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นายองอาจ ซึมรัมย์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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APPLICATIONS OF GROUP ANALYSIS TO STOCHASTIC FLUID DYNAMICS EQUATIONS

Ongart Sumrum

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics

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APPLICATIONS OF GROUP ANALYSIS TO STOCHASTIC FLUID DYNAMICS EQUATIONS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วิทยานิพนธ์ฉบับนี้ได้แสดงการประยุกต์ของการวิเคราะห์เชิงกลุ่มกับการสร้างผลเฉลยยืนยงของสมการเชิงอนุพันธ์เฟ้นสุ่มของแก๊สและสมการเชิงอนุพันธ์เฟ้นสุ่มของของไหลเป็นครั้งแรก ซึ่งถ้าผลเฉลยยืนยงไม่มีตัวแปลอิสระ t ในตัวแปรอิสระยืนยง ระบบที่ลดรูปจะประกอบด้วยสมการเชิงกำหนดและปริพันธ์เฟ้นสุ่มจะหายไปจากผลเฉลยยืนยง กรณีที่มีตัวแปลอิสระเวลาอยู่ในตัวแปรอิสระยืนยง ระบบสมการลดรูปจะมีส่วนที่เป็นระบบสมการเฟ้นสุ่มเชิงอนุพันธ์ ในอีกกรณีหนึ่งถ้ามีตัวแปรอิสระเวลาเป็นตัวยืนยงแล้วบราวเนียนโมชันไม่เปลี่ยนแปลง ในทางตรงข้ามบราวเนียนโมชันในระบบที่ลดรูปจะเปลี่ยน ซึ่งเปรียบเทียบได้กับบราวเนียนโมชันในระบบสมการเริ่มแรก เค้าโครงหลักอย่างอื่นของระบบสมการลดรูปที่เป็นปริพันธ์ในส่วนของปริพันธ์อีโต้ต้องมีรูปแบบบางส่วนสัมพันธ์กับกลุ่มของลีที่ยอมรับ

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ADMITTED LIE GROUP/STOCHASTIC FLUID DYNAMICS EQUATIONS/ INVARIANT SOLUTION

This present thesis demonstrates a first experience in the application of the group analysis method for constructing invariant solutions of stochastic differential equations of gas and hydrodynamics. It is obtained by example that if in the representation of an invariant solution the time t is not contained in the invariant independent variable, then the reduced system consists of deterministic equations, and the stochastic integral has to vanish on the invariant solution. In the case where the time is included in the invariant independent variable, then the reduced system is kept as a system of stochastic differential equations. In the latter case, if the time itself is invariant, then the Brownian motion is not changed, whereas otherwise the Brownian motion in the reduced system is changed compared with the Brownian motion in the original system of equations. Another feature of obtaining the reduced system is that the form of the integrand in the Itô integral has to have a particular form related with the admitted Lie group.

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CHAPTER I

INTRODUCTION

Lie group analysis is a well developed method for studying properties of ordinary and partial differential equations, and obtaining exact solutions of them. A general survey of this method can be found in Ovsiannikov (1982) and Ibragimov (1994, 1995, 1996). It involves the study of symmetries of equations, by which one means local groups of transformations mapping solutions of a given system of equations to solutions of the same system. Symmetries make it possible to reduce the number of dependent and independent variables, and also to construct new solutions from known solutions. Point transformations are the most commonly used types, involving both, the independent and dependent variables in the transformations. Other types of transformations have also been applied with success, one may for example choose to include derivatives of various orders in the transformations.

The requirement that an infinitesimal transformation maps every solution of a system of equations to a solution of the same system gives rise to the concept of an admitted Lie group for a given system. The handbooks by Ibragimov (1994, 1995, 1996) provide excellent references on the vast collection of differential equations for which admitted Lie groups are known and solutions have been obtained. Hydrodynamics equations were the first object of application of group analysis in Ovsiannikov (1958). The recent PODMODELI (SUBMODELS) program in Ovsiannikov (1994) was aimed at an exhaustive use of the group analysis method for studying the gas dynamics equations. Results of this study are summerized in Ovsiannikov (1999). The first application of the group analysis method to the Navier-Stokes equations was done in Pukhnachov (1960). A review of exact solutions of the Navier-Stokes equations can be found in Pukhnachov (2006).

In contrast to deterministic equations, there have been few attempts to apply symmetry techniques to stochastic ordinary and partial differential equations. In general, the change of variables in stochastic differential equations differs from the change of variables in ordinary differential equations, as the Itô formula takes the place of the chain rule of differentiation. Exploiting the Itô formula and the requirement that the solution of a stochastic differential equation is mapped into a solution of the same equation, the determining equations of an admitted Lie group can be obtained.

In the first attempts, the transformation of time t was only a function of time and the group parameter. This approach has been applied to stochastic dynamical system in Misawa (1994), Albeverio and Fei (1995), Alexandrova (2005, 2006), to the Fokker-Plank equation in Gaeta and Quinter (1999), Gaeta (2004), Unal (2003), Unal and Sun (2004) and Ibragimov (2004), to scalar second-order stochastic ordinary differential equations in Mahomed and Wafo Soh (2001), and to the Hamiltonian-Stratonovich dynamical control system in Unal and Sun (2004). It has also been applied to stochastic partial differential equations in Melnick (2003). The latter is the only known result on the application of group analysis to partial stochastic differential equations.

Another approach in Srihirun, Meleshko, and Schulz (2006, 2007) includes the dependent variables in the transformation of time as well,

$$\bar{x} = \varphi(t, x, a), \quad \bar{t} = H(t, x, a).$$

In particular, the transformation of Brownian motion is defined through the transformation of the dependent and independent variables. Generalizing the change of the time formula in Oksendal (1998), it was proven in Srihirun, Meleshko, and

Schulz (2006) that the transformed Brownian motion

$$\overline{B}(t) = \int_0^t \eta^2(s, X(s), a) dB(s)$$

(where $\eta(t, x, a) \neq 0$) satisfies again the properties of Brownian motion. This transformation of Brownian motion is a logical generalization of the time change in the Itô integral to the case where the stochastic process is included in the change. Exploiting the Itô formula, this transformation of Brownian motion, and the requirement that a solution of the stochastic differential equation is mapped into a solution of the same equation, and finally equating the Riemann and Itô integrands, the determining equations of an admitted Lie group were obtained. The definition of an admitted Lie group for stochastic ordinary differential equations was given using these determining equations.

We first extend this discussion to systems of second order stochastic partial differential equations of the form

$$dv^i(t, y) = A^i(t, y, v, v_{y_k}, v_{y_k y_l})dt + \sum_{j=1}^{m} B^{ij}(t, y, v)dw^j(t), \quad (i = \overline{1, n}, \quad k, l = \overline{1, N}).$$

We construct determining equations for admitted Lie groups of transformations which involve both the independent and the dependent variables, and transform the Wiener processes w^j as well. This should serve as a model for the correct generalization of the group analysis method to stochastic partial differential equations. We then apply our result to some of the fundamental equations of fluid dynamics with stochastic parts, namely the Kardar-Parisi-Zhang equation, the gas dynamics equations and the Navier-Stokes equations, to obtain invariant solutions and more generally, systems of stochastic differential equations of reduced complexity.

This thesis is organized as follows. Chapter II introduces some necessary knowledge from stochastic processes. Chapter III introduces notations of group analysis. Applications of the group analysis method to the deterministic gas and hydro dynamics equations are considered in Chapter IV. Chapters V and VI present the results of our investigations. Transformations for stochastic partial differential equations and the application of group analysis to constructing determining equations for admitted Lie groups of transformations for stochastic differential equations are studied in Chapter V. Chapter VI is devoted to developing a new knowlege of application of group analysis to stochastic fluid dynamic differential equations. The last Chapter is the conclusion.

CHAPTER II

STOCHASTIC PROCESSES

This chapter is devoted to introducing the tools from the theory of stochastic processes which are used throughout this thesis. In particular, it discusses stochastic integrals with respect to Wiener processes, Wiener processes as time change and Itô's formula. We assume that the reader is familiar with the fundamental measure theoretic concepts of probability, as described in standard textbooks.

2.1 Stochastic Processes and Wiener Processes

Let Ω be a given set of elementary events ω ; $\mathcal{F}$ a σ-algebra of subsets of Ω and $\mathcal{P}$ a probability measure on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, \mathcal{P})$ is called a probability space. It is assumed that the σ-algebra $\mathcal{F}$ is generated by a family of σ-algebras $\mathcal{F}_t$ $(t \geq 0)$ such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \quad \forall s \leq t.$$

The nondecreasing family of σ-algebras $\mathcal{F}_t$ is also called a filtration and the σ-algebra $\mathcal{F}$ is denoted by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$. The triple $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ is called a filtered probability space.

A stochastic process X on $(\Omega, \mathcal{F}, \mathcal{P})$ is a collection of random variables $\{X(t)\}_{t\geq 0}$. The process $\{X(t)\}_{t\geq 0}$ is said to be adapted to $(\mathcal{F}_t)_{t\geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each t. Denoting the Borel σ-algebra on an interval I by $\mathcal{B}(I)$, the process X is called measurable if $(t, \omega) \longmapsto X(t, \omega)$ is a $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$-measurable mapping. The process X is said to be continuous if the trajectories $t \longmapsto X(t, \omega)$

are continuous for almost all $\omega \in \Omega$. It is called progressively measurable if $X : [0,t] \times \Omega \longmapsto \mathbb{R}$ is a $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable mapping for each $0 \le t < \infty$. Note that a progressively measurable process is measurable and adapted.

Proposition 2.1. An adapted process that is left-continuous or right-continuous is progressively measurable.

Proof. See Albin (2001), p.58, Theorem 8.8.

Recall that a Wiener processes, also called Brownion motion, is a real-valued stochastic processes $\{W(t)\}_{t\geq 0}$ satisfying the properties

(1) continuity: the map $s \to W(s,\omega)$ is continuous almost surely.

(2) independent increments: if $s \le t$, then $W(t) - W(s)$ is independent of (the past) $\mathcal{F}_s = \sigma(W(u) : u \le s)$.

(3) stationary increments: if $s \le t$, then $W(t) - W(s)$ and $W(t-s) - W(0)$ have the same distribution functions. Increments $W(t) - W(s)$ are normally distributed with mean zero and variance $t - s$.

A Wiener processes is said to be standard if it satisfies $W(0,\omega) = 0$ almost surely.

2.1.1 The Itô Integral for Simple Processes

From now on, unless stated otherwise, we let $\{W(t)\}_{t\geq 0}$ be a standard Wiener process applied to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Fix $T \in (0,\infty]$. A process $\{X_t\}_{t\geq 0}$ is said to be simple on $[0,T]$ (denoted $X \in S_T$) if it can be written as follows: There exist a partition $0 = t_0 < t_1 < ... < t_n = T$ of $[0,T]$ and random variables

$Y_0, Y_1, ..., Y_{n-1}$ which are adapted to $\mathcal{F}_0$, $\mathcal{F}_{t_1}$,..., $\mathcal{F}_{t_{n-1}}$ respectively and satisfy the conditions $E\{Y_0^2\}$, ... , $E\{Y_{n-1}^2\} < \infty$, such that

$$X(t) = Y_0 I_{\{0\}}(t) + \sum_{i=1}^{n} Y_{i-1} I_{(t_{i-1},t_i]}(t)$$

for $t \in [0, T]$. Here, I_S denotes the characteristic function of a set S. In the case $T = \infty$, there is one more requirement for a simple process: $Y_{n-1} = 0$.

Definition 2.1. Let $X \in S_T$, say

$$X(t) = X_0 I_{\{0\}}(t) + \sum_{i=1}^{n} X_{i-1} I_{(t_{i-1},t_i]}(t),$$

where $\{t_0, t_1, ..., t_n\}$ is partition of $[0, T]$. The Itô integral of $\{X(t)\}_{t\in[0,T]}$ is the process defined by

$$\int_0^t X(s)dW(s) = X_m\big(W(t) - W(t_m)\big) + \sum_{i=1}^{m} X_i\big(W(t_i) - W(t_{i-1})\big), \qquad (2.1)$$

where $(t_m, t_{m+1}]$ is the partition interval containing t.

2.1.2 The Itô Integral for Square Integrable Processes

A stochastic process $\{X(t)\}_{t\geq 0}$ is said to belong to the class E_T if it is measurable and adapted to $(\mathcal{F}_t)_{t\geq 0}$ with

$$\|X\|_2^2 = E\{\int_0^T X^2(r)dr\} < \infty.$$

It turns out that S_T is dense in E_T in this mean square norm, and that the Itô integral (2.1) is a linear isometry of S_t into $L^2(\Omega, \mathcal{F}_t)$, for each $0 < t \leq T$. Thus, it can be extended to class E_T as follows.

Definition 2.2. For a process $X \in E_T$, the Itô integral of $\{X(t)\}_{t\in[0,T]}$ is defined in the sense of convergence in the mean square (and hence in the mean).

$$\int_0^t X(s)dW(s) = \lim_{n\to\infty} \int_0^t X_n(s)dW(s), \quad t \in [0, T],$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequence of simple processes on $[0, T]$ such that

$$\lim_{n\to\infty} \|X_n - X\|_2^2 = \lim_{n\to\infty} \int_0^T E[\big(X_n(s) - X(s)\big)^2]ds = 0.$$

2.1.3 The Itô Integral for Predictable Processes

A stochastic process $\{X(t)\}_{t\geq 0}$ is said to belong to the class P_T of predictable processes on $[0, T]$ if it is measurable and adapted to $(\mathcal{F}_t)_{t\geq 0}$ with

$$\mathcal{P}\{\int_0^T X^2(r)dr < \infty\} = 1.$$

Note that $S_T \subset E_T \subset P_T$. Every process X in P_T is the limit of a sequence $\{X_n\}_{n=1}^{\infty} \subseteq E_T$ in the sense of convergence in probability.

Definition 2.3. For a process $X \in P_T$, the Itô integral of $\{X(t)\}_{t\in[0,T]}$ can be defined in the sense of convergence in probability,

$$\int_0^t X(s)dW(s) = \lim_{n\to\infty} \int_0^t X_n(s)dW(s), \quad t \in [0, T],$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequence of processes which belong to the class E_T such that

$$\lim_{n\to\infty} \int_0^T \big(X_n(s) - X(s)\big)ds = 0,$$

with limit in the sense of convergence in probability.

Remark 2.1. In Albin (2001) it is proven that processes $X, Y \in P_T$ satisfy the following:

1. The Itô integral $\displaystyle\int_0^t X(\tau)dW(\tau)$ is well-defined for $0 \leq t \leq T$,

2. $E\Big(\big(\displaystyle\int_0^t X(\tau)dW(\tau)\big)^2\Big) = \displaystyle\int_0^t E\big(X^2(\tau)\big)d\tau$ for $0 \leq t \leq T$ (Itô isometry property), provided that $X \in E_T$,

3. $E\Big(\int_0^t X(\tau)dW(\tau)\int_0^t Y(\tau)dW(\tau)\Big) = \int_0^t E\big(X(\tau)Y(\tau)\big)d\tau$ for $0 \le t \le T$, provided that $X \in E_T$,

4. $\int_0^t \big(aX(\tau) + bY(\tau)\big)dW(\tau) = a\int_0^t X(\tau)dW(\tau) + b\int_0^t Y(\tau)dW(\tau)$ a.s. for all $a, b \in \mathbb{R}$ and $0 \le t \le T$,

5. $\int_0^t X(\tau)dW(\tau)$ is $\mathcal{F}_t$-measurable for $0 \le t \le T$,

6. $\Big\{\int_0^t X(\tau)dW(\tau)\Big\}_{t\ge 0}$ is continuous with probability one and progressively measurable.

2.2 Stochastic Integrals as Time Change of Wiener Processes

In this section, we prepare the mathematical tools required for defining transformation of Wiener processes.

The constructions below are similar to (Oksendal (1998), Section 8.5) and described in (Srihirun (2005), Section 3). Let $\eta(t, x, a)$ be a sufficiently many times continuously differentiable function and $\{X(t)\}_{t\ge 0}$ a continuous and adapted stochastic process. Since $\eta^2(t, x, a)$ is continuous, $\eta^2(t, X(t, \omega), a)$ is also an adapted process. Define

$$\beta(t, \omega, a) = \int_0^t \eta^2(s, X(s, \omega), a)ds, \quad t \ge 0. \tag{2.2}$$

For brevity we write $\beta(t)$ instead of $\beta(t, \omega, a)$. The function $\beta(t)$ is called a random time change with time change rate $\eta^2(t, X(t, \omega), a)$ and $\beta(t)$ is an adapted process.

Suppose now that $\eta(t, x, a) \neq 0$ for all (t, x, a). Then for each ω, the map $t \longmapsto \beta(t)$ is strictly increasing. Next define

$$\alpha(t, \omega, a) = \inf_{s \geq 0}\{s : \beta(s, \omega, a) > t\}, \tag{2.3}$$

and for brevity, write $\alpha(t)$ instead of $\alpha(t, \omega, a)$. For all ω, the map $t \longmapsto \alpha(t)$ is nondecreasing and continuous. One easily shows that for all ω, and for all $t \geq 0$,

$$\beta(\alpha(t)) = t = \alpha(\beta(t)). \tag{2.4}$$

Since $\beta(t)$ is an $\mathcal{F}_t$-adapted process, one has

$$\{\omega : \alpha(t) \leq s\} = \{\omega : t \leq \beta(s)\} \in \mathcal{F}_s, \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

Hence $t \longmapsto \alpha(t)$ is an $\mathcal{F}_s$-stopping time for each t.

Definition 2.4.(Stopping time) A nonnegative random variable τ, which is allowed to take the value ∞, is called stopping time with respect to the filtration $\mathcal{F}_t = (\mathcal{F}_s)_{s \leq t}$ if for each t, the event $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$.

The following theorem will be crucial for defining the transformation of a Wiener process.

Theorem 2.1. Let $\eta(t, x, a)$ and $\{X(t)\}_{t \geq 0}$ be as above and $\{W(t)\}_{t \geq 0}$ a standard Wiener processes. Define

$$\overline{W}(t) = \int_0^t \eta(s, X(s, \omega), a)dW(s), \quad t \geq 0. \tag{2.5}$$

Then $(\overline{W}_{\alpha(t)}, \mathcal{F}_{\alpha(t)})$ is a standard Wiener processes, where

$$\mathcal{F}_{\alpha(t)} = \{A \in \mathcal{F} : A \cap \{\omega : \alpha(t) \leq s\} \in \mathcal{F}_s, \text{for all } s \geq 0\}.$$

Proof. See Srihirun (2011), p.17, Theorem 3.1.

2.3 Itô's Formula

Let $X(t)$ be a stochastic process. We say that $X(t)$ has stochastic differential

$$dX(t) = f(t)dt + G(t)dW(t), \tag{2.6}$$

if

$$X(t) = X(t_0) + \int_{t_0}^{t} f(s)ds + \int_{t_0}^{t} G(s)dW(s),$$

for some Wiener Process $W(t)$. Similarly, let $X(t) = (X_1(t), ..., X_n(t))^T$ be a vector valued process, $W(t) = (W_1(t), ..., W_m(t))^T$ an m-dimensional Wiener Process with independent components, $f(t) = (f_1(t), ..., f_n(t))^T$ a measurable vector function, and $G(t) = \{g_{ij}(t)\}$ a measurable $n \times m$ matrix function with components g_{ij}. We say that $X(t)$ has stochastic differential

$$dX(t) = f(t)dt + G(t)dW(t),$$

if for each $i = 1, ..., n$

$$dX_i(t) = f_i(t)dt + \sum_{j=1}^{m} g_{ij}(t)dW_j(t).$$

that is

$$X_i(t) = X_i(t_0) + \int_{t_0}^{t} f_i(s)ds + \sum_{j=1}^{m} \int_{t_0}^{t} g_{ij}(s)dW_j(s).$$

Itô's formula is a stochastic version of the chain rule and allows us to express $F(t, X_1(t), ..., X_n(t))$ as a stochastic differential:

$$\begin{aligned} dF = \left(\frac{\partial F}{\partial t} + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} f_i + \sum_{i,j=1}^{n} \sum_{k=1}^{m} \frac{1}{2} \frac{\partial^2 F}{\partial x_i \partial x_j} g_{ik} g_{jk} \right) dt \\ + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \sum_{j=1}^{m} g_{ij} dW_j. \end{aligned} \tag{2.7}$$

Hence, F is assumed to be continuously, differentiable in t, and twice continuously, differentiable in $x = (x_1, ..., x_n)$.

CHAPTER III

GROUP ANALYSIS

In this Chapter, the group analysis method is discussed. A general introduction to this method can be found in common textbooks (cf. Ovsiannikov (1978), Ibragimov (1899), Handbook of Lie Group Analysis (1994), (1995), (1996)).

3.1 Local Lie Group

We consider invertible point transformations

$$\bar{z}^i = g^i(z; a), \tag{3.1}$$

where $i = 1, 2, ..., N, z \in V \subset R^N$ and $a \in \Delta$ is a parameter. The set V is an open set in R^N, and Δ is an interval in R^1 symmetric w.r.t. zero.

For differential equations the variable z is separated into two parts, $z = (x, u) \in V \subset R^n \times R^m, N = n + m$. Here $x = (x_1, x_2, ..., x_n) \in R^n$ is considered as the independent variable, $u = (u^1, u^2, ..., u^m) \in R^m$ is considered as the dependent variable. For the transformations we use

$$\bar{x}_i = \varphi^i(x, u; a),\ \bar{u}^j = \psi^j(x, u; a), \tag{3.2}$$

where $i = 1, 2, ..., n,\ j = 1, 2, ..., m,\ (x, u) \in V$.

3.1.1 One-Parameter Lie Group of Transformations

Definition 1. A set of transformations (3.1) is called a local one-parameter Lie group if it has the following properties:

1. $g(z;0) = z$ for all $z \in V$.
2. $g(g(z;a), b) = g(z; a+b)$ for all $a, b, a+b \in \Delta, z \in V$.
3. If for $a \in \Delta$ we have $g(z;a) = z$ for all $z \in V$, then $a = 0$.
4. $g \in C^\infty(V, \Delta)$.

Let us define the functions

$$\xi^{x_i}(x,u) = \left.\frac{\partial \varphi^i(x,u;a)}{\partial a}\right|_{a=\ 0}, \quad \eta^{u^j}(x,u) = \left.\frac{\partial \psi^j(x,u;a)}{\partial a}\right|_{a=\ 0},$$

and set

$$X = \xi^{x_i}(x,u)\partial_{x_i} + \eta^{u^j}(x,u)\partial_{u^j}. \tag{3.3}$$

The operator X is called an infinitesimal generator or the generator of the Lie group of transformations (3.2). The coefficients ξ^{x_i}, η^{u^j} are called the coefficients of the generator.

A local Lie group of transformations (3.2) can be completely determined by the solution of a Cauchy problem of ordinary differential equations, which are called the Lie equations:

$$\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}a} = \xi^{x_i}(\bar{x}, \bar{u}), \quad \frac{\mathrm{d}\bar{u}^j}{\mathrm{d}a} = \eta^{u^j}(\bar{x}, \bar{u}) \tag{3.4}$$

with the initial data

$$\bar{x}_i|_{a=\ 0} = x_i, \ \ \bar{u}^j\big|_{a=\ 0} = u^j. \tag{3.5}$$

Theorem 1 (Lie)**.** Given a vector field $\zeta = (\xi, \eta) : V \to R^N$ of class $C^\infty(V)$ with $\zeta(z_0) \neq 0$ for some $z_0 \in V$, then the solution of the Cauchy problem (3.4), (3.5) generates a local Lie group with the infinitesimal generator $X = \xi^{x_i}(x,u)\partial_{x_i} + \eta^{u^j}(x,u)\partial_{u^j}$. Conversely, let functions $\varphi^i(x,u;a),\ i = 1, ..., n$ and $\psi^j(x,u;a),\ j = 1, ..., m$ satisfy the properties of a Lie group and have the expansion

$$\bar{x}_i = \varphi^i(x,u;a) \approx x_i + \xi^{x_i}(x,u)a,$$

$$\bar{u}^j = \psi^j(x,u;a) \approx u^j + \eta^{u^j}(x,u)a$$

where

$$\xi^{x_i}(x,u) = \left.\frac{\partial \varphi^i(x,u;a)}{\partial a}\right|_{a=\ 0}, \quad \eta^{u^j}(x,u) = \left.\frac{\partial \psi^j(x,u;a)}{\partial a}\right|_{a=\ 0}.$$

Then the functions $\varphi^i(x,u;a)$, $\psi^j(x,u;a)$ solve the Cauchy problem (3.4), (3.5).

Thus, Lie's theorem establishes a one-to-one correspondence between Lie groups of transformations and infinitesimal generators.

3.1.2 Prolongation of a Lie Group

Let $Z = R^n(x) \times R^m(u)$ denote the space of independent and dependent variables. We want to "prolong" the group of transformations by including some derivatives in the transformations. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a multi-index, and set $|\alpha| \equiv \alpha_1 + \alpha_2 + ... + \alpha_n$ and $\alpha, i \equiv (\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, ..., \alpha_n)$. The variable p^k_α will denote the derivative $\frac{\partial^{|\alpha|} u^k}{\partial x^\alpha}$ defined as

$$p^k_\alpha = \frac{\partial^{|\alpha|} u^k}{\partial x^\alpha} = \frac{\partial^{|\alpha|} u^k}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}},$$

The space J^l of the variables

$$x = (x_i), u = (u^k), p = (p^k_\alpha)$$

$$(i = 1, 2, ..., n;\ k = 1, 2, ..., m;\ |\alpha| \leq l)$$

is called the l-th prolongation of the space Z. This space can be provided with a manifold structure. For convenience we agree that $J^0 \equiv Z$.

Definition 2. The generator

$$X^l = X + \sum_{j,\alpha} \eta^j_\alpha \partial_{p^j_\alpha}$$

with the coefficients

$$\eta^j_{\alpha,k} = D_k \eta^j_\alpha - \sum_i p^j_{\alpha,i} D_k \xi^i \tag{3.6}$$

is called the l-th prolongation of the generator X.

Here the operators

$$D_k = \frac{\partial}{\partial x_k} + \sum_{j,\alpha} p^j_{\alpha,k} \frac{\partial}{\partial p^j_\alpha}$$

are operators of the total derivatives with respect to x_k, where

$$k = 1, 2, ..., n; \ j = 1, 2, ..., m; \ |\alpha| \le l, \ \xi^i = \xi^{x_i}, \ \eta^j_0 = \eta^{u^j},$$

and $\xi^{x_i}, \ \eta^{u^j}$ are defined as in (3.3).

For a simple illustration of using the prolongation formulae (3.6), let us study the first prolongation of the generator X with $n = m = 1$. In this case, the generator X^1 induces a local Lie group of transformations in the space J^1:

$$\bar{x} = \varphi(x, u; a), \ \bar{u} = \psi(x, u; a), \ \bar{p} = f(x, u, p; a), \tag{3.7}$$

with the generator

$$X^1 = \xi^x(x, u)\partial_x + \eta^u(x, u)\partial_u + \zeta^p(x, u, p)\partial_p \ , \tag{3.8}$$

where

$$\zeta^p = D_x(\eta^u) - pD_x(\xi^x), \ p = \frac{du}{dx} \ , \ \bar{p} = \frac{d\bar{u}}{d\bar{x}} \ .$$

Notice that the coefficients $\xi^x, \ \eta^u$ are defined as in (3.3). Let us show in the following why the coefficient ζ^p must be of this form. Let a function $u_0(x)$ be given. Substituting it into the first equation of (3.7), one obtains

$$\bar{x} = \varphi(x, u_0(x); a).$$

Since $\varphi(x, u_0(x); 0) = x$, the Jacobian at $a = 0$ is

$$\left.\frac{\partial \bar{x}}{\partial x}\right|_{a=\,0} = \left.\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x}\right)\right|_{a=\,0} = 1.$$

Thus, by virtue of the inverse function theorem, in some neighborhood of $a = 0$ one can express x as a function of $\bar{x}$ and a,

$$x = \phi(\bar{x}, a). \tag{3.9}$$

Note that after substituting (3.9) into the first equation (3.7), one has the identities

$$\begin{aligned} \bar{x} &= \varphi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \\ x &= \phi(\varphi(x, u_0(x); a), a) \end{aligned} \tag{3.10}$$

Substituting (3.9) into the second equation of (3.7), one obtains the transformed function

$$u_a(\bar{x}) = \psi(\phi(\bar{x}, a), u_0(\phi(\bar{x}, a)); a). \tag{3.11}$$

Differentiating the function $u_a(\bar{x})$ with respect to $\bar{x}$, one gets

$$\bar{u}_{\bar{x}} = \frac{\partial u_a(\bar{x})}{\partial \bar{x}} = \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \psi}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}u_0'\right)\frac{\partial \phi}{\partial \bar{x}},$$

where the derivative $\frac{\partial \phi}{\partial \bar{x}}$ can be found by differentiating equation (3.10) with respect to $\bar{x}$,

$$1 = \frac{\partial \varphi}{\partial x}\frac{\partial \phi}{\partial \bar{x}} + \frac{\partial \varphi}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\right)\frac{\partial \phi}{\partial \bar{x}}.$$

Since

$$\frac{\partial \varphi}{\partial x}(\phi(\bar{x}, 0), u_0(\phi(\bar{x}, 0)); 0) = 1, \ \frac{\partial \varphi}{\partial u}(\phi(\bar{x}, 0), u_0(\phi(\bar{x}, 0)); 0) = 0, \tag{3.12}$$

one has $\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0' \neq 0$ in some neighborhood of $a = 0$. Thus,

$$\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\right)^{-1},$$

and

$$\bar{u}_{\bar{x}} = \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}u_0'\right)\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial u}u_0'\right)^{-1} =: g(x, u_0, u_0'; a). \tag{3.13}$$

Transformation (3.7) together with

$$\bar{u}_{\bar{x}} = g(x, u, u_x; a)$$

is called the prolongation of (3.7). Now, we define the coefficient ζ^p as follows:

$$\zeta^p(x,u,p) = \left.\frac{\partial g(x,u,p;a)}{\partial a}\right|_{a=0}, \quad g|_{a=0} = p. \tag{3.14}$$

Equation (3.13) can be rewritten

$$g(x,u,p;a)\left(\frac{\partial \varphi(x,u;a)}{\partial x} + p\frac{\partial \varphi(x,u;a)}{\partial u}\right) = \left(\frac{\partial \psi(x,u;a)}{\partial x} + p\frac{\partial \psi(x,u;a)}{\partial u}\right).$$

Differentiating this equation with respect to the group parameter a and substituting $a = 0$, one finds

$$\left.\left(\frac{\partial g}{\partial a}\left(\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial u}\right) + g\left(\frac{\partial^2 \varphi}{\partial x \partial a} + p\frac{\partial^2 \varphi}{\partial u \partial a}\right)\right)\right|_{a=0} = \left.\left(\frac{\partial^2 \psi}{\partial x \partial a} + p\frac{\partial^2 \psi}{\partial u \partial a}\right)\right|_{a=0}$$

or

$$\zeta^p(x,u,p) = \left.\frac{\partial g}{\partial a}\right|_{a=0}\left.\left(\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial u}\right)\right|_{a=0}.$$

Since by (3.12)

$$\left.\left(\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial u}\right)\right|_{a=0} = 1,$$

then

$$\begin{aligned}
\zeta^p(x,u,p) &= \left.\left(\frac{\partial^2 \psi}{\partial x \partial a} + p\frac{\partial^2 \psi}{\partial u \partial a}\right)\right|_{a=0} - g|_{a=0}\left.\left(\frac{\partial^2 \varphi}{\partial x \partial a} + p\frac{\partial^2 \varphi}{\partial u \partial a}\right)\right|_{a=0} \\
&= \left(\frac{\partial \eta^u}{\partial x} + p\frac{\partial \eta^u}{\partial u}\right) - p\left(\frac{\partial \xi^x}{\partial x} + p\frac{\partial \xi^x}{\partial u}\right) \\
&= D_x(\eta^u) - pD_x(\xi^x)
\end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + p_x\frac{\partial}{\partial p} + \dots, \quad \xi^x = \left.\frac{\partial \varphi}{\partial a}\right|_{a=0}, \quad \eta^u = \left.\frac{\partial \psi}{\partial a}\right|_{a=0}, \quad \zeta^p = \left.\frac{\partial g}{\partial a}\right|_{a=0}.$$

Thus, the first prolongation of the generator (3.3) is given by

$$X^{(1)} = X + \zeta^p(x,u,p)\partial_p.$$

Similarly one can obtain prolongation formulae for any order prolongation of an infinitesimal generator.

3.1.3 Lie Groups Admitted by Differential Equations

Admitted Lie groups of transformations are related with differential equations by the following.

Consider a manifold M which is defined by a system of partial differential equations

$$F^k(x, u, p) = 0, \quad (k = 1, 2, ..., s). \tag{3.15}$$

Hence

$$M = \{(x, u, p) \mid \quad F^k(x, u, p) = 0, \quad (k = 1, ..., s)\}.$$

Here x is the independent variable, u is the dependent variable and p are arbitrary partial derivatives of u with respect x. The manifold M is assumed to be regular, i.e.

$$rank \left(\frac{\partial(F)}{\partial\ (u, p)} \right) = s.$$

Definition 3. A manifold M is said to be invariant with respect to the group of transformations (3.2), if these transformations carry every point of the manifold M along this manifold, i.e.

$$F^k(\bar{x}, \bar{u}, \bar{p}) = 0, \quad (k = 1, 2, ..., s).$$

Accordingly, equations (3.15) are not changed under the Lie group of transformations and we say that the Lie group of transformations (3.2) is admitted by equations (3.15).

In order to find an infinitesimal generator of a Lie group admitted by differential equations (3.15) one can use the following theorem.

Theorem 2. A system of equations (3.15) is not changed with respect to the Lie group of transformations (3.2) with the infinitesimal generator

$$X = \xi^i \partial_{x_i} + \eta^j \partial_{u^j}$$

if and only if

$$X^{(p)}F^k\mid_{M=0} = 0,\ \ (k = 1, ..., s). \tag{3.16}$$

Equations (3.16) are called determining equations.

The determining equations (3.16) can be computed. These equations are linear homogeneous differential for the unknown coefficients $\xi^i(x,u)$, $\eta^j(x,u)$. Since the coefficients of a generator X do not depend on the derivatives p^k_α the determining equations can be split with respect to the parametric derivatives. The split system of equations is an overdetermined system. The general solution of the determining equations generates a principal Lie algebra of the system (3.15). The set of transformations, which is finitely generated by one-parameter Lie groups corresponding to the generators X is called the principal Lie group admitted by the system (3.15). Later this approach will be applied for stochastic differential equations for obtaining their determining equations.

Definition 4. A function $J(x,u)$ is called an invariant of a Lie group if

$$J(\bar{x},\bar{u}) = J(x,u).$$

Theorem 3. A function $J(x,u)$ is an invariant of the Lie group with the generator X if and only if,

$$XJ(x,u) = 0. \tag{3.17}$$

Definition 5. A vector function $\vec{J}(x,u)$ defines a relative invariant if the manifold defined by the equation $\vec{J}(x,u) = 0$ is an invariant manifold.

Using theorem 2, one obtains the following theorem.

Theorem 4. The functions $J^k(x,u)$ are relative invariant of Lie group with the generator X if and only if,

$$XJ^k\mid_{\vec{J}=0} = 0,$$

where J^k denotes the components of $\vec{J}$.

Let H be a Lie group of transformations, admitted by a system of differential equations (3.15).

Definition 6. A solution $u = \varphi(x)$ of the system (3.15) is called an H-invariant solution if the manifold $u = \varphi(x)$ is an invariant manifold with respect to any transformation of the group H.

Invariant solutions are constructed as follows. First, one finds all the independent invariants $J^k = J^k(x, u),\ (k = 1, ..., m + \sigma)$. Here $\sigma = n - r_*$ is the number of independent variables, which is called the rank of the invariant solution. The rank of the Jacobi matrix $\dfrac{\partial(J^1, ..., J^{m+\sigma})}{\partial(u^1, ..., u^m)}$ has to be equal to m. Without loss of generality one can choose the first invariants $J^1, ..., J^m$ such that the rank of the Jacobi matrix $\dfrac{\partial(J^1, ..., J^m)}{\partial(u^1, ..., u^m)}$ is equal to m. At the next step one supposes that the first m invariants $J^k,\ (k = 1, ...m)$ depend on the remaining invariants $J^k,\ (k = m + 1, ...m + \sigma)$,

$$J^k = \varphi^k(J^{m+1}, ..., J^{m+\sigma}), \quad (k = 1, ...m) \tag{3.18}$$

Equations (3.18) should be such that they can be solved with respect to all dependent variables $u^i,\ (i = 1, ..., m)$. After substituting the representation of the functions u^i into the initial system of partial differential equations, one obtains the system of equations for the unknown functions $\varphi^k,\ (k = 1, ..., m)$. This system involves a smaller number of independent variables.

CHAPTER IV

APPLICATION OF GROUP ANALYSIS TO THE DETERMINISTIC GAS AND HYDRO DYNAMICS EQUATIONS

This chapter deals with the deterministic gas and hydrodynamics equations. Hydrodynamics equations were the first object of application of group analysis in Ovsiannikov (1958). Recently the PODMODELI (SUBMODELS) program in Ovsiannikov (1994) aimed for an exhaustive use of the group analysis method for studying solutions of the gas dynamics equations. Results of this study are summarized in Ovsiannikov (1999). Group analysis was first applied to the Navier-Stokes equations in Pukhnachov (1960). A review of exact solutions of the Navier-Stokes equations can be found in Andreev, Kaptsov, Pukhnachov and Rodionov (1998), and in Pukhnachov (2006).

4.1 One-Dimensional Gas Dynamics Equations

The one-dimensional gas dynamics equations have the form

$$\begin{gathered}\rho_t + u\rho_x + \rho u_x = 0,\\ \rho(u_t + uu_x) + p_x = 0,\\ p_t + up_x + A(p,\rho)u_x = 0,\end{gathered} \tag{4.1}$$

where ρ is the density, p is the pressure, and u is the velocity of the gas in the direction x. The function $A(p, \rho)$ is related with the state equation of the gas. For exanple, for a polytropic gas $A = \gamma p$, where γ is constant.

4.1.1 Admitted Lie Group

For arbitrary $A(p,\rho)$, the symmetry Lie algebra is four-dimensional and thus denoted L_4, and is spanned by the generators (Ovsiannikov(1962))

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x},$$
$$X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

For a polytropic gas $A = \gamma p$ this algebra extends to a six-dimensional Lie algebra L_6 by the additional generators

$$X_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + 2\rho\frac{\partial}{\partial \rho}, \quad X_6 = p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}.$$

If $\gamma = 3$ which corresponds to mono-atomic gas, then there is one more extension to the Lie algebra L_7 by the generator

$$X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tu)\frac{\partial}{\partial u} - 3tp\frac{\partial}{\partial p} - t\rho\frac{\partial}{\partial \rho}.$$

In this thesis we compare our results with the class of solutions which are called self-similar solutions. The class of self-similar solutions is used for the explanation of many physical phenomena in continuum mechanics (Sedov (1993)).

4.1.2 Invariant Solutions

A self-similar solution is an invariant solution of an admitted Lie group, which is related with scaling of the variables. In this section, representations of self-similar solutions are shown. Later, similar representations of invariant solutions of the stochastic gas dynamics equations are shown.

1. Invariants of the subalgebra spanned by the operator

$$X = (2+\beta)X_6 - X_5 = -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + (2+\beta)p\frac{\partial}{\partial p} + \beta\frac{\partial}{\partial \rho}$$

are

$$ut, \quad pt^{(\beta+2)}, \quad \rho t^{\beta}, \quad x.$$

The representation of the corresponding invariant solution has the form

$$u = t^{-1}\phi^1(x), \quad p = t^{-(\beta+2)}\phi^2(x), \quad \rho = t^{-\beta}\phi^3(x).$$

The reduced system of equations is obtained by substituting the representation of the invariant solution into the original system of the gas dynamics equations

$$\begin{gathered}
\phi^3(\phi^1_x - \beta) + \phi^3_x\phi^1 = 0, \\
\phi^1\phi^2_x + \phi^2(\gamma\phi^1_x - (\beta+2)) = 0, \\
\phi^1 - \left(\phi^1\phi^1_x + \frac{\phi^2_x}{\phi^3}\right) = 0.
\end{gathered}$$

2. Invariants of the subalgebra spanned by the operator

$$\begin{gathered}
X = (2(\tilde{\alpha}+1)+\beta)X_6 + \tilde{\alpha}X_3 - (1+\tilde{\alpha})X_5 \\
= -t\frac{\partial}{\partial t} + \tilde{\alpha}x\frac{\partial}{\partial x} + (2(\tilde{\alpha}+1)+\beta)p\frac{\partial}{\partial p} + \beta\rho\frac{\partial}{\partial \rho} - (1+\tilde{\alpha})u\frac{\partial}{\partial_u}
\end{gathered}$$

are

$$ut, \quad pxt^{(\beta+2)}, \quad \rho t^{\beta}, \quad x^{\alpha}t,$$

where $\tilde{\alpha} \neq 0$ and $\alpha = 1/\tilde{\alpha}$.

The representation of the corresponding invariant solution has the form

$$u = xt^{-1}\phi^1(x^{\alpha}t), \quad p = x^{-2}t^{(-\beta-2)}\phi^2(x^{\alpha}t), \quad \rho = t^{-\beta}\phi^3(x^{\alpha}t).$$

The reduced system of equations is obtained by substituting the representation of the invariant solution into the original system of the gas dynamics equations

$$\begin{gathered}
\alpha\phi^1_z\phi^3 z + \phi^3_z z(\alpha\phi^1 + 1) + \phi^3(\phi^1 - \beta) = 0, \\
\alpha\phi^1_z\phi^2\gamma z + \phi^2_z z(\alpha\phi^1 + 1) + \phi^2(-\beta + \phi^1\gamma + 2\phi^1 - 2) = 0, \\
\phi^1_z + \left(\alpha z^2\left(\phi^1\phi^1_z + \frac{\phi^2_z}{\phi^3}\right) - (\phi^1)^2 + 2\frac{\phi^2}{\phi^3}\right) = 0.
\end{gathered}$$

3. Invariants of the subalgebra spanned by the operator

$$X = X_3 - X_5 + \beta X_6 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + \beta p\frac{\partial}{\partial p} + (\beta-2)\rho\frac{\partial}{\partial \rho}$$

are

$$ux^{-1}, \quad px^{-\beta}, \quad \rho x^{(2-\beta)}, \quad t.$$

The representation of the corresponding invariant solution has the form

$$u = x\phi^1(t), \quad p = x^{\beta}\phi^2(t), \quad \rho = x^{(\beta-2)}\phi^3(t).$$

The reduced system of equations is

$$\phi^3_t + (\beta - 1)\phi^1\phi^3 = 0,$$
$$\phi^2_t + (\beta + 3)\phi^1\phi^2 = 0,$$
$$\phi^1_t + (\phi^1)^2 + \beta\phi^2/\phi^3 = 0.$$

4.2 The Two-Dimensional Navier-Stokes Equations

The two-dimensional Navier-Stokes equations have the form

$$\rho(u_t + uu_x + vu_y) = -p_x + \mu(u_{xx} + u_{yy}),$$
$$\rho(v_t + uv_x + vv_y) = -p_y + \mu(v_{xx} + v_{yy}),$$
$$u_x + v_y = 0,$$

where t is time, ρ is the density, p is the pressure, μ is the coefficient of viscosity, and u, v are the velocities in the direction x and y respectively.

It is useful to write the Navier-Stokes equations in a dimensionless form. Let $\bar{u}, \bar{p}, \bar{x}, \bar{y}, \bar{t}$ be the dimensionless variables, which are related by the formulae

$$u = V\bar{u}, \quad p = Q\bar{p}, \quad x = L\bar{x}, \quad y = L\bar{y}, \quad t = T\bar{t},$$

where V, Q, L and T are velocity, pressure, length and time units, respectively. After dropping the symbol $\bar{\cdot}$, and choosing appropriate units L, V, T, Q, one obtains the equations

$$
\begin{aligned}
u_t + uu_x + vu_y &= -p_x + u_{xx} + u_{yy},\\
v_t + uv_x + vv_y &= -p_y + v_{xx} + v_{yy},\\
u_x + v_y &= 0.
\end{aligned}
\tag{4.2}
$$

In the cylindrical coordinate system

$$
r = \sqrt{x^2+y^2}, \qquad \theta = \arctan\frac{x}{y},
$$

$$
U = u\cos\theta + v\sin\theta, \qquad V = v\cos\theta - u\sin\theta
$$

these equations have the form

$$
\begin{gathered}
U_t + UU_r + \frac{V}{r}U_\theta - \frac{V^2}{r} = -p_r + (U_{rr} + \frac{1}{r}U_r - \frac{U}{r^2} + \frac{1}{r^2}U_{\theta\theta} - \frac{2}{r^2}V_\theta),\\
V_t + UV_r + \frac{V}{r}V_\theta + \frac{UV}{r} = -\frac{1}{r}p_\theta + (V_{rr} + \frac{1}{r}V_r - \frac{V}{r^2} + \frac{1}{r^2}V_{\theta\theta} + \frac{2}{r^2}U_\theta),\\
U_r + \frac{U}{r} + \frac{1}{r}V_\theta = 0.
\end{gathered}
$$

4.2.1 Admitted Lie Group

The symmetry Lie algebra of (4.2) is infinite dimensional and is generated by the generators (Pukhnachev (1960) in the two-dimensional case, Buchnev (1971))

$$
\begin{gathered}
X_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - 2p\frac{\partial}{\partial p},\\
X_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial t},\\
X_4 = \psi_1(t)\frac{\partial}{\partial x} + \psi_1'(t)\frac{\partial}{\partial u} - x\psi_1''(t)\frac{\partial}{\partial p}, \quad X_5 = \frac{\partial}{\partial x},\\
X_6 = \psi_2(t)\frac{\partial}{\partial y} + \psi_2'(t)\frac{\partial}{\partial v} - y\psi_2''(t)\frac{\partial}{\partial p}, \quad X_7 = \frac{\partial}{\partial t}, \quad X_8 = \varphi(t)\frac{\partial}{\partial p},
\end{gathered}
$$

where $\psi_1(t)$, $\psi_2(t)$ and $\varphi(t)$ are arbitrary functions, and $\psi_i'(t)$, $\psi_i''(t)$ are the first and the second-order derivatives of $\psi_i(t)$.

4.2.2 Invariant Solutions

The group classification of the Navier-Stokes equations in the two-dimensional case has been done in (Pukhnachev (1960)). Many invariant solutions of the Navier-Stokes equations are collected in the Handbook of Lie Group Analysis of Differential Equations Vol.2 (Ibragimov (1995)). We use the following solutions.

1. Invariants of the subalgebra spanned by the operators X_1, and X_3 are

$$Ur, \quad Vr, \quad pr^2, \quad \theta.$$

The representation of the corresponding invariant solution has the form

$$U = \frac{\phi^1(\theta)}{r}, \quad V = \frac{\phi^2(\theta)}{r}, \quad p = \frac{\phi^3(\theta)}{r^2}.$$

The reduced system of equations is

$$\phi^1_{\theta\theta} - \phi^2\phi^1_\theta + (\phi^1)^2 + (\phi^2)^2 + 2\phi^3 = 0,$$

$$2\phi^1_\theta - \phi^3_\theta = 0, \quad \phi^2_\theta = 0.$$

2. Invariants under the subalgebra spanned by the generators X_1, and X_2 are

$$Ur, \quad Vr, \quad pr^2, \quad tr^{-2}.$$

The representation of the corresponding invariant solution has the form

$$U = \frac{\phi^1(tr^-2)}{r}, \quad V = \frac{\phi^2(tr^-2)}{r}, \quad p = \frac{\phi^3(tr^-2)}{r^2}.$$

The reduced system of equations is

$$2z\phi^3_z + (\phi^1)^2 + (\phi^2)^2 + 2\phi^3 = 0,$$

$$2z(2z\phi^2_{zz} + (\phi^1 + 4)\phi^2_z) - \phi^2_z = 0, \quad \phi^1_z = 0.$$

CHAPTER V

GROUP ANALYSIS APPLIED TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

In this chapter, the transformation and group analysis method is discussed as it applies to stochastic partial differential equations. In Melnick (2003), a criterion of invariance of a scalar stochastic partial differential equation with respect to a one-parameter group of transformations was given. We will generalize this to systems of equations. A general of the group analysis method can be found in textbooks (cf. Ovsiannikov (1978), Ibragimov (1899), Handbook of Lie Group Analysis of Differential Equations (1994), (1995), (1996)).

5.1 Transformations of Stochastic Partial Differential Equations

On a complete probability space $(\Omega, \mathcal{F}, P)$ consider the stochastic Cauchy problem

$$v(r,y) = v(r_0,y) + \int_{r_0}^{r} A(\tau,y)\,d\tau + \int_{r_0}^{r} B(\tau,y)\,dw(\tau) \tag{5.1}$$

where

$$A(r,y) = A(r,y,v(r,y),v_{y_k}(r,y),v_{y_ky_l}(r,y))$$

is an n-random vector,

$$B(r,y) = B(r,y,v(r,y))$$

an $n \times m$ random matrix, $w(r)$ a vector of m independent standard Wiener processes, $v(r_0, y) = v_0(y)$ a vector of random functions, $r \in [r_0, T]$ and $y = (y_1, \ldots, y_N)^T \in \mathbb{R}^N$. In component form,

$$v^i(r, y) = v^i(r_0, y) + \int_{r_0}^{r} A^i(\tau, y)\, d\tau + \sum_{j=1}^{m} \int_{r_0}^{r} B^{ij}(\tau, y)\, dw^j(\tau), \quad (i = \overline{1, n}) \quad (5.2)$$

where by convention, superscripts denote the vector components of a dependent variable or the entries of a matrix, numeric subscripts the vector components of an independent variable, and alphabetic subscripts the partial derivatives with respect to the given variables.

Consider a change of the dependent and independent variables

$$r = \alpha(t), \quad y = h(t, x), \quad v = g(t, x, u), \qquad (5.3)$$

where the functions $\alpha(t)$, $h(t, x)$ and $g(t, x, u)$ are sufficiently smooth and locally invertible with respect to r, y and v. This means that there exist functions $t = \hat{\alpha}(r)$, $x = \hat{h}(r, y)$ and $u = \hat{g}(r, y, v)$ such that

$$r = \alpha(\hat{\alpha}(r))$$

$$y = h(\hat{\alpha}(r), \hat{h}(r, y)),$$

$$v = g(\hat{\alpha}(r), \hat{h}(r, y), \hat{g}(r, y, v)),$$

and

$$t = \hat{\alpha}(\alpha(t))$$

$$x = \hat{h}(\alpha(t), h(t, x)),$$

$$u = \hat{g}(\alpha(t), h(t, x) g(t, x, u)).$$

To be precise, we are in fact considering local groups of transformations depending on some parameter a which, as it is not required at this stage, is omitted.

The change of variables (5.3) will map a function $v(r, y)$ to a new function

$$u(t, x) = \hat{g}\big(\, \alpha(t), h(t, x), v(\alpha(t), h(t, x)\, \big). \qquad (5.4)$$

Assuming that $v(r, y)$ is a solution of the Cauchy problem (5.1), let us derive the stochastic differential equation for the random function $u(t, x)$. This will be achieved in two steps: First a stochastic differential equation for the function $v\big(\alpha(t), h(t, x)\big)$ will be derived, followed by an application of Itô's formula.

Begin by changing the time variable, using $r = \alpha(t)$. Introducing the notation $\tilde{v}(t, y) = v(\alpha(t), y)$ and $\tilde{v}(t_0, y) = v(\alpha(t_0), y)$, then (5.2) becomes

$$\tilde{v}^i(t, y) = \tilde{v}^i(t_0, y) + \int_{t_0}^{t} \bar{A}^i(\tau, y)\, d\tau + \sum_{j=1}^{m} \int_{t_0}^{t} \bar{B}^{ij}(\tau, y)\, dW^j(\tau) \tag{5.5}$$

where

$$\bar{A}^i(t, y) = A^i(\alpha(t), y, \tilde{v}(t, y), \tilde{v}_{y_k}(t, y), \tilde{v}_{y_k y_l}(t, y)) \cdot \alpha'(t)$$
$$\bar{B}^{ij}(t, y) = B^{ij}(\alpha(t), y, \tilde{v}(t, y)) \cdot \sqrt{\alpha'(t)}.$$

Here each $W^j(t)$ is a new Wiener process such that (see Melnick, 2003)

$$w^j(\alpha(t_2)) - w^j(\alpha(t_1)) = \int_{t_1}^{t_2} \sqrt{\alpha'(s)}\, dW^j(s).$$

Differentiating equation (5.5) with respect to each component y_k of the independent variable y, one obtains

$$\tilde{v}^i_{y_k}(t, y) = \tilde{v}^i_{y_k}(t_0, y) + \int_{t_0}^{t} \bar{A}^i_{y_k}(\tau, y)\, d\tau + \sum_{j=1}^{m} \int_{t_0}^{t} \bar{B}^{ij}_{y_k}(\tau, y)\, dW^j(\tau) \qquad (k = \overline{1, N}). \tag{5.6}$$

Here we have used the property

$$\frac{\partial}{\partial y_k}\left(\int_{t_0}^{t} \bar{A}^i(\tau, y)\, d\tau\right) = \int_{t_0}^{t} \bar{A}^i_{y_k}(\tau, y)\, d\tau \qquad (k = \overline{1, N}),$$

and have assumed the property

$$\frac{\partial}{\partial y_k}\left(\int_{t_0}^{t} \bar{B}^{ij}(\tau, y)\, dW^j(\tau)\right) = \int_{t_0}^{t} \bar{B}^{ij}_{y_k}(\tau, y)\, dW^j(\tau) \qquad (k = \overline{1, N}).$$

Substituting $y = h(t,x)$ into equations (5.6), multiplying each by $h_t^k(t,x)$, and integrating with respect to t, they become

$$\int_{t_0}^{t} h_t^k(\tau,x)\tilde{v}^i_{y_k}(\tau,h(\tau,x))\,d\tau = \int_{t_0}^{t} h_t^k(\tau,x)\tilde{v}^i_{y_k}(t_0,h(\tau,x))\,d\tau$$
$$+\int_{t_0}^{t} h_t^k(\tau,x)\left(\int_{t_0}^{\tau}\bar{A}^i_{y_k}(s,h(\tau,x))ds\right)d\tau$$
$$+\sum_{j=1}^{m}\int_{t_0}^{t} h_t^k(\tau,x)\left(\int_{t_0}^{\tau}\bar{B}^{ij}_{y_k}(s,h(\tau,x))\,dW^j(s)\right)d\tau \qquad (k=\overline{1,N}). \tag{5.7}$$

Summing the first terms on the right-hand side of equations (5.7) over k, we obtain

$$\sum_{k=1}^{l}\int_{t_0}^{t} h_t^k(\tau,x)\tilde{v}^i_{y_k}(t_0,h(\tau,x))\,d\tau = \int_{t_0}^{t} d\tilde{v}^i(t_0,h(\tau,x))$$
$$= \tilde{v}^i(t_0,h(t,x)) - \tilde{v}^i(t_0,h(t_0,x)).$$

Summing the second terms on the right-hand side of equations (5.7) over k and using Fubini's theorem gives

$$\sum_{k=1}^{l}\int_{t_0}^{t} h_t^k(\tau,x)\left(\int_{t_0}^{\tau}\bar{A}^i_{y_k}(s,h(\tau,x))ds\right)d\tau$$
$$= \sum_{k=1}^{l}\int_{t_0}^{t}\left(\int_{s}^{t} h_t^k(\tau,x)\bar{A}^i_{y_k}(s,h(\tau,x))d\tau\right)ds$$
$$= \int_{t_0}^{t}\left(\int_{s}^{t} d_\tau\bar{A}^i(s,h(\tau,x))\right)ds$$
$$= \int_{t_0}^{t}\big(\bar{A}^i(s,h(t,x)) - \bar{A}^i(s,h(s,x))\big)\,ds.$$

Summing the third terms on the right-hand side of equations (5.7) over k and using Fubini's theorem for the Itô integral, (Medvedyev (2007), p. 328, Corollary 5.28) we obtain

$$\begin{aligned}\sum_{k=1}^{l}\int_{t_0}^{t} h_t^k(\tau,x)\left(\int_{t_0}^{\tau}\bar{B}^{ij}_{y_k}(s,h(\tau,x))\,dW^j(s)\right)d\tau \\ &= \sum_{k=1}^{l}\int_{t_0}^{t}\left(\int_{s}^{t} h_t^k(\tau,x)\bar{B}^{ij}_{y_k}(s,h(\tau,x))\,d\tau\right)dW^j(s)\\ &= \int_{t_0}^{t}\left(\int_{s}^{t} d_\tau\bar{B}^{ij}(s,h(\tau,x))\right)dW^j(s)\\ &= \int_{t_0}^{t}\left(\bar{B}^{ij}(s,h(t,x))-\bar{B}^{ij}(s,h(s,x))\right)dW^j(s).\end{aligned}$$

Thus, from (5.7) we obtain

$$\begin{aligned}\sum_{k=1}^{l}\int_{t_0}^{t} h_t^k(\tau,x)\tilde{v}^i_{y_k}(\tau,h(\tau,x))\,d\tau &= \tilde{v}^i(t_0,h(t,x))-\tilde{v}^i(t_0,h(t_0,x))\\ &+\left(\int_{t_0}^{t}\bar{A}^i(s,h(t,x))\,ds+\sum_{j=1}^{m}\int_{t_0}^{t}\bar{B}^{ij}(s,h(t,x))\,dW^j(s)\right)\\ &-\left(\int_{t_0}^{t}\bar{A}^i(s,h(s,x))\,ds+\sum_{j=1}^{m}\int_{t_0}^{t}\bar{B}^{ij}(s,h(s,x))\,dW^j(s)\right).\end{aligned}$$

Hence, the substitution of $y=h(t,x)$ into equation (5.5) yields

$$\begin{aligned}\tilde{v}^i(t,h(t,x)) &= \tilde{v}^i(t_0,h(t_0,x))+\int_{t_0}^{t}\bar{A}^i(\tau,h(\tau,x))\,d\tau\\ &+\sum_{k=1}^{l}\int_{t_0}^{t} h_t^k(\tau,x)\tilde{v}^i_{y_k}(\tau,h(\tau,x))\,d\tau+\sum_{j=1}^{m}\int_{t_0}^{t}\bar{B}^{ij}(\tau,h(\tau,x))\,dW^j(\tau)\end{aligned} \tag{5.8}$$

and shows that $\tilde{v}^i$ is an Itô process. This equation can be rewritten in differential form as

$$d\hat{v}^i=\hat{A}^i\,dt+\sum_{j=1}^{m}\hat{B^{ij}}\,dW^j(t),$$

where

$$\hat{v}^i(t,x) = \tilde{v}^i(t,h(t,x)) = v^i(\alpha(t),h(t,x)),$$

$$\hat{A}^i(t,x) = \bar{A}^i(t,h(t,x)) + \sum_{k=1}^{l} \tilde{v}^i_{y_k}(t,h(t,x))h^k_t(t,x)$$

$$= A^i(\alpha(t),h(t,x))\cdot\alpha'(t) + \sum_{k=1}^{l} v^i_{y_k}(\alpha(t),h(t,x))h^k_t(t,x),$$

$$\hat{B}^{ij}(t,x) = \bar{B}^{ij}(t,h(t,x)) = B^{ij}(\alpha(t),h(t,x))\cdot\sqrt{\alpha'(t)}.$$

In this notation, the components of (5.4) my be written as

$$u^i(t,x) = \tilde{g}^i\big(\,t,x,\hat{v}(t,x)\,\big)$$

where we have set

$$\tilde{g}\big(\,t,x,\hat{v}(t,x)\,\big) = \tilde{g}\big(\,t,x,\tilde{v}(t,h(t,x))\,\big) = \hat{g}\big(\,\alpha(t),h(t,x),v(\alpha(t),h(t,x))\,\big).$$

Applying Itô's formula one obtains

$$du^i = \left[\tilde{g}^i_t + \sum_{j=1}^{n}\hat{A}^j\tilde{g}^i_{\hat{v}^j} + \frac{1}{2}\sum_{k=1}^{n}\sum_{\sigma=1}^{n}\Big(\sum_{j=1}^{m}\hat{B}^{kj}\hat{B}^{\sigma j}\Big)\tilde{g}^i_{\hat{v}^k\hat{v}^\sigma}\right]dt + \sum_{k=1}^{n}\tilde{g}^i_{\hat{v}^k}\sum_{j=1}^{m}\hat{B}^{kj}\,dW^j(t)$$

or equivalently,

$$\begin{aligned} du^i = \Big[\alpha'\hat{g}^i_r + \sum_{k=1}^{l}h^k_t\hat{g}^i_{y_k} + \sum_{j=1}^{n}\left(\alpha' A^j + \sum_{k=1}^{l}v^j_{y_k}h^k_t\right)\hat{g}^i_{v_j} \\ +\frac{\alpha'}{2}\sum_{k=1}^{n}\sum_{\sigma=1}^{n}\left(\sum_{j=1}^{m}B^{kj}B^{\sigma j}\right)\hat{g}^i_{v^k v^\sigma}\Big]dt + \sqrt{\alpha'}\sum_{k=1}^{n}\hat{g}^i_{v^k}\sum_{j=1}^{m}B^{kj}\,dW^j(t). \end{aligned} \tag{5.9}$$

One needs to express $\hat{g}_r$, $\hat{g}_{y_k}$, $\hat{g}_{v^k}$, $\hat{g}_{v^k v^\sigma}$, A and B in equation (5.9) in terms of the functions α, h, g, u and their derivatives.

For this purpose, consider the identity

$$u^i = \hat{g}^i(\alpha(t),h(t,x),g(t,x,u)) \qquad (i=\overline{1,n}). \tag{5.10}$$

Differentiating equations (5.10) with respect to each u^j

$$\frac{\partial}{\partial u^1}: \quad \hat{g}^i_{v^1} g^1_{u^1} + \hat{g}^i_{v^2} g^2_{u^1} + \hat{g}^i_{v^3} g^3_{u^1} + ... + \hat{g}^i_{v^n} g^n_{u^1} = 0$$

$$\frac{\partial}{\partial u^2}: \quad \hat{g}^i_{v^1} g^1_{u^2} + \hat{g}^i_{v^2} g^2_{u^2} + \hat{g}^i_{v^3} g^3_{u^2} + ... + \hat{g}^i_{v^n} g^n_{u^2} = 0$$

$$...$$

$$\frac{\partial}{\partial u^i}: \quad \hat{g}^i_{v^1} g^1_{u^i} + \hat{g}^i_{v^2} g^2_{u^i} + \hat{g}^i_{v^3} g^3_{u^i} + ... + \hat{g}^i_{v^n} g^n_{u^i} = 1$$

$$...$$

$$\frac{\partial}{\partial u^n}: \quad \hat{g}^i_{v^1} g^1_{u^n} + \hat{g}^i_{v^2} g^2_{u^n} + \hat{g}^i_{v^3} g^3_{u^n} + ... + \hat{g}^i_{v^n} g^n_{u^n} = 0$$

one obtains

$$\hat{g}^i_{v^j} = \det(M)^{-1} \det(M^{i,j}) \qquad (j = \overline{1, n}), \tag{5.11}$$

where M and M^{ij} have different j-th columns,

$$M = \begin{pmatrix} g^1_{u^1} & . & . & . & g^n_{u^1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g^1_{u^n} & . & . & . & g^n_{u^n} \end{pmatrix}, \qquad M^{i,j} = \begin{pmatrix} g^1_{u^1} & . & 0 & . & g^n_{u^1} \\ . & . & 0 & . & . \\ . & . & 1_{i,j} & . & . \\ g^1_{u^n} & . & 0 & . & g^n_{u^n} \end{pmatrix}.$$

On the other hand, differentiating equations (5.10) with respect to t and x_k one has

$$\frac{\partial}{\partial t}: \quad \hat{g}^i_r \alpha' + \hat{g}^i_{y_1} h^1_t + \hat{g}^i_{y_2} h^2_t + ... + \hat{g}^i_{y_N} h^N_t = -\sum_{j=1}^{n} \hat{g}^i_{v^j} g^j_t$$

$$\frac{\partial}{\partial x_1}: \quad 0 + \hat{g}^i_{y_1} h^1_{x_1} + \hat{g}^i_{y_2} h^2_{x_1} + ... + \hat{g}^i_{y_N} h^N_{x_1} = -\sum_{j=1}^{n} \hat{g}^i_{v^j} g^j_{x_1}$$

$$...$$

$$\frac{\partial}{\partial x_N}: \quad 0 + \hat{g}^i_{y_1} h^1_{x_N} + \hat{g}^i_{y_2} h^2_{x_N} + ... + \hat{g}^i_{y_N} h^N_{x_N} = -\sum_{j=1}^{n} \hat{g}^i_{v^j} g^j_{x_N}$$

$$\hat{g}^i_r = \det(\Delta)^{-1} \det(\Delta^0),$$

$$\hat{g}^i_{y_k} = \det(\Delta)^{-1} \det(\Delta^{k+1}) \qquad (k = \overline{1, N}),$$

where the $(k+1)$-th columns of Δ and Δ^{k+1} differ,

$$\Delta = \begin{pmatrix} \alpha' & h_t^1 & . & . & h_t^N \\ 0 & h_{x_1}^1 & . & . & h_{x_1}^N \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & h_{x^N}^1 & . & . & h_{x^N}^N \end{pmatrix}, \qquad \Delta^{k+1} = \begin{pmatrix} \alpha' & h_t^1 & . & \lambda^0 & . & h_t^N \\ 0 & h_{x_1}^1 & . & \lambda^1 & . & h_{x_1}^N \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & h_{x_N}^1 & . & \lambda^N & . & h_{x_N}^N \end{pmatrix}$$

and where

$$\lambda^0 = -\sum_{j=1}^n \hat{g}^i_{v^j} g^j_t, \qquad \lambda^k = -\sum_{j=1}^n \hat{g}^i_{v^j} g^j_{x_k}.$$

Differentiating the j-th equation in (5.11) with respect to u^j

$$\frac{\partial}{\partial u^1}: \quad \hat{g}^i_{v^j v^1} g^1_{u^1} + \hat{g}^i_{v^j v^2} g^2_{u^1} + \hat{g}^i_{v^j v^3} g^3_{u^1} + ... + \hat{g}^i_{v^j v^n} g^n_{u^1} = \frac{\partial}{\partial u^1}\ \hat{g}^i_{v^j}$$

$$\frac{\partial}{\partial u^2}: \quad \hat{g}^i_{v^j v^1} g^1_{u^2} + \hat{g}^i_{v^j v^2} g^2_{u^2} + \hat{g}^i_{v^j v^3} g^3_{u^2} + ... + \hat{g}^i_{v^j v^n} g^n_{u^2} = \frac{\partial}{\partial u^2}\ \hat{g}^i_{v^j}$$

$$...$$

$$\frac{\partial}{\partial u^n}: \quad \hat{g}^i_{v^j v^1} g^1_{u^n} + \hat{g}^i_{v^j v^2} g^2_{u^n} + \hat{g}^i_{v^j v^3} g^3_{u^n} + ... + \hat{g}^i_{v^j v^n} g^n_{u^n} = \frac{\partial}{\partial u^n}\ \hat{g}^i_{v^j}$$

one obtains

$$\hat{g}^i_{v^j v^k} = \det(M)^{-1} \det(\bar{M}^{i,k}), \qquad (i, k = \overline{1, n})$$

where

$$\bar{M}^{i,k} = \begin{pmatrix} g^1_{u^1} & . & \Lambda^{i,1} & . & g^n_{u^1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g^1_{u^n} & . & \Lambda^{i,n} & . & g^n_{u^n} \end{pmatrix},$$

and the entries in the k-th column are given by

$$\Lambda^{i,\sigma} = \frac{\partial}{\partial u^\sigma}\ \hat{g}^i_{v^j}.$$

The derivatives $v^i_{y_k}(\alpha(t), h(t,x))$ are obtained from the relations

$$v^i(\alpha(t), h(t,x)) = g^i(t, x, u(t,x)). \tag{5.12}$$

Differentiating equation (5.12) with respect to t and each x_k, respectively,

$$\frac{\partial}{\partial t}: \quad v^i_r\alpha' + v^i_{y_1}h^1_t + v^i_{y_2}h^2_t + ... + v^i_{y_N}h^N_t = g^i_t + \sum_{j=1}^n g^i_{u^j}u^j_t$$

$$\frac{\partial}{\partial x_1}: \quad 0 + v^i_{y_1}h^1_{x_1} + v^i_{y_2}h^2_{x_1} + ... + v^i_{y_N}h^N_{x_1} = g^i_{x_1} + \sum_{j=1}^n g^i_{u^j}u^j_{x_1}$$

$$...$$

$$\frac{\partial}{\partial x_N}: \quad 0 + v^i_{y_1}h^1_{x_N} + v^i_{y_2}h^2_{x_N} + ... + v^i_{y_N}h^N_{x_N} = g^i_{x_N} + \sum_{j=1}^n g^i_{u^j}u^j_{x_N}$$

one obtains

$$v^i_{y_k} = \det(\Delta)^{-1}\det(\Theta^{i,k}), \tag{5.13}$$

where

$$\Theta^{i,k} = \begin{pmatrix} \alpha' & h^1_t & . & \theta^0 & . & h^N_t \\ 0 & h^1_{x_1} & . & \theta^1 & . & h^N_{x_1} \\ . & . & . & . & . & . \\ . & . & . & . & . & \\ 0 & h^1_{x_N} & . & \theta^N & . & h^N_{x_N} \end{pmatrix},$$

and the entries in the $(k+1)$-th column are

$$\theta^0 = g^i_t + \sum_{j=1}^n g^i_{u^j}u^j_t, \qquad \theta^\sigma = g^i_{x_\sigma} + \sum_{j=1}^n g^i_{u^j}u^j_{x_\sigma}.$$

For simplicity, equations (5.9) are now rewritten as

$$du^i = F^i(t, x, u, u_{x_k}, u_{x_kx_l})\,dt + \sum_{j=1}^m G^{ij}(t,x,u)\,dW^j(t) \qquad (i = \overline{1,n}).$$

5.2 Determining Equations

According to the definition, invariance of equation (5.1) with respect to a given group of transformations is equivalent to the equalities

$$F^i(t,x,u,u_{x_k},u_{x_kx_l},a) = A^i(t,x,u,u_{x_k},u_{x_kx_l})$$
$$G^{ij}(t,x,u,a) = B^{ij}(t,x,u)$$

for all t, x, u, u_{x_k}, $u_{x_kx_l}$, a, and $i=\overline{1,n}$, $j=\overline{1,m}$, $k,l=\overline{1,N}$.

Differentiating F and G with respect to the parameter a one has

$$\frac{\partial F^i}{\partial a}\Big|_{a=0} = \widetilde{X}(A^i - u^i_t)$$
$$\frac{\partial G^{ij}}{\partial a}\Big|_{a=0} = \widetilde{X}B^{ij} + \frac{B^{ij}}{2}\psi_t - \sum_{k=1}^{n} B^{kj}\zeta^{u^i}_{u^k}$$

where

$$X = \psi(t)\partial_t + \xi^{x_k}(t,x)\partial_{x_k} + \zeta^{u^i}(t,x,u)\partial_{u^i},$$

$\widetilde{X}$ is the prolonged generator of X and

$$\zeta^{u^i_t} = \sum_{j=1}^{n}\left(A^j\psi_t + A^j\zeta^i_{u^j} + \frac{1}{2}\sum_{k=1}^{n}\left(\sum_{\sigma=1}^{m} B^{j\sigma}B^{k\sigma}\right)\zeta^i_{u^ju^k}\right)$$

is substituted into $\widetilde{X}u^i_t$. We can find determining equations of the stochastic partial differential equations by

$$\begin{gathered}\widetilde{X}(A^i - u^i_t) = 0\\ \widetilde{X}B^{ij} + \frac{B^{ij}}{2}\psi_t - \sum_{k=1}^{n} B^{kj}\zeta^{u^i}_{u^k} = 0\end{gathered} \tag{5.14}$$

5.3 Example : The Kardar-Parisi-Zhang Equation

The Kardar-Parisi-Zhang (KPZ) equation is

$$du = \big(u_{xx} + \frac{1}{2}(u_x)^2\big)\,dt + B(t,x,u)\,dW(t). \tag{5.15}$$

Infinitesimal generators of the admitted Lie group are sought in the form

$$X = \psi(t)\partial_t + \xi(t,x)\partial_x + \zeta^u(t,x,u,p,\rho)\partial_u$$

with prolongation

$$\widetilde{X} = X + \zeta^{u_t}\partial_{u_t} + \zeta^{u_x}\partial_{u_x} + \zeta^{u_{xx}}\partial_{u_{xx}}.$$

Applying the operator $\widetilde{X}$ to equations (5.15), one obtains the determining equations

$$\widetilde{X}\left[u_{xx} + \frac{1}{2}(u_x)^2\right] = 0,$$
$$\widetilde{X}B + \frac{B}{2}\psi_t - B\zeta^u_u = 0.$$

Solving this system of equations, one obtains

$$\psi = c_4 + c_5 t, \qquad \xi = c_3 + c_2 t + \frac{1}{2}c_5 x, \qquad \zeta^u = c_1 - c_2 x + \varphi(t,x)e^{-u/2},$$

where

$$8(\varphi_{xx}(t,x) - \varphi_t(t,x)) = B^2(t,x,u)\varphi(t,x).$$

The generator corresponding to these coefficients is

$$X = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + X_\infty$$

with

$$X_1 = \partial_u, \quad X_2 = t\partial_x - x\partial_u, \quad X_3 = \partial_x, \quad X_4 = \partial_t, \quad X_5 = 2t\partial_t + x\partial_x,$$
$$X_\infty = e^{-u/2}\varphi(t,x)\partial_u.$$

We first construct the solutions which are invariant under the operator X_3. The Lie group of transformations corresponding to this basis generator is

$$X_3 \ : \ \bar{t} = t, \quad \bar{x} = x + a, \quad \bar{u} = u$$

$$X_3 J = J_x = 0.$$

Invariants of this subalgebra are

$$u, \quad t.$$

The representation of the corresponding invariant solution has the form

$$u = \phi(t). \tag{5.16}$$

To find the function B, as

$$X_3 B + \frac{B}{2}\psi_t - B\zeta_u^u = B_x = 0$$

then $B = \mu(t, u)$. Substituting this function B and (5.16) into the original equation (5.15), one arrives at a simple stochastic ordinary differential equation.

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} \mu(t, \phi) dW(\tau)$$

for the solutions which are invariant under X_3.

Next we construct the solutions which are invariant under the operator $X = \alpha X_1 + X_4$. The Lie group of transformations corresponding to this basis generator is

$$X \ : \ \bar{t} = t + a, \quad \bar{x} = x, \quad \bar{u} = u + \alpha a$$

$$XJ = \alpha J_u + J_t = 0.$$

Invariants of this subalgebra are

$$u - \alpha t, \quad x.$$

The representation of the corresponding invariant solution has the form

$$u = \phi(x) + \alpha t,$$

To find the function B, as

$$XB + \frac{B}{2}\psi_t - B\zeta_u^u = \alpha B_u + B_t = 0$$

then $B = \mu(x, u - \alpha t)$. Substituting into (5.15) gives

$$\phi_{xx} + \frac{(\phi_x)^2}{2} - \alpha = \frac{-1}{t - t_0} \int_{t_0}^{t} \mu(x, \phi) dW(\tau)$$

that is, the deterministic second order ordinary differential equation

$$\phi_{xx} + \frac{(\phi_x)^2}{2} - \alpha = \frac{-\mu(x, \phi)}{t - t_0}(W(t) - W(t_0)).$$

Since the left-hand side is independent of t, the $\mu(x, \phi) = 0$ and $\phi_{xx} + \dfrac{(\phi_x)^2}{2} = \alpha$.

CHAPTER VI

GROUP ANALYSIS OF STOCHASTIC FLUID DYNAMICS EQUATIONS

We now apply the above techniques to the group analysis of some stochastic fluid dynamics equations. Most of the substantial computations were performed with the help of the REDUCE symbolic software.

6.1 One-Dimensional Gas Dynamics Stochastic Equations

Consider the gas dynamic partial differential equations with stochastic part,

$$\begin{aligned}
&\rho_t + u\rho_x + \rho u_x = 0,\\
&p_t + up_x + \gamma p u_x = 0, \qquad (6.1)\\
&du = -(uu_x + \frac{1}{\rho}p_x)\,dt + B(t,x,u,p,\rho)\,dW(t),
\end{aligned}$$

where γ is constant. Infinitesimal generators are sought in the form

$$X = \psi(t)\partial_t + \xi(t,x)\partial_x + \zeta^u(t,x,u,p,\rho)\partial_u + \zeta^p(t,x,u,p,\rho)\partial_p + \zeta^\rho(t,x,u,p,\rho)\partial_\rho.$$

The prolonged generator is

$$\widetilde{X} = X + \zeta^{u_t}\partial_{u_t} + \zeta^{u_x}\partial_{u_x} + \zeta^{p_t}\partial_{p_t} + \zeta^{p_x}\partial_{p_x} + \zeta^{\rho_t}\partial_{\rho_t} + \zeta^{\rho_x}\partial_{\rho_x},$$

where

$$\zeta^{u_t} = A\zeta^u_u + \frac{1}{2}(B)^2\zeta^u_{uu},$$

and ζ^{u_x}, ζ^{p_t}, ζ^{ρ_t} are the usual prolonged forms. Applying the operator $\widetilde{X}$ to equations (6.1) and substituting $\rho_t = -(u\rho_x + \rho u_x)$, $p_t = -(up_x + \gamma p u_x)$, one obtains the determining equations of the stochastic gas dynamics equations.

$$
\begin{aligned}
\widetilde{X}(\rho_t + u\rho_x + \rho u_x) &= 0, \\
\widetilde{X}(p_t + up_x + \gamma p u_x) &= 0, \\
\widetilde{X}(u_t + uu_x + \frac{1}{\rho}p_x) &= 0, \\
\widetilde{X}B + \frac{B}{2}\psi_t - B\zeta^u_u &= 0.
\end{aligned} \tag{6.2}
$$

We distinguish two cases.

In case $\gamma = 3$ one obtains

$$
\begin{aligned}
\psi &= c_7 t^2 + (c_5 + c_3)t + c_1, \\
\xi &= c_7 tx + c_4 t + c_3 x + c_2, \\
\zeta^u &= -c_7 tu - c_5 u + c_7 x + c_4, \\
\zeta^p &= p(c_6 - 3c_7 t), \\
\zeta^\rho &= \rho(c_6 + 2c_5 - c_7 t).
\end{aligned}
$$

The generator corresponding to these coefficients is

$$
X = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 + c_7X_7
$$

with

$$
X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_t + x\partial_x, \quad X_4 = t\partial_x + \partial_u, \quad X_5 = t\partial_t - u\partial_u + 2\rho\partial_\rho,
$$
$$
X_6 = p\partial_p + \rho\partial_\rho, \quad X_7 = t^2\partial_t + tx\partial_x + (x - tu)\partial_u - 3pt\partial_p - \rho t\partial_\rho.
$$

Let us consider the class of solutions invariant under the operator $X = X_3 - X_5 + \beta X_6$ (Ibragimov (1994), p.258). The Lie group of transformations corresponding to this basis generator is

$$
X \;:\; \bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = ue^a, \quad \bar{p} = pe^{\beta a}, \quad \bar{\rho} = \rho e^{(\beta-2)a}
$$

$$
(X_3 - X_5 + \beta X_6)J = xJ_x + uJ_u + \beta pJ_p + (\beta - 2)\rho J_\rho = 0.
$$

Invariants of this subalgebra are

$$ux^{-1}, \quad px^{-\beta}, \quad \rho x^{(2-\beta)}, \quad t.$$

The representation of the corresponding invariant solution has the form

$$u = x\phi^1(t), \qquad p = x^{\beta}\phi^2(t), \qquad \rho = x^{(\beta-2)}\phi^3(t). \tag{6.3}$$

To find the function B in the Itô part, as

$$(X_3 - X_5 + \beta X_6)B - B\zeta^u_u = xB_x + uB_u + \beta pB_p + (\beta-2)\rho B_\rho - B = 0$$

then B must be of the form

$$B = x\mu\Big(t, ux^{-1}, px^{-\beta}, \rho x^{(2-\beta)}\Big). \tag{6.4}$$

Substituting (6.3) and (6.4) into (6.1), one arrives at the system of stochastic differential equations

$$\phi^3_t + (\beta-1)\phi^1\phi^3 = 0,$$

$$\phi^2_t + (\beta+3)\phi^1\phi^2 = 0,$$

$$d\phi^1(t) = -\big((\phi^1)^2 + \beta\phi^2/\phi^3\big)\,dt + \mu(t, \phi^1, \phi^2, \phi^3)\,dW(t).$$

On the other hand,when $\gamma \neq 3$ one obtains

$$\psi = (c_5 + c_3)t + c_1,$$

$$\xi = c_4 t + c_3 x + c_2,$$

$$\zeta^u = -c_5 u + c_4,$$

$$\zeta^p = pc_6,$$

$$\zeta^\rho = \rho(c_6 + 2c_5).$$

The generator corresponding to these coefficients is

$$X = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6$$

with

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = t\partial_t + x\partial_x, \qquad X_4 = t\partial_x + \partial_u,$$

$$X_5 = t\partial_t - u\partial_u + 2\rho\partial_\rho, \qquad X_6 = p\partial_p + \rho\partial_\rho.$$

We consider the class of solutions invariant under the subalgebra spanned by $X = (2(\tilde{\alpha}+1)+\beta)X_6 + \tilde{\alpha}X_3 - (\tilde{\alpha}+1)X_5$ and calculated by

$$XJ = -tJ_t + \tilde{\alpha}xJ_x + (\tilde{\alpha}+1)uJ_u + (2(\tilde{\alpha}+1)+\beta)pJ_p + \beta\rho J_\rho = 0.$$

If $\tilde{\alpha} = 0$, then the Lie group of transformations corresponding to this basis generator is

$$X \;:\; \bar{t} = te^{-a}, \quad \bar{x} = x \quad \bar{u} = ue^{a}, \quad \bar{p} = pe^{(2+\beta)a}, \quad \bar{\rho} = \rho e^{\beta a}$$

$$XJ = -tJ_t + uJ_u + (2+\beta)pJ_p + \beta\rho J_\rho = 0.$$

Invariants of this subalgebra are

$$ut, \quad pt^{(\beta+2)}, \quad \rho t^{\beta}, \quad x.$$

The representation of the corresponding invariant solution has the form

$$u = t^{-1}\phi^1(x), \qquad p = t^{-(\beta+2)}\phi^2(x), \qquad \rho = t^{-\beta}\phi^3(x). \tag{6.5}$$

To find the necessary form of the function B in the Itô part, as

$$XB + \frac{B}{2}\psi_t - B\zeta_u^u = -tB_t + uB_u + (2+\beta)pB_p + \beta\rho B_\rho - \frac{3}{2}B = 0$$

then

$$B = t^{-\frac{3}{2}}\mu(x, tu, pt^{(\beta+2)}, pt^{\beta}). \tag{6.6}$$

Substituting (6.5) and (6.6) into (6.1), one arrives at the reduced system of stochastic partial differential equations

$$\phi^3(\phi^1_x - \beta) + \phi^3_x \phi^1 = 0$$

$$\phi^1 \phi^2_x + \phi^2(\gamma\phi^1_x - (\beta+2)) = 0$$

$$\left(\phi^1 - \left(\phi^1\phi^1_x + \frac{\phi^2_x}{\phi^3}\right)\right) = \mu(x,\phi^1,\phi^2,\phi^3)\left[\frac{tt_0}{t_0 - t}\int_{t_0}^{t} s^{-\frac{3}{2}} dW(s)\right].$$

Since the left-hand side of the third equation is independent of t, then

$$\mu(x,\phi^1,\phi^2,\phi^3) = 0, \quad \phi^1\phi^1_x + \frac{\phi^2_x}{\phi^3} = \phi^1.$$

If $\tilde{\alpha} \neq 0$, then let $\tilde{\alpha} = \frac{1}{\alpha}$, and the Lie group of transformations corresponding to the above basis generator is

$$X \ : \ \bar{t} = te^{-a}, \quad \bar{x} = xe^{\frac{a}{\alpha}}, \quad \bar{u} = ue^{(\frac{1}{\alpha}+1)a}, \quad \bar{p} = pe^{(2(\frac{1}{\alpha}+1)+\beta)a}, \quad \bar{\rho} = \rho e^{\beta a}$$

$$XJ = -tJ_t + \frac{1}{\alpha}xJ_x + (\frac{1}{\alpha}+1)uJ_u + (2(\frac{1}{\alpha}+1)+\beta)pJ_p + \beta\rho J_\rho = 0,$$

Invariants of this subalgebra are

$$utx^{-1}, \quad pt^{\beta+2}, \quad \rho t^{\beta}, \quad x^{\alpha}t.$$

The representation of the corresponding invariant solution has the form

$$u = xt^{-1}\phi^1(x^\alpha t), \qquad p = x^{-2}t^{(-\beta-2)}\phi^2(x^\alpha t), \qquad \rho = t^{-\beta}\phi^3(x^\alpha t). \tag{6.7}$$

To find the required form of the function B in the Itô part, as

$$XB + \frac{B}{2}\psi_t - B\zeta^u_u$$
$$= -tB_t + \frac{1}{\alpha}xB_x + (\frac{1}{\alpha}+1)uB_u + (2(\frac{1}{\alpha}+1)+\beta)pB_p + \beta\rho B_\rho - (\frac{1}{\alpha}+\frac{3}{2})B = 0$$

then

$$B = xt^{-3/2}\mu(x^\alpha t, utx^{-1}, pt^{\beta+2}, \rho t^{\beta}). \tag{6.8}$$

Substituting (6.7) and (6.8) into (6.1), one arrives at the reduced system

$$\alpha\phi^1_z\phi^3 z + \phi^3_z z(\alpha\phi^1 + 1) + \phi^3(\phi^1 - \beta) = 0$$

$$\alpha\phi^1_z\phi^2\gamma z + \phi^2_z z(\alpha\phi^1 + 1) + \phi^2(-\beta + \phi^1\gamma + 2\phi^1 - 2) = 0$$

$$\begin{aligned} z^{-1}\phi^1(z) = z_0^{-1}\phi^1(z_0) - \int_{z_0}^{z} s^{-2}\Big(\alpha s\Big(\phi^1\phi^1_z + \frac{\phi^2_z}{\phi^3}\Big) - (\phi^1)^2 + 2\frac{\phi^2}{\phi^3}\Big)ds \\ + \int_{z_0}^{z} s^{-3/2}\mu(s, \phi^1, \phi^2, \phi^3)\, d\omega(s), \end{aligned}$$

where

$$v(z) = \frac{z}{x^\alpha} = t, \qquad z = x^\alpha t$$

$$\omega(z) = \int_0^t \frac{1}{\sqrt{v'(\tau)}}\, dW(\tau) = \int_0^t x^{\frac{\alpha}{2}}\, dW(\tau) = x^{\alpha/2} W(z/x^\alpha)$$

is scaled Brownian motion W.

6.2 The Two-Dimensional Navier-Stokes Stochastic Differential Equations

We next discuss the two-dimensional Navier Stokes stochastic partial differential equations,

$$\begin{aligned} du^1 &= \big[u^1_{x_1x_1} + u^1_{x_2x_2} - (u^1u^1_{x_1} + u^2u^1_{x_2} + p_{x_1})\big]\, dt + B^{11}dW_1(t) + B^{12}dW^2(t), \\ du^2 &= \big[u^2_{x_1x_1} + u^2_{x_2x_2} - (u^1u^2_{x_1} + u^2u^2_{x_2} + p_{x_2})\big]\, dt + B^{21}dW^1(t) + B^{22}dW^2(t), \\ & u^1_{x_1} + u^2_{x_2} = 0. \end{aligned} \tag{6.9}$$

where $B^{ij} = B^{ij}(t, x, u, p)$, $1 \le i, j \le 2$. The infinitesimal generator is

$$\begin{aligned} X = \psi(t)\partial_t + \xi^{x_1}(t, x)\partial_{x_1} + \xi^{x_2}(t, x)\partial_{x_2} + \zeta^{u^1}(t, x, u, p)\partial_{u^1} \\ + \zeta^{u^2}(t, x, u, p)\partial_{u^2} + \zeta^{p}(t, x, u, p)\partial_{p} \end{aligned}$$

and the prolonged generator is

$$\widetilde{X} = X + \zeta^{u_t^1}\partial_{u_t^1} + \zeta^{u_t^2}\partial_{u_t^2} + \zeta^{p_t}\partial_{p_t} + \zeta^{u_{x_1}^1}\partial_{u_{x_1}^1} + \zeta^{u_{x_1}^2}\partial_{u_{x_1}^2} + \zeta^{p_{x_1}}\partial_{p_{x_1}} + \zeta^{u_{x_2}^1}\partial_{u_{x_2}^1}$$
$$+ \zeta^{u_{x_2}^2}\partial_{u_{x_2}^2} + \zeta^{p_{x_2}}\partial_{p_{x_2}} + \zeta^{u_{tt}^1}\partial_{u_{tt}^1} + \zeta^{u_{tx_1}^1}\partial_{u_{tx_1}^1} + \zeta^{u_{tx_2}^1}\partial_{u_{tx_2}^1} + \zeta^{u_{tp}^1}\partial_{u_{tp}^1}$$
$$+ \zeta^{u_{x_1x_1}^1}\partial_{u_{x_1x_1}^1} + \zeta^{u_{x_1x_2}^1}\partial_{u_{x_1x_2}^1} + \zeta^{u_{x_1p}^1}\partial_{u_{x_1p}^1} + \zeta^{u_{x_2x_2}^1}\partial_{u_{x_2x_2}^1} + \zeta^{u_{x_2p}^1}\partial_{u_{x_2p}^1}$$
$$+ \zeta^{u_{pp}^1}\partial_{u_{pp}^1} + \zeta^{u_{tt}^2}\partial_{u_{tt}^2} + \zeta^{u_{tx_1}^2}\partial_{u_{tx_1}^2} + \zeta^{u_{tx_2}^2}\partial_{u_{tx_2}^2} + \zeta^{u_{tp}^2}\partial_{u_{tp}^2} + \zeta^{u_{x_1x_1}^2}\partial_{u_{x_1x_1}^2}$$
$$+ \zeta^{u_{x_1x_2}^2}\partial_{u_{x_1x_2}^2} + \zeta^{u_{x_1p}^2}\partial_{u_{x_1p}^2} + \zeta^{u_{x_2x_2}^2}\partial_{u_{x_2x_2}^2} + \zeta^{u_{x_2p}^2}\partial_{u_{x_2p}^2} + \zeta^{u_{pp}^2}\partial_{u_{pp}^2}.$$

where

$$\zeta^{u_t^1} = A_1\zeta^{u^1}_{u_1} + A_2\zeta^{u^1}_{u_2} + \tfrac{1}{2}((B_{11})^2 + (B_{12})^2)\zeta^{u^1}_{u_1u_1} + (B_{11}B_{21} + B_{12}B_{22})\zeta^{u^1}_{u_1u_2}$$
$$+\tfrac{1}{2}((B_{21})^2 + (B_{22})^2)\zeta^{u^1}_{u_2u_2}$$
$$\zeta^{u_t^2} = A_1\zeta^{u^2}_{u_1} + A_2\zeta^{u^2}_{u_2} + \tfrac{1}{2}((B_{11})^2 + (B_{12})^2)\zeta^{u^2}_{u_1u_1} + (B_{11}B_{21} + B_{12}B_{22})\zeta^{u^2}_{u_1u_2}$$
$$+\tfrac{1}{2}((B_{21})^2 + (B_{22})^2)\zeta^{u^2}_{u_2u_2}$$

and $\zeta^{u_{x_1}^1}$, $\zeta^{u_{x_2}^1}$, $\zeta^{u_{x_1}^2}$, $\zeta^{u_{x_2}^2}$, ζ^{p_t} are the usual prolonged forms. Applying the operator $\widetilde{X}$ to equation (6.9) and substituting $u_{x_2}^2 = -u_{x_1}^1$, $u_{x_1x_2}^2 = -u_{x_1x_1}^1$, $u_{x_2x_2}^2 = -u_{x_1x_2}^1$, the determining equations of Navier Stokes stochastic differential equations are

$$\widetilde{X}\big(u_{x_1x_1}^1 + u_{x_2x_2}^1 - (u^1u_{x_1}^1 + u^2u_{x_2}^1 + p_{x_1})\big) = 0,$$
$$\widetilde{X}\big(u_{x_1x_1}^2 + u_{x_2x_2}^2 - (u^1u_{x_1}^2 + u^2u_{x_2}^2 + p_{x_2})\big) = 0,$$
$$\widetilde{X}\big(u_{x_1}^1 + u_{x_2}^2\big) = 0,$$
$$\widetilde{X}B^{11} + \frac{B^{11}}{2}\psi_t - B^{11}\zeta^{u^1}_{u^1} - B^{21}\zeta^{u^1}_{u^2} = 0, \quad \widetilde{X}B^{12} + \frac{B^{12}}{2}\psi_t - B^{12}\zeta^{u^1}_{u^1} - B^{22}\zeta^{u^1}_{u^2} = 0,$$
$$\widetilde{X}B^{21} + \frac{B^{21}}{2}\psi_t - B^{11}\zeta^{u^2}_{u^1} - B^{21}\zeta^{u^2}_{u^2} = 0, \quad \widetilde{X}B^{22} + \frac{B^{22}}{2}\psi_t - B^{12}\zeta^{u^2}_{u^1} - B^{22}\zeta^{u^2}_{u^2} = 0.$$

One obtains

$$\psi = 2c_1t + c_3, \qquad \xi^{x_1} = c_1x_1 - c_2x_2 + c_4\varphi_1(t), \qquad \xi^{x_2} = c_1x_2 + c_2x_1 + c_5\varphi_2(t),$$
$$\zeta^{u^1} = -c_1u^1 - c_2u^2 + c_4\varphi_1'(t), \qquad \zeta^{u^2} = -c_1u^2 - c_2u^1 + c_5\varphi_2'(t),$$
$$\zeta^p = -2c_1p - c_4x_1\varphi^1_{tt}(t) - c_5x_2\varphi^2_{tt}(t) + c_6\mu(t).$$

The generator corresponding to these coefficients is

$$X = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6$$

with

$$\begin{aligned}
&X_1 = 2t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2} - u^1\partial_{u^1} - u^2\partial_{u^2} - 2p\partial_p,\\
&X_2 = -x_2\partial_{x_1} + x_1\partial_{x_2} - u^2\partial_{u^1} + u^1\partial_{u^2}, \qquad X_3 = \partial_t,\\
&X_4 = \varphi_1(t)\partial_{x_1} + \varphi_1'(t)\partial_{u^1} - x_1\varphi_{tt}^1(t)\partial_p,\\
&X_5 = \varphi_2(t)\partial_{x_2} + \varphi_2'(t)\partial_{u^2} - x_2\varphi_{tt}^2(t)\partial_p, \qquad X_6 = \mu(t)\partial_p.
\end{aligned}$$

Let us continue with the form of the operators $X_4,\ X_5,\ X_6$ as given in Pukhachov (1960) (see also Ibragimov (1995)), that is $\varphi_1(t) = \varphi_2(t) = t$ and $\mu(t) = 1$. Along with the Cartesian coordinates x_1, x_2, u^1, u^2, one uses here the cylindrical coordinates

$$r = \sqrt{(x_1)^2 + (x_2)^2}, \qquad \theta = \arctan\frac{x_2}{x_1},$$
$$U = u^1\cos\theta + u^2\sin\theta, \qquad V = u^2\cos\theta - u^1\sin\theta.$$

This change of coordinates transforms equations (6.9) to

$$\begin{aligned}
dU &= \left[\left(U_{rr} + \frac{1}{r}U_r - \frac{U}{r^2} + \frac{1}{r^2}U_{\theta\theta} - \frac{2}{r^2}V_\theta\right) - \left(UU_r\frac{V}{r}U_\theta - \frac{V^2}{r} + p_r\right)\right]dt\\
&\qquad + b^{11}(t,r,\theta,U,V,p)\,dW^1(t) + b^{12}(t,r,\theta,U,V,p)\,dW^2(t),\\
dV &= \left[\left(V_{rr} + \frac{1}{r}V_r - \frac{V}{r^2} + \frac{1}{r^2}V_{\theta\theta} + \frac{2}{r^2}U_\theta\right) - \left(UV_r + \frac{V}{r}V_\theta + \frac{UV}{r} + \frac{1}{r}p_\theta\right)\right]dt\\
&\qquad + b^{21}(t,r,\theta,U,V,p)\,dW^1(t) + b^{22}(t,r,\theta,U,V,p)\,dW^2(t),\\
&U_r + \frac{U}{r} + \frac{1}{r}V_\theta = 0.
\end{aligned} \tag{6.10}$$

The operators with respect to (6.10) are

$$X_1 = 2t\partial_t + r\partial_r - U\partial_U - V\partial_V - 2p\partial_p, \qquad X_2 = -\partial_\theta, \qquad X_3 = \partial_t,$$

$$X_4 = \varphi_1(t)\ \cos\theta\partial_r + (\varphi_1'(t)\cos\theta - \frac{\varphi_1(t)}{r}\ V\sin\theta)\partial_U$$
$$+(\frac{\varphi_1(t)}{r}U\sin\theta - \phi_1'(t)\sin\theta)\partial_V - \frac{\varphi_1(t)}{r}\sin\theta\partial_\theta,$$
$$X_5 = \varphi_2(t)\sin\theta\partial_r + (\varphi_2'(t)\sin\theta + \frac{\varphi_2(t)}{r}\ V\cos\theta)\partial_U$$
$$+(-\frac{\varphi_2(t)}{r}U\cos\theta + \varphi_2'(t)\cos\theta)\partial_V + \frac{\varphi_2(t)}{r}\cos\theta\partial_\theta,$$
$$X_6 = \mu(t)\partial_p.$$

Since

$$X = \psi^t\partial_t + \xi^\theta\partial_\theta + \xi^r\partial_r + \zeta^U\partial_U + \zeta^V\partial_V + \zeta^p\partial_p$$

then

$$\psi^t = 2c_1t + c_3,$$
$$\xi^\theta = -c_3 - c_4\frac{\varphi_1(t)}{r}\sin\theta + c_5\frac{\varphi_2(t)}{r}\cos\theta,$$
$$\xi^r = c_1r + c_4\varphi_1(t)\cos\theta + c_5\varphi_2(t)\sin\theta,$$
$$\zeta^U = -c_1U + c_4(\varphi_1'(t)\cos\theta - \frac{\varphi_1(t)}{r}\ V\sin\theta) + c_5(\varphi_2'(t)\sin\theta + \frac{\varphi_2(t)}{r}\ V\cos\theta),$$
$$\zeta^V = -c_1V + c_4(\frac{\varphi_1(t)}{r}\ U\sin\theta - \varphi_1'(t)\sin\theta) + c_5(\varphi_2'(t)\cos\theta + \frac{\varphi_2(t)}{r}\ U\cos\theta),$$
$$\zeta^p = -2c_1p + c_6\mu(t).$$

We first consider the class of solutions invariant under the subalgebra spanned by the operators X_1 and X_3. The Lie group of transformations corresponding to these basis generators is

$$X\ :\ \bar{t} = t, \quad \bar{r} = re^a, \quad \bar{\theta} = \theta, \quad \bar{U} = Ue^{-a}, \quad \bar{V} = Ve^{-a}, \quad \bar{p} = pe^{-2a}$$
$$X_3J = J_t = 0,$$

so that $J = J(r, \theta, U, V, p)$, and

$$X_1 J = rJ_r - UJ_U - VJ_V - 2pJ_p = 0.$$

Invariants of this subalgebra are

$$Ur, \quad Vr, \quad pr^2, \quad \theta.$$

The representation of the corresponding invariant solution has the form

$$U = \frac{\phi^1(\theta)}{r}, \qquad V = \frac{\phi^2(\theta)}{r}, \qquad p = \frac{\phi^3(\theta)}{r^2}. \tag{6.11}$$

To find the necessary form of the functions b^{ij} in the Itô part, since

$$X_3 b^{ij} = b^{ij}_t = 0$$

then $b^{ij} = b^{ij}(r, \theta, U, V, p)$, and

$$X_1 b^{ij} + 2b^{ij} = rb^{ij}_r - Ub^{ij}_U - Vb^{ij}_V - 2pb^{ij}_p + 2b^{ij} = 0$$

so that

$$b^{ij} = \frac{\psi^{ij}(\theta, Ur, Vr, pr^2)}{r^2}. \tag{6.12}$$

Substituting (6.11) and(6.12) into (6.10), one arrives at

$$\begin{aligned}
\frac{\phi^1}{r} = \frac{\phi^1}{r} &+ \int_{z_0}^{z} \frac{1}{r}\big(\phi^1_{\theta\theta} - \phi^2\phi^1_\theta - 2\phi^2_\theta + (\phi^1)^2 + (\phi^2)^2 + 2\phi^3\big)ds \\
&+ \int_{z_0}^{z} \frac{\psi^{11}}{r} d\omega^1(s) + \int_{z_0}^{z} \frac{\psi^{12}}{r} d\omega^2(s) \\
\frac{\phi^2}{r} = \frac{\phi^2}{r} &+ \int_{z_0}^{z} \frac{1}{r}\left(\phi^2_{\theta\theta} - \phi^2\phi^2_\theta + 2\phi^1_\theta - \phi^3_\theta\right) ds \\
&+ \int_{z_0}^{z} \frac{\psi^{21}}{r} d\omega^1(s) + \int_{z_0}^{z} \frac{\psi^{22}}{r} d\omega^2(s) \\
\phi^2_\theta &= 0
\end{aligned}$$

that is,

$$-\big(\phi^1_{\theta\theta} - \phi^2\phi^1_\theta + (\phi^1)^2 + (\phi^2)^2 + 2\phi^3\big)(z - z_0)$$
$$= \psi^{11}\big(\omega^1(z) - \omega^1(z_0)\big) + \psi^{12}\big(\omega^2(z) - \omega^2(z_0)\big)$$
$$\big(-2\phi^1_\theta + \phi^3_\theta\big)(z - z_0) = \psi^{21}\big(\omega^1(z) - \omega^1(z_0)\big) + \psi^{22}\big(\omega^2(z) - \omega^2(z_0)\big)$$
$$\phi^2_\theta = 0$$

where

$$v(z) = zr^2 = t, \qquad z = \frac{t}{r^2}$$
$$\omega(z) = \int_0^t \frac{1}{\sqrt{v'}}\, dW(\tau) = \int_0^t \frac{1}{r}\, dW(\tau) = \frac{1}{r}W(r^2 z) \tag{6.13}$$

is scaled Brownian motion W. Then

$$\phi^1_{\theta\theta} - \phi^2\phi^1_\theta + (\phi^1)^2 + (\phi^2)^2 + 2\phi^3 = 0, \;\; 2\phi^1_\theta - \phi^3_\theta = 0.$$

It is possible to show that in the case of independent Brownian motion W^1 and W^2 the equation

$$h^1 dW^1(z) + h^2 dW^2(z) = 0$$

leads to $h^1 = h^2 = 0$. (Communicated by Bruno Bouchard in private discussion)

Next we consider the class of solutions invariant under the subalgebra spanned by the operators X_1 and X_2. The Lie group of transformations corresponding to these basis generators is

$$X \;:\; \bar{t} = te^{2a}, \quad \bar{r} = re^a, \quad \bar{\theta} = \theta, \quad \bar{U} = Ue^{-a}, \quad \bar{V} = Ve^{-a}, \quad \bar{p} = pe^{-2a}$$
$$X_2 J = J_\theta = 0$$

so that $J = J(t, r, U, V, p)$ and

$$X_1 J = 2tJ_t + rJ_r - UJ_U - VJ_V - 2pJ_p = 0.$$

Invariants of this subalgebra are

$$Ur, \quad Vr, \quad pr^2, \quad tr^{-2}.$$

Set $z = \dfrac{t}{r^2}$. The representation of the corresponding invariant solution has the form

$$U = \frac{\phi^1(z)}{r}, \qquad V = \frac{\phi^2(z)}{r}, \qquad p = \frac{\phi^3(z)}{r^2}. \tag{6.14}$$

To find the required form of the functions b^{ij} in the Itô part, since

$$X_2 b^{ij} = b^{ij}_\theta = 0$$

then $b^{ij} = b^{ij}(t, r, U, V, p)$, and thus

$$X_1 b^{ij} + 2b^{ij} = 2tJ_t + rb^{ij}_r - Ub^{ij}_U - Vb^{ij}_V - 2pb^{ij}_p + 2b^{ij} = 0,$$

yields

$$b^{ij} = \frac{\psi^{ij}(z, Ur, Vr, pr^2)}{r^2}. \tag{6.15}$$

Substituting (6.14) and (6.15) into (6.10), one arrives at the reduced system

$$\begin{aligned}
\frac{\phi^1(z)}{r} &= \frac{\phi^1(z_0)}{r} + \int_{z_0}^{z} \frac{1}{r}\left(2s\phi^3_z + (\phi^1)^2 + (\phi^2)^2 + 2\phi^3\right) ds \\
&\quad + \int_{z_0}^{z} \frac{\psi^{11}}{r} d\omega^1(s) + \int_{z_0}^{z} \frac{\psi^{12}}{r} d\omega^2(s) \\
\frac{\phi^2(z)}{r} &= \frac{\phi^2(z_0)}{r} + \int_{z_0}^{z} \frac{2s}{r}(2s\phi^2_{zz} + (\phi^1 + 4)\phi^2_z) ds \\
&\quad + \int_{z_0}^{z} \frac{\psi^{21}}{r} d\omega^1(s) + \int_{z_0}^{z} \frac{\psi^{22}}{r} d\omega^2(s) \\
&\phi^1_z = 0,
\end{aligned}$$

that is,

$$\left(2z\phi^3_z + (\phi^1)^2 + (\phi^2)^2 + 2\phi^3\right) dz = \psi^{11} d\omega^1(z) + \psi^{12} d\omega^2(z)$$

$$d\phi^2 = 2z(2z\phi^2_{zz} + (\phi^1 + 4)\phi^2_z)dz + \psi^{21} d\omega^1(z) + \psi^{22} d\omega^2(z)$$

$$\phi^1_z = 0,$$

and where the vector $\omega(z)$ of Wiener processes is determined as in (6.13).

CHAPTER VII

CONCLUSION

This thesis constitutes a study by group analysis of systems of stochastic partial differential equations

$$dv^i(r,y) = A^i(r,y,v,v_{y_k},v_{y_ky_l})\,dt + \sum_{j=1}^m B^{ij}(r,y,v)\,dw^j(t),$$
$$(i=\overline{1,n},\ k,l=\overline{1,N}), \tag{7.1}$$

by employing invertible transformations of the independent and dependent variables of the form

$$r=\alpha(t),\ y=h(t,x),\ v=g(t,x,u). \tag{7.2}$$

The main goals of the thesis were to construct determining equations for such stochastic differential equations, and to apply the developed theory to stochastic fluid dynamics equations.

For solving the problem of the thesis the following steps were used.

1. Transform a system of stochastic partial differential equations (7.1) using a local one-parameter group of transformations of type (7.2).

2. Differentiating with respect to the group parameter, construct determining equations for admitted Lie groups of transformations for the stochastic differential equations (7.1).

3. Apply the developed theory for constructing determining equations of admitted Lie groups for stochastic fluid dynamics equations.

The found admitted Lie groups of the stochastic fluid dynamics equations were applied for constructing their invariant solutions. This present thesis demonstrates a first experience in the application of the group analysis method for constructing invariant solutions of stochastic differential equations of gas and hydrodynamics. Our results show that the stochastic part of the reduced system depends on how the variable t is included in the collection of invariant independent variables, as summarized in Table 7.1.

Table 7.1 The stochastic part in the reduced system, depending on t

Invariant Solution	Reduced system
Has no t	Deterministic
Include t	Deterministic + Itô part
Has t only	Deterministic + Itô part

Another feature of obtaining the reduced system is that the integrand in the Itô integral has to have a particular form which is related with the admitted Lie group.

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