# Rotationally Invariant and Partially Invariant Flows of a Viscous Incompressible Fluid and a Viscous Gas

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**Abstract**. In this manuscript we study the Navier-Stokes equations and viscous gas dynamics equations. These equations play a central role in much of the research within applied mathematics, physics and engineering. One of the questions that we study here is the existence of solutions of special vortex type for the Navier-Stokes equations and viscous gas dynamics equations. This type of solution for the inviscid gas and fluid dynamics equations was introduced by L.V.Ovsiannikov [1]. Note that this solution is partially invariant with respect to group of rotations $O(3)$. Another part of our study is devoted to the group classification of spherically symmetric viscous gas dynamics equations. The approach used is classical group analysis. We use the notions of invariant and partially invariant solutions.

**Key words:** Invariant and partially invariant solutions, group classification, Navier-Stokes and viscous gas equations.

## 1 Introduction

The mathematical models of many real world phenomena are formulated in the form of differential equations. One of the methods for studying the properties of differential equations is group analysis. Differential equations usually contain parameters or functions that are determined experimentally and hence are not strictly fixed. Group analysis not only helps to construct exact solutions, but also to classify the differential equations with respect to these arbitrary elements.

Many of the invariant solutions of the Navier-Stokes equations have been known for a long time; however their systematic analysis became possible only with the development[1] of the modern methods for the group analysis of differential equations [4]. The first group classification of the Navier-Stokes equations in the three-dimensional case was done in [5]. It was shown that the Lie group admitted by the Navier-Stokes equations is infinite-dimensional. There is still no classification of

[1]A historical review of a group analysis develpoment can be found in [2]. Many results of the group analysis are collected in [3]

this group. Several papers [6–11][2] are devoted to invariant solutions of the Navier-Stokes equations. Partially invariant solutions of the Navier–Stokes equations have been less studied [6, 15]. At the same time there has been progress in studying such classes of solutions of inviscid gas dynamics equations [4, 16, 17]. Recently, L.V.Ovsiannikov [1] found one class of partially invariant solutions, called a special vortex. This solution is based on the group of rotations $O(3)$. An ideal fluid and an inviscid gas have the same class of solutions. Therefore, it is natural to investigate the existence of special vortex type solutions for the Navier-Stokes equations and viscous gas dynamics equations.

As is well-known, the main difficulty in the study of partially invariant solutions is the analysis of the compatibility [18, 19] of the appearing overdetermined systems. The analysis of compatibility can be reduced to the consecutive performance of algebraic operations of symbolic nature. These operations are connected with a prolongation of the system, substitution of composite expressions (transition onto manifold), and finding ranks of matrices. Typically, the compatibility study of systems of partial differential equations requires a large amount of analytical calculations, and it is necessary to use a computer system for these calculations. Here we used the system REDUCE [20].

Another part of our study is devoted to the group classification of spherically symmetric viscous gas dynamics equations. The group classification problem consists of searching for admitted groups of transformations admitted by the system for all arbitrary elements and all specifications of arbitrary elements. By special choice of the arbitrary elements one can extend the admitted group.

After finding the admitted group one can try to construct exact solutions: every subgroup of the admitted group can be a source of invariant or partially invariant solutions. There is an infinite number of subgroups[3], even in cases where the admitted groups are finite-dimensional. But if two subgroups are similar, i.e., they are connected with each other by a symmetry transformation, then their corresponding invariant solutions are connected with each other by the same transformation. Since the set of subgroups can be divided into classes of similar subgroups, therefore, it is sufficient to find only one representative solution from each similar class of subgroups. A set of representatives of equivalent subgroup classes is called an optimal system of subgroups. In this manuscript we give representations of all invariant solutions with respect to subgroups of two-dimensional admitted groups of spherically symmetric viscous gas dynamics equations.

We should also note here that, as for the Navier-Stokes equations, many of the invariant solutions of the viscous gas dynamics equations have been obtained by other methods [21–29]. The group classification of the viscous gas equations (in case when the first $\lambda$ and the second $\mu$ coefficients of viscosity are related by the equation $\lambda = -2\mu/3$) was done in [30]. For some models of viscous gas dynamics equations, group analysis was used in [31, 32]. There also exist other similar approaches for

[2]Short reviews devoted to invariant solutions of the Navier-Stokes equations can be found in [6, 12, 13, 14].

[3]Because there is a one-to-one correspondence between groups and Lie algebras one can study the Lie algebra of the admitted group.

constructing exact solutions of the Navier–Stokes equations. We note here two of them: nonclassical symmetry reductions [33, 14] and linear profile of velocity [34].

# 2 Viscous Gas Equations

## 2.1 Coordinateless form of viscous gas equations

In this manuscript we study unsteady viscous gas dynamics equations. These equations govern a three-dimensional motion of a compressible, thermal conductive, Newtonian viscous gas flow

$$\frac{d\mathbf{v}}{dt} = \tau\, div(P),\ \frac{d\tau}{dt} - \tau div(\mathbf{v}) = 0,$$

$$\frac{d\varepsilon}{dt} = \tau\, P : D + \tau\, div(\kappa\nabla T).$$

Here $\tau = 1/\rho$ is a specific volume, $\rho$ is a density, $\mathbf{v}$ is a velocity, $P$ is a stress tensor, $D = \frac{1}{2}\left(\frac{\partial\mathbf{v}}{\partial\mathbf{x}} + (\frac{\partial\mathbf{v}}{\partial\mathbf{x}})^*\right)$ is a rate-of-strain tensor, $\varepsilon$ is an internal energy, $T$ is a temperature, $\kappa$ is a coefficient of a heat conductivity. The Stokes axioms for a viscous gas give

$$P = (-p + \lambda div(v))I + 2\mu D,$$

where $p$ is a pressure, $\lambda$ and $\mu$ are the first and the second coefficients of viscosity, respectively. A viscous gas is a two parametric media. As the main thermodynamic variables we choose the pressure $p$ and specific volume $\tau$: the entropy $\eta$, the internal energy $\varepsilon$ and the temperature $T$ are functions of the pressure and specific volume

$$\eta = \eta(p,\tau),\ \varepsilon = \varepsilon(p,\tau),\ T = T(p,\tau).$$

The first and the second thermodynamic laws require for these functions to satisfy the equations

$$\eta_p = \frac{\varepsilon_p}{T},\ \eta_\tau = \frac{\varepsilon_\tau + p}{T},\ 3\lambda + 2\mu \geq 0,\ \mu \geq 0,\ \kappa \geq 0.$$

For the simplicity of classification we study case, which corresponds to an essentially viscous and heat conductive gas

$$\mu \neq 0,\ \kappa \neq 0.$$

Thus, the studying viscous gas dynamics equations are

$$\frac{d\mathbf{v}}{dt} + \tau\nabla p = \tau\left((\lambda+\mu)\nabla(\, div(\mathbf{v})) + (div(\mathbf{v}))\nabla\lambda + \mu\triangle\mathbf{v} + 2D(\nabla\mu)\right), \quad (1)$$

$$\frac{d\tau}{dt} - \tau div(\mathbf{v}) = 0,$$

$$\frac{dp}{dt} + A(p,\tau)div(\mathbf{v}) = B(p,\tau)\left(\lambda(\, div(\mathbf{v}))^2 + 2\mu D : D + (\nabla\kappa)(\nabla T) + \kappa\triangle T\right)$$

with the functions

$$A = \frac{\tau(\varepsilon_\tau + p)}{\varepsilon_p}, \ B = \frac{\tau}{\varepsilon_p}.$$

Note that the internal energy and entropy can be expressed through the functions $A = A(p, \tau), \ B = B(p, \tau)$ by formulae

$$\varepsilon_p = \frac{\tau}{B}, \ \varepsilon_\tau = \frac{A}{B} - p, \ \eta_p = \frac{\tau}{BT}, \ \eta_\tau = \frac{A}{BT}.$$

The conditions $\varepsilon_{p\tau} = \varepsilon_{\tau p}, \ \eta_{p\tau} = \eta_{\tau p}$ lead to the restrictions

$$\begin{gathered} \tau B_\tau + BA_p - AB_p = B^2 + B, \\ \tau T_\tau = AT_p - TB. \end{gathered} \tag{2}$$

In the case of an ideal gas (i.e., the gas that obeys the Clapeyron equation $T = R^{-1}p\tau$) there are $B = B(\tau p), \ A = p(1 + B(\tau p))$ with an arbitrary function $B(\tau p)$. For a polytropic gas $\varepsilon = (\gamma - 1)^{-1}\tau p$ and this one more simplifies the functions $A$ and $B$: $B = (\gamma - 1), \ A = \gamma p$. Here $R$ is the gas constant and $\gamma$ is a polytropic exponent. Note also that the Navier-Stokes equations are obtained from the viscous gas equations by assuming that the second coefficient of viscosity $\mu$ and the density $\rho$ (or the specific volume $\tau$) are constants.

**Remark**. In the case of constant $\tau$ and $\mu$ system (1) is split on two systems: the Navier-Stokes equations and the energy equation.

## 2.2 Spherical coordinate system

Equations (1) are written in coordinateless form. For applications one needs to write them in some coordinate system. Because our goal is to study solutions of viscous gas equations connected with the group of rotations, then it is convenient to use a spherical coordinate system.

The spherical coordinates $(r, \theta, \varphi)$ and the Cartesian coordinates $(x, y, z)$ of the point $\mathbf{x} \in R^3$ are introduced by the formulae

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

Corresponding physical components of the velocity vector $\mathbf{v}$ in the spherical coordinate system $(U, V, W)$ and $(u, v, w)$ in the Cartesian coordinate system are related by the expressions

$$u = U\sin\theta\cos\varphi + V\cos\theta\cos\varphi - W\sin\varphi,$$

$$v = U\sin\theta\sin\varphi + V\cos\theta\sin\varphi - W\cos\varphi,$$

$$w = U\sin\theta - V\sin\theta.$$

Note that the vector $(V, W)$ can be described by its modules $H$ and by the angle $\omega$:

$$V = H\cos\omega, \ W = H\sin\omega. \tag{3}$$

For the spherical coordinate system the fundamental tensor is diagonal:

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix};\ (g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix};\ |g| = det(g_{ij}) = r^4 \sin^2\ \theta.$$

The Christoffel's symbols

$$\Gamma^l_{ij} = \frac{1}{2} g^{ls} \left( \frac{\partial g_{is}}{\partial K^j} + \frac{\partial g_{js}}{\partial K^i} - \frac{\partial g_{ij}}{\partial K^s} \right)$$

are (we write down only nonvanishing symbols)

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},\ \Gamma^1_{22} = -r,\ \Gamma^3_{23} = \Gamma^3_{32} = \cot\ \theta,$$

$$\Gamma^1_{33} = -r\ \sin^2\ \theta,\ \Gamma^2_{33} = -\sin\ \theta\ \cos\ \theta.$$

Here $K^1 = r;\ K^2 = \theta;\ K^3 = \varphi$ and there is a summation with respect to a repeat index.

Tensor components of the vector $\mathbf{v}$ are

$$(v^1, v^2, v^3) = (U, \frac{V}{r}, \frac{W}{r\ \sin\ \theta}),\quad (v_1, v_2, v_3) = (U, rV, r\ \sin\ \theta\, W).$$

Coordinates of a gradient of any scalar function $F$ are

$$(\nabla F)_1 = (\nabla F)^1 = \frac{\partial F}{\partial r},\quad (\nabla F)_2 = \frac{\partial F}{\partial \theta},\ (\nabla F)^2 = \frac{1}{r^2} \frac{\partial F}{\partial \theta},$$

$$(\nabla F)_3 = \frac{\partial F}{\partial \varphi},\quad (\nabla F)^3 = \frac{1}{r^2\ \sin^2\ \theta} \frac{\partial F}{\partial \varphi}.$$

A matrix of the covariant derivatives is (here $i$ is a number of a row, $j$ is a number of a column)

$$\left( v^i_{,j} \right) = \begin{bmatrix} \frac{\partial U}{\partial r} & \frac{\partial U}{\partial \theta} - V & \frac{\partial U}{\partial \varphi} - \sin\theta W \\ \frac{1}{r}\frac{\partial V}{\partial r} & \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r} & \frac{1}{r}\frac{\partial V}{\partial \varphi} - \frac{\cos\theta}{r} W \\ \frac{1}{r\sin\theta}\frac{\partial W}{\partial r} & \frac{1}{r\sin\theta}\frac{\partial W}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial W}{\partial \varphi} + \frac{U}{r} + \frac{\cot\theta}{r} V \end{bmatrix}$$

Coordinates of the rate-of-strain tensor $D = \frac{1}{2}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + (\frac{\partial \mathbf{v}}{\partial \mathbf{x}})^*\right)$ are

$$2D^{j\cdot}_{\cdot i} = v^j_{,i} + v^\alpha_{,\beta} g_{i\alpha} g^{j\beta}.$$

Hence,

$$2D : D = v^j_{,i} v^i_{,j} + v^i_{,j} v^\alpha_{,\beta} g_{i\alpha} g^{j\beta}.$$

For the divergency there is

$$div\,\mathbf{v} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial K^i} \left( \sqrt{|g|} v^i \right) = \frac{1}{r^2} \frac{\partial (r^2\, U)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \sin\theta\, V}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial W}{\partial \varphi}.$$

The Laplace operator of a scalar function is

$$\triangle F = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial K^i} \left( \sqrt{|g|} g^{is} \frac{\partial F}{\partial K^s} \right) =$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 F}{\partial \varphi^2}.$$

The Laplace operator $\triangle \mathbf{v}$ of the vector $\mathbf{v}$ has coordinates

$$(\triangle \mathbf{v})^l = \triangle \left( v^l \right) + 2\, g^{ij} \Gamma^l_{is} \frac{\partial v^s}{\partial K^j} + g^{ij} \left( \frac{\partial \Gamma^l_{ip}}{\partial K^j} + \Gamma^s_{ip} \Gamma^l_{js} - \Gamma^s_{ij} \Gamma^l_{ps} \right) v^p,$$

which for the spherical coordinate system are

$$(\triangle \mathbf{v})^1 = \triangle (U) - \frac{2}{r} \frac{\partial V}{\partial \theta} - \frac{2}{r^2 \sin\theta} \frac{\partial W}{\partial \varphi} - \frac{2U}{r^2} - \frac{2 \cot\theta}{r^2} V,$$

$$(\triangle \mathbf{v})^2 = \triangle \left( \frac{V}{r} \right) + \frac{2}{r^2} \frac{\partial V}{\partial r} + \frac{2}{r^3} \frac{\partial U}{\partial \theta} - \frac{2 \cot\theta}{r^3 \sin\theta} \frac{\partial W}{\partial \varphi} - \frac{V}{r^3 \sin^2\theta},$$

$$(\triangle \mathbf{v})^3 = \triangle \left( \frac{W}{r \sin\theta} \right) + \frac{2}{r \sin\theta} \frac{\partial}{\partial r} \left( \frac{W}{r} \right) + \frac{2 \cot\theta}{r^3} \frac{\partial}{\partial \theta} \left( \frac{W}{\sin\theta} \right) +$$
$$+ \frac{2}{r^3 \sin^2\theta} \frac{\partial U}{\partial \varphi} + \frac{2 \cot\theta}{r^3 \sin^2\theta} \frac{\partial V}{\partial \varphi}.$$

The acceleration vector $\dfrac{d\mathbf{v}}{dt}$ has the components

$$\left( \frac{d\mathbf{v}}{dt} \right)^1 = \mathcal{D}(U) - \frac{W^2 + V^2}{r}, \quad \left( \frac{d\mathbf{v}}{dt} \right)^2 = \frac{1}{r} \mathcal{D}(V) + \frac{UV - \cot\theta\, W^2}{r^2},$$

$$\left( \frac{d\mathbf{v}}{dt} \right)^3 = \frac{1}{r \sin\theta} \mathcal{D}(W) + \frac{UW + \cot\theta\, WV}{r^2 \sin\theta},$$

where

$$\mathcal{D}(f) = \frac{\partial f}{\partial t} + U \frac{\partial f}{\partial r} + \frac{V}{r} \frac{\partial f}{\partial \theta} + \frac{W}{r \sin\theta} \frac{\partial f}{\partial \varphi}.$$

A substitution of the presented coordinates of the tensors and the vectors into system (1) gives equations of a viscous thermoconductive gas in the spherical coordinate system. These equations are very cumbersome[4].

[4]All symbolic calculations for the coordinates of the tensors and the vectors were made on computer with the help of the system REDUCE [20].

# 3 Partially Invariant Solution

In the space of variables $t, r, \theta, \varphi, U, H, \omega, \tau, p$ the group of rotations $O(3)$ has the generators [1]

$$X = -\sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi + \cos\varphi(\sin\theta)^{-1}\partial_\omega,$$
$$Y = \cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi + \sin\varphi(\sin\theta)^{-1}\partial_\omega,$$
$$Z = \partial_\varphi.$$

Invariants of this group are $t, r, U, H, \tau, p$.

The rank of the Jacobi matrix of the invariants with respect to the dependent functions is equal to four. Therefore, according to [4] there are no nonsingular invariant solutions that are invariant with respect to group of rotations $O(3)$. A minimal possible defect of a partially invariant with respect to $O(3)$ solution is equal to one. In this case a representation of the partially invariant solution is

$$\tau = \tau(t,r),\ U = U(t,r),\ H = H(t,r),\ p = p(t,r),\ \omega = \omega(t,r,\theta,\varphi). \qquad (4)$$

The function $\omega(t, r, \theta, \varphi)$ is "superfluous": it depends on all independent variables. Note that if $H = 0$, then by virtue of (3) the tangent component of the velocity vector is equal to zero and it corresponds to spherically symmetric flows that are considered in the next section. In this section it is assumed that $H \neq 0$.

All analytic calculations for the viscous gas dynamics and the Navier-Stokes equations are done in the REDUCE system [20]. The result of these calculation is: class of solutions that is partially invariant with respect to $O(3)$ is confined by spherically symmetric solutions.

## 3.1 Analysis of compatibility of partially invariant solutions

For the sake of simplicity we present here analysis of compatibility of partially invariant solution for the Navier-Stokes equations, i.e., when $\tau$ and $\mu$ are constants. Analysis of compatibility for the viscous gas dynamics equations is similar, but it needs more cumbersome symbolic calculations.

After substituting the representation of the partially invariant solution (4) into the Navier-Stokes equations[5] and some combinations of the second and the third equations the initial system can be split on two subsystems: the invariant system

$$D_0U + p_r = r^{-1}H^2 + (U_{rr} + 4r^{-1}U_r + 2r^{-2}U) \qquad (5)$$

with the operator $D_0 = \partial_t + U\partial_r$ and the supplementary system

$$\begin{gathered} D_0(rH) = (rH)_{rr} - (r\sin^2\theta)^{-1}H - rH(\omega_r^2 + r^{-2}\omega_\theta^2 + \\ +(r\sin\theta)^{-2}\omega_\varphi^2 + 2(r^2\sin\theta)^{-1}\cot\theta\omega_\varphi), \\ D_0\omega + (r\sin\theta)^{-1}H(\sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi + \cos\theta\sin\omega) = \\ = \omega_{rr} + 2(rH)^{-1}(rH)_r\omega_r + r^{-2}\omega_{\theta\theta} + r^{-2}\cot\theta\omega_\theta + (r\sin\theta)^{-2}\omega_{\varphi\varphi}, \\ \sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi = \cos\theta\cos\omega + \sin\theta(rH)^{-1}(r^2U)_r. \end{gathered} \qquad (6)$$

[5]Here we use dimensionless representation of the Navier-Stokes equations in which one can account that $\mu = 1$ and $\tau = 1$.

For the analysis of compatibility of system (5),(6) it is convenient to use implicit representation for the function $\omega = \omega(t,r,\theta,\varphi)$ in the form

$$F(\omega,t,r,\theta,\varphi) = 0, \quad (F_\omega \neq 0).$$

All derivatives of the function $\omega(t,r,\theta,\varphi)$ can be calculated through the derivatives of the function $F(\omega,t,r,\theta,\varphi)$. For example, for the first derivatives we have

$$\omega_t = -F_t/F_\omega, \ \omega_r = -F_r/F_\omega, \ \omega_\theta = -F_\theta/F_\omega, \ \omega_\varphi = -F_\varphi/F_\omega.$$

Then the last equation of (6) becomes

$$\sin\theta\sin\omega F_\theta - \cos\omega F_\varphi + F_\omega(\cos\theta\cos\omega + k\sin\theta) = 0,$$

where the function $k = (rH)^{-1}(r^2U)_r$ only depends on $t$ and $r$. Note that for a viscous gas dynamics equations there is the same equation with the function $k(t,r) = (Hr\tau)^{-1}(-rD_0\tau + \tau(r^2U)_r$. The general solution of the last equation is

$$F = \Phi\left(\varphi + arctan(\frac{\sin\omega}{k\sin\theta + \cos\theta\cos\omega}), \sin\theta\cos\omega - k\cos\theta, t, r\right).$$

Here the function $\Phi = \Phi(y_1, y_2, t, r)$ is an arbitrary function of the arguments $t, r$ and

$$y_1 = \varphi + arctan(\frac{\sin\omega}{k\sin\theta + \cos\theta\cos\omega}), \ y_2 = \sin\theta\cos\omega - k\cos\theta.$$

All further intermediate calculations in studying the compatibility of overdetermined system ( 5), ( 6) were made on computer in the system REDUCE [20]. Here we give the way of computing and the final results.

Note that the Jacobian $\frac{\partial(y_1,y_2,\theta,t,r)}{\partial(\omega,\theta,\varphi,t,r)} \neq 0$, therefore one can choose $(y_1, y_2, \theta, t, r)$ as the new independent variables. All derivatives of the function $\omega(t,r,\theta,\varphi)$ can be written through the derivatives of the function $\Phi(y_1, y_2, t, r)$. After that the second equation of (8) accepts the form

$$\sin\omega G_1(y_1, y_2, t, r, \theta) + G_2(y_1, y_2, t, r, \theta) = 0,$$

where the functions $G_1(y_1, y_2, t, r, \theta)$ and $G_2(y_1, y_2, t, r, \theta)$ do not include $\omega$ and its derivatives. In the last equation $\sin\omega$ can be excluded by using the trigonometry identity:

$$G_1^2(1 - (y_2 + k\cos\theta)^2) - G_2^2(1 - \cos^2\theta) = 0,$$

where the equality $\cos\omega = \sin^{-1}\theta(y_2 + k\cos\theta)$ found from the representation of $y_2$ was applied.

Further calculations show that the last equation depends on $\theta$ as the polynomial of the degree 8 with respect to $\cos\theta$:

$$P_8 = \sum_{k=0}^{8} a_k \cos^k\theta = 0.$$

The coefficients $a_k$, $(k = 0, 1, \dots, 8)$ only depend on $y_1, y_2, t, r$ and do not depend on $\theta$. This allows splitting the equation with respect to $\cos\theta$: $a_k = 0$, $(k = 1, 2, \dots, 8)$.

The equality $a_8 = 0$ gives

$$D_0 h = h_{rr} + h(k^2+1)^{-1} h_r, \tag{7}$$

where $h = rH$. Substituting $h_t$ found from (7) into $a_6 = 0$, we obtain

$$k_r \left( (k^2+1)\Phi_r + k k_r y_2 \Phi_{y_2} \right) = 0. \tag{8}$$

If $(k^2+1)\Phi_r + k k_r y_2 \Phi_{y_2} = 0$, then the equation $a_4 = 0$ gives the equation $y_2^2 - (k^2+1) = 0$ or

$$(\sin\theta\cos\omega - k\cos\theta)^2 = k^2 + 1.$$

Note that substituting the representation of the function $\omega(t, r, \theta, \varphi)$ found from this equation into ( 5), ( 6) and splitting them with respect to $\cos\theta$ gives the expression $H = 0$ that contradicts the assumption about $H$.

For the second case in (8), when $k_r = 0$ we will obtain a contradiction with the help of the first equation of (6). Really, the same study of the first equation of (6) as for the second equation leads to the polynomial of the degree 10 with respect to $\cos\theta$:

$$P_{10} = \sum_{k=0}^{10} b_k \cos^k\theta = 0,$$

where the coefficients $b_k$, $(k = 0, 1, ..., 10)$ only depend on $y_1, y_2, t, r$. The equality $b_{10} = 0$ gives

$$k_t = r^{-2} h(k^2+1). \tag{9}$$

By virtue of $k_r = 0$, ( 9) and the definition of $k = (r^2 U)_r / h$ one can obtain that

$$h(t, r) = 3c(t) r^2, \quad r^2 U(t, r) = k(t) c(t) r^3 + \lambda(t),$$

where $c(t) = (k^2(t)+1)^{-1} k'(t)/3$. Substitution of this representation into ( 7) and splitting it with respect to $r$ gives $c(t) = 0$ that contradicts the assumption $H \neq 0$.

Similar calculations have been done for the viscous gas dynamics equations.

The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal incompressible inviscid fluid dynamics equations, are only spherically symmetric solutions.

# 4 Spherically Symmetric Flows of a Viscous Gas

The case $H = 0$ corresponds to a spherically symmetric flow of a viscous gas. According to the definitions of the group analysis it is a singular invariant solution with respect to group of rotations $O(3)$. The viscous gas dynamics equations in this case are

$$\begin{aligned} D_0\tau - \tau(U_r + 2r^{-1}U) = 0, \\ D_0U + \tau p_r = \tau(\lambda + 2\mu)(U_{rr} + 2r^{-1}U_r - 2r^{-2}U) + 6\tau(\mu_\tau\tau_r + \mu_p p_r) + \\ +\tau(U_r + 2r^{-1}U)(\lambda_\tau\tau_r + \lambda_p p_r), \\ D_0p + A(U_r + 2r^{-1}U) = B[\lambda(U_r + 2r^{-1}U)^2 + 2\mu(U_r^2 + 2r^{-2}U^2) + \\ +\kappa(T_{\tau\tau}\tau_r^2 + 2T_{\tau p}\tau_r p_r + T_{pp}p_r^2 + T_\tau(\tau_{rr} + 2r^{-1}\tau_r) + \\ +T_p(p_{rr} + 2r^{-1}p_r) + (\kappa_p p_r + \kappa_\tau\tau_r)(T_\tau\tau_r + T_p p_r)], \end{aligned} \tag{10}$$

where $D_0 = \partial_t + U\partial_r$. In this section we study a group classification of equations (10) with respect to the arbitrary elements $A$ , $B$, $\lambda$, $\mu$, $\kappa$, $T$.

## 4.1 Equivalence transformations

The first stage of group classification requires determining a group of equivalence transformations of equations (10). An equivalence transformation is a nondegenerate change of dependent and independent variables and arbitrary elements, which transforms any system of differential equations of a given class to the system of equations of the same class. It allows using the simplest representation of given equations. Here we give a construction of the group of equivalence transformations without restrictions on the representation of equivalence transformations [4]. We follow the approach for the calculation of equivalence transformations developed in [35].

Since arbitrary elements satisfy restrictions (2) and $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$, then for calculating an equivalence group of transformations we have to append the equations

$$\begin{gathered} A_r = 0, A_t = 0, A_U = 0, B_r = 0, B_t = 0, B_U = 0, \\ \lambda_r = 0, \lambda_t = 0, \lambda_U = 0, \mu_r = 0, \mu_t = 0, \mu_U = 0, \\ \kappa_r = 0, \kappa_t = 0, \kappa_U = 0, T_r = 0, T_t = 0, T_U = 0 \end{gathered}$$

to equations (10). All coefficients of the infinitesimal generator of the equivalence group

$$X^e = \zeta^r\partial_r + \zeta^t\partial_t + \zeta^U\partial_U + \zeta^\tau\partial_\tau + \zeta^p\partial_p + \zeta^A\partial_A + \zeta^B\partial_B + \zeta^\lambda\partial_\lambda + \zeta^\mu\partial_\mu + \zeta^\kappa\partial_\kappa + \zeta^T\partial_T$$

are dependent on all independent, dependent variables and arbitrary elements

$$r,\ t,\ U,\ \tau,\ p,\ A,\ B,\ \lambda,\ \mu,\ \kappa,\ T.$$

With the following notation:

$$u^1 = U,\ u^2 = \tau,\ u^3 = p,\ a^1 = A,\ a^2 = B,\ a^3 = \lambda,\ a^4 = \mu,\ a^5 = \kappa,\ a^6 = T$$

and

$$z^1 = r,\ z^2 = t,\ z^3 = U,\ z^4 = \tau,\ z^5 = p, a^k_\beta = \frac{\partial a^k}{\partial z^\beta}, a^k_{j\beta} = \frac{\partial^2 a^k}{\partial z^j \partial z^\beta},$$

the coefficients of the prolonged operator

$$\bar{X}^e = X^e + \sum_i (\zeta^{u^i_r}\partial_{u^i_r} + \zeta^{u^i_t}\partial_{u^i_t}) + \sum_{k,j} \zeta^{a^k_{zj}}\partial_{a^k_{zj}} + \dots$$

can be constructed with the prolongation formulae:

$$\zeta^{u^i_r} = D_r\zeta^{u^i} - u^i_r D_r\zeta^r - u^i_t D_r\zeta^t,\ \zeta^{u^i_t} = D_t\zeta^{u^i} - u^i_r D_t\zeta^r - u^i_t D_t\zeta^t,$$

$$\zeta^{u^i_{rr}} = D_r\zeta^{u^i_r} - u^i_{rr} D_r\zeta^r - u^i_{rt} D_r\zeta^t.$$

$$\zeta^{a^k_\beta} = D^e_{z^\beta}\zeta^{a^k} - \sum_{\alpha=1}^{5} a^k_\alpha D^e_{z^\beta}\zeta^{z^\alpha},\ \zeta^{a^k_{j\beta}} = D^e_{z^\beta}\zeta^{a^k_j} - \sum_{\alpha=1}^{5} a^k_{j\alpha} D^e_{z^\beta}\zeta^{z^\alpha}.$$

Here the operators $D_r,\ D_t$ denote the total derivative operators with respect to $r$ and $t$, respectively. For example,

$$D_r = \partial_r + \sum_\alpha u^\alpha_r\partial_{u^\alpha} + \sum_i (a^i_r + \sum_j a^i_{u^j} u^j_r)\partial_{a^i} + \dots$$

When we use the operator $D^e_{z^j}$ we consider $z^1, \dots, z^5$ as independent variables and $a^1, ..., a^6$ as dependent variables, we obtain:

$$D^e_{z^j} = \partial_{z^j} + \sum_i a^i_{z^j}\partial_{a^i} + \dots .$$

All necessary calculations here as in the previous sections were carried on a computer using the symbolic manipulation program REDUCE [20].

The calculations showed that the group of equivalence transformations of equations (10) corresponds to Lie algebra with generators

$$X^e_1 = \partial_t,\ X^e_2 = \partial_p,\ X^e_3 = r\partial_r + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa,$$
$$X^e_4 = r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa,\ X^e_5 = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

**Remark**. If instead of the functions $A(p,\tau), B(p,\tau)$ one considers the internal energy $\varepsilon(p,\tau)$, then the operators $X^e_2, X^e_4$, and $X^e_5$ are changed to

$$X^e_2 = \partial_p - \tau\partial_\varepsilon,\ X^e_4 = r\partial_r + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa + 2\varepsilon\partial_\varepsilon,$$

$$X^e_5 = -\tau\partial_\tau + p\partial_p + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

and there is one more generator $X^e_6 = \partial_\varepsilon$.

**Remark**. By a direct checking one can obtain that in the general case[6] (equations (1)) the equivalence group includes the transformations with the generators

$$X^e_1 = \partial_t,\ X^e_2 = \partial_p,$$
$$X^e_3 = \mathbf{x}\partial_\mathbf{x} + t\partial_t + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa,$$
$$X^e_4 = \mathbf{x}\partial_\mathbf{x} + u\partial_u + 2\tau\partial_\tau + 2\kappa\partial_\kappa,$$
$$X^e_5 = -\tau\partial_\tau + p\partial_p + A\partial_A + \lambda\partial_\lambda + \mu\partial_\mu + \kappa\partial_\kappa.$$

There are also other generators, for example, that correspond to the Galilei transformations and to the rotations in the three-dimensional case.

[6]Group classification of three-dimensional viscous gas dynamics equations with $\lambda = -2\mu/3$ was studied in [30].

## 4.2 Admitted group

Finding an admitted group consists of seeking solutions of determining equations [4]. We are looking for the generator

$$X = \zeta^r \partial_r + \zeta^t \partial_t + \zeta^U \partial_U + \zeta^\tau \partial_\tau + \zeta^p \partial_p$$

with the coefficients depending on $r, t, U, \tau, p$. Calculations lead to the following result.

The kernel of the fundamental Lie algebra is made up of the generator

$$X = \partial_t.$$

Extension of the kernel of the main Lie algebra occurs by specializing the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. Note that the functions $A = A(p,\tau), B = B(p,\tau), T = T(p,\tau)$ have to satisfy equations (2). There are three types of the generators admitted by system (10). Further $\alpha, \beta$ and $\delta$ are arbitrary constants.

**Type (a)**. If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{gathered} \alpha\tau A_\tau + A_p = 0,\ \alpha\tau B_\tau + B_p = 0, \\ \alpha\tau\mu_\tau + \mu_p = \beta\mu,\ \alpha\tau\lambda_\tau + \lambda_p = \beta\lambda, \\ \alpha\tau T_\tau + T_p = \delta T,\ \alpha\tau\kappa_\tau + \kappa_p = (-\delta + \alpha + \beta)\kappa, \end{gathered} \tag{11}$$

then there is one more admitted generator:

$$Y_a = \alpha U \partial_U + 2\alpha\tau\partial_\tau + 2\partial_p + (\alpha + 2\beta) r \partial_r + 2\beta t \partial_t.$$

The general solution of equations (11) is

$$\begin{gathered} A = A(\tau e^{-\alpha p}),\ B = B(\tau e^{-\alpha p}),\ \mu = e^{\beta p} M(\tau e^{-\alpha p}),\ \lambda = e^{\beta p}\Lambda(\tau e^{-\alpha p}), \\ T = e^{\delta p}\Theta(\tau e^{-\alpha p}),\ \kappa = e^{(-\delta+\alpha+\beta)p} K(\tau e^{-\alpha p}), \end{gathered}$$

where the functions $A(z), B(z)$ and $\Theta(z)$ satisfy the equations $(z \equiv \tau e^{-\alpha p})$

$$-\alpha z B A' + z B'(1 + \alpha A) = B^2 + B, \quad (1 + \alpha A) z \Theta' = (\delta A - B)\Theta. \tag{12}$$

The internal energy is represented by the formula

$$\varepsilon = e^{\alpha p}(\varphi(z) - zp) + \psi(p),\ \psi'(p) = C e^{\alpha p},$$

where the function $\varphi(z)$ and the constant $C$ can be accounted as arbitrary and they are related with the functions $A(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{A(z)}{B(z)}, \quad C = z + \frac{z}{B(z)} + \alpha z \varphi'(z) - \alpha\varphi(z).$$

In this case the function $\Theta(z)$ has to satisfy the equation

$$(C - z + \alpha\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z).$$

**Type (b)**. If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{gathered}\alpha\tau A_\tau + pA_p = A,\ \alpha\tau B_\tau + pB_p = 0,\\ \alpha\tau\mu_\tau + p\mu_p = (\beta+1)\mu,\ \alpha\tau\lambda_\tau + p\lambda_p = (\beta+1)\lambda,\\ \alpha\tau T_\tau + pT_p = \delta T,\ \alpha\tau\kappa_\tau + pT_p = (-\delta+2+\alpha+\beta)\kappa,\end{gathered} \tag{13}$$

then there is an extension by the generator

$$Y_b = (1+\alpha)U\partial_U + 2\alpha\tau\partial_\tau + 2p\partial_p + (\alpha+2\beta+1)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (13) is

$$\begin{gathered}A = p\widehat{A}(\tau p^{-\alpha}),\ B = B(\tau p^{-\alpha}),\ \mu = p^{\beta+1}M(\tau p^{-\alpha}),\ \lambda = p^{\beta+1}\Lambda(\tau p^{-\alpha}),\\ T = p^\delta\Theta(\tau p^{-\alpha}),\ \kappa = p^{-\delta+\alpha+\beta+2}K(\tau p^{-\alpha}),\end{gathered}$$

where the functions $\widehat{A}(z), B(z)$ and $\Theta(z)$ satisfy the equations ($z \equiv \tau p^{-\alpha}$)

$$-\alpha zB\widehat{A}' + zB'(1+\alpha\widehat{A}) = B^2 + B - B\widehat{A}, \quad (1+\alpha\widehat{A})z\Theta' = (\delta\widehat{A} - B)\Theta. \tag{14}$$

The internal energy is represented by the formula

$$\varepsilon = p^{(\alpha+1)}(\varphi(z) - z) + \psi(p),\ \psi'(p) = Cp^\alpha,$$

where the function $\varphi(z)$ and the constant $C$ are arbitrary and they are related with the functions $\widehat{A}(z)$ and $B(z)$ by the formulae

$$\varphi'(z) = \frac{\widehat{A}(z)}{B(z)},\ C = z + \frac{z}{B(z)} + \alpha z\varphi'(z) - (\alpha+1)\varphi(z)$$

The function $\Theta(z)$ is represented through the function $\varphi(z)$ by the formula

$$(C - z + (\alpha+1)\varphi(z))\,\Theta'(z) = (\delta\varphi'(z) - 1)\Theta(z)$$

Note that an ideal gas belongs to this type in case of $\delta = \alpha+1$ and the function $\varphi(z)$ satisfies the equation

$$\delta(z\varphi' - \varphi) = C.$$

**Type (c)**. If the functions $A(\tau,p), B(\tau,p), \lambda(\tau,p), \mu(\tau,p), \kappa(\tau,p), T(\tau,p)$ satisfy the equations

$$\begin{gathered}A_\tau = 0,\ B_\tau = 0,\ \tau\mu_\tau = \beta\mu,\ \tau\lambda_\tau = \beta\lambda,\\ \tau T_\tau = \delta T,\ \tau\kappa_\tau = (-\delta+1+\beta)\kappa,\end{gathered} \tag{15}$$

then there is one more admitted generator:

$$Y_c = U\partial_U + 2\tau\partial_\tau + (1+2\beta)r\partial_r + 2\beta t\partial_t.$$

The general solution of equations (15) is

$$\begin{gathered}A = A(p),\ B = B(p),\ \mu = \tau^\beta M(p),\ \lambda = \tau^\beta\Lambda(p),\\ T = \tau^\delta\Theta(p),\ \kappa = \tau^{-\delta+\beta+1}K(p),\end{gathered}$$

where the functions $A(p), B(p)$ and $\Theta(p)$ satisfy the equations

$$BA' - AB' = B^2 + B, \quad A\Theta' = (\delta + B)\Theta. \tag{16}$$

The internal energy is represented by the formula

$$\varepsilon = \tau\varphi(p) - \tau p,$$

where the function $\varphi(p)$ is an arbitrary function and it is related with the functions $A(p)$ and $B(p)$ by the formula

$$\varphi(p) = \frac{A(p)}{B(p)}.$$

In this case the function $\Theta(p)$ is related with the function $\varphi(z)$ by the formula

$$\varphi(p)\Theta'(p) = (1 - \delta + \delta\varphi'(p))\Theta(p).$$

Note that if $\delta = 1$ and $\varphi = Cp$, then the gas is ideal.

The final results of the group classification are presented in Table I. In this table the first column means the type of the extension of the algebra $\{X\}$: the types $a$, $b$, or $c$, respectively. The last column means conditions for the state functions.

Therefore, there are three kinds of admitted by equations (10) groups, which depend on the specifications of the functions $A = A(p,\tau), B = B(p,\tau), \lambda = \lambda(p,\tau), \mu = \mu(p,\tau), \kappa = \kappa(p,\tau), T = T(p,\tau)$. These groups are one-dimensional, two-dimensional and three-dimensional.

The two-dimensional admitted groups are groups with the generators either $\{X, Y_a\}$ or $\{X, Y_b\}$ or $\{X, Y_c\}$. The three-dimensional admitted groups are the groups with the generators either $\{X, Y_a, Y_b\}$ or $\{X, Y_a, Y_c\}$ or $\{X, Y_b, Y_c\}$.

The group with the generators $\{X, Y_a, Y_b\}$ is admitted by equations (10) if

$$A = A_0\tau^\alpha,\ B = -1,\ \mu = \mu_0\tau^{\beta+\alpha},\ \lambda = \lambda_0\tau^{\beta+\alpha},\ \kappa = \kappa_0\tau^{\beta+2\alpha},\ T = T_0\tau,\ \alpha \neq 0.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0\int\tau^\alpha\,d\tau)$. Instead the operators $Y_a$ and $Y_b$ one can use their linear combinations:

$$\widehat{Y}_a = \partial_p,\ \widehat{Y}_b = (1+\alpha)U\partial_U + 2\tau\partial_\tau + (\alpha + 2\beta + 1)r\partial_r + 2\beta t\partial_t.$$

The algebra of the type $\{X, Y_a, Y_c\}$ is admitted by equations (10) if

$$A = A_0,\ B = -1,\ \mu = \mu_0\tau^\beta e^{\alpha p},\ \lambda = \lambda_0\tau^\beta e^{\alpha p},\ \kappa = \kappa_0\tau^{\beta - A_0\sigma}e^{(\alpha-\sigma)p},\ T = T_0\tau^{1+A_0\sigma}e^{\sigma p}.$$

In this case the internal energy is $\varepsilon = -(\tau p + A_0\tau)$ and by taking linear combinations of the operators $Y_a$ and $Y_c$ one obtains another basis of the generators:

$$\widehat{Y}_a = \partial_p + \alpha(r\partial_r + t\partial_t),\ \widehat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta+1)r\partial_r + 2\beta t\partial_t.$$

The third type of the algebras $\{X, Y_b, Y_c\}$ is admitted by (10) if

$$A = \gamma p,\ B = \gamma - 1,\ \mu = \mu_0\tau^\beta p^{1+\alpha},\ \lambda = \lambda_0\tau^\beta p^{1+\alpha},$$
$$\kappa = \kappa_0\tau^{\gamma(1-\alpha)+\beta}p^{\alpha-\delta+2},\ T = T_0\tau^{\gamma(\delta-1)+1}p^\delta,\ \gamma \neq 1.$$

The internal the energy in this case is

$$\varepsilon = \frac{\tau p}{\gamma - 1}$$

and linear combinations of the operators $Y_b$ and $Y_c$ are:

$$\widehat{Y}_b = U\partial_U + 2p\partial_p + (2\alpha+1)r\partial_r + 2\alpha t\partial_t,\ \widehat{Y}_c = U\partial_U + 2\tau\partial_\tau + (2\beta+1)r\partial_r + 2\beta t\partial_t$$

Note that a polytropic gas belongs to the last case of gases, where $\gamma$ is a polytropic exponent.

In the formulas above $A_0, \mu_0, \lambda_0, \kappa_0, T_0, \alpha, \beta, \gamma, \delta, \sigma$ are arbitrary constants; the commutators

$$[\widehat{Y}_a, \widehat{Y}_b] = 0,\ [\widehat{Y}_a, \widehat{Y}_c] = 0,\ [\widehat{Y}_b, \widehat{Y}_c] = 0.$$

## 4.3 Optimal systems of subalgebras

Here we study subalgebras of the two-dimensional admitted algebras $\{X, Y_a\}$, $\{X, Y_b\}$, $\{X, Y_c\}$.

The commutator $[X, Y]$ of the generators $X$ and $Y$ is

$$[X, Y] = zX.$$

Here either $Y = Y_a$ or $Y = Y_b$ or $Y = Y_c$ and $z = 2\beta$. Automorphisms are recovered by the table of commutators and consists of the automorphisms

$$\begin{array}{c} A_1:\ x' = x + zya_1,\ y' = y, \\ A_2:\ x' = e^{-za_2}x,\ y' = y, \end{array}$$

where $x$ and $y$ are coordinates of the operator $Z = xX + yY$, $x'$ and $y'$ are coordinates of the operator $Z'$ after actions of the automorphisms, and $a_1,\ a_2$ are parameters of the automorphisms. There is also one involution

$$E:\ x' = -x,\ y' = y,$$

which corresponds to the change of the variables $t \to -t$ and $U \to -U$ without changes of equations (10). Note that if $z = 0$, then the automorphisms are identical transformations. This leads to two optimal systems of subalgebras.

If $z = 0$ (or $\beta = 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\},\ \{Y + hX\},\ \{X, Y\},$$

where $h$ is an arbitrary positive constant.

If $z \neq 0$ (or $\beta \neq 0$), then the optimal system of subalgebras consists of the subalgebras

$$\{X\},\ \{Y\},\ \{X, Y\}.$$

Therefore, one can summarize: optimal systems of subalgebras for the two-dimensional algebras are described by the following system of subalgebras

$$\{X\},\ \{Y + hX\},\ \{X, Y\},\ \beta h = 0. \tag{17}$$

## 4.4 Representations of invariant solutions

A next step in the construction of representations of invariant solutions is a finding universal invariants. Note that invariant solutions corresponding to the case of the subalgebra $\{X\}$ are the well-known stationary solutions. The universal invariants for the other subalgebras of the optimal system (17) of the algebras $\{X, Y_a\}$, $\{X, Y_b\}$ and $\{X, Y_c\}$ are presented in Tables II, III and IY, respectively.

According to the theory of the group analysis [4] on the next step in constructing of invariant solutions one needs to separate the universal invariant on two parts: one part has to be solvable with respect to the dependent variables $U, \tau, p$. After that the representations of invariant solutions are obtained by supposing that the first part of the universal invariant depends on the second part. Because of this requirement there are no invariant solutions in the cases: $a.1$ if $h = 0$, $a.3$, $b.1$, $b.5$ and $c.3$. The cases $a.5$, $b.6$ and $c.4$ correspond to the special cases of stationary solutions, which we also exclude from our consideration[7].

All possible representations of invariant solutions of equations (10) are presented in Table Y, where the functions $f^u$, $f^\tau$, $f^p$ are functions of one independent variable presented in the last column. These functions must satisfy ordinary differential equations, which are obtained after substituting the representation of solution into system (10).

**Remark**. Invariant solutions $a.3$, $b.2$, $b.4$, $c.2$ are self-similar solutions.

**Remark**. One of the well-known solutions of the Boltzmann equation (the BKW-solution[8]) has the representation [36, 37]

$$f = \phi(|u|e^{ct}),$$

where $f$ is a distribution function, $|u|$ is a modulus of the velocity. The invariant solution of the viscous gas equations, which corresponds to the case $b.3$ gives

$$|u|e^{-t(\alpha+1)/h} = qf^u(q),$$

with $q = re^{-t(\alpha+1)/h}$. Therefore this solution can correspond to the BKW-solution and generalize it on molecules with an exponent intermolecular potentials. For the molecules with an exponent intermolecular potentials the coefficients of viscosity and conductivity are [38]

$$\mu = \mu_0 T^k, \kappa = \kappa_0 T^k,$$

where $T = T_0 p\tau$, $k = (n-1)/m + 1/2$, $n$ is dimension of the problem, $m$ is the exponent of intermolecular potentials. In this case $\alpha = 1/k - 1 = (m + 2n - 2)^{-1}(m - 2n + 2)$. For the Maxwell molecules, for which the BKW-solution was constructed, the exponent of intermolecular potentials is $m = 4$, and hence, in the three-dimensional case $\alpha = 0$ and $k = 1$.

[7] If an universal invariant is three-dimensional (consists of three invariants), such as in the cases of $a.5$, $b.6$, $c.4$, then the representation of the invariant solution is obtained by supposing that all invariants of the universal invariant are constants.

[8] This solution is constructed for the Maxwell molecules.

## 5 Spherically Symmetric Flows of the Navier-Stokes Equations

For the complete consideration of solutions connected with the group of rotations $O(3)$ we present solutions of the Navier-Stokes equations with spherical symmetry[9]. Substituting the value of $V = 0,\ W = 0$ in the last equation of (6) one obtains that $r^2U = h(t, \theta, \varphi)$. From the remained equations of (5),(6) all space derivatives of the pressure can be found

$$p_r = r^{-4}(\cot\theta h_\theta + \sin^{-2}\theta h_{\varphi\varphi} + h_{\theta\theta} - r^2 h_t + 2r^{-1}h^2),$$
$$p_\theta = 2r^{-3}h_\theta,\ p_\varphi = 2r^{-3}h_\varphi,$$

where $g(t)$ and $h(t)$ are arbitrary functions.

Using symmetry property of the mixed derivatives $p_{\theta r} = p_{r\theta},\ p_{\varphi r} = p_{r\varphi},\ p_{\varphi\theta} = p_{\theta\varphi}$ and spliting these equalities with respect to $r$ one can get that $h = h(t)$ and the general solution of the Navier-Stokes equations in this case is

$$p = r^{-1}h'(t) - r^{-4}h^2(t)/2 + g(t),\ U = r^{-2}h(t),\ V = 0,\ W = 0.$$

## 6 Conclusion

The analysis that has been done proves that the partially invariant solutions of the studied class for the both types of equations (the Navier-Stokes equations and the full viscous gas dynamics equations), in contrast to inviscid gas and ideal fluid dynamics equations, are spherically symmetric solutions. For the completeness of consideration of partially invariant solutions that are connected with the group of rotations $O(3)$ the group classification of the full viscous gas dynamics equations with spherical symmetry has been done.

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[9]These solutions are irregular invariant solutions of the Navier-Stokes equations with respect to rotations. We think that they are known, but, unfortunately, we do not know any reference on this subject.

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Table I. Group classification.

| | $\lambda$ | $\mu$ | $T$ | $\kappa$ | $A$ | $B$ | $z$ | Cond. |
|---|---|---|---|---|---|---|---|---|
| $a$ | $e^{\beta p}\Lambda(z)$ | $e^{\beta p}M(z)$ | $e^{\delta p}\Theta(z)$ | $e^{(-\delta+\alpha+\beta)p}K(z)$ | $A(z)$ | $B(z)$ | $\tau e^{-\alpha p}$ | (12) |
| $b$ | $p^{\beta+1}\Lambda(z)$ | $p^{\beta+1}M(z)$ | $p^{\delta}\Theta(z)$ | $p^{-\delta+\alpha+\beta+2}K(z)$ | $p\widehat{A}(z)$ | $B(z)$ | $\tau p^{-\alpha}$ | (14) |
| $c$ | $\tau^{\beta}\Lambda(p)$ | $\tau^{\beta}M(p)$ | $\tau^{\delta}\Theta(p)$ | $\tau^{-\delta+\beta+1}K(p)$ | $A(p)$ | $B(p)$ | $p$ | (16) |

Table II. Universal invariants of subalgebras of the algebra $\{X, Y_a\}$.

| N | Subalgebra | consts | Universal invariant |
|---|---|---|---|
| $a.1$ | $Y_a + hX$ | $\beta = 0, \alpha = 0$ | $U, \tau, t - hp/2, r$ |
| $a.2$ | $\beta h = 0$ | $\beta = 0, \alpha \neq 0$ | $Ur^{-1}, \tau r^{-2}, p - 2\alpha^{-1}\ln r, t - h\alpha^{-1}\ln r$ |
| $a.3$ | | $\beta \neq 0$ | $Ut^{-(\alpha+1)/(2\beta)}, \tau t^{-\alpha/\beta}, pt^{-1/\beta}, rt^{-(\alpha+2\beta+1)/(2\beta)}$ |
| $a.4$ | $X, Y_a$ | $\alpha + 2\beta = 0$ | $Ue^{-\alpha p/2}, \tau e^{-\alpha p}, r$ |
| $a.5$ | | $\alpha + 2\beta \neq 0$ | $Ur^{-\alpha/(\alpha+2\beta)}, \tau r^{-2\alpha/(\alpha+2\beta)}, p - 2(\alpha+2\beta)^{-1}\ln r$ |

Table III: Universal invariants of subalgebras of the algebra $\{X, Y_b\}$ ($k \equiv \alpha+2\beta+1$).

| N | Subalgebra | consts | Universal invariant |
|---|---|---|---|
| $b.1$ | $Y_b + hX$ | $\beta = 0, \alpha = -1, h = 0$ | $U, p/\tau, r, t$ |
| $b.2$ | $\beta h = 0$ | $\beta = 0, \alpha \neq -1, h = 0$ | $Ur^{-1}, \tau r^{-2\alpha/(\alpha+1)}, pr^{-2/(\alpha+1)}, t$ |
| $b.3$ | | $\beta = 0, h \neq 0$ | $Ue^{-t(\alpha+1)/h}, \tau e^{-2t\alpha/h}, pe^{-2t/h}, re^{-t(\alpha+1)/h}$ |
| $b.4$ | | $\beta \neq 0$ | $Ut^{-(\alpha+1)/(2\beta)}, \tau t^{-\alpha/\beta}, pt^{-1/\beta}, rt^{-k/(2\beta)}$ |
| $b.5$ | $X, Y_b$ | $k = 0$ | $Up^{-(\alpha+1)/2}, \tau p^{-\alpha}, r$ |
| $b.6$ | | $k \neq 0$ | $Ur^{-(\alpha+1)/k}, \tau r^{-2\alpha/k}, pr^{-2/k}$ |

Table IV. Universal invariants of subalgebras of the algebra $\{X, Y_c\}$.

| N | Subalgebra | consts | Universal invariant |
|---|---|---|---|
| $c.1$ | $Y_c + hX$ | $\beta = 0$ | $Ur^{-1}, \tau r^{-2}, p, t - h\ln r$ |
| $c.2$ | $\beta h = 0$ | $\beta \neq 0$ | $Ut^{-1/(2\beta)}, \tau t^{-1/\beta}, p, rt^{-(2\beta+1)/(2\beta)}$ |
| $c.3$ | $X, Y_c$ | $2\beta + 1 = 0$ | $U\tau^{-1/2}, p, r$ |
| $c.4$ | | $2\beta + 1 \neq 0$ | $Ur^{-1/(2\beta+1)}, \tau r^{-2/(2\beta+1)}, p$ |

Table V. Representations of invariant solutions.

| N | Representation of invariant solution | Ind. variable | Model |
|---|---|---|---|
| 1 | $U = f^u, \tau = f^\tau, p = 2th^{-1} + f^p$ | $r$ | $a.1$ |
| 2 | $U = rf^u, \tau = r^2 f^\tau, p = 2\alpha^{-1}\ln r + f^p$ | $t - h\alpha^{-1}\ln r$ | $a.2$ |
| 3 | $U = t^{(\alpha+1)/(2\beta)}f^u, \tau = t^{\alpha/\beta}f^\tau, p = t^{1/\beta}f^p$ | $rt^{-(\alpha+2\beta+1)/(2\beta)}$ | $a.3$ |
| 4 | $U = rf^u, \tau = r^{2\alpha/(\alpha+1)}f^\tau, p = r^{2/(\alpha+1)}f^p$ | $t$ | $b.2$ |
| 5 | $U = e^{t(\alpha+1)/h}f^u, \tau = e^{2t\alpha/h}f^\tau, p = e^{2t/h}f^p,$ | $re^{-t(\alpha+1)/h}$ | $b.3$ |
| 6 | $U = t^{(\alpha+1)/(2\beta)}f^u, \tau = t^{\alpha/\beta}f^\tau, p = t^{1/\beta}f^p$ | $rt^{-(\alpha+2\beta+1)/(2\beta)}$ | $b.4$ |
| 7 | $U = rf^u, \tau = r^2 f^\tau, p = f^p$ | $t - h\ln r$ | $c.1$ |
| 8 | $U = t^{1/(2\beta)}f^u, \tau = t^{1/\beta}f^\tau, p = f^p$ | $rt^{-(2\beta+1)/(2\beta)}$ | $c.2$ |