Isogeometric Interpolation by Generalized Splines

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Abstract. It is well known that polynomial splines generally do not retain the geometric properties of the given data. This paper defines a class of functions $I(V)$ having shape properties ("isogeometry") determined by a given set of points $V = \{P_i = (x_i, f_i) \in R^2 : x_0 < x_1 < \cdots < x_N\}$. Based on the definition, necessary and sufficient inequality conditions on V are given in order that $I(V)$ be non-empty. A local algorithm for convex and monotone interpolation by C^2 generalized splines is obtained. Its application enables us to give a complete solution to the isogeometric interpolation problem for data of arbitrary form, and to isolate the sections of linearity, the angles and the breaks.

One of the most widespread approximation methods for mesh functions is the interpolation by C^2 cubic splines. Sufficient for many applications smoothness properties of such splines combine with the simplicity of their computer realization and high accuracy of results obtained. But in a number of cases the behavior of cubic spline does not agree with qualitative characteristics of the initial data. Visually it is displayed in the presence of jumps, oscillations, various deviations not characteristic for a given set of points. These features may be expressed mathematically as nonmonotonicity and presence of the inflection points in the data monotonicity and convexity intervals.

The attempts to improve the shape properties of cubic splines were undertaken long before. For this purpose various generalizations of cubic splines were introduced. One of the first such papers seems to be [19] devoted to cubic spline in tension. The papers [2-4,16,20] are also worth mentioning. The authors of all these investigations introduced several parameters into the structure of the spline to control the qualitative behavior of the obtained curve. They did not offer, however, any procedure for automatical choice of these parameters giving only recommendations on their choice in the dialogue

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regime. Naturally, it was connected with insufficient formality of the cubic spline shape behavior notion and with the difficulties in the development of algorithms for free parameters automatic choice.

The problem of spline construction with given geometric characteristics seems to be formalized for the first time in the papers by A.I.Grebennikov [9,10], where it was referred to as the problem of isogeometric approximation. In particular, based on the local B-spline approximation methods, there was shown that on sufficiently detailed mesh the cubic spline preserves the geometric properties of the approximated function.

In [7] had given an effective partial solution to the isogeometric interpolation problem using the Hermite cubic spline. This algorithm is local and produce the C^1 monotone curve. However it does not retain the data convexity. Comparatively recently this result was improved in [6]. Then by several authors [12,18,21] the problem of isogeometric interpolation was also solved using parabolic splines. The same problem was treated by special rational splines in [5,11,13].

Certain progress during a solution to the isogeometric interpolation problem by C^2 splines was the invention of variable order cubic splines having additional knots [17]. Such splines include as a particular case the piecewise linear interpolation. Based on the investigation of the defining equations system for generalized C^2 cubic spline, in [15] sufficient conditions were obtained which guarantee the preservation of the data monotonicity and convexity properties. In particular, it enables to automatize the process of the generalized cubic spline parameters searching for rational splines and splines with additional knots. Unfortunately, this algorithm works only on the data monotonicity and/or convexity intervals not involving the case of arbitrary data. The algorithm of the parameters choice for isogeometric exponential splines is given in [14].

By the development of algorithms for isogeometric interpolation we obtain essential advantages using the parametrization. It permits to improve the visual correspondance of the initial data and the resulting curve. We can also extend the well-known algorithms to non-unique data [8,22].

In the present paper the definition of the functions with isogeometry different from that in [6] is proposed. The necessary and sufficient conditions for the existence of functions with isogeometry are clarified. By exhibiting of C^2 cubic rational splines with additional knots we give a complete solution to the isogeometric interpolation problem for arbitrary form data. As the initial step the standard interpolating cubic spline or local approximation methods can be used. The efficiency of the proposed method is illustrated by several examples.

§1. THE CLASS OF FUNCTIONS WITH ISOGEOMETRY

Let the sequence of points $V = \{P_i \,|\, i = 0, 1, \ldots, N\}$, $P_i = (x_i, f_i)$, on the plane $\mathbb{R}^2$ be fixed, where $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ forms a partition of the interval $[a, b]$. We introduce the notation for first two devided differences $\Delta_i f = (f_{i+1} - f_i)/h_i$, $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, N-1$; $\delta_i f = \Delta_i f - \Delta_{i-1} f$, $i = 1, 2, \ldots, N-1$. As usual, we shall say that the initial data increases monotonically (decreases monotonically) on the subinterval $[x_n, x_k]$, $n > k$, if $\Delta_i f > 0$ ($\Delta_i f < 0$), $i = n, \ldots, k-1$. We say it is convex down (up) on $[x_n, x_k]$, $k > n+1$ if $\delta_i f > 0$ ($\delta_i f < 0$), $i = n, \ldots, k-2$.

We call the problem of searching for a sufficiently smooth function $S(x)$ such that $S(x_i) = f_i$, $i = 0, 1, \ldots, N$, and $S(x)$ preserves the form of the initial data an isogeometric interpolation problem. It means that $S(x)$ should monotonically increases or decreases if the data has the same behaviour. Analogously $S(x)$ should also be convex (concave) in data convexity (concavity) intervals.

Evidently the solution of the isogeometric interpolation problem is not unique. We formalize the class of functions in which we search for the solution.

Definition 1.1. *The set of functions $I(V)$ is called the class of the functions with the isogeometry if for any $S(x) \in I(V)$ the following conditions are met:*

(1) $S(x) \in C^2[a, b]$;
(2) $S(x_i) = f_i$, $i = 0, 1, \ldots, N$;
(3) $S'(x)\Delta_i f \geq 0$ *if* $\Delta_i f \neq 0$ *and* $S'(x) = 0$ *if* $\Delta_i f = 0$ *for all* $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$; *and*
(4) $S''(x_i)\delta_i f \geq 0$, $i = 1, 2, \ldots, N-1$; $S''(x)\delta_j f \geq 0$, $x \in [x_i, x_{i+1}]$, $j = i, i+1$ *if* $\delta_i f \delta_{i+1} f \geq 0$; $S(x)$ *has no more than one inflection point* $\overline{x}$ *in the interval* (x_i, x_{i+1}) *if* $\delta_i f \delta_{i+1} f < 0$ *and also* $S''(x)\delta_i f \geq 0$ *for* $x \in [x_i, \overline{x}]$ *and the number of inflection points in the interval* (x_{i-1}, x_{i+1}) *does not exceed the number of sign changes in the sequence* $\delta_{i-1} f, \delta_i f, \delta_{i+1} f$.

Remark. When counting the number of sign changes in the sequence $\delta_{i-1} f, \delta_i f, \delta_{i+1} f$, the zeros are omitted.

The formulated definition enables us to choose the class of interpolants $I(V)$ having geometric properties determined by a given set of points V. It differs from the approach [10] in which for the functions with definite analytical structure the approximations preserving several shape properties are considered.

It is to be pointed that piecewise linear interpolation of the initial data V fulfills all mentioned requirements except belonging to the $C^2[a,b]$ class.

We will find necessary and sufficient inequality conditions on the set of points V in order that the class of functions with the isogeometry $I(V)$ be non-empty. Preliminarily we prove a number of auxiliary propositions which characterize the properties of the functions with the isogeometry.

Lemma 1.1. *If $\Delta_{i-1}f\Delta_i f \le 0$, then for the function $S(x)$ to be isogeometric, it is necessary that $S'(x_i)=0$.*

The assertion of the Lemma 1.1 is evident by virtue of the condition (3) in the Definition 1.1.

Lemma 1.2. *If $\delta_i f = 0$ and $\delta_{i-1}f\delta_{i+1}f \ge 0$, then the unique function with the isogeometry on the interval $[x_{i-1}, x_{i+1}]$ is the straight line passing through the points P_{i-1}, P_i, P_{i+1}.*

Proof: The equality $\delta_i f = 0$ implies that the points P_{i-1}, P_i, P_{i+1} are situated on the straight line $f(x) = f_i + \Delta_i f(x - x_i)$. Then by the condition (4) from the Definition 1.1 the function with the isogeometry $S(x)$ in the interval $[x_{i-1}, x_{i+1}]$ has no inflection points, i.e. $S''(x) \ge 0$, $x \in [x_{i-1}, x_{i+1}]$ or $S''(x) \le 0$, $x \in [x_{i-1}, x_{i+1}]$. It is obvious that the function $u(x) = S(x) - f(x)$ is the solution to the boundary-value problem $u''(x) = S''(x)$, $u(x_{i-1}) = 0$, $u(x_{i+1}) = 0$. So we have

$$u(x) = \frac{1}{x_{i+1} - x_i} \int_{x_{i-1}}^{x_{i+1}} G(x,v) S''(v) dv,$$

where $G(x,v) = (v - x_{i-1})(x - x_{i+1})$ if $v \le x$, $G(x,v) = (v - x_{i+1})(x - x_{i-1})$ if $v \ge x$. There, if $S''(x)$ has constant sign in $[x_{i-1}, x_{i+1}]$ and $S''(x) \not\equiv 0$, then $u(x_i) \ne 0$. The last contradicts the condition of interpolation $S(x_i) = f_i$. Hence, $S''(x) \equiv 0$ holds and $S(x)$ coincides with $f(x)$ that was to be proved.

Lemma 1.2 immediately implies

Corollary 1.1. *If $\delta_i f = \delta_{i+1} f = 0$, then the unique function with the isogeometry in the interval $[x_{i-1}, x_{i+1}]$ is the straight line passing through the points P_j, $j = i-1, i, i+1, i+2$.*

Lemma 1.3. *If $\delta_i f = 0$ and $\delta_{i-1}f\delta_{i+1}f < 0$, then for $S(x) \in I(V)$ it is necessary that one of the following conditions be met:*

(1) $S'(x_i)\delta_{i-1}f > \Delta_i f\delta_{i-1}f$, $S''(x_i) = 0$;
(2) $S'(x) = \Delta_i f$, $S''(x) = 0$ *for all* $x \in [x_{i-1}, x_{i+1}]$.

Proof: By Assumption the sequence $\delta_j f$, $j = i-1, i, i+1$ contains one sign changes that together with the condition $\delta_i f\delta_j f = 0$, $j = i-1, i+1$ and property (4) from the Definition 1.1 leads to a unique possible inflection points x_i, i.e. $S''(x_i) = 0$.

Let $S'(x_i)\delta_{i-1}f < \Delta_i f\delta_{i-1}f$ and $\delta_{i-1}f > 0$ for definiteness. Then by the condition (4) on $I(V)$ we have $S''(x) \geq 0$ for $x \in [x_{i-1}, x_i]$. It contradicts the inequality $S'(x_i) < \Delta_i f$. If we now consider the case $S'(x_i) = \Delta_i f$, it will be obvious that in the interval $[x_{i-1}, x_{i+1}]$ the function $S(x)$ is linear. The Lemma 1.3 is proved.

Lemma 1.4. *Let* $\delta_i f \neq 0$ *and* $S''(x_i)S''(x) \geq 0$ *for all* $x \in [z_1, z_2]$, $z_1, z_2 \in [x_i, x_{i+1}]$. *Then for* $S(x) \in I(V)$ *it is necessary that one of the following conditions be met:*
(1) $S'(z_1) < \Delta_z S < S'(z_2)$ *for* $\delta_i f > 0$,
(2) $S'(z_1) > \Delta_z S > S'(z_2)$ *for* $\delta_i f < 0$,
(3) $S'(x) = \Delta_z S$, $S''(x) = 0$ *for all* $x \in [z_1, z_2]$,
where $\Delta_z S = (S(z_2) - S(z_1))/(z_2 - z_1)$.

Proof: By the property (4) of the $I(V)$ class we have $S''(x_i)\delta_i f \geq 0$. Therefore, $S''(x)\delta_i f \geq 0$ for all $x \in [z_1, z_2]$, $z_1, z_2 \in [x_i, x_{i+1}]$. Let $\delta_i f > 0$. Then in the interval $[z_1, z_2]$ the function $S(x)$ is convex down, i.e. $S''(x) \geq 0$. If in addition $S''(x) \not\equiv 0$, then we obtain the condition (1) of the Lemma 1.4. If $S''(x) \equiv 0$ we arrive at the equalities of the Lemma 1.4. Analogously, from the condition $\delta_i f < 0$ we arrive at either the condition (2) or (3) of the Lemma 1.4.

From Lemma 1.4 follows

Corollary 1.2. *If* $\delta_i f\delta_{i+1}f > 0$ *and* $S'(x_j) \neq \Delta_i f$, $j = i, i+1$, *then for* $S(x) \in I(V)$ *it is necessary that the condition*

$$S'(x_i)\delta_i f < \Delta_i f\delta_i f < S'(x_{i+1})\delta_i f$$

holds.

Corollary 1.3. *If* $\delta_{i-1}f\delta_i f > 0$ *and* $\delta_i f\delta_{i+1}f > 0$, *then for* $S(x)$ *to be isogeometric it is necessary that the inequalities hold:*

$$\min(\Delta_{i-1}f, \Delta_i f) \leq S'(x_i) \leq \max(\Delta_{i-1}f, \Delta_i f).$$

Lemma 1.5. *If $S'(x_i) = 0$, then for $S(x)$ to be isogeometric it is necessary that $S''(x_i)\Delta_i f \geq 0$, $S''(x_i)\Delta_{i-1} f \leq 0$.*

Proof: Assume the contrary, i. e. $S'(x_i) = 0$ but $S''(x_i)\Delta_i f < 0$. Then by the continuity of $S'(x)$ we find $\varepsilon > 0$ such that $S'(x)\Delta_i f < 0$ for $x \in (x_i, x_{i+1} + \varepsilon)$. However, it contradicts the condition (3) from the Definition 1.1. Therefore, $S''(x_i)\Delta_i f \geq 0$. Analogously, the condition $S''(x_i)\Delta_{i-1} f \leq 0$ is deduced.

Theorem 1.1. *For the existence of a function with the isogeometry it is necessary and sufficient that none of the following conditions hold:*
(1) $\Delta_{i-1} f \Delta_i f \leq 0$, $\Delta_{i-1} f \neq 0$, $\delta_{i-2} f \delta_i f \geq 0$, $\delta_{i-1} f = 0$, $i = 3, \ldots, N-1$,
(2) $\Delta_{i-1} f \Delta_i f \leq 0$, $\Delta_i f \neq 0$, $\delta_i f \delta_{i+2} f \geq 0$, $\delta_{i+1} f = 0$, $i = 1, \ldots, N-3$,
(3) $\delta_i f \neq 0$, $\delta_{i-1} f = \delta_{i+1} f = 0$, $\delta_i f \delta_k f \geq 0$, $k = i-2, i+2$; $i = 3, \ldots, N-3$.

Proof: Necessity. Let $S(x) \in I(V)$ and the conditions (1)–(3) of the Theorem be satisfied. From the first condition we have according to Lemma 1.1 that $S'(x_i) = 0$. By Lemma 1.1 (Corollary 1.1) the function $S(x)$ should be linear in the interval $[x_{i-2}, x_i]$ (analogously, in the case of the second condition $S(x)$ should be linear in $[x_i, x_{i+2}]$). Since by the first condition $\Delta_{i-1} f \neq 0$ (by the second one $\Delta_i f \neq 0$), then at the point x_i the derivative $S'(x)$ is discontinuous and hence the function $S(x) \notin I(V)$. The satisfaction of the third condition implies the matching at the point x_i of two straight lines with different slopes and hence we also obtain the contradiction to the assumption.

The proof of the sufficiency consists in constructing the function with the isogeometry $S(x)$ interpolating arbitrary initial data, for which the conditions (1)–(3) of the Theorem 1.1 are not satisfied. In the case, when the initial points are in the straight line, i.e. $\Delta_0 f = \Delta_1 f = \cdots = \Delta_{N-1} f$ it is obvious that the unique function with the isogeometry is this straight line.

Assume that not all the points P_i, $i = 0, 1, \ldots, N$, are on the straight line and consider various intervals $[x_{m_1}, x_{m_2}]$, $0 \leq m_1 \leq m_2 \leq N$, where $x_{m_1}, x_{m_2} \in \Delta$, for which one of the following relations be met:
(1) $m_2 = m_1 + 1$, $\Delta_{m_1} f = 0$;
(2) $m_2 = m_1 + 2$, $m_1 \geq 1$, $m_2 \leq N-1$, $\delta_{m_1+1} f = 0$, $\delta_{m_1} f \delta_{m_2} f \geq 0$;
(3) $m_2 = m_1 + 3$, $m_1 \leq N-3$, $m_2 \geq 3$, $\delta_{m_1+1} f = \delta_{m_1+2} f = 0$.

It is not difficult to convince that the function with the isogeometry $S(x)$ in such intervals is linear, i.e. $S(x) = f_{m_1} + \Delta_{m_1} f (x - x_{m_1})$, $x \in [x_{m_1}, x_{m_2}]$. In particular, $S'_k = \Delta_{m_1} f$, $S''_k = 0$, $k = m_1, m_2$ and by the conditions of the

Theorem 1.1 there take place the inequalities:

$$S'_{m_1}\delta_{m_1-1}f \geq 0 \quad \text{if} \quad m_1 \geq 1,$$
$$S'_{m_2}\delta_{m_2}f \geq 0 \quad \text{if} \quad m_2 \leq N-1.$$

(here and in what follows the notations $S_j^{(r)} = S^{(r)}(x_j)$, $r = 1, 2$ be used).

For any two intervals $[x_{m_1}, x_{m_2}]$ and $[x_{m_3}, x_{m_4}]$, where the function $S(x)$ is linear, one of two cases be possible:

$$[x_{m_1}, x_{m_2}] \cap [x_{m_3}, x_{m_4}] = \emptyset \quad \text{or} \quad [x_{m_1}, x_{m_2}] \cap [x_{m_3}, x_{m_4}] \neq \emptyset.$$

In the last case by the conditions of the Theorem 1.1 $\Delta_{m_1}f = \Delta_{m_3}f$, i.e. $S(x)$ is linear in $[x_{m_1}, x_{m_4}]$.

Therefore, we have isolated the $S(x)$ linearity intervals. In those knots x_i of the mesh Δ, where $\Delta_{i-1}f\Delta_i f < 0$, we put $S'_i = 0$. It is not contradict to the above as by the conditions of the Theorem 1.1 in intervals of linearity such knots do not arise.

We define now the admissible values $S_i^{(r)}$, $r = 1, 2$, in those knots of the mesh Δ, where they were not given earlier. The choice of these values to the following constraints be subjected:

$$\min(\Delta_{i-1}f, \Delta_i f) < S'_i < \max(\Delta_{i-1}f, \Delta_i f) \quad \text{and} \quad \delta_i f S''_i \geq 0 \qquad \text{if} \quad \delta_i f \neq 0, \quad 1 \leq i \leq N-1; \tag{1.1}$$

$$(S'_i - \Delta_i f)\delta_{i-1}f > 0,\ S'_i\Delta_i f \geq 0,\ S''_i = 0 \qquad \text{if} \quad \delta_i f = 0,\ \delta_{i-1}f\delta_{i+1}f < 0, \quad 2 \leq i \leq N-2; \tag{1.2}$$

$$\begin{gathered}(S'_1 - \Delta_1 f)\delta_2 f < 0,\ S'_1\Delta_1 f \geq 0,\ S''_1 = 0 \quad \text{if} \quad \delta_1 f = 0,\\ (S'_{N-1} - \Delta_{N-1}f)\delta_{N-2}f > 0,\ S'_{N-1}\Delta_{N-1}f \geq 0,\ S''_{N-1} = 0 \\ \text{if} \ \delta_{N-1}f = 0;\end{gathered} \tag{1.3}$$

$$\begin{gathered}(\Delta_0 f - S'_0)\delta_1 f > 0,\ S'_0\Delta_0 f \geq 0\,(\Delta_0 f \neq 0),\ S''_0\delta_1 f \geq 0 \ \text{if} \ \delta_1 f \neq 0,\\ (\Delta_0 f - S'_0)\delta_2 f < 0,\ S'_0\Delta_0 f \geq 0\,(\Delta_0 f \neq 0),\ S''_0\delta_2 f \leq 0 \\ \text{if} \quad \delta_1 f = 0,\ \delta_2 f \neq 0;\end{gathered} \tag{1.4}$$

$$(S'_N - \Delta_{N-1}f)\delta_{N-1}f > 0,\ S'_N\Delta_{N-1}f \ge 0\,(\Delta_{N-1}f \ne 0),\ S''_N\delta_{N-1}f \ge 0 \quad \text{if} \quad \delta_{N-1}f \ne 0,$$
$$(S'_N - \Delta_{N-1}f)\delta_{N-2}f < 0,\ S'_N\Delta_{N-1}f \ge 0\,(\Delta_{N-1}f \ne 0),\ S''_N\delta_{N-2}f \le 0 \quad \text{if} \quad \delta_{N-1}f = 0, \delta_{N-2}f \ne 0. \tag{1.5}$$

It is not difficult to convince that the realization of these constraints is consistent with necessary conditions of Lemmas 1.3, 1.4 and Corollaries 1.2, 1.3.

As a result for the termination of the isogeometric function $S(x)$ constructing it is sufficient to eliminate from the consideration the intervals of the $S(x)$ linearity and to define $S(x)$ in arbitrary subinterval $[x_i, x_{i+1}]$ for the following possible configurations of the data:
(A) $\delta_i f\delta_{i+1}f > 0,\ \ 0 \le i \le N-1$;
(B) $\delta_i f = 0,\ \ \delta_{i-1}f\delta_{i+1}f < 0,\ 1 \le i \le N-1$;
(C) $\delta_i f\delta_{i+1}f < 0,\ \ 1 \le i \le N-2$
(if $i = 0, N$, then we formally set $\delta_i f = S''_i$).

Case (A). Let $\delta_i f > 0$. Then $\delta_{i+1}f > 0$ and thus $\Delta_{i-1}f < \Delta_i f < \Delta_{i+1}f$. According to (1.1) we have $\Delta_{i-1}f < S'_i < \Delta_i f < S'_{i+1} < \Delta_{i+1}f$, i.e.

$$\min(S'_i, S'_{i+1}) < \Delta_i f < \max(S'_i, S'_{i+1}). \tag{1.6a}$$

The condition $S'_i = 0$ if $\Delta_{i-1}f\Delta_i f < 0$ and the constraints (1.1) lead to the inequality $\Delta_i f S'_i \ge 0$. Analogously we obtain the restriction $\Delta_i f S'_{i+1} \ge 0$, i.e.

$$\Delta_i f S'_j \ge 0, \quad j = i, i+1. \tag{1.6b}$$

Since $(S'_{i+1} - S'_i)(\Delta_i f - \Delta_{i-1}f) > 0$, i.e. $(S'_{i+1} - S'_i)\delta_i f > 0$ and by virtue of (1.1) $\delta_j f S''_j \ge 0$, $j = i, i+1$, then

$$S''_j/(S'_{i+1} - S'_i) \ge 0, \quad j = i, i+1. \tag{1.7}$$

The conditions (1.6a), (1.6b) and (1.7) validity for $\delta_i f < 0$ is established by the same way.

In addition, according to the Definition 1.1 the relations should be satisfied:

$$S''(x)S''(x_j) \ge 0, \quad j = i, i+1; \quad x \in [x_i, x_{i+1}]. \tag{1.8}$$

Hence, in this case the problem of the isogeometric function constructing is reduced to the solution in $[x_i, x_{i+1}]$ of the Hermite interpolation problem by the given values $S_j^{(r)}$, $r = 0, 1, 2$; $j = i, i+1$ with the additional restrictions (1.6)–(1.8) and the function monotonicity and convexity requirement in this interval.

In the case (B) it is easy to veryfy that in each of two considered here intervals $[x_{i-1}, x_i]$, $[x_i, x_{i+1}]$ by virtue of the constraints (1.1)–(1.5) the conditions (1.6) and (1.7) are satisfied and we arrive at the same Hermite interpolation problem with the constraints (1.6)–(1.8).

By introducing on the straight line, joining the points P_i, P_{i+1}, the additional inflection points extending the mesh Δ the case (C) is reduced to the case (B).

We can point out sufficiently many methods for the solution of the formulated above Hermite interpolation problem with the constraints (1.6)–(1.8). One of the effective methods is presented in the next section that enables us to finish the proof of the Theorem 1.1.

§2. THE SOLUTION OF THE HERMITE INTERPOLATION PROBLEM WITH CONSTRAINTS

The question of the isogeometric function $S_f(x)$ local constructing is solved by with the help of generalized cubic splines [20,23].

To solve the Hermite interpolation problem with constraints on the interval $[x_i, x_{i+1}]$ let us define the function

$$S(x) = \begin{cases} S(x, x_i, x_{i1}) & \text{if} \quad x \in [x_i, x_{i1}]; \\ S(x, x_{i1}, x_{i+1}) & \text{if} \quad x \in [x_{i1}, x_{i+1}], \end{cases}$$

which should satisfy to the interpolation and smoothness conditions

$$S^{(r)}(x_j) = f_j^{(r)}, \quad S^{(r)}(x_{i1} - 0) = S^{(r)}(x_{i1} + 0), \quad r = 0, 1, 2; \quad j = i, i+1. \tag{2.1}$$

On the interval $[x_j, x_{j+1}]$ we set

$$\begin{aligned} S(x, x_j, x_{j+1}) =& [S_j - \varphi_j(0) h_j^2 S_j''](1 - t) + [S_{j+1} - \psi_j(1) h_j^2 S_{j+1}'']t \\ &+ \varphi_j(t) h_j^2 S_j'' + \psi_j(t) h_j^2 S_{j+1}'', \end{aligned} \tag{2.2}$$

where $t = (x - x_j)/h_j$ and the functions $\varphi_j(t)$, $\psi_j(t)$ satisfy to the constraints

$$\varphi_j^{(r)}(1) = \psi_j^{(r)}(0) = 0, \quad r = 0, 1, 2; \quad \varphi_j''(0) = \psi_j''(1) = 1.$$

We assume that $\varphi_j''(t)$, $\psi_j''(t)$ are continuous monotonic functions by the argument $t \in [0,1]$ values and

$$\varphi_j(t) = \varphi(p_j, t), \quad \psi_j(t) = \varphi(q_j, 1-t), \quad p_j, q_j \geq 0. \tag{2.3}$$

According to the inequalities (1.6) and (1.7) we assume that $S_i' S_{i+1}' \geq 0$ and

$$\min(S_i', S_{i+1}') < \Delta_i f < \max(S_i', S_{i+1}'), \tag{2.4}$$

$$S_j''/(S_{i+1}' - S_i') \geq 0, \quad j = i, i+1. \tag{2.5}$$

The function $S(x)$ with taking into account the parameter x_{i1} defining the position of the spline matching point has 9 parameters. The interpolation and smoothness requirements (2.1) also lead to the system of 9 equations. The remaining free parameters p_j, q_j, $j = i, i1$ are used for the isogeometry conditions satisfaction, the calculating formulae simplicity and minimization of the approximation error.

Let us introduce the notations

$$h_{i1} = x_{i1} - x_i, \quad \mu_{i1} = 1 - \lambda_{i1} = h_{i1}/h_i, \quad \tau_i = \frac{S_{i+1}' - \Delta_i f}{S_{i+1}' - S_i'},$$

$$\alpha_i = \frac{S_{i1} - f_i}{h_{i1}}, \quad \beta_i = \frac{f_{i+1} - S_{i1}}{h_i - h_{i1}}, \quad \sigma_j = \frac{h_i S_j''}{S_{i+1}' - S_i'}, \quad j = i, i+1.$$

According to these notations and by the inequalities (2.4) we have

$$\Delta_i f = \tau_i S_i' + (1 - \tau_i) S_{i+1}', \quad 0 < \tau_i < 1. \tag{2.6}$$

Using the formula (2.2) we obtain

$$\begin{aligned}
\alpha_i &= \frac{1}{\psi_i'(1)} \Big\{ h_{i1} T_i^{-1} S_i'' - [\psi_i(1) - \psi_i'(1)] S_i' + \psi_i(1) S_{i1}' \Big\}, \\
\beta_i &= \frac{1}{\varphi_{i1}'(0)} \Big\{ (h_i - h_{i1}) T_{i1}^{-1} S_{i+1}'' - \varphi_{i1}(0) S_{i1}' + [\varphi_{i1}(0) + \varphi_{i1}'(0)] S_{i+1}' \Big\}, \\
T_j^{-1} &= [\varphi_j(0) + \varphi_j'(0)][\psi_j(1) - \psi_j'(1)] - \varphi_j(0)\psi_j(1), \quad j = i, i1.
\end{aligned} \tag{2.7}$$

By the continuity of the spline second derivative in the knot x_{i1} we have the equation

$$S'_{i1} = \left[\frac{\lambda_{i1}}{\psi'_i(1)} - \frac{\mu_{i1}}{\varphi'_{i1}(0)}\right]^{-1}\left[\frac{\lambda_{i1}}{\psi'_i(1)}S'_i - \frac{\mu_{i1}}{\varphi'_{i1}(0)}S'_{i+1} \right. \\ \left. - \frac{\varphi'_i(0)}{\psi'_i(1)}\lambda_{i1}h_{i1}S''_i + \frac{\psi'_{i1}(1)}{\varphi'_{i1}(0)}\mu_{i1}(h_i - h_{i1})S''_{i+1}\right]. \tag{2.8}$$

Now taking into account the identity $\mu_{i1}\alpha_i + \lambda_{i1}\beta_i = \Delta_i f$ and substituting here the expressions for α_i, β_i, S'_{i1} from (2.7) and (2.8) we arrive at the equation with respect to μ_{i1}

$$\Phi_i(\mu_{i1}) = A_i\mu_{i1}^3 + B_i\mu_{i1}^2 + C_i\mu_{i1} + D_i = 0, \tag{2.9}$$

where

$$A_i = \left[\left(\frac{\psi_i(1)}{\psi'_i(1)} + \frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)}\right)\varphi'_i(0) - \left(\frac{1}{\psi'_i(1)} + \frac{1}{\varphi'_{i1}(0)}\right)T_i^{-1}\right]\frac{\sigma_i}{\psi'_i(1)} \\ - \left[\left(\frac{\psi_i(1)}{\psi'_i(1)} + \frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)}\right)\psi'_{i1}(1) + \left(\frac{1}{\psi'_i(1)} + \frac{1}{\varphi'_{i1}(0)}\right)T_{i1}^{-1}\right]\frac{\sigma_{i+1}}{\varphi'_{i1}(0)},$$

$$B_i = \left[-\left(\frac{\psi_i(1)}{\psi'_i(1)} + 2\frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)}\right)\varphi'_i(0) + \frac{T_i^{-1}}{\psi'_i(1)}\right]\frac{\sigma_i}{\psi'_i(1)} \\ + \left[\left(\frac{\psi_i(1)}{\psi'_i(1)} + 2\frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)}\right)\psi'_{i1}(1) + \left(\frac{3}{\psi'_i(1)} + \frac{2}{\varphi'_{i1}(0)}\right)T_{i1}^{-1}\right]\frac{\sigma_{i+1}}{\varphi'_{i1}(0)} \\ + [\varphi_{i1}(0) + \varphi'_{i1}(0) - \psi_i(1) + \psi'_i(1)][\varphi'_{i1}(0)\psi'_i(1)]^{-1},$$

$$C_i = \frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)}\frac{\varphi'_i(0)}{\psi'_i(1)}\sigma_i - \left[\frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)}\psi'_{i1}(1) + \left(\frac{3}{\psi'_i(1)} + \frac{1}{\varphi'_{i1}(0)}\right)T_{i1}^{-1}\right]\frac{\sigma_{i+1}}{\varphi'_{i1}(0)} \\ - \left(1 + 2\frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)}\right)\frac{1}{\psi'_i(1)} + \left(\frac{1}{\psi'_i(1)} + \frac{1}{\varphi'_{i1}(0)}\right)\tau_i,$$

$$D_i = \frac{1}{\psi'_i(1)}\left[\frac{T_{i1}^{-1}}{\varphi'_{i1}(0)}\sigma_{i+1} + \frac{\varphi_{i1}(0)}{\varphi'_{i1}(0)} + \tau_i\right].$$

In particular

$$\Phi_i(1) = \frac{1}{\varphi'_{i1}(0)}\left[-\frac{T_i^{-1}}{\psi'_i(1)}\sigma_i - \frac{\psi_i(1)}{\psi'_i(1)} + 1 - \tau_i\right].$$

If now to set $p_{i1} = q_i$ then according to (2.3) we have $\varphi_{i1}^{(r)}(0) = (-1)^r \psi_i^{(r)}(1)$, $r = 0, 1$. Therefore in (2.9) the coefficient $A_i = 0$ and to define μ_{i1} we obtain the quadratic equation

$$\Phi_i(\mu_{i1}) = \frac{1}{\psi_i'(1)} \left[\hat{B}_i \mu_{i1}^2 + \hat{C}_i \mu_{i1} + \hat{D}_i \right] = 0, \tag{2.10}$$

where

$$\begin{aligned} \hat{B}_i =& [\psi_i(1)\varphi_i'(0) + T_i^{-1}]\sigma_i + [\psi_i(1)\psi_{i1}'(1) - T_{i1}^{-1}]\sigma_{i+1}, \\ \hat{C}_i =& -\psi_i(1)\varphi_i'(0)\sigma_i + [-\psi_i(1)\psi_{i1}(1) + 2T_{i1}^{-1}]\sigma_{i+1} + 2\psi_i(1) - \psi_i'(1), \\ \hat{D}_i =& -T_{i1}^{-1}\sigma_{i+1} - \psi_i(1) + \tau_i\psi_i'(1). \end{aligned}$$

Since

$$\begin{aligned} \Phi_i(0) =& \tau_i - \frac{1}{\psi_i'(1)} \left[T_{i1}^{-1}\sigma_{i+1} + \psi_i(1) \right], \\ \Phi_i(1) =& -(1 - \tau_i) + \frac{1}{\psi_i'(1)} \left[T_i^{-1}\sigma_i + \psi_i(1) \right], \end{aligned} \tag{2.11}$$

we can find such $\overline{p}_i$, $\overline{q}_i$, $\overline{q}_{i1}$, that for all $p_i \geq \overline{p}_i$, $q_i \geq \overline{q}_i$, $q_{i1} \geq \overline{q}_{i1}$ according to (2.6) we have $\Phi_i(0) > 0$, $\Phi_i(1) < 0$. Thus the equation (2.10) has a unique root $\mu_{i1} \in (0, 1)$.

The choice of remaining free parameters p_i, q_i, q_{i1} we realise taking in mind to satisfy the isogeometry conditions and the raising of the approximation order.

As $p_{i1} = q_i$ we can rewrite the equation (2.8) in the form

$$S_{i1}' = S_i' + \mu_{i1}(S_{i+1}' - S_i') - \lambda_{i1}\mu_{i1}h_i[\varphi_i'(0)S_i'' + \psi_{i1}'(1)S_{i+1}'']. \tag{2.12}$$

Considering $f(x)$ as a sufficiently smooth function we assume that $S_j^{(r)} - f_j^{(r)} = O(h_i^{k+1-r})$, $r = 1, 2$; $j = i, i+1$; $k = 2$ or $k = 3$. Then

$$S_{i+1}' - S_i' = f_{i+1}' - f_i' + O(h_i^k) = h_i f''(x_i) + \frac{h_i^2}{2} f'''(x_i) + O(h_i^k).$$

Using (2.12) we obtain

$$\begin{aligned} S_{i1}' - f'(x_{i1}) =& S_i' - f_i' - [\varphi_i'(0) + \psi_{i1}'(1)]\lambda_{i1}\mu_{i1}h_i f_i'' \\ & + h_{i1}(h_i - h_{i1})[1/2 - \psi_{i1}'(1)]f_i''' + O(h_i^k). \end{aligned}$$

It implies that the approximation error order in the point x_{i1} is raising for the derivative of the spline if according to (2.3) we set $q_{i1} = p_i$.

With $p_{i1} = q_i$, $q_{i1} = p_i$ and the equalities (2.7) and (2.12) we obtain

$$S(x_{i1}) - f(x_{i1}) = f_i + \alpha_i h_{i1} - f(x_{i1}) = \left(-\frac{1}{2} + \frac{T_i^{-1} + \psi_i(1)}{\psi_i'(1)} \right) h_{i1}^2 f_i'' + \left[-\frac{h_{i1}}{6} + \frac{\psi_i(1)}{\psi_i'(1)} \left(\frac{h_i}{2} + (h_i - h_{i1})\varphi_i'(0) \right) \right] h_{i1}^2 f_i''' + O(h_i^{k+1}). \tag{2.13}$$

The general requirement for the choice of the parameters p_i, q_i may consist in the minimization of the coefficient at f_i'' in (2.13). For commonly used choices of the functions $\varphi_i(t)$, $\psi_i(t)$ according to (2.13) the maximal approximation order $O(h_i^4)$ in the point x_{i1} is reached by $p_i, q_i = O(h_i^2)$. If $p_i, q_i = O(1)$, then the approximation order is decreased to $O(h_i^2)$.

We consider now the question of the isogeometry for the generalized spline $S(x)$ in the interval $[x_i, x_{i+1}]$. The following criterion is valid.

Theorem 2.1. *By the fulfillment the restrictions*

$$\frac{\psi_i(1)}{\psi_i'(1)} - \varphi_i'(0)\sigma_i < 1 - \tau_i, \quad \frac{\psi_i(1)}{\psi_i'(1)} - \varphi_i'(0)\sigma_{i+1} < \tau_i, \tag{2.14}$$

the unique generalized cubic spline with the isogeometry $S(x)$ exists solving the Hermite interpolation problem with restrictions (1.6)–(1.8).

Proof: The conditions (1.6) and (1.7) are fulfilled by the construction. The requirement (1.8) means the absence on $[x_i, x_{i+1}]$ inflection points for the function with isogeometry $S_f(x)$. Let us show that for the spline $S(x)$ this condition will be fulfilled if the inequalities are valid

$$\begin{aligned} \min(\alpha_i, \beta_i) < S_{i1}' < \max(\alpha_i, \beta_i), \\ \min(S_i', \Delta_i f) < \alpha_i < \max(S_i', \Delta_i f), \\ \min(S_{i+1}', \Delta_i f) < \beta_i < \max(S_{i+1}', \Delta_i f). \end{aligned}$$

It is convenient to rewrite these inequalities in the form

$$\begin{aligned} \alpha_i(S_{i+1}' - S_i')^{-1} < S_{i1}'(S_{i+1}' - S_i')^{-1} < \beta_i(S_{i+1}' - S_i')^{-1}, \\ S_i'(S_{i+1}' - S_i')^{-1} < \alpha_i(S_{i+1}' - S_i')^{-1} < \Delta_i f(S_{i+1}' - S_i')^{-1}, \\ \Delta_i f(S_{i+1}' - S_i')^{-1} < \beta_i(S_{i+1}' - S_i')^{-1} < S_{i+1}'(S_{i+1}' - S_i')^{-1}. \end{aligned} \tag{2.15}$$

From (2.7) and (2.12) we find

$$\alpha_i = S_i' + \mu_{i1}(S_{i+1}' - S_i')\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right], \quad (2.16)$$

$$\beta_i = S_{i+1}' - \lambda_{i1}(S_{i+1}' - S_i')\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1} - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right].$$

It enables us to write the conditions (2.15) in the form

$$\begin{gathered}
\frac{\psi_i(1)}{\psi_i'(1)}\big[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\big] < 1 + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i), \\
\frac{\psi_i(1)}{\psi_i'(1)}\big[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1} - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\big] < 1 - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i), \\
0 < \mu_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\big[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\big] < 1 - \tau_i, \\
0 < \lambda_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\big[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_{i+1} - \mu_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\big] < \tau_i.
\end{gathered} \quad (2.17)$$

To fulfill these inequalities and the conditions $\Phi_i(0) > 0$, $\Phi_i(1) < 0$ it is sufficiently to choose the parameters p_i, q_i in such a way that the restrictions (2.14) are satisfied.

Indeed if $p_{i1} = q_i$, $q_{i1} = p_i$ according to (2.7) we have $T_{i1} = T_i$ and the inequality $T_i^{-1}/\psi_i'(1) < -\varphi_i'(0)$ is equivalent to the obvious relation $\varphi_i'(0)\psi_i(1) - \varphi_i(0)\psi_i'(1) < 0$. By the formulae (2.11) the conditions $\Phi_i(0) > 0$ and $\Phi_i(1) < 0$ are equivalent to the inequalities

$$\begin{gathered}
\frac{1}{\psi_i'(1)}\big[T_i^{-1}\sigma_{i+1} + \psi_i(1)\big] < -\varphi_i'(0)\sigma_{i+1} + \psi_i(1)/\psi_i'(1) < \tau_i, \\
\frac{1}{\psi_i'(1)}\big[T_i^{-1}\sigma_i + \psi_i(1)\big] < -\varphi_i'(0)\sigma_i + \psi_i(1)/\psi_i'(1) < 1 - \tau_i,
\end{gathered}$$

which are satisfied by the fulfillment of the restrictions (2.14).

Let us prove the first inequality in (2.17). Let $\sigma_{i+1} - \sigma_i \le 0$. Then according to (2.14) we have

$$\begin{aligned}
\frac{\psi_i(1)}{\psi_i'(1)}\left[1 + \frac{T_i^{-1}}{\psi_i(1)}\sigma_i + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i)\right] &< \frac{\psi_i(1)}{\psi_i'(1)} - \varphi_i'(0)\sigma_i \\
&+ \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i) < 1 + \lambda_{i1}\varphi_i'(0)(\sigma_{i+1} - \sigma_i).
\end{aligned}$$

If $\sigma_{i+1}-\sigma_i>0$, i.e. $\sigma_{i+1}>\sigma_i\geq 0$ we obtain

$$\begin{aligned}\frac{\psi_i(1)}{\psi_i'(1)}&+\left(\frac{T_i^{-1}}{\psi_i'(1)}+\lambda_{i1}\varphi_i'(0)\left[1-\frac{\psi_i(1)}{\psi_i'(1)}\right]\right)\sigma_i-\lambda_{i1}\varphi_i'(0)\left[1-\frac{\psi_i(1)}{\psi_i'(1)}\right]\sigma_{i+1}\\&<\frac{\psi_i(1)}{\psi_i'(1)}-\varphi_i'(0)\sigma_i\left[\mu_{i1}+\lambda_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\right]-\lambda_{i1}\varphi_i'(0)\left[1-\frac{\psi_i(1)}{\psi_i'(1)}\right]\sigma_{i+1}\\&<\frac{\psi_i(1)}{\psi_i'(1)}-\varphi_i'(0)\sigma_{i+1}<1.\end{aligned}$$

Thus the first inequality in (2.17) is again fulfilled. Analogously the second inequality in (2.17) is proved.

Let us prove the third inequality in (2.17). It is necessary to point that as for the generalized cubic spline the estimate

$$\max\left(\frac{\psi_i(1)}{\psi_i'(1)},-\frac{\varphi_i(0)}{\varphi_i'(0)}\right)\leq 1/3,$$

is valid we have

$$\begin{aligned}T_i^{-1}&=\varphi_i'(0)[\psi_i(1)-\psi_i'(1)]-\varphi_i(0)\psi_i'(1)\\&=\varphi_i'(0)[2\psi_i(1)-\psi_i'(1)]-\varphi_i'(0)\psi_i(1)-\varphi_i(0)\psi_i'(1)\\&>\varphi_i'(0)[2\psi_i(1)-\psi_i'(1)]+2\varphi_i(0)\psi_i(1)-\varphi_i(0)\psi_i'(1)\\&=[2\psi_i(1)-\psi_i'(1)][\varphi_i(0)+\varphi_i'(0)]>0.\end{aligned}$$

Let $\sigma_{i+1}-\sigma_i\leq 0$. Then it is obvious that

$$1+\frac{T_i^{-1}}{\psi_i(1)}\sigma_i+\lambda_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i)>0.$$

By summation the inequalities (2.14) we have

$$1+\varphi_i'(0)(\sigma_i+\sigma_{i+1})>2\frac{\psi_i(1)}{\psi_i'(1)}.$$

Therefore if $\sigma_{i+1}-\sigma_i>0$ we obtain

$$1+\frac{T_i^{-1}}{\psi_i(1)}\sigma_i+\lambda_{i1}\varphi_i'(0)(\sigma_{i+1}-\sigma_i)>1+\varphi_i'(0)(\sigma_i+\sigma_{i+1})>2\frac{\psi_i(1)}{\psi_i'(1)}>0.$$

Finally

$$\mu_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\left[1+\left(\frac{T_i^{-1}}{\psi_i(1)}-\lambda_{i1}\varphi_i'(0)\right)\sigma_i+\lambda_{i1}\varphi_i'(0)\sigma_{i+1}\right]$$
$$<\frac{\psi_i(1)}{\psi_i'(1)}-\varphi_i'(0)\sigma_i\mu_{i1}\left[1+\lambda_{i1}\frac{\psi_i(1)}{\psi_i'(1)}\right]$$
$$<\frac{\psi_i(1)}{\psi_i'(1)}-\varphi_i'(0)\sigma_i<1-\tau_i.$$

By the same way we can prove the validity of the last two inequalities in (2.17).

According to (2.2)

$$S_i'=\alpha_i+[\varphi_i(0)+\varphi_i'(0)]h_{i1}S_i''-\psi_i(1)h_{i1}S_{i1}''.$$

Then by substituting here the expression for α_i from (2.16) we have

$$S_{i1}''=\frac{S_{i+1}'-S_i'}{h_i\psi_i'(1)}\big[1+\varphi_i'(0)(\mu_{i1}\sigma_i+\lambda_{i1}\sigma_{i+1})\big]. \tag{2.18}$$

We multiply the inequality (2.14) by μ_{i1} and λ_{i1} respectively. By summation the resulting equations we obtain

$$0<\frac{\psi_i(1)}{\psi_i'(1)}+\mu_{i1}\tau_i+\lambda_{i1}(1-\tau_i)<1+\varphi_i'(0)(\mu_{i1}\sigma_i+\lambda_{i1}\sigma_{i+1}).$$

So, if the inequalities (2.14) are fulfilled, the expression in square parentheses in (2.18) is positive and $S_{i1}''(S_{i+1}'-S_i')\geq 0$. As $S_j''(S_{i+1}'-S_i')\geq 0$, $j=i,i+1$, we conclude from here that $S_{i1}''S_j''\geq 0,\quad j=i,i+1$.

From (2.2) on the interval $[x_i,x_{i1}]$ we have

$$S''(x)=S_i''\varphi_i''(t)+S_{i1}''\psi_i''(t).$$

Since $\varphi_i''(t)\geq 0$, $\psi_i''(t)\geq 0$ for $t\in[0,1]$, then

$$S''(x)S_j''\geq 0,\quad j=i,i1\quad \text{for}\quad x\in[x_i,x_{i1}].$$

The analogous conclusion we arrive at considering the subinterval $[x_{i1},x_{i+1}]$. As a result the function $S''(x)$ is convex in the interval $[x_i,x_{i+1}]$ and $S'(x)$ is monotone. Because of the assumption $S_i'S_{i+1}'\geq 0$ the function $S(x)$ has the monotonicity property. The theorem is proved.

The given constructing completes the proof of the Theorem 1.1 sufficiency conditions from the previous section.

§3. NUMERICAL ALGORITHM FOR THE CONSTRUCTION OF THE FUNCTION WITH THE ISOGEOMETRY

Within the proof of the main Theorem 1.1 it was described the procedure for the construction of the generalized cubic splines set which satisfies to the isogeometry conditions. Now it is necessary to define more exactly how one should choose the spline derivatives values in the knots of the mesh Δ and the additional inflection points reaching the uniqueness in such constructing. Naturally, we aim at obtaining the generalized spline $S(x)$ lying as close as possible to the function with the isogeometry $S_f(x)$.

Let us consider the Lagrange polynomial of the n-th degree with respect to the points (x_j, f_j), $j = i, \ldots, i+n$:

$$\begin{aligned} L_{n,i}(x) =& f_i + f[x_i, x_{i+1}](x - x_i) + \\ & \cdots + f[x_i, \ldots, x_{i+n}](x - x_i)\cdots(x - x_{i+n-1}), \quad n \geq 1, \end{aligned} \tag{3.1}$$

where the usual notation for the divided differences [23] is used
$f[x_i, \ldots, x_{i+k}] = (f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}])/(x_{i+k} - x_i)$.

Based on the constraints (1.1)–(1.5), we shall define more exactly the choice of the derivatives values in the interior knots of the mesh Δ. If $\delta_i f \neq 0$, then we set

$$S_i^{(k)} = L_{3,i-2}^{(k)}(x_i) \quad \text{or} \quad S_i^{(k)} = L_{3,i-1}^{(k)}(x_i), \quad k = 1, 2 \tag{3.2}$$

depending on which of these values satisfy the inequalities (1.1). It yields a $O(h^{4-k})$ derivative approximation, where $h = \max_i h_i$ (cf. [23]).

If by means of such $S_i^{(k)}$ choice, it is impossible to satisfy the constraints (1.1), then we set $S_i^{(k)} = L_{2,i-1}^{(k)}(x_i)$, $k = 1, 2$. The last automatically guarantees the inequalities (1.1) satisfaction since

$$L'_{2,i-1}(x_i) = (h_i \Delta_{i-1} f + h_{i-1} \Delta_i f)/(h_{i-1} + h_i), \quad L''_{2,i-1}(x_i) = 2\overline{\delta}_i f,$$

where $\overline{\delta}_i f = \delta_i f/(h_{i-1} + h_i)$.

Although such decrease of the Lagrange interpolating polynomial degree generally diminishes the derivative approximation order on the unit, the S'_i accuracy is retained if $\delta_j f \delta_{j+1} f > 0$, $j = i-1, i$. In fact since

$$\begin{aligned} L'_{3,i-2}(x_i) &= \Delta_{i-1} f + \overline{\delta}_{i-1} f h_{i-1} + \frac{h_{i-1}(h_{i-2} + h_{i-1})}{h_{i-2} + h_{i-1} + h_i}(\overline{\delta}_i f - \overline{\delta}_{i-1} f), \\ L'_{3,i-1}(x_i) &= \Delta_i f - \overline{\delta}_{i+1} f h_i + \frac{(h_i + h_{i+1})h_i}{h_{i-1} + h_i + h_{i+1}}(\overline{\delta}_{i+1} f - \overline{\delta}_i f), \end{aligned} \tag{3.3}$$

then

$$(L'_{3,i-2}(x_i) - \Delta_{i-1}f)\delta_i f > 0 \quad \text{if} \quad \delta_{i-1}f\delta_i f > 0,$$
$$(\Delta_i f - L'_{3,i-1}(x_i))\delta_i f > 0 \quad \text{if} \quad \delta_i f\delta_{i+1} f > 0.$$

Hence, $(L'_{3,i-2}(x_i) - \Delta_{i-1}f)(\Delta_i f - L'_{3,i-1}(x_i)) > 0$ and either $L'_{3,i-2}(x_i) > \Delta_{i-1}f$, $L'_{3,i-1}(x_i) < \Delta_i f$, or $L'_{3,i-2}(x_i) < \Delta_{i-1}f$, $L'_{3,i-1}(x_i) > \Delta_i f$. We assume that setting S'_i by means of (3.2) is inadmissible by virtue of the inequalities (1.1) violation. Therefore, taking into account the above either $L'_{3,i-2}(x_i) > \max(\Delta_{i-1}f, \Delta_i f)$, $L'_{3,i-1}(x_i) < \min(\Delta_{i-1}f, \Delta_i f)$, or conversely $L'_{3,i-2}(x_i) < \min(\Delta_{i-1}f, \Delta_i f)$, $L'_{3,i-1}(x_i) > \max(\Delta_{i-1}f, \Delta_i f)$. In both cases we can find such number $\alpha \in (0,1)$, that $L'_{2,i-1}(x_i) = \alpha L'_{3,i-2}(x_i) + (1-\alpha)L'_{3,i-1}(x_i)$, i.e. the S'_i accuracy setting is $O(h^3)$.

If $\delta_i f = 0$, $\delta_{i-1}f\delta_{i+1}f < 0$, $2 \le i \le N-1$, then we set

$$S''_i = 0, \quad S'_i = \begin{cases} L'_{4,i-2}(x_i) & \text{if } L'_{4,i-2}(x_i)\Delta_i f \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

where according to (3.1) the expression for the derivative value of the fourth degree Lagrange polynomial has the form

$$L'_{4,i-2}(x_i) = \Delta_{i-1}f + \bar{\delta}_{i-1}f \frac{h_{i-1}h_i(h_i + h_{i+1})}{(h_{i-2} + h_{i-1} + h_i)(h_{i-2} + h_{i-1} + h_i + h_{i+1})} - \bar{\delta}_{i+1}f \frac{h_{i-1}h_i(h_{i-2} + h_{i-1})}{h_{i-2} + h_{i-1} + h_i + h_{i+1}}.$$

Since here $\Delta_{i-1}f = \Delta_i f$ we have that $(L'_{4,i-2}(x_i) - \Delta_i f)\delta_{i-1}f > 0$ and the constraints (1.2) are satisfied. Note that for a sufficiently smooth function with the isogeometry $S_f(x)$ the inequality be met $|S'_i - L'_{4,i-2}(x_i)| \le |S'_f(x_i) - L'_{4,i-2}(x_i)| = O(h^4)$.

If $\delta_1 f = 0$ then we set

$$S''_1 = 0, \quad S'_1 = \begin{cases} L'_{3,0}(x_1) & \text{if } L'_{3,i}(x_1)\Delta_1 f \ge 0, \\ 0, & \text{otherwise,} \end{cases}$$

and if $\delta_{N-1}f = 0$

$$S''_{N-1} = 0, \quad S'_{N-1} = \begin{cases} L'_{3,N-3}(x_{N-1}) & \text{if } L'_{3,N-3}(x_{N-1})\Delta_{N-1} f \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Using (3.3) it is not difficult to verify that the constraints (1.3) are satisfied.

The correction of the first and second derivative values in the knots x_0, x_N is given by the formulae:

if $\delta_1 f \neq 0$

$$S_0'' = L_{2,0}''(x_0), \quad S_0' = \begin{cases} L_{2,0}'(x_0) & \text{if } L_{2,0}'(x_0)\Delta_0 f \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

if $\delta_1 f = 0$

$$S_0'' = L_{3,0}''(x_0), \quad S_0' = \begin{cases} L_{3,0}'(x_0) & \text{if } L_{3,0}'(x_0)\Delta_0 f \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

if $\delta_{N-1} f \neq 0$

$$S_N'' = L_{2,N-2}''(x_N), \quad S_N' = \begin{cases} L_{2,N-2}'(x_N) & \text{if } L_{2,N-2}'(x_N)\Delta_{N-1} f \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

if $\delta_{N-1} f = 0$

$$S_N'' = L_{3,N-3}''(x_N), \quad S_N' = \begin{cases} L_{3,N-3}'(x_N) & \text{if } L_{3,N-3}'(x_N)\Delta_{N-1} f \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

The direct testing of the constraints (1.4) and (1.5) is not difficult.

As a result in all knots of the mesh Δ the satisfying isogeometry conditions derivative values $S_i^{(k)}$, $k = 1, 2$ are given.

Let us consider now the question of the additional inflection points choice. As was pointed out at the proof of the Theorem 1.1, if the initial data are such that $\delta_i f \delta_{i+1} f < 0$, then in the interval $[x_i, x_{i+1}]$ in addition to the knots of the mesh Δ it is necessary to introduce the inflection point $\overline{x}$. At this inflection point it is necessary to set the value $S'(\overline{x})$ (by the Definition we put $S''(\overline{x}) = 0$).

Thus, let $\delta_i f \delta_{i+1} f < 0$. In the interval $[x_i, x_{i+1}]$ we consider the Hermite cubic polynomial

$$S_{3,2}(x) = S_i + h_i t^2 (3 - 2t)\Delta_i f + h_i t(1-t)^2 S_i' - h_i t^2 (1-t) S_{i+1}',$$

where $t = (x - x_i)/h_i$. From the requirement $S_{3,2}''(x) = 0$ we find

$$x = \overline{x}_0 = x_i + t^* h_i, \quad t^* = \frac{1}{3}(3\Delta_i f - 2S_i' - S_{i+1}')/(2\Delta_i f - S_i' - S_{i+1}').$$

By virtue of the condition $\delta_i f \delta_{i+1} f < 0$ in this case

$$\Delta_i f < \min(S_i', S_{i+1}') \quad \text{or} \quad \Delta_i f > \max(S_i', S_{i+1}'). \tag{3.4}$$

Then $t^* \in (0,1)$ and hence the inflection point $\overline{x}_0 \in (x_i, x_{i+1})$.

We consider consequently the following situations:

(a) If the condition

$$\min(S_i, S_{i+1}) < S_{3,2}(\overline{x}_0) < \max(S_i, S_{i+1}), \tag{3.5}$$

is satisfied, then we take $\overline{x}_0$ as an inflection point and set

$$S'(\overline{x}_0) = \begin{cases} S_{3,2}'(x_0) & \text{if } S_{3,2}'(\overline{x}_0)\Delta_i f \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) The requirement (3.5) is not satisfied. From the condition $S_{3,2}'(x) = 0$ we find

$$\overline{x}_{1,2} = \tilde{x}_0 \pm \tilde{t} h_i, \quad \tilde{t} = \frac{1}{3}\sqrt{(S_i' + S_{i+1}' - 3\Delta_i f)^2 - S_i' S_{i+1}'}/(S_i' + S_{i+1}' - 2\Delta_i f),$$

and as an inflection point we takes that of the points $\overline{x}_1$, $\overline{x}_2$, for which the inequalities (3.5) are satisfied. In this point we set $S'(\overline{x}_k) = 0$.

(c) If by means of the points $\overline{x}_{1,2}$ we cannot satisfy the constraints (3.5), then as an inflection point we take the intersection point of the function $S_{3,2}(x)$ graph with the straight line interval joining the points (x_i, S_i), (x_{i+1}, S_{i+1}). We find $\overline{x}_3 = x_i + \hat{t} h_i$, $\hat{t} = (\Delta_i f - S_i')/(2\Delta_i f - S_i' - S_{i+1}')$,where by virtue of (3.4) $\hat{t} \in (0,1)$ and hence $\overline{x}_3 \in (x_i, x_{i+1})$. In this case we set $S'(\overline{x}_3) = 0$. It should be noted that

$$\overline{x}_3 - \overline{x}_0 = (\hat{t} - t^*)h_i = \frac{h_i}{3}(S_{i+1}' - S_i')/(2\Delta_i f - S_i' - S_{i+1}'),$$

and therefore the points $\overline{x}_3$, $\overline{x}_0$ coincide only if $S_i' = S_{i+1}'$.

Recall [23] that the Hermite cubic polynomial gives the approximation of the interpolating function with the order $O(h^4)$. By virtue of the constructing this order takes place for all three cases (a) – (c).

It remains to convince that in these intervals $[x_i, x_{i+1}]$, on which ends the derivatives values were not corrected, the corresponding Hermite cubic polynomial $S_{3,2}(x)$ is monotone (the convexity takes place by virtue of the

$S''_{3,2}(x)$ linearity). It is reached by means of checking the Fritsch-Carlson [7] necessary and sufficient monotonicity conditions:

$$0 \le d_i, e_i \le 3 \quad \text{or} \quad d_i - \frac{1}{3}\frac{(2d_i + e_i - 3)^2}{(d_i + e_i - 2)} \ge 0,$$

where $d_i = S'_i/\Delta_i f$, $e_i = S'_{i+1}/\Delta_i f$. If these conditions are not satisfied, then in $[x_i, x_{i+1}]$ it is necessary to consider constructing the generalized cubic spline. The same constructing is realized, if at the ends of the subinterval $[x_i, x_{i+1}]$ the derivative values are changed.

We have described constructing the generalized cubic spline $S(x)$ retaining the isogeometry. Now we summarize the main steps of the algorithm basing on the possibility of its refinement.

Step 1. Construction the spline $\tilde{S}(x)$ interpolating arbitrary initial data on the mesh Δ and belonging to the class C^2. It can be the usual cubic spline or for example a locally approximating spline. In the last case it is required to replace in the knots of the mesh Δ the values f_i by $\tilde{S}(x_i)$.

Step 2. The correction of the spline $\tilde{S}(x)$ for the initial derivative approximations in the knots of mesh Δ according to the isogeometry requirements and the choice of the additional inflection points extending the mesh Δ.

Step 3. The control of the monotonicity and convexity conditions in the intervals, where the derivative values are not changed. If these conditions are not satisfied or we have the case of the end values correction, then in the corresponding subintervals the piecewise generalized cubic spline is constructed.

After the realization of this algorithm we obtain the generalized spline being the isogeometric function.

§4. PROGRAM REALIZATION OF THE ALGORITHM

The algorithm of isogeometric interpolation was realized as a complex of programs in Fortran, which enables to calculate the values of generalized spline and its first and second derivatives. If the conditions of the existence theorem for functions with isogeometry are fulfilled the constructing spline will belong to the class $C^2[a, b]$. In the nodes where the conditions of this theorem are not satisfied the interpolany will have the break of the first derivative. On the whole of the interval of the initial data the spline will be the function with isogeometry.

In the program the calculation of the spline values and the values of its first and second derivatives is realized locally, as by the constructing of the spline on the interval $[x_i, x_{i+1}]$ only the points $P_j = (x_j, f_j)$, $j = i-3, i-2, \ldots, i+4$ are used and the values of the first derivative of the spline in the nodes x_i, x_{i+1}.

The values of the first and second derivatives of the spline with isogeometry in the node x_i $(1 \le i \le N-1)$ are restricted to the constraints stipulated as the initial data set and the algorithm of its construction. The possible types of the restrictions in the node x_i are classified with the help of integer index k_i $(0 \le k_i \le 6)$:

index	constraints
$k_i = 0$	$\min(\Delta_{i-1}f, \Delta_i f) < S_i' < \max(\Delta_{i-1}f, \Delta_i f),\ S_i''\delta_i f \ge 0;$
$k_i = 1$	$S_i' = 0,\ S_i''\delta_i f \ge 0,$ where if $\Delta_{i-1}f = 0$ or $\Delta_i f = 0$, then $S_i'' = 0;$
$k_i = 2$	$S_i' = \Delta_{i-1}f,\ S_i'' = 0;$
$k_i = 3$	$S_i' = \Delta_i f,\ S_i'' = 0;$
$k_i = 4$	$S_i' = \begin{cases} \Delta_{i-1}f & \text{if } x < x_i, \\ 0 & \text{otherwise,} \end{cases} \quad S_i'' = 0;$
$k_i = 5$	$S_i' = \begin{cases} 0 & \text{if } x < x_i, \\ \Delta_i f & \text{otherwise,} \end{cases} \quad S_i'' = 0;$
$k_i = 6$	$S_i' = \begin{cases} \Delta_{i-1}f & \text{if } x < x_i, \\ \Delta_i f & \text{otherwise,} \end{cases} \quad S_i'' = 0;$

In the program the following algorithm is used to calculate the index k_i values in the knot x_i $(1 \le i \le N-1)$:

1. $k_i = 0$ (initialization);
2. if $\Delta_{i-1}f\Delta_i f < 0$ we set $k_i = 1$;
3. for $\delta_i f = 0$, $1 < i < N-1$:
   if $\delta_{i-1}f\delta_{i+1}f \ge 0$ and $\Delta_{i-1}f = 0$, we set $k_i = 2$,
   if $\delta_{i-1}f\delta_{i+1}f \ge 0$ and $\Delta_i f = 0$, we set $k_i = 3$;
4. for $\delta_{i-1}f = 0$, $i \ge 3$:
   if $\delta_{i-2}f\delta_i f \ge 0$ we set
   $$k_i = \begin{cases} 4 & \text{if } k_i = 1 \text{ and } \Delta_{i-1}f \ne 0, \\ 2 & \text{otherwise.} \end{cases}$$
5. for $\delta_{i+1}f = 0$, $i \le N-3$:
   if $\delta_i f\delta_{i+2}f \ge 0$ we set
   $$k_i = \begin{cases} 3 & \text{if } k_i = 1 \text{ or } \Delta_i f = 0, \\ 5 & \text{if } k_i = 1 \text{ and } \Delta_i f \ne 0, \\ 6 & \text{if } k_i > 1. \end{cases}$$

For $k_i \geq 1$ and fixed interval the values of the first derivative in the node x_i is uniquely defined and for $k_i = 0$ if it is necessary the values of the first and second derivatives are corrected by the formulae from Section 4.

As was shown in the calculations, the comparisons of the first and second differences with zero, used in the algorithm of the node index calculation, should be realized with the accuracy $O(h^3)$ and $O(h^2)$ respectively. It is necessary to coordinate these comparisons with the approximation order of the corresponding derivative of the interpolating function by cubic spline.

The minimization of the coefficient at f_i'' in (2.13), to decrease the values of the second derivative of the spline in the additional node, is realized only by the parameter p_i.

An access to the main function program has the form

$$CALL \quad RSPIZG(IN, N, X, Y, B, U, K).$$

The initial parameters:

N The number of interpolation points ($N > 2$).

X The array of N elements containing the strictly increasing sequence of the initial data x-coordinates.

B The array of N elements of first derivative values of cubic spline (calculated already by the program $SPLINE$).

IN The indication of the initial data variation of the problem. If only one or more elements of the arrays X, Y, B or their length N is changed the parameter IN should be increased by 1. Initially it is necessary to set $IN = 1$.

U X-coordinate where the value of the spline is calculated.

K $= 0, 1, 2$ The order of the calculated derivative of the spline. If $K = 0$ the value of the spline by itself is sought for.

Output data:

$RSPIZG$ The value of k-th derivative of the isogeometric spline.

§5. NUMERICAL EXAMPLES

The described algorithm of the isogeometric interpolation was tested on a number of the examples. For constructing the function with isogeometry the rational splines due to Späth [20] were used that corresponds to choose in (2.3)

$$\varphi(p_i, t) = P_i(1-t)^3/(1+p_i t), \quad P_i^{-1} = 2(3 + 3p_i + p_i^2).$$

The data for the first example were taken from [1]. We consider the interpolation of the function $f(x) = 1 - [exp(100x) - 1]/[exp(100) - 1]$, $x \in [0, 1]$, on the uniform mesh: $x_i = i/10$, $i = 0, \ldots, 10$. In Figure 1 (and further) the dotted and solid lines show the graphs of the usual cubic spline $S_3(x)$ of the C^2 class and of the isogeometric rational spline $S(x)$, respectively. In both cases there are used the boundary conditions $S_0' = 0$, $S_{10}' = -100$. The spline $S_3(x)$ gives the unacceptable oscillations. It is possible to decrease their amplitude by the introducing of the nonuniform mesh with the concentration of the knots in the domain of large gradient or by the choice of the appropriate parameterization. At the same time the maximal deviation of the rational spline $S(x)$ from the interpolated function in this example does not exceed 0.078 and also the $S(x)$ behaviour agrees with $f(x)$ by the monotonicity and convexity.

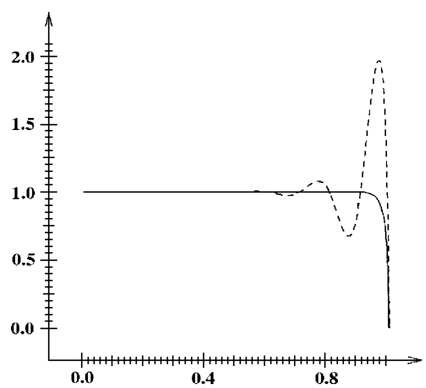

Figure 1. Exponential boundary-layer type data [23].
Profiles of interpolation and isogeometric splines.

In a number of the papers devoted to the isogeometric interpolation (cf. for example [5,6]) the algorithms are tested on the data of [11] given in the Table 1. For this data there are given in Figure 2 the graphs of the splines $S_3(x)$ and $S(x)$. The last has the inflection point $\overline{x}$ in $[x_8, x_9]$ and one additional knot in the intervals $[x_i, x_{i+1}]$, $i = 5, 6$, $[x_8, \overline{x}]$, $[\overline{x}, x_9]$. In comparison with the profile given in [5,6], in addition to retaining of the monotonocity and convexity properties of the initial data the spline $S(x)$ is also nearer to $S_3(x)$.

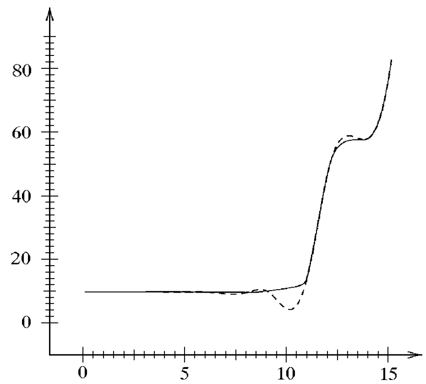

Figure 2. Data obtained by Akima [1]. Typical behaviour of interpolation and isogeometric splines, given fast- and slow-change sections of data.

Table 1. Data for Fig. 2.

i	0	1	2	3	4	5	6	7	8	9	10
x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	56	60	85

In [11] it is considered the function $f(x) = 2 - \sqrt{x(2-x)}$, $0 \le x \le 2$, defining the semicircle. This function is interpolated on the mesh uniform in x (Figure 3) and by the arc length (Figure 4). In both cases it was taken 13 points of the interpolation and were used the boundary conditions $S'_0 = -50$, $S'_{12} = 50$. It is seen that the transition to the mesh with the constant step along the arc length enables one to reduce the oscillations of the spline $S-3(x)$ but it does not remove them. The rational spline $S(x)$ retains again the monotonicity and convexity properties of the initial data.

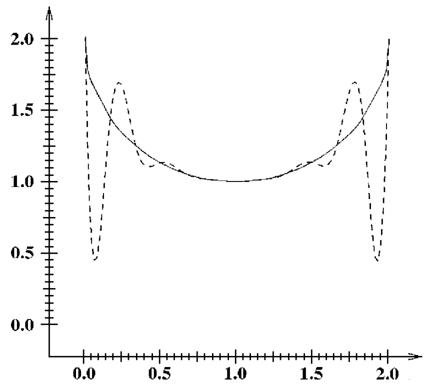

Figure 3. Interpolation of a semicircle by the data uniform in x-coordinate.

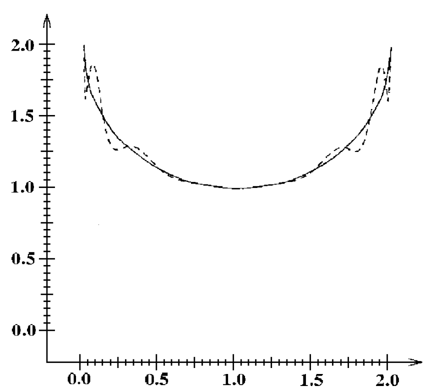

Figure 4. Interpolation of a semicircle by the data uniform by the arc length.

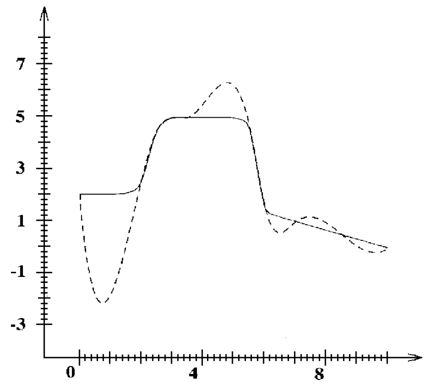

Figure 5. Data obtained by Späth [20]. The isogeometric spline is not sensitive to such outliers and automatically adjust boundary conditions.

The Figure 5 illustrates the data of [20] (Table 2). Here it is used the variant of the algorithm when the conditions $\delta_{N-1} = 0$, $\delta_{N-2} \neq 0$ and $\Delta_{N-3}\Delta_{N-2} > 0$ imply the linearity of the spline in the interval $[x_{N-2}, x_N]$.

Table 2. Data for Fig. 5.

i	0	1	2	3	4	5	6	7	8
x_i	0	2.0	2.5	3.5	5.5	6.0	7	8.5	10
f_i	2	2.5	4.5	5.0	4.5	1.5	1	0.5	0

The obtained profiles of the rational spline retaining the properties of the interpolated data characterize the algorithm proposed in the paper as sufficiently universal.

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