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GROUP CLASSIFICATION OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS IN THE FORM OF A CUBIC POLYNOMIAL IN THE FIRST DERIVATIVE

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Applied Mathematics

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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for a Master's Degree.

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งานวิจัยนี้ต้องศึกษาการจำแนกกลุ่มลีของสมการเชิงอนุพันธ์สามัญอันดับสอง

$$y''+a(x,y)y'^3+3b(x,y)y'^2+3c(x,y)y'+d(x,y)=0$$

โดยใช้เทคนิคของโอฟเซียนนิโคฟในการแก้ปัญหา ซึ่งแนวทางดังกล่าวเกี่ยวข้องกับการทำให้ตัวก่อกำเนิดง่ายขึ้นและการหาฟังก์ชันสมทบ ขั้นตอนการทำวิจัยประกอบด้วย การใช้การแปลงสมมูลทำให้ตัวก่อกำเนิดและฟังก์ชันที่ปรากฏในสมการง่ายขึ้น แล้วทำการจำแนกกลุ่มได้เกือบบริบูรณ์

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา_______________________________

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SOKKHEY PHAUK : GROUP CLASSIFICATION OF SECOND -ORDER ORDINARY DIFFERENTIAL EQUATIONS IN THE FORM OF A CUBIC POLYNOMIAL IN THE FIRST DERIVATIVE. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D. 65 PP.

EQUIVALENCE TRANSFORMATION / ADMITTED LIE GROUP / ADMITTED GENERATOR / GROUP CLASSIFICATION

The purpose of this research is the Lie group classification of a second-order ordinary differential equation:

$$y'' + a(x,y)y'^3 + 3b(x,y)y'^2 + 3c(x,y)y' + d(x,y) = 0.$$

For solving the problem of the thesis, Ovsiannikov's approach was used. This approach involves simplifying one admitted generator and finding associated functions. First, equivalence transformations are used to simplify the generator and the functions presented in the equation. Then, an almost complete group classification is given.

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CHAPTER I

INTRODUCTION

1.1 Background and History

Since the mathematical models of many physical phenomena of the real world are formulated in the form of differential equations, it is clear that the methods of solving differential equations are essential in applications.

The group analysis method is one of the general methods of constructing exact solutions of partial and ordinary differential equations. According to their admitted Lie groups, all equations are separated into equivalence classes. S. Lie himself classified all second-order ordinary differential equations with respect to complex Lie algebras. For this classification, he used a list of all non-similar Lie algebras.

Classification of all non-similar Lie algebras in the real domain has been done in the papers of Gonzalez-Lopez, Kamran, and Olver (1992), and Nesterenko (2006). Lie group classification of second-order ordinary differential equations in the real domain was presented by Mahomed and Leach (1989), where the classification was considered in an indirect way. This is because the direct way involves solving the determining equations, which in the general case of second-order ordinary differential equations cannot be done. There are now some studies of solving the determining equations for particular second-order ordinary differential equations. In particular, Lie (1883) studied group classification of all second order ordinary differential equations of the form $y'' = f(x, y)$. Later, this equation was also studied by Ovsiannikov (2004) using a different approach. The results of this group classification are presented in Table 1.1, where the first column lists the nonequivalent forms of the function f, and $f(y)$ is an arbitrary function. The remaining three columns are the basic operators of the admitted Lie algebras.

Table 1.1 The group classification of $y'' = f(x, y)$.

f	X_1	X_2	X_3
$f(y)$	∂_x	0	0
e^y	∂_x	$x\partial_x - 2\partial_y$	0
$y^k, k \neq -3$	∂_x	$(k-1)x\partial_x - 2y\partial_y$	0
$\pm y^{-3}$	∂_x	$2x\partial_x + y\partial_y$	$x^2\partial_x + xy\partial_y$
$x^{-2}f(y)$	$x\partial_x$	0	0

More general than the equation $y'' = f(x, y)$ is the second-order ordinary differential equation of the form:

$$y'' = P_3(x, y, y'), \tag{1.1}$$

where

$$P_3(x, y, y') = a(x, y)y'^3 + 3b(x, y)y'^2 + 3c(x, y)y' + d(x, y),$$

and $a(x, y)$, $b(x, y)$, $c(x, y)$, and $d(x, y)$ are arbitrary functions.

Equation of the form (1.1) attracted the attention of many scientists starting from Lie. For instance, Lie proved that any second-order ordinary differential equation which is equivalent to a linear second-order ordinary differential equation has to be of the form (1.1).

Another attractive property of Equation (1.1) is that its form is not changed under any point transformation:

$$t = \varphi(x, y),\ u = \psi(x, y), \tag{1.2}$$

where the Jacobian $\Delta = \varphi_x\psi_y - \varphi_y\psi_x$ is assumed to not vanish. Indeed, under the change (1.2), equation (1.1) is transformed to the following equation:

$$u'' + \tilde{a}(t,u)u'^3 + 3\tilde{b}(t,u)u'^2 + 3\tilde{c}(t,u)u' + \tilde{d}(t,u) = 0, \tag{1.3}$$

with the coefficients,

$$\begin{aligned}
a &= \Delta^{-1}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + \varphi_y^3\tilde{d} + 3\varphi_y^2\psi_y\tilde{c} + 3\varphi_y\psi_y^2\tilde{b} + \psi_y^3\tilde{a}),\\
b &= \Delta^{-1}(3^{-1}(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)) + \varphi_x\varphi_y^2\tilde{d}\\
&\quad + \varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x)\tilde{c} + (\varphi_x\psi_y^2 + 2\varphi_x\psi_x\psi_y)\tilde{b} + \psi_x\psi_y^2\tilde{a}),\\
c &= \Delta^{-1}(3^{-1}(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)) + \varphi_x^2\varphi_y\tilde{d}\\
&\quad + (\varphi_x^2\psi_y + 2\varphi_x\varphi_y\psi_x))\tilde{c} + (\varphi_y\psi_x^2 + 2\varphi_x\psi_x\psi_y)\tilde{b} + \psi_y\psi_x^2\tilde{a}),\\
a &= \Delta^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x + \varphi_x^3\tilde{d} + 3\varphi_x^2\psi_x\tilde{c} + 3\varphi_x\psi_x^2\tilde{b} + \psi_x^3\tilde{a}),
\end{aligned} \tag{1.4}$$

where

$$\varphi_x = \frac{\partial\varphi}{\partial x}\ , \quad \varphi_{xx} = \frac{\partial^2\varphi}{\partial x^2}\ , \quad \varphi_{xy} = \frac{\partial^2\varphi}{\partial x\partial y}\ ,$$

and similarly for function ψ.

Since the form of equation is not changed, the problem of studying invariants of transformation of the Equation (1.2) arrives naturally under the change of the Equation (1.2).

For the study presented in this thesis, let us introduce some of the invariants. Lie discovered that the functions:

$$\begin{aligned}
L_1 &= -\frac{\partial\Pi_{11}}{\partial u} + \frac{\partial\Pi_{12}}{\partial t} - b\Pi_{11} - d\Pi_{22} + 2c\Pi_{12},\\
L_2 &= -\frac{\partial\Pi_{12}}{\partial u} + \frac{\partial\Pi_{22}}{\partial t} - a\Pi_{11} - c\Pi_{22} + 2b\Pi_{12},
\end{aligned}$$

play a key role in the linearization problem of the second-order ordinary differential equation $y'' = f(x,y,y')$. Here,

$$\Pi_{11} = 2(c^2 - bd) + c_t - d_u,$$

$$\Pi_{22} = 2(b^2 - ac) + a_t - b_u,$$

$$\Pi_{12} = \Pi_{21} = bc - ad + b_t - c_u.$$

As obtained in Lie (1883), any equation of the form (1.3) is linearizable if and only if $L_1 = 0, L_2 = 0$.

The first investigation of invariants of Equation (1.3) was done by Liouville (1889) and Tresse (1894). Liouville found the invariant:

$$v_5 = L_2(L_1L_{2t} - L_2L_{1t}) + L_1(L_2L_{1u} - L_1L_{2u}) - aL_1^3 + 3bL_1^2L_2 - 3cL_1L_2^2 + dL_2^3.$$

It has the property that if

$$v_5 = 0,$$

then $v_5 = 0$ after any change of the Equation (1.2). Liouville also discovered another semi invariant:

$$w_1 = \frac{1}{L_1^4}[-L_1^3(\Pi_{12}L_1 - \Pi_{11}L_2) - R_1(L_1^2)_t - L_1^2R_{1t} + L_1R_1(cL_1 - dL_2)],$$

where

$$R_1 = L_1L_{2t} - L_2L_{1t} + bL_1^2 - 2cL_1L_2 + dL_2^2,$$

and here it is assumed that $L_1 \neq 0$.

Barbich and Bordag (1998) proved that an equation of the form (1.3) is equivalent to an equation of the form $y'' = f(x, y)$, if and only if $v_5 = 0$, and $w_1 = 0$.

1.2 Statement of the Problem

This thesis is devoted to the group classification of second-order ordinary differential equations of the form:

$$y'' + a(x,y)y'^3 + 3b(x,y)y'^2 + 3c(x,y)y' + d(x,y) = 0. \tag{1.5}$$

Notice that the group classification is invariant with respect to the change of dependent and independent variables. By using transformations of the Equation

(1.2), such that φ and ψ satisfy the equations:

$$a = \Delta^{-1}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y + \varphi_y^3\tilde{d} + 3\varphi_y^2\psi_y\tilde{c} + 3\varphi_y\psi_y^2\tilde{b} + \psi_y^3\tilde{a}) = 0,$$

$$b = \Delta^{-1}(3^{-1}(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)) + \varphi_x\varphi_y^2\tilde{d}$$
$$+ \varphi_y(2\varphi_x\psi_y + \varphi_y\psi_x)\tilde{c} + (\varphi_x\psi_y^2 + 2\varphi_x\psi_x\psi_y)\tilde{b} + \psi_x\psi_y^2\tilde{a}) = 0,$$

$$k = -3c(x, y),$$

$$f = -d(x, y),$$

Equation (1.5) can be transformed to an equivalent equation of the form:

$$y'' = k(x, y)y' + f(x, y). \tag{1.6}$$

Hence, for the group classification of Equation (1.5), one can study the equivalent Equation (1.6). The main goal of this thesis is to do group classification of equation (1.6) with $k_y \neq 0$. One notes that for $k_y = 0$, the semi invariants v_5 and w_1 vanish. According to Barbich and Bordag (1998), this means that Equation (1.5) is equivalent to the case studied by Lie (1883) and Ovsiannikov (2004).

For solving the problem of the thesis, the approach considered in Ovsiannikov (2004) is used, where the criteria of equivalence Lie group of transformations and admitted Lie group are applied. This approach contains the following steps.

(1) Separate Equation (1.5) into classes according to the form of the admitted generator by using the concept of equivalence transformations.

(2) Simplify the functions $k(x, y)$ and $f(x, y)$ by equivalence transformations.

(3) Solve the determining equations for the chosen functions $k(x, y)$ and $f(x, y)$.

Since each step needs a huge amount of analytical calculations, it is necessary to use a computer for these calculations. A brief review of computer systems of symbolic manipulations can be found, for example, in Davenport (1993). In our calculations the system REDUCE (cf. Hearn (1999)) was used.

This thesis is organized as follows. Chapter II introduces some background knowledge of Lie group analysis, which is necessary for our study. Chapter III presents the equivalence Lie group of transformations and the determining equations of the admitted Lie group of Equation (1.6). All possible solutions of the determining equations are given in Chapter IV.

CHAPTER II

GROUP ANALYSIS

Group analysis is a powerful method for analyzing differential equations. One part of the group analysis method involves equivalence transformations. An introduction to this method can be found in textbooks (cf. Ovsiannikov (1978), Olver (1984), Ibragimov (1999), Meleshko (2006)). Many results obtained by this method are collected in the Handbooks of Lie Group Analysis (1994), (1995), (1996)).

2.1 Local Lie Group

In this section, one reviews some background knowledge of Lie group analysis, which is necessary for the study.

One considers invertible point transformations:

$$\bar{z}^i = g^i(z; a), \tag{2.1}$$

where $i = 1, 2, ..., N, z \in V \subset R^N$ and a is a parameter, $a \in \Delta$. The set V is an open set in R^N, and Δ is a symmetric interval in R^1 with respect to zero.

For differential equations, the variable z is separated into two parts, $z = (x, u) \in V \subset R^n \times R^m, N = n + m$. Here, $x = (x_1, x_2, ..., x_n) \in R^n$ is considered as the independent variable, $u = (u^1, u^2, ..., u^m) \in R^m$ is considered as the dependent variable. Then the transformations of the Equation (2.1) can be decomposed as:

$$\bar{x}_i = \varphi^i(x, u; a), \ \bar{u}^j = \psi^j(x, u; a), \tag{2.2}$$

where $i = 1, 2, ..., n, \ j = 1, 2, ..., m, \ (x, u) \in V$.

2.1.1 One-parameter Lie Group of Transformations

Definition 1. A set of transformation of the Equation (2.1) is called a local one-parameter Lie group if it has the following properties:

(1) $g(z;0) = z$ for all $z \in V$.

(2) $g(g(z;a), b) = g(z; a+b)$ for all $a, b, a+b \in \Delta, z \in V$.

(3) If for $a \in \Delta$ we have $g(z;a) = z$ for all $z \in V$, then $a = 0$.

(4) $g \in C^{\infty}(V, \Delta)$.

This definition of Lie group is called local, because we only require that V is an open neighborhood of some z_0, and Δ is a small symmetric interval around zero.

Define

$$\xi^i(x,u) = \left.\frac{\partial \varphi^i(x,u;a)}{\partial a}\right|_{a=0}, \eta^j(x,u) = \left.\frac{\partial \psi^j(x,u;a)}{\partial a}\right|_{a=0},$$

and,

$$X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u^j}. \tag{2.3}$$

The operator X is called an infinitesimal generator or a generator of the Lie group of transformations of the Equation (2.2), and the functions ξ^i, η^j are called the coefficients of the generator.

A local Lie group of transformations (2.2) can be completely determined by the solution of the Cauchy problem of a system of ordinary differential equations, which are called Lie equations:

$$\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}a} = \xi^i(\bar{x},\bar{u}), \quad \frac{\mathrm{d}\bar{u}^j}{\mathrm{d}a} = \eta^j(\bar{x},\bar{u}), \tag{2.4}$$

with the initial data:

$$\bar{x}_i|_{a=0} = x_i, \ \ \bar{u}^j\big|_{a=0} = u^j. \tag{2.5}$$

Theorem 1 (Lie)**.** Let a vector field $\zeta = (\xi, \eta) : V \to R^N$ of class $C^{\infty}(V)$ with $\zeta(z_0) \neq 0$ for some $z_0 \in V$ be given. Then the solution of the Cauchy problem of the Equations (2.4), (2.5) generates a local Lie group with the infinitesimal generator $X = \xi^i(x,u)\partial_{x_i} + \eta^j(x,u)\partial_{u^j}$. Conversely, let functions $\varphi^i(x,u;a),\ i = 1,...,n$ and $\psi^j(x,u;a),\ j = 1,...,m$ satisfy the properties of a Lie group and have the expansion:

$$\bar{x}_i = \varphi^i(x,u;a) \approx x_i + \xi^i(x,u)a,$$

$$\bar{u}^j = \psi^j(x,u;a) \approx u^j + \eta^j(x,u)a,$$

where

$$\xi^i(x,u) = \left.\frac{\partial\varphi^i(x,u;a)}{\partial a}\right|_{a=\ 0}, \eta^j(x,u) = \left.\frac{\partial\psi^j(x,u;a)}{\partial a}\right|_{a=\ 0},$$

then the functions $\varphi^i(x,u;a),\ \psi^j(x,u;a)$ solve the Cauchy problem as described of the Equations (2.4), and (2.5).

Precisely, Lie's theorem establishes a one-to-one correspondence between Lie groups of transformations and infinitesimal generators.

2.1.2 Prolongation of a Lie Group

Given $Z = R^n \times R^m$, the space Z is prolonged by introducing the additional variables $p = (p^k_\alpha)$. Here $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index. For a multi-index the notations $|\alpha| \equiv \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\alpha, i \equiv (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_n)$ are used. The variable p^k_α plays a role of the derivative,

$$p^k_\alpha = \frac{\partial^{|\alpha|}u^k}{\partial x^\alpha} = \frac{\partial^{|\alpha|}u^k}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}.$$

The space J^l of the variables:

$$x = (x_i), u = (u_k), p = (p^k_\alpha),$$

$$(i = 1,2,...,n;\ k = 1,2,...,m;\ |\alpha| \le l)$$

is called the l-th prolongation of the space Z. This space can be provided with a manifold structure. For convenience one agrees that $J^0 \equiv Z$.

Definition 2. The generator

$$X^l = X + \sum_{j,\alpha} \eta^j_\alpha \partial_{p^j_\alpha}, \qquad (j = 1, ..., m,\ |\alpha| \leq l),$$

with the coefficients:

$$\eta^j_{\widetilde{\alpha},k} = D_k \eta^j_{\widetilde{\alpha}} - \sum_i p^j_{\widetilde{\alpha},i} D_k \xi^i, \qquad (|\widetilde{\alpha}| \leq l-1), \tag{2.6}$$

is called the l-th prolongation of the generator X.

Here the operators:

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha} p^j_{\alpha,k} \frac{\partial}{\partial p^j_\alpha}\ , \qquad (k = 1, 2, ..., n),$$

are operators of the total derivatives with respect to x_k, and $\eta^j_0 = \eta^j$, where $\xi^i,\ \eta^j$ are defined as in the Equation (2.3).

For a simple illustration of using the prolongation formula as described in the Equation (2.6), let us study the first prolongation of the generator X with $n = m = 1$. In this case, the generator X^1 induces a local Lie group of transformations in the space J^1:

$$\bar{x} = \varphi(x, u; a),\ \bar{u} = \psi(x, u; a),\ \bar{p} = f(x, u, p; a), \tag{2.7}$$

with the generator:

$$X^1 = \xi^x(x, u)\partial_x + \eta^u(x, u)\partial_u + \zeta^p(x, u, p)\partial_p\ , \tag{2.8}$$

where

$$\zeta^p = D_x(\eta^u) - pD_x(\xi^x),\ p = \frac{du}{dx}\ .$$

Notice that the coefficients $\xi^x,\ \eta^u$ are defined as in the Equation (2.3). Let us show in the following text why the coefficient ζ^p must be of this form. Let a function $u_0(x)$ be given. Substituting it into the Equation (2.7), one obtains:

$$\bar{x} = \varphi(x, u_0(x); a).$$

Since $\varphi(x, u_0(x); 0) = x$, the Jacobian at $a = 0$ is

$$\left.\frac{\partial \bar{x}}{\partial x}\right|_{a=\ 0} = \left.\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}\frac{du_0}{dx}\right)\right|_{a=\ 0} = 1.$$

Thus, by virtue of the inverse function theorem, in some neighborhood of $a = 0$ one can express x as a function of $\bar{x}$ and a,

$$x = \phi(\bar{x}, a). \tag{2.9}$$

Note that after substituting (2.9) into the Equation (2.7), one has the identity:

$$\bar{x} = \varphi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \tag{2.10}$$

Substituting (2.9) into the Equation (2.7), one obtains the transformed function:

$$u_a(\bar{x}) = \psi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \tag{2.11}$$

Differentiating the function $u_a(\bar{x})$ with respect to $\bar{x}$, one gets:

$$\bar{u}_{\bar{x}} = \frac{\partial u_a}{\partial \bar{x}}(\bar{x}) = \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \psi}{\partial u}\frac{du_0}{dx}\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}u_0'\right)\frac{\partial \phi}{\partial \bar{x}},$$

where the derivative $\frac{\partial \phi}{\partial \bar{x}}$ can be found by differentiating Equation (2.10) with respect to $\bar{x}$,

$$1 = \frac{\partial \varphi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \varphi}{\partial u}\frac{du_0}{dx}\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\right)\frac{\partial \phi}{\partial \bar{x}}.$$

Since

$$\frac{\partial \varphi}{\partial x}(\phi(\bar{x}, 0), u_0(\phi(\bar{x}, 0)); 0) = 1,\ \ \frac{\partial \varphi}{\partial u}(\phi(\bar{x}, 0), u_0(\phi(\bar{x}, 0)); 0) = 0, \tag{2.12}$$

one has $\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0' \neq 0$ in some neighborhood of $a = 0$. Thus,

$$\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\right)^{-1},$$

and

$$\bar{u}_{\bar{x}} = \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}u_0'\right)\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\right)^{-1} =: g(x, u_0, u_0'; a). \tag{2.13}$$

Transformation as shown in the Equation (2.7) together with:

$$\bar{u}_{\bar{x}} = g(x, u, u'; a), \quad \bar{p} = \frac{d\bar{u}}{d\bar{x}}.$$

is called the prolongation of the Equation (2.7). Now, one defines the coefficient ζ^p as follows:

$$\zeta^p(x, u, p) = \left.\frac{\partial g(x, u, p; a)}{\partial a}\right|_{a=0}, \quad g|_{a=0} = p. \tag{2.14}$$

Equation (2.13) can be rewritten:

$$g(x, u, p; a)\left(\frac{\partial \varphi(x, u; a)}{\partial x} + p\frac{\partial \varphi(x, u; a)}{\partial u}\right) = \left(\frac{\partial \psi(x, u; a)}{\partial x} + p\frac{\partial \psi(x, u; a)}{\partial u}\right).$$

Differentiating this equation with respect to the group parameter a and substituting $a = 0$, one finds:

$$\left.\left(\frac{\partial g}{\partial a}\left(\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial u}\right) + g\left(\frac{\partial^2 \varphi}{\partial x \partial a} + p\frac{\partial^2 \varphi}{\partial u \partial a}\right)\right)\right|_{a=0} = \left.\left(\frac{\partial^2 \psi}{\partial x \partial a} + p\frac{\partial^2 \psi}{\partial u \partial a}\right)\right|_{a=0}$$

or

$$\begin{aligned}
\zeta^p(x, u, p) &= \left.\frac{\partial g}{\partial a}\right|_{a=0}\left.\left(\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial u}\right)\right|_{a=0}, \text{ since by (2.12) } \left.\left(\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial u}\right)\right|_{a=0} = 1 \\
&= \left.\left(\frac{\partial^2 \psi}{\partial x \partial a} + p\frac{\partial^2 \psi}{\partial u \partial a}\right)\right|_{a=0} - g|_{a=0}\left.\left(\frac{\partial^2 \varphi}{\partial x \partial a} + p\frac{\partial^2 \varphi}{\partial u \partial a}\right)\right|_{a=0} \\
&= \left(\frac{\partial \eta^u}{\partial x} + p\frac{\partial \eta^u}{\partial u}\right) - p\left(\frac{\partial \xi^x}{\partial x} + p\frac{\partial \xi^x}{\partial u}\right) \\
&= D_x(\eta^u) - pD_x(\xi^x)
\end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + p_x\frac{\partial}{\partial p} + \ldots, \quad \xi^x = \left.\frac{\partial \varphi}{\partial a}\right|_{a=0}, \quad \eta^u = \left.\frac{\partial \psi}{\partial a}\right|_{a=0}, \quad \zeta^p = \left.\frac{\partial g}{\partial a}\right|_{a=0}.$$

Thus, the first prolongation of the generator (2.3) is given by:

$$X^{(1)} = X + \zeta^p(x, u, p)\partial_p.$$

Similarly one can obtain prolongation formula for any order prolongation of an infinitesimal generator.

Admitted Lie groups of transformations are related with differential equations by the following.

2.1.3 Lie Groups Admitted by Differential Equations

Consider a manifold M which is defined by a system of partial differential equations:

$$F^k(x, u, p) = 0, \quad (k = 1, 2, ..., s). \tag{2.15}$$

Hence,

$$M = \{(x, u, p) \mid \quad F^k(x, u, p) = 0, \quad (k = 1, ..., s)\}.$$

Here, x is the independent variable, u is the dependent variable, and p are arbitrary partial derivatives of u with respect to x. The manifold M is assumed to be regular, i.e.

$$rank \left(\frac{\partial(F)}{\partial\ (u, p)}\right) = s.$$

Definition 3. A manifold M is said to be invariant with respect to the group of transformations as shown in Equation (2.2), if these transformations carry every point of the manifold M along this manifold, i.e.

$$F^k(\bar{x}, \bar{u}, \bar{p}) = 0, \quad (k = 1, 2, ..., s).$$

Accordingly, Equation (2.15) is not changed under the Lie group of transformations or, in other words, the Lie group of transformations as in the Equation (2.2) is admitted by the Equation (2.15).

In order to find an infinitesimal generator of a Lie group admitted by differential Equation (2.15) one can use the following theorem.

Theorem 2. A system of the Equation (2.15) is not changed with respect to the Lie group of transformations as in Equation (2.2) with the infinitesimal generator:

$$X = \xi^i \partial_{x_i} + \eta^j \partial_{u^j}$$

if and only if,

$$X^{(p)} F^k \mid_{M=0} = 0, \ (k = 1, ..., s). \tag{2.16}$$

Equation (2.16) is called determining equation.

2.2 Equivalence Lie Group

Consider a system of differential equation:

$$F^k(x,u,p,\theta)=0, \quad (k=1,2,...,s). \tag{2.17}$$

Here $\theta=\theta(x,u)$ are arbitrary elements of system as described in the Equation (2.17), $(x,u)\in V\subset R^{n+m}$, and $\theta:V\to R^t$.

A nondegenerate change of dependent and independent variables, which transforms a system of differential Equation (2.17) to a system of equations of the same class or same differential structure is called an equivalence transformation.

The problem of finding a Lie group of equivalence transformations consists of constructing a transformation of the space $R^{n+m+t}(x,u,\theta)$ that preserves the equations, only changing their representative $\theta=\theta(x,u)$. For this purpose a one parameter Lie group of transformations of the space R^{n+m+t} with the group parameter a is used. Assume that the transformations:

$$\begin{aligned}\bar{x}&=f^x(x,u,\theta;a),\\ \bar{u}&=f^u(x,u,\theta;a),\\ \bar{\theta}&=f^\theta(x,u,\theta;a),\end{aligned} \tag{2.18}$$

compose a Lie group of equivalence transformations. So the infinitesimal generator of this group (2.18) has the form:

$$X^e=\xi^{x_i}\partial_{x_i}+\zeta^{u^j}\partial_{u^j}+\zeta^{\theta^k}\partial_{\theta^k},$$

with the coefficients:

$$\begin{aligned}\xi^{x_i}&=\frac{\partial f^{x_i}(x,u,\theta;a)}{\partial a}\Big|_{a=0},\\ \zeta^{u^j}&=\frac{\partial f^{u^j}(x,u,\theta;a)}{\partial a}\Big|_{a=0},\\ \zeta^{\theta^k}&=\frac{\partial f^{\theta^k}(x,u,\theta;a)}{\partial a}\Big|_{a=0},\end{aligned}$$

where $(i = 1, ..., n; j = 1, ..., m; k = 1, ..., t)$.

We use the main requirement for the Lie group of equivalence transformations that any solution $u_0(x)$ of the system as in Equation (2.17) with the functions $\theta(x, u)$ is transformed by the Equation (2.18) into a solution $u = u_a(x')$ of the system as in Equation (2.17) of the same equations F^k, but with other (transformed) functions $\theta_a(x, u)$. The functions $\theta_a(x, u)$ are defined as follows. Solving the relations:

$$\bar{x} = f^x(x, u, \theta(x, u); a), \quad \bar{u} = f^u(x, u, \theta(x, u); a),$$

for (x, u), one obtains:

$$x = g^x(\bar{x}, \bar{u}; a), \quad u = g^u(\bar{x}, \bar{u}; a). \tag{2.19}$$

The transformed function is:

$$\theta_a(\bar{x}, \bar{u}) = f^\theta(x, u, \theta(x, u); a),$$

where, instead of (x, u) one has to substitute their expressions of the Equation (2.19). Because of the definition of the function $\theta(x, u)$, there is the following identity with respect to x and u:

$$(\theta \circ (f^x, f^u))(x, u, \theta(x, u); a) = f^\theta(x, u, \theta(x, u); a).$$

The transformed solution $T_a(u) = u_a(x)$ is obtained by solving the relations:

$$\bar{x} = f^x(x, u_0(x), \theta_a(x, u_0(x)); a),$$

with respect to x, then obtaining $x = \psi^x(\bar{x}; a)$. Substituting $x = \psi^x(\bar{x}; a)$ into the Equation (2.18), one obtains the transformed function:

$$u_a(\bar{x}) = f^u(x, u_0(x), \theta_a(x, u_0(x)); a).$$

Notice that, there is the identity with respect to x:

$$(u_a \circ f^x)((x, u_0(x), \theta_a(x, u_0(x)); a)) = f^u(x, u_0(x), \theta_a(x, u_0(x)); a). \tag{2.20}$$

Formulae for transformations of partial derivatives are obtained by differentiating of the Equation (2.20) with respect to $\bar{x}$.

Lemma 1. The transformations $T_a(u) = u_a(x)$ constructed in this way form a Lie group.

Because the transformed function $u_a(\bar{x})$ is a solution of system in the Equation (2.17) with transformed arbitrary elements $\theta_a(\bar{x}, \bar{u})$, then the equations:

$$F^k(\bar{x}, u_a(\bar{x}), \bar{p}_a(\bar{x}), \theta_a(\bar{x}, u_a(\bar{x}))) = 0, (k = 1, 2, \ldots, s)$$

must be satisfied for an arbitrary $\bar{x}$. Because of a one–to–one correspondence between x and $\bar{x}$ one has:

$$F^k(f^x(z(x), a), f^u(z(x), a), f^p(z_p(x), a), f^\theta(z(x)))) = 0, (k = 1, 2, \ldots, s) \tag{2.21}$$

where $z(x) = (x, u_0(x), \theta(x, u_0(x)))$, $z_p(x) = (x, u_0(x), \theta(x, u_0(x)), p_0(x), \ldots)$.

After differentiating the Equation (2.21) with respect to the group parameter a, one obtains an algorithm for finding equivalence transformation as in the Equation (2.18). The differences in the algorithms for obtaining an admitted Lie group and equivalence group are only in the prolongation formulae of the infinitesimal generator.

In agreement with the construction, after differentiating Equation (2.21) with respect to the group parameter a, one obtains the determining Equation:

$$\widetilde{X}^e F^k(x, u, \theta)\,|_{F=0} = 0, \quad k = 1, 2, ..., s, \tag{2.22}$$

with the prolonged operator $\widetilde{X}^e$,

$$\widetilde{X}^e = X^e + \zeta^{u^j_{x_i}} \partial_{u^j_{x_i}} + \zeta^{\theta^k_{x_i}} \partial_{\theta^k_{x_i}} + \zeta^{\theta^k_{u^j}} \partial_{\theta^k_{u^j}} + \ldots .$$

Here the coefficients $\zeta^{u^j_{x_i}}, \zeta^{\theta^k_{x_i}}, \zeta^{\theta^k_{u^j}}, ...$ are expressed by the following:

$$\begin{aligned}
\zeta^{u^j_{x_i}} &= D^e_{x_i}\zeta^{u^j} - u^j_{x_\beta}D^e_{x_i}\xi^{x_\beta}, \\
\zeta^{\theta^k_{x_i}} &= \widetilde{D}^e_{x_i}\zeta^{\theta^k} - \theta^k_{x_\beta}\widetilde{D}^e_{x_i}\xi^{x_\beta} - \theta^k_{u^j}\widetilde{D}^e_{x_i}\zeta^{u^j}, \\
\zeta^{\theta^k_{u^j}} &= \widetilde{D}^e_{u^j}\zeta^{\theta^k} - \theta^k_{x_i}\widetilde{D}^e_{u^j}\xi^{x_i} - \theta^k_{u^\beta}\widetilde{D}^e_{u^j}\zeta^{u^\beta},
\end{aligned}$$

where

$$\begin{aligned}
D^e_{x_i} &= \partial_{x_i} + u^j_{x_i}\partial_{u^j} + (\theta^k_{x_i} + \theta^k_{u^j}u^j_{x_i})\partial_{\theta^k} + ..., \\
\widetilde{D}^e_{x_i} &= \partial_{x_i} + \theta^k_{x_i}\partial_{\theta^k} + ..., \\
\widetilde{D}^e_{u^j} &= \partial_{u^j} + \theta^k_{u^j}\partial_{\theta^k} + ...
\end{aligned}$$

The solution of the determining Equation (2.22) gives us the coefficients of an infinitesimal generator. By solving the Lie equations, one can obtain the equivalence group of transformation as illustrated in Equation (2.18).

CHAPTER III

COMPUTATIONAL PROCEDURE

Considering second-order ordinary differential equation that is written in the form:

$$y'' = k(x, y)y' + f(x, y). \tag{3.1}$$

Suppose the equation admits the generator

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y. \tag{3.2}$$

This thesis is devoted to finding coefficients $\xi(x, y)$ and $\eta(x, y)$, which give all possible generators that are admitted by Equation (3.1). The procedure is to proceed as follows:

(1) find the equivalence transformations of Equation (3.1),

(2) employ equivalence transformations to obtains generators of the Lie group of the Equation (3.1) in the simple form; and

(3) obtain the further generators of the admitted Lie group of the Equation (3.1), and the corresponding functions $k(x, y)$ and $f(x, y)$.

Step 1 and 2 are performed in this chapter, while step 3 is performed in the next chapter

3.1 Equivalence Transformation of Equation (3.1)

First, one has to find an equivalence group of transformations for Equation (3.1), that is to find a Lie group of transformations, which transforms Equation (3.1) into an equation with the same differential structure.

The arbitrary elements in Equation (3.1) are the functions $k(x,y)$ and $f(x,y)$, here the generator of the equivalence Lie group is written in the form:

$$X^e = \xi\partial_x + \eta\partial_y + \zeta^k\partial_k + \zeta^f\partial_f,$$

with the coefficients

$$\xi = \xi(x,y,k,f), \eta = \eta(x,y,k,f), \zeta^k = \zeta^k(x,y,k,f), \zeta^f = \zeta^f(x,y,k,f).$$

The prolonged operator is:

$$\tilde{X}^e = X^e + \zeta^{y'}\partial_{y'} + \zeta^{y''}\partial_{y''} + \zeta^{k_x}\partial_{k_x} + \zeta^{k_y}\partial_{k_y} + \zeta^{f_x}\partial_{f_x} + \zeta^{f_y}\partial_{f_y}.$$

The coefficients of the prolonged generator are:

$$\zeta^{y'} = D_x^e\eta - y'D_x^e\xi, \quad \zeta^{y''} = D_x^e\zeta^{y'} - y''D_x^e\xi,$$

$$\zeta^{k_x} = \tilde{D}_x^e\zeta^k - k_x\tilde{D}_x^e\xi - k_y\tilde{D}_x^e\eta, \quad \zeta^{k_y} = \tilde{D}_y^e\zeta^k - k_x\tilde{D}_y^e\xi - k_y\tilde{D}_y^e\eta,$$

$$\zeta^{f_x} = \tilde{D}_x^e\zeta^f - f_x\tilde{D}_x^e\xi - f_y\tilde{D}_x^e\eta, \quad \zeta^{f_y} = \tilde{D}_y^e\zeta^f - f_x\tilde{D}_y^e\xi - f_y\tilde{D}_y^e\eta.$$

Here, the operators $D_x^e, \tilde{D}_x^e$ and $\tilde{D}_y^e$ are:

$$D_x^e = \partial_x + y'\partial_{y'} + y''\partial_{y''} + (k_x + y'k_y)\partial_k + (f_x + y'f_y)\partial_f$$

$$+ (k_{xx} + y'k_{xy})\partial_{k_x} + (f_{xx} + y'f_{xy})\partial_{f_x},$$

$$\tilde{D}_x^e = \partial_x + (k_x + y'k_y)\partial_k + (f_x + y'f_y)\partial_f,$$

$$\tilde{D}_y^e = \partial_y + k_y\partial_k + f_y\partial_f.$$

The determining equations of the equivalence Lie group become:

$$[\zeta^{y''} - y'(\zeta^k + k_x\zeta^{k_x} + k_y\zeta^{k_y}) - k\zeta^{y'} - (\zeta^f + f_x\zeta^{f_x} + f_y\zeta^{f_y})]\Big|_{[S]} = 0.$$

After substitutions of $\zeta^{y''}, \zeta^{y'}, \zeta^{k_x}, \zeta^{k_y}, \zeta^{f_x}, \zeta^{f_y}$ and transition onto the manifold $[S]$: $y'' = ky' + f$, the equation can be split with respect to the variables $y'', y', k_x, k_y, f_x, f_y$. As a result, the determining equations are obtained:

$$\xi_y = 0, \xi_k = 0, \xi_f = 0, \eta_{yy} = 0, \eta_k = 0, \eta_f = 0,$$

$$\zeta^k = 2\eta_{xy} - \xi_{xx} - k\xi_x,$$

$$\zeta^f = \eta_{xx} - f\eta_y - k\eta_x - 2f\xi_x.$$

The general solution of the last equations is:

$$\begin{aligned}
X_\xi &= \xi(x)\partial_x - (\xi''(x) + k\xi'(x))\partial_k - 2f\xi'(x)\partial_f,\\
X_\eta &= \eta(x)\partial_y + (\eta''(x) - k\eta'(x))\partial_f,\\
X_\zeta &= y\zeta(x)\partial_y + 2\zeta'(x)\partial_k + (y\zeta''(x) - yk\zeta'(x) + f\zeta(x))\partial_f.
\end{aligned} \tag{3.3}$$

The Lie group of transformations corresponding to these generators are:

$$\begin{aligned}
&X_\xi:\ t = x + a\xi(x),\ u = y,\\
&X_\eta:\ t = x,\ u = y + a\eta(x),\\
&X_\zeta:\ t = x,\ u = ye^{a\zeta(x)}.
\end{aligned}$$

It can be rewritten in general form as:

$$\begin{aligned}
&X_\xi:\ t = \varphi(x),\ u = y,\\
&X_\eta:\ t = x,\ u = y + \tilde{\eta}(x),\\
&X_\zeta:\ t = x,\ u = y\tilde{\zeta}(x).
\end{aligned} \tag{3.4}$$

3.2 Admitted Lie Group of the Equation (3.1)

Infinitesimal generators of one-parameter Lie groups admitted by Equation (3.1) are sought in the form:

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y, \tag{3.5}$$

with simple coefficients ξ and η.

The prolonged infinitesimal generator of (3.5) is:

$$X^{(2)} = X + \eta^{(1)}\partial_{y'} + \eta^{(2)}\partial_{y''},$$

with the coefficients,

$$\eta^{(1)} = D_x\eta - y'D_x\xi, \quad \eta^{(2)} = D_x\eta^{(1)} - y''D_x\xi,$$

where D_x is the operator of the total derivative

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'}.$$

The generator of the Equation (3.5) is admitted by Equation (3.1), if and only if,

$$[X^{(2)}(y'' - ky' - f)]\big|_{[S]} = 0.$$

The last equation becomes:

$$[\eta^{(2)} - k\eta^{(1)} - \xi(k_x y' + f_x) - \eta(k_y y' + f_y)]\big|_{[S]} = 0. \tag{3.6}$$

Here $[S]$ is the manifold defined by the relation $y'' = ky' + f$.

After substituting $y'' = ky' + f$ into Equation (3.6), and splitting it with respect to y', one has:

$$\eta_{xx} = k\eta_x - f\eta_y + f_x\xi + f_y\eta + 2f\xi_x, \tag{3.7}$$

$$2\eta_{xy} = k_x\xi - k_y\eta + \xi_{xx} + k\xi_x + 3f\xi_y, \tag{3.8}$$

$$\eta_{yy} = 2(\xi_{xy} + k\xi_y), \tag{3.9}$$

$$\xi_{yy} = 0. \tag{3.10}$$

From equation (3.10) the general solution is:

$$\xi(x, y) = ay + b,$$

where $a = a(x)$ and $b = b(x)$ are arbitrary functions of the integration.

Applying the equivalence transformation as in the Equation (3.4):

$$t = \varphi(x), \quad u = \psi(x)y + h(x), \tag{3.11}$$

generator as in the Equation (3.5) is transformed to the generator of the same form:

$$\tilde{X} = \tilde{\xi}\partial_t + \tilde{\eta}\partial_u,$$

where

$$\tilde{\xi} = Xt = \frac{\varphi' a}{\psi}u + \varphi'(b - \frac{ah}{\psi}), \quad \tilde{\eta} = Xu. \tag{3.12}$$

Lemma. The following properties are valid:

(i) if $a \neq 0$, then the generator of the Equation (3.5) is equivalent to the generator:

$$X = y\partial_x + 6g(x,y)\partial_y,$$

where $g(x,y)$ satisfies:

$$yf_x + 6gf_y = 6(fg_y + g_{xx} - 3g_x g_{yy});$$

(ii) if $a = 0$ and $b \neq 0$, the generator (3.5) is equivalent to the generator:

$$X = \partial_x;$$

(iii) if $a = 0$ and $b = 0$, the generator (3.5) is equivalent to the generator:

$$X = xy\partial_y.$$

Proof.

(i) Case $a \neq 0$.

Using the equivalence transformation as in the Equation (3.8) with:

$$\psi = 1, \quad \varphi' = \frac{1}{a}, \quad h = \frac{b}{a},$$

one can assume that $\xi = y$. Using a function $g(x,y)$ such that $\eta = 6g(x,y)$, determining equations (3.7) and (3.8) become:

$$k = 3g_{yy}, f = 4g_{xy} - yg_{xyy} - gg_{yy},$$

and

$$yf_x + 6gf_y = 6(fg_y + g_{xx} - 3g_x g_{yy}).$$

In this case, the admitted generator is given in the form:

$$X = y\partial_x + 6g(x,y)\partial_y. \tag{3.13}$$

(ii) Case $a = 0$, $b \neq 0$.

From Equation (3.9), where $a = 0$, the function η is linear with respect to y:

$$\eta = y\alpha(x) + \beta(x),$$

and from relation of the Equation (3.12) can be rewritten in the form:

$$\tilde{\xi} = Xt = \varphi' b,$$
$$\tilde{\eta} = Xu = (\frac{b\psi'}{\psi} + \alpha)u + bh' + \beta\psi - h(\frac{b\psi'}{\psi} + \alpha).$$

Again, simplifying the coefficients by choosing:

$$\varphi' = \frac{1}{b}, \quad \psi' + \psi\frac{\alpha}{b} = 0, \quad h' + \beta\frac{\psi}{b} = 0,$$

the generator X can be reduced to the form:

$$X = \partial_x. \tag{3.14}$$

(iii) Case $a = 0$, $b = 0$.

Substituting $a = 0, b = 0$ into the relation of the Equation (3.9), one gets:

$$\xi = 0, \quad \eta = \alpha(x)y + \beta(x).$$

Substituting these functions into Equations (3.8) and (3.9), one has:

$$2\alpha' + k_y = 0,$$
$$(\alpha'' y + \beta'') - k(\alpha' y + \beta') + \alpha f - f_y(\alpha y + \beta) = 0.$$

If $\alpha = 0$, then $\beta \neq 0$, otherwise it will give generator $X = 0$. Solving the above equations, one gets $k = k(x)$ and $f = \frac{\beta''(x) - k(x)\beta(x)}{\beta(x)} y + \mu(x)$ which lead to $L_1 = 0, L_2 = 0$. This means that the studied equation is linearizable. Hence, one has to assume that $\alpha \neq 0$. Then one can choose the Equation (3.11)

$$\varphi = \alpha, \quad \psi = 1, \quad h = \frac{\beta}{\alpha}. \tag{3.15}$$

Substituting relation of the Equation (3.15) into the relation of the Equation (3.12), one gets $\tilde{\xi} = 0$ and $\tilde{\eta} = \alpha u = tu$.The generator X has been reduced to:

$$X = xy\partial_y. \tag{3.16}$$

CHAPTER IV

GROUP CLASSIFICATION

In the previous chapter, it is evident that Equation (3.1) admitted a generator of one of the following forms: $X = \partial_x$, or $X = y\partial_x + 6g(x, y)\partial_y$, or $X = xy\partial_y$. In this chapter, one continues with the group classification of Equation (3.1) by performing the following scheme:

Step S_1: Find the general form of the functions $k(x, y)$ and $f(x, y)$ such that Equation (3.1) admits the generator either $X = \partial_x$, or $X = y\partial_x + 6g(x, y)\partial_y$, or $X = xy\partial_y$. Notice that if $k_y = 0$, then by virtue of the results of Babich and Bordag the studied equation is reduced to the form $y'' = f(x, y)$ which was studied by Lie (1883) and later by Ovsianikov (2004), so one shall assume that $k_y \neq 0$.

Step S_2: Simplify the functions $k(x, y)$ and $f(x, y)$ and substitute them into the determining Equations (3.7)-(3.9). Solving the determining equations, additional generators are found, which one called extension of the generators.

4.1 Extension of the Generator $X = \partial_x$

Substituting the coefficients of the generator $X = \partial_x$ into the determining Equations (3.7)-(3.9) and solving them, one obtains:

$$k = k(y), \quad f = f(y).$$

Substituting the functions k and f into the Equation (3.7)-(3.9), one gets:

$$\xi(x, y) = \xi_1(x)y + \xi_0(x),$$

and

$$\begin{aligned}\eta_{xx} &= \eta_x k - \eta_y f + \eta f_y + 2\xi_0^{'} f + 2\xi_1^{'} f y,\\ \eta_{xy} &= 2^{-1}(\eta k_y + \xi_0^{''} + \xi_0^{'} k + \xi_1^{''} y + \xi_1^{'} k y + 3f\xi_1),\\ \eta_{yy} &= 2(\xi_1^{'} + k\xi_1),\end{aligned} \tag{4.1}$$

where $\xi_0(x)$ and $\xi_1(x)$ are arbitrary functions of the integration.

Comparing the mixed derivatives $(\eta_{xx})_y = (\eta_{xy})_x$ and $(\eta_{yy})_x = (\eta_{xy})_y$, one derives the equations:

$$\begin{aligned}\eta_x = &-(\eta(2f_{yy} + k_y k) - 4f_y\xi_0^{'} - 4f_y\xi_1^{'} y - \xi_0^{'} k^2\\ &+ \xi_0^{(3)} + \xi_1^{(3)} y + 3\xi_1^{'} f - \xi_1^{'} k^2 y + fk\xi_1)/k_y,\\ \eta_y = &(-\eta k_{yy} - 3f_y\xi_1 - k_y\xi_0^{'} - k_y\xi_1^{'} y + 3\xi_1^{''} + 3\xi_1^{'} k)/k_y.\end{aligned} \tag{4.2}$$

Substituting the Equation (4.2) into (4.1), and equating the mixed derivatives $(\eta_x)_y = (\eta_y)_x$, one finds:

$$\begin{aligned}J_1(y)\eta + k_1(x,y) = 0,\\ J_2(y)\eta + k_2(x,y) = 0,\\ J_3(y)\eta + k_3(x,y) = 0,\\ J_4(y)\eta + k_4(x,y) = 0,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}J_1 &= (-4f_{yy}^2 - 6f_{yy}k_y k + f_y k_y^2 + k_{yy}k_y f - 2k_y^2 k^2)/k_y^2,\\ J_2 &= (4f_{yyy}k_y - 8f_{yy}k_{yy} - 2k_{yy}k_y k + 3k_y^3)/(2k_y^2),\\ J_3 &= (-4f_{yy}k_{yy} - 2k_{yy}k_y k + k_y^3)/(2k_y^2),\\ J_4 &= (k_{yyy}k_y - 2k_{yy}^2)/k_y^2,\end{aligned}$$

$$
\begin{aligned}
k_1 = (&-8f_{yy}f_y\xi_0^{'} - 8f_{yy}f_y\xi_1^{'}y + 2f_{yy}\xi_0^{(3)} - 2f_{yy}\xi_0^{'}k^2 \\
&+ 2f_{yy}\xi_1^{(3)}y + 6f_{yy}\xi_1^{'}f - 2f_{yy}\xi_1^{'}k^2y + 2f_{yy}fk\xi_1 \\
&- 8f_yk_y\xi_0^{'}k + 4f_yk_y\xi_1^{''}y - 8f_yk_y\xi_1^{'}ky + 3f_yk_yf\xi_1 \\
&+ 3k_y^2\xi_0^{'}f + 3k_y^2\xi_1^{'}fy - k_y\xi_0^{(4)} + 2k_y\xi_0^{(3)}k + k_y\xi_0^{''}k^2 \\
&- 2k_y\xi_0^{'}k^3 - k_y\xi_1^{(4)}y + 2k_y\xi_1^{(3)}ky - 6k_y\xi_1^{''}f + k_y\xi_1^{''}k^2y \\
&+ 2k_y\xi_1^{'}fk - 2k_y\xi_1^{'}k^3y + 2k_yfk^2\xi_1 + 4f_yk_y\xi_0^{''})/k_y^2, \\
k_2 = (&-12f_{yy}f_y\xi_1 + 4f_{yy}k_y\xi_0^{'} + 4f_{yy}k_y\xi_1^{'}y + 12f_{yy}\xi_1^{''} \\
&+ 12f_{yy}\xi_1^{'}k - 8f_yk_{yy}\xi_0^{'} - 8f_yk_{yy}\xi_1^{'}y + 2f_yk_y\xi_1^{'} \\
&- 8f_yk_yk\xi_1 + 2k_{yy}\xi_0^{(3)} - 2k_{yy}\xi_0^{'}k^2 + 2k_{yy}\xi_1^{(3)}y + 6k_{yy}\xi_1^{'}f \\
&- 2k_{yy}\xi_1^{'}k^2y + 2k_{yy}fk\xi_1 + k_y^2\xi_0^{''} + 3k_y^2\xi_0^{'}k + k_y^2\xi_1^{''}y \\
&+ 3k_y^2\xi_1^{'}ky + k_y^2f\xi_1 - 2k_y\xi_1^{(3)} + 6k_y\xi_1^{''}k + 8k_y\xi_1^{'}k^2)/(2k_y^2), \\
k_3 = (&-8f_yk_{yy}\xi_0^{'} - 8f_yk_{yy}\xi_1^{'}y + 6f_yk_y\xi_1^{'} + 2k_{yy}\xi_0^{(3)} \\
&+ 2k_{yy}\xi_1^{(3)}y + 6k_{yy}\xi_1^{'}f - 2k_{yy}\xi_1^{'}k^2y + 2k_{yy}fk\xi_1 \\
&+ 3k_y^2\xi_0^{''} + k_y^2\xi_0^{'}k + 3k_y^2\xi_1^{''}y + k_y^2\xi_1^{'}ky - 2k_{yy}\xi_0^{'}k^2 \\
&+ 3k_y^2f\xi_1 - 6k_y\xi_1^{(3)} - 6k_y\xi_1^{''}k)/(2k_y^2), \\
k_4 = (&3f_{yy}k_y\xi_1 - 6f_yk_{yy}\xi_1 - k_{yy}k_y\xi_0^{'} - k_{yy}k_y\xi_1^{'}y \\
&+ 6k_{yy}\xi_1^{''} + 6k_{yy}\xi_1^{'}k + 2k_y^2k\xi_1)/k_y^2.
\end{aligned}
$$

Notice that if one of the functions $J_i, (i = 1, 2, 3, 4)$ is not equal to zero, then one can find the coefficient η of the admitted generator. Since the simplest expression of these functions is J_4, further analysis proceeds by considering different cases of J_4.

4.1.1 Case $\mathbf{J}_4 = 0$

The equation $J_4 = 0$ can be rewritten as:

$$\beta''\beta - 2\beta'^2 = 0,$$

where $\beta = k_y \neq 0$. The general solution of this equation is:

$$k_y = \frac{1}{C_1 y + C_2}, \tag{4.4}$$

where C_1 and C_2 are constant. Integration of the Equation (4.4) depends on the value of C_1.

4.1.1.1 Case $C_1 \neq 0$

In this case, applying the equivalence transformations related with shifting and scaling y, one simplifies the function $k(y)$:

$$k = \ln y.$$

From the last equation of equations (4.3), one has:

$$\xi_1'' = 6^{-1}\xi_1 y(3yf_{yy} + 6f_y + 2y\ln y) + 6^{-1}\xi_1'(1 - 6\ln y) + \frac{\xi_0'}{6y}.$$

Differentiating this equation with respect to y, one gets:

$$\xi_1' = 6^{-1}\xi_1(3f_{yyy}y^2 + 9f_{yy}y + 2) - \frac{\xi_0'}{6y}.$$

Differentiating ξ_1' with respect to y, one finds

$$\xi_0' = -3\xi_1 y^2(f_{yyyy}y^2 + 5f_{yyy}y + 3f_{yy}).$$

Differentiation of ξ_0' with respect to y yields:

$$\xi_1\mu(y) = 0, \tag{4.5}$$

where

$$\mu(y) = y^3 f_{yyyyy} + 9y^2 f_{yyyy} + 18yf_{yyy} + 6f_{yy}.$$

I. Case $\mu(y) = 0$

In this case, one finds that

$$f = \frac{c_1}{y} + c_2 \ln y + c_3 y \ln y + c_4 + c_5 y,$$

where c_i, $(i = 1, 2, 3, 4, 5)$ are constant.

Substituting f into the determining Equations (4.3) and splitting them with respect to $\ln y$, one finds $\xi_1 = 0$, $\eta = 0$ and $\xi_0' = 0$ which does not give an extension of the generator ∂_x.

II. Case $\mu(y) \neq 0$

In this case $\xi_1 = 0$, and system as in the Equation (4.3) becomes:

$$\begin{aligned}
&\eta(4y^3 f_{yy}^2 + 6y^2 \ln y f_{yy} - y f_y + 2y \ln^2 y + f) = 0, \\
&\eta(4y^2 f_{yyy} + 8y f_{yy} + 2 \ln y + 3) = 0, \\
&\eta(2y^2 f_{yyy} + 2y f_{yy} + 1) = 0, \\
&\eta(4y f_{yy} + 2 \ln y + 1) = 0.
\end{aligned} \tag{4.6}$$

Here, one has to assume that $\eta \neq 0$, otherwise there is no extension of the generator ∂_x. From the Equation (4.6), one obtains:

$$f = -\frac{y}{4}(\ln^2 y - \ln y + 1) + c_1 y + c_0.$$

Equation (4.3) yield $c_0 = 0$. Solving Equation (4.2) for η, another admitted generator $X = e^x y \partial_y$ is obtained.

4.1.1.2 Case $C_1 = 0$

Similar integration of the Equation (4.4) gives:

$$k = y.$$

Substituting k into the last equation of Equation (4.3), one has:

$$\xi_1(3 f_{yy} + 2y) = 0.$$

Notice that if $3f_{yy} + 2y = 0$, then,

$$f = -\frac{y^3}{9} + cy + d,$$

where c and d are constant. In this case, the studied equation is linearizable. Hence, $3f_{yy} + 2y \neq 0$, and $\xi_1 = 0$.

From the Equation (4.3), one has:

$$\eta = -3\xi_0'' - \xi_0' y.$$

Substitution of η into Equation (4.3) yields

$$\begin{aligned} &\xi_0^{(4)} - 2\xi_0^{(3)}(f_{yy} + y) + \xi_0''(12f_{yy}^2 + 18f_{yy}y + f_y + 7y^2) \\ &\quad + \xi_0'(-4f_{yy}^2 y + 8f_{yy}f_y - 4f_{yy}y^2 + 9f_y y - 3f) = 0, \\ &\xi_0''(3f_{yyy} + 2) + \xi_0'(f_{yyy}y - f_{yy}) = 0. \end{aligned} \tag{4.7}$$

Let us start with the analysis of Equation (4.7) above.

I. Case $3f_{yyy} + 2 = 0$

One obtains that

$$f = -\frac{y^3}{9} + by^2 + cy + d,$$

where a, $b \neq 0$ and c are constant. Substitution of f into Equation (4.7) yields $\xi_0' = 0$ and $\eta = 0$, which does not provide new admitted generators.

II. Case $3f_{yyy} + 2 \neq 0$

From the Equation (4.7), one gets:

$$\xi_0'' = -\frac{\xi_0'(f_{yyy}y - f_{yy})}{(3f_{yyy} + 2)}.$$

Differentiating it with respect to y, one has:

$$\xi_0' f_{yyyy} = 0.$$

Since for $\xi_0' = 0$, there is no extension of the generator ∂_x, one obtains that $f_{yyyy} = 0$ or

$$f = ay^3 + by^2 + cy + d,$$

where $a \neq -\frac{1}{9}$, b, c, and d are constant.

Substituting f into Equation (4.7) and η into the Equation(4.2), one gets the conditions:

$$d = 3\alpha^2 b,$$

$$c = \alpha^2(27a + 4).$$

and

$$\xi_0'' = \alpha\xi_0',$$

$$\eta = -3\xi_0'' - \xi_0' y,$$

where $\alpha = \frac{b}{9a+1}$.

In this case, the functions k and f are:

$$k = y, \quad f = ay^3 + by^2 + cy + d,$$

and the extension of the generator $X = \partial_x$ is given by the generators:

$$\alpha \neq 0 : X = e^{\alpha x}(\partial_x - (3\alpha^2 + \alpha y)\partial_y),$$

$$\alpha = 0 : X = x\partial_x - y\partial_y.$$

4.1.2 Case $\mathbf{J}_4 \neq 0$

From the Equation (4.3), one gets:

$$\eta = -\frac{k_4(x, y)}{J_4(y)}.$$

Substitution of η into the Equation (4.2) gives:

$$\xi_0' J_6 + a\xi_1'' + b\xi_1' + c\xi_1 = 0, \tag{4.8}$$

where

$$J_6 = k_y(k_{yyyy}k_{yy}k_y - 2k_{yyy}^2k_y + k_{yyy}k_{yy}^2),$$

$$a = (-2k_{yyyy}k_{yy} + 3k_{yyy}^2)/J_4^2,$$

$$b = (k_{yyyy}k_{yy}k_y y - 6k_{yyyy}k_{yy}k - 2k_{yyy}^2k_y y$$

$$+ 9k_{yyy}^2k + k_{yyy}k_{yy}^2y + 5k_{yyy}k_{yy}k_y - 10k_{yy}^3)/J_4^2,$$

$$c = (3f_{yyy}k_{yyy}k_y - 6f_{yyy}k_{yy}^2 - 3f_{yy}k_{yyyy}k_y + 9f_{yy}k_{yyy}k_{yy}$$

$$+ 6f_yk_{yyyy}k_{yy} - 9f_yk_{yyy}^2 - 2k_{yyyy}k_y^2k + 12k_{yyy}k_{yy}k_yk$$

$$+ 2k_{yyy}k_y^3 - 12k_{yy}^3k - 4k_{yy}^2k_y^2)/J_4^2.$$

4.1.2.1 Case $J_6 = 0$

Since $k_y \neq 0$, then

$$k_{yyyy}k_{yy}k_y - 2k_{yyy}^2k_y + k_{yyy}k_{yy}^2 = 0.$$

Here, $k_{yyy} = 0$ satisfies this equation, then one has to study different cases of k_{yyy}.

4.1.2.1.1 Case $k_{yyy} \neq 0$

Then,

$$k_{yyyy}k_{yy}k_y - 2k_{yyy}^2k_y + k_{yyy}k_{yy}^2 = 0.$$

Dividing this equation by $k_{yyy}k_{yy}k_y$ and integrating it twice with respect to y, one obtains:

$$h_y - c_1h^{c_0} = 0, \tag{4.9}$$

where $h = k_y$ and c_1, c_0 are constant. Since $k_{yyy} \neq 0$, one has to assume that $c_0c_1 \neq 0$.

Integration of Equation (4.9) depends on the value of c_0.

I. Case $c_0 = 1$

In this case,

$$h = c_1e^{c_2y},$$

where $c_2 \neq 0$ is constant. Scaling y and using involution if necessary, one obtains that

$$k_y = h = e^y.$$

Integration of k_y with respect to y gives:

$$k = e^y + \alpha,$$

where α is constant.

Substituting the function k into Equation (4.8), one obtains:

$$\xi_1'' = 3^{-1}(\xi_1'(2e^y - 3\alpha) + \xi_1(3f_{yyy} - 6f_{yy} + 3f_y + 4e^{2y} + 2e^y\alpha)). \tag{4.10}$$

Differentiating of the Equation (4.10) with respect to y, one gets:

$$\xi_1' = \xi_1\mu(y), \tag{4.11}$$

where

$$\mu(y) = -\frac{3}{2}e^{-y}(f_{yyyy} - 2f_{yyy} + f_{yy}) - 4e^y - \alpha.$$

Differentiating Equation (4.11) with respect to y, one gets

$$\xi_1\mu'(y) = 0.$$

I.1 Case $\mu'(y) \neq 0$

In this case $\xi_1 = 0$. From the Equation (4.3), one obtains:

$$\xi_0'' = \xi_0'\mu_1(y), \tag{4.12}$$

where

$$\mu_1(y) = -2e^{-y}(f_{yyy} - 2f_{yy}) + \alpha.$$

Differentiating Equation (4.12) with respect to y, one gets:

$$\xi_0'\mu_1'(y) = 0.$$

Since $\eta = -\xi_0'$, for the existence of an extension, one has to assume that $\xi_0' \neq 0$.

Therefore, $\mu_1'(y) = 0$ or

$$f = c_1 e^y + c_2 e^{2y} + c_3 y + c_4,$$

where c_i, $(i = 1, 2, 3, 4)$ are constant and $c_2 \neq -\frac{2}{3}$ because of the assumption that $\mu'(y) \neq 0$.

Substituting f into the Equation (4.3), the function f is reduced to the form:

$$f = c_2 e^{2y} - \alpha e^y - \alpha^2,$$

and the extension of the generator $X_0 = \partial_x$ is:

$$\alpha \neq 0 : X = e^{\alpha x}(\partial_x + \alpha \partial_y),$$

$$\alpha = 0 : X = x\partial_x - \partial_y.$$

I.2 Case $\mu'(y) = 0$

In this case,

$$\xi_1 = c e^{\mu(x)},$$

and

$$f = e^y(c_1 y^2 + c_2 y + c_3) + c_4 y + c_5 - \frac{2e^{2y}}{3},$$

where c_i, $(i = 1, 2, 3, 4, 5)$ are constant.

Substituting f into the Equation (4.3), splitting them with respect to e^y, and then with respect to y, one obtains $c = 0$, $\xi_0'' = 0$, and $c_i = 0$, $(i = 1, 2, 3, 4, 5)$.

In this case $f = -\frac{2}{3}e^{2y}$ and $\alpha = 0$. The additional admitted generator is:

$$X = x\partial_x - \partial_y.$$

Notice that combining the results of this case and the previous case with $\alpha = 0$, one has that for the functions $k = e^y$ and $f = c_2 e^{2y}$ with arbitrary c_2, the admitted Lie group is defined by the generators ∂_x, $x\partial_x - \partial_y$.

II. Case $c_0(c_0 - 1) \neq 0$

Integrating Equation (4.9), one has:

$$h = (c_1 y + c_2)^{c_0 - 1}.$$

Scaling and shifting y, one obtains:

$$k = y^{\lambda} + \alpha,$$

where α, $\lambda = c_0 - 1$ are constant and $\lambda(\lambda-1)(\lambda-2) \neq 0$ because of the condition $k_{yyy} \neq 0$.

From the Equation (4.2), one can find ξ_1''. Differentiating it with respect to y, one obtains:

$$2y^{\lambda}\xi_1'\lambda(\lambda+3)(\lambda-1) + \xi_1 h_1(y) = 0, \tag{4.13}$$

where

$$\begin{aligned} h_1(y) =& 3y^3 f_{yyyy} - 6(\lambda-3)y^2 f_{yyy} + 3(\lambda^2 - 5\lambda + 6)y f_{yy} \\ &+ 4y^{2\lambda}\lambda^2(2\lambda+1) + 2y^{\lambda}\lambda^2\alpha(\lambda+1), \end{aligned}$$

and $\lambda \neq 0, 1, 2$.

II.1 Case $\lambda = -3$

Equation (4.13) is reduced to:

$$\xi_1 h_1(y) = 0.$$

II.1.1 Case $h_1(y) \neq 0$

In this case $\xi_1 = 0$. From the last equation of determining equations (4.3), one obtains:

$$\xi_0'' = -\frac{2}{9}\xi_0'(y^5 f_{yyy} + 7y^4 f_{yy}) + \alpha.$$

Differentiating ξ_0'' with respect to y, one gets:

$$\xi_0'(y^5 f_{yyy} + 7y^4 f_{yy})' = 0.$$

Here, one has to assume that $\xi_0' \neq 0$, otherwise there is extension. Then

$$f = \frac{c_1}{6y^2} + \frac{c_2}{30y^5} + c_3 + c_4 y,$$

where c_i, $(i = 1, 2, 3, 4)$ are constant.

Substituting the function f into Equation (4.3), and splitting them with respect to y, one obtains $c_3 = 0$, $c_1 = 6\alpha$, $c_4 = 2\alpha^2$.

Hence, $\xi_0'' = -3\alpha\xi_0'$, and $\eta = \frac{\xi_0'}{3}y$. Thus, one has the additional generator:

$$\alpha \neq 0 : X = e^{-3\alpha x}(\partial_x - \alpha y \partial_y),$$

$$\alpha = 0 : X = 3x\partial_x + y\partial_y.$$

II.1.2 Case $h_1(y) = 0$

The general solution of $h_1(y) = 0$ is

$$f = \frac{1}{y^5} + \frac{c_1}{20y^4} + \frac{c_2}{12y^3} + \frac{\alpha}{y^2} + c_3 + c_4 y.$$

Substituting the function f into the Equation (4.3), and splitting them with respect to y, one obtains $\xi_1 = 0$, $\eta = 0$ and $\xi_0' = 0$, which does not give new extensions.

II.2 Case $\lambda \neq -3$

From Equation (4.13) one finds ξ_1' and differentiating it with respect to y, one gets:

$$\xi_1(\frac{h_1}{y^\lambda})' = 0. \tag{4.14}$$

Since $\xi_1 \neq 0$, then $(\frac{h_1}{y^\lambda})' = 0$. For solving this equation, one needs to consider different cases of

$$\Lambda = (\lambda + 1)(\lambda + 2)(2\lambda + 1).$$

II.2.1 Case $\Lambda = 0$

Substituting each solution λ of equation $\Lambda = 0$ into the Equation (4.14) and solving it, one can find the function f. Substituting the obtained function f

into Equation (4.3), and splitting them with respect to y, one gets $\xi_1 = 0$, $\xi_0' = 0$ and $\eta = 0$, which means that there is extension.

II.2.2 Case $\Lambda \neq 0$

Solving Equation (4.14), one obtains:

$$f = c_1 + c_2 y + c_3 y^{\lambda} + c_4 y^{\lambda-1} + c_5 y^{\lambda+1} - \frac{2\lambda y^{2\lambda+1}}{3(\lambda+1)(\lambda+2)},$$

where c_i, (i=1,2,3,4) are constant.

Substituting f into the Equation (4.3), and splitting them with respect to y, one gets $\xi_1 = 0$, $\xi_0' = 0$ and $\eta = 0$ which means that there is no extension.

4.1.2.1.2 Case $k_{yyy} = 0$

Since $J_4 \neq 0$, then the function k can be represented as:

$$k = y^2 + \alpha y + \beta,$$

where α and β are constant.

From the the Equation (4.2), one obtains:

$$\xi_1' = -\frac{\xi_1 \mu(y)}{10}, \tag{4.15}$$

where

$$\mu(y) = 3f_{yyy} + 20y(\alpha + y) + 2(\alpha^2 + 6\beta)$$

Differentiating the Equation (4.15) with respect to y, one gets the equation:

$$\xi_1 \mu'(y) = 0.$$

I. Case $\mu'(y) \neq 0$

In this case $\xi_1 = 0$, and the Equation (4.3) becomes:

$$\xi_0'' = -\xi_0' \mu_1(y), \tag{4.16}$$

where

$$\mu_1(y) = f_{yyy} - \frac{6}{(\alpha + 2y)} f_{yy} + \alpha^2 - 4\beta.$$

Differentiating the Equation (4.16) with respect to y, one finds:

$$\xi_0' \mu_1'(y) = 0.$$

Since for the existence of an extension one has to assume that $\xi_0' \neq 0$, one gets:

$$\mu_1'(y) = 0.$$

The general solution of the equation $\mu_1'(y) = 0$ is:

$$f = c_1(\alpha + 2y)^5 + c_2(\alpha + 2y)^3 + c_3 y + c_4,$$

where $c_1 \neq -\frac{1}{288}$ because of the assumption that $\mu' \neq 0$, and

$$\lambda = \frac{\alpha^2 - 4\beta}{8}, \quad c_2 = \frac{\lambda}{8}, \quad c_3 = -\frac{3\lambda^2}{16}, \quad c_4 = -\frac{3\alpha\lambda^2}{64}.$$

Substituting the function f into the Equation (4.3), and solving them, one obtains that $\xi_0'' = \lambda\xi_0'$ and $\eta = \frac{\alpha+2y}{4}\xi_0'$. Hence, the extension of the generator ∂_x is defined by the generator:

$$\lambda \neq 0 : X = e^{\lambda x}(4\partial_x - \lambda(\alpha + 2y)\partial_y),$$

$$\lambda = 0 : X = 4x\partial_x - (\alpha + 2y)\partial_y.$$

II. Case $\mu'(y) = 0$

The general solution of this equation is:

$$f = -\frac{1}{288}(\alpha + 2y)^5 + c_2 y^3 + c_3 y^2 + c_4 y + c_5,$$

where

$$\lambda = \frac{\alpha^2 - 4\beta}{8}, \quad c_2 = \frac{\lambda}{2}, \quad c_3 = \frac{3\alpha\lambda}{4}, \quad c_4 = \frac{3\lambda(3\alpha^2 + 4\beta)}{32}, \quad c_5 = \frac{3\alpha\lambda(3\alpha^2 + 12\beta)}{64}.$$

Substituting into the Equation (4.3), and splitting with respect to y, one gets that $\xi_0'' = \lambda\xi_0'$ and $\eta = \frac{\alpha+2y}{4}\xi_0'$. Hence, the extension of the generator ∂_x is defined by the generators:

$$\lambda \neq 0 : X = e^{\lambda x}(4\partial_x - \lambda(\alpha + 2y)\partial_y),$$

$$\lambda = 0 : X = 4x\partial_x - (\alpha + 2y)\partial_y.$$

Notice that combining the result of this case and the previous case, one has that for the functions:

$$k = y^2 + \alpha y + \beta, \quad f = c_1(\alpha + 2y)^5 + \frac{\lambda}{8}(\alpha + 2y)^3 - \frac{3\lambda^2}{16}y - \frac{3\alpha\lambda^2}{64},$$

with $\lambda = \frac{\alpha^2 - 4\beta}{8}$, and arbitrary constant c_1, the admitted Lie group is defined by the generators:

$$\lambda \neq 0 : X = e^{\lambda x}(4\partial_x - \lambda(\alpha + 2y)\partial_y), X_1 = \partial_x$$

$$\lambda = 0 : X = 4x\partial_x - (\alpha + 2y)\partial_y, X_1 = \partial_x.$$

4.1.2.2 Case $J_6 \neq 0$

From equation (4.8), one gets:

$$\xi_0' = \frac{a}{J_6}\xi_1'' + \frac{b}{J_6}\xi_1' + \frac{c}{J_6}\xi_1. \tag{4.17}$$

One has to assume that $\xi_1 \neq 0$, otherwise, it leads to the nonexistence of an extension.

Differentiation of Equation (4.17) with respect to y yields:

$$u\xi_1'' + v\xi_1' + w\xi_1 = 0, \tag{4.18}$$

where

$$u = (\frac{a}{J_6})', \quad v = (\frac{b}{J_6})', \quad w = (\frac{c}{J_6})'.$$

Differentiating the Equation (4.18) once and twice with respect to x, one gets a system of equations:

$$\begin{aligned} u\xi_1^{(3)} + v\xi_1'' + w\xi_1' &= 0, \\ u\xi_1^{(4)} + v\xi_1^{(3)} + w\xi_1'' &= 0. \end{aligned} \tag{4.19}$$

The system of the Equation (4.18) and (4.19) can be written in matrix form:

$$AB = 0,$$

where

$$\mathbf{A}=\begin{pmatrix}\xi_1'' & \xi_1' & \xi_1\\ \xi_1^{(3)} & \xi_1'' & \xi_1'\\ \xi_1^{(4)} & \xi_1^{(3)} & \xi_1''\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}u\\ v\\ w\end{pmatrix}.$$

I. Case $detA \neq 0$

Then $B = 0$, or

$$u=0,\quad v=0,\quad w=0.$$

Integrating the equation $u = 0$, one has:

$$a-\gamma J_6=0,$$

which can be rewritten in the form:

$$k_{yyyy}=\frac{k_{yyy}(2k_{yyy}k_y\gamma+9k_{yyy}-\gamma k_{yy}^2)}{k_{yy}(\gamma k_y+6)},$$

where γ is constant of the integration.

Substitution of k_{yyyy} into $v = 0$ yields:

$$(4k_y\gamma+9)k_{yyy}=5k_{yy}^2\gamma. \tag{4.20}$$

Since $k_{yyy} \neq 0$, then $\gamma \neq 0$. The general solution of (4.20) up to scaling and shifting of y has the form:

$$k(y)=\frac{1}{y^3}+\alpha y+h_0,$$

where $\alpha \neq 0$ and h_0 are constant.

Substituting the function k into $w = 0$, one finds:

$$y^2f_{yyyy}+12yf_{yyy}+30f_{yy}-\frac{12h_0}{y^4}-\frac{60}{y^7}+4\alpha(5h_0+7\alpha y)=0.$$

The general solution of this equation is:

$$f=\frac{1}{y^5}+\frac{\beta_1}{y^4}+\frac{\beta_2}{y^3}+\frac{h_0}{y^2}+\beta_3+\beta_4 y-\frac{\alpha h_0}{3}y^2-\frac{\alpha^2}{9}y^3,$$

where $\beta_i, (i = 1, 2, 3, 4)$ are constant.

Substituting k and f into the Equation (4.3), one obtains $\xi_1 = 0$ or $\alpha = 0$ which is a contradiction.

II. Case $detA = 0$

Let us consider the minor,

$$\mathbf{A_1} = \begin{pmatrix} \xi_1' & \xi_1 \\ \xi_1'' & \xi_1' \end{pmatrix}.$$

II.1 Case $detA_1 = 0$

In this case,

$$\xi_1' = \alpha \xi_1, \tag{4.21}$$

where α is constant.

Substituting the Equation (4.21) into the Equation (4.13), one gets:

$$\xi_0' = \beta \xi_1,$$

where β is constant. Substituting these relations into equation (4.2), one gets

$$k_y \eta_x + \eta(2f_{yy} + k_y k) = \xi_1 f(k + 3\alpha) + \xi_1(\alpha^2 - k^2 - 4f_y)(\alpha y + \beta), \tag{4.22}$$

and

$$(k_y \eta)_y = \xi_1(3\alpha^2 + 3\alpha k - 3f_y - k_y(\alpha y + \beta)). \tag{4.23}$$

Integrating Equation (4.23) with respect to y, one has:

$$k_y \eta = \xi_1(3\alpha^2 y + 4\alpha g_y - 3f - k(\alpha y + \beta)) + \lambda, \tag{4.24}$$

where $\lambda = \lambda(x)$, and the function $g = g(y)$ is such that $g_{yy} = k$.

Substitution of η into the Equation (4.22) gives:

$$\lambda' = a\lambda + b\xi_1, \tag{4.25}$$

where

$$
\begin{aligned}
a &= -(2f_{yy} + k_y k)/k_y,\\
b &= 2f_{yy}(3f - 3\alpha^2 y + \alpha k y - \alpha g_y + k\beta)/k_y - 4f_y(\alpha y + \beta)\\
&\quad + 4fk - \alpha^3 y - 2\alpha^2 k y + \alpha^2\beta - 4\alpha^2 g_y + 6\alpha f + \alpha k\beta - 4\alpha k g_y.
\end{aligned}
$$

Differentiating the Equation (4.25) with respect to y, one gets

$$a'\lambda + b'\xi_1 = 0. \tag{4.26}$$

First, let us consider a particular case where

$$\lambda = \alpha_0\xi_1,$$

and α_0 is constant.

Substituting λ into Equation (4.24), one gets

$$f_{yy}J - \varphi = 0, \tag{4.27}$$

where

$$
\begin{aligned}
\varphi &= 4fk - 4f_y(\alpha y + \beta) - 2\alpha^3 y - 4\alpha^2 g_y - 2\alpha^2 k y\\
&\quad + 6\alpha f - 4\alpha g_y k + \alpha k\beta - \alpha\alpha_0 + \alpha^2\beta - \alpha_0 k,\\
J &= 2(-3\alpha^2 y - 4\alpha g_y + \alpha k y + 3f + k\beta - \alpha_0)/k_y.
\end{aligned}
$$

II.1.1 Case $J \neq 0$

From Equation (4.27), one can derive:

$$f_{yy} = \frac{\varphi(y)}{J}.$$

From Equation (4.24), one gets:

$$\eta = \xi_1\phi(y),$$

where $\phi = \phi(y)$, and

$$f = (\phi k_y + 3\alpha^2 y - \alpha y k + 4\alpha g_y - \beta k + \alpha_0)/3.$$

From the Equation (4.3), one obtains:

$$2\alpha\phi_y = \alpha(4\alpha y + \beta + 4g_y) + \alpha_0. \quad (4.28)$$

II.1.1.1 Case $\alpha \neq 0$

Integration of the Equation (4.28) gives:

$$\phi = 2g + \alpha y^2 + (\beta + \frac{\alpha_0}{\alpha})\frac{y}{2} + \alpha_1,$$

where α_1 is constant.

From the Equation (4.3), one finds g_{yyyy}. Substituting g_{yyyy} into (4.27), one obtains:

$$g_{yyy} - g_{yy} = 0.$$

Solving this equation, one gets:

$$k = g_{yy} = c_0 e^y,$$

which leads to a contradiction of the assumption that $J_4 \neq 0$.

II.1.1.2 Case $\alpha = 0$

In this case, one gets $\eta = 0$, and $\xi_0' = \beta\xi_1$, where ξ_1 is constant.

Hence, the admitted generator is:

$$X = (y + \beta x)\partial_x.$$

Substitution of these coefficients into Equation (4.3), one finds:

$$k = \frac{\phi_{yy}}{2}, \quad f = -\frac{\phi_{yyy} + \phi_{yy}\beta}{6},$$

where ϕ satisfies the relation

$$\phi_{yyyy}\phi_{yy} + 3\phi_{yyy}\phi\beta - \phi_{yy}\beta(\phi_y + 2\beta) = 0.$$

II.1.2 Case $J = 0$

Equation $J = 0$ can be rewritten in the form:

$$f = (-g_{yy}(\alpha y + \beta) + 4\alpha g_y + 3\alpha^2 y + \alpha_0)/3. \tag{4.29}$$

Substitution of $k = g_{yy}$ and f into equations (4.3) yields:

$$\begin{aligned} & g_{yyyy}(\beta + \alpha y)^2 - 4{g_{yy}}^2(\alpha y + \beta) + 4\alpha g_{yy} g_y \\ & + g_{yy}(-\alpha^2 y - 15\alpha\beta + \alpha_0) + 12\alpha^2 g_y + 3\alpha(-3\alpha\beta + \alpha_0) = 0. \end{aligned} \tag{4.30}$$

and

$$\eta = 0.$$

Thus, Equation (3.1) with $k = g_{yy}$ and the function f defined by the Equation (4.29) admits the generators:

$$X = \partial_x, \quad X = e^{\alpha x}(\alpha y + 1)\partial_x.$$

Notice that the function $g = g(y)$ satisfies Equation (4.30).

Now let us analyze Equation (4.26).

If $a' \neq 0$, equation (4.26) gives:

$$\lambda = -\frac{b'}{a'}\xi_1.$$

Since $\xi_1 \neq 0$, one has to assume that $\frac{a'}{b'}$ is constant. This particular case has already been studied.

If $a' = 0$, then $b' = 0$ or $a = a_0$, and $b = b_0$ are constant.

Since the general solution of equation (4.30) is $\xi_1' = \alpha\xi_1$, Equation (4.25) become:

$$\lambda' = a_0\lambda + b_0 c_0 e^{\alpha x}. \tag{4.31}$$

The general solution of (4.31) depends on value of $(a_0 - \alpha)$.

Case $a_0 = \alpha$

The general solution of the Equation (4.31) for this case is:

$$\lambda = (b_0 x + \gamma_0)\xi_1.$$

From the Equation (4.3) after substituting λ, and splitting them with respect to x, one obtains $b_0 = 0$, which leads to a case already studied.

Case $a_0 \neq \alpha$

In this case, the general solution of (4.31) is:

$$\lambda = (c_1 e^{(a_0-\alpha)x} + \frac{b_0}{\alpha - a_0})\xi_1.$$

From the Equation (4.3) after splitting it with respect to $e^{(a_0-\alpha)x}$, one obtains $c_1 = 0$, which also leads to the studied a case already studied.

II.2 Case $det A_1 \neq 0$

Since $det A = 0$, one obtains:

$$\xi_1'' = \alpha\xi_1' + \beta\xi_1.$$

where α and β are constant.

Substituting ξ_1'' into Equation (4.17), one gets:

$$\xi_0' = \alpha_0\xi_1' + \beta_0\xi_1,$$

where α_0 and β_0 are constant.

Substituting these relations into Equation (4.2), one gets:

$$\begin{aligned} k_y\eta_x = {} & \xi_1'(3f + (\alpha^2 + \beta - 4f_y)(\alpha_0 + y) \\ & + \alpha\beta_0 - k^2(\alpha_0 + y) + \xi_1(fk - 4f_y\beta_0) \\ & + \alpha\beta(\alpha_0 + y) + \beta_0(\beta - k^2)) + \eta(-2f_{yy} - k_y k), \end{aligned} \tag{4.32}$$

and

$$(k_y\eta)_y = \xi_1'(3\alpha - \alpha_0 k_y - k_y y + 3k) + \xi_1(3\beta - 3f_y - \beta_0 k_y). \tag{4.33}$$

Integrating Equation (4.33) with respect to y, one obtains:

$$k_y\eta = \xi_1'(3y\alpha - \alpha_0 k - ky + 4g_y) + \xi_1(3\beta y - 3f - \beta_0 k) + \lambda, \tag{4.34}$$

where $\lambda = \lambda(x)$, and the function $g = g(y)$ is such that $g_{yy} = k$.

Substitution of η into the Equation (4.32) gives:

$$\lambda' = a\lambda + b\xi_1 + c\xi_1', \tag{4.35}$$

where

$$\begin{aligned} a &= -(2f_{yy} + k_y k)/k_y, \\ b &= 2f_{yy}(3f - 3\beta y + k\beta_0)/k_y - (4f_y\beta_0 - \alpha\beta\alpha_0 \\ &\quad + \alpha\beta y + 4\beta g_y - \beta k\alpha_0 + \beta ky + \beta\beta_0 - 4fk), \\ c &= 2f_{yy}(k\alpha_0 - 3\alpha y + ky - 4g_y)/k_y - (4f_y\alpha_0 + 4f_y y \\ &\quad + 4\alpha g_y - \alpha^2\alpha_0 + 2\alpha^2 y - \alpha k\alpha_0 + 2\alpha ky - \alpha\beta_0 \\ &\quad + 2\beta y - \beta\alpha_0 - 6f - \beta_0 + 4kg_y). \end{aligned}$$

Differentiating Equation (4.35) with respect to y, one gets:

$$a'\lambda + b'\xi_1 + c'\xi_1' = 0. \tag{4.36}$$

First, let us consider the particular case where

$$\lambda = \alpha_1\xi_1' + \beta_1\xi_1,$$

with constant α_1 and β_1.

Substituting λ into Equation (4.34), one gets η. Substituting η into the Equation (4.3) and solving them, one gets $k_y = 0$ which is a contradiction.

Now let us analyze Equation (4.35).

If $a' \neq 0$, then λ has the form:

$$\lambda = -\frac{b'}{a'}\xi_1' - \frac{c'}{a'}\xi_1.$$

Since $\xi_1 \neq 0$ and $detA_1 \neq 0$, one has to assume that $\frac{b'}{a'}$ and $\frac{c'}{a'}$ are constant. This case has already been studied.

If $a' = 0$. Since $detA_1 \neq 0$, then from Equation (4.36) with the condition $detA_1 \neq 0$, one gets $b' = 0$ and $c' = 0$ or $a = a_0$, $b = b_0$ and $c = c_0$ are constant.

Equation (4.35) can be rewritten as:

$$\lambda' = a_0\lambda + b_0\xi_1 + c_0\xi_1'.$$

From the last equation of (4.3), one obtains that

$$\lambda = \alpha_1\xi_1' + \beta_1\xi_1.$$

This case has also been studied already.

4.2 Extension of the Generator $X = y\partial_x + 6g(x, y)\partial_y$

In this case, the functions $k(x, y)$ and $f(x, y)$ are:

$$k = 3g_{yy}, \quad f = -yg_{xyy} + 4g_{xy} - 6gg_{yyy},$$

where the function $g = g(x, y)$ satisfies the relation

$$\begin{aligned} &12ygg_{xyyy} - 6yg_yg_{xyy} - 18g_yg_{xyy} + 24g_yg_{xy} + y^2g_{xxyy} \\ &+ 6g_{xx} + 6yg_xg_{yyy} - 18g_xg_{yy} + 36g^2g_{yyyyy} = 0. \end{aligned} \tag{4.37}$$

Notice that $g_{yyyyy} \neq 0$, because otherwise Equation (3.1) can be reduced to the form $y'' = f(x, y)$, which is excluded from the study.

Assume that there is an extension of the generator $y\partial_x + 6g\partial_y$:

$$X_1 = (\xi_1 y + \xi_0)\partial_x + \eta\partial_y,$$

where $\xi_1' \neq 0$.

Substituting the functions k and f into (3.9), one obtains:

$$\eta_{yy} = 6g_{yy}\xi_1 + 2\xi_1''.$$

Integrating this equation, one gets:

$$\eta = 6\xi_1 g + y^2\xi_1' + y\mu_0 + \mu_1,$$

where $\mu_0 = \mu_0(x)$ and $\mu_1 = \mu_1(x)$ are arbitrary functions of the integration.

Substituting η into the Equation (3.8), and integrating it twice with respect to y, one has:

$$\begin{aligned} &6\xi_0 g_x + 6(\mu_1 y + \mu_0 + \xi_1' y^2)g_y + 6(\xi_0' - 3\xi_1' y - 2\mu_1)g \\ &+ (\xi_0'' - 2\mu_1')y^2 - \xi_1'' y^3 + 2\mu_3 y + 2\mu_4 = 0, \end{aligned} \tag{4.38}$$

where $\mu_3 = \mu_3(x)$ and $\mu_4 = \mu_4(x)$ are arbitrary functions.

Equation (4.38) is a first-order quasilinear partial differential equation with respect to the function $g(x,y)$. The characteristic system of this equation is:

$$\begin{aligned} \frac{dx}{6\xi_0} &= \frac{dy}{6(\mu_1 y + \mu_0 + \xi_1' y^2)} \\ &= -\frac{dg}{6(\xi_0' - 3\xi_1' y - 2\mu_1)g + (\xi_0'' - 2\mu_1')y^2 - \xi_1'' y^3 + 2\mu_3 y + 2\mu_4}. \end{aligned}$$

If $\xi_0 \neq 0$, then for finding integrals of the characteristic system of equations, one needs to solve a Riccati type equation. In order to overcome this difficulty, the further study is separated into two cases: $\xi_0 = 0$, and $\xi_0 \neq 0$.

4.2.1 Case $\xi_0 \neq 0$

In this case, one introduces the change of the independent and dependent variables:

$$t = \varphi(x), \quad u = y\psi(x). \tag{4.39}$$

The transformation of the Equation (4.39) transforms the generator X to the same form:

$$\tilde{X}_1 = (\tilde{\xi}_1 u + \tilde{\xi}_0)\partial_t + \tilde{\eta}\partial_u, \tag{4.40}$$

where

$$(\tilde{\xi_1}u + \tilde{\xi_0}) = X_1 t = X_1\varphi' = \xi_1 y\varphi' + \xi_0\varphi'.$$

Requiring that transformation of the Equation (4.39) transform the generator X into the same form, one has that $\varphi' = \psi$.

From the Equation (4.40), one can choose the function ψ such that $\tilde{\xi_0} = \psi\xi_0 = 1$. Hence, one can assume that $\xi_0 = 1$.

In order to overcome the problem of solving a Riccati type equation, the following analysis is performed.

Substituting g_x found from the Equation (4.38) into the Equation (3.7), (4.37), and taking their linear combination, one gets the equation;

$$ag_{yyy} + bg_{yy} + cg_y + d = 0, \tag{4.41}$$

where

$$\begin{aligned}
b_2 &= \mu_1' - \mu_0\xi_1',\\
a &= 3(b_2y^2 - \mu_0'y - \mu_1\mu_1 y - \mu_0^2 - 2\mu_3 y - 2\mu_4),\\
b &= 3(b_2y + \mu_0'),\\
c &= -12b_2,\\
d &= 2b_2\mu_0' + 3\mu_0'' - 3b_2'y - 3\mu_0'y\xi_1' + 2\xi_1'\mu_0^2 + 4\mu_3'.
\end{aligned}$$

Differentiating Equation (4.41) twice with respect to y, one gets:

$$\begin{aligned}
&(b_2y^2 - \mu_0'y + \mu_0\mu_1 y - \mu_0^2 - 2\mu_3 y - 2\mu_4)g_{yyyyy}\\
&+ (5b_2y - \mu_0' - 2\mu_0\mu_1 - 4\mu_3)g_{yyyy} = 0.
\end{aligned} \tag{4.42}$$

4.2.1.1 Case $b_2 \neq 0$

Introducing the functions:

$$\begin{aligned}
b_0 &= -(\mu_0^2 + 2\mu_4)/b_2,\\
b_1 &= -(\mu_0' + \mu_0\mu_1 + 2\mu_3)/(2b_2),\\
b_3 &= \mu_0'/b_2,
\end{aligned} \tag{4.43}$$

Equation (4.42) becomes:

$$\frac{g_{yyyyy}}{g_{yyyy}} = -\frac{5y + 4b_0 + b_3}{y^2 + 2yb_1 + b_0}. \tag{4.44}$$

Integration of the right hand side of the Equation (4.44) depends on the denominator.

I. Case $b_1^2 - b_0 = v^2 > 0$

Integrating equation (4.44) with respect to y, one gets:

$$g_{yyyy} = \alpha \frac{(b_1 - v + y)^{h_1}}{(b_1 + v + y)^{h_1+5}}, \tag{4.45}$$

where $h_1 = (2v)^{-1}(b_1 - b_3 - 5v)$, and $\alpha \neq 0$.

Equating mixed derivatives $(g_x)_{yyyy} = (g_{yyyy})_x$, and splitting with respect to y, one obtains that h_1 is constant,

$$h_1(h_1 + 5)(v' - v(\mu_1 - b_1\xi_1')) = 0.$$

In integration of the Equation (4.45) one needs to consider different cases of:

$$\lambda = h_1(h_1 + 1)(h_1 + 2)(h_1 + 3)(h_1 + 4)(h_1 + 5).$$

I.1 Case $\lambda \neq 0$

Integrating of the Equation (4.45) four times with respect to y, one finds:

$$g = \alpha \frac{(b_1 - v + y)^{h_1+4}}{16v^4(b_1 + v + y)^{h_1+1}(h_1 + 1)(h_1 + 2)(h_1 + 3)(h_1 + 4)} + P_3(y), \tag{4.46}$$

where $P_3(y)$ is a cubic polynomial of y.

Substitution of the Equation (4.46) into equation (4.37) gives:

$$\Pi_1 u^8 \left(\frac{u - 2v}{u}\right)^{3h_1} + \Pi_2 u^3 \left(\frac{u - 2v}{u}\right)^{h_1} + \Pi_3 u + \Pi_4 = 0, \tag{4.47}$$

where $u = b_1 + v + y$, $\Pi_i = \Pi_i(\alpha, v, h_1), (i = 1, 2, 3)$.

For splitting Equation (4.47) with respect to y, one needs to consider $\Lambda = 0$ and $\Lambda \neq 0$, where

$$\Lambda = (h_1 - 1)(h_1 - 2)(2h_1 - 1)(2h_1 - 3)(2h_1 - 5)$$
$$(3h_1 - 8)(3h_1 - 7)(3h_1 - 5)(3h_1 - 4)(3h_1 - 2)(3h_1 - 1).$$

I.1.1 Case $\Lambda \neq 0$

In this case Equation (4.47) yields $\Pi_i = 0, (i = 1, 2, 3)$, which implies the contradiction $\alpha = 0$.

I.1.2 Case $\Lambda = 0$

Substituting each case of h_1 solving the equation $\Lambda = 0$ into the determining Equations (3.8)-(3.11), and splitting them with respect to y, one gets $\alpha = 0$, which is a contradiction.

I.2 Case $\lambda = 0$

Substituting each case of h_1 into Equation (4.45) and integrating four times with respect to y, one gets:

$$g = \alpha g_0(y) + P_3(y),$$

where $P_3(y)$ is the cubic polynomial of the integration, and the expression of the function $g_0(y)$ depends on h_1.

Substituting the found function g into Equations (3.8)-(3.11) and splitting them with respect to y, one obtains $\alpha = 0$, which is a contradiction.

II. Case $b_1^2 - b_0 = -v^2 < 0$

The study of this case is similar to the case where $b_1^2 - b_0 = v^2 > 0$.

Integrating Equation (4.44) with respect to y, one gets g_{yyyy}, which after changing of the independent variables $u = \arctan \frac{b_1+y}{v}$ (or $y = v\tan(u) - b_1$) becomes:

$$g_{yyyy} = \alpha \frac{e^{\lambda u}}{v^4(\tan^2(u)+1)^2\sqrt{\tan^2(u)+1}} \tag{4.48}$$

where $\lambda = \frac{b_1 - b_3}{v}$.

Using the chain rule, the right hand side of the Equation (4.48) can be integrated four times with respect to y:

$$g(y(u)) = \alpha \frac{e^{\lambda u}(\tan^2(u)+1)^{3/2}}{v(\lambda^2+1)(\lambda^2+9)} + P_3(y(u)),$$

where α is constant, and $P_3(y(u))$ is the cubic polynomial of integration.

Substituting the found function g into Equations (3.8)-(3.11) and splitting them with respect to $e^{\lambda u}$ and $\tan(u)$, one obtains that $\alpha = 0$, which is a contradiction.

III. Case $b_1^2 - b_0 = 0$

Equation (4.44) becomes:

$$\frac{g_{yyyyy}}{g_{yyyy}} = -\frac{5y + 4b_1^2 + b_3}{(y+b_1)^2}. \tag{4.49}$$

For integration of the Equation (4.49), one needs to consider different cases of the value of b_1.

III.1 Case $b_1 \neq 0$

Integrating the Equation (4.49) with respect to y, one gets:

$$g_{yyyy} = \alpha \frac{e^{\lambda y}}{(b_1+y)^5}, \tag{4.50}$$

where $\lambda = \frac{b_1 - b_3}{b_1(b_1+y)}$.

Notice that for integration of Equation (4.50), one needs to consider different cases of λ.

III.1.1 Case $\lambda = 0$

Integrating of the Equation (4.50) four times with respect to y, one obtains:

$$g = \alpha \frac{y}{24b_1(y+b_1)} + P_3(y), \tag{4.51}$$

where $P_3(y)$ is the cubic polynomial of integration.

Substituting the found function g into Equations (3.8)-(3.11) and splitting them with respect to y, one obtains $\xi_1' = 0$, which is a contradiction.

III.1.2 Case $\lambda \neq 0$

Integrating of the Equation (4.50) four times with respect to y, one obtains:

$$g = \alpha \frac{e^{(b_1(b_1+y))^{-1}(b_1-b_3)y}(b_1+y)^3}{(b_1-b_3)^4} + P_3(y), \tag{4.52}$$

where $P_3(y)$ is the cubic polynomial of integration.

Substituting the found function g into Equations (3.8)-(3.11) and splitting them with respect to e^y and y, one obtains $b_1 = 0$, which is a contradiction.

III.2 Case $b_1 = 0$

Equation (4.49) becomes:

$$\frac{g_{yyyyy}}{g_{yyyy}} = -\frac{5}{y} - b_3. \tag{4.53}$$

Further study is similar to the case $b_1 \neq 0$. Finally, one arrives at the contradiction $b_2 = 0$.

4.2.1.2 Case $b_2 = 0$

Introducing the functions:

$$b_1 = -(\mu_0' + \mu_0\mu_1 + 2\mu_3),$$

$$b_0 = -(\mu_0^2 + 2\mu_4),$$

Equations (4.42) takes the form:

$$\frac{g_{yyyyy}}{g_{yyyy}} = -\frac{2b_1 + \mu_0'}{b_1 y + b_0}. \tag{4.54}$$

I. Case $b_1 \neq 0$

In this case Equation (4.54) is rewritten:

$$\frac{g_{yyyyy}}{g_{yyyy}} = \frac{b_4}{y + b_3}, \tag{4.55}$$

where

$$b_3 = -b_0/b_1, \quad b_4 = 2 + \mu_0'/b_1.$$

Integrating of the Equation (4.55) with respect to y, one obtains:

$$g_{yyyy} = \alpha \frac{1}{(y+b_3)^{b_4}}, \tag{4.56}$$

where α is constant.

To integrate Equation (4.56) for finding the function g, one needs to consider different cases of b_4.

I.1 Case $(b_4-1)(b_4-2)(b_4-3)(b_4-4) \neq 0$

Integrating of the Equation (4.56) four times with respect to y, one has:

$$g = \alpha g_0 + P_3(y),$$

where

$$g_0 = \frac{(y+b_3)^4}{(y+b_3)^{b_4}(b_4-1)(b_4-2)(b_4-3)(b_4-4)}$$

and $P_3(y)$ is the cubic polynomial of integration.

Substituting g into Equations (3.7)-(3.9) and splitting them with respect to y, one comes to the contradiction $\xi_1' = 0$.

I.2 Case $(b_4-1)(b_4-2)(b_4-3)(b_4-4) = 0$

Substituting into the Equation (4.56) each case of b_4 solving $(b_4-1)(b_4-2)(b_4-3)(b_4-4) = 0$ and integrating it four times with respect to y, one gets:

$$g = \alpha g_0(y) + P_3(y),$$

where $P_3(y)$ is the cubic polynomial of integration, and the expression of the function $g_0(y)$ depends on b_4.

Substituting the found function g into Equations (3.8)-(3.11) and splitting them with respect to y, one obtains the contradiction $\alpha = 0$.

II. Case $b_1 = 0$

Equation (4.54) becomes:

$$b_0 g_{yyyyy} + \mu_0' g_{yyyy} = 0. \tag{4.57}$$

The analysis of this case is similar to the case defined by $b_1 \neq 0$ and also leads to the contradiction $\alpha = 0$.

Notice that this case is solved with assumption that one of b_0 and μ_0' do not vanish. The further study will consider for case that both of them are vanish.

4.2.2 Case $\xi_0 = 0$

In this case, Equation (4.38) becomes:

$$\begin{aligned} &6(\xi_1' y^2 + \mu_1 y + \mu_0) g_y - 6(3\xi_1' y + 2\mu_1) g \\ &- 2\mu_1' y^2 - \xi_1'' y^3 + 2\mu_3 y + 2\mu_4 = 0, \end{aligned} \tag{4.58}$$

Since $\xi_1' \neq 0$, one can introduce new functions:

$$a_1 = \mu_1/(2\xi_1'),$$

$$a_0 = \mu_0/\xi_1'.$$

Equation (4.12) becomes:

$$g_y = \frac{4a_1 + 3y}{y^2 + 2a_1 y + a_2} g + \frac{\xi_1'' y^3 + 4(a_1 \xi_1')' y^2 - 2\mu_3 y - 2\mu_4}{6\xi_1'(y^2 + 2a_1 y + a_2)}. \tag{4.59}$$

For integration of the Equation (4.59), one needs to consider different cases of $({a_1}^2 - a_2)$.

I. Case ${a_1}^2 - a_2 = v^2 > 0$

The general solution of the homogenous Equation (4.59) is:

$$g = g_0 \frac{(2v(h_1 - 2) + y)^{h_1}}{(2v(h_1 - 1) + y)^{h_1 - 3}},$$

where $h_1 = \frac{a_1 + 3v}{2v}$, and g_0 is constant.

For finding the general solution of the Equation (4.59), one needs to consider different cases of:

$$\lambda_0 = h_1(h_1 - 1)(h_1 - 2)(h_1 - 3).$$

I.1 Case $\lambda_0 \neq 0$

Solving the Equation (4.59), one gets the general solution of g in the form

$$g = c_0 u^{h_1} P_{13}(y) + P_{23}(y),$$

where $u = \frac{a_1 - v + y}{a_1 + v + y}$, $c_0 \neq 0$ is constant, and $P_{13}(y)$ and $P_{23}(y)$ are cubic polynomials of y.

Substituting g into Equation (4.37), and splitting it with respect to u, one obtains that $v = 0$ or $\xi_1' = 0$ which is a contraction.

I.2 Case $\lambda_0 = 0$

Solving of the Equation (4.59) for different cases of h_1 such that $\lambda_0 = 0$, one gets the general solution g. Substitution of the obtained function g into (4.37) yields ξ_1'' such that $g_{yyyy} = 0$.

II. Case ${a_1}^2 - a_2 = -v^2 < 0$

The general solution of the homogenous Equation (4.59) is:

$$g = g_0 v^4 e^{\lambda u} (\tan^2(u) + 1)^2 \sqrt{\tan^2(u) + 1} + P_3(y),$$

where $\lambda = \frac{a_1}{v}$ and $u = \arctan \frac{a_1 + y}{v}$, $g_0 \neq 0$ is constant and $P_3(y)$ is a cubic polynomial of y.

Substituting g into Equation (4.37) and splitting it with respect to y, one obtains that $g_0 = 0$, which is a contraction.

III. Case ${a_1}^2 - a_2 = 0$

The general solution of the homogenous Equation (4.59) is

$$g = g_0 e^{y(a_1+y)^{-1}} (a_1 + y)^3,$$

where g_0 is an arbitrary constant of the integration.

For finding the general solution of the Equation (4.59), one needs to consider different values of a_1.

III.1 Case $a_1 \neq 0$

The general solution of (4.59) has the form:

$$g = c_0 e^{y(a_1+y)^{-1}}(a_1+y)^3 + P_3(y),$$

where c_0 is constant, and $P_3(y)$ is a cubic polynomial with coefficients written through ξ_1', a_1, μ_0, μ_1, μ_3, μ_4, and their derivatives.

Substituting g into the Equation (4.59), and splitting it with respect to $e^{y(a_1+y)^{-1}}$ and y, one gets $c_0 = 0$ which implies that $g_{yyyy} = 0$.

III.2 Case $a_1 = 0$

The general solution of (4.59) takes the form:

$$g = \frac{36\xi_1' g_0 y^4 - 6\xi_1'' y^3 + 4\mu_3 y + 3\mu_4}{36\xi_1' y}.$$

Substituting g into the Equation (4.59) and splitting it with respect to y, one gets $\mu_4 = 0$ which implies that $g_{yyyy} = 0$.

4.3 Extension of the Generator $X = xy\partial_y$

Substituting the coefficients of the generator $X = xy\partial_y$ into Equations (3.7)-(3.9), and solving them, one obtains:

$$k = \frac{2}{x}\ln y, \quad f = y(\mu - \frac{1}{x^2}\ln^2 y), \tag{4.60}$$

where $\mu = \mu(x)$ is an arbitrary function of the integration.

For finding the admitted Lie group of Equation (3.1) the Equation (4.60) are substituted into the determining Equations (3.7)-(3.9). From these equations, the derivatives η_{xx}, η_{xy}, and η_{yy} are found. Comparing the mixed derivatives $(\eta_{xx})_y = (\eta_{xy})_x$ and $(\eta_{yy})_x = (\eta_{xy})_y$, one finds:

$$\begin{aligned}\eta_x = &(2\eta x + y(\xi_0^{(3)} x^3 + 4\xi_0' x(\ln y - \mu x^2) + \xi_1^{(3)} x^3 y - 3\xi_1' xy \ln^2 y \\ &+ 4xy\xi_1' \ln y - \xi_1' \mu x^3 y - 2\mu' \xi_0 x^3 - \mu' \xi_1 x^3 y - 2\xi_1 y \ln^3 y \\ &+ 2\xi_1 y \ln^2 y - 4\xi_0 \ln y + 2\xi_1 \mu x^2 y \ln y - 4\xi_1 y \ln y))/(2x^2), \qquad (4.61) \\ \eta_y = &(2\eta x + y(-2\xi_0' x + 3\xi_1'' x^2 y + 6xy\xi_1' \ln y - 2\xi_1' xy + 3\xi_1 y \ln^2 y \\ &+ 2\xi_0 - 3\xi_1 \mu x^2 y + 2\xi_1 y))/(2xy).\end{aligned}$$

Substituting η_x and η_y into Equation (3.9), and splitting them with respect to $\ln y$, one gets that $\xi_0 = cx$, and $\xi_1 = 0$, where c is constant.

Equation (3.8) becomes:

$$cx(x\mu' + 2\mu) = 0.$$

If μ is an arbitrary function, then $c = 0$ and $\eta = xy$, which gives no extension of the Lie algebra with the generator $xy\partial_y$. If $\mu = \frac{\gamma}{x^2}$, then this case provides an extension of generator $xy\partial_y$ by the generator $X = x\partial_x$.

CHAPTER V

CONCLUSION

This thesis deals with the group classification of second-order ordinary differential equations of the form:

$$y'' = P_3(x, y, y'), \tag{5.1}$$

where

$$P_3(x, y, y') = a(x, y)y'^3 + 3b(x, y)y'^2 + 3c(x, y)y' + d(x, y).$$

5.1 Problems

The problems considered in this thesis are related with finding all possible admitted generators of equation • of the form (5.1).

For solving the problem of the thesis the approach considered in Ovsiannikov (2004) is used, where the criteria of equivalence Lie group of transformations and admitted Lie group are applied. This approach contains the following steps.

(1) Separate Equation (5.1) into classes according to the form of the admitted generator using the concept of equivalence transformations.

(2) Simplify the functions $k(x, y)$ and $f(x, y)$ by equivalence transformations.

(3) Solve the determining equations for the chosen functions $k(x, y)$ and $f(x, y)$.

5.2 Results

The results of the group classification of Equation (5.1) are presented in Table 5.1, where the first and second columns present the form of functions $k(x, y)$

and $f(x,y)$ of the studied equation, the third column lists the simplified generator obtained in Chapter III, and the last column presents the extension of the simplified generator, if there is one. Thus, the admitted Lie group have dimension one or two only.

Table 5.1 The group classification of $y'' = k(x,y)y' + f(x,y)$.

k	f	X_1	X_2
$k(y)^*$	$f(y)^*$	∂_x	
$\ln y$	$-(\ln^2 y - \ln y + 1)\frac{y}{4} + c_1 y$	∂_x	$e^x y\partial_y$
y	$ay^3 + by^2 + cy + d,$	∂_x	$e^{\alpha x}(\partial_x - (3\alpha^2 + \alpha y)\partial_y)$
	$a \neq -\frac{1}{9}, b \neq 0$		
y	$ay^3 + by^2, (a \neq -\frac{1}{9}, b \neq 0)$	∂_x	$x\partial_x - y\partial_y$
$e^y + \alpha$	$c_2 e^{2y} - \alpha e^y - \alpha^2$	∂_x	$e^{\alpha x}(\partial_x + \alpha\partial_y)$
e^y	$c_2 e^{2y}$	∂_x	$x\partial_x - \partial_y$
$y^{\lambda_0} + \alpha$	$\frac{\alpha}{y^2} + \frac{1}{y^2} + 2\alpha^2 y$	∂_x	$e^{-3\alpha x}(\partial_x - \alpha y\partial_y)$
y^{λ_0}	$\frac{c_0}{y^2}$	∂_x	$3x\partial_x + y\partial_y$
$y^2 + \alpha y + \beta$	$c_1(\alpha + 2y)^5 + \frac{\lambda}{8}(\alpha + 2y)^3$	∂_x	$e^{\lambda x}(4\partial_x - \lambda(2y + \alpha)\partial_y)$
	$-\frac{3\lambda^2}{16}y - \frac{3\alpha\lambda^2}{64}$		
$y^2 + \alpha y + \frac{\alpha^2}{4}$	$-\frac{(\alpha+2y)^5}{288}$	∂_x	$4x\partial_x - (2y + \alpha)\partial_y$
$\frac{\phi_{yy}}{2}$	$-\frac{\phi_{yyy} + \phi_{yy}\beta}{6}$	∂_x	$(y + \beta x)\partial_x$
g_{0yy}	$\frac{4\alpha g_{0y} - g_{0yy}(\alpha y + \beta) + 3\alpha^2 y + \alpha_0}{3}$	∂_x	$e^{\alpha x}(\alpha y + 1)\partial_x$
$3g_{yy}$	$4g_{xy} - yg_{xyy} - 6gg_{yyy}$	$y\partial_x$	
		$+6g\partial_y$	
$\frac{2}{x}\ln y$	$y(\mu(x)^* - \frac{1}{x^2}\ln^2 y)$	$xy\partial_y$	
$\frac{2}{x}\ln y$	$\frac{y}{x^2}(\gamma - \ln^2 y)$	$xy\partial_y$	$x\partial_x$

The asterisk indicates that the functions are arbitrary, $\lambda_0 \neq 0, 1, 2,$ $\lambda = \frac{\alpha^2 - 4\beta}{8}$, and the functions g_0, g, and ϕ satisfy the conditions:

$$g_{0yyyy}(\alpha y+\beta)^2 - 4{g_{0yy}}^2(\alpha y+\beta) + g_{0yy}(-\alpha^2 y - 15\alpha\beta + \alpha_0)$$
$$+ 4\alpha g_{0yy}g_{0y} + 12\alpha^2 g_{0y} + 3\alpha(-3\alpha\beta + \alpha_0) = 0,$$
$$12ygg_{xyyy} - 6yg_y g_{xyy} - 18g_y g_{xyy} + 24g_y g_{xy} + 6g_{xx}$$
$$+ y^2 g_{xxyy} + 6yg_x g_{yyy} - 18g_x g_{yy} + 36g^2 g_{yyyy} = 0,$$
$$\phi_{yyyy}\phi_{yy} + 3\phi_{yyy}\phi\beta - \phi_{yy}\beta(\phi_y + 2\beta) = 0.$$

5.3 Further Research

The further study will concern with the Equation (4.57), where b_0 and μ_0' will be vanished together.

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