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Quantum Electrodynamics of Čerenkov Radiation at Finite Temperature

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An exact, to order α, study of Čerenkov radiation in QED is carried out at *finite* temperatures $T \neq 0$ in isotropic homogeneous media for the first time. By avoiding the method of combining denominators of Feynman propagators in parametric form, which has led to approximations in the past due to the complexity of the resulting integrals, we use instead a complex integration method and automatically evaluate the $\hbar^2\omega^2/E^2$ contribution to the quantum power spectrum and settle the ambiguity associated with this term which has been known to exist at $T = 0$. We show that complex integration over a so-called pinching singularity actually simplifies the problem tremendously over the usual method of combining the denominators of the propagators. In particular, the imaginary part of the electron self-energy satisfies the correct underlying boundary condition and *no contact* term is needed in its evaluation. QED, unlike its classical counterpart, introduces automatically a *cutoff* for higher frequencies, emphasizing the importance of the *quantum* treatment.

1. INTRODUCTION

The history of Čerenkov radiation began with its discovery in 1936 (Čerenkov, 1936; see also Čerenkov, 1934) and its first theoretical explanation in 1937 (Tamm and Frank, 1937). Recent years, have seen numerous experimental studies (e.g., Wang *et al.*, 1991; Garate *et al.*, 1990; Efimov and Skurov, 1979; Hartman *et al.*, 1979), applications (Ferbel, 1987; Prothers and Patterson, 1984), and theoretical investigations for radiating particles (Manoukian and Bantitadawit, 1999, Manoukian *et al.* 1997a,b, Šoln, 1997; Zhevago and Glebov, 1997; Orisa, 1995; Manoukian, 1994, 1993; Pratap *et al.* 1993; Fulop, 1993; Ginzburg *et al.*, 1993; Manoukian, 1991; Pardy, 1989; Schwinger *et al.*, 1976) and strings (Manoukian, 1991). In simplest terms, Čerenkov radiation is the radiation emitted by a charged particle in a medium

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when its speed exceeds the speed of light in the medium. Although most recent theoretical studies have been classical, many have also dealt at the quantum level (e.g., Orisa, 1995; Fulop, 1993; Schwinger *et al.*, 1976; Kong, 1975) following earlier attempts (Tidman, 1956; Taniuti, 1951; Beck, 1948; Sokolov, 1940) at the quantum level. One of the clearest and most detailed quantum treatments of the problem is that of Schwinger *et al.* (1976). The latter deals with the full QED, to order α, in an isotropic homogeneous medium *at* $T = 0$. Unfortunately, due to the method of combining the denominators of the propagators in parametric form, the resulting integrals are *exceedingly* complicated and approximations were necessarily made. This left, in particular, the contribution $\hbar^2\omega^2/E^2$ to the quantum correction undetermined. The ambiguity associated with the latter part of the quantum correction is well known.

The purpose of this work is to carry out to order α with *no* further approximations a study of the power for Čerenkov radiation emission in QED at *finite* (e.g., Manoukian, 1990) temperatures $T \neq 0$ for the first time and in the process obtain the full quantum correction to the spectrum in an isotropic homogeneous medium. We use the method of complex integration directly on the electron self-energy *without* combining the denominators of the underlying propagators. This, as we will see, simplifies the problem tremendously over the more conventional method of combining denominators of the propagators in parametric form. This allows us to obtain a *closed* expression for the integral in question for the power. We justify rigorously integrating over the complex domain by deriving lower bounds on the singularities involved through Lemmas 1 and 2. The method of complex integration brings us into contact with studies of the analytical properties of Feynman diagrams (e.g., Cutkosky, 1960; Mandelstam, 1958; Eden *et al.*, 1966) dealing with so-called pinching singularities, as will be discussed in the text. The imaginary part of the self-energy of the electron satisfies the correct underlying boundary condition and *no contact* term needs to be introduced. The inclusion of temperature in quantum field theory (e.g., Manoukian, 1990; Ahmed and Masood, 1991; Johansson *et al.*, 1986; Bechler, 1981) is well known, but to our knowledge, this has not been carried out in the QED of Čerenkov radiation. Also in the present problem, one is involved in evaluating the *imaginary* rather than the real part of a self-energy. Although quantum effects may be difficult to detect, one of the most pleasing aspects of the *quantum* treatment is that QED, unlike its classical counterpart, introduces automatically a *cutoff* for higher frequencies beyond which the power is necessarily zero, as emphasized in the text. Section 2 deals in detail with the intricacies of the $T = 0$ case. Section 3 then extends the theory to finite temperatures. Our notation is summarized in the Appendix.

2. POWER SPECTRUM IN QED: $T = 0$

The expression for the decay rate of an electron in a medium is given by (Schwinger *et al.*, 1976)

$$\Gamma = -\frac{2\mathrm{m}}{E}\,\mathrm{Im}(\bar{u}\,\Sigma u) \tag{2.1}$$

where u is a Dirac spinor (see Appendix) and Σ is the electron self-energy in a medium

$$\Sigma(p) = ie^2 \int \frac{(dq)}{(2\pi)^4}\,\mathrm{Tr}[\gamma^\alpha S(p-q)\gamma^\beta]D_{\alpha\beta}(q), \qquad \varepsilon \to +0 \tag{2.2}$$

where

$$S(p-q) = \frac{-\gamma(p-q)+m}{(p-q)^2+m^2-i\varepsilon}, \qquad p^2+m^2=0 \tag{2.3}$$

is the electron propagator and

$$D_{\alpha\beta}(q) = \mu\,\frac{[g_{\alpha\beta}+(1-1/n^2)\eta_\alpha\eta_\beta}{\vec{q}^{\,2}-n^2q^{02}-i\epsilon} \tag{2.4}$$

is the photon propagator in the Lorentz gauge, $\eta^\alpha = (1, \vec{0})$ is a timelike unit vector, with μ and $n = \sqrt{\mu\kappa}$ denoting, respectively, the permeability and the index of refraction of the medium.

To obtain the power spectrum for photon emission each with energy ω, we insert in (2.2) the identity operation

$$1 = \int_0^\infty d\omega\,\delta\!\left(\omega - \frac{|\vec{q}|}{n}\right) \tag{2.5}$$

leading to the following expression for the power spectrum:

$$P(\omega) = -\frac{2m}{E}\,n\omega\mu e^2 \int \frac{(dq)}{(2\pi)^4}\,\delta(|\vec{q}| - n\omega)\,\mathrm{Im}\left[\frac{i\bar{u}Nu}{D}\right] \tag{2.6}$$

where

$$N = \gamma^\alpha(-\gamma(p-q)+m)\gamma^\beta\left[g_{\alpha\beta}+\left(1-\frac{1}{n^2}\right)\eta_\alpha\eta_\beta\right] \tag{2.7}$$

and

$$D = (q^2 - 2pq - i\varepsilon)(\vec{q}^{\,2} - n^2q^{02} - i\varepsilon) \tag{2.8}$$

Carrying out the $|\vec{q}|$ and the angular ϕ integrations in (2.6) gives

$$P(\omega) = -\frac{2m}{E}\,\omega^3\mu n e^2 \frac{(2\pi)}{(2\pi)^4}\int_{-1}^{1} d(\cos\theta)\int_{-\infty}^{\infty} dq^0\,\mathrm{Im}\left[i\,\frac{\bar{u}Nu}{D_\varepsilon}\right] \tag{2.9}$$

with

$$\bar{u}Nu = \frac{2E^2\beta^2}{m}\left\{1 - \frac{1}{n^2\beta^2} - \frac{q^0}{2\beta^2 E}\left(3 - \frac{1}{n^2}\right) + \frac{n\omega}{2E\beta}\left(1 + \frac{1}{n^2}\right)\cos\theta\right\} \tag{2.10}$$

Here $\beta = |\vec{p}|\,/\,E = v/c$ for the electron of energy E and

$$D_\varepsilon = [q^{0^2} - 2q^0E - n^2\omega^2 + 2n\beta\omega E\cos\theta + i\varepsilon][q^{0^2} - \omega^2 + i\varepsilon] \tag{2.11}$$

The $+i\varepsilon$ rather than $-i\varepsilon$ occurs in D_ε as we have factored out a minus sign from each of the two products in (2.8). We have also factored out n^2 from the second factor in (2.8).

We consider the singularities occurring in $1/D_\varepsilon$. To this end, the roots of $D_\varepsilon = 0$ as a function of q^0 are

$$q^0_{1\pm} = E(1 \pm A(\theta)) \mp i\varepsilon \tag{2.12}$$

$$q^0_{2\pm} = \pm\,\omega \mp i\varepsilon \tag{2.13}$$

where

$$A^2(\theta) = \left(1 + \frac{n^2\omega^2}{E^2} - 2n\beta\,\frac{\omega}{E}\cos\theta\right) > 0 \tag{2.14}$$

and for $0 \leq \beta < 1$, $A^2(\theta)$ is strictly positive as indicated. This is as a consequence of the following result:

Lemma 1. For $0 \leq \beta < 1$,

$$A(\theta) \geq \sqrt{1 - \beta^2} > 0 \tag{2.15}$$

for all θ in $[0, \pi]$.

To establish (2.15), we note that

$$A^2(\theta) = 1 + \frac{n^2\omega^2}{E^2} - 2n\beta\,\frac{\omega}{E}\cos\theta$$

$$\geq 1 + \frac{n^2\omega^2}{E^2} - 2n\beta\,\frac{\omega}{E} \tag{2.16}$$

Let $n\omega/E = x$. Then the right-hand side of the above inequality is $f(x) = 1 + x^2 - 2x\beta$. It is easily verified that the minimum of $f(x)$ occurs for $x = \beta$, i.e., $A^2(\theta) \geq 1 - \beta^2 > 0$.

We also need the following result:

Lemma 2. For $n > 1$, $0 \leq \beta < 1$,

$$E(1 + A(\theta)) > \omega \tag{2.17}$$

for all $0 \leq \theta \leq \pi$ as a *strict* inequality.

To prove this, suppose the converse is true. That is, as an initial hypothesis suppose that

$$\omega \geq E(1 + A(\theta)) \tag{2.18}$$

for some θ in $[0, \pi]$. Let $\omega/E = x$. Then (2.18) implies that

$$x \geq 1 + A(\theta) > 1 \tag{2.19}$$

where we have used Lemma 1. That is,

$$x - 1 \geq A(\theta) > 0 \tag{2.20}$$

Upon squaring the latter, this leads to the inequality

$$2n\beta x \cos\theta \geq (n^2 - 1)x^2 + 2x \tag{2.21}$$

or

$$\begin{aligned}\cos\theta &\geq \frac{1}{n\beta}\left[1 + \frac{n^2 - 1}{2}x\right] > \frac{1}{n\beta}\left[1 + \frac{n^2 - 1}{2}\right] \\ &= \frac{n^2 + 1}{2n\beta} > \frac{1}{\beta} > 1\end{aligned} \tag{2.22}$$

where in the first strict inequality we used (2.19), and the next one follows from $n^2 + 1 > 2n$ for $n > 1$. The contradictory statement in (2.22) for a cosine function implies that the initial hypothesis in (2.18) is false for all θ in $[0, \pi]$. That is, since $E(1 + A(\theta))$ is some real number it must satisfy (2.17).

We rewrite D_ε as

$$D_\varepsilon = (q^0 - q^0_{1+})(q^0 - q^0_{1-})(q^0 - q^0_{2+})(q^0 - q^0_{2-}) \tag{2.23}$$

We close the q^0-contour in the complex q^0-plane from below (clockwise) by noting that D_ε has enough powers in q^0 to make sure that the infinite semicircle gives no contribution to the resulting integral. The poles in the lower complex plane are q^0_{1+} and q^0_{2+}.

From the residue theorem, we have for the integral

$$\int_{-\infty}^{\infty} dq^0 \left[\frac{i\bar{u}Nu}{D_\varepsilon}\right] = 2\pi\left[\frac{\bar{u}Nu|_{q^0=q^0_{1+}}}{D_{1\varepsilon}} + \frac{\bar{u}Nu|_{q^0=q^0_{2+}}}{D_{2\varepsilon}}\right], \qquad \varepsilon \to +0 \tag{2.24}$$

where

$$D_{1\varepsilon} = (q^0_{1+} - q^0_{1-})(q^0_{1+} - q^0_{2+})(q^0_{1+} - q^0_{2-}) \tag{2.25}$$

$$D_{2\varepsilon} = (q^0_{2+} - q^0_{1+})(q^0_{2+} - q^0_{1-})(q^0_{2+} - q^0_{2-}) \tag{2.26}$$

We explicitly have

$$\lim_{\varepsilon\to+0} D_{1\varepsilon} = 2A(\theta)E[E^2(1 + A(\theta))^2 - \omega^2] \neq 0 \quad (>0) \tag{2.27}$$

by using Lemmas 1 and 2. That is, in the limit $\varepsilon \to +0$, the first term in (2.24) is real and gives no contribution to $P(\omega)$ in (2.9).

For $D_{2\varepsilon}$, we explicitly obtain

$$D_{2\varepsilon} = 2(\omega - i\varepsilon)\{(\omega - E)^2 - E^2A^2(\theta) - i[\omega - E(1 + A(\theta))]\varepsilon\} \tag{2.28}$$

Since by Lemma 2, $\omega < E(1 + A(\theta))$ as a strict inequality, this gives an overall + sign to the coefficient of $i\varepsilon$ in the second factor in (2.28):

$$D_{2\varepsilon} = 2(\omega - i\varepsilon)[(\omega - E)^2 - E^2A^2(\theta) + i\varepsilon] \tag{2.29}$$

In detail,

$$D_{2\varepsilon} = 2(\omega - i\varepsilon)[\omega^2 - 2\omega E - n^2\omega^2 + 2n\omega E\beta \cos\theta + i\varepsilon] \tag{2.30}$$

and for $\omega E > 0$, we have

$$\operatorname{Im}\left(\frac{1}{D_{2\varepsilon}}\right) = -\frac{1}{4n\omega^2E\beta}\,\delta\left(\cos\theta - \frac{1}{n\beta}\left(1 + \frac{n^2 - 1}{2}\frac{\omega}{E}\right)\right), \qquad \varepsilon \to +0 \tag{2.31}$$

The delta function over $\cos\theta$ puts the following constraint for the *non*vanishing of $P(\omega)$ in (2.9):

$$n\beta > 1 + \frac{n^2 - 1}{2}\frac{\omega}{E} \tag{2.32}$$

At $q^0 = \omega$, the delta function over $\cos\theta$ leads to

$$\bar{u}Nu = \frac{2E^2\beta^2}{m}\left\{1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{E}(n^2 - 1)\right) + \frac{\omega^2}{4E^2}\frac{n^4 - 1}{n^2\beta^2}\right\} \tag{2.33}$$

All told, we have for the power in (2.9)

$$P(\omega) = \alpha\omega\beta\mu\left\{1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{E}(n^2 - 1)\right) - \frac{\omega^2}{4E^2}\frac{n^4 - 1}{n^2\beta^2}\right\} \tag{2.34}$$

where $\alpha = e^2/4\pi$. Equation (2.34) is valid with the threshold condition given through (2.32) for the emission of radiation. The task, however, is not complete. It remains to verify that $P(\omega)$ is indeed strictly *positive* under this constraint and that no further restrictions are necessary. To this end, we note that (2.34) may be rewritten in the more convenient form

$$P(\omega) = \alpha\omega\beta\mu\left\{1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{2E}(n^2 - 1)\right)^2 - \frac{\omega^2}{2E^2}\frac{n^2 - 1}{\beta^2}\right\} \tag{2.35}$$

and with the constraint in (2.32), there is no question of the strict positivity of $P(\omega)$ for $\omega > 0$. Finally we note that for $n = 1$, the statement in (2.32) is empty and *no contact* term was needed to derive the expression for $P(\omega)$ in (2.35).

It is interesting to dwell further on why the pole at $q^0_{2+} = \omega - i\varepsilon$ contributes and the pole at $q^0_{1+} = E(1 + A(\theta)) - i\varepsilon$ does not. To this end, we note that the condition $E(1 - A(\theta)) = \omega$ is in the domain of integration over cos θ. Accordingly, at this point, the upper pole q^0_{1-} is just above the lower pole q^0_{2+}. For $\varepsilon \to +0$ they pinch the q^0-contour and no deformation of the latter is possible at this point to avoid the q^0_{2+} pole in the lower complex plane. The poles q^0_{1-} and q^0_{2-}, however, never coincide with q^0_{1+} for $\varepsilon \to +0$ according to Lemmas 1 and 2.

3. POWER SPECTRUM IN QED: $T \neq 0$

The inclusion of temperature amounts in the following replacements:

$$\begin{aligned}((p - q)^2 + m^2 - i\varepsilon)^{-1} &\\ \to ((p - q)^2 + m^2 - i\varepsilon)^{-1} &\\ - 2\pi i\,\delta((p - q)^2 + m^2)\left(1 + \exp \rho\sqrt{(\vec{p} - \vec{k})^2 + m^2}\right)^{-1} &\end{aligned} \tag{3.1}$$

$$\begin{aligned}(\vec{q}^{\,2} - n^2 q^{0 2} - i\varepsilon)^{-1} &\\ \to (\vec{q}^{\,2} - n^2 q^{0 2} - i\varepsilon)^{-1} &\\ - 2\pi i\,\delta(\vec{q}^{\,2} - n^2 q^{0 2})(1 - \exp \rho|q^0|)^{-1} &\end{aligned} \tag{3.2}$$

in the denominators of the propagators in question, where $\rho = 1/kT$ and k

is the Boltzmann constant. To obtain the temperature correction $\Delta_T P(\omega)$ for radiation emission to the integrals (2.6), (2.9), we use

$$\delta(q^{0^2} - \omega^2) = \frac{1}{2\omega}[\delta(q^0 - \omega) + \delta(q^0 + \omega)] \tag{3.3}$$

$\Delta_T P(\omega)$ for radiation *emission* of frequency $\omega > 0$ is then

$$\Delta_T P(\omega) = -\frac{m}{E} n\omega^2 \mu e^2 \frac{2\pi}{(2\pi)^4} \int_{-1}^{1} d(\cos\theta) \int_{-\infty}^{\infty} dq^0 \, [\bar{u}Nu] \times (i)(-2\pi i)(2\pi) \times \delta(q^{0^2} - n^2\omega^2 - 2n\beta E\omega\cos\theta - 2\,Eq^0)\,\delta(q^0 - \omega) F_T(q^0) \tag{3.4}$$

where

$$F_T(q^0) = \tfrac{1}{2}(1 - \exp\rho\omega)^{-1} + \tfrac{1}{2}(1 + \exp\rho\sqrt{E^2 + \omega^2 - 2Eq^0})^{-1} - (1 - \exp\rho\omega)^{-1}(1 + \exp\rho\sqrt{E^2 + \omega^2 - 2Eq^0})^{-1} \tag{3.5}$$

The delta function over $\cos\theta$ in (3.4) sets the threshold condition (2.32) as before. Upon integrating out (3.4) and adding the resulting integral to the expression in (2.35), we obtain for the power spectrum of photon emission

$$P_T(\omega) = \alpha\omega\beta\mu\left\{1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{2E}(n^2 - 1)\right)^2 + \frac{\omega^2}{2E^2}\frac{n^2 - 1}{\beta^2}\right\}A_T(\omega) \tag{3.6}$$

where

$$A_T(\omega) = \frac{e^{\omega/kT}}{e^{\omega/kT} - 1}\left[\frac{\exp(|E - \omega|/kT) - \exp(-\omega/kT)}{(\exp(|E - \omega|/kT) + 1)}\right] \tag{3.7}$$

which is strictly positive, with (3.6) holding only with the threshold condition

$$n\beta > 1 + (n^2 - 1)\omega/2E \tag{3.8}$$

satisfied, otherwise $P_T(\omega)$ is zero.

For given $0 < \beta < 1$, $n > 1$ (with necessarily $n\beta > 1$), Eq. (3.8) provides a *cutoff* for higher frequencies:

$$\omega < \omega_c \tag{3.9}$$

with

$$\omega_c = 2\,\frac{n\beta - 1}{(n^2 - 1)}\,E = \frac{2(n\beta - 1)}{(n^2 - 1)}\,\frac{m}{\sqrt{1 - \beta^2}} \tag{3.10}$$

beyond which the power of radiation emission is zero. This cutoff upper limit ω_c is still bounded above by the electron energy E. That is, $\omega < E$, a result which is expected on physical grounds. The proof of the latter bound follows from the following inequalities:

$$\omega_c = 2\,\frac{n\beta - 1}{n^2 - 1}\,E < 2\,\frac{n - 1}{n^2 - 1}\,E = \frac{2}{n + 1}\,E < E \tag{3.11}$$

since $(n + 1) > 2$. That is, necessarily, $\omega < E$. The latter also means that the absolute value sign in $|E - \omega|$ appearing in (3.7) may be *removed*.

The coefficient $A_T(\omega)$ has the following asymptotic behavior at low temperatures $kT \ll \omega$ and at high temperatures $kt \gg E$, respectively:

$$A_T(\omega) \sim 1 \tag{3.12}$$

$$A_T(\omega) \sim \frac{E}{2\omega} \tag{3.13}$$

In particular,

$$P_T(\omega) \underset{kT \gg E}{\sim} \frac{\alpha\beta\mu E}{2}\left\{1 - \frac{1}{n^2\beta^2}\left(1 + \frac{\omega}{2E}(n^2 - 1)\right)^2 + \frac{\omega^2}{2E^2}\,\frac{n^2 - 1}{\beta^2}\right\} \tag{3.14}$$

and the power of emission is enhanced for $\omega < E/2$ and suppressed for $E/2 < \omega < E$ at high temperatures.

APPENDIX

Our notation is $\hbar = 1$, $c = 1$, $g^{\mu\nu} = \mathrm{diag}[-1, 1, 1, 1]$; $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, $(\gamma p + m)u = 0$, $\bar{u}u = 1$, $u^\dagger u = p^0/m$; $\gamma^\mu\gamma_\sigma\gamma^\mu = 2\gamma_\sigma$, $\gamma^\mu\gamma_\mu = -4I$, $\bar{u}\vec{\gamma}u = \vec{p}/m$.

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