpha-1}}\varphi(z), \quad z = x^{\alpha-1}t^{\alpha}, \quad \alpha \neq 1,$$

substituting this into equation (7.3), the reduced equation is given by

$$\begin{aligned}&2(1-\alpha)\alpha z\varphi'(z) + 2\left[1 \pm \int_0^{\infty} z^{\frac{2-\alpha}{\alpha-1}}\varphi(z)\,dz\right]\varphi(z)\\ &+(1-\alpha)z^{\frac{1}{\alpha-1}}\int_0^1 \varphi(z(1-s)^{\alpha-1})\varphi(zs^{\alpha-1})\,ds = 0.\end{aligned}$$

7.5.3 Invariant Solutions with $\{X_1 + X_2, X_4\}$

The invariants of the subalgebra $\{X_1 + X_2, X_4\}$ are t, xf, the representation of the invariant solution is $f = x^{-1}\varphi(t)$, substituting this into equation (7.3), notice that the integral $\int_0^{\infty} x^{-1}\,dx$ is divergent. Hence, an invariant solution can not be found in this case.

7.5.4 Invariant Solutions with $\{X_1 + X_2 + X_3, X_4\}$

The invariants of the subalgebra $\{X_1 + X_2 + X_3, X_4\}$ are xe^{-t}, fe^{t}, the representation of the invariant solution is

$$f = e^{-t}\varphi(z), \quad z = xe^{-t},$$

substituting this into equation (7.3), one obtains the reduced equation

$$2z\varphi'(z) + 2\left[1 - \int_0^{\infty}\varphi(z)\,dz\right]\varphi(z) + z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds = 0.$$

7.5.5 Invariant Solutions with $\{X_1 + X_2 - X_3, X_4\}$

The invariants of the subalgebra $\{X_1 + X_2 - X_3, X_4\}$ are xe^t, fe^{-t}, the representation of the invariant solution is

$$f = e^t\varphi(z), \quad z = xe^t,$$

substituting this into equation (7.3), the reduced equation is

$$2z\varphi'(z) + 2\left[1 + \int_0^\infty \varphi(z)\,dz\right]\varphi(z) - z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds = 0.$$

7.5.6 Invariant Solutions with $\{X_1 + \alpha X_2, X_3\}, 1 - p \neq \frac{1}{\alpha}$

The invariants of the subalgebra $\{X_1 + \alpha X_2, X_3\}$ are $x^{\alpha(1-p)-1}y^\alpha, x^{\frac{1}{\alpha}}f$, the representation of the invariant solution is

$$f = x^{-\frac{1}{\alpha}}\varphi(z), \quad z = x^{\alpha(1-p)-1}y^\alpha, \quad 1 - p \neq \frac{1}{\alpha},$$

substituting this into equation (7.3), the reduced equation is presented by

$$2\alpha\sigma z^\mu\varphi'(z) - \sigma z^\lambda \int_0^1 [(1-s)s]^{-\frac{1}{\alpha}}\varphi(z(1-s)^\sigma)\varphi(zs^\sigma)\,ds \pm 2\varphi(z)\int_0^\infty z^\gamma\varphi(z)\,dz = 0,$$

where $\sigma = \alpha(1-p) - 1, \mu = \frac{(p-1)(1-\alpha)}{\alpha(1-p)-1}, \lambda = \frac{\alpha-1}{\alpha[\alpha(1-p)-1]}, \gamma = \frac{2\alpha-\alpha^2(1-p)-1}{\alpha[\alpha(1-p)-1]}, 1 - p \neq \frac{1}{\alpha}$.

7.5.7 Invariant Solutions with $\{X_1, X_3 + \beta X_4\}$

The invariants of the subalgebra $\{X_1, X_3 + \beta X_4\}$ are $x, (y - \beta t)f$, the representation of the invariant solution is

$$f = (y - \beta t)^{-1}\varphi(x),$$

substituting this into equation (7.3), one obtains the reduced equation

$$2\left[\beta - x^p + \int_0^\infty \varphi(x)\,dx\right]\varphi(x) - x\int_0^1 \varphi(x(1-s))\varphi(xs)\,ds = 0.$$

7.5.8 Invariant Solutions with $\{X_2, X_4\}$

The invariants of the subalgebra $\{X_2, X_4\}$ are xt, f, the representation of the invariant solution is

$$f = \varphi(z), \quad z = xt,$$

substituting this into equation (7.3), one obtains the reduced equation

$$2z\varphi'(z) + 2\varphi(z)\int_0^\infty \varphi(z)\,dz - z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds = 0.$$

7.5.9 Invariant Solutions with $\{X_2, X_3\}$

The invariants of the subalgebra $\{X_2, X_3\}$ are $x^{1-p}y, f$, the representation of the invariant solution is

$$f = \varphi(z), \quad z = x^{1-p}y, \quad p \neq 1,$$

substituting this into equation (7.3), one obtains the reduced equation

$$2(1-p)z^{\frac{1}{1-p}}\varphi'(z) \pm 2\varphi(z)\int_0^\infty z^{\frac{p}{1-p}}\varphi(z)\,dz$$
$$+(p-1)z^{\frac{1}{1-p}}\int_0^1 \varphi(z(1-s)^{1-p})\varphi(zs^{1-p})\,ds = 0.$$

7.5.10 Invariant Solutions with $\{X_2 \pm X_4, X_3\}$

The invariants of the subalgebra $\{X_2 \pm X_4, X_3\}$ are $x^{1-p}((p-1)y \pm 1), f$, the representation of the invariant solution is

$$f = \varphi(z), \quad z = x^{1-p}((p-1)y \pm 1), \quad p \neq 1,$$

substituting this into equation (7.3), one obtains the reduced equation

$$2(1-p)^2 z^{\frac{1}{1-p}}\varphi'(z) \pm 2\varphi(z)\int_0^\infty z^{\frac{p}{1-p}}\varphi(z)\,dz$$
$$+(1-p)z^{\frac{1}{1-p}}\int_0^1 \varphi(z(1-s)^{1-p})\varphi(zs^{1-p})\,ds = 0.$$

7.5.11 Invariant Solutions with $\{X_3, X_4\}$

The invariants of the subalgebra $\{X_3, X_4\}$ are x, f, the representation of the invariant solution is

$$f = \varphi(x),$$

substituting this into equation (7.3), one obtains the reduced equation

$$x\int_0^1 \varphi(x(1-s))\varphi(xs)\,ds - 2\varphi(x)\int_0^\infty \varphi(x)\,dx = 0.$$

7.5.12 Invariant solutions with $\{X_2 \pm X_4\}$, $p = 1$

The invariants of the subalgebra $\{X_2 \pm X_4\}$ with $p = 1$ are xt, $te^{\pm y}$, f, the representation of the invariant solution is

$$f = \varphi(u, v), \quad u = xt, \quad v = te^{\pm y},$$

substituting this solution into equation (7.3), the reduced equation is given by

$$u\varphi_u + (1 \pm u)v\varphi_v - \frac{1}{2}u\int_0^1 \varphi(u(1-s), v)\varphi(us, v)\,ds + \varphi\int_0^\infty \varphi(u, v)\,du = 0.$$

7.5.13 Invariant solutions with $\{X_2\}$, $p \neq 1$

The invariants of the subalgebra $\{X_2\}$ with $p \neq 1$ are xt, $t^{p-1}y$, f, the representation of the invariant solution is

$$f = \varphi(u, v), \quad u = xt, \quad v = t^{p-1}y,$$

substituting this solution into equation (7.3), the reduced equation is presented by

$$u\varphi_u + ((p-1)v + u^p)\varphi_v - \frac{1}{2}u\int_0^1 \varphi(u(1-s), v)\varphi(us, v)\,ds + \varphi\int_0^\infty \varphi(u, v)\,du = 0.$$

7.5.14 Invariant solutions with $\{X_4\}$

The invariants of the subalgebra $\{X_4\}$ are $x,\ t,\ f$, the representation of the invariant solution is

$$f = \varphi(x,t),$$

substituting this solution into equation (7.3), the reduced equation is given by

$$\varphi_t - \frac{1}{2}x\int_0^1 \varphi(x(1-s),t)\varphi(xs,t)\,ds + \varphi\int_0^\infty \varphi(x,t)\,dx = 0.$$

In Chapter V, an explicit analytic physical solution of the latter equation is presented by $\varphi(x,t) = 2\beta t^{-2}e^{-\beta x t^{-1}},\ \beta > 0$, where β is constant. Hence, an explicit analytic physical solution of equation (7.3) is given by

$$f(x,y,t) = 2\beta t^{-2}e^{-\beta x t^{-1}}, \quad \beta > 0.$$

7.5.15 Invariant solutions with $\{X_3 + \alpha X_4\}$

The invariants of the subalgebra $\{X_3 + \alpha X_4\}$ are $x,\ y - \alpha t,\ f$, the representation of the invariant solution is

$$f = \varphi(x,v), \quad v = y - \alpha t,$$

substituting this solution into equation (7.3), the reduced equation is given by

$$(x^p - \alpha)\varphi_v - \frac{1}{2}x\int_0^1 \varphi(x(1-s),v)\varphi(xs,v)\,ds + \varphi\int_0^\infty \varphi(x,v)\,dx = 0.$$

7.5.16 Invariant solutions with $\{X_1 + X_2 \pm X_3\}$

The invariants of the subalgebra $\{X_1 + X_2 \pm X_3\}$ are $xe^{\mp t},\ ye^{\mp pt},\ fe^{\pm t}$, the representation of the invariant solution is

$$f = e^{\mp t}\varphi(u,v), \quad u = xe^{\mp t}, \quad v = ye^{\mp pt},$$

substituting this solution into equation (7.3), the reduced equation is given by

$$(u^p \mp pv)\varphi_v \mp u\varphi_u \mp \varphi - \frac{1}{2}u\int_0^1 \varphi(u(1-s),v)\varphi(us,v)\,ds + \varphi\int_0^\infty \varphi(u,v)\,du = 0.$$

7.5.17 Invariant solutions with $\{X_1+\alpha X_2\}, (1-p)\alpha \neq 1, \alpha \neq 1$

The invariants of the subalgebra $\{X_1+\alpha X_2\}$ are $xt^{\frac{\alpha}{\alpha-1}}$, $ty^{\frac{\alpha-1}{1-(1-p)\alpha}}$, $t^{\frac{1}{1-\alpha}}f$, the representation of the invariant solution is

$$f = t^{\frac{1}{\alpha-1}}\varphi(u,v), \quad u = xt^{\frac{\alpha}{\alpha-1}}, \quad v = ty^{\frac{\alpha-1}{1-(1-p)\alpha}},$$

substituting this solution into equation (7.3), the reduced equation is provided by

$$\frac{\alpha}{\alpha-1}u\varphi_u + \left(v + \frac{\alpha-1}{1-(1-p)\alpha}u^p v^{\frac{(2-p)\alpha-2}{\alpha-1}}\right)\varphi_v + \frac{1}{\alpha-1}\varphi$$
$$-\frac{1}{2}u\int_0^1 \varphi(u(1-s),v)\varphi(us,v)\,ds + \varphi\int_0^\infty \varphi(u,v)\,du = 0.$$

7.6 Invariant Solutions of Equation (7.4)

7.6.1 Invariant Solutions with $\{X_1, X_2, X_3\}$

The invariant of the subalgebra $\{X_1, X_2, X_3\}$ is $x^2 f$. Thus, the invariant solution is $f = cx^{-2}$, where c is constant. Substituting this into equation (7.4), notice that the integral $\int_0^\infty x^{-2}\,dx = \infty$ is divergent, hence an invariant solution can not be found in this case.

7.6.2 Invariant Solutions with $\{X_2, X_3\}$

The invariants of the subalgebra $\{X_2, X_3\}$ are x, f, the representation of the invariant solution is

$$f = \varphi(x),$$

substituting this into equation (7.4), one obtains the reduced equation

$$\varphi'(x) - \frac{1}{2}x\int_0^1 \varphi(x(1-s))\varphi(xs)\,ds + \varphi(x)\int_0^\infty \varphi(x)\,dx = 0.$$

7.6.3 Invariant Solutions with $\{X_1, X_3\}$

The invariants of the subalgebra $\{X_1, X_3\}$ are $xt^{-1}, t^2 f$, the representation of the invariant solution is

$$f = t^{-2}\varphi(z), \quad z = xt^{-1},$$

substituting this into equation (7.4), the reduced equation is

$$2z\varphi'(z) + 2\left(1 - \int_0^\infty \varphi(z)\,dz\right)\varphi(z) + z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds = 0.$$

7.6.4 Invariant Solutions with $\{X_1, X_2\}$

The invariants of the subalgebra $\{X_1, X_2\}$ are $x^{p+1}y^{-1}, x^2 f$, the representation of the invariant solution is

$$f = x^{-2}\varphi(z), \quad z = x^{p+1}y^{-1},$$

substituting this into equation (7.4), one obtains the reduced equation

$$\begin{aligned} 2(p+1)(z-p-1)z\varphi'(z) + \left[4(p+1) - 2\int_0^\infty \varphi(z)\,dz\right]\varphi(z) \\ +(p+1)\int_0^1 (1-s)^{-2}s^{-2}\varphi(z(1-s)^{p+1})\varphi(zs^{p+1})\,ds = 0. \end{aligned}$$

7.6.5 Invariant solutions with $\{X_2 + \alpha X_3\}$

The invariants of the subalgebra $\{X_2 + \alpha X_3\}$ are $x, y - \alpha t, f$, the representation of the invariant solution is

$$f = \varphi(x, z), \quad z = y - \alpha t,$$

substituting this into equation (7.4), one obtains the reduced equation

$$(\alpha - x^p)\varphi_z - \varphi_x + \frac{1}{2}x\int_0^1 \varphi(x(1-s),z)\varphi(xs,z)\,ds - \varphi\int_0^\infty \varphi(x,z)\,dx = 0.$$

7.6.6 Invariant solutions with $\{X_3\}$

The invariants of the subalgebra $\{X_3\}$ are x, t, f, the representation of the invariant solution is

$$f = \varphi(x,t),$$

substituting this into equation (7.4), one obtains the reduced equation

$$\varphi_t + \varphi_x - \frac{1}{2}x\int_0^1 \varphi(x(1-s),t)\varphi(xs,t)\,ds + \varphi\int_0^\infty \varphi(x,t)\,dx = 0.$$

7.6.7 Invariant solutions with $\{X_1\}$

The invariants of the subalgebra $\{X_1\}$ are $xt^{-1}, yt^{-p-1}, t^2 f$, the representation of the invariant solution is

$$f = t^{-2}\varphi(u,v), \quad u = xt^{-1}, v = yt^{-p-1},$$

substituting this solution into equation (7.4), the reduced equation is given by

$$(1-u)\varphi_u - (p+1)v\varphi_v + \left(\int_0^\infty \varphi(u,v)\,du - 2\right)\varphi$$
$$-\frac{1}{2}u\int_0^1 \varphi(u(1-s),v)\varphi(us,v)\,ds = 0.$$

7.7 Solving the Determining Equation (7.10)

If $k = 0$, then equation (7.10) becomes

$$x^p\frac{\partial p_0}{\partial y} + \frac{\partial p_0}{\partial t} = 0, \quad p > 0.$$

For $k=1$, equation (7.10) is

$$(x^p\frac{\partial p_1}{\partial y}+\frac{\partial p_1}{\partial t})be^{-bx}+(x^p\frac{\partial h_0}{\partial y}+\frac{\partial h_0}{\partial t}-x^{2p}\frac{\partial r_0}{\partial y}-x^p\frac{\partial r_0}{\partial t}-pq_0x^{p-1})bce^{-bx}$$
$$+(x^p\frac{\partial q_0}{\partial y}+\frac{\partial q_0}{\partial t})b^2e^{-bx}-bxe^{-bx}\int_0^1 p_0(xs)e^{bsx}\,ds+be^{-bx}\int_0^\infty p_0(x)\,dx+p_0=0.$$

Splitting with respect to c, the latter equation leads to

$$x^p\frac{\partial h_0}{\partial y}+\frac{\partial h_0}{\partial t}-x^{2p}\frac{\partial r_0}{\partial y}-x^p\frac{\partial r_0}{\partial t}-pq_0x^{p-1}=0. \qquad (7.21)$$

$$(x^p\frac{\partial p_1}{\partial y}+\frac{\partial p_1}{\partial t})be^{-bx}+(x^p\frac{\partial q_0}{\partial y}+\frac{\partial q_0}{\partial t})b^2e^{-bx}$$
$$-bxe^{-bx}\int_0^1 p_0(xs)e^{bsx}\,ds+be^{-bx}\int_0^\infty p_0(x)\,dx+p_0=0.$$

Splitting with respect to b in the latter equation, one has

$$x^p\frac{\partial p_1}{\partial y}+\frac{\partial p_1}{\partial t}=0,\quad x^p\frac{\partial q_0}{\partial y}+\frac{\partial q_0}{\partial t}=0,\quad p_0=0. \qquad (7.22)$$

For $k=2$, equation (7.10) becomes

$$(x^p\frac{\partial p_2}{\partial y}+\frac{\partial p_2}{\partial t})be^{-bx}+(x^p\frac{\partial q_1}{\partial y}+\frac{\partial q_1}{\partial t})b^2e^{-bx}+\frac{1}{2}bp_1x+(\frac{1}{2}b^2x-\frac{1}{2}b)q_0$$
$$+(x^p\frac{\partial h_1}{\partial y}+\frac{\partial h_1}{\partial t})cbe^{-bx}-(x^p\frac{\partial r_1}{\partial y}+\frac{\partial r_1}{\partial t})cbx^pe^{-bx}+(1-\frac{1}{2}bx)(x^p\frac{\partial r_0}{\partial y}+\frac{\partial r_0}{\partial t})$$
$$-pcq_1bx^{p-1}e^{-bx}+(bx-1)ch_0+(b^{-p}\Gamma(p+1)-\frac{b}{p+1}x^{p+1})cr_0$$
$$-x\int_0^1[bp_1(xs)+b^2q_0(xs)+cb(h_0(xs)-x^ps^pr_0(xs))]\,ds$$
$$+\int_0^\infty[bp_1(x)+b^2q_0(x)+cb(h_0(x)-x^pr_0(x))]e^{-bx}\,dx=0.$$

Splitting it with respect to c, one obtains

$$(x^p\frac{\partial h_1}{\partial y}+\frac{\partial h_1}{\partial t})be^{-bx}-(x^p\frac{\partial r_1}{\partial y}+\frac{\partial r_1}{\partial t})bx^pe^{-bx}$$
$$+(bx-1)h_0-pq_1bx^{p-1}e^{-bx}+(b^{-p}\Gamma(p+1)-\frac{b}{p+1}x^{p+1})r_0$$
$$-x\int_0^1 b(h_0(xs)-x^ps^pr_0(xs))\,ds+\int_0^\infty b(h_0(x)-x^pr_0(x))e^{-bx}\,dx=0. \qquad (7.23)$$

$$
\begin{aligned}
(x^p\frac{\partial p_2}{\partial y}+\frac{\partial p_2}{\partial t})be^{-bx}+(x^p\frac{\partial q_1}{\partial y}+\frac{\partial q_1}{\partial t})b^2e^{-bx}\\
+(\frac{1}{2}b^2x-\frac{1}{2}b)q_0+\frac{1}{2}bp_1x+(1-\frac{1}{2}bx)(x^p\frac{\partial r_0}{\partial y}+\frac{\partial r_0}{\partial t})\\
-x\int_0^1[bp_1(xs)+b^2q_0(xs)]\,ds+\int_0^\infty[bp_1(x)+b^2q_0(x)]e^{-bx}\,dx=0. \qquad (7.24)
\end{aligned}
$$

Substituting (7.11) into the last integrals of (7.23) and (7.24), one has

$$
\begin{aligned}
(x^p\frac{\partial h_1}{\partial y}+\frac{\partial h_1}{\partial t})b^{p+1}e^{-bx}-(x^p\frac{\partial r_1}{\partial y}+\frac{\partial r_1}{\partial t})b^{p+1}x^pe^{-bx}-pq_1b^{p+1}x^{p-1}e^{-bx}\\
+(bx-1)b^ph_0+(\Gamma(p+1)-\frac{b^{p+1}}{p+1}x^{p+1})r_0-b^{p+1}x\int_0^1(h_0(xs)-x^ps^pr_0(xs))\,ds\\
+h_{00}b^p-r_{00}\Gamma(p+1)+b^p\sum_{i=1}^{\infty}\left(h_{0i}i!-r_{0i}\Gamma(p+1+i)b^{-p}\right)b^{-i}=0. \qquad (7.25)
\end{aligned}
$$

$$
\begin{aligned}
(x^p\frac{\partial p_2}{\partial y}+\frac{\partial p_2}{\partial t})be^{-bx}+(x^p\frac{\partial q_1}{\partial y}+\frac{\partial q_1}{\partial t})b^2e^{-bx}+(\frac{1}{2}b^2x-\frac{1}{2}b)q_0\\
+\frac{1}{2}bp_1x+(1-\frac{1}{2}bx)(x^p\frac{\partial r_0}{\partial y}+\frac{\partial r_0}{\partial t})-x\int_0^1[bp_1(xs)+b^2q_0(xs)]\,ds\\
+p_{10}+q_{01}+bq_{00}+\sum_{i=1}^{\infty}(p_{1i}+(i+1)q_{0i+1})\,i!b^{-i}=0. \qquad (7.26)
\end{aligned}
$$

Equating the coefficients of b^0 and $b^{-i}(i\geq 1)$ in (7.26) to zero , we also used here the expansion of the exponent e^{-bx}, one obtains

$$
x^p\frac{\partial r_0}{\partial y}+\frac{\partial r_0}{\partial t}+p_{10}+q_{01}=0,\quad p_{1i}=-(i+1)q_{0i+1},\quad i\geq 1. \qquad (7.27)
$$

$$
\begin{aligned}
(x^p\frac{\partial p_2}{\partial y}+\frac{\partial p_2}{\partial t})e^{-bx}+(x^p\frac{\partial q_1}{\partial y}+\frac{\partial q_1}{\partial t})be^{-bx}+(\frac{1}{2}q_0-\int_0^1q_0(xs)\,ds)bx\\
+q_{00}-\frac{1}{2}q_0+\frac{1}{2}p_1x-\frac{1}{2}x(x^p\frac{\partial r_0}{\partial y}+\frac{\partial r_0}{\partial t})-x\int_0^1p_1(xs)\,ds=0.
\end{aligned}
$$

Splitting with respect to b, the latter equation leads to

$$
x^p\frac{\partial p_2}{\partial y}+\frac{\partial p_2}{\partial t}=0,\quad x^p\frac{\partial q_1}{\partial y}+\frac{\partial q_1}{\partial t}=0,\quad \frac{1}{2}q_0-\int_0^1q_0(xs)\,ds=0. \qquad (7.28)
$$

$$
q_{00}-\frac{1}{2}q_0+\frac{1}{2}p_1x-\frac{1}{2}x(x^p\frac{\partial r_0}{\partial y}+\frac{\partial r_0}{\partial t})-x\int_0^1p_1(xs)\,ds=0.
$$

Substituting (7.11) into the last equation of (7.28), and using equation (7.22) and the second equation of (7.27), one has

$$p_1 = c_1, \quad q_0 = c_2 x, \tag{7.29}$$

where c_1, c_2 are constant.

Notice that p is a positive real number and function $g = h_0(x, y, t) - x^p r_0(x, y, t)$ is bounded. Dividing by b and letting $b \to \infty$ in (7.23), one derives that

$$h_0 - \frac{1}{p+1} x^p r_0 - \int_0^1 (h_0(xs) - x^p s^p r_0(xs))\, ds = 0. \tag{7.30}$$

Substituting (7.11) into equation (7.30), one obtains

$$\sum_{i=1}^{\infty} \left[\frac{i}{i+1} h_{0i}(y,t) x^i - \frac{i}{(p+1)(p+i+1)} r_{0i}(y,t) x^{p+i} \right] = 0. \tag{7.31}$$

Assume that p is a natural number, equation (7.31) can be rewritten

$$\sum_{i=1}^{\infty} \left[\frac{p+i}{p+i+1} h_{0p+i}(y,t) - \frac{i}{(p+1)(p+i+1)} r_{0i}(y,t) \right] x^{p+i} + \sum_{j=1}^{p} \frac{j}{j+1} h_{0j}(y,t) x^j = 0. \tag{7.32}$$

Equation (7.32) implies that

$$h_{0j} = 0, \; j = 1, \cdots, p, \quad h_{0p+i} = \frac{i}{(p+1)(p+i)} r_{0i}, \;\; i \geq 1. \tag{7.33}$$

Equating the coefficients of b^0 in (7.25) to zero, one obtains

$$r_0 - r_{00} + h_{0p} = 0. \tag{7.34}$$

Hence, equations (7.33) and (7.34) imply that $h_0 = h_{00}(y,t), \; r_0 = r_{00}(y,t)$.

Assume that p is not a natural number, equating the coefficients of b^0 in (7.25) to zero, one obtains $r_0 = r_{00}(y,t)$. Therefore, equations (7.31) leads to $h_0 = h_{00}(y,t)$.

In summary, for arbitrary positive real number p, one has

$$h_0 = h_{00}(y,t), \quad r_0 = r_{00}(y,t) \tag{7.35}$$

and equation (7.23) becomes

$$x^p\frac{\partial h_1}{\partial y} + \frac{\partial h_1}{\partial t} - (x^p\frac{\partial r_1}{\partial y} + \frac{\partial r_1}{\partial t})x^p - pq_1x^{p-1} = 0.$$

Hence, from equations (7.29), (7.35), (7.21) and the first equation of (7.27) one derives that

$$r_0 = -(c_1+c_2)t + c_3, \quad h_0 = ((p-1)c_2 - c_1)y + c_4, \tag{7.36}$$

where c_3, c_4 are constant.

If $k = l+3 (l \geq 1)$, then equation (7.10) becomes

$$\begin{aligned}
&[x^p\frac{\partial p_{l+3}}{\partial y} + \frac{\partial p_{l+3}}{\partial t} + b(x^p\frac{\partial q_{l+2}}{\partial y} + \frac{\partial q_{l+2}}{\partial t})]b^2e^{-(l+2)bx} + (\frac{l+2}{2}bx - l - 1)bp_{l+2}e^{-(l+1)bx}\\
&\quad+(\frac{l+2}{2}bx - \frac{2l+3}{2})b^2q_{l+1}e^{-(l+1)bx} + [x^p\frac{\partial h_{l+2}}{\partial y} + \frac{\partial h_{l+2}}{\partial t} - pq_{l+2}x^{p-1}]cb^2e^{-(l+2)bx}\\
&\quad+[(\frac{l+3}{2}bx - l - 2)ch_{l+1} + (1-\frac{1}{2}bx)(x^p\frac{\partial r_{l+1}}{\partial y} + \frac{\partial r_{l+1}}{\partial t})]be^{-(l+1)bx} + [b^{-p}\Gamma(p+1)\\
&\quad-\frac{pl+p+l+3}{2(p+1)}bx^{p+1} + (l+1)x^p]bcr_{l+1}e^{-(l+1)bx} - (x^p\frac{\partial r_{l+2}}{\partial y} + \frac{\partial r_{l+2}}{\partial t})cb^2x^pe^{-(l+2)bx}\\
&\quad+[(l+1)bx - \frac{l+1}{4}b^2x^2 - \frac{2l+1}{2}]r_le^{-lbx} - x\int_0^1[(p_{l+2}(xs) + bq_{l+1}(xs))b^2e^{-bxs}\\
&\quad+cb^2(h_{l+1}(xs) - x^ps^pr_{l+1}(xs))e^{-bxs} + (1-\frac{1}{2}bxs)br_l(xs)]e^{-lbsx}\,ds + \int_0^\infty[(p_{l+2}(x)\\
&\quad+bq_{l+1}(x) + c(h_{l+1}(x) - x^pr_{l+1}(x)))b^2e^{-(l+2)bx} + (1-\frac{1}{2}bx)br_l(x)e^{-(l+1)bx}]\,dx = 0.
\end{aligned}$$

Splitting this equation with respect to c, it follows that

$$\begin{aligned}
&[x^p\frac{\partial p_{l+3}}{\partial y} + \frac{\partial p_{l+3}}{\partial t} + b(x^p\frac{\partial q_{l+2}}{\partial y} + \frac{\partial q_{l+2}}{\partial t})]b^2e^{-(l+2)bx}\\
&\quad+[(\frac{l+2}{2}bx - \frac{2l+3}{2})b^2q_{l+1} + (b-\frac{1}{2}b^2x)(x^p\frac{\partial r_{l+1}}{\partial y} + \frac{\partial r_{l+1}}{\partial t})]e^{-(l+1)bx}\\
&\quad+(\frac{l+2}{2}bx - l - 1)bp_{l+2}e^{-(l+1)bx} + [(l+1)bx - \frac{l+1}{4}b^2x^2 - \frac{2l+1}{2}]r_le^{-lbx}\\
&\quad-x\int_0^1[(p_{l+2}(xs) + bq_{l+1}(xs))b^2e^{-(l+1)bxs} + b(1-\frac{1}{2}bxs)r_l(xs)e^{-lbsx}]\,ds\\
&\quad+\int_0^\infty[(p_{l+2}(x) + bq_{l+1}(x))b^2e^{-(l+2)bx} + b(1-\frac{1}{2}bx)r_l(x)e^{-(l+1)bx}]\,dx = 0,
\end{aligned} \tag{7.37}$$

$$\left[x^p\frac{\partial h_{l+2}}{\partial y}+\frac{\partial h_{l+2}}{\partial t}-x^p(x^p\frac{\partial r_{l+2}}{\partial y}+\frac{\partial r_{l+2}}{\partial t})-pq_{l+2}x^{p-1}\right]b^{p+1}e^{-(l+2)bx}$$
$$+\Big[(\frac{l+3}{2}bx-l-2)b^p h_{l+1}+r_{l+1}((l+1)b^p x^p-\frac{pl+p+l+3}{2(p+1)}b^{p+1}x^{p+1}$$
$$+\Gamma(p+1))\Big]e^{-(l+1)bx}-b^{p+1}x\int_0^1(h_{l+1}(xs)-x^p s^p r_{l+1}(xs))e^{-(l+1)bxs}\,ds$$
$$+b^{p+1}\int_0^\infty\left(h_{l+1}(x)-x^p r_{l+1}(x)\right)e^{-(l+2)bx}\,dx=0. \quad (7.38)$$

Substituting (7.11) into the last integral of equation (7.38), equation (7.38) can be rewritten in the form

$$\left[x^p\frac{\partial h_{l+2}}{\partial y}+\frac{\partial h_{l+2}}{\partial t}-x^p(x^p\frac{\partial r_{l+2}}{\partial y}+\frac{\partial r_{l+2}}{\partial t})-pq_{l+2}x^{p-1}\right]b^{p+1}e^{-(l+2)bx}$$
$$+\Big[(\frac{l+3}{2}bx-l-2)b^p h_{l+1}+r_{l+1}((l+1)b^p x^p-\frac{pl+p+l+3}{2(p+1)}b^{p+1}x^{p+1}$$
$$+\Gamma(p+1))\Big]e^{-(l+1)bx}-b^{p+1}x\int_0^1(h_{l+1}(xs)-x^p s^p r_{l+1}(xs))e^{-(l+1)bxs}\,ds$$
$$+\sum_{i=1}^\infty\left[\frac{i!}{(l+2)^{i+1}}h_{(l+1)i}b^{p-i}-\frac{\Gamma(p+1+i)}{(l+2)^{p+i+1}}r_{(l+1)i}b^{-i}\right]$$
$$+\frac{h_{(l+1)0}}{l+2}b^p-\frac{\Gamma(p+1)}{(l+2)^{p+1}}r_{(l+1)0}=0. \quad (7.39)$$

Substituting (7.11) into equation (7.37), one obtains

$$\sum_{i=0}^\infty\frac{(-1)^i}{i!}x^i\Big[(x^p\frac{\partial p_{l+3}}{\partial y}+\frac{\partial p_{l+3}}{\partial t}+b(x^p\frac{\partial q_{l+2}}{\partial y}+\frac{\partial q_{l+2}}{\partial t}))(l+2)^i b^{i+2}$$
$$+((\frac{l+2}{2}bx-\frac{2l+3}{2})b^2 q_{l+1}+(b-\frac{1}{2}b^2x)(x^p\frac{\partial r_{l+1}}{\partial y}+\frac{\partial r_{l+1}}{\partial t}))(l+1)^i b^i$$
$$+(\frac{l+2}{2}bx-l-1)p_{l+2}(l+1)^i b^{i+1}+((l+1)bx-\frac{l+1}{4}b^2x^2-\frac{2l+1}{2})r_l l^i b^i$$
$$-x\int_0^1((p_{l+2}(xs)+bq_{l+1}(xs))(l+1)^i b^{i+2}s^i+(1-\frac{1}{2}bxs)r_l(xs)l^i b^{i+1}s^i)\,ds\Big]$$
$$+\frac{bp_{(l+2)0}+b^2q_{(l+1)0}}{l+2}+\frac{p_{(l+2)1}+bq_{(l+1)1}}{(l+2)^2}+\frac{2q_{(l+1)2}}{(l+2)^3}+\frac{(2l+1)r_{l0}}{2(l+1)^2}$$
$$+\sum_{i=1}^\infty\left[\frac{(i+1)!p_{(l+2)i+1}}{(l+2)^{i+2}}+\frac{(i+2)!q_{(l+1)i+2}}{(l+2)^{i+3}}+\frac{(2l+1)i!r_{li}}{2(l+1)^{i+2}}\right]b^{-i}=0. \quad (7.40)$$

Equating the coefficients of b^0 on the left-hand side of (7.40) to zero , one has

$$r_l(x,y,t)=\frac{2}{2l+1}\left(\frac{p_{(l+2)1}(y,t)}{(l+2)^2}+\frac{2q_{(l+1)2}(y,t)}{(l+2)^3}+\frac{(2l+1)r_{l0}(y,t)}{2(l+1)^2}\right),\quad l\geq 1,$$

which implies that

$$r_l(x,y,t) = r_{l0}(y,t), \quad r_{li}(y,t) = 0, \quad l \geq 1, \quad i \geq 1, \tag{7.41}$$

where

$$r_{l0} = \frac{2(l+1)^2}{l(l+2)(2l+1)} \left(\frac{p_{(l+2)1}}{(l+2)^2} + \frac{2q_{(l+1)2}}{(l+2)^3} \right), \quad l \geq 1.$$

Equating the coefficients of b, b^2 and $b^{-i} (i \geq 1)$ in (7.40) to zero, respectively, one has

$$p_{l+2} = \frac{1}{l+1} \left[x^p \frac{\partial r_{(l+1)0}}{\partial y} + l r_{l0} x + \frac{\partial r_{(l+1)0}}{\partial t} + \frac{p_{(l+2)0}}{l+2} + \frac{q_{(l+1)1}}{(l+2)^2} \right], \tag{7.42}$$

$$\begin{aligned} q_{l+1} = & \frac{2x^p}{(2l+3)(l+2)} [\frac{\partial^2 r_{(l+2)0}}{\partial y^2} x^p + \frac{\partial r_{(l+1)0}}{\partial y} (l+1)x + \frac{\partial^2 r_{(l+2)0}}{\partial y \partial t} \\ & + \frac{1}{l+3} \frac{\partial p_{(l+3)0}}{\partial y} + \frac{1}{(l+3)^2} \frac{\partial q_{(l+2)1}}{\partial y}] + \frac{2}{(2l+3)(l+2)} [\frac{\partial^2 r_{(l+2)0}}{\partial t \partial y} x^p \\ & + \frac{\partial r_{(l+1)0}}{\partial t} (l+1)x + \frac{1}{l+3} \frac{\partial p_{(l+3)0}}{\partial t} + \frac{\partial^2 r_{(l+2)0}}{\partial t^2} + \frac{1}{(l+3)^2} \frac{\partial q_{(l+2)1}}{\partial t}] \\ & - (x^p \frac{\partial r_{(l+1)0}}{\partial y} + \frac{\partial r_{(l+1)0}}{\partial t})x + \frac{2l^2+5l+4}{(2l+3)(l+1)} x [l r_{l0} x + \frac{\partial r_{(l+1)0}}{\partial t} \\ & + x^p \frac{\partial r_{(l+1)0}}{\partial y} + \frac{p_{(l+2)0}}{l+2} + \frac{q_{(l+1)1}}{(l+2)^2}] - \frac{2x}{(2l+3)(l+1)} [\frac{1}{2} l r_{l0} x + \frac{\partial r_{(l+1)0}}{\partial t} \\ & + \frac{x^p}{p+1} \frac{\partial r_{(l+1)0}}{\partial y} + \frac{p_{(l+2)0}}{l+2} + \frac{q_{(l+1)1}}{(l+2)^2}] - \frac{(l+1)l}{2} r_{l0} x^2 + \frac{2q_{(l+1)0}}{(2l+3)(l+2)}, \end{aligned} \tag{7.43}$$

$$q_{(l+1)i+2} = -\frac{l+2}{i+2} p_{(l+2)i+1}, \quad l \geq 1, \quad i \geq 1. \tag{7.44}$$

The obtained conditions (7.41)-(7.44) are used for analyzing equation (7.39). Thus, equation (7.39) becomes

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{(-x)^i}{i!} \Big[& (x^p \frac{\partial h_{l+2}}{\partial y} + \frac{\partial h_{l+2}}{\partial t} - x^p (x^p \frac{\partial r_{(l+2)0}}{\partial y} + \frac{\partial r_{(l+2)0}}{\partial t}) - p q_{l+2} x^{p-1})(l+2)^i b^{p+i+1} \\ & + ((\frac{l+3}{2} bx - l - 2) h_{l+1} + r_{(l+1)0}((l+1)x^p - \frac{pl+p+l+3}{2(p+1)} bx^{p+1}))(l+1)^i b^{p+i} \\ & + (l+1)^i (\frac{b^{p+i+1}}{p+i+1} r_{(l+1)0} x^{p+1} - b^{p+i+1} x \int_0^1 s^i h_{l+1}(xs)\, ds + \Gamma(p+1) r_{(l+1)0} b^i) \Big] \end{aligned}$$

$$+\frac{h_{(l+1)0}}{l+2}b^p - \frac{\Gamma(p+1)}{(l+2)^{p+1}}r_{(l+1)0} + \sum_{i=1}^{\infty}\frac{i!}{(l+2)^{i+1}}h_{(l+1)i}b^{p-i} = 0. \qquad (7.45)$$

As considered earlier, one needs to study cases where p is natural number or real number. Assume that p is a natural number, equating the coefficients with respect to $b^0, b^{-i}(i \geq 1)$ and b^p in (7.45) to zero, respectively, one obtains

$$h_{(l+1)p} = (1-(l+2)^{p+1})r_{(l+1)0},\ h_{(l+1)i} = 0,\ i \geq p+1,$$

$$h_{l+1} = \frac{1}{l+2}[l+1+(-l-1)^p]r_{(l+1)0}x^p + \frac{1}{(l+2)^2}h_{(l+1)0}.$$

These conditions mean that $h_{l+1} = 0, r_{(l+1)0} = 0(l \geq 1)$. Substitution of them into equation (7.45) leads to $q_{l+2} = 0(l \geq 1)$. Hence, one has

$$h_{l+1} = 0, \quad r_{(l+1)0} = 0, \quad q_{l+2} = 0, \quad l \geq 1.$$

Assuming that p is not a natural number, equating the coefficients with respect to b^0 in equation (7.45) to zero, it follows that

$$\Gamma(p+1)(1-\frac{1}{(l+2)^{p+1}})r_{(l+1)0} = 0, \quad l \geq 1,$$

which provides that

$$r_{(l+1)0} = 0, \quad l \geq 1.$$

Therefore, equation (7.45) becomes

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i\Big[(x^p\frac{\partial h_{l+2}}{\partial y} + \frac{\partial h_{l+2}}{\partial t} - pq_{l+2}x^{p-1})(l+2)^i b^{i+1}$$
$$+h_{l+1}(\frac{l+3}{2}bx - l - 2)(l+1)^i b^i - (l+1)^i b^{i+1} x\int_0^1 s^i h_{l+1}(xs)\,ds\Big]$$
$$+\frac{h_{(l+1)0}}{l+2} + \sum_{i=1}^{\infty}\frac{i!}{(l+2)^{i+1}}h_{(l+1)i}b^{-i} = 0. \qquad (7.46)$$

Equating the coefficients with respect to b^0 and $b^{-i}(i \geq 1)$ in equation (7.46) to zero, one obtains

$$h_{l+1} = \frac{1}{(l+2)^2} h_{(l+1)0}, \quad h_{(l+1)i} = 0, \quad i \geq 1, \quad l \geq 1,$$

these equations and (7.46) imply that

$$q_{l+2} = 0, \quad h_{l+1} = 0, \quad l \geq 1.$$

Hence, equations (7.41)-(7.44) are reduced to

$$p_{(l+2)0} = \frac{q_{(l+1)1}}{(l^2+3l+1)(l+2)}, \quad p_{(l+2)1} = \frac{l r_{l0}}{l+1}, \quad p_{(l+2)i} = 0, \quad i \geq 2, \quad l \geq 1,$$

$$r_{l0} = \frac{4(l+1)^3}{l^2(l+2)^3(l+3)(2l+1)} q_{(l+1)2}, \quad q_{(l+1)i} = 0, \quad i \geq 3, \quad l \geq 1,$$

$$p_{(l+2)1} = -\frac{2}{l+2} q_{(l+1)2}, \quad q_2 = \frac{4}{405} q_{22} x^2 + \frac{3}{20} q_{21} x + \frac{2}{15} q_{20}, \quad l \geq 2,$$

which means that

$$p_{l+2} = 0, \quad q_{l+1} = 0, \quad h_{l+1} = 0, \quad r_l = 0, \quad l \geq 1. \tag{7.47}$$

Finally, consider $k = 3$. Notice that by virtue of (7.36) and (7.47), the determining equation (7.10) in this case is

$$[(bx-1)p_2 + (b^2x - \frac{3}{2}b)q_1 + (\frac{3}{2}bx - 2)ch_1]e^{-bx} - x\int_0^1 [bp_2(xs)$$
$$+ b^2 q_1(xs) + cbh_1(xs)]e^{-bxs}\, ds + \int_0^\infty [bp_2(x) + b^2 q_1(x) + cbh_1(x)]e^{-2bx}\, dx = 0.$$

Splitting it with respect to c, one obtains

$$[(bx-1)p_2 + (b^2x - \frac{3}{2}b)q_1]e^{-bx} - x\int_0^1 [bp_2(xs) + b^2 q_1(xs)]e^{-bxs}\, ds$$
$$+ \int_0^\infty [bp_2(x) + b^2 q_1(x)]e^{-2bx}\, dx = 0. \tag{7.48}$$

$$(\frac{3}{2}bx - 2)h_1 e^{-bx} - x\int_0^1 bh_1(xs)e^{-bxs}\, ds + \int_0^\infty bh_1(x)e^{-2bx}\, dx = 0.$$

Substituting (7.11) into the last integral of the latter equation, one has

$$(\frac{3}{2}bx-2)h_1e^{-bx}-x\int_0^1 bh_1(xs)e^{-bxs}\,ds+\frac{1}{2}h_{10}+\sum_{i=1}^{\infty}2^{-(i+1)}i!h_{1i}b^{-i}=0.$$

Equating the coefficients of $b^{-i}(i\geq 1)$ to zero, one obtains

$$h_1=h_{10},\quad \left[(3bx-2)e^{-bx}-1\right]h_{10}=0.$$

By virtue of the arbitrariness of b and that function h_{10} does not depend on b, one derives that

$$h_1=0.$$

By means of (7.11), equation (7.48) is rewritten

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i\Big[p_2(bx-1)b^i+q_1(b^2x-\frac{3}{2}b)b^i+\frac{1}{2}q_{10}b+\frac{1}{2}p_{20}+\frac{1}{4}q_{11}$$

$$-x\int_0^1(p_2(xs,t)+q_1(xs,t)b)b^{i+1}s^i\,ds\Big]+\sum_{i=1}^{\infty}\frac{i!}{2^{i+1}}\left(\frac{i+1}{2}q_{1i+1}+p_{2i}\right)b^{-i}=0.\quad(7.49)$$

Equating the coefficients of b^0 and $b^{-i}(i\geq 1)$ on the left-hand side of (7.49) to zero, one has

$$p_2(x,y,t)=\frac{1}{2}p_{20}(y,t)+\frac{1}{4}q_{11}(y,t),\qquad q_{1i+1}=-\frac{2}{i+1}p_{2i},\quad i\geq 1,$$

which means that

$$p_2=p_{20}=\frac{1}{2}q_{11},\quad q_1=q_{10}+q_{11}x,\quad p_{2i}=0,\quad i\geq 1.$$

Thus, equation (7.48) is reduced to

$$e^{-bx}(b^2q_{11}x^2+b^2q_{10}x-\frac{1}{2}bq_{10}+q_{11})-\frac{1}{2}bq_{10}-q_{11}=0.$$

Because of the arbitrariness of b the latter equation leads to

$$p_2=0,\quad q_1=0.$$

In summary, the general solution of the determining equation (7.10) is

$$\xi^x=c_2x,\quad \xi^t=-(c_1+c_2)t+c_3,\quad \xi^y=((p-1)c_2-c_1)y+c_4,\quad \eta^f=c_1f.$$

CHAPTER VIII

CONCLUSION

The PBEs (Ramkrishna, 2000) have received an unprecedented amount of attention during the past few years from both academic and industrial practitioners because of their applicability to a wide variety of particulate processes (Randolph and Larson, 1988). In general the application of initial conditions and auxiliary equations to the population balance results in a set of integro-partial-differential equations of multi-dimension.

Solving the PBE is of significance for applications in physics, chemistry, engineering and mathematics. However, for the PBE it is typically impossible to find exact solutions in most realistic cases. The classical group analysis method was developed for obtaining solutions of partial differential equations. Recently this analysis was extended for equations with nonlocal terms, in particular, for integro-differential equations (Meleshko, 2005; Grigoriev et al., 2010). This dissertation was devoted to the investigation of analytical solutions of the PBEs by using the developed group analysis method (Meleshko, 2005; Grigoriev et al., 2010) and the method of moments. It is separated in four parts.

The one-dimensional homogeneous PBE with time dependent but size independent growth rate and time dependent nucleation rate was investigated in the first part of this dissertation, and involves both a partial differential equation and integro-differential equation. We started from applications of scaling transformations and the method of moments for constructing exact solutions. Then the complete group analysis was also applied to investigate exact solutions of the one-dimensional homogeneous PBE with size-independent growth. All invariant solutions were considered. Analysis of the reduced equations is provided.

In the second part of this dissertation, we considered Smoluchowski's coagulation equation with constant kernel $K = k_0$

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}k_0 x \int_0^1 f(x(1-s),t)f(xs,t)\,ds - k_0 f(x,t)\int_0^\infty f(x,t)\,dx,$$

using scaling group an explicit analytical physical solution of this equation was presented by

$$f(x,t) = 2\frac{\beta}{k_0}t^{-2}e^{-\beta x t^{-1}},$$

where β is a positive real number. Using the developed group analysis method for integro-differential equations is complicated due to the difficulty of solving the determining equations. Moreover, the approach of solving the determining equations depends on the studied integro-differential equations. Hence, the study of the new integro-differential equations was begun by considering the self-similar solutions. The explicit physical solutions are useful for verifying numerical programs and checking the accuracy of numerical solutions of PBEs. However, analytic solutions of PBEs are seldom available; this challenging question has motivated us to develop specialized approaches for finding explicit physical solutions of the PBEs.

In the third part of this dissertation, the developed group analysis method was presented for the one-dimensional PBE for aggregation in a well-mixed batch system including a crystal growth term. The determining equations were solved, the optimal system, invariant solutions and all the reduced equations were obtained, and the numerical solutions for the reduced equation were determined by the homotopy perturbation method. Furthermore, finding the determining equation by use of the preliminary group classification was also considered.

The last part of this dissertation was devoted to the group analysis of the one-dimensional PBE involving aggregation, breakage, and growth processes with one internal coordinate (particle volume or mass) and one external coordinate (the locations of particles). The coefficients of group generator in the determining equation were represented by the Taylor series and the determining equation is successfully solved. For constant growth rate G, the symmetries of dimensionless equation (1.5) with homogeneous kernel (1.8) and the spatial velocity (1.6) were presented. The complete group classification is provided, and all invariant solutions were obtained by using the optimal system of one and two-dimensional subalgebras. In particular, an explicit analytical solution of equation

$$\frac{\partial f(x,y,t)}{\partial t} + x^p \frac{\partial f(x,y,t)}{\partial y} = \frac{1}{2} x \int_0^1 f(x(1-s),y,t) f(xs,y,t)\, ds - f(x,y,t) \int_0^\infty f(x,y,t)\, dx$$

is given by

$$f(x,y,t) = 2\beta t^{-2} e^{-\beta x t^{-1}},$$

where β is a positive real number.

REFERENCES

REFERENCES

Akhatov, I. S., Gazizov, R. K., and Ibragimov N. H. (1989). Nonlocal symmetries. Heuristic approach [Russian]. In: Itogi Nauki i Tekhniki, Ser. Sovremennye Problemy Matematiki, Noveishie Dostizheniya Vol. 34, VINITI, Moscow. English translation in: **J. Sov. Math.** 55(1): 1401-1450, 1991.

Bourbaki, N. (1896). **Eléments d'histoire des mathématiques.** Paris: Masson Editeur. English translated from the French by John Meldrum, Elements of the History of Mathematics, published by Springer, Berlin, 1994.

Birkhoff, G. (1950). **Hydrodynamics: A Study in Logic, Fact and Similitude** (1st ed.). Princeton: Princeton University Press.

Bobylev, A. V., and Ibragimov, N. H. (1989). Interconnectivity of symmetry properties for equations of dynamics, kinetic theory of gases, and hydrodynamics. **Matem. Mod.** 1(3): 100-109. [Russian]

Briesen, H. (2006). Simulation of crystal size and shape by means of a reduced two-dimensional population balance model. **Chem. Eng. Sci.** 61(1): 104-112.

Bunimovich, A. I., and Krasnoslobodtsev, A. V. (1982). Invariant-group solutions of kinetic equations. **Mekhanika Zhidkosti i Gaza** (4): 135-140. [Russian]

Bunimovich, A. I., and Krasnoslobodtsev, A. V. (1983). On some invariant transformations of kinetic equations. **Vestnik Moscow State Univ. Ser. 1 Matemat. Mechan.** (4): 69-72. [Russian]

Bridgman, P. W. (1922). **Dimensional analysis**. New Haven and London: Yale University Press.

Batchelor, G. K. (1967). **An Introduction to Fluid Dynamics**. New York: Cambridge University Press.

Bobylev, A. V. (1975). On exact solutions of the Boltzmann equation. **Dokl. AS USSR.** 225(6): 1296-1299.

Cameron, I. T., Wang, F. Y., Immanuel, C. D., and Stepanek, F. (2005). Process systems modelling and applications in granulation: A review. **Chem. Eng. Sci.** 60(14): 3723-3750.

Chetverikov, V. N., and Kudryavtsev A. G. (1995a). A method for computing symmetries and conservation laws of integro-differential equations. **Acta Appl. Math.** 41: 45-56.

Chetverikov, V. N., and Kudryavtsev A. G. (1995b). Modeling integro-differential equations and a method for computing their symmetries and conservation laws. In V. V. Lychagin (ed.). The Interplay Between Differential Geometry and Differential Equations. **Amer. Math. Soc. Transl. Ser. 2** 167(2): 1-22. Translated by the authors.

Dette, H., and Studden, W. J. (1997). **The Theory of Canonical Moments with Applications in Statistics, Probability, and Analysis.** New York: Wiley.

Flood, A. E. (2002). Thoughts on recovering particle size distributions from the moment form of the population balance. **Developments in Chem. Eng. Min. Proc.** 10(5-6): 501-519.

Flood, A. E. (2009). **Industrial Crystallization from Solution: A Primer.** Thailand: School of Chemical Engineering, Suranaree University of Technology.

Fournier, N., and Laurençot, P. (2006). Well-posedness of Smoluchowski's coagulation equation for a class of homogeneous kernels. **J. Funct. Anal.** 233(2): 351-379.

Grigoriev, Y. N., and Meleshko, S. V. (1986). **Investigation of Invariant Solutions of the Boltzmann Kinetic Equation and Its Models.** Preprint of Institute of Theoretical and Applied Mechanics.

Grigoriev, Y. N., and Meleshko, S. V. (1987). Group analysis of the integro-differential Boltzman equation. **Dokl. AS USSR.** 297(2): 323-327.

Grigoriev, Y. N., and Meleshko, S. V. (1995). Group analysis of kinetic equations. **Russ. J. Numer. Anal. Math. Model.** 10(5): 425-447.

Grigoriev, Y. N., and Meleshko, S. V. (1998). Bobylev-Krook-Wu models for multicomponent gas mixture. **Phys. Rev. Lett.** 81(1): 93-95.

Grigoriev, Y. N., Ibragimov, N. H., Kovalev, V. F., and Meleshko, S. V. (2010). **Symmetries of Integro-Differential Equations: With Applications in Mechanics and Plasma Physics**. New York: Springer.

Hulburt, H. M., and Katz, S. (1964). Some problems in particle technology. A statistical mechanical formulation. **Chem. Eng. Sci.** 19(8): 555-574.

He, J. H. (2000). A coupling method of a homotopy technique and a perturbation technique for non-linear problems. **Int. J. Non-Linear Mech.** 35(1): 37-43.

He, J. H. (1999). Homotopy perturbation technique. **Comput. Method Appl. Mech. Eng.** 178(3-4): 257-262.

He, J. H. (2003). Homotopy perturbation method: a new nonlinear analytical technique. **Appl. Math. Comput.** 135(1): 73-79.

Ibragimov, N. H. (1983). **Transformation Groups Applied to Mathematical Physics**. Moscow: Nauka. English translation, published by D. Reidel Publishing Company, Dordrecht, 1985.

Ibragimov, N. H. (1999). **Elementary Lie Group Analysis and Ordinary Differential Equations**. Chichester: John Wiley & Sons.

Ibragimov, N. H. **CRC Handbook of Lie Group Analysis of Differential Equations**. Volume 1 (1994), Volume 2 (1995) and Volume 3 (1996). Boca Raton: CRC Press.

Kumar, S., and Ramkrishna, D. (1997). On the solution of population balance equations by discretization-III. Nucleation, growth and aggregation of particles. **Chem. Eng. Sci.** 52(24): 4659-4679.

Krasilshchik, I. S., and Vinogradov, A. M. (1984). Nonlocal symmetries and the theory of coverings: An addendum to A. M. Vinogradov's 'local symmetries and conservation laws'. **Acta Appl. Math.** 2(1): 79-96.

Krook, M., and Wu, T. T. (1977). Exact solutions of the Boltzmann equation. **Phys. Fluids** 20(10): 1589-1595.

Lie, S. (1888). **Theorie der Transformationsgruppen I, II and III**. Leipzig: B. G. Teubner. Reprinted by Chelsea Publishing Company, New York, 1970.

Langhaar, H. L. (1951). **Dimensional Analysis and Theory of Models**. New York: John Wiley and Sons.

Lie, S. (1871). Om en Classe geometriske Transformationer. **Ges. der Wiss. zu Christiania.** 1870: 506–509.

Lie, S. (1896). **Geometrie der Berührungstransformationen**. Leipzig: B. G. Teubner. Reprinted by Chelsea Publishing Company, New York, 1977.

Liao, S. J. (1995). An approximate solution technique not depending on small parameters: a special example. **Int. J. Non-linear Mech.** 30(3): 371-380.

Liao, S. J. (1997). Boundary element method for general nonlinear differential operators. **Eng. Anal. Bound. Elem.** 20(2): 91-99.

Lakshmikantham, V., and Rama Mohana Rao, M. (1995). **Theory of Integro-Differential Equations**. Switzerland: Gordon and Breach Science Publishers.

Meleshko, S. V. (2005). **Methods for Constructing Exact Solutions of Partial Differential Equations: Mathematical and Analytical Techniques with Applications to Engineering**. New York: Springer.

Maxwell, J. C. (1867). On the dynamical theory of gases. **Philos. Trans. R. Soc. Lond.** 157: 49-88.

McCoy, B. J. (2002). A population balance framework for nucleation, growth, and aggregation. **Chem. Eng. Sci.** 57(12): 2279-2285.

Mastan, E., and Zhu, S. (2015). Method of moments: A versatile tool for deterministic modeling of polymerization kinetics. **Eur. Polym. J.** 68: 139-160.

Nyvlt, J., Sohnel, O., Matuchova, M., and Broul, M. (1985). **Kinetics of Industrial Crystallization**. Netherlands, Amsterdam: Elsevier.

Nonnenmacher, T. F. (1984). Application of the similarity method to the nonlinear Boltzmann equation. **Z. Angew. Math. Phys.** 35(5): 680-691.

Ovsiannikov, L. V. (1978). **Group Analysis of Differential Equations**. Moscow: Nauka. English translation, Ames, W. F., Ed., published by Academic Press, New York, 1982.

Olver, P. J. (1993). **Applications of Lie Groups to Differential Equations** (2nd ed.). New York: Springer.

Ramkrishna, D. (2000). **Population Balances: Theory and Applications to Particulate Systems in Engineering**. San Diego: Academic Press.

Randolph, A. D., and Larson, M. A. (1988). **Theory of Particulate Processes: Analysis and Techniques of Continuous Crystallization** (2nd ed.). New York: Academic Press.

Randolph, A. D., and Larson, M. A. (1971). **Theory of Particulate Processes: Analysis and Techniques of Continuous Crystallization** (1st ed.). New York: Academic Press.

Randolph, A. D., and Larson, M. A. (1964). A population balance for countable entities. **Can. J. Chem. Eng.** 42(6): 280-281.

Rawlings, J. B., Witkowski, W. R., and Eaton, J. W. (1992). Modelling and control of crystallizers. **Powder Technol.** 69(1): 3-9.

Smoluchowski, M. Z. (1917). Versuch einer mathematischen Theorie der Koagulation rinetisch kolloider Lösungen. **Zeit. Phys. Chem.** 92: 129-168.

Szirtes, T. (2007). **Applied Dimensional Analysis and Modeling** (2nd ed.). Burlington: Butterworth-Heinemann.

Sedov, L. I. (1993). **Similarity and Dimensional Methods in Mechanics** (10th ed.). Boca Raton: CRC Press.

Scott, W. T. (1968). Analytic studies of cloud droplet coalescence I. **J. Atmos. Sci.** 25(1): 54-65.

Schumann, T. E. W. (1940). Theoretical aspects of the size distribution of fog particles. **Quart. J. Roy. Meteorol. Soc.** 66(285): 195-208.

Smit, D. J., Hounslow, M. J., and Paterson, W. R. (1994). Aggregation and gelation-I. Analytical solutions for CST and batch operation. **Chem. Eng. Sci.** 49(7): 1025-1035.

Smith, M., and Matsoukas, T. (1998). Constant-number Monte-Carlo simulation of population balances. **Chem. Eng. Sci.** 53(9): 1777-1786.

Suriyawichitseranee, A., Grigoriev, Y. N., and Meleshko, S. V. (2015). Group analysis of the Fourier transform of the spatially homogeneous and isotropic Boltzmann equation with a source term. **Commun. Nonlinear Sci. Numer. Simulat.** 20(3): 719-730.

Taranov, V. B. (1976). On symmetry of one-dimensional high frequency motions of collisionless plasma. **J. Tech. Phys.** 46(6): 1271-1277.

William, M. J. (2001). **Crystallization** (4th ed.). London: Butterworth-Heinemann.

CURRICULUM VITAE

NAME: Fubiao Lin

GENDER: Male

NATIONALITY: Chinese

DATE OF BIRTH: Dec. 20, 1978

PLACE OF BIRTH: Guizhou, China

E-MAIL: linfubiao0851@163.com

AFFILIATION AND ADDRESS: School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

EDUCATIONS:

M. Sc. School of Mathematics and Computer Science, Guizhou Normal University, China, 2009.

B. Sc. School of Mathematics and Computer Science, Guizhou Normal University, China, 2004.

CONFERENCES:

On the exact solutions of population balance equation by the method of moments and Lie group analysis. **The 41st Congress on Science and Technology of Thailand (STT41)**. November 6-8, 2015, Suranaree University of Technology, Nakhonratchasima, Thailand.

PUBLICATIONS:

F. B. Lin, A. E. Flood, and S. V. Meleshko. Exact solutions of population balance equation. **Communications in Nonlinear Science and Numerical Simulation**. 2016, 36: 378-390.