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OPTION PRICING MODEL FOR A STOCHASTIC VOLATILITY LEVY PROCESS WITH STOCHASTIC INTEREST RATE

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วิทยานิพนธ์ฉบับนี้เสนอการพิจารณาตัวแบบความผันผวนสโทแคสติกเลวีที่มีอัตราดอกเบี้ยเป็นแบบสโทแคสติก โดยในลำดับแรกได้กล่าวถึงพลศาสตร์ของราคาสินทรัพย์ภายใต้ฟอร์เวิร์ดเมเชอร์และได้สร้างสูตรเพื่อการกำหนดราคาของพันธบัตรที่ไม่ระบุดอกเบี้ยโดยมีข้อสมมุติที่ว่าอัตราดอกเบี้ยเป็นสโทแคสติกและสอดคล้องกับกระบวนการวาสิเช็กภายใต้ฟอร์เวิร์ดเมเชอร์ ต่อจากนั้นได้มีการนำเสนอวิธีการสร้างสูตรสำหรับสิทธิเลือกที่จะซื้อแบบยุโรปด้วยวิธีหาฟังก์ชันลักษณะเฉพาะของสินทรัพย์อ้างอิง นอกจากนั้นได้มีการประมาณค่าพารามิเตอร์ของตัวแบบการแพร่อย่างกระโดดซึ่งความผันผวนและอัตราดอกเบี้ยเป็นแบบสโทแคสติก (SVJSI) โดยใช้เทคนิควิธี GMM และในท้ายที่สุดได้แสดงวิถีของตัวอย่างของแบบจำลอง SVJSI สำหรับดัชนี SET50 และวิถีของตัวอย่างของแบบจำลองทีเอฟวาสิเช็กสำหรับตั๋วเงินคลังอายุ 3 เดือน

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SARISA PINKHAM : OPTION PRICING MODEL FOR A STOCHASTIC VOLATILITY LEVY PROCESS WITH STOCHASTIC INTEREST RATE. THESIS ADVISOR : PROF. PAIROTE SATTAYATHAM, Ph.D. 104 PP.

STOCHASTIC VOLATILITY/STOCHASTIC INTEREST RATE/LEVY PROCESS/ZERO COUPON BOND/OPTION PRICING/GENERALIZED METHOD OF MOMENT

A stochastic volatility Lévy model with stochastic interest rate (SVLSI) is considered. Firstly, we present the dynamics of asset prices under the T-forward measure. Secondly, we derive an explicit formula for pricing zero coupon bonds whose interest rate is stochastic and driven by a Vasicek process under T-forward measure (TF-Vasicek). Thirdly, an explicit formula for the European call option is presented using the technique based on the characteristic functions of an underlying asset. Fourthly, we estimate parameters of the stochastic volatility jump diffusion model with stochastic interest rate (SVJSI) by using the GMM technique. Finally, a simulation example shows a sample path of the SVJSI model for SET50 index and sample path of the TF-Vasicek model for 3 month Treasury bill of Thailand as compared to actual data.

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CHAPTER I

INTRODUCTION

1.1 An Introduction to Option Pricing Problem

An option is a contract in which the writer (seller) promises that the contract buyer has the right, but not the obligation, to buy or sell a certain security at a certain price (the exercise price or strike price) K on or before a certain expiration date T. The asset in the contract is referred to as the *underlying asset*. An option giving the buyer the right to buy at a certain price is called a call option *call option*, while one that gives the buyer the right to sell is called a *put option*.

The price of the option is the *premium*. When the option trades on an organized market, the premium is quoted by the market. Otherwise, the problem is to price the option. Also, even if the option is traded in an organized market, it can be instructive to detect some possible abnormalities in the market.

The *style* of an option is a general term denoting the class in which the option belongs, which in turn is defined by the dates on which the option may be exercised. The two major option styles are the *European* and the *American* style, with the difference being the time when the holder can exercise the option. The European style allows the holder to exercise only on the expiration date, while the American style allows exercise *at any time* before the expiration date.

The writer of an option needs to specify:

- *the type of option*: i.e., call and put option,
- *the underlying asset*: typically, it can be a stock, a bond, a currency, an

index and so on,

- *the exercise price*: the price at which the transaction is done if the option is exercised,
- *the amount of an underlying asset* to be purchased or sold, and
- *the expiration date.*

Let us consider the case of a European call option on a stock price, where the stock price at time t is denoted by S_t. If $K \geq S_T$, then the holder has no interest whatsoever in exercising the option. But if $K < S_T$, the holder stands to make a profit $S_T - K$ by exercising the option by purchasing the stock at price K and selling for prices S_T on the market. Hence, the price of the European call option at expiration date is given by

$$(S_T - K)_+ = \max(S_T - K, 0).$$

The case of European put option is similar.

When an option is exercised, the writer must be able to deliver the stock price at price K, implying that the amount $\max(S_T - K, 0)$ is generated at expiration date. At the time of writing the option (as the origin of time), S_T is unknown, provoking two questions:

1. How do we model the underlying asset specific on stock price?
2. How should we price the stock option at time $t = 0$ with an asset worth $\max(S_T - K, 0)$ at time $t = T$?

This is the option pricing problem.

1.2 The Behavior of Asset Prices

In the certainty case, the stock price at the expiration date T equals the future value of the stock price when continuously compounded at the risk-free interest rate r:

$$S_T = S_0 e^{rT},$$

where S_0 is the stock price at the present time. One way to think about this is that the future value is the end result of a dynamic process: the stock price starts at the present time and evolves through time to its future value. The formal expression that describes how the stock price moves through time in the certainty case can be obtained by employing the differential equation

$$\frac{dS_t}{dt} = rS_t,$$

which describes the dynamic stock price process in a world with certainty. Multiplying both sides by dt and rearranging, we can rewrite the above equation as:

$$dS_t = rS_t dt.$$

In this form, r denotes instantaneous rate of return on the stock.

In the uncertainty case, Fisher Black, Robert Merton and Myron Scholes (1973) determined the prices of the European and American options. They assumed that the behaviour of the stock price is determined by the following stochastic differential equation :

$$dS_t = S_t \left(\mu dt + \sigma dW_t\right), \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, $\mu \in \mathbb{R}$ is the instantaneous expected return of the stock (possibly adjusted by a dividend yield), and $\sigma > 0$ is the volatility of stock price returns. The equation (1.1) is known as Black - Scholes model. However, the Black-Scholes model fails to reflect the following three empirical phenomena. First, the distribution of the stock price return is non-normal with a higher peak and heavier tail, but in the Black-Scholes model the distribution is normal. Secondly, volatility is not constant as assumed in Black-Scholes model and thirdly, stock prices show large and sudden movements and are not always continuous as depicted in the Black-Scholes model.

To improve upon the pricing option of the Black-Scholes model, the stock price has been modeled as diffusion with stochastic volatility (Heston, 1993): as jump diffusion (Merton, 1976; Kou and Wang , 2004): or both (Bates, 1996 or Yan and Hanson, 2006).

The option pricing in Black-Scholes model follows these assumptions:

1. We have frictionless markets with continuous trading.
2. There are no transaction costs or taxes and no dividends during the life of the option.
3. There are no arbitrage opportunities.
4. The risk free interest rate is deterministic and equal to $r > 0$.
5. Under the real-world measure the stock price process follows equation (1.1).

Assumption (4) is harmless in most situations since interest rate variability is usually negligible compared to the variability observed in the market. When pricing an option with a long expiration date, however, the stochastic feature of the interest rate has a stronger impact on the option price. In such cases it is advisable to relax the assumption of deterministic rates. This is the topic of our research.

1.3 Outline of This Thesis

We now provide a brief outline of how we intend to proceed and what each chapter contains . The thesis is organized as follows:

In Chapter II, we introduce notation, terminology and some mathematical tools to be used in subsequent chapters.

Chapter III presents an option pricing when the stock price follows a stochastic volatility Lévy model with stochastic interest rate. We introduce the stock price dynamics under the T-forward measure, and then in the next section, derive an explicit formula for European call option by using a technique based on characteristic functions of an underlying asset. These models and formulas for option price requires values for fixed parameters to be discussed in the next chapter.

In chapter IV, we study the techniques of parameter estimation in the special case of the stock price model in Chapter III, namely stochastic volatility jump diffusion model with stochastic interest rate (SVJSI). We employ the technique of Generalized Method of Moment (GMM) in this research. By using the unconditional characteristic function, we obtain the moment of the underlying asset forming the SVJSI model. Next we use the data from observations SET50 index and 3 month Treasury Bills of Thailand to estimate the parameter and demonstrate a simulation example. Finally, in application to a financial problem, the option prices are presented by using the closed form formula and the Monte Carlo simulation for a fixed parameter.

The conclusion of the thesis and suggestions for the further work are contained in the last chapter.

CHAPTER II

PRELIMINARIES

In this chapter, we introduce some notation, terminology and some mathematical tools for the further use.

2.1 Pricing Options in the Black-Scholes Model

In 1997 Black and Scholes considered the problem of pricing and hedging a European option (call or put) on a non-dividend paying stock. In this section we briefly explain the main results and the assumptions of the Black- Scholes model as listed in Section 1.2.

We note that an *arbitrage opportunity* is the opportunity to buy an asset at a low price, then immediately selling it on a different market for higher price. Less rigorously, an arbitrage opportunity is a free lunch that allows investors to make a gain with no risk.

Suppose that the above assumptions hold. Standard derivative pricing theory offers two ways for computing the fair value $C(t, S_t)$ of a European call option at time $t \leq T$. Under the partial differential equation (PDE) approach the function $C(t, s)$ is computed by solving the PDE,

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} + rs\frac{\partial C}{\partial s} - rC = 0, \quad \text{for} \quad t \in [0, T]\,. \tag{2.1}$$

This is the famous Black-Scholes PDE of European call option.

In order to obtain a unique solution for the Black-Scholes PDE we must consider final and boundary conditions. We will restrict our attention to a European call option, $C(t,s)$. At maturity, $t=T$, a call option is worth:

$$C(T,s)=\max(s_T-K,0),$$

where K is the exercise price. So this will serve as the final condition.

The asset price boundary conditions are applied at $s=0$ and also as $s\to\infty$.

If $s=0$ then ds is also zero and therefore s can never change. This implies that when $s=0$ we have:

$$C(t,0)=0.$$

Obviously, if the asset price increases without bound as $s\to\infty$, the value of the option becomes that of the asset:

$$C(t,s)\approx s, \quad s\to\infty.$$

The European call option $C(t,S_t)$ is computed by solving the final boundary value problem:

$$\begin{cases} \frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 C}{\partial s^2}+rs\frac{\partial C}{\partial s}-rC=0, \quad \text{for } t\in[0,T] \\ C(t,0)=0; \quad C(t,s)\approx s, s\to\infty \\ C(T,s)=\max(s_T-K,0). \end{cases} \tag{2.2}$$

Alternatively, the value $C(t,S_t)$ can be computed as the expectation of the discounted pay-off under the risk-neutral measure $\mathbb{Q}$, the so-called risk-neutral pricing approach. Under $\mathbb{Q}$, the process S_t satisfies the stochastic differential equation (SDE)

$$dS_t=S_t\left(rdt+\sigma d\tilde{W}_t\right), \tag{2.3}$$

for a standard Brownian motion $\tilde{W}_t$. In particular, the drift μ in equation (1.1) has been replaced by risk-free interest rate r. The risk-neutral pricing rule now states that

$$C(t,S_t)=\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(S_T-K,0\right)|\mathcal{F}_t,\right] \tag{2.4}$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes expectation with respect to $\mathbb{Q}$.

To obtain the analytical formula for the option price, we compute this expectation which is in fact the computation of an integral.

$$\begin{aligned} C(t,S_t) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\max\left(S_T-K,0\right)|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\int_K^{\infty}(S_T-K)f_{S_T}(s)ds, \end{aligned} \tag{2.5}$$

where $f_{S_T}(\cdot)$ is the probability density function of S_T under the risk-neutral probability.

The solution of PDE (2.2), or the risk-neutral value of stock price obtained from equation (2.5), is given by

$$C(t,S_t;r,\sigma,T,K) := S_t\Phi(d_{t,1}) - Ke^{-r(T-t)}\Phi(d_{t,2}), \tag{2.6}$$

where

$$\begin{aligned} d_{t,1} &= \frac{\ln S_t - \ln K + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_{t,2} &= d_{t,1} - \sigma\sqrt{T-t} \end{aligned}$$

and Φ is the cumulative distribution function for the standard normal distribution. The equation (2.6) is known as Black-Scholes formula for a European call option. Similarly, the price for a European put option is:

$$P(t,S_t;r,\sigma,T,K) := -S_t\Phi(-d_{t,1}) + Ke^{-r(T-t)}\Phi(-d_{t,2}).$$

2.2 Historical Volatility and Implied Volatility

In theory, volatility should not depend on the method of measurement. However, in practice this is not the case. Volatility σ can be empirically measured by two methods: historical volatility and implied volatility. Historical volatility reflects the past price movement of the underlying asset, while implied volatility is a measure of market expectations regrading the asset's future volatility.

2.2.1 Historical Volatility

Historical volatility is calculated from empirical asset price data $S_1, S_2, ..., S_N$. Historical volatility is also referred to as the asset's actual or realized volatility. To estimate the historical volatility $\hat{\sigma}$ we calculate the annualized standard deviation for log return $R_i = ln(S_{i+1}/S_i)$ observed on a given time, for N days :

$$\hat{\sigma} = \sqrt{\frac{252}{N-1} \sum_{i=1}^{N} \left(R_i - \bar{R}\right)}, \tag{2.7}$$

where the sample mean $\bar{R} = \sum\limits_{i=1}^{N} R_i/N$. The factor 252 is based on the supposition that there are approximately 252 business days in a year.

2.2.2 Implied Volatility

Implied Volatility of an asset price is computed using an option pricing model such Black-Scholes. In contrast to historical volatility being a measure of price return in the past, implied volatility reflects expectations regarding the asset or future volatility of the markets. It can also help to evaluate whether the options are cheap or expensive. Rising implied volatility causes option prices to rise or become more expensive; falling implied volatility results in lower option premiums. Therefore, with everything else being equal, when implied volatility on an option is high, it is better to sell that option. If implied volatility is low, the option is more suitable for buying.

More precisely, using Black-Scholes option pricing, the call option C is a function $C(t, S; r, \sigma, T, K)$ where t is the time at which C is being priced, T is the expiration date, r is the risk-free rate of return, and K is the exercise price. Note that all the independent variables are indeed observable except σ. Since the quoted option price C_{obs} itself is observable, using the Black-Scholes formula we can therefore compute the volatility that is consistent with the quoted option prices and observed variables. We can therefore define implied volatility V by:

$$C(t, S; r, V, T, K) = C_{obs},$$

where C is the option price calculated by the Black-Scholes equation (2.6).

2.3 An Extension of Black-Scholes Model

It is well known that the Black-Scholes models of asset prices fails to reflect the following three empirical phenomena:

- non normal features, that is the return distribution is skewed negative and leptokurtic (higher peak and heavy tail),
- the volatility smile: implied volatility not constant as assumed in Black-Scholes model,
- large and sudden movement in prices such as crashes and rallies.

Therefore, many financial engineering studies have been undertaken to modify and improve the Black-Scholes formula to explain some or all of the above three empirical phenomena.

Definition 2.1. (Brownian motion). A stochastic process $W = (W_t)_{t\geq 0}$ is a *Brownian motion* (or Wiener process) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if

- the process has stationary increments,
- the process has independent increments,
- for $0 \leq s < t$ the random variable $W_t - W_s$ follows a Normal distribution $N(0, t-s)$.

Proposition 2.2. (Property of Brownian motion)

(i) *Martingale property: Brownian motion is one of the simplest example of a martingale. We have, for all* $0 \leq s \leq t$,

$$E[W_t|\mathcal{F}_s] = E[W_t|W_s] = W_s.$$

(ii) *Path property: Brownian motion has continuous paths, i.e.*$W = (W_t)_{t\geq 0}$ *is a continuous function of* t. *However, the paths of Brownian motion are very erratic. Moreover, the paths of Brownian motion are of infinite variation, i.e. their variation is infinite on every interval. Another property , we have that*

$$P\left(\sup_{t\geq 0} W_t = +\infty \text{ and } \inf_{t\geq 0} W_t = -\infty\right) = 1.$$

This mean that the Brownian path will keep oscillating between positive and negative values.

(iii) *Scaling Property: for every* $c \neq 0$, $\tilde{W} = \left(\tilde{W}_t = cW_{t/c^2}\right)_{t\geq 0}$ *is also Brownian motion.*

Definition 2.3. (Poisson Process) Let $(\tau_i)_{i\geq 1}$ be a sequence of exponential random variables with parameter $\lambda > 0$ and let $T_n = \sum_{i=1}^{n} \tau_i$. The process

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n}, \tag{2.8}$$

is called the Poisson process with parameter (or intensity) λ.

Proposition 2.4. (Properties of the Poisson process)

1. *For all* $t \geq 0$, *the sum in equation (2.8) is finite a.s.*

2. *The trajectories of* N *are piecewise constant with jumps of size 1 only.*

3. *The trajectories are cadlag, i.e., right continuous with left limits.*

4. *For all* $t \geq 0$, $N_{t-} = N_t$ *with probability 1.*

5. *For all $t \geq 0$, N_t follows the Poisson law with parameter λt :*

$$P(N_t = n) = \frac{(\lambda t)^n \, e^{-\lambda t}}{n!}.$$

6. *The characteristic function of the Poisson process is*

$$E\left[e^{iuN_t}\right] = \exp\left[\lambda t\left(e^{iu} - 1\right)\right].$$

2.3.1 Jump-Diffusion Model

In addition to the volatilities smile, there is evidence that the assumption of the pure normal distribution (also called pure diffusion) for the stock return is not accurate. Some authors attempt try to explain the volatility smile and the leptokurticity by changing the underlying stock distribution from a diffusion process to a jump-diffusion process.

For jump-diffusion models, the normal evolution of price is given by a diffusion process, punctuated by jumps on random intervals, representing rare events, crashes and large drawdowns. Merton (1976) was first to actually introduce jumps in a stock distribution:

$$S_t = S_0 \exp\left(\mu t + \sigma W_t + \sum_{n=1}^{N_t} Y_n\right) \tag{2.9}$$

where $(N_t)_{t\in[0,T]}$ is Poisson process with intensity λ, with independent jumps normally distributed with mean m and variance δ^2. Further, he assumed that the Poisson process and the jumps are independent of the Brownian Motion W_t. This model is called the *Merton jumps diffusion model with Gaussian jumps* (known as an exponential Lévy model).

Recently, Kou (2002) proposed double exponential jump-diffusion models by using the same idea to explain both the existence of fat tails and the volatility smile.

In analogy to the Black-Scholes model, the parameter μ in the Merton model stands for the expected asset return with σ, the volatility of regular shocks to the stock return. The jump component can be interpreted as a model for crashes, with the parameter λ denoting the expected number of crashes per year and m and δ^2 determining the distribution of a single jump.

2.3.2 A Stochastic Volatility Model

Asset price models with stochastic volatility are useful because they explain in a self consistent way why it is that options with different strikes and expirations have different Black-Scholes implied volatilities - the *volatility smile*. Several different stochastic processes have been suggested for the volatility, i.e.,

- Ornstein-Uhlenbeck (OU) process:

$$dv_t = (-\theta v_t)dt + \xi d\tilde{W}_t \qquad (2.10)$$

- Cox-Integersoll-Ross (CIR) process:

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}d\tilde{W}_t \qquad (2.11)$$

- GARCH process :

$$dv_t = (\omega - \theta v_t)dt + \xi v_t d\tilde{W}_t \qquad (2.12)$$

- the 3/2 process :

$$dv_t = (\omega {v_t}^2 - \theta v_t)dt + \xi {v_t}^{3/2} d\tilde{W}_t \qquad (2.13)$$

where ω, θ and ξ are parameters and $\tilde{W}_t$ is standard Brownian motion.

Heston (1993) and Stein and Stein (1991) were among those who suggested using the stochastic volatility process. Using Ito's formula, we can see that the asset price model with variance $v_t = {\sigma_t}^2$ satisfying a CIR process is given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t, \\ dv_t &= (\omega - \theta v_t)dt + \xi v_t d\tilde{W}_t, \end{aligned} \tag{2.14}$$

where $dW_t d\tilde{W}_t = \rho dt$ with $\omega = \beta^2, \theta = 2\alpha$ and $\xi = 2\beta$.

In general, the stochastic volatility models generalize to

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t, \\ dv_t &= f(v_t, t)dt + g(v_t, t) d\tilde{W}_t, \end{aligned} \tag{2.15}$$

where $f(v_t, t)$and $g(v_t, t)$ are some functions of v_t while $d\tilde{W}_t$ is another standard Brownian motion that is correlated with dW_t with constant correlation ρ.

In order to incorporate a volatility effect to cause the volatility parameter of the Black-Scholes model to be stochastic in a suitable way, the new model called the stochastic volatility model as mentioned above has been implemented by Scott (1987), Hull and White (1987) and Heston (1993). Another variation is to use time change stochastic process as proposed by Carr et al. (2003) which we will introduce in Section 2.3.4.

2.3.3 A Stochastic Volatility Model with Jumps

Bates (1996) introduced the *jump diffusion stochastic volatility model* by adding proportional log normal jumps to the Heston model (equation (2.14)) as follows. In the original formulation the model is:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t + S_t dJ_t, \\ dv_t &= (\omega - \theta v_t)dt + \xi v_t d\tilde{W}_t, \end{aligned} \tag{2.16}$$

where W_t and $\tilde{W}_t$ are Brownian motions with correlation ρ, driving price and volatility, and J_t is a compound Poisson process with intensity λ and log-normal distribution of jumps size such that if k is its jumps size then $ln(1+k) \sim N(ln(1+\bar{k}) - \frac{1}{2}\delta^2, \delta^2)$.

2.3.4 A Stochastic Time-Change Process

A stochastic time change process was first studied in Bochner (1949) and was later introduced to the finance literature in Clark (1973) who modelled the asset price as a geometric Brownian motion subordinated by an independent Lévy subordinator. Today, it can be regarded as one of the standard tools for building financial models (see also Geman (2005)).

Let $X = (X_t)_{t \geq 0}$ denote a stochastic process and let $T = (T_s)_{s \geq 0}$ denote a non negative, non decreasing stochastic process not necessarily independent of X. The time changed process is then defined as $Y = (Y_s)_{s \geq 0}$, where

$$Y_s = X_{T_s}. \tag{2.17}$$

The process X is said to evolve in operational time. The process T is referred to as *time change, stochastic clock or business time.* It reflects the varying speed of Y.

The use of time change stochastic process in finance is closely linked to the concept of stochastic volatility model for asset price. Numerous empirical studies have shown that asset price volatility tends to be time varying and tends to display clustering effects. The concept of stochastic volatility in continuous time asset price models can be introduced by two ways.

One is to use the time-change stochastic process as in equation (2.17). The other way is to use a stochastic integral of the form

$$Y_t = \int_0^t \sigma_s dX_s, \tag{2.18}$$

where $\sigma = (\sigma_t)_{t \geq 0}$ is a stochastic volatility process. The models in equation (2.17) and (2.18) lead to equivalent models.

We now provide an overview of a stochastic process which can serve to model the rate of time change. Since time needs to increase, the process modeling the rate time of time change needs to be positive, as for example, the Integrated Cox-Ingersoll-Ross (CIR) time change. Carr et al. (2003) uses as the rate of time change as the classical example of a mean reverting positive stochastic process: the CIR process $(v_t)_{t\geq 0}$ that solves the SDE,

$$dv_t = (\omega - \theta v_t)dt + \xi\sqrt{v_t}d\tilde{W}_t, \tag{2.19}$$

where $(W_t)_{t\geq 0}$ is standard Brownian motion with $\omega = \kappa\eta, \theta = \kappa$ and $\xi = \lambda$. The parameter η is interpreted as the long run rate of time change, κ is the rate of mean reversion, and λ governs the volatility of the time change.

The mean and variance of v_t given v_0 are given by

$$E\left[v_t|v_0\right] = v_0\exp(-\kappa t) + \eta\left(1 - \exp(-\kappa t)\right),$$
$$Var\left[v_t|v_0\right] = v_0\tfrac{\lambda^2}{\kappa}\left(\exp\left(-\kappa t\right) - \exp\left(-2\kappa t\right)\right) + \tfrac{\eta\lambda^2}{2\kappa}\left(1 - \exp(-\kappa t)\right)^2.$$

The economic time elapsed in t units of calendar time is then given by the integrated CIR process, $(T_t)_{t\geq 0}$, where

$$T_t = \int_0^t v_s ds. \tag{2.20}$$

Since $(v_t)_{t\geq 0}$ is a positive process, $(T_t)_{t\geq 0}$ is an increasing process. The characteristic function of T_t (given v_0) is explicitly known (see Cox et al., 1985 or Elliot and Kopp, 1999, Theorem 9.6.3):

$$\varphi(u,t;\kappa,\eta,\lambda,v_0) = \exp\left(\frac{\kappa^2\eta t}{\lambda^2} + \frac{2v_0 iu}{\kappa\coth\left(\gamma t/2\right)}\right)\left(\cosh\left(\gamma t/2\right) + \frac{\kappa}{\gamma}\sinh\left(\gamma t/2\right)\right)^{-2\kappa\eta/\lambda^2}, \tag{2.21}$$

where $\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}$.

From this the mean of T_t given v_0 is given by

$$E\left[T_t|v_0\right] = \eta t + \kappa^{-1}\left(v_0 - \eta\right)\left(1 - \exp(-\kappa t)\right).$$

2.3.5 Lévy Process

Lévy processes are a class of stochastic processes with discontinuous paths which are at the same time simple enough to study and rich enough for applications, or at least to be used as building blocks of more realistic models.

2.3.5.1 Stochastic Calculus for Lévy Process

In this section, we shall review the notation of the Lévy process and some properties. For more details see Cont and Tankov (2004) and Oksendal and Sulem (2009).

Definition 2.5. (Lévy process) A stochastic process $X = (X_t)_{t\geq 0}$ is a Lévy process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if it is *cadlag* (i.e. right continuous with left limits), satisfies $X_0 = 0$ and pocesseses the following properties:

- The process has stationary increments i.e. the distribution of the increment $X_{t+s} - X_t$ over interval $[t, t+s]$ does not depend on t, but only on the length s of the interval.

- The process has independent increments, i.e. if $r < s \leq t < u, X_u - X_t$ and $X_s - X_r$ are independent random variables.

The *jump* of Lévy process X_t at time $t \geq 0$ is defined by

$$\triangle X_t = X_t - X_{t-}.$$

Let B_0 be the family of Borel sets $U \subset \mathbb{R}$ which closure $\bar{U} \subset \mathbb{R}_0 \equiv \mathbb{R} \setminus \{0\}$. For $U \in B_0$. we define

$$N\left([0,t], U\right) = N(t,U) = \sum_{s:0<s\leq t} 1_U\left(\Delta X_s\right). \tag{2.22}$$

In other words , $N(t,U)$ is the number of jumps of size $\Delta X_s \in U$ which occur before or at time t. $N(t,U)$ is called the *Poisson random measure (or jump measure)* of $X(\cdot)$.

Remark 2.6. Note that $N(t,U)$ is finite for all $U \in B_0$.

Theorem 2.7.
- *The set function $U \to N(t,U,\omega)$ defines a σ-finite measure on B_0 for each fixed t. The differential form of this measure is written $N(t,dz)$.*
- *The set function $[a,b)\times U \to N(b,U,\omega)-N(a,U,\omega)$ with $[a,b) \subset [0,\infty)$, $U \in B_0$ defines a σ-finite measure for each fixed ω. The differential form of this measure is written $N(dt,dz)$.*
- *The set function $v(U) = E[N([0,1],U)]$ where E is expectation under probability measure $\mathbb{P}$, also defines a σ-finite measure on B_0, called the Lévy measure of X_t.*

For example, the compound Poisson process $Z = (Z_t)_{t\geq 0}$ is defined by

$$Z_t = \sum_{i=1}^{N_t} Y_i,$$

where $(Y_i)_{i\geq 1}$ is a sequence of i.i.d random variables with distribution $\mu_Y = \mu_{Y_1}$ and $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ independent from $(Y_i)_{i\geq 1}$. An increment of this process is given by

$$Z_s - Z_t = \sum_{i=N_t+1}^{N_S} Y_i, \quad s > t.$$

This is independent of $Y_1, Y_2, ..., Y_{N_t}$: Its distribution depends only on the difference $(s-t)$ and on the distribution of Y_1. Thus Z_t is a Lévy process.

To find the Lévy measure of Z_t note that if $U \in B_0$ then

$$v(U) = E\left[N\left([0,1],U\right)\right] = E\left[\sum_{0<s\leq 1} 1_U(\Delta Z_s)\right]$$
$$= E\left[N_1 1_U(\Delta Z_s)\right] = E\left[N_1 1_U(Z_s - Z_{s-})\right] = E\left[N_1 1_U\left(Y_1\right)\right].$$

By the fact that N_t and Y_i are independent, we get

$$v(U) = E[N_1]\, E[1_U(Y_1)] = \lambda\mu_{Y_1}(U) = \lambda\mu_Y(U).$$

We conclude that $v(\cdot) = \lambda\mu_Y(\cdot)$.

Definition 2.8. (Infinitely Divisible). A probability distribution F on $\mathbb{R}^d$ is said to be infinitely divisible if for any integer $n \geq 2$, there exist n i.i.d. random variables $Y_1, Y_2, ..., Y_n$ such that $Y_1 + Y_2 + ... + Y_n$ has distribution F.

Thorem 2.9. (Infinite divisibility and Lévy process).

Let $X = (X_t)_{t\geq 0}$ be a Lévy process. Then for every t, X_t has an infinitely divisible distribution F. Conversely, if F is an infinitely divisible distribution, then there exists a Lévy process such that the distribution of X_1 is given by F.

Further, we can write the characteristic function of a Lévy process by

$$\phi_{X_t}(u) = E[\exp(iuX_t)] = \exp(t\psi(u))$$

where $\psi(u) = log(\phi(u))$ is the characteristic exponent. The characteristic exponent of a Lévy process satisfies the following *Lévy Khinchine* formula:

Theorem 2.10. (The Lévy-Khintchine formula). *Let $(X_t)_{t\geq 0}$ be a Lévy process with Lévy measure v. Then $\int_{-\infty}^{\infty} \min\{1, x^2\}\, v(dx) < \infty$ and*

$$E(e^{iuX_t}) = \exp(t\psi(u)), \quad u \in R, \tag{2.23}$$

where

$$\psi(u) = iau - \frac{b^2u^2}{2} + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux1_{|x|<1}\right) v(dx), \tag{2.24}$$

Conversely, given constants a, b^2, and a measure v on B_0 such that

$$\int_{-\infty}^{\infty} \min\left\{1, x^2\right\} v(dx) < \infty,$$

there exists a Lévy process X_t such that (2.23) and (2.24) hold.

From the Lévy-Khinchine formula, one sees that generally a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part, and a pure jump part. We will call the characteristic $(a, b^2, v(dx))$ in the above representation *the triplet of* X_t. From equation (2.24), we can see that the Brownian motion such that the characteristic function is given by

$$\phi_{X_t}(u) = \exp\left(i\mu tu - \frac{\sigma^2 u^2 t}{2}\right),$$

is a Lévy process such that the triplet is $(\mu, \sigma^2, 0)$.

Under the assumption above the Itô-Lévy decomposition theorem states that any Lévy process has the form

$$X_t = at + bW_t + \int_0^t \int_{\mathbb{R}} z\tilde{N}(dt, dz),$$

where W_t is a Brownian motion, $\tilde{N}(dt, dz) = N(dt, dz) - v(dz)dt$, and a, b are constants. More generally, we study the Ito-Lévy process, which is a process of the form

$$X_t = X_0 + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dW_s + \int_0^t \int_{\mathbb{R}} c(s,z,\omega)\tilde{N}(ds,dz), \qquad (2.25)$$

where $\int_0^t |a(s)|\, ds + \int_0^t b^2(s)dW_s + \int_0^t \int_{\mathbb{R}} c(s,z)v(dz)ds < \infty$ and $a(t), b(t)$ and $c(t,z)$ are predictable processes (predictable with respect to the filtration $\mathcal{F}_t$ generated by X_s for t).

The differential form of equation (2.25) is

$$dX_t = a(t)dt + b(t)dW_t + \int_{\mathbb{R}} c(t,z)\tilde{N}(dt,dz).$$

The financial models with jumps fall into two categories: *jump diffusion* and *infinite activity Lévy processes*. In jump diffusion process, jumps are considered rare events, and in any given finite interval there are only finitely many jumps. Examples of jump diffusion models in finance include Merton's model as in equation (2.8) in which the jump sizes haves normal distribution. For infinite activity Lévy processes, there are infinitely many jumps in any finite time interval.

Moreover, we can construct new Lévy processes by using three basic types of transformations, under which the class of Lévy processes is invariant: linear transformations, subordination (time change of a Lévy process with another increasing Lévy process) and exponential tilting of the Lévy measure.

2.3.5.2 The Itô Formula and its Extensions

We now review Itô's formula and its extensions.

Lemma 2.11. (Itô's formula)

Assume that the process $X = (X_t)_t \geq 0$ *has stochastic differential given by*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

where $\mu(t, X_t)$ *and* $\sigma(t, X_t)$ *are adapted processes, and let* f *be a* $C^{1,2}$*- function. Define the process* $Y_t = f(t, X_t)$*. Then* Y *has a stochastic differential given by*

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW_t. \tag{2.26}$$

Note that the term $\mu \frac{\partial f}{\partial x}$, for example, is shorthand notation for

$$\mu\left(t, X_t\right) \frac{\partial f}{\partial x}\left(t, X_t\right),$$

and correspondingly for the other terms.

In fact Itô's formula provides a derivative chain rule for stochastic functions, clarifying the relationship between a stochastic process and a function of that stochastic process. Itô's formula has many extensions. The following Itô's formulas are the key step in establishing the main theorem of our thesis (for the proof see Cont and Tankov (2004), pages 275-277).

Lemma 2.12. (Itô's formula for jump-diffusion processes)

Let $X = (X_t)_t \geq 0$ be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process given by

$$X_t = X_0 + \int_0^t \mu\,(s, X_s)ds + \int_0^t \sigma\,(s, X_s)dW_s + \sum_{i=1}^{N_t} \Delta X_i,$$

where $\mu(s, X_s)$ and $\sigma(s, X_s)$ are continuous non anticipating processes with

$$E\left[\int_0^T \sigma^2\,(t, X_t)\,dt\right] < \infty.$$

Then, for any $C^{1,2}$- function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, the process $Y_t = f(t, X_t)$.can be represented as:

$$df(t, X_t) = \left\{\tfrac{\partial f}{\partial t}(t, X_t) + \mu(t, X_t)\tfrac{\partial f}{\partial x}(t, X_t) + \tfrac{\sigma^2(t,X_t)}{2}\tfrac{\partial^2 f}{\partial x^2}(t, X_t)\right\} dt$$
$$+\sigma(t, X_t)\tfrac{\partial f}{\partial x}(t, X_t)dW_t + f(t, X_{t-} + \Delta X_t) - f(t, X_{t-}).$$

Lamma 2.13. (Itô's formula for Lévy process)

Let $X = (X_t)_t \geq 0$ be a d-dimensional Lévy process with the generating triplet (A, γ, v). Then, for any $C^{1,2}$ function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, the process $Y_t = f(t, X_t)$ can be represented as:

$$\begin{aligned} f\,(t, X_t) &= f\,(t, X_0) + \textstyle\int_0^t \frac{\partial f}{\partial s}\,(s, X_s)\,ds + \int_0^t \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\,(s, X_s)\,dX_s^i \\ &+ \tfrac{1}{2}\textstyle\int_0^t \sum_{i,j=1}^{d} A_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\,(s, X_s)\,ds \\ &+ \textstyle\int_0^t \int_d \left[f\,(s, X_s + z) - f\,(s, X_s) - \sum_{i=1}^{d} z_i \frac{\partial f}{\partial x_i}\,(s, X_s)\right] N(dz, ds) \end{aligned} \tag{2.27}$$

where $N(dz, ds)$ is the Poisson random measure associated with $X = (X_t)_{t\geq 0}$.

2.3.6 A Time Change Lévy Model

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space and $\mathbb{Q}$ a risk neutral probability measure. We consider the asset price process before time change with non dividend payment as the following Lévy process,

$$S_t = S_0 \exp(rt + (\sigma W_t - \frac{1}{2}\sigma^2 t) + (J_t - \xi t)). \tag{2.28}$$

Equation (2.28) decomposes the asset price S_t into three components. The first component, rt, forms the instantaneous drift, which is determined by no arbitrage. The second component, $(\sigma W_t - \frac{1}{2}\sigma^2 t)$, comes from the diffusion, with the $\frac{1}{2}\sigma^2 t$ as the concavity adjustment, The last term, $(J_t - \xi t)$, represents the contribution from the jump component, with ξ as the analogous concavity adjustment for J_t .

The stochastic volatility in Lévy model as equation (2.28) comes from applying a stochastic time change to the diffusion component or the jump component, or both.

- *Stochastic volatility from diffusion*: If we apply a stochastic time change only to the diffusion component of the model (2.28), that is , $(\sigma W_t - \frac{1}{2}\sigma^2 t) \rightarrow (\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t)$, and leave the jump component $(J_t - \xi t)$ unchanged, stochastic volatility aries solely from the diffusion component. The time change Lévy model is given by

$$S_t = S_0 \exp(rt + (\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t) + (J_t - \xi t)). \tag{2.29}$$

  Examples using this specification include Bakshi et al. (1997) and Bates (1996). Under this specification, whenever the asset price movement becomes more volatile, it is due to an increase in the diffusive movement in the asset price. The frequency of large events remains constant. Thus, the relative weight of the diffusion and jump components in the return process varies over time. The relative weight of the jump component declines as the total volatility of the return process increase.

- *Stochastic volatility from jump*: Alternatively, if we apply a stochastic time change only to the jump component of the model (2.28), that is $(J_t - \xi t) \rightarrow (J_{T_t} - \xi T_t)$, but leave the diffusion component unchanged, stochastic volatility comes solely from the time variation in the arrival rate of jumps. The time change Lévy model is given by

$$S_t = S_0 \exp(rt + (\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t) + (J_{T_t} - \xi T_t)). \tag{2.30}$$

Under this specification, an increase in the return volatility is due solely to an increase in the discontinuous movements in the asset price. Hence the relative weight of the jump component increases with the return volatility. The models proposed in Carr et al. (2003) are degenerate examples of this stochastic volatility from jump category because they apply stochastic time changes to pure jump Lévy process.

- *Joint contribution form jump and diffusion*: To model the situation in which stochastic volatility comes simultaneously from both the diffusion and jump components of the model (2.28), we can apply the same stochastic time change T_t to both $(\sigma W_t - \frac{1}{2}\sigma^2 t)$ and $(J_t - \xi t)$. The time change Lévy model is now given by

$$S_t = S_0 \exp\left(rt + (\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t) + (J_{T_t} - \xi T_t)\right). \tag{2.31}$$

In this case, the instantaneous variance of the diffusion and the arrival rate of jumps vary synchronously over time. Under joint contribution from jump and diffusion, the relative proportions of the diffusion and jump component are constant, even though the return volatility varies over time. The recent affine models in Bates (2000) and Pan (2002) are variations of this category. In these models, both the arrival rate of the Poisson jump and instantaneous variance of the diffusion component are driven by one stochastic process.

2.4 The Girsanov Theorem and The Risk-Neutral Probability

Thorem 2.14. (Girsanov theorem). *Let $(W_t)_{t\in[0,T]}$ be a Brownian motion on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $(q_t)_{t\in[0,T]}$, be an adapted process. Define*

$$M(t) = \exp\left(-\int_0^t q_u dW_u - \frac{1}{2}\int_0^t q_u^2 du\right), \tag{2.32}$$

and

$$\tilde{W}_t = W_t + \int_0^t q_u du, \tag{2.33}$$

and assume that

$$E\left[\int_0^T q_u^2 M^2(u) du\right] < \infty. \tag{2.34}$$

Set $M = M(T)$. Then $E[M] = 1$ and under probability measure $\mathbb{Q}$ given by

$$d\mathbb{Q} = M d\mathbb{P}, \tag{2.35}$$

the process $(\tilde{W}_t)_{0\leq t\leq T}$ is a Brownian motion.

By virtue of this theorem, let S be a diffusion process defined by equation (1.1). If we make the assumption that the risk-free interest rate r is constant, we can choose

$$q = \frac{r-\mu}{\sigma}. \tag{2.36}$$

In this case, we can easily show that the process

$$M_q(t) = \exp\left(\int_0^t q dW_s - \frac{1}{2}\int_0^t q^2 ds\right), \tag{2.37}$$

is a martingale. From Girsanov's Theorem 1.1 above, the process

$$\begin{aligned} d\tilde{W}_t &= dW_t - q dt \\ &= dW_t - \left(\frac{r-\mu}{\sigma}\right) dt, \end{aligned} \tag{2.38}$$

is a Wiener process under the probability measure $\mathbb{Q}$. The process S is thus

$$\begin{aligned} dS_t &= S_t\left(\mu + \sigma\left(\frac{r-\mu}{\sigma}\right)\right)dt + \sigma S_t d\tilde{W}_t \\ &= S_t\left(rdt + \sigma d\tilde{W}_t\right). \end{aligned} \tag{2.39}$$

The probability $\mathbb{Q}$ is called the *risk-neutral probability*. Under this measure, the asset has expected rate of return equivalent to the risk-free rate. More generally, in a risk-neutralized world, the expected return of any asset is the risk-free rate r.

2.5 Interest Rate Models

Interest rate models can be used to model the dynamics of the yield curve, which is vital in pricing and hedging of fixed-income securities and also of great importance from a macro economical point of view as we have stated earlier. Traditionally these models specify a stochastic process for the term structure dynamics in a continuous time setting. A stochastic process means that the outcome depends both on a deterministic component and a random component and thus has the form

$$dr_t = f(t, r_t)dt + g(t, r_t)dW_t, \tag{2.40}$$

where $f(t, r_t)$ and $g(t, r_t)$ are suitably chosen drift and diffusion coefficients, and W_t is the Brownian motion. Models of this type are referred to as *one-factor models*, as there is only one stochastic driver. Models with multiple stochastic drivers are called *multi-factor models*.

Interest rate models can be divided into *equilibrium* models and *no-arbitrage models*. Equilibrium models are also referred to as endogenous term structure models because the term structure of interest rate is an output of, rather than an input to, these models. If we have the initial zero coupon bond curve from the market, the parameters of the equilibrium models are chosen such that the model produces a zero coupon bond curve as close as possible to the one observed in the market. Since the equilibrium models cannot reproduce exactly the initial yield curve, most traders have very little confidence in using these models to price complex interest rate derivatives. Hence, no-arbitrage models designed to exactly match the current term structure of interest rate are more popular. It is not possible to arbitrage using simple interest rate instruments in this type of no arbitrage models. Two most important no-arbitrage models are the Hull-White process (1990) and the Black-Karasinski (1991) model.

In this section we present four classical interest rate process models: the Vasicek (1977), the Dothan (1978), the Cox, Ingersoll and Ross (1985) and Hull-White (1990) models. The Vasicek model used in this thesis is one of the first term structure models appearing on the market in 1977. For more details see Brigo and Mercurio (2001), pp. 52-135.

2.5.1 The Vasicek Model

In the Vasicek (1977) model, the model formulation under the risk-neutral measure $\mathbb{Q}$ is

$$dr_t = (\alpha - \beta r_t)\, dt + \sigma_r dW_t, \tag{2.41}$$

where r_0, α, β and σ_r are positive constants. This model is an Ornstein Uhlenbeck process, the first interest rate model that incorporates mean reversion. The parameter σ_r determines the overall level of volatility and the mean reversion parameter, β, the relative volatility of long and short rates. A high value of β causes short term rate movements to damp out quickly, so long term volatility is reduced. The probability distribution of all rates at all time is normal. However, with the normal distribution, short rates can be negative with positive probability, which is a major drawback of the Vasieck model. Nevertheless, the analytic tractability resulting from the Gaussian distribution is this model's nicest feature. The rate under the real world measure evolves as an Ornstein Uhlenbeck process with constant coefficients.

Letting $X_t = r_t e^{\beta t}$ and using Itô's formula (2.26) and isometry, we derive by integrating equation (2.41):

$$r_t = r_s e^{-\beta(t-s)} + \frac{\alpha}{\beta}\left(1 - e^{-\beta(t-s)}\right) + \sigma_r \int_s^t e^{-\beta(t-u)} dW_u, \tag{2.42}$$

for $s \leq t$. Whereas the distribution of the short rate is Gaussian with mean and variance,

$$E\left[r_t\right] = r_s e^{-\beta(t-s)} + \frac{\alpha}{\beta}\left(1 - e^{-\beta(t-s)}\right), \tag{2.43}$$

$$Var\left[r_t\right] = \frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta(t-s)}\right). \tag{2.44}$$

From equation (2.43), we can see that as $t \to \infty, E[r_t] \to \frac{\alpha}{\beta}$. Hence the interest rate is mean reverting and $\frac{\alpha}{\beta}$ can be regarded as the long term level of the interest rate.

2.5.2 The Dothan Model

Dothan (1978) started from driftless geometric Brownian motion as the interest rate process under the risk neutral measure $\mathbb{Q}$ as follows:

$$dr_t = \beta r_t dt + \sigma_r r_t dW_t, \tag{2.45}$$

where r_0, β and σ_r are positive constants. The dynamics are easily integrated as follows

$$r_t = r_s \exp\left\{\left(\beta - \frac{\sigma_r^2}{2}\right)(t-s) + \sigma_r\left(W_t - W_s\right)\right\}, \tag{2.46}$$

for $s \leq t$. Hence, r_t is log-normal distributed with mean and variance

$$E\left[r_t\right] = r_s e^{\beta(t-s)},$$

$$Var\left[r_t\right] = r_s^2 e^{2\beta(t-s)}\left(e^{\sigma_r^2(t-s)} - 1\right).$$

2.5.3 The Cox-Ingersoll and Ross (CIR) Model

The general equilibrium approach developed by Cox, Ingersoll and Ross (1985) led to the introduction of a square root term in the diffusion coefficient of the instantaneous interest rate dynamic proposed by Vasieck (1977). The resulting model has since been a benchmark because of its analytical tractability and the fact that, contrary to the Vasicek model, the instantaneous interest rate is always positive.

The model formulation under the risk-neutral measure $\mathbb{Q}$ is

$$dr_t = \gamma\left(\kappa - r_t\right) dt + \sigma_r \sqrt{r_t} dW_t, \tag{2.47}$$

where r_0, γ, κ and σ_r are positive constants. The condition $2\kappa\gamma > \sigma_r^2$ has to be imposed to ensure that the origin is inaccessible to the process (2.47), so that we can be sure that r remains positive. The mean and variance of r_t are given by

$$E\left[r_t\right] = r_s e^{-\gamma(t-s)} + \kappa e^{-\gamma(t-s)},$$
$$Var\left[r_t\right] = r_s \tfrac{\sigma_r^2}{\gamma}\left(e^{-\gamma(t-s)} - e^{-2\gamma(t-s)}\right) + \tfrac{\kappa\sigma_r^2}{2\gamma}\left(1 - e^{-\gamma(t-s)}\right)^2.$$

2.5.4 The Hull-White Model

The Hull-White model (1990) is one of the no-arbitrage models that is designed to be exactly consistent with the observed bond prices or the term structure of interest rate. According to the Hull-White model, also referred to as the extended-Vasicek model, the interest rate process evolves under the risk neutral measure as follows:

$$dr_t = \left(\alpha(t) - \beta r_t\right) dt + \sigma_r dW_t, \tag{2.48}$$

where β and σ_r are positive constants. The time deterministic function $\alpha(t)$ is chosen so that the model fits the initial term structure of interest rate. Hence, the Hull-White (1990) model can be characterized as Vasieck model with a time dependent reversion level. As in the Vasicek model, The Hull-White model as equation (2.48) can be integrated to give

$$r_t = r_s e^{-\beta(t-s)} + \xi_t - \xi_s e^{-\beta(t-s)} + \sigma_r \int_s^t e^{-\beta(t-u)} dW_u,$$

where $\xi_t = f_{0,t} + \frac{\sigma_r^2}{2\beta^2}\left(1 - e^{-\beta t}\right)^2$. Hence, r_t is normally distributed with mean and variance, for $s \leq t$

$$E\left[r_t\right] = r_s e^{-\beta(t-s)} + \xi_t - \xi_s e^{-\beta(t-s)},$$
$$Var\left[r_t\right] = \frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta(t-s)}\right).$$

2.6 Pricing of Zero Coupon Bond

In this section we shall describe the zero coupon bond and some of its properties. For more details see Privault (2008) and Brigo and Mercurio (2001).

Definition 2.15. (Zero-coupon bond). A *zero-coupon bond* with maturity $T > 0$ is a contract that guarantees the holder a cash payment of one unit on the date T. The price at time $t \in [0, T]$ of a zero-coupon bond with maturity T is denoted by $P(t, T)$. At time t, the time to maturity is $T - t$.

The computation of the arbitrage price of a zero coupon bond based on an underlying interest rate process r_t is a fundamental issue in interest rate modelling. We may distinguish three different situations:

- The interest rate is a deterministic constant $r > 0$. In this case $P(t, T)$ should satisfy the equation

  $$e^{r(T-t)} P(t, T) = P(T, T) = 1,$$

  which leads to $P(t, T) = e^{-r(T-t)}, \quad 0 \leq t \leq T.$

- The interest rate is a time dependent and deterministic function $r(t)$. In this case, an argument similar to the above show that

$$P(t,T) = \exp\left(-\int_t^T r\,(s)ds)\right), \quad 0 \leq t \leq T. \tag{2.49}$$

- The interest rate is a stochastic process r_t.

We focus now on the stochastic interest rate and the pricing of the zero coupon bond $P(t,T)$ with the following steps.

2.6.1 Absence of Arbitrage and the Markov Property

Given previous experience with Black Scholes pricing (see Privault (2008) proposition 2.2), it seems natural to write $P(t,T)$ as a conditional expectation under martingale measure. On the other hand and with respect to interest rate as a stochastic process, the use of conditional expectation appears natural in this framework since it can help us filter out the future information past time t contained in equation (2.49). Thus we assume that

$$P(t,T) = E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right]. \tag{2.50}$$

Under some martingale (also called risk-neutral measure) measure $\mathbb{Q}$ yet to be determined, Equation (2.50) makes sense as the best possible estimate of the future quantity $\exp\left(-\int_t^T r_s ds\right)$ given information known up to time t.

From now on, we assume that the underlying interest rate process is the solution to the SDE

$$dr_t = f(t,r_t)dt + g(t,r_t)dW_t, \tag{2.51}$$

where W_t is a standard Brownian motion under probability measure $\mathbb{P}$. Consider a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and given by its density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^\infty K_s dW_s - \frac{1}{2}\int_0^\infty |K_s|^2 ds\right)$$

where K_s is a sufficiently integrable adapted process. By the Girsanov's theorem it is known that

$$\hat{W}_t := W_t + \int_0^t K_s ds$$

is a standard Brownian motion under $\mathbb{Q}$, thus equation (2.51) can be written as

$$dr_t = \tilde{f}(t, r_t)dt + g(t, r_t)d\hat{W}_t, \tag{2.52}$$

where $\tilde{f}(t, r_t) := f(t, r_t) - g(t, r_t)K_t$. The process K_t is called the *market price risk* and it needs to be specified, usually via statistical estimation based on market data.

In the sequel we will assume for simplicity that $K = 0$: in other words, we assume $\mathbb{P}$ is the martingale used by the market.

The Markov property states that the future after time t of a Markov process X_t depends only on its present state t and not on the whole history of the process up to time t. It can be stated as follows using conditional expectations:

$$E\left[f(X_{t_1}, ..., X_{t_n})|\mathcal{F}_t\right] = E\left[f(X_{t_1}, ..., X_{t_n})|X_t\right],$$

for all times $t_1, t_2, ..., t_n$ greater than t and all sufficiently integrable function f on $\mathbb{R}^n$, see Privault (2008), pp. 138-139.

Proposition 2.16. *All solutions of stochastic differential equation (2.51) have the Markov property.*

As a consequence, the arbitrage price $P(t, T)$ satisfies

$$P(t, T) = E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right] = E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)|r_t\right],$$

and depends on r_t only instead of depending on all information available in $\mathcal{F}_t$ up to time t. As such, it becomes a function $F(t, r_t)$ of r_t:

$$P(t, T) = F(t, r_t).$$

Thus the pricing problem can now be formulated as a search for the function $F(t,r)$.

2.6.2 Absence of Arbitrage and the Martingale Property

Our objective is now to apply Itô calculus to $P(t,T) = F(t,r_t)$ in order to derive a PDE satisfied by $F(t,r)$. From Itô's formula, we have

$$
\begin{aligned}
& d\left[\exp\left(-\int_0^t r_s ds\right) P(t,T)\right] = e(t,r_t)\left[-r_t P(t,T)dt + dP(t,T)\right] \\
& = e(t,r_t)\left[-r_t F(t,r_t)dt + dF(t,r_t)\right] \\
& = e(t,r_t)\left[-r_t F(t,r_t)dt + \left(\tfrac{\partial F}{\partial t}(t,r_t)dt + \tfrac{\partial}{\partial r}F(t,r_t)dr_t + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial r^2}(t,r_t)\left(dr_t\right)^2\right)\right] \\
& = e(t,r_t)\left[\begin{array}{l} -r_t F(t,r_t)dt + \frac{\partial F}{\partial t}(t,r_t)dt + \frac{\partial F}{\partial r}(t,r_t)\left(\tilde{f}(r_t,t)dt + g(r_t,t)d\hat{W}_t\right) \\ +\frac{1}{2}\frac{\partial^2 F}{\partial r^2}(t,r_t)\left(\tilde{f}(r_t,t)dt + g(r_t,t)d\hat{W}_t\right)^2 \end{array}\right] \\
& = e(t,r_t)\left[\begin{array}{l} -r_t F(t,r_t)dt + \frac{\partial F}{\partial t}(t,r_t)dt + \frac{\partial F}{\partial r}(t,r_t)\left(\tilde{f}(r_t,t)dt + g(r_t,t)d\hat{W}_t\right) \\ +\frac{1}{2}\frac{\partial^2 F}{\partial r^2}(t,r_t)g^2(r_t,t)dt \end{array}\right] \\
& = e(t,r_t)\left[\begin{array}{l} \left(-r_t F(t,r_t) + \frac{\partial F}{\partial t}(t,r_t) + \tilde{f}(t,r_t)\frac{\partial F}{\partial r}(t,r_t) + \frac{1}{2}g^2(t,r_t)\frac{\partial^2 F}{\partial r^2}(t,r_t)\right)dt \\ +\frac{\partial F}{\partial r}(t,r_t)g(r_t,t)d\hat{W}_t \end{array}\right].
\end{aligned}
\tag{2.53}
$$

where $e(t,r_t) = \exp\left(-\int_0^t r_s ds\right)$. Note that we have

$$
\begin{aligned}
& \exp\left(-\int_0^t r_s ds\right) P(t,T) = \exp\left(-\int_0^t r_s ds\right) E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds_t\right) |\mathcal{F}_t\right] \\
& = E_{\mathbb{Q}}\left[\exp\left(-\int_0^t r_s ds\right)\exp\left(-\int_t^T r_s ds_t\right) |\mathcal{F}_t\right] \\
& = E_{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds_t\right) |\mathcal{F}_t\right].
\end{aligned}
$$

Hence $t \mapsto \exp\left(-\int_0^t r_s ds\right) P(t,T)$ is martingale since for any $0 < u < t$ we have

$$E_{\mathbb{Q}}\left[\exp\left(-\int_0^t r_s ds_t\right) P(t,T)|\mathcal{F}_u\right] = E_{\mathbb{Q}}\left[E_{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds_t\right) P(t,T)|\mathcal{F}_t\right]|\mathcal{F}_u\right]$$
$$= E_{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds_t\right)|\mathcal{F}_u\right]$$
$$= E_{\mathbb{Q}}\left[\exp\left(-\int_0^u r_s ds_t\right)\exp\left(-\int_u^T r_s ds_t\right)|\mathcal{F}_u\right]$$
$$= \exp\left(-\int_0^u r_s ds_t\right) E_{\mathbb{Q}}\left[\exp\left(-\int_u^T r_s ds_t\right)|\mathcal{F}_u\right]$$
$$= \exp\left(-\int_0^u r_s ds_t\right) P(u,T).$$

As a consequence the above expression of $d\left[\exp\left(-\int_0^t r_s ds\right) P(t,T)\right]$ should contain terms in $d\hat{W}_t$ only, meaning that all terms in dt should vanish inside equation (2.53). This leads to the identity

$$-r_t F(t,r_t) + \frac{\partial F}{\partial t}(t,r_t) + \tilde{f}(t,r_t)\frac{\partial F}{\partial r}(t,r_t) + \frac{1}{2}g^2(t,r_t)\frac{\partial^2 F}{\partial r^2}(t,r_t) = 0,$$

which can be rewritten as in the next proposition.

Proposition 2.17. *The bond pricing PDE for $P(t,T) = F(t,r_t)$ is written as*

$$-r_t F(t,r_t) + \frac{\partial F}{\partial t}(t,r_t) + \tilde{f}(t,r_t)\frac{\partial F}{\partial r}(t,r_t) + \frac{1}{2}g^2(t,r_t)\frac{\partial^2 F}{\partial r^2}(t,r_t) = 0,$$

subject to terminal condition $F(T,r_T) = 1$.

2.7 Forward Measure and Option Pricing

In a standard Black-Scholes framework with a riskless account yielding interest at the interest rate r_t, the call option price at time t of a contingent claim with payoff $F = \max(S_T - K, 0)$ at maturity time T, is the conditional expectation

$$E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\max(S_T - K, 0)|\mathcal{F}_t\right], \tag{2.54}$$

under a risk neutral probability measure $\mathbb{Q}$. When the interest rate process r_t is a deterministic function of time, the expression becomes

$$\exp\left(-\int_t^T r_s ds\right) E_{\mathbb{Q}}\left[\max(S_T - K, 0)|\mathcal{F}_t\right].$$

And when r_t equals a deterministic constant r, we get

$$\exp\left(-r(T-t)\right) E_{\mathbb{Q}}\left[\max(S_T - K, 0)|\mathcal{F}_t\right].$$

In most interest rate models the interest rate r_t is a random process forbidding the above manipulation so that we will have to evaluate expression of the form (2.54) where r_t is a random process, compounding another level of complexity in comparison with the standard Black-Schole model as in section 2.1.

Definition 2.18. The T-forward measure is the probability measure $\mathbb{Q}^T$ defined as

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{\exp\left(-\int_0^T r_s ds\right)}{P(0,T)}. \tag{2.55}$$

The expectation under $\mathbb{Q}^T$ will be denoted by $E^T_{\mathbb{Q}}$.

The following proposition will allow us to price call option under the T-forward measure $\mathbb{Q}^T$.

Proposition 2.19. *For all sufficiently integrable random variables F we have*

$$E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) F|\mathcal{F}_t\right] = P(t,T) E_{\mathbb{Q}^T}\left[F|\mathcal{F}_t\right], \quad 0 \le t \le T. \tag{2.56}$$

Proof. See Privault (2008) page 74. □

As consequence of this proposition the computation of $E_{\mathbb{Q}^T}\left[\exp\left(-\int_t^T r_s ds\right) F|F_t\right]$ can be replaced by that of $P(t,T)E_{\mathbb{Q}^T}[F|\mathcal{F}_t]$ under T-forward measure $\mathbb{Q}^T$.

CHAPTER III

OPTION PRICING FOR A STOCHASTIC VOLATILITY LEVY MODEL WITH STOCHASTIC INTEREST RATE

3.1 Introduction

Let us assume that a risk-neutral probability measure $\mathbb{Q}$ exists so that we will consider all processes in section 3.1 and 3.2 under risk-neutral measure $\mathbb{Q}$.

In the Black-Scholes model, the price of a risky asset S_t under a risk-neutral measure $\mathbb{Q}$ and with non dividend payment as follows

$$S_t = S_0 \exp(\tilde{L}_t) = S_0 \exp\left(rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)\right), \tag{3.1}$$

where $r \in \mathbb{R}$ is risk-free interest rate and $\sigma \in \mathbb{R}$ is the volatility coefficient of the asset price.

Instead of modelling the log returns $\tilde{L}_t = rt + \sigma W_t - \frac{1}{2}\sigma^2 t$, with a normal distribution, we now replace it with a more sophisticated process L_t, a Lévy process of the form

$$L_t = rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) + (J_t - \xi t), \tag{3.2}$$

where J_t denotes a pure Lévy jump component, (i.e. a Lévy process with no Brownian motion part), and ξ the concavity adjustment. We assume that the processes W_t and J_t are independent.

To incorporate the volatility effect to the model (3.2), we follow the technique of Carr and Wu (2004) by subordinating a part of diffusion, $(\sigma W_t - \frac{\sigma^2}{2} t)$ and a part of jump $(J_t - \xi t)$ to the time integral of a mean reverting Cox Ingersoll Ross (CIR) process

$$T_t = \int_0^t v_s ds,$$

where v_t follows the CIR process

$$dv_t = \gamma(\kappa - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v. \tag{3.3}$$

Here W_t^v is a standard Brownian motion which corresponds to the process v_t . The constant $\gamma \in \mathbb{R}$ is the rate at which the process v_t reverts toward its long term mean, κ is the long term variance, and $\sigma_v > 0$ is the volatility coefficient of the process v_t .

Hence, the model (3.2) becomes

$$L_t = rt + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + \left(J_{T_t} - \xi T_t\right), \tag{3.4}$$

and this new process is called a *stochastic volatility Lévy process.* One can interpret T_t as the stochastic clock process with activity rate process v_t. By replacing $\tilde{L}_t$ in (3.2) with L_t, we obtain a model of an underlying asset under the risk-neutral measure $\mathbb{Q}$ with stochastic volatility as follows:

$$S_t = S_0 \exp(L_t) = S_0 \exp\left[rt + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + (J_{T_t} - \xi T_t)\right]. \tag{3.5}$$

In this chapter, we shall consider the problem of finding a formula for European call options based on the underlying asset model (3.5) for which the constant interest rate r is replaced by the stochastic interest rate r_t, so that the model under our consideration is given by

$$S_t = S_0 \exp\left[r_t t + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + (J_{T_t} - \xi T_t)\right]. \tag{3.6}$$

Here, we assume that r_t follows the Vasicek process

$$dr_t = (\alpha - \beta r_t)dt + \sigma_r dW_t^r, \tag{3.7}$$

where W_t^r is a standard Brownian motion with respect to the process r_t and $dW_t^r dW_t = 0$. The constant $\beta \in \mathbb{R}$ is the rate at which the interest rate reverts toward its long term mean: $\sigma_r > 0$ is the volatility coefficient of the interest rate process (3.7): the constant $\alpha > 0$ is a speed reversion. We also assume that the interest rate process r_t and the activity rate process v_t are independent.

The problem of option pricing under stochastic interest rate has been investigated for quite a while. Kim (2001) constructed the option pricing formula based on the Black-Scholes model under several stochastic interest rate processes, i.e., Vasicek, CIR, Ho-Lee type. He found that incorporating stochastic interest rate into the Black-Scholes model for a short maturity option does not improve the performance of the original Black-Scholes' pricing formula. Brigo and Mercurio (2001, pp. 883) mention that the stochastic feature of interest rate has a stronger impact on the option price when pricing for a long maturity option. Carr and Wu (2004) continued this study by giving an option pricing formula based on a time-changed Lévy process, with constant interest rate remaining in the model.

In this chapter, we give an analysis of the option pricing model based on a time change Lévy process with stochastic interest rates.

The dynamics under the forward measure is described in Section 3.2, with the option pricing formula given in Section 3.3. Finally, the closed form solution for a European call option in term of the characteristic function is in Section 3.4.

3.2 The Dynamics under the Forward Measure

We begin by briefly reviewing the definition of correlated Brownian motion and some of its properties (for more details see Brummelhuis (2009), pp. 70). A standard Brownian motion in $\mathbb{R}^n$ is a stochastic process $(\vec{Z}_t)_{t\geq 0}$ whose value at time t is simply a vector of n independent Brownian motions at t :

$$\vec{Z}_t = (Z_{1,t}, ...Z_{n,t}).$$

We use Z instead of W, since we would like to reserve the latter for the more general case of *correlated Brownian motion*, which will be defined as follows:

Let $\rho = (\rho_{ij})_{1\leq i,j\leq n}$ be a (constant) positive symmetric matrix satisfying $\rho_{ii} = 1$ and $-1 \leq \rho_{ij} \leq 1$. By *Cholesky's decomposition* theorem, one can find an upper triangular $n \times n$ matrix $\mathrm{H} = (h_{ij})$ such that

$$\rho = \mathrm{HH}^t,$$

where H^t is the transpose of the matrix H. Letting $\vec{Z}_t = (Z_{1,t}, ...Z_{n,t})$ be a standard Brownian motion as introduced above, we define a new vector-valued process $\vec{W}_t = (W_{1,t}, ..., W_{n,t})$ by

$$\vec{W}_t = \mathrm{H}\vec{Z}_t,$$

or, in term of components,

$$W_{i,t} = \sum_{j=1}^{n} h_{ij} Z_{j,t}, i = 1, ..., n.$$

The process $(\vec{W}_t)_{t\geq 0}$ is called a *correlated Brownian motion* with a (constant) correlation matrix ρ. Each component-process $(W_{i,t})_{t\geq 0}$ is itself a standard Brownian motion. Note that if $\rho = Id$ (the identity matrix) then $\vec{W}_t$ is a standard Brownian motion. For example, if we consider a symmetric matrix

$$\rho = \begin{bmatrix} 1 & \rho_v & 0 \\ \rho_v & 1 & 0 \\ 0 & 0 & 1. \end{bmatrix}, \tag{3.8}$$

then ρ has a Cholesky decomposition of the form $\rho = HH^T$, where H is an upper triangular matrix of the form

$$H = \begin{bmatrix} \sqrt{1-\rho_v^2} & \rho_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\vec{Z}_t = (Z_t, Z_t^r, Z_t^v)$ be three independent Brownian motions, then $\vec{W}_t = (W_t, W_t^r, W_t^\nu)$ defined by $\vec{W}_t = \mathrm{H}\vec{Z}_t$, is a correlated Brownian motion with correlation matrix ρ as given in (3.8), or in terms,

$$\begin{aligned} W_t &= (\sqrt{1-\rho_v^2})Z_t + \rho_v Z_t^\nu, \\ W_t^\nu &= Z_t^\nu, \\ W_t^r &= Z_t^r. \end{aligned} \tag{3.9}$$

Now let us turn to equation (3.6). Note that, by Ito's formula, the model (3.6) has the dynamic given by

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma S_t dW_{T_t} + S_{t-} dJ_{T_t}^*, \\ dr_t &= (\alpha - \beta r_t)dt + \sigma_r dW_t^r, \\ dv_t &= \gamma(\kappa - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v, \end{aligned} \tag{3.10}$$

where $dJ_{T_t}^* = dJ_{T_t} + \left(e^{\Delta J_{T_t}} - 1 - \Delta J_{T_t}\right), dW_t dW_t^r = dW_t^r dW_t^v = 0$, and $dW_t dW_t^v = \rho_v dt$.

We can re-write the system (3.10) in terms of three independent Brownian motions (Z_t, Z_t^ν, Z_t^r) as follows:

$$dS_t = r_t S_t dt + \sigma S_t \left(\rho_v dZ_{T_t}^v + \sqrt{1-\rho_v^2} dZ_{T_t}\right) + S_{t-} dJ_{T_t}^*, \tag{3.11}$$

$$dr_t = (\alpha - \beta r_t)dt + \sigma_r dZ_t^r, \tag{3.12}$$

$$dv_t = \gamma(\kappa - v_t)dt + \sigma_v \sqrt{v_t} dZ_t^v. \tag{3.13}$$

This decomposition makes it easier to perform a measure transformation. In fact, for any fixed maturity T, let us denote by $\mathbb{Q}^T$ the T-forward measure, i.e. the probability measure that is defined by the Radon-Nikodym derivative,

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{\exp\left(-\int\limits_0^T r_u du\right)}{P(0,T)}. \tag{3.14}$$

Here, $P(t,T)$ is the price at time t of a zero-coupon bond with maturity T and is defined as

$$P(t,T) = E_{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}|\mathcal{F}_t\right]. \tag{3.15}$$

Lemma 3.1. *The process r_t following the dynamics in (3.12) can be written in the form*

$$r_t = x_t + w(t), \quad 0 \le t \le T, \tag{3.16}$$

where the process x_t satisfies

$$dx_t = -\beta x_t dt + \sigma_r dZ_t^r, \; x_0 = 0. \tag{3.17}$$

Moreover, the function w is deterministic and well defined in the time interval $[0,T]$ which satisfied

$$w(t) = r_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right). \tag{3.18}$$

In particular, $w(0) = r_0$.

Proof. To solve the solution of SDE (3.12), let $g(t,r) = e^{\beta t} r$, by using Itô's formula, we have

$$dg = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial r}dr + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}(dr)^2.$$

Then

$$\begin{aligned} de^{\beta t} r &= \beta e^{\beta t} r dt + e^{\beta t}\left((\alpha - \beta r)dt + \sigma_r dZ^r\right) \\ &= \beta e^{\beta t} r dt + \alpha e^{\beta t} dt - \beta e^{\beta t} r dt + \sigma_r e^{\beta t} dZ^r \\ &= \alpha e^{\beta t} dt + \sigma_r e^{\beta t} dZ^r. \end{aligned} \tag{3.19}$$

Integrating on both side the above equation from 0 to t where $0 < t \leqslant T$ and simplifying, we obtain

$$\begin{aligned}\int_0^t de^{\beta u} r_u &= \alpha \int_0^t e^{\beta u} du + \sigma_r \int_0^t e^{\beta u} dZ_u^r \\ r_t &= r_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r.\end{aligned}$$

By using the definition of $w(t)$ form equation (3.18),we get

$$r_t = w(t) + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r.$$

Note that the solution of equation (3.17) is

$$x_t = x_0 e^{-\beta t} + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r = \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r. \tag{3.20}$$

Hence, $r_t = w(t) + x_t$, for each t. The proof is now complete. □

Now we are ready to calculate the Radon-Nikodym derivative as it appears in equation (3.14). By virtue of Lemma 3.1, $r_t = x_t + w(t)$ and the price of zero coupon bond at time $t = 0$ with maturity time T in Vasicek model which satisfy (see, Privault (2008) pages. 38-39)

$$P(t,T) = \exp\left[A(t,T) + r_t B(t,T)\right],$$

where $B(t,T) = \frac{e^{-\beta(T-t)}-1}{\beta}$ and $A(t,T) = \left(\frac{\alpha}{\beta} - \frac{\sigma_r^2}{2\beta^2}\right)(B(t,T) - T + t) + \frac{\sigma_r^2}{4\beta} B^2(t,T)$. Substituting r_t and $P(0,t)$ into equation (3.14), we have

$$\begin{aligned}\frac{d\mathbb{Q}^T}{d\mathbb{Q}} &= \frac{\exp\left(-\int_0^T x_u + w(u)\, du\right)}{\exp(a(0,T) + b(0,T) r_0)} \\ &= \exp\left(-\int_0^T x_u du - \frac{\sigma^2}{2\beta^2}\int_0^T \left(1 - e^{-\beta(T-u)}\right)^2 du\right).\end{aligned} \tag{3.21}$$

Stochastic integration by parts implies

$$\int_0^T x_u du = T x_T - \int_0^T u dx_u = \int_0^T (T-u)\, dx_u. \tag{3.22}$$

By substituting the expression for dx_u from equation (3.17),

$$\int_0^T (T-u)\, dx_u = -\beta \int_0^T (T-u) x_u du + \sigma_r \int_0^T (T-u) dZ_u^r. \tag{3.23}$$

Moreover by substituting the expression for x_u from equation (3.20), the first integral on the right hand side of equation (3.23) becomes

$$-\beta \int_0^T (T-u) x_u du = -\beta\sigma_r \int_0^T ((T-u) \int_0^u e^{-\beta(u-s)} dZ_u^r) du. \tag{3.24}$$

Using integration by parts, we have

$$\begin{aligned}
&-\beta\sigma_r \int_0^T \left((T-u) \int_0^u e^{-\beta(u-s)} dZ_u^r \right) du = -\beta\sigma_r \int_0^T \left(\int_0^u e^{\beta s} dZ_s^r \right) (T-u) e^{-\beta u} du \\
&= -\beta\sigma_r \int_0^T \left(\int_0^u e^{\beta s} dZ_s^r \right) d \left(\int_0^u (T-v) e^{-\beta v} dv \right) \\
&= -\beta\sigma_r \left[\left(\int_0^T e^{\beta u} dZ_u^r \right) \left(\int_0^T (T-v) e^{-\beta v} dv \right) - \int_0^T \left(\int_0^u (T-v) e^{-\beta v} dv \right) e^{\beta u} dZ_u^r \right] \\
&= -\beta\sigma_r \left[\int_0^T e^{\beta u} \left(\int_u^T (T-v) e^{-\beta v} dv \right) dZ_u^r \right] \\
&= -\tfrac{\sigma_r}{\beta} \left[\int_0^T \left(e^{-\beta(T-u)} - 1 \right) dZ_u^r \right] - \sigma_r \int_0^T (T-u) dZ_u^r.
\end{aligned} \tag{3.25}$$

Substituting equation (3.25) into equation (3.23), we obtain

$$\int_0^T (T-u)\, dx_u = -\frac{\sigma_r}{\beta} \left[\int_0^T \left(e^{-\beta(T-u)} - 1 \right) dZ_u^r \right].$$

Hence,

$$\int_0^T x_u du = -\frac{\sigma_r}{\beta} \left[\int_0^T \left(e^{-\beta(T-u)} - 1 \right) dZ_u^r \right]. \tag{3.26}$$

Substituting equation (3.26) into equation (3.21), we obtain

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \exp \left(-\frac{\sigma_r}{\beta} \int_0^T \left(1 - e^{-\beta(T-u)} \right) dZ_u^r - \frac{\sigma_r^2}{2\beta^2} \int_0^T \left(1 - e^{-\beta(T-u)} \right)^2 du \right). \tag{3.27}$$

Hence, by Girsanov's theorem, three processes Z_t^{rT}, Z_t^{vT} and Z_t^T defined by

$$\begin{aligned} dZ_t^{rT} &= dZ_t^r + \tfrac{\sigma_r}{\beta}\left(1 - e^{-\beta(T-t)}\right)dt, \\ dZ_t^{vT} &= dZ_t^v, \\ dZ_t^T &= dZ_t, \end{aligned} \tag{3.28}$$

are three independent Brownian motions under the measure $\mathbb{Q}^T$. Therefore, the dynamics of r_t, v_t and S_t under $\mathbb{Q}^T$ are given by

$$dS_t = r_t S_t dt + \sigma S_t \left(\rho_v dZ_{T_t}^{vT} + \sqrt{1-\rho_v^2} dZ_{T_t}^T\right) + S_{t-} dJ_{T_t}^*, \tag{3.29}$$

$$dr_t = \left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right) dt + \sigma_r dZ_t^{rT}, \tag{3.30}$$

$$dv_t = \gamma(\kappa - v_t)dt + \sigma_v \sqrt{v_t} dZ_t^{vT}. \tag{3.31}$$

From now on, this system will be called the stochastic volatility Lévy model with stochastic interest rate (SVLSI) and the equation (3.30) is TF-Vasieck model.

3.3 The Pricing of European Call Option on a Given Asset

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modeled as a stochastic process on a filtered probability space $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}^T\right)$, where $\mathcal{F}_t$ is usually the price history up to time t. All processes in this section will be defined on this space. We denote by C the price at time t of a European call option on the current price of an underlying asset S_t with strike price K and expiration time T.

The terminal payoff of a European option on the underlying stock S_t with strike price K is

$$\max(S_T - K, 0). \tag{3.32}$$

This means the holder will exercise his right only if $S_T > K$ and then his gain would be $S_T - K$. Otherwise, if $S_T \leq K$ then the holder will buy the underlying asset from the market and the value of the option is zero.

We would like to find a formula for pricing a European call option with strike price K and maturity T based on the model (3.29)-(3.31). Consider a continuous-time economy where interest rate are stochastic and the price of the European call option at time t under the forward measure $\mathbb{Q}^T$ is

$$\begin{aligned} C(t, S_t, r_t, v_t; T, K) &= P^*(t,T) E_{\mathbb{Q}^T}\left(\max\left(S_T - K, 0\right) | S_t, r_t, v_t\right) \\ &= P^*(t,T) \int\limits_0^\infty \max\left(S_T - K, 0\right) p_{\mathbb{Q}^T}(S_T | S_t, r_t, v_t) dS_T \end{aligned}$$

where $E_{\mathbb{Q}^T}$ is the expectation with respect to the T-forward probability measure, $p_{\mathbb{Q}^T}$ is the corresponding conditional density given (S_t, v_t, r_t) and P^* is a zero coupon bond defined by

$$P^*(t,T) := E_{Q^T}\left[\exp\left(-\int\limits_t^T r_s ds\right) | \mathcal{F}_t\right]. \tag{3.33}$$

Changing variable, $X_t = \ln S_t$,

$$\begin{aligned} &C(t, S_t, r_t, v_t; T, K) = P^*(t,T) \int\limits_{-\infty}^{\infty} \max\left(e^{X_T} - K, 0\right) p_{Q^T}\left(X_T | X_t, r_t, v_t,\right) dX_T \\ &= P^*(t,T) \int\limits_{\ln K}^{\infty} \left(e^{X_T} - K\right) 1_{X_T \geq \ln K} p_{Q^T}\left(X_T | X_t, r_t, v_t\right) dX_T \\ &= P^*(t,T) \int\limits_{\ln K}^{\infty} e^{X_T} p_{Q^T}\left(X_T | X_t, r_t, v_t\right) dX_T - K P^*(t,T) \int\limits_{\ln K}^{\infty} p_{Q^T}\left(X_T | X_t, r_t, v_t\right) dX_T \\ &= e^{X_t}\left(\frac{1}{E_{Q^T}(e^{X_T} | S_t, r_t, v_t)} \int\limits_{\ln K}^{\infty} e^{X_T} p_{Q^T}\left(X_T | X_t, r_t, v_t\right) dX_T\right) \\ &\quad - K P^*(t,T) \int\limits_{\ln K}^{\infty} p_{Q^T}\left(X_T | X_t, r_t, v_t\right) dX_T \\ &= e^{X_t}\left(\int\limits_{\ln K}^{\infty} e^{X_T} \frac{p_{Q^T}(X_T | X_t, r_t, v_t)}{E_{Q^T}(e^{X_T} | S_t, r_t, v_t)} dX_T\right) - K P^*(t,T) \int\limits_{\ln K}^{\infty} p_{Q^T}\left(X_T | X_t, r_t, v_t\right) dX_T. \end{aligned} \tag{3.34}$$

With the first integral in equation (3.34) being positive and integrating up to one, it therefore defines a new probability measure that we denote by $q_{\mathbb{Q}^T}$ below

$$\begin{aligned} &C(t, S_t, r_t, v_t; T, K) \\ &= e^{X_t} \int\limits_{\ln K}^{\infty} q_{\mathbb{Q}^T}\left(X_T | X_t, r_t, v_t\right) dX_T - K P^*(t,T) \int\limits_{\ln K}^{\infty} p_{\mathbb{Q}^T}\left(X_T | X_t, r_t, v_t\right) dX_T \\ &:= e^{X_t} \mathrm{P}_1\left(t, X_t, r_t, v_t; T, K\right) - K P^*(t,T) \mathrm{P}_2\left(t, X_t, r_t, v_t; T, K\right) \\ &= e^{X_t} \Pr\left(X_T > \ln K | X_t, r_t, v_t\right) - K P^*(t,T) \Pr\left(X_T > \ln K | X_t, r_t, v_t\right) \end{aligned} \tag{3.35}$$

where those probabilities in equation (3.35) are calculated under the probability measure $\mathbb{Q}^T$.

The European call option for log asset price $X_t = \ln S_t$ will be denoted by

$$\hat{C}(t, X_t, r_t, v_t; T, \kappa^*) = e^{X_t}\tilde{P}_1(t, X_t, r_t, v_t; T, \kappa^*) - e^{\kappa^*}P^*(t,T)\tilde{P}_2(t, X_t, r_t, v_t; T, \kappa^*), \tag{3.36}$$

where $\kappa^* = \ln K$ and $\tilde{\mathrm{P}}_j\left(t, X_t, r_t, v_t; T, \kappa\right) := \mathrm{P}_j\left(t, X_t, r_t, v_t; T, K\right), j = 1, 2$.
Note that we do not have a closed form solution for these probabilities. However, these probabilities are related to characteristic functions which have closed form solutions as will be seen in Lemma 3.4.

Next let us consider a continuous-time economy where interest rates are stochastic and satisfy equation (3.30). Since the SDE in equation (3.30) satisfies all the necessary conditions of Theorem 32, see Protter (2005) pp. 238, the solution of equation (3.30) has the Markov property. As a consequence, the zero coupon bond price at time t under the forward measure $\mathbb{Q}^T$ in equation (3.33) satisfies

$$P^*(t,T) = E_{Q^T}\left[\exp\left(-\int_t^T r_s ds\right)|r_t\right]. \tag{3.37}$$

Note that since $P^*(t,T)$ depends on r_t, it becomes a function $F(t, r_t)$ of r_t, i.e. $P^*(t,T) = F(t, r_t)$. This means that the calculation of $P^*(t,T)$ can now be formulated as a search for the function $F(t, r_t)$.

Lemma 3.2. *The price of a zero coupon bond can be derived by computing the expectation (3.37). We obtain*

$$P^*(t,T) = \exp\left(a(t,T) + b(t,T)r_t\right), \tag{3.38}$$

where $b(t,T) = \frac{1}{\beta}\left(\mathrm{e}^{-\beta(T-t)} - 1\right)$ *and*
$a(t,T) = \left(\frac{3\sigma_r^2}{2\beta^4} - \frac{\alpha}{\beta}\right)\left(b(t,T) + (T-t)\right) - \frac{3\sigma_r^2}{4\beta^4}b^2(t,T)$.

Proof. Under the T-forward measure $\mathbb{Q}^T$, the interest rate is given by the following stochastic differential equation (SDE):

$$dr_t = \left(\alpha - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) - \beta r_t\right) dt + \sigma_r dZ_t^{rT}. \tag{3.39}$$

The specification of the interest rate means that the model (3.39) belong to the affine class of interest rate models. Thus the bond price at time t with maturity T is of the form

$$P^*(t,T) = \exp\left(a(t,T) + b(t,T)r_t\right), \tag{3.40}$$

where $a(t,T)$ and $b(t,T)$ are functions to be determined under the condition $a(T,T) = 0$ and $b(T,T) = 0$. We will now find explicit formulas for the functions $a(t,T)$ and $b(t,T)$ in equation (3.38).

By Proposition 2.18, the zero coupon bond price PDE satisfies

$$\begin{aligned}\tfrac{\partial F(t,r_t)}{\partial t} + \left(\alpha - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) - \beta r_t\right)\tfrac{\partial F(t,r_t)}{\partial r_t} \\ + \tfrac{1}{2}\tfrac{\partial^2 F(t,r_t)}{\partial r_t^2}\sigma_r^2 - r_t F(t,r_t) = 0.\end{aligned} \tag{3.41}$$

Note that $F(t,r_t) = P^*(t,T)$. We substitute the value $F(t,r_t)$ from equation (3.40) into the above equation and after canceling some common factors, we have

$$\begin{aligned}\left(\tfrac{\partial a(t,T)}{\partial t} + r_t\tfrac{\partial b(t,T)}{\partial t}\right) + \left(\alpha - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) - \beta r_t\right) b(t,T) \\ + \tfrac{1}{2}b^2(t,T)\sigma_r^2 - r_t = 0,\end{aligned}$$

so that we can reduce it to two ordinary differential equations

$$\frac{\partial a(t,T)}{\partial t} + \frac{\sigma_r^2}{2}b^2(t,T) + \left(\alpha - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right) b(t,T) = 0, \tag{3.42}$$

$$\frac{\partial b(t,T)}{\partial t} - \beta b(t,T) - 1 = 0, \tag{3.43}$$

with boundary conditions $a(T,T) = 0$ and $b(T,T) = 0$.

Firstly, we note that the solution of (3.43) satisfying the boundary conditions $b(T,T) = 0$ is

$$b(t,T) = \frac{1}{\beta}\left(\mathrm{e}^{-\beta(T-t)} - 1\right). \tag{3.44}$$

Secondly, we try to solve equation (3.42). Note that

$$\int_t^T \frac{\partial a(u,T)}{\partial u} du = [a(u,T)]_{u=t}^{u=T} = a(T,T) - a(t,T) = -a(t,T).$$

Thus

$$\begin{aligned}
a(t,T) &= \tfrac{\sigma_r^2}{2} \int_t^T b^2(u,T)du + \int_t^T b(u,T)\left(\alpha - \tfrac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-u)}\right)\right)du \\
&= \tfrac{\sigma_r^2}{2} \int_t^T b^2(u,T)du + \int_t^T \alpha b(u,T)du - \sigma_r^2 \int_t^T b(u,T)\tfrac{1}{\beta}\left(1-e^{-\beta(T-u)}\right)du \\
&= \tfrac{\sigma_r^2}{2} \int_t^T b^2(u,T)du + \int_t^T \alpha b(u,T)du + \sigma_r^2 \int_t^T b^2(u,T)du \\
&= \tfrac{3\sigma_r^2}{2} \int_t^T b^2(u,T)du + \int_t^T \alpha b(u,T)du.
\end{aligned} \tag{3.45}$$

Note that the first integral on the right hand side of equation (3.45) is equal to

$$\begin{aligned}
\tfrac{3\sigma_r^2}{2} \int_t^T b(u,T)^2 du &= \tfrac{3\sigma_r^2}{2\beta^2} \int_t^T \left(\mathrm{e}^{-\beta(T-u)} - 1\right)^2 du \\
&= -\tfrac{3\sigma_r^2}{2\beta^2}\left[\tfrac{3}{2\beta^3} + \tfrac{e^{-2\beta(T-t)}}{2\beta^3} - \tfrac{2e^{-2\beta(T-t)}}{\beta^3} - \tfrac{(T-t)}{\beta^2}\right] \\
&= -\tfrac{3\sigma_r^2}{2\beta^2}\left[\tfrac{1}{2\beta^3}\left(3 + e^{-2\beta(T-t)} - 2e^{-\beta(T-t)}\right) - \tfrac{e^{-\beta(T-t)}}{\beta^3} - \tfrac{(T-t)}{\beta^2}\right] \\
&= -\tfrac{3\sigma_r^2}{2\beta^2}\left[\tfrac{1}{2\beta}\left(\tfrac{e^{-\beta(T-t)}-1}{\beta}\right)^2 + \tfrac{1}{\beta^2}\left(\tfrac{1-e^{-\beta(T-t)}}{\beta}\right) - \tfrac{(T-t)}{\beta^2}\right] \\
&= -\tfrac{3\sigma_r^2}{2\beta^4}\left[\tfrac{1}{2}b^2(t,T) - b(t,T) - (T-t)\right].
\end{aligned} \tag{3.46}$$

The second integral on the right hand side of equation (3.45) is equal to

$$\begin{aligned}
\alpha \int_t^T b(u,T)du &= \tfrac{\alpha}{\beta} \int_t^T \left(\mathrm{e}^{-\beta(T-u)} - 1\right)du \\
&= \tfrac{\alpha}{\beta}\left(\tfrac{1-\mathrm{e}^{-\beta(T-t)}}{\beta} - (T-t)\right) \\
&= -\tfrac{\alpha}{\beta}\left(b(t,T) + (T-t)\right).
\end{aligned} \tag{3.47}$$

Summing up the two expressions on the right hand side of equation (3.46) and (3.47) and simplifying them, we get

$$\begin{aligned}
a(t,T) &= -\frac{3\sigma_r^2}{2\beta^4}\left[\frac{1}{2}b^2(t,T) - b(t,T) - (T-t)\right] - \frac{\alpha}{\beta}\left(b(t,T) + (T-t)\right) \\
&= -\frac{3\sigma_r^2}{2\beta^4}\left[\frac{1}{2}b^2(t,T)\right] + \left[\frac{3\sigma_r^2}{2\beta^4} - \frac{\alpha}{\beta}\right]\left[b(t,T) + (T-t)\right].
\end{aligned}$$

The proof is now complete. □

The following Lemma shows the relationship between $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of equation (3.36).

Lemma 3.3. *Assume the Lévy density exists. The functions* $\tilde{P}_1$ *and* $\tilde{P}_2$ *in the option values of the equation (3.36) satisfy the PIDEs:*

$$\begin{aligned} 0 &= \frac{\partial \tilde{P}_1}{\partial t} + A[\tilde{P}_1] + (\rho_v \sigma v \sigma_v) \frac{\partial \tilde{P}_1}{\partial v} \\ &+ v \int_{-\infty}^{\infty} (e^y - 1) \left(\tilde{P}_1(x+y, t, r, v; T, \kappa^*) - \tilde{P}_1(x, t.r, v; T, \kappa^*) \right) k(y) dy, \end{aligned} \tag{3.48}$$

subject to the boundary condition at expiration $t = T$*:*

$$\tilde{P}_1(T, x, r, v; T, \kappa^*) = 1_{x > \kappa^*}. \tag{3.49}$$

Moreover, $\tilde{P}_2$ *satisfies the equation*

$$\begin{aligned} 0 &= \frac{\partial \tilde{P}_2}{\partial t} + A[\tilde{P}_2] + \tilde{P}_2 \left[b(t,T) \left(\alpha - \beta r - \frac{\sigma_r^2}{\beta} \left(1 - e^{-\beta(T-t)} \right) \right) \right] \\ &+ \sigma_r^2 b(t,T) \frac{\partial \tilde{P}_2}{\partial r} - \sigma^2 v \frac{\partial \tilde{P}_2}{\partial x} + \tilde{P}_2 \left[\frac{\partial}{\partial t} \left(a(t,T) + b(t,T) r \right) - r + \frac{\sigma_r^2}{2} b^2(t,T) \right], \end{aligned} \tag{3.50}$$

subject to the boundary condition at expiration $t = T$*:*

$$\tilde{P}_2(T, x, r, v; T, \kappa^*) = 1_{x > \kappa^*}. \tag{3.51}$$

Here

$$\begin{aligned} A[\tilde{P}_i] &= \left(r + \tfrac{1}{2}\sigma^2 v \right) \frac{\partial \tilde{P}_i}{\partial x} + \gamma(\kappa - v) \frac{\partial \tilde{P}_i}{\partial v} + \frac{\sigma^2 v}{2} \frac{\partial^2 \tilde{P}_i}{\partial x^2} + \frac{\sigma_r^2}{2} \frac{\partial^2 \tilde{P}_i}{\partial r^2} + \frac{\sigma_v^2 v}{2} \frac{\partial^2 \tilde{P}_i}{\partial v^2} \\ &+ \left(\alpha - \beta r - \frac{\sigma_r^2}{\beta} \left(1 - e^{-\beta(T-t)} \right) \right) \frac{\partial \tilde{P}_i}{\partial r} + (v \sigma \sigma_v \rho_v) \frac{\partial \tilde{P}_i}{\partial v \partial x} \\ &+ v \int_{-\infty}^{\infty} \left(\left(\tilde{P}_i(t, x+y, r, v; T, \kappa^*) - \tilde{P}_i(t, x, r, v; T, \kappa^*) \right) - (e^y - 1) \frac{\partial \tilde{P}_i}{\partial x} \right) k(y) dy. \end{aligned} \tag{3.52}$$

Note that $1_{x > \kappa^*} = 1$ *if* $x > \kappa^*$ *and otherwise* $1_{x > \kappa^*} = 0$.

Proof. By Ito's formula for Lévy process (Lemma 2.13), $\hat{C}(t, x, r, v)$ follows the partial integro - differential equation (PIDE)

$$\frac{\partial \hat{C}}{\partial t} + L_t^D \hat{C} + L_t^J \hat{C} = 0, \tag{3.53}$$

where

$$L_t^D\hat{C} = \left(r - \tfrac{1}{2}\sigma^2 v\right)\tfrac{\partial \hat{C}}{\partial x_t} + \left(\alpha - \beta r - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\tfrac{\partial \hat{C}}{\partial r} + \gamma(\kappa - v)\tfrac{\partial \hat{C}}{\partial v}$$
$$+\tfrac{\sigma_v^2 v}{2}\tfrac{\partial^2 \hat{C}}{\partial v^2} + \tfrac{\sigma^2 v}{2}\tfrac{\partial^2 \hat{C}}{\partial x^2} + \tfrac{\sigma_r^2}{2}\tfrac{\partial^2 \hat{C}}{\partial r^2} + (\rho_v \sigma v \sigma_v)\tfrac{\partial^2 \hat{C}}{\partial x \partial v} - r\hat{C}$$

and

$$L_t^J\hat{C} = v\int_{-\infty}^{\infty}\left(\hat{C}(t, x+y, r, v) - \hat{C}(t, x, r, v) - \frac{\partial \hat{C}}{\partial x}(e^y - 1)\right)k(y)dy,$$

where $k(y)$ is the Lévy density.

We plan to substitute equation (3.36) into equation (3.10). Firstly, we compute

$$\begin{aligned}
\tfrac{\partial \hat{C}}{\partial t} &= e^x \tfrac{\partial \tilde{P}_1}{\partial t} - e^\kappa P^*(t,T)\left[\tfrac{\partial \tilde{P}_2}{\partial t} + \tilde{P}_2 \tfrac{\partial}{\partial t}\left(a(t,T) + b(t,T)r\right)\right],\\
\tfrac{\partial \hat{C}}{\partial x} &= e^x\left(\tfrac{\partial \tilde{P}_1}{\partial x} + \tilde{P}_1\right) - e^\kappa P^*(t,T)\tfrac{\partial \tilde{P}_2}{\partial x},\\
\tfrac{\partial \hat{C}}{\partial v} &= e^x \tfrac{\partial \tilde{P}_1}{\partial v} - e^\kappa P^*(t,T)\tfrac{\partial \tilde{P}_2}{\partial v},\\
\tfrac{\partial \hat{C}}{\partial r} &= e^x \tfrac{\partial \tilde{P}_1}{\partial r} - e^\kappa P^*(t,T)\left(\tfrac{\partial \tilde{P}_2}{\partial r} + \tilde{P}_2 b(t,T)\right),\\
\tfrac{\partial^2 \hat{C}}{\partial x^2} &= e^x\left(\tfrac{\partial^2 \tilde{P}_1}{\partial x^2} + 2\tfrac{\partial \tilde{P}_1}{\partial x} + \tilde{P}_1\right) - e^\kappa P^*(t,T)\tfrac{\partial^2 \tilde{P}_2}{\partial x^2},\\
\tfrac{\partial^2 \hat{C}}{\partial v^2} &= e^x \tfrac{\partial^2 \tilde{P}_1}{\partial v^2} - e^\kappa P^*(t,T)\tfrac{\partial^2 \tilde{P}_2}{\partial v^2},\\
\tfrac{\partial^2 \hat{C}}{\partial r^2} &= e^x \tfrac{\partial^2 \tilde{P}_1}{\partial r^2} - e^\kappa P^*(t,T)\left(\tfrac{\partial^2 \tilde{P}_2}{\partial r^2} + 2b(t,T)\tfrac{\partial \tilde{P}_2}{\partial r} + \tilde{P}_2 b^2(t,T)\right),\\
\tfrac{\partial^2 \hat{C}}{\partial v \partial x} &= e^x\left(\tfrac{\partial \tilde{P}_1}{\partial v \partial x} + \tfrac{\partial \tilde{P}_1}{\partial v}\right) - e^\kappa P^*(t,T)\tfrac{\partial \tilde{P}_2}{\partial v \partial x},
\end{aligned}$$

$$
\begin{aligned}
&C(t,x+y,r,v;T,\kappa^*) - C(t,x,r,v;T,\kappa^*)\\
&\quad= \left[e^{x+y}\tilde{P}_1(t,x+y,r,v;T,\kappa^*) - e^{\kappa *}P^*(t,T)\tilde{P}_1(x+y,t,r,v;T,\kappa^*)\right]\\
&\qquad - \left[e^{x}\tilde{P}_2(t,x,r,v;T,\kappa^*) - e^{\kappa *}P^*(t,T)\tilde{P}_2(t,x,r,v;T,\kappa^*)\right]\\
&\quad= e^{x+y}\tilde{P}_1(t,x+y,r,v;T,\kappa^*) - e^{x}\tilde{P}_2(t,x,r,v;T,\kappa^*)\\
&\qquad + e^{\kappa *}P^*(t,T)\tilde{P}_2(t,x,r,v;T,\kappa^*) - e^{\kappa *}P^*(t,T)\tilde{P}_1(x+y,t,r,v;T,\kappa^*)\\
&\quad= e^{x}\left[e^{y}\tilde{P}_1(t,x+y,r,v;T,\kappa^*) - \tilde{P}_2(t,x,r,v;T,\kappa^*)\right]\\
&\qquad - e^{\kappa *}P^*(t,T)\left[\tilde{P}_2(t,x+y,r,v;T,\kappa^*) - \tilde{P}_1(x,t,r,v;T,\kappa^*)\right]\\
&\quad= e^{x}\left[\begin{array}{l} e^{y}\tilde{P}_1(t,x+y,r,v;T,\kappa^*) - \tilde{P}_2(t,x+y,r,v;T,\kappa^*)\\ +\tilde{P}_2(t,x+y,r,v;T,\kappa^*) - \tilde{P}_2(t,x,r,v;T,\kappa^*)\end{array}\right]\\
&\qquad - e^{\kappa *}P^*(t,T)\left[\tilde{P}_2(t,x+y,r,v;T,\kappa^*) - \tilde{P}_1(x,t,r,v;T,\kappa^*)\right]\\
&\quad= e^{x}\left[\begin{array}{l} \left(e^{y}-1\right)\tilde{P}_1(t,x+y,r,v;T,\kappa^*) + \tilde{P}_2(t,x+y,r,v;T,\kappa^*)\\ -\tilde{P}_2(t,x,r,v;T,\kappa^*)\end{array}\right]\\
&\qquad - e^{\kappa *}P^*(t,T)\left[\tilde{P}_2(t,x+y,r,v;T,\kappa^*) - \tilde{P}_1(t,x,r,v;T,\kappa^*)\right].
\end{aligned}
$$

Substitute all terms above into equation (3.53) and separate it by assumed independent terms of $\tilde{P}_1$ and $\tilde{P}_2$. This gives two PIDEs for the forward probability for $\tilde{P}_j(t,x,r,v;T,\kappa),\ j=1,2:$

$$
\begin{aligned}
0 = {}& \tfrac{\partial \tilde{P}_1}{\partial t} + \left(r + \tfrac{1}{2}\sigma^2 v\right)\tfrac{\partial \tilde{P}_1}{\partial x} + \left(\gamma(\kappa - v) + (\rho_v \sigma v \sigma_v)\right)\tfrac{\partial \tilde{P}_1}{\partial v} + \tfrac{\sigma_r^2}{2}\tfrac{\partial^2 \tilde{P}_1}{\partial r^2} + \tfrac{\sigma_v^2 v}{2}\tfrac{\partial^2 \tilde{P}_1}{\partial v^2}\\
&+ \left(\alpha - \beta r - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\tfrac{\partial \tilde{P}_1}{\partial r} + \tfrac{\sigma^2 v_t}{2}\tfrac{\partial^2 \tilde{P}_1}{\partial x^2} + (v\sigma\sigma_v\rho_v)\tfrac{\partial^2 \tilde{P}_1}{\partial x \partial v}\\
&+ v\int_{-\infty}^{\infty}\left(\left(\tilde{P}_1(t,x+y,r,v) - \tilde{P}_1(t,x,r,v)\right) - \tfrac{\partial \tilde{P}_1}{\partial x}(e^y - 1)\right)k(y)dy\\
&+ v\int_{-\infty}^{\infty}\left((e^y - 1)\left(\tilde{P}_1(t,x+y,r,v) - \tilde{P}_1(t,x,r,v)\right)\right)k(y)dy,
\end{aligned}
\tag{3.54}
$$

and subject to the boundary condition at the expiration time $t = T$ according to equation (3.49).

By using the notation in (3.52), equation (3.54) becomes

$$\begin{aligned}0 &= \tfrac{\partial \tilde{P}_1}{\partial t} + A[\tilde{P}_1] + (\rho_v \sigma v \sigma_v)\,\tfrac{\partial \tilde{P}_1}{\partial v}\\ &+ v\int_{-\infty}^{\infty} (e^y - 1)\Big(\tilde{P}_1(x+y,t,r,v;T,K) - \tilde{P}_1(x,t.r,v;T,K)\Big)k(y)dy \\ &:= \tfrac{\partial \tilde{P}_1}{\partial t} + A_1[\tilde{P}_1].\end{aligned} \tag{3.55}$$

For $\tilde{P}_2(t,x,r,v;T,\kappa)$:

$$\begin{aligned}0 &= \tfrac{\partial \tilde{P}_2}{\partial t} + \left(r - \tfrac{1}{2}\sigma^2 v\right)\tfrac{\partial \tilde{P}_2}{\partial x} + \gamma(\kappa - v)\tfrac{\partial \tilde{P}_2}{\partial v} + (\rho_v \sigma v \sigma_v)\,\tfrac{\partial \tilde{P}_2}{\partial v \partial x} + \tfrac{\sigma^2 v}{2}\tfrac{\partial^2 \tilde{P}_2}{\partial x^2}\\ &+ \left(\alpha - \beta r - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) + 2b(t,T)\tfrac{\sigma_r^2}{2}\right)\left(\tfrac{\partial \tilde{P}_2}{\partial r}\right) + \tfrac{\sigma_v^2 v}{2}\tfrac{\partial^2 \tilde{P}_2}{\partial v^2} + \tfrac{\sigma_r^2}{2}\tfrac{\partial^2 \tilde{P}_2}{\partial r^2}\\ &+ \tilde{P}_2\left(\tfrac{\partial a(t,T)}{\partial t} + r\tfrac{\partial b(t,T)}{\partial t} + \left(\alpha - \beta r - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)b(t,T) + \tfrac{\sigma_r^2}{2}b^2(t,T) - r\right)\\ &+ v\int_{-\infty}^{\infty}\left(\tilde{P}_2(t,x+y,r,v;T,\kappa) - \tilde{P}_2(t,x,r,v;T,\kappa) - \tfrac{\partial \tilde{P}_2}{\partial x}(e^y - 1)\right)k(y)dy,\end{aligned} \tag{3.56}$$

subject to the boundary condition at expiration time $t = T$ according to equation (3.51). Again, by using the notation (3.52), equation(3.56) becomes

$$\begin{aligned}0 &= \tfrac{\partial \tilde{P}_2}{\partial t} + A[\tilde{P}_2] + \tilde{P}_2\left[b(t,T)\left(\alpha(t) - \beta r - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right) + \tfrac{\sigma_r^2}{2}b^2(t,T)\right]\\ &+ \sigma_r^2 b(t,T)\tfrac{\partial \tilde{P}_2}{\partial r} - \sigma^2 v\tfrac{\partial \tilde{P}_2}{\partial x}\tilde{P}_2\left[\tfrac{\partial}{\partial t}\left(a(t,T) + b(t,T)r\right) - r\right] := \tfrac{\partial \tilde{P}_2}{\partial t} + A_2[\tilde{P}_2].\end{aligned} \tag{3.57}$$

The proof is now complete. □

3.4 The Closed-Form Solution for European Call Option

For $j = 1, 2$, the characteristic functions for $\tilde{P}_j(t,x,r,v;T,\kappa^*)$, with respect to the variable κ^*, are defined by

$$f_j(t,x,r,v;T,u) := -\int_{-\infty}^{\infty} e^{-iu\kappa^*}d\tilde{P}_j(t,x,r,v;T,\kappa^*), \tag{3.58}$$

with a minus sign to account for the negativity of the measure $d\tilde{P}_j$. Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t} + A_j\left[f_j\right](t,x,r,v;T,\kappa^*) = 0, \tag{3.59}$$

with the respective boundary conditions

$$\begin{aligned} f_j(T,x,r,v;T,u) &= -\int_{-\infty}^{\infty} e^{iu\kappa^*} d\tilde{P}_j(t,x,r,v;T,\kappa^*) \\ &= -\int_{-\infty}^{\infty} e^{iu\kappa^*}(-\delta(\kappa^*-x))d\kappa^* = e^{iux}, \end{aligned}$$

since $d\tilde{P}_j(t,x,r,v;T,\kappa^*) = d1_{x>\kappa^*} = -\delta(\kappa^*-x)d\kappa^*$.

The following lemma shows how to calculate the characteristic functions for $\tilde{P}_1$ and $\tilde{P}_2$ as they appeared in Lemma 3.3.

Lemma 3.4. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ can be calculated by the inverse Fourier transformations of the characteristic function, i.e.*

$$\tilde{P}_j(t,x,r,v;T,\kappa) = \frac{1}{2} + \frac{1}{\pi}\int_{0+}^{\infty} \mathrm{Re}\left[\frac{e^{-iu\kappa} f_j(t,x,r,v;T,u)}{iu}\right] du,$$

for $j = 1,2$ with Re[.] denoting the real component of a complex number. By letting $\tau = T-t$, the characteristic function f_j becomes

$$f_j(t,x,r,v;t+\tau,u) = \exp\left(iux + B_j(\tau) + rC_j(\tau) + vE_j(\tau) - (j-1)\ln P^*(t,t+\tau)\right),$$

where

$$\begin{aligned} &B_j(\tau) = \left[\tfrac{\gamma\kappa\zeta_j}{b_1}\ln\left(\tfrac{(b_{2j}+\zeta_j)e^{\tau\zeta}-(b_{2j}-\zeta_j)}{2\zeta_j}\right) - \tfrac{\tau\zeta_j\gamma\kappa(b_{2j}+\zeta_j)}{2b_1}\right] + \tfrac{(iu-j+1)\alpha}{\beta^2}\left(e^{-\beta\tau}-1+\tau\beta\right) \\ &+ \tfrac{\sigma_r^2}{2\beta^3}\left(\tfrac{(iu-j+1)^2}{2} - (iu-j+1)\right)\left(\left(e^{-\beta\tau}-2\right)^2 - 7 - 2\beta\tau\right), \\ &C_j(\tau) = \tfrac{iu-(j-1)}{\beta}(1-e^{-\beta\tau}), \quad E_j(\tau) = \tfrac{\left(b_{2j}^2-\zeta_j^2\right)\left(e^{\zeta\tau}-1\right)}{2b_1\left((b_{2j}-\zeta_j)-(b_{2j}+\zeta_j)e^{\zeta\tau}\right)}, \\ &b_{0j} = \tfrac{\sigma^2}{2}(iu-u^2) + \int_{-\infty}^{\infty}\left(e^{(iu+2-j)y} - (2-j)e^y - iu(e^y-1)\right)k(y)dy, \\ &b_1 = \tfrac{\sigma_v^2}{2}, \quad b_{2j} = \left(\sigma\sigma_v\rho_v(iu+2-j)-\gamma\right), \quad \varsigma_j = \sqrt{b_{2j}^2 - 4b_{0j}b_1}. \end{aligned}$$

Proof. To solve the characteristic function explicitly, letting $\tau = T-t$ be the time-to-go, we conjecture that the function f_1 is given by

$$f_1(t,x,r,v;t+\tau,u) = \exp\left(iux + B_1(\tau) + rC_1(\tau) + vE_1(\tau),\right) \tag{3.60}$$

with the boundary conditions $B_1(0) = C_1(0) = E_1(0) = 0$. This conjecture exploits the linearity of the coefficients in PIDEs (3.59).

Note that the characteristic function of f_1 always exists. In order to substitute equation (3.60) into equation (3.59), firstly, we compute

$$\tfrac{\partial f_1}{\partial t} = -\left(B_1'(\tau) + rC_1'(\tau) + vE_1'(\tau)\right) f_1,\ \tfrac{\partial f_1}{\partial x} = iuf_1,\ \tfrac{\partial f_1}{\partial r} = C_1(\tau) f_1,\ \tfrac{\partial f_1}{\partial v} = E_1(\tau) f_1,$$
$$\tfrac{\partial^2 f_1}{\partial x^2} = -u^2 f_1,\ \tfrac{\partial^2 f_1}{\partial v^2} = E_1^2(\tau) f_1,\ \tfrac{\partial^2 f_1}{\partial r^2} = C_1^2(\tau) f_1,\ \tfrac{\partial^2 f_1}{\partial v \partial x} = iuE_1(\tau) f_1,$$
$$f_1(t, x+y, r, v; t+\tau, u) - f_1(t, x, r, v; t+\tau, u) = (e^{iuy} - 1) f_1(t, x, r, v; t+\tau, u).$$

Substituting all the above terms into equation (3.59), after canceling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned} 0 = &-\left(B_1'(\tau) + rC_1'(\tau) + vE_1'(\tau)\right) + \left(r + \tfrac{\sigma^2 v}{2}\right) iu + \left(\gamma(\kappa - v) + (v\sigma\sigma_v\rho_v)\right) E_1(\tau) \\ &+ \tfrac{\sigma_r^2}{2} C_1^2(\tau) + \left(\alpha - \beta r - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right) C_1(\tau) + \tfrac{v\sigma_v^2}{2} E_1^2(\tau) - \tfrac{vu^2\sigma^2}{2} \\ &+ iuv\sigma\sigma_v\rho_v E_1(\tau) + v \int_{-\infty}^{\infty} \left(e^{(iu+1)y} - e^y - iu(e^y - 1)\right) k(y) dy. \end{aligned}$$

By separating the order r, v and ordering the remaining terms, we can reduce it to three ordinary differential equations (ODEs) as follows:

$$C_1'(\tau) = -\beta C_1(\tau) + iu, \tag{3.61}$$

$$\begin{aligned} E_1'(\tau) &= \frac{\sigma_v^2}{2} E_1^2(\tau) + \left(\rho_v \sigma\sigma_v (1 + iu) - \gamma\right) E_1(\tau) \\ &+ \frac{\sigma^2}{2}\left(iu - u^2\right) + \int_{-\infty}^{\infty} \left(e^{iuy+y} - e^y - iu(e^y - 1)\right) k(y) dy \end{aligned} \tag{3.62}$$

$$B_1'(\tau) = \left(\alpha - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right) C_1(\tau) + \gamma\kappa E_1(\tau) + \frac{\sigma_r^2}{2} C_1^2(\tau). \tag{3.63}$$

It is clear from equation (3.61) and $C(0) = 0$ that

$$C_1(\tau) = \frac{iu}{\beta}(1 - e^{-\beta\tau}). \tag{3.64}$$

Let $b_0 = \frac{\sigma^2}{2}(iu - u^2) + \int_{-\infty}^{\infty} \left(e^{(iu+1)y} - e^y - iu(e^y - 1)\right) k(y) dy$, $b_1 = \frac{\sigma_v^2}{2}$, and $b_2 = \rho_v \sigma\sigma_v(1 + iu) - \gamma$.

Substitute these constants into equation (3.62), one gets

$$\begin{aligned} E_1'(\tau) &= b_1 E_1^2(\tau) + b_2 E_1(\tau) + b_0 \\ &= b_1 \left(E_1(\tau) - \tfrac{-b_2 + \sqrt{b_2^2 - 4b_0 b_1}}{2b_1} \right) \left(E_1(\tau) - \tfrac{-b_2 - \sqrt{b_2^2 - 4b_0 b_1}}{2b_1} \right). \end{aligned} \tag{3.65}$$

By method of variable separation, we have

$$\frac{dE_1(\tau)}{\left(E_1(\tau) - \tfrac{-b_2 + \sqrt{b_2^2 - 4b_0 b_1}}{2b_1} \right) \left(E_1(\tau) - \tfrac{-b_2 - \sqrt{b_2^2 - 4b_0 b_1}}{2b_1} \right)} = b_1 d\tau.$$

Let $\varsigma = \sqrt{b_2^2 - 4b_0 b_1}$ and using partial fraction on the left hand side, one obtains

$$\left[\frac{1}{\left(E_1(\tau) + \frac{b_2 + \zeta}{2b_1} \right)} - \frac{1}{\left(E_1(\tau) + \frac{b_2 - \zeta}{2b_1} \right)} \right] dE_1(\tau) = \zeta d\tau.$$

Integrating both sides and simplify, we have

$$\begin{aligned} \int \tfrac{1}{\left(E_1(\tau) + \frac{b_2 + \zeta}{2b_1} \right)} - \tfrac{1}{\left(E_1(\tau) + \frac{b_2 - \zeta}{2b_1} \right)} dE_1(\tau) &= \int \zeta d\tau \\ \ln \left(E_1(\tau) + \tfrac{b_2 + \zeta}{2b_1} \right) - \ln \left(E_1(\tau) + \tfrac{b_2 - \zeta}{2b_1} \right) &= \zeta\tau + E_0 \\ \ln \left(\tfrac{E_1(\tau) + \frac{b_2 + \zeta}{2b_1}}{E_1(\tau) + \frac{b_2 - \zeta}{2b_1}} \right) &= \zeta\tau + E_0. \end{aligned}$$

Using boundary condition $E_1(\tau = 0) = 0$, we get

$$E_0 = \ln \left(\frac{b_2 + \zeta}{b_2 - \zeta} \right).$$

Solving of $E_1(\tau)$, we obtain

$$E_1(\tau) = \frac{(b_2^2 - \zeta^2)\left(e^{\zeta\tau} - 1\right)}{2b_1 \left((b_2 - \zeta) - (b_2 + \zeta) e^{\zeta\tau} \right)}. \tag{3.66}$$

In order to solve $B_1(\tau)$ explicitly, we substitute $C_1(\tau)$ and $E_1(\tau)$ in equation (3.64) and (3.66) into equation (3.63) to get

$$\begin{aligned} B_1'(\tau) &= \frac{\gamma\kappa(b_2^2 - \zeta^2)\left(e^{\zeta\tau} - 1\right)}{2b_1 \left((b_2 - \zeta) - (b_2 + \zeta) e^{\zeta\tau} \right)} + \left(\alpha - \frac{\sigma_r^2}{\beta} \left(1 - e^{-\beta\tau)} \right) \right) \frac{iu}{\beta}(1 - e^{-\beta\tau}) \\ &\quad + \frac{\sigma_r^2}{2} \left(\frac{iu}{\beta}(1 - e^{-\beta\tau}) \right)^2 \\ &= \frac{\gamma\kappa\left(b_2^2 - \zeta^2\right)}{2b_1} \frac{\left(e^{\zeta\tau} - 1\right)}{\left((b_2 - \zeta) - (b_2 + \zeta) e^{\zeta\tau} \right)} - \frac{\sigma_r^2}{\beta^2} \left(\frac{u^2}{2} - iu \right) \left(1 - e^{-\beta\tau} \right)^2 \\ &\quad + \frac{iu\alpha}{\beta}(1 - e^{-\beta\tau}). \end{aligned} \tag{3.67}$$

Note that

$$\int \frac{\left(e^{\zeta\tau}-1\right)}{\left((b_2-\zeta)-(b_2+\zeta)e^{\zeta\tau}\right)}d\tau$$
$$= \left[-\frac{\tau\zeta}{(b_2-\zeta)} + \left(\frac{1}{(b_2-\zeta)} - \frac{1}{(b_2+\zeta)}\right)\ln\left((b_2+\zeta)\,e^{\tau\zeta} - (b_2-\zeta)\right)\right] + \zeta_0.$$

And

$$\int \left(1-e^{-\beta\tau}\right)^2 d\tau = \int \left(1-2e^{-\beta\tau}+e^{-2\beta\tau}\right)d\tau = \left(\tau + \frac{2e^{-\beta\tau}}{\beta} - \frac{e^{-2\beta\tau}}{2\beta}\right) + \zeta_1.$$

Integrating both side of equation (3.67), we get

$$B_1(\tau) = \frac{\gamma\kappa\left(b_2^2-\zeta^2\right)}{2b_1}\left[-\frac{\tau\zeta}{(b_2-\zeta)} + \left(\frac{1}{(b_2-\zeta)} - \frac{1}{(b_2+\zeta)}\right)\ln\left((b_2+\zeta)\,e^{\tau\zeta} - (b_2-\zeta)\right)\right]$$
$$-\frac{\sigma_r^2}{\beta^2}\left(\frac{u}{2}-iu\right)\left(\tau + \frac{2e^{-\beta\tau}}{\beta} - \frac{e^{-2\beta\tau}}{2\beta}\right) + \frac{iu\alpha}{\beta}\left(\tau + \frac{e^{-\beta\tau}}{\beta}\right) + B_0.$$

Using boundary condition $B_1(0) = 0$, we get

$$B_0 = -\frac{\gamma\kappa\left(b_2^2-\zeta^2\right)}{2b_1}\left[\left(\frac{1}{(b_2-\zeta)} - \frac{1}{(b_2+\zeta)}\right)\ln(2\zeta)\right] + \frac{3\sigma_r^2}{2\beta^3}\left(\frac{u}{2}-iu\right) - \frac{iu\alpha}{\beta^2}.$$

Hence,

$$B_1(\tau) = \left[\frac{\gamma\kappa\zeta}{b_1}\ln\left(\frac{(b_2+\zeta)e^{\tau\zeta}-(b_2-\zeta)}{2\zeta}\right) - \frac{\tau\zeta\gamma\kappa(b_2+\zeta)}{2b_1}\right] + \frac{iu\alpha}{\beta^2}\left(e^{-\beta\tau}-1+\tau\beta\right)$$
$$+\frac{\sigma_r^2}{2\beta^3}\left(\frac{u}{2}-iu\right)\left(\left(e^{-\beta\tau}-2\right)^2 - 7 - 2\beta\tau\right).$$

The details of proof for the characteristic function f_2 are similar to f_1. Hence, we have

$$f_2(t,x,r,v;T-\tau,u) = \exp\left[iux + B_2(\tau) + rC_2(\tau) + vE_2(\tau) - \ln P^*(t,t+\tau)\right],$$

where $B_2(\tau), C_2(\tau)$ and $E_2(\tau)$ are as given in the lemma and $P^*(t,t+\tau)$ is given in Lemma 3.2.

We can thus evaluate the characteristic function in closed form. However, we are more interested in the probability, which can be obtained by investing the characteristic functions by performing the following integration

$$\tilde{P}_j(x,r,v,t;\kappa^*,T) = \frac{1}{2} + \frac{1}{\pi}\int_{0+}^{\infty} \text{Re}\left(\frac{e^{-iu\kappa^*}f_j(t,x,v.r;T,u)}{iu}\right)du, \qquad (3.68)$$

for $j = 1, 2$. where $X_t = \ln S_t$ and $\kappa^* = \ln K$. To verify equation (3.68), first we note that

$$
\begin{aligned}
&E\left[e^{iu(\ln S_t - \ln K)} | \ln S_t = X_t, r_t = r, v_t = v\right] \\
&\quad = E\left[e^{iu(x-\kappa^*)} | \ln S_t = X_t, r_t = r, v_t = v\right] \\
&\quad = \int_{-\infty}^{\infty} e^{iu(x-\kappa^*)} dP_j(t, x, v, r; T, \kappa^*) \\
&\quad = e^{-iu\kappa^*} \int_{-\infty}^{\infty} e^{iux} dP_j(t, x, v, r; T, \kappa^*) \\
&\quad = -e^{-iu\kappa^*} \int_{-\infty}^{\infty} e^{iux} \delta\left(x - \kappa^*\right) d\kappa^* \\
&\quad = -e^{-iu\kappa^*} \int_{-\infty}^{\infty} e^{iuK^*} \delta\left(x - \kappa\right) d\kappa^* = e^{-iu\kappa^*} f_j(t, x, r, v; T, u).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{e^{-iu\kappa^*} f_j(t,x,r,v;T,u)}{iu}\right] du \\
&\quad = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{E\left[e^{iu(x-\kappa^*)} | \ln S_t = X_t, r_t = r, v_t = v\right]}{iu}\right] du \\
&\quad = E\left[\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{e^{iu(x-\kappa^*)}}{iu}\right] du | \ln S_t = X_t, r_t = r, v_t = v\right] \\
&\quad = E\left[\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\sin(u(x-\kappa^*))}{u} du | \ln S_t = X_t, r_t = r, v_t = v\right] \\
&\quad = E\left[\frac{1}{2} + \mathrm{sgn}(x - \kappa^*) \frac{1}{\pi} \int_0^{\infty} \frac{\sin(u)}{u} du | \ln S_t = X_t, r_t = r, v_t = v\right] \\
&\quad = E\left[\frac{1}{2} + \mathrm{sgn}(x - \kappa^*) | \ln S_t = X_t, r_t = r, v_t = v\right] \\
&\quad = E\left[1_{x \geq \ln K} | \ln S_t = X_t, r_t = r, v_t = v\right],
\end{aligned}
$$

where we have used the wellknown Dirichlet formula $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 1$ and the signum (sgn) function, defined as $sgn(x) = 1$ if $x > 0, 0$ if $x = 0$ and -1 if $x < 0$. The proof is now complete. □

Thus we have proved the following main theorem.

Thorem 3.5. *The value of a European call option of SDE (3.29) is*

$$
C(t, S_t, r_t, v_t; T, K) = S_t \tilde{P}_1\left(t, X_t, r_t, v_t; T, \kappa^*\right) - K P^*(t, T) \tilde{P}_2\left(t, X_t, r_t, v_t; T, \kappa^*\right)
$$

where $\tilde{P}_1$ and $\tilde{P}_2$ are given in Lemma 3.4 and $P^*(t, T)$ is given in Lemma 3.2.

CHAPTER IV

PARAMETER ESTIMATION AND APPLICATIONS IN FINANCE

4.1 Introduction

This chapter estimates a model which is a special case of SVLSI model as described in Chapter III, i.e., pure jump Lévy process being compound Poisson process with normal jump, so that we apply a stochastic time change only to the diffusion component:

$$S_t = S_0 \exp\left(r_t t + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + (J_t - \xi t)\right), \tag{4.1}$$

since J_t is compound Poisson process $\sum_{i=1}^{N_i} Y_i$ where N_t is Poisson process with intensity λ and Y_i are i.i.d. random variable with normal density with parameters (μ_J, σ_J^2). Since the concavity adjustment ξ is equal to $\lambda E[e^Y - 1] := \lambda m$, model (4.1) becomes

$$S_t = S_0 \exp\left(r_t t + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + (J_t - \lambda m t)\right), \tag{4.2}$$

where r_t satisfies TF-Vasicek process as in equation (3.30) and T_t the integrated CIR process as in section 3.2. By Itô's formula for jump diffusion process, the model (4.2) has the dynamic

$$\begin{aligned} dS_t &= S_t\left((r_t - \lambda m)\,dt + \rho_v \sigma dZ^v_{T_t} + \sigma\sqrt{1-\rho_v^2}\,dZ_{T_t}\right) \\ &\quad + S_{t-}\left(e^Y - 1\right) dN_t, \end{aligned} \tag{4.3}$$

where

$$dr_t = \left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)dt + \sigma_r dZ_t^r, \quad (4.4)$$

$$dv_t = \gamma(\kappa - v_t)dt + \sigma_v\sqrt{v_t}dZ_t^v. \quad (4.5)$$

We name this the SVJSI model for stochastic volatility jump diffusion model with stochastic interest rate and assumed dN_t is statistically independent of $Y_t, dZ_t^v, dZ_t^r, dZ_t$ and dZ_t^v, dZ_t^r, dZ_t are uncorrelated.

Our model requires values for the fixed parameters which determine the variation of interest rate and stock price volatility. In this thesis, we are interested in using the Generalized Method of Moments (GMM) technique to estimates parameters, so we set the parameter values so that selected moment from the model are close to sample moments computed from the interest rate and log return.

Chapter IV is structured as follows. The GMM technique is described in section 4.2 and the underlying model of asset return and interest rate is specified with the statistic properties in section 4.3. Section 4.4 contains the GMM estimator of asset prices and interest rate. In section 4.5, we use the data of daily price observation for SET50 index and daily observations for rates on 3 month Treasury Bills of Thailand to calibrates the SVJSI model and apply results to finance problems.

4.2 Parameter Estimation Method

4.2.1 Moment

Definition 4.1. Let X be a random variable with probability density function $f(x)$. The rth moment about the mean of a random variable X, denote by m_r, is the expected value of $(X-\mu)^r$:

$$m_r = E\left[(X-\mu)^r\right] = \sum_x (x-\mu)^r f(x), \tag{4.6}$$

for $r = 1, 2, 3, \ldots$ when X is discrete and

$$m_r = E\left[(X-\mu)^r\right] = \int_{-\infty}^{\infty} (x-\mu)^r f(x)dx \tag{4.7}$$

when X is continuous, provided that the finiteness conditions hold.

Note that the second moment of a random variable about the mean is the *variance* of the random variable. We can express by simply writing out the binomial expansion of $(X-\mu)^r$:

$$m_r = E\left[(X-\mu)^r\right] = \sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} E\left[X^j\right](-\mu)^{r-j}. \tag{4.8}$$

4.2.2 The Generalized Method of Moments

Almost all economic models contain unknown parameters, which need to be estimated before we can use them. Typically, this estimation is done by taking a random sample of observations and using those observations to estimates the unknown parameters. In the estimation, an important idea is the choice of sample representing the population from which it has been drawn. For the same parameter of a population we can apply different methods of estimation, for example, the methods of maximum likelihood (ML), moments (MM) and the generalized method of moments (GMM).

In this thesis, we focus on the technique of GMM as described by Hansen (1982). The GMM is historically one of the oldest methods with the big advantage that its use does not require the knowledge of the distribution of x_t. GMM has been used by Heston (1988), Longstaff and Schwartz (1992) for estimating parameters of one and two factor CIR process and Vetzal (1998) has used GMM to estimate the Chan-Karolyi-Longstaff-Sanders (CKLS) process.

Suppose we have a set of observations $\{x_t\}_{t=1,...,N}$, whose evolution depends upon a set of parameters $\delta = (\delta_1, \delta_2, ..., \delta_k) \in \Theta \subset \mathbb{R}^k$. The estimation problem is to find the true value of this parameter, δ_0, or at least a reasonably close estimate. In order to apply GMM there should exist functions $f_i(x_t, \delta), i = 1, 2, ..., m$, called *condition functions* such that

$$E\left[f_i(x_t, \delta_0)\right] = 0. \tag{4.9}$$

The functions $E\left[f_i(x_t, \delta)\right], i = 1, ..., m$ are called moment condition functions. Given such a set of such functions one may compute the sample estimate of $E[f_i(x_t, \delta)], i = 1, ..., m$. This sample estimate is defined by $\frac{1}{N}\sum_{t=1}^{N} f_i(x_t, \hat{\delta}), i = 1, ..., m$. The GMM estimates $\hat{\delta}$ of δ are those values of the set of sample estimates as close to zero as possible.

In the classic Method of Moments (MM), the number of parameters are equal to the number of moment condition functions when $m = k$ so that it should be possible to set $f_i(x_t, \delta)$ exactly to zero. Hence, one can find the GMM estimates $\hat{\delta}$ of δ by solving the following equation:

$$\frac{1}{N}\sum_{t=1}^{N} f_i(x_t, \hat{\delta}) = 0, i = 1, ..., m. \tag{4.10}$$

Let us relax the assumption that $m = k$. Set $f = (f_1(x_t, \delta), ..., f_m(x_t, \delta))'$ and define $\hat{\delta}$ to be

$$\hat{\delta} = \arg\min_{\delta \in \Theta} \left(\hat{f}' W \hat{f}\right), \tag{4.11}$$

where $\hat{f} =: \frac{1}{N}\sum_{t=1}^{N} f(x_t, \delta)$ is the *sample moment condition function*, and W is a positive definite weighting matrix. This is GMM estimate of δ, contingent upon W and f. To obtain an optimal choice of W, in the sense that the variance of the estimator $\hat{\delta}$ is minimum, we set

$$W = \hat{S}^{-1}, \tag{4.12}$$

where $\hat{S}$ is an estimate of the spectral density matrix of moment condition function. This important result is due to Hansen (1982).

The spectral density or long run covariance matrix is defined

$$S = \lim_{N \to \infty} E\left[\frac{1}{N}\sum_{t=1}^{N}\sum_{s=1}^{N} f_t f_s'\right]. \tag{4.13}$$

The spectral density matrix allows for serial correlation and heteroscedasticity in the observations of the moment function. A popular consistent estimate of the spectral density matrix is the Newey-West estimator (Newey and West (1987)):

$$\hat{S} = \hat{S}_0 + \sum_{j=1}^{q}\left(1 - \frac{j}{q+1}\right)\left(\hat{S}_j + \hat{S}_j'\right), \tag{4.14}$$

where

$$\hat{S}_j = \frac{1}{N}\sum_{t=j+1}^{N} f_t f_{t-j}',$$

and $f_t = (f_{t,1}, f_{t,2}, ..., f_{t,m})$, $f_{t,i} = f_i\left(x_t|\delta\right)$.

A disadvantage of the GMM is that $\hat{\delta}$ depend on the chosen functions $f_i(x_t, \delta)$ and when the number of the moment condition functions is greater than the number of parameters, the model is said to be *over identified*. Over identification allows us to check whether the model's moment conditions match the data or not. Conceptually we can check whether $\hat{f}(x_i, \hat{\delta})$ is sufficiently close to zero to suggest that the model fits the data well. Since the GMM technique has then replaced the problem of solving $\hat{f}(x_t, \delta) = 0$, which choose δ to match the restriction exactly by minimization calculation. The minimization can always be conducted even when no δ_0 exists such that $f(x_t, \delta_0) = 0$. This is what *J-test* does.

We consider two hypothesis

$$H_0 \quad : \quad f(x_t, \delta_0) = 0$$
$$H_1 \quad : \quad f(x_t, \delta) \neq 0 \quad , \forall \delta \in \Theta.$$

Under hypothesis H_0, the following so called $J-$ statistic is asymptotically χ^2 with with $m - k$ degrees of freedom. Define J as

$$J = N\hat{f}'(x_t, \hat{\delta})W\hat{f}(x_t, \hat{\delta}), \tag{4.15}$$

where $\hat{\delta}$ is the GMM estimator of δ_0 and W the optimal weighting matrix. If the test statistic shows rejection, then the underlying model that generated the system of moment conditions is declared invalid.

We will next discuss the statistical properties of the log return and interest rate to derive the moment condition function to find the GMM estimator.

4.3 Statistical Properties

In this section, the model concerned is described following the SDE (4.3) - (4.5). For the statistical properties for this model, we rely on the Kolmogorov backward equation to solve for the conditional characteristic functions of the log return and to derive the moment conditions.

Lemma 4.2. *Given the interest rate process r_t defined in equation (4.4), the characteristic function is given by*

$$\phi(u; r_t) = \exp\left[\frac{iu}{\beta}\left[\alpha - \frac{\sigma_r^2}{2\beta}\right] - \frac{u^2\sigma_r^2}{4\beta}\right].$$

Sometimes called the unconditional characteristic function (UCF).

Proof. We consider the interest rate process

$$dr_t = \left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)dt + \sigma_r dZ_t^{rT}.$$

We specialize the definition of the conditional characteristic function (CCF) for the process r_t as follows

$$f(r_t, t) := E\left[e^{iur_T}|r_t\right]. \tag{4.16}$$

Applying Itô's formula to $Y_t = f(r_t, t)$ yields

$$dY = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}dr + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}(dr)^2.$$

Substituting equation (4.4) into above equation and simplifying, we get

$$\begin{aligned}
dY &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}\left(\left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)dt + \sigma_r dZ_t^{rT}\right) \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}\left(\left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)dt + \sigma_r dZ_t^{rT}\right)^2 \\
&= \frac{\partial f}{\partial t}dt + \left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\frac{\partial f}{\partial r}dt + \sigma_r\frac{\partial f}{\partial r}dZ_t^{rT} \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}\left(\sigma_r^2 dZ_t^{rT}dZ_t^{rT}\right) \\
&= \left[\frac{\partial f}{\partial t} + \left(\alpha - \beta r_t - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\frac{\partial f}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 f}{\partial r^2}\right]dt + \sigma_r\frac{\partial f}{\partial r}dZ_t^{rT}.
\end{aligned}$$

Since $f(r_t, t)$ is a martingale, by setting the drift term to zero, we obtain

$$\frac{\partial f}{\partial t} + \left(\alpha - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\frac{\partial f}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 f}{\partial r^2} = 0. \tag{4.17}$$

We guess that the solution to equation (4.17) is also a linear function of the form

$$f(u; r, \tau) = \exp(A(u, \tau) + B(u, \tau)r), \tag{4.18}$$

where $\tau = T - t$ and the solution satisfies the boundary condition

$$f(u; r, T) = exp(iur)$$

implies that as $\tau \to 0, A(u, 0) = 0$ and $B(u, 0) = iu$.

Next, we compute

$$\frac{\partial f}{\partial t} = -f\left(A'(\tau) + rB'(\tau)\right), \quad \frac{\partial f}{\partial r} = fB, \quad \frac{\partial^2 f}{\partial r^2} = fB^2. \tag{4.19}$$

Substituting all the terms above into equation (4.17), understanding that $f(u;r,\tau) \neq 0$, we obtain:

$$-f\left(A'(\tau)+rB'(\tau)\right)+\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}(1-e^{-\beta\tau})\right)fB+\frac{\sigma_r^2}{2}fB^2=0.$$

After canceling the common factor of f, we obtain:

$$-\left(A'(\tau)+rB'(\tau)\right)+\left(\alpha-\beta r-\frac{\sigma_r^2}{\beta}(1-e^{-\beta\tau})\right)B+\frac{\sigma_r^2}{2}B^2=0. \tag{4.20}$$

Equations (4.20) must be true for all r. So by setting $r=1$ and $r=0$ respectively, we see the following equations must be true

$$\begin{aligned} A'(\tau) &= \left(\alpha-\frac{\sigma_r^2}{\beta}(1-e^{-\beta\tau})\right)B+\frac{\sigma_r^2}{2}B^2, & (4.21)\\ B'(\tau) &= -\beta B, & (4.22)\end{aligned}$$

with boundary condition, $A(u,0)=0$ and $B(u,0)=iu$.

Consider equation (4.22) with boundary condition $B(u,0)=iu$, it is clear that

$$B(u,\tau)=iu\exp(-\beta\tau). \tag{4.23}$$

Substituting $B(u,\tau)$ into equation (4.21), we have

$$A'(\tau)=iu\left(\alpha-\frac{\sigma_r^2}{\beta}(1-e^{-\beta\tau})\right)\exp(-\beta\tau)-\frac{\sigma_r^2}{2}u^2\exp(-2\beta\tau).$$

By the method of separation of variables, we have

$$dA=\left(iu\left(\alpha-\frac{\sigma_r^2}{\beta}(1-e^{-\beta\tau})\right)\exp(-\beta\tau)-\frac{\sigma_r^2}{2}u^2\exp(-2\beta\tau)\right)d\tau.$$

Integrating both sides and simplifying, we obtain

$$\begin{aligned} A(u,\tau) &= \int\left(iu\alpha e^{-\beta\tau}-iu\frac{\sigma_r^2}{\beta}e^{-\beta\tau}+iu\frac{\sigma_r^2}{\beta}e^{-2\beta\tau}-\frac{\sigma_r^2}{2}u^2e^{-2\beta\tau}\right)d\tau\\ &= \left[iu\alpha-iu\frac{\sigma_r^2}{\beta}\right]\int\left(e^{-\beta\tau}\right)d\tau+\left[iu\frac{\sigma_r^2}{\beta}-\frac{\sigma_r^2}{2}u^2\right]\int\left(e^{-2\beta\tau}\right))\,d\tau\\ &= -\frac{iu}{\beta}\left[\alpha-\frac{\sigma_r^2}{\beta}\right]e^{-\beta\tau}-\left[\frac{iu}{\beta}-\frac{u^2}{2}\right]\frac{\sigma_r^2}{2\beta}e^{-2\beta\tau})+A_0.\end{aligned}$$

With the boundary condition $A(u, 0) = 0$, we have

$$A_0 = \frac{iu}{\beta}\left[\alpha - \frac{\sigma_r^2}{\beta}\right] + \left[\frac{iu}{\beta} - \frac{u^2}{2}\right]\frac{\sigma_r^2}{2\beta}.$$

Therefore

$$A(u, \tau) = \frac{iu}{\beta}\left[\alpha - \frac{\sigma_r^2}{\beta}\right](1 - e^{-\beta\tau}) + \left[\frac{iu}{\beta} - \frac{u^2}{2}\right]\frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta\tau}\right).$$

Finally, the conditional characteristic function of the interest rate process is

$$f(u; r_t) = \exp\left[\frac{iu}{\beta}\left[\alpha - \frac{\sigma_r^2}{\beta}\right](1 - e^{-\beta\tau}) + \left[\frac{iu}{\beta} - \frac{u^2}{2}\right]\frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta\tau}\right) + iue^{-\beta\tau}r_t)\right].$$

In the case that r_t is stationary, the characteristic function of the process can be derived from conditional characteristic function, namely through

$$\phi(u; r_t) =: \lim_{\tau\to\infty} f(u, r_t).$$

Hence, the characteristic function of r_t is given by

$$\phi(u; r_t) = \exp\left[\frac{iu}{\beta}\left[\alpha - \frac{\sigma_r^2}{2\beta}\right] - \frac{u^2\sigma_r^2}{4\beta}\right]. \tag{4.24}$$

Therefore, r_t is a normal probability distribution process with mean $\frac{1}{\beta}\left[\alpha - \frac{\sigma_r^2}{2\beta}\right]$ and variance $\frac{\sigma_r^2}{2\beta}$. The proof is now complete. □

Next, we consider the given stochastic process as defined in equation (4.3). Applying Itô's formula for jump diffusion, the log asset price process is given by

$$dX_t = \left(r_t - \lambda m - \frac{1}{2}\sigma^2 v_t\right)dt + \sigma\rho_v dZ_{T_t}^{vT} + \sigma\sqrt{1 - \rho_v^2}dZ_{T_t}^{T} + YdN_t. \tag{4.25}$$

The following lemma shows the joint characteristic function of the log return process $R_{t+\tau} = X_{t+\tau} - X_t$ and interest rate process by using the joint conditional characteristic function.

Lemma 4.3. *Given the stochastic process defined in equation (4.25), (4.4) and (4.5), the joint characteristic function of log return $R_{t+\tau} = X_{t+\tau} - X_t$ and interest rate $r_{t+\tau}$ can be derived as*

$$\phi(u_1, u_2, u_3; R_{t+\tau}, r_{t+\tau}) = \exp\begin{pmatrix} A(\tau, u_1, 0, u_3) - \frac{2\gamma\kappa}{\sigma_r^2}\ln\left(1 - \frac{B(\tau,u_1,0,u_3)\sigma_r^2}{2\gamma}\right) \\ +\frac{C(\tau,u_1,0,u_3)+iu_3}{\beta}\left(\alpha - \frac{\sigma_r^2}{2\beta}\right) \\ -\frac{(-iC(\tau,u_1,0,u_3)+u_3)^2\sigma_r^2}{4\beta} + \varphi(u_1, \Delta J_t) \end{pmatrix},$$

where

$$\begin{aligned} A(\tau, u_1, u_2, u_3) = &-\tfrac{1}{\beta}\left\{u_3\sigma_r^2\left(\tfrac{u_1}{\beta} - u_3\right)\right\}\left(1 - e^{-\beta\tau}\right) + \tfrac{1}{2\beta}i\left(\tfrac{u_1}{\beta} - u_3\right)\tfrac{\sigma_r^2}{\beta} \\ &+\tfrac{1}{\beta}\left\{\left(\tfrac{iu_1}{\beta} - iu_3\right)\left(\alpha - \tfrac{\sigma_r^2}{\beta}\right) - \tfrac{\sigma_r^2}{\beta}\left(\tfrac{iu_1}{\beta} - 2iu_3\right) + \sigma_r^2\left(\tfrac{u_1}{\beta} - u_3\right)^2\right\}\left(1 - e^{-\beta\tau}\right) \\ &-\left(\tfrac{u_1}{\beta} - u_3\right)^2\tfrac{\sigma_r^2}{2}\left(1 - e^{-2\beta\tau}\right) - \left(\left(\alpha - \tfrac{\sigma_r^2}{\beta}\right)\left(\tfrac{iu_1}{\beta} - 2iu_3\right)\tfrac{\sigma_r^2}{2}\left(\tfrac{u_1}{\beta} - u_3\right)^2\right)\tau \\ &+u_3\sigma_r^2\tau^2\left(\tfrac{u_1}{\beta} - u_3\right) + \left(u_1^2\sigma^2 - u_3^2\sigma_r^2\right)\tfrac{\tau}{2} - \left(\lambda\mu_J iu_1 - \gamma\kappa iu_2\right)\tau \\ &-\tfrac{2\kappa\gamma n_3}{(n_2-\nabla)^2}\ln\left[4\nabla^2\left(e^{\tau\nabla}\left(n_2 - \nabla\right) - \left(n_2 + \nabla\right)\right)^2\right] - \tfrac{2\kappa\gamma n_3\tau}{(n_2-\nabla)}, \end{aligned}$$

$$B(\tau, u_1, u_2, u_3) = \frac{2n_3\left[e^{\tau\nabla} - 1\right]}{(n_2 + \nabla) - e^{\tau\nabla}(n_2 - \nabla)}, \quad n_2 = iu_1\sigma\sigma_v\rho_v + \sigma_v^2 iu_2 - \gamma,$$

$$C(\tau, u_1, u_2, u_3) = \left(\frac{iu_1}{\beta} - iu_3\right)\left(e^{-\beta\tau} - 1\right), \quad \varphi(u_1, \Delta J_t) = \lambda\Delta\left(e^{iu_1\mu_J - \frac{1}{2}u_1^2\sigma_J^2} - 1\right),$$

$$n_3 = -u_1u_2\sigma\sigma_v\rho_v - \gamma iu_2 - \frac{iu_1\sigma^2}{2} - \frac{\sigma_v^2}{2}u_2^2, \quad \nabla = \sqrt{n_2^2 - 2n_3\sigma_v^2}.$$

Proof. We specialize the definition of the joint conditional characteristic function of the continuous component for the process X_t, r_t and v_t following Jiang and Knight (2002), as

$$\phi(u_1, u_2, u_3 : X_{t+\tau}, v_{t+\tau}, r_{t+\tau}|X_t, v_t, r_t) = E\left[e^{iu_1X_{t+\tau}+iuv_{t+\tau}+iur_{t+\tau}}|X_t, r_t, v_t\right], \tag{4.26}$$

so that we apply Itô's formula to $Y_t = \varphi(X_t, r_t, v_t, t)$:

$$\begin{aligned} dY = &\tfrac{\partial\varphi}{\partial t}dt + \tfrac{\partial\varphi}{\partial x}d + \tfrac{\partial\varphi}{\partial r}dr + \tfrac{\partial\varphi}{\partial v}dv + \tfrac{1}{2}\tfrac{\partial^2\varphi}{\partial x^2}(dx)^2 + \tfrac{1}{2}\tfrac{\partial^2\varphi}{\partial r^2}(dr)^2 + \tfrac{1}{2}\tfrac{\partial^2\varphi}{\partial v^2}(dv)^2 \\ &+\tfrac{\partial^2\varphi}{\partial x\partial r}(dxdr) + \tfrac{\partial^2\varphi}{\partial x\partial v}(dxdv) + \tfrac{\partial^2\varphi}{\partial v\partial r}(dvdr). \end{aligned}$$

Substitute equation (4.25), (4.4) and (4.5) into above equation and simplifying, we get

$$\begin{aligned} dY &= \left[\begin{array}{l} \frac{\partial \varphi}{\partial t} + \left(r - \lambda m - \frac{1}{2}\sigma^2 v\right)\frac{\partial \varphi}{\partial x} + \left(\alpha - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta\tau}\right)\right)\frac{\partial \varphi}{\partial r} \\ +\gamma(\kappa - v)\frac{\partial \varphi}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 \varphi}{\partial v^2} + \sigma\sigma_v\rho_v v \frac{\partial^2 \varphi}{\partial x \partial v} \end{array} \right] dt \\ &\quad + \frac{\partial \varphi}{\partial x}\left[\sigma\rho_v\sqrt{v}dZ^{vT} + \sigma\sqrt{v(1-\rho_v^2)}dZ^T\right] + \frac{\partial \varphi}{\partial r}\left[\sigma_r dZ^{rT}\right] + \frac{\partial \varphi}{\partial v}\left[\sigma_v\sqrt{v}dZ^{vT}\right]. \end{aligned}$$

Using its martingale property, we can derive the Kolmogorov backward equation:

$$\begin{aligned} 0 &= \frac{\partial \varphi}{\partial t} + \left(r - \lambda m - \frac{1}{2}\sigma^2 v\right)\frac{\partial \varphi}{\partial x} + \left(\alpha - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta\tau}\right)\right)\frac{\partial \varphi}{\partial r} \\ &\quad +\gamma(\kappa - v)\frac{\partial \varphi}{\partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\sigma_r^2}{2}\frac{\partial^2 \varphi}{\partial r^2} + \frac{\sigma_v^2 v}{2}\frac{\partial^2 \varphi}{\partial v^2} + \sigma\sigma_v\rho_v v\frac{\partial^2 \varphi}{\partial x \partial v}. \quad (4.27) \end{aligned}$$

The joint conditional characteristic function should satisfy the following PDE (4.27). The usual practice in solving this kind of PDE (4.27) is to guess the general form of the solution. Inspired by work of Heston (1993) and Duffie as well as Pan and Singleton (2000), we guess the solution is following structure

$$\varphi(\vec{u}; x, v, , r, t) = \exp(A(\tau, \vec{u}) + B(\tau, \vec{u})v + C(\tau, \vec{u})r + iu_1 x + iu_2 v + iu_3 r), \quad (4.28)$$

where $\vec{u} = (u_1, u_2, u_3), \tau = T - t$. the solution satisfies the boundary condition

$$\varphi(\vec{u}; x, v, , r, T) = \exp(iu_1 x_T + iu_2 v_T + iu_3 r_T),$$

which implies that $A(0, \vec{u}) = B(0, \vec{u}) = C(0, \vec{u}) = 0$.

Firstly, we compute

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} = \varphi\left(-\frac{\partial A}{\partial \tau} - v\frac{\partial B}{\partial \tau} - r\frac{\partial C}{\partial \tau}\right), \quad \frac{\partial \varphi}{\partial x} = iu_1\varphi, \quad \frac{\partial \varphi}{\partial v} = \varphi\left(B + iu_2\right), \\ &\frac{\partial \varphi}{\partial r} = \varphi\left(C + iu_3\right), \quad \frac{\partial^2 \varphi}{\partial x^2} = -u_1^2\varphi, \quad \frac{\partial^2 \varphi}{\partial v^2} = \varphi\left(B + iu_2\right)^2 \\ &\frac{\partial^2 \varphi}{\partial r^2} = \varphi\left(C + iu_3\right)^2, \quad \frac{\partial^2 \varphi}{\partial x \partial v} = iu_1(B + iu_2)\varphi. \end{aligned}$$

Then substitute all term above into equation (4.27), understanding that

$\varphi(\vec{u}; x, v, , r, t) \neq 0$:

$$0 = (-A'(\tau) - vB'(\tau) - rC'(\tau)) + \left(r - \lambda m - \tfrac{1}{2}\sigma^2 v\right) iu_1 + \gamma(\kappa - v)(B + iu_2)$$
$$+ \left(\alpha - \beta r - \tfrac{\sigma_r^2}{\beta}\left(1 - e^{-\beta\tau}\right)\right)(C + iu_3) - \tfrac{\sigma^2 v}{2} u_1^2 + \tfrac{\sigma_r^2}{2}(C + iu_3)^2$$
$$+ \tfrac{\sigma_v^2 v}{2}(B + iu_2)^2 + \sigma\sigma_v\rho_v v iu_1(B + iu_2). \tag{4.29}$$

Equation (4.29) must be true for all r_t, v_t, so we can transform it to three ordinary differential equations as follows:

$$A'(\tau) = \left(\alpha - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta\tau}\right)\right)(C + iu_3) + \gamma\kappa B + \frac{\sigma_r^2}{2}(C + iu_3)^2 - \lambda m i u_1 + \gamma\kappa i u_2, \tag{4.30}$$

$$C'(\tau) = -\beta C + i(u_1 - \beta u_3) \tag{4.31}$$

$$B'(\tau) = \frac{\sigma_v^2}{2}B^2 + \left(\sigma_v^2 i u_2 + \sigma\sigma_v\rho_v i u_1 - \gamma\right)B - u_1 u_2 \sigma\sigma_v\rho_v - \gamma i u_2 - \frac{\sigma_v^2 u_2^2}{2} - \frac{\sigma^2}{2}\left(iu_1 + u_1^2\right). \tag{4.32}$$

Under the conditions $A(0, \vec{u}) = B(0, \vec{u}) = C(0, \vec{u}) = 0$.

Considering equation (4.31) with initial condition $C(0, \vec{u})$, it is clear that

$$C(\tau, \vec{u}) = \left(\frac{iu_1}{\beta} - iu_3\right)\left(e^{-\beta\tau} - 1\right). \tag{4.33}$$

Let $n_1 = \frac{\sigma_v^2}{2}$, $n_2 = iu_1\sigma\sigma_v\rho_v + \sigma_v^2 i u_2 - \gamma$, and $n_3 = -u_1 u_2 \sigma\sigma_v\rho_v - \gamma i u_2 - \frac{\sigma^2}{2}\left(iu_1 + u_1^2\right) - \frac{\sigma_v^2}{2}u_2^2$.

Substituting them into equation (4.32), we get

$$B'(\tau) = n_1\left(B + \frac{n_2 - \sqrt{n_2^2 - 4n_3 n_1}}{2n_1}\right)\left(B + \frac{n_2 + \sqrt{n_2^2 - 4n_3 n_1}}{2n_1}\right).$$

Separating variable yeilds:

$$\frac{dB}{\left(B + \frac{n_2 - \sqrt{n_2^2 - 4n_3 n_1}}{2n_1}\right)\left(B + \frac{n_2 + \sqrt{n_2^2 - 4n_3 n_1}}{2n_1}\right)} = n_1 d\tau. \tag{4.34}$$

Using the partial fraction decomposition, equation (4.34) becomes

$$\left(\frac{1}{\left(B+\frac{n_2-\sqrt{n_2^2-4n_3n_1}}{2n_1}\right)}-\frac{1}{\left(B+\frac{n_2+\sqrt{n_2^2-4n_3n_1}}{2n_1}\right)}\right)dB=\sqrt{n_2^2-4n_3n_1}d\tau.$$

Integrating both sides, we obtain:

$$\ln\left(\frac{B+\frac{n_2-\sqrt{n_2^2-4n_3n_1}}{2n_1}}{B+\frac{n_2+\sqrt{n_2^2-4n_3n_1}}{2n_1}}\right)=\tau\sqrt{n_2^2-4n_3n_1}+B_0. \tag{4.35}$$

Using the boundary condition $B(0,\vec{u})=0$, we obtain

$$B_0=\ln\left(\frac{n_2-\sqrt{n_2^2-4n_3n_1}}{n_2+\sqrt{n_2^2-4n_3n_1}}\right)$$

.

Letting $\nabla=\sqrt{n_2^2-4n_3n_1}$ and substituting B_0 into equation (4.35), we get

$$B(\tau,\vec{u})=\frac{2n_3\left[e^{\tau\nabla}-1\right]}{(n_2+\nabla)-e^{\tau\nabla}(n_2-\nabla)}.$$

To Solve for $A(\tau,\vec{u})$, we substitute the value of $B(\tau,\vec{u})$ and $C(\tau,\vec{u})$ into equation (4.31)

$$\begin{aligned}A'(\tau)=&\left\{\left(\tfrac{iu_1}{\beta}-iu_3\right)\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)-\tfrac{\sigma_r^2}{\beta}\left(\tfrac{iu_1}{\beta}-2iu_3\right)+\tfrac{\sigma_r^2}{2}\left(\tfrac{u_1}{\beta}-u_3\right)^2 2\right\}e^{-\beta\tau}\\&-u_3\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)e^{-\beta\tau}+\left\{i\left(\tfrac{u_1}{\beta}-u_3\right)\tfrac{\sigma_r^2}{\beta}-\left(\tfrac{u_1}{\beta}-u_3\right)^2\tfrac{\sigma_r^2}{2}\right\}e^{-2\beta\tau}\\&+\tfrac{2\kappa\gamma n_3\left[e^{\tau\nabla}-1\right]}{(n_2+\nabla)-e^{\tau\nabla}(n_2-\nabla)}-\lambda\mu_J iu_1+u_3\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)+i\gamma\kappa u_2-\tfrac{u_1^2\sigma^2}{2}-\tfrac{u_3^2\sigma_r^2}{2}\\&-\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)\left(\tfrac{iu_1}{\beta}-2iu_3\right)-\tfrac{\sigma_r^2}{2}\left(\tfrac{u_1}{\beta}-u_3\right)^2.\end{aligned}$$

Again separating variables, we obtain

$$dA=\left[\begin{array}{l}\left\{\left(\tfrac{iu_1}{\beta}-iu_3\right)\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)-\tfrac{\sigma_r^2}{\beta}\left(\tfrac{iu_1}{\beta}-2iu_3\right)\right\}e^{-\beta\tau}+u_3\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)\\\left\{\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)^2-u_3\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)\right\}e^{-\beta\tau}+i\gamma\kappa u_2-\tfrac{u_1^2\sigma^2}{2}-\tfrac{u_3^2\sigma_r^2}{2}\\+\left\{i\left(\tfrac{u_1}{\beta}-u_3\right)\tfrac{\sigma_r^2}{\beta}-\left(\tfrac{u_1}{\beta}-u_3\right)^2\tfrac{\sigma_r^2}{2}\right\}e^{-2\beta\tau}-\tfrac{\sigma_r^2}{2}\left(\tfrac{u_1}{\beta}-u_3\right)^2\\+\tfrac{2\kappa\gamma n_3\left[e^{\tau\nabla}-1\right]}{(n_2+\nabla)-e^{\tau\nabla}(n_2-\nabla)}-\lambda\mu_J iu_1-\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)\left(\tfrac{iu_1}{\beta}-2iu_3\right)\end{array}\right]d\tau. \tag{4.36}$$

Let us note that

$$\int \frac{2\kappa\gamma n_3\left[e^{\tau\nabla}-1\right]}{(n_2+\nabla)-e^{\tau\nabla}(n_2-\nabla)}d\tau = 2\kappa\gamma n_3 \int \frac{\left[e^{\tau\nabla}-1\right]}{(n_2+\nabla)-e^{\tau\nabla}(n_2-\nabla)}d\tau$$
$$= -\frac{2\kappa\gamma n_3}{(n_2-\nabla)}\left(\frac{2\ln\left(e^{\tau\nabla}(n_2-\nabla)-(n_2+\nabla)\right)}{(n_2-\nabla)}+\tau\right)+H_0.$$

Integrating both sides of equation (4.36), we obtain

$$\begin{aligned}
A(\tau,\vec{u}) = &-\tfrac{1}{\beta}\left\{\left(\tfrac{iu_1}{\beta}-iu_3\right)\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)-\tfrac{\sigma_r^2}{\beta}\left(\tfrac{iu_1}{\beta}-2iu_3\right)\right\}e^{-\beta\tau}\\
&-\tfrac{1}{\beta}\left\{+\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)^2-u_3\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)\right\}e^{-\beta\tau}+u_3\sigma_r^2\tau^2\left(\tfrac{u_1}{\beta}-u_3\right)\\
&-\tfrac{1}{2\beta}\left\{i\left(\tfrac{u_1}{\beta}-u_3\right)\tfrac{\sigma_r^2}{\beta}-\left(\tfrac{u_1}{\beta}-u_3\right)^2\tfrac{\sigma_r^2}{2}\right\}e^{-2\beta\tau}-\left(\lambda miu_1-\gamma\kappa iu_2\right)\tau\\
&-\left(\tfrac{u_1^2\sigma^2}{2}+\tfrac{u_3^2\sigma_r^2}{2}+\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)\left(\tfrac{iu_1}{\beta}-2iu_3\right)+\tfrac{\sigma_r^2}{2}\left(\tfrac{u_1}{\beta}-u_3\right)^2\right)\tau\\
&-\tfrac{2\kappa\gamma n_3}{(n_2-\nabla)}\left(\tfrac{2\ln\left(e^{\tau\nabla}(n_2-\nabla)-(n_2+\nabla)\right)}{(n_2-\nabla)}+\tau\right)+A_0.
\end{aligned}$$

Using the boundary condition $A(0,\vec{u})=0$:

$$\begin{aligned}
A_0 = &\tfrac{1}{\beta}\left\{\left(\tfrac{iu_1}{\beta}-iu_3\right)\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)-\tfrac{\sigma_r^2}{\beta}\left(\tfrac{iu_1}{\beta}-2iu_3\right)+\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)^2\right\}\\
&+\tfrac{1}{2\beta}\left\{i\left(\tfrac{u_1}{\beta}-u_3\right)\tfrac{\sigma_r^2}{\beta}-\left(\tfrac{u_1}{\beta}-u_3\right)^2\tfrac{\sigma_r^2}{2}\right\}+\tfrac{2\kappa\gamma n_3\ln(2\nabla)^2}{(n_2-\nabla)^2}-\tfrac{u_3\sigma_r^2}{\beta}\left(\tfrac{u_1}{\beta}-u_3\right).
\end{aligned}$$

Hence

$$\begin{aligned}
A(\tau,\vec{u}) = &\tfrac{1}{\beta}\left\{\left(\tfrac{iu_1}{\beta}-iu_3\right)\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)-\tfrac{\sigma_r^2}{\beta}\left(\tfrac{iu_1}{\beta}-2iu_3\right)\right\}\left(1-e^{-\beta\tau}\right)\\
&+\tfrac{1}{\beta}\left\{\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)^2-u_3\sigma_r^2\left(\tfrac{u_1}{\beta}-u_3\right)\right\}\left(1-e^{-\beta\tau}\right)+u_3\sigma_r^2\tau^2\left(\tfrac{u_1}{\beta}-u_3\right)\\
&+\tfrac{1}{2\beta}\left\{i\left(\tfrac{u_1}{\beta}-u_3\right)\tfrac{\sigma_r^2}{\beta}-\left(\tfrac{u_1}{\beta}-u_3\right)^2\tfrac{\sigma_r^2}{2}\right\}\left(1-e^{-2\beta\tau}\right)-\left(\lambda miu_1-\gamma\kappa iu_2\right)\tau\\
&-\left(\tfrac{u_1^2\sigma^2}{2}+\tfrac{u_3^2\sigma_r^2}{2}+\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)\left(\tfrac{iu_1}{\beta}-2iu_3\right)+\tfrac{\sigma_r^2}{2}\left(\tfrac{u_1}{\beta}-u_3\right)^2\right)\tau\\
&-\tfrac{2\kappa\gamma n_3}{(n_2-\nabla)^2}\ln\left[4\nabla^2\left(e^{\tau\nabla}\left(n_2-\nabla\right)-\left(n_2+\nabla\right)\right)^2\right]-\tfrac{2\kappa\gamma n_3\tau}{(n_2-\nabla)}.
\end{aligned} \tag{4.37}$$

Now, if we define $R_{t+\tau}=X_{t+\tau}-X_t$ as a log return, then the joint CCF can be derived as

$$\begin{aligned}\varphi(u_1,u_2,u_3;R_{t+\tau},v_{t+\tau},r_{t+\tau}) &= E\left[e^{iu_1R_{t+\tau}+iu_2v_{t+\tau}+iu_3r_{t+\tau}}|X_t,r_t,v_t\right]\\ &= E\left[e^{iu_1(X_{t+\tau}-X_t)+iu_2v_{t+\tau}+iu_3r_{t+\tau}}|X_t,r_t,v_t\right]\\ &= e^{-iu_1(X_t)}E\left[e^{iu_1(X_{t+\tau})+iu_2v_{t+\tau}+iu_3r_{t+\tau}}|X_t,r_t,v_t\right]\\ &= e^{-iu_1(X_t)}e^{A(u_1,u_2,u_3,\tau)+B(u_1,u_2,u_3,\tau)v_t+C(u_1,u_2,u_3,\tau)r_t+iu_1X_t+iu_2v_t+iu_3r_t}\\ &= e^{A(u_1,u_2,u_3,\tau)+B(u_1,u_2,u_3,\tau)v_t+C(u_1,u_2,u_3,\tau)r_t+iu_2v_t+iu_3r_t}\end{aligned} \tag{4.38}$$

and furthermore, the joint CCF of $R_{t+\tau}, r_t$ can be derived as

$$\begin{aligned}\varphi(u_1,u_3;R_{t+\tau},r_{t+\tau}|X_t,v_t,r_t) &= \varphi(u_1,0,u_3;R_{t+\tau},v_{t+\tau},r_{t+\tau}|X_t,v_t,r_t)\\ &= E\left(\exp(iu_1R_{t+\tau}+iu_3r_{t+\tau})|X_t,v_t,r_t\right)\\ &= \exp(A(\tau,u_1,0,u_3)+B(\tau,u_1,0,u_3)v_t+C(\tau,u_1,0,u_3)r_t+iu_3r_t).\end{aligned} \tag{4.39}$$

Next, the joint characteristic function of the continuous component for the process R_t and r_t is given by

$$\begin{aligned}&\phi(u_1,u_3;R_{t+\tau},r_{t+\tau}) = \phi(u_1,0,u_3;R_{t+\tau},v_{t+\tau},r_{t+\tau})\\ &= E\left[\exp(iu_1R_{t+\tau}+iu_3r_{t+\tau})\right]\\ &= E\left[E\left[\exp(iu_1R_{t+\tau}+iu_3r_{t+\tau})|X_t,r_t,v_t\right]\right]\\ &= E\left[\exp(A(\tau,u_1,0,u_3)+B(\tau,u_1,0,u_3)v_t+C(\tau,u_1,0,u_3)r_t+iu_3r_t)\right]\\ &= \exp(A(\tau,u_1,0,u_3))E\left[\exp(B(\tau,u_1,0,u_3)v_t+\left(C(\tau,u_1,0,u_3)+iu_3\right)r_t)\right]\\ &= \exp(A(\tau,u_1,0,u_3))E\left[\exp(i(-iB(\tau,u_1,0,u_3))v_t)\right]\\ &\quad E\left[\exp\left(i\left(-iC(\tau,u_1,0,u_3)+u_3\right)r_t\right)\right]\\ &= \exp(A(\tau,u_1,0,u_3))\phi(-iB(\tau,u_1,0,u_3),v_t)\phi\left(-iC(\tau,u_1,0,u_3)+u_3,r_t\right).\end{aligned} \tag{4.40}$$

Now using the fact that v_t follows a Gamma distribution with $f(v_t)=\frac{\theta^{\omega}}{\Gamma(\omega)}v_t^{\omega-1}e^{-\theta v_t}$ where $\theta=\frac{2\gamma}{\sigma_v^2}$ and $\omega=\frac{2\kappa\gamma}{\sigma_v^2}$. Since the characterisitc function of v_t in turn is given by

$$\phi\left(u; v_t\right) = \left(1 - \frac{iu\sigma_v^2}{2\gamma}\right)^{-\frac{2\gamma\kappa}{\sigma_v^2}} = \exp\left(-\frac{2\gamma\kappa}{\sigma_v^2}\ln\left(1 - \frac{iu\sigma_v^2}{2\gamma}\right)\right), \tag{4.41}$$

Using the Characteristic function of r_t from Lemma 4.1, we obtain

$$\phi(u_1, u_2, u_3; R_{t+\tau}, r_{t+\tau}) = \exp\left(\begin{array}{l} A(\tau, u_1, 0, u_3) - \frac{2\gamma\kappa}{\sigma_r^2}\ln\left(1 - \frac{B(\tau,u_1,0,u_3)\sigma_r^2}{2\gamma}\right) \\ +\frac{C(\tau,u_1,0,u_3)+iu_3}{\beta}\left(\alpha - \frac{\sigma_r^2}{2\beta}\right) - \frac{(-iC(\tau,u_1,0,u_3)+u_3)^2\sigma_r^2}{4\beta} \end{array}\right) \tag{4.42}$$

To incorporate the jump term, since jumps are homogeneous and independent for the continuous part, we need only to multiply the characteristic function that we have obtained by the characteristic function of the jumps as follows.

The characteristic function of compound Poisson process is explicitly given by

$$\phi(u, J_t) = \exp\left[\lambda t\left(\phi(u, Y)\right) - 1\right], \tag{4.43}$$

where $\phi(u, Y)$ is the characteristic function of jump size random variable, i.e.

$$\phi(u, Y) = \exp\left[iu\mu_J - \frac{1}{2}u^2\sigma_J^2\right]. \tag{4.44}$$

Since the compound Poisson process is stationary incremental, then

$$\phi(u, \Delta J_t) = \exp\left[\lambda\Delta\left(\phi(u, Y)\right) - 1\right] = \exp\left[\lambda\Delta\left(\exp\left[iu\mu_J - \frac{1}{2}u^2\sigma_J^2\right] - 1\right)\right]. \tag{4.45}$$

Finally, the joint characteristic function of the log return and the interest rate is given by

$$\phi(u_1, u_2, u_3; R_{t+\tau}, r_{t+\tau}) = \exp\left(\begin{array}{l} A(\tau, u_1, 0, u_3) - \frac{2\gamma\kappa}{\sigma_r^2}\ln\left(1 - \frac{B(\tau,u_1,0,u_3)\sigma_r^2}{2\gamma}\right) \\ +\frac{C(\tau,u_1,0,u_3)+iu_3}{\beta}\left(\alpha - \frac{\sigma_r^2}{2\beta}\right) - \frac{(-iC(\tau,u_1,0,u_3)+u_3)^2\sigma_r^2}{4\beta} \\ +\lambda\Delta\left(\exp\left[iu_1\mu_J - \frac{1}{2}u_1^2\sigma_J^2\right] - 1\right). \end{array}\right)$$

The proof is now complete. □

Lemma 4.4. *The log asset price X_t defined in equation (4.13) are stationary processes with the log return $R_{t+\tau} = X_{t+\tau} - X_t$, with the following two moments*

$$\begin{aligned}
E\left[R_{t+\tau}\right] &= \tfrac{3\sigma_r^2}{2\beta^3}\left(1-e^{-\beta\tau}\right) - \tfrac{\sigma_r^2}{2\beta^3}(1-e^{-2\beta\tau}) - \kappa\sigma^2\left(e^{\tau\gamma}+1-\gamma\tau\right)\\
&\quad -\tfrac{\tau}{\beta}\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right) - \tfrac{\kappa\sigma^2}{2\gamma}(1-e^{-\gamma\tau})\\
V[R_{t+\tau}] &= \tfrac{\sigma_r^2}{2\beta^3}\left((e^{-\beta\tau}-2)^2-1\right) - \left(\sigma^2+\tfrac{\sigma_r^2}{\beta^2}\right)\tau - \tfrac{\sigma_r^2}{2\beta^2}\left(e^{-\beta\tau}-1\right)^2\\
&\quad -\tfrac{\sigma^3\sigma_v\rho\kappa}{\gamma^2}\left(1-(1+\tau\gamma)e^{-\gamma\tau}\right) - \lambda\tau\left(\sigma_J^2+\mu_J^2\right)\\
&\quad +\tfrac{\kappa\sigma^4\sigma_v^2}{4\gamma^3}\left(1-\sigma e^{-2\gamma\tau}-2\tau\gamma e^{-\gamma\tau}\right) - \tfrac{\kappa\sigma_v^2\sigma^4}{8\gamma^3}\left(1-e^{-\tau\gamma}\right)^2 + \varpi_0
\end{aligned}$$

where

$$\varpi_0 = \tfrac{\sigma^4\sigma_v^2}{2\gamma^4}\left(1-\ln(2\gamma)+\tfrac{1}{4}\gamma\right) + \left(2\ln(2\gamma)-1+\tfrac{\gamma}{2}(\tau-1)\right)\tfrac{\sigma^3\sigma_v\rho}{\gamma^3}.$$

Proof. The moment of the log return $R_{t+\tau}$ can be derived form the marginal characteristic function as given in Lemma 4.3. Suppose $E\left[|R_t|^k\right] < \infty$, then

$$E\left[R_{t+\tau}^k\right] = i^{-k}\,\frac{d^k}{du_1^k}\phi(u_1;R_{t+\tau})\Big|_{u_1=0}, \tag{4.46}$$

where $\phi(u_1;R_{t+\tau}) = \phi(u_1,0,0;R_{t+\tau},r_{t+\tau})$, that is

$$\phi(u_1,0,0;R_{t+\tau},r_{t+\tau}) = \exp\begin{pmatrix} A(\tau,u_1)+C(\tau,u_1)\left(\frac{\alpha}{\beta}-\frac{\sigma_r^2}{2\beta^2}\right)+\frac{(C(\tau,u_1))^2\sigma_r^2}{4\beta}\\ -\frac{2\gamma\kappa}{\sigma_v^2}\ln\left(1-\frac{B(\tau,u_1)\sigma_v^2}{2\gamma}\right)\\ +\lambda\Delta\left(\exp\left[iu_1\mu_J-\frac{1}{2}u_1^2\sigma_J^2\right]-1\right)\end{pmatrix}, \tag{4.47}$$

$$\begin{aligned}
A(\tau,u_1) &= \tfrac{1}{\beta}\left\{\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)\left(\tfrac{iu_1}{\beta}\right)-\tfrac{\sigma_r^2}{\beta^2}\left(iu_1+u_1^2\right)\right\}\left(1-e^{-\beta\tau}\right)-\lambda\mu_J iu_1\tau\\
&\quad +\tfrac{\sigma_r^2}{2\beta^3}\left\{iu_1-\tfrac{u_1^2}{2}\right\}\left(1-e^{-2\beta\tau}\right)+\left(-\tfrac{u_1^2\sigma^2}{2}-\left(\alpha-\tfrac{\sigma_r^2}{\beta}\right)\left(\tfrac{iu_1}{\beta}\right)-\tfrac{\sigma_r^2u_1^2}{2\beta^2}\right)\tau\\
&\quad -\tfrac{2\kappa\gamma n_3}{(n_2-\nabla)^2}\ln\left[4\nabla^2\left(e^{\tau\nabla}(n_2-\nabla)-(n_2+\nabla)\right)^2\right]-\tfrac{2\kappa\gamma n_3\tau}{(n_2-\nabla)},\\
B(\tau,u_1) &= \tfrac{2n_3\left[e^{\tau\nabla}-1\right]}{(n_2+\nabla)-e^{\tau\nabla}(n_2-\nabla)},\ C(\tau,u_1)=\left(\tfrac{iu_1}{\beta}\right)\left(e^{-\beta\tau}-1\right),\\
\nabla &= \sqrt{n_2^2-2n_3\sigma_v^2},\ n_2=iu_1\sigma\sigma_v\rho_v-\gamma,\ n_3=-\tfrac{iu_1\sigma^2}{2}.
\end{aligned}$$

Now, the first moment will be found by using equation (4.47) with k =1:

$$E\left[R_{t+\tau}\right] = i^{-1}\,\frac{d}{du_1}\phi(u_1;R_{t+\tau})\Big|_{u_1=0}, \tag{4.48}$$

where

$$\frac{d}{du_1}\phi(u_1;R_{t+\tau})|_{u_1=0}$$
$$= \phi(0;R_{t+\tau})\frac{d}{du_1}\left(\begin{array}{l} A(\tau,u_1)+C(\tau,u_1)\left(\frac{\alpha}{\beta}-\frac{\sigma_r^2}{2\beta^2}\right) \\ +\frac{(C(\tau,u_1))^2\sigma_r^2}{4\beta}-\frac{2\gamma\kappa}{\sigma_v^2}\ln\left(1-\frac{B(\tau,u_1)\sigma_v^2}{2\gamma}\right) \\ +\lambda\tau\left(\exp\left[iu_1\mu_J-\frac{1}{2}u_1^2\sigma_J^2\right]-1\right) \end{array}\right)_{u_1=0}. \tag{4.49}$$

Note that, we calculate the following the first derivative at $u_1 = 0$:

$$\begin{aligned} \frac{d}{du_1}A(\tau,u_1)\Big|_{u_1=0} &= \frac{i}{\beta^2}\left(1-e^{-\beta\tau}\right)\left(\alpha-\frac{2\sigma_r^2}{\beta}\right)+\frac{\sigma_r^2 i}{2\beta^3}\left(1-e^{-2\beta\tau}\right)-\frac{i\tau}{\beta}\left(\alpha-\frac{\sigma_r^2}{\beta}\right) \\ &\quad -\lambda mi\tau-\frac{i\sigma^2\kappa}{2\gamma}\left(\tau\gamma+2\ln(2\gamma)\right)-2i\kappa\tau\sigma^2\gamma^2, \\ \frac{d}{du_1}C(\tau,u_1)\Big|_{u_1=0} &= \frac{i}{\beta}\left(e^{-\beta\tau}-1\right), \quad \frac{d}{du_1}C^2(\tau,u_1)\Big|_{u_1=0}=0, \\ \frac{d}{du_1}\lambda\tau\left(\exp\left[iu_1m-\frac{1}{2}u_1^2\sigma_J^2\right]-1\right)\Big|_{u_1=0} &= im\lambda\tau, \\ -\frac{2\gamma\kappa}{\sigma_v^2}\frac{d}{du_1}\ln\left[1-\frac{B(\tau,u_1,0,0)\sigma_v^2}{2\gamma}\right]\Big|_{u_1=0} &= -\frac{i\kappa\sigma^2}{2\gamma}(1-e^{-\gamma\tau}). \end{aligned}$$

Substitute all term above into equation (4.49) we obtain

$$\begin{aligned} \frac{d}{du_1}\phi(u_1;R_{t+\tau})\Big|_{u_1=0} &= i\left(\frac{1}{\beta^2}\left(1-e^{-\beta\tau}\right)\left(2\alpha-\frac{3\sigma_r^2}{2\beta}\right)-\frac{\tau}{\beta}\left(\alpha-\frac{\sigma_r^2}{\beta}\right)\right) \\ &\quad -i\left(\frac{\sigma^2\kappa}{2\gamma}\left(2\ln(2\gamma)+\tau\nabla+e^{-\gamma\tau}-1\right)\right)-2\kappa\tau\sigma^2\gamma\Big). \end{aligned}$$

Therefore

$$\begin{aligned} E\left[R_{t+\tau}\right] &= \frac{1}{\beta^2}\left(1-e^{-\beta\tau}\right)\left(2\alpha-\frac{3\sigma_r^2}{2\beta}\right)-\frac{\tau}{\beta}\left(\alpha-\frac{\sigma_r^2}{\beta}\right)-2\kappa\tau\sigma^2\gamma \\ &\quad -\frac{\sigma^2\kappa}{2\gamma}\left(2\ln(2\gamma)+\tau\nabla+e^{-\gamma\tau}-1\right)). \end{aligned}$$

Next, we will compute the variance of $R_{t+\tau}$ by using the relation between the cumulant function $\eta(u)$ and the characteristic function, i.e., $\eta(u) = \ln\phi(u)$ such that the variance of $R_{t+\tau}$ can be derived by

$$\begin{aligned} V[R_{t+\tau}] &= \frac{d^2}{du_1^2}\ln\left[\phi(u_1;R_{t+\tau})\right]\Big|_{u_1=0} \\ &= \frac{d^2}{du_1^2}\left(\begin{array}{l} A(\tau,u_1)+C(\tau,u_1)\left(\frac{\alpha}{\beta}-\frac{\sigma_r^2}{2\beta^2}\right)+\frac{(C(\tau,u_1))^2\sigma_r^2}{4\beta} \\ -\frac{2\gamma\kappa}{\sigma_v^2}\ln\left(1-\frac{B(\tau,u_1)\sigma_v^2}{2\gamma}\right)+\lambda\tau\left(\exp\left[iu_1\mu_J-\frac{1}{2}u_1^2\sigma_J^2\right]-1\right) \end{array}\right)\Bigg|_{u_1=0}. \end{aligned} \tag{4.50}$$

We calculate the derivative as follows:

$$\frac{d^2}{du_1^2}C(\tau,u_1)\Big|_{u_1=0}=0, \quad \frac{d^2}{du_1^2}C^2(\tau,u_1)\Big|_{u_1=0}=-\frac{2}{\beta}\left(e^{-\beta\tau}-1\right)^2,$$

$$\frac{d^2}{du_1^2}B(\tau,u_1)\Big|_{u_1=0}=\frac{\rho\sigma_v\sigma^3}{\gamma^2}\left(1-e^{-\gamma\tau}-\tau\gamma e^{-\gamma\tau}\right)-\frac{\sigma^4\sigma_v^2}{4\gamma^3}\left(1-\sigma e^{-2\gamma\tau}-2\tau\gamma e^{-\gamma\tau}\right),$$

$$\frac{d^2}{du_1^2}\lambda\tau\left(\exp\left[iu_1\mu_J-\frac{1}{2}u_1^2\sigma_J^2\right]-1\right)\Big|_{u_1=0}=-\lambda\tau\left(\sigma_J^2+\mu_J^2\right),$$

$$\frac{d^2}{du_1^2}A((\tau,u_1))\Big|_{u_1=0}=\frac{\sigma_r^2}{2\beta^3}\left((e^{-\beta\tau}-2)^2-1\right)-\left(\sigma^2+\frac{\sigma_r^2}{\beta^2}\right)\tau+\frac{\sigma^4\sigma_v^2}{2\gamma^4}\left(1-\ln(2\gamma)+\frac{1}{4}\gamma\right)$$
$$+\left(2\ln(2\gamma)-1+\frac{\gamma}{2}(\tau-1)\right)\frac{\sigma^3\sigma_v\rho}{\gamma^3},$$

$$-\frac{2\gamma\kappa}{\sigma_v^2}\frac{d^2}{du_1^2}\ln\left[1-\frac{B(\tau,u_1)\sigma_v^2}{2\gamma}\right]_{u_1=0}=-\frac{\kappa\rho\sigma_v\sigma^3}{\gamma^2}\left(1-e^{-\gamma\tau}-\tau\gamma e^{-\gamma\tau}\right)$$
$$+\frac{\kappa\sigma^4\sigma_v^2}{4\gamma^3}\left(1-\sigma e^{-2\gamma\tau}-2\tau\gamma e^{-\gamma\tau}\right)-\frac{\kappa\sigma_v^2\sigma^4}{8\gamma^3}\left(1-e^{-\tau\gamma}\right)^2.$$

Substituting all the terms above into equation (4.50), we get

$$V[R_{t+\tau}]=\frac{\sigma_r^2}{2\beta^3}\left((e^{-\beta\tau}-2)^2-1\right)-\left(\sigma^2+\frac{\sigma_r^2}{\beta^2}\right)\tau+\frac{\sigma^4\sigma_v^2}{2\gamma^4}\left(1-\ln(2\gamma)+\frac{1}{4}\gamma\right)$$
$$+\left(2\ln(2\gamma)-1+\frac{\gamma}{2}(\tau-1)\right)\frac{\sigma^3\sigma_v\rho}{\gamma^3}-\frac{\sigma_r^2}{2\beta^2}\left(e^{-\beta\tau}-1\right)^2-\frac{\kappa\sigma_v^2\sigma^4}{8\gamma^3}\left(1-e^{-\tau\gamma}\right)^2$$
$$-\frac{\sigma^3\sigma_v\rho\kappa}{\gamma^2}\left(1-(1+\tau\gamma)e^{-\gamma\tau}\right)+\frac{\kappa\sigma^4\sigma_v^2}{4\gamma^3}\left(1-\sigma e^{-2\gamma\tau}-2\tau\gamma e^{-\gamma\tau}\right)-\lambda\tau\left(\sigma_J^2+\mu_J^2\right).$$

The proof is now complete. □

4.4 The GMM Estimator for Stochastic Volatility Jump Diffusion Model with Stochastic Interest Rate

In this section, we shall estimate the parameters for the SVJSI model as in equation (4.3)-(4.5) using the GMM technique. The GMM estimates $\hat{\delta}$ of δ depend on the chosen condition function $f(x_t,\delta)$. So, we shall present the moment selection for the interest rate process in section 4.4.1 and for asset price in the next section.

4.4.1 The Process of Selecting Moments for Interest Rate

In this section, we concentrate on the GMM estimator of the interest rate model satisfying the TF-Vasicek process as in equation (4.4), by using the same idea of Chan, Karolyi, Longstaff and Sanders (1992).

Firstly, we have to apply the Euler discretization scheme to discretize the model as follows:

$$r_{t+1} - r_t = (\alpha - \beta r_t - \psi) + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \equiv \sigma_r N(0,1), \tag{4.51}$$

where $\psi = -\frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)$ and $N(0,1)$ is normal variable with zero mean and unit variance.

The parameter set is $\delta = (\alpha, \beta, \sigma_r)'$ and set $\varepsilon_{t+1} = r_{t+1} + (\beta - 1)r_t + \psi - \alpha$ with $\varepsilon_{t+1} \sim N(0, \sigma_r^2)$, giving us an abundance of moment conditions:

$$E\left[\varepsilon_{t+1}\right] = 0, E\left[\varepsilon_{t+1}^2 - \sigma_r^2\right] = 0, E\left[\left(\varepsilon_{t+1}^2 - \sigma_r^2\right) r_t\right] = 0, E\left[\varepsilon_{t+1} r_t\right] = 0,$$

that is

$$f(r_t, \delta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^2 - \sigma_r^2 \\ \varepsilon_{t+1} r_t \\ \left(\varepsilon_{t+1}^2 - \sigma_r^2\right) r_t \end{bmatrix}. \tag{4.52}$$

Given a value for δ, we can compute the sample moments condition function. For instance, sample moments as in equation (4.52) are

$$\hat{f}\left(r_t, \delta\right) = \begin{bmatrix} \frac{1}{N}\sum_{t=1}^{N} \varepsilon_{t+1} \\ \frac{1}{N}\sum_{t=1}^{T} \left(\varepsilon_{t+1}^2 - \sigma_r^2\right) \\ \frac{1}{N}\sum_{t=1}^{T} \varepsilon_{t+1} r_t \\ \frac{1}{N}\sum_{t=1}^{T} \left(\varepsilon_{t+1}^2 - \sigma_r^2\right) r_t \end{bmatrix}. \tag{4.53}$$

We will use the moment function as in equation (4.52) and the sample moment condition function as in (4.53) in the GMM procedure which discussed in section 4.2 to estimate parameters α, β and σ_r.

4.4.2 Selecting Moment for Asset Price Process

In the previous section, we discuss the moment selection to estimate parameter α, β and σ_r . These fixed parameters will be used in model (4.3)-(4.5) where the set of parameters $\delta = (\sigma, \rho, \gamma, \kappa, \sigma_v, \mu_J, \sigma_J, \lambda)$ has not yet been estimated. As shown in section 4.3, moments can be derived from the joint characteristics function to derive the GMM estimates $\hat{\delta}$ for δ .

Let $Z_t = R_{t+1} - E\left[R_{t+1}\right],\ t = 1, 2, 3, ..., N$ be the demeaned log return process, with expectation of Z_t calculated as in Lemma 4.4. Furthermore, we define

$$f(R_t, \delta) = \begin{bmatrix} Z_t^n - E\left[Z_t^n\right] \\ Z_t Z_{t+1} - E\left[Z_t Z_{t+1}\right] \\ Z_t^2 Z_{t+k}^2 - E\left[Z_t^2 Z_{t+k}^2\right] \end{bmatrix} \tag{4.54}$$

as a vector of various chosen moment conditions function with $n = 1, 2, 3, 4, 5$ and $k = 1, 2, 3$ such that $E[f(R_t, \delta)] = 0$.

The moment condition functions $f(R_t, \delta)$ are chosen with further considerations to estimation efficiency of the model. First, the jump component is only reflected in the moments. Also, the stochastic volatility and random jump both allow for skewness and kurtosis, so that the first group of moment condition function, i.e. $f_n(R_t, \delta) = Z_t^n - E[Z_t^n]$ for $n = 1, 2, 3, 4, 5$, is important for estimation the parameter. Secondly, since the autocorrelation of log return and squared log return is determined by the activity rate process v_t and its correlation with asset price, the joint moments of log return and squared log returns (second and third group in equation (4.54)) are important for the identification of the SVJSI process. As autocorrelation varies over time, we can use these moment condition functions with different lags, namely $k = 1, 2, 3$.

4.5 The Estimation Results

The data for sample moment conditions functions include the daily price observations for the SET50 index and daily observations for rates on three month Treasury bill of Thailand from January 4, 2010 to October 26, 2011. The data set for these stock price and Treasury bill rate were obtained from http://www.set.or.th. and http://www.thaibma.or.th. Figure 4.1 shows the daily asset price and the log return of the asset in this period, and the daily rate of 3 month Treasury bills in the same period is shown in Figure 4.2.

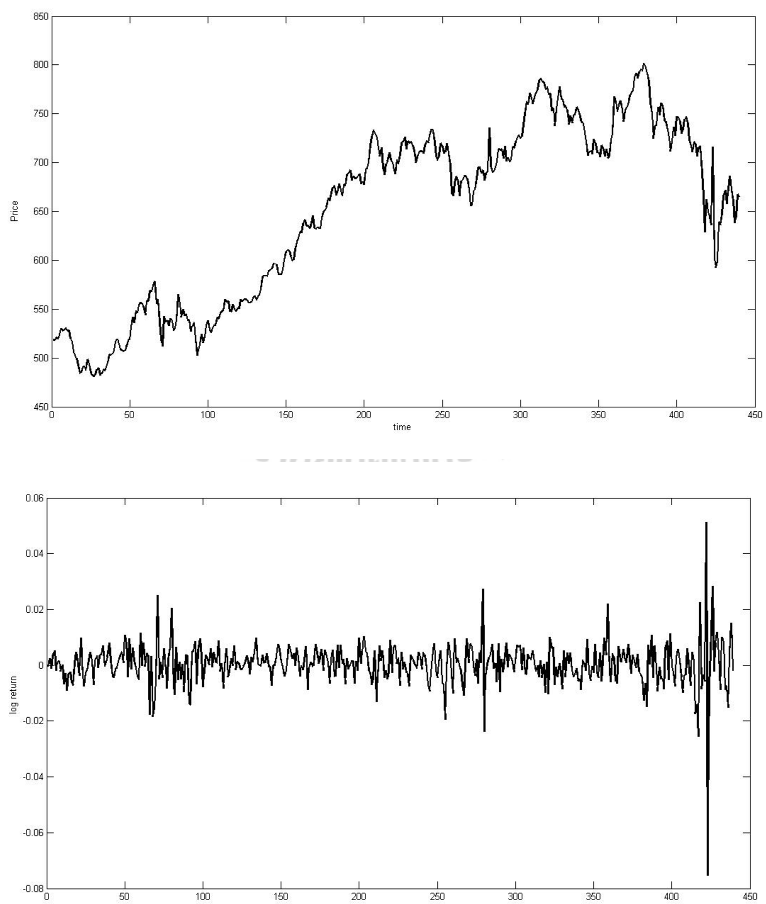

Figure 4.1 *The daily price of SET50 index (up) and the log return on SET50 index (down) between January 4, 2010 and October 26, 2011.*

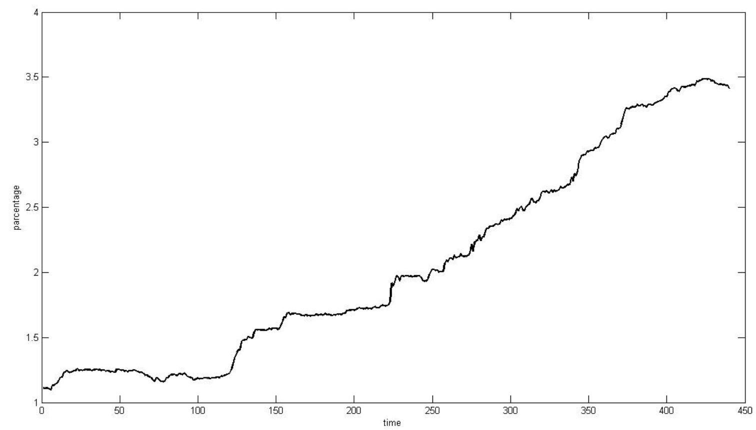

Figure 4.2 *The daily rate of 3 month Treasury bill of Thailand between January 4, 2010 and October 26, 2011.*

The summary of the statistical properties of the daily SET50 index, log return and 3-month Treasury bill rate are reported in Panel A of Table 4.1. We can see that the log return of daily SET50 index is skewed and possesses positive excess kurtosis, with its dynamic properties displayed in Panel B of Table 4.1, where we can see that the autocorrelation for the SET50 log returns and squared SET50 log returns with lag equal to 1 are -0.1341 and 0.3624, which is statistically significant (95%) in the sample of 439 observations.

Figure 4.3 shows the historical volatility, that is the annualized standard derivation of the log return as in equation (2.7), indicating that the historical volatility of the log return on SET50 index is not constant over time.

Table 4.1 *The statistic and dynamic property of data set.*

(a) Statistical property

	Asset price	Log return	Treasury bill
Sample size	440	439	440
Mean	649.497	2.160×10^{-4}	2.0566
Std	90.872	7.751×10^{-3}	0.7794
Skewness	-0.3384	-1.3920	0.5223
Kurtosis	1.7357	27.8365	1.9097
Maximum	801.440	0.0513	3.487
Minimum	480.600	-0.0754	1.097

(b) The dynamics property

	Asset price	Log return	Square log return
Sample size	440	439	440
ACF with lag k			
$\rho(1)$	0.9893*	-0.1341*	0.3624*
$\rho(2)$	0.983*	-0.0218	0.0037
$\rho(3)$	0.9774*	-0.1363*	0.1050*
$\rho(4)$	0.9736*	0.1550*	0.0683*
$\rho(5)$	0.9377*	-0.1554*	0.1003*
$\rho(10)$	0.9497*	0.0022	0.0093
$\rho(20)$	0.8917*	-0.0149	-0.0110

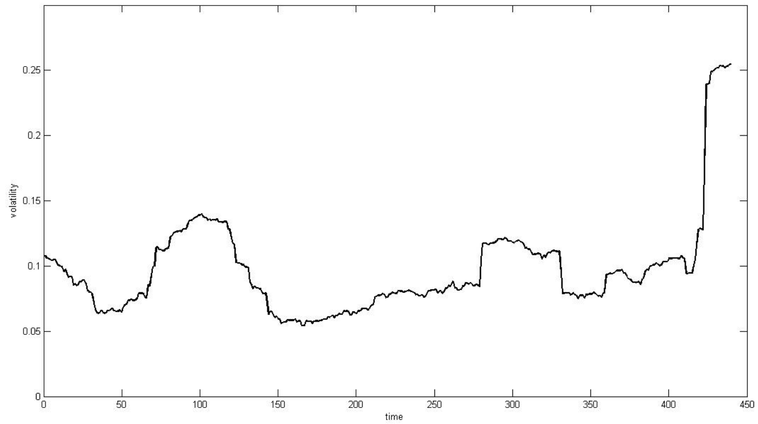

Figure 4.3 *The 50 day historical volatility of the log return on SET50 index between January 4, 2010 and October 26, 2011.*

For comparative purposes, we compute the Average Relative Percentage Error (ARPE), which is by definition

$$ARPE = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{w_i - \hat{w}_i}{w_i}\right| \times 100,$$

where N is the number of data, $w = (w_i)_{i\geq 1}$ is the empirical data and $\hat{w} = (\hat{w}_i)_{i\geq 1}$ is the model price.

We used MATLAB to obtain the results of GMM parameter estimation (see code in www.risklabbkk.com). The estimation routine was executed by running the RunEstimation.m (uses GMMestimation.m, GMMobjective.m, GMMweightsNW.m and MomentsJacobia.m).

First, the parameter estimates for $\delta = (\alpha, \beta, \sigma_r)$ are shown in table 4.2. The statistic J-test shows it accept the hypothesis at 95 % confidence level, that means the model is valid. After working 500 simulations with these fixed parameters with initial value $r_0 = 1.110$, we choose a sample path with the smallest ARPE, i.e., ARPE = 5.187%. Figure 4.4 shows that the empirical data of 3 month Treasury bill closing price compared to the price simulated by the TF-Vasicek model:

Table 4.2 *GMM estimation results of interest rate model.*

Parameter	α	β	σ_r	ARPE	J-test
TF Vasieck	1.0764	0.0706	0.4673	5.187	0.8098

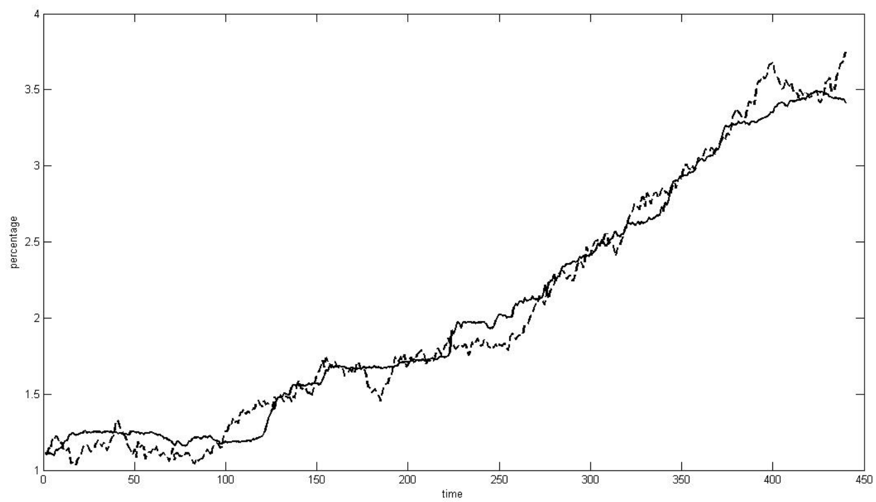

Figure 4.4 *The rate behavior of 3 month Treasury bills of Thailand, between January 4, 2010 and October 26, 2011, as compared with a scenario simulated from TF Vasicek model (solid line:= empirical data, dash line:= simulation data) with N = 440 and ARPE = 5.187%.*

Second, the SVJSI model parameters are $\sigma = -0.381, \sigma_v = 0.801, \rho = 0.347, \mu_J = 0.0002, \sigma_J = 0.030, \kappa = 2.218, \gamma = 0.525, \alpha = 1.0764, \beta = 0.7543, \lambda = 2.949$ and $\sigma_r = 0.4673$. After working 500 simulations with initial values $S_0 = 518.54, v_0 = 0.149, r_0 = 1.110$ and $N = 440$, we choose the smallest ARPE's sample path and show the price simulation as compared to the empirical data of SET50 index closing price in Figure 4.5.

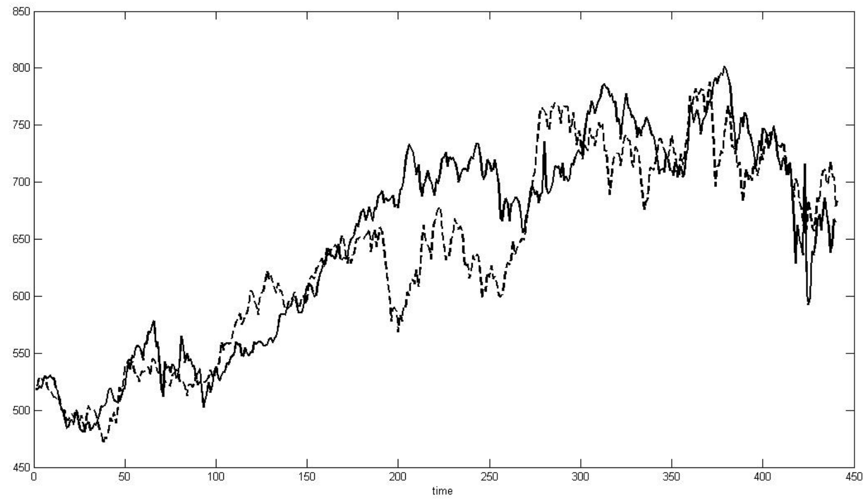

Figure 4.5 *The index behavior of SET50, between January 4, 2010 and October 26, 2011, as compared to a scenario simulated from SVJSI model (solid line:= empirical data, dash line:= simulation data) with N = 440 and ARPE = 4.912%.*

4.6 Application to a Financial Problem

In this section we use the SVJSI model as shown in equation (4.3)-(4.5) and the parameter estimates as in section 4.4 to analyze the valuation of stock options on SET50 index. The initial stock price was taken to be equal to 681.44, the interest rate, 3.4386%, and the option's maturity, 44 days. For our consideration, the option price is presented by using the closed form solution and Monte Carlo simulation. The closed form solution for SVJSI model referring to the closed form solution in Chapter III is given by

$$C(t, S_t, r_t, v_t; T, \kappa) = S_t P_1(t, x, r, v; T, K) - KP^*(t, T) P_2(t, x, r, v; T, K) \quad (4.55)$$

where

$$P_j(t, x, r, v; T, K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-iu \ln K} f_j(t, x, r, v; T, u)}{iu}\right] du, \quad (4.56)$$

$$
\begin{aligned}
&f_j(t,x,r,v;T,u) = \exp\left[iux + B_j(\tau) + rC_j(\tau) + vE_j(\tau) - (j-1)\ln P^*(t,t+\tau)\right], \\
&P^*(t,T) = \exp\left(a(t,T) + b(t,T)r_t\right), b(t,T) = \tfrac{1}{\beta}\left(\mathrm{e}^{-\beta(T-t)} - 1\right), \\
&a(t,T) = \left(\tfrac{3\sigma_r^2}{2\beta^4} - \tfrac{\alpha}{\beta}\right)\left(b(t,T) + (T-t)\right) - \tfrac{3\sigma_r^2}{4\beta^4} b^2(t,T), \\
&B_j(\tau) = \left[\tfrac{\gamma\kappa\zeta_j}{b_1}\ln\left(\tfrac{(b_{2j}+\zeta_j)e^{\tau\zeta} - (b_{2j}-\zeta_j)}{2\zeta_j}\right) - \tfrac{\tau\zeta_j\gamma\kappa(b_{2j}+\zeta_j)}{2b_1}\right] + \tfrac{(iu-j+1)\alpha}{\beta^2}\left(e^{-\beta\tau} - 1 + \tau\beta\right) \\
&\quad + \tfrac{\sigma_r^2}{2\beta^3}\left(\tfrac{(iu-j+1)^2}{2} - (iu-j+1)\right)\left(\left(e^{-\beta\tau} - 2\right)^2 - 7 - 2\beta\tau\right) + \tilde{B}_j(\tau), \\
&C_j(\tau) = \tfrac{iu-(j-1)}{\beta}(1 - e^{-\beta\tau}), E_j(\tau) = \tfrac{\left(b_{2j}^2 - \zeta_j^2\right)\left(e^{\zeta\tau} - 1\right)}{2b_1\left((b_{2j}-\zeta_j) - (b_{2j}+\zeta_j)e^{\zeta\tau}\right)}, \\
&\tilde{B}_j(\tau) = \tau\left(e^{(iu+2-j)\mu_J + \frac{(iu+2-j)^2\sigma_J^2}{2}} - (2-j+iu)e^{\mu_J + \frac{\sigma_J^2}{2}} + iu\right), \varsigma_j = \sqrt{b_{2j}^2 - 4b_0b_1} \\
&b_0 = \tfrac{\sigma^2}{2}\left(iu - u^2\right), b_1 = \tfrac{\sigma_v^2}{2}, \quad b_{2j} = \left(\sigma\sigma_v\rho_v(iu+2-j) - \gamma\right).
\end{aligned}
$$

The proof for this closed form solution is similar to Theorem 3.5. Even though the formula looks complicated, it is really quite explicit and easy to evaluate in MATLAB. The slight difficulty lies in the limits of the integral in equation (4.59), so that the integral cannot be evaluated exactly, but can be approximated with reasonable accuracy by using some numerical integration technique e.g., Gauss Lagendre of Gauss Labatto integration. (see MATLAB code in Appendix A1).

For the same options, we now estimates the call option by using Monte Carlo Simulation technique(see MATLAB code in Appendix A2):

$$C(t,S_t,r_t,v_t;T,\kappa) = P^*(t,T)E\left[\max(S_T - K,0)|S_t,v_t,r_t\right].$$

Table 4.3 shows the market price and the estimate prices by using both technique with a comparison of the speed of both. The estimated zero coupon bond price is displayed in Table 4.4 .

Table 4.3 *Computing time of alternative numerical techniques for SVJSI model with $S = 681.44$, $r = 3.4386$ % and trading day: 27 Oct, 2011 and contract month: Dec, 2011. Numbers in parentheses are standard errors for the estimates of the option price.*

Exercise price	Market price	Closed form solution	Monte Carlo simulation for call option with N simulation			
			N = 50	N = 500	N = 2000	N = 10000
660	45.00	46.76	41.67 (4.989)	39.23 (1.529)	41.14 (0.873)	41.23 (0.156)
670	39.00	42.76	35.74 (3.867)	32.34 (1.245)	33.45 (0.645)	34.89 (0.233)
680	35.50	38.78	30.19 (3.921)	31.34 (1.233)	31.15 (0.648)	31.23 (0.189)
690	30.00	36.70	24.83 (3.459)	26.35 (1.259)	25.39 (0.523)	25.41 (0.197)
700	26.00	32.10	20.56 (2.336)	22.48 (0.829)	21.95 (0.528)	21.87 (0.198)
710	22.40	28.32	18.10 (2.381)	21.79 (0.821)	19.83 (0.643)	20.02 (0.197)
Computing time (sec)		0.07	0.19	2.59	32.45	852.47

Table 4.4 *Computing time of alternative numerical techniques for TF-Vasicek model with initial rate $r = 3.4386\,\%$. Numbers in parentheses are standard errors for the estimates of bond price.*

	Closed form solution	Monte Carlo simulation for zero coupon bond price with N simulation			
		N = 50	N = 500	N = 2000	N = 10000
	0.985	0.987 (1.83×10^{-4})	0.975 (6.3×10^{-5})	0.981 (2.9×10^{-5})	0.983 (1.7×10^{-5})
Computing time (sec)	2×10^{-5}	0.23	2.47	30.85	729.45

CHAPTER V

CONCLUSION AND RESEARCH POSSIBILITY

5.1 Conclusion

This thesis has proposed the use of asset price dynamics to accommodate the stochastic volatility Lévy model with the stochastic interest rate as driven by Vasicek process. To incorporate the volatility effect to the model, we applied the stochastic time change process, i.e., the Integrated CIR process, to the diffusion part and jump part. In pricing an option when the interest rate is a stochastic process, we have to consider the asset price dynamic under the T-forward measure, i.e., the probability measure that is defined by the Radon-Nikodym derivative. Using the Girsanov's Theorem, we obtained the dynamic under the T-forward measure.

Under T-forward measure, the formula of European call options was formulated by inverting the characteristic function of the model. In order to solve the characteristic function explicitly, we proved the lemma that established relationship between stochastic volatility, stochastic interest rate and partial differential equations. We then derived the explicit formula of characteristic function and the probability distribution function by inverting the characteristic function. Moreover, the formula of the European option can be expressed in term of the probability function. Hence by using the technique based on the characteristic function of an underlying asset, an approximate formula of a European call options is derived explicitly.

To apply our work to finance, first of all we need to estimate the parameters of the SVJSI model, a special case of SVLSI model, by using the GMM technique. Since the GMM technique required a moment condition function, we constructed a moment condition function by deriving a formula for the joint characteristic function of log return and interest rate. Consequently, we compute the moment of log return and interest rate by using the joint characteristic function to get the moment condition functions used in GMM procedure. Here, MATLAB is used to obtain the result of GMM estimator.

With the obtained estimates parameters, we simulated the SVLSI model and TF-Vasicek models to display the sample path against the actual data. Moreover, we calculated the European call option by using the closed form solution and Monte Carlo simulation.

5.2 Research Possibility

In this section, we provide possible extension of the stochastic volatility Lévy model with stochastic interest rate. for further exploration.

1. The stochastic interest rate in this research satisfies Vasicek process under T-forward measure, however in practice, the behavior of interest rate may better be modeled by other processes such as the CIR process, Hull-White process, or Ho-Lee process under T-forward measure. Thus it may be possible to change the interest rate process to more general processes.

2. In order to study the numerical solution of a European option, we may be able to apply the Discrete Fourier transform (DFT) or the fast Fourier transform (FFT) for higher accuracy.

3. In this thesis, the fixed parameter, used to find the option prices, are estimated by using the information from the asset price. Moreover, we may discover the parameters by using market option prices; this is the inverse problem. This approach is very popular so it is the next problem for a future research.

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APPENDIX

APPENDIX A

COMPUTER PROGRAMS

This appendix contains a copy of the programs written in MATLAB to implement the approximation in Chapter IV.

A.1 European Call Option using Numerical Integration

```
function call = callSVJSI(S,X,tau,r,v,sigma,sigmar,sigmav,sigmaJ,rho,...
    ,gamma,alpha,beta,lambda,muJ,kappa,P)
% S= Asset price ,X = strike , % tau = time to mat
% P=zero coupon bond price
% Call price calculate
vP1 = 0.5 + 1/pi * quadl(@P1f,0,200,[],[],S,X,tau,r,v,sigma,sigmar,...
    sigmav,sigmaJ,rho,gamma,alpha,beta,lambda,muJ,kappa,P);
vP2 = 0.5 + 1/pi * quadl(@P2f,0,200,[],[],S,X,tau,r,v,sigma,sigmar,...
    sigmav,sigmaJ,rho,gamma,alpha,beta,lambda,muJ,kappa,P);
call = S * vP1 - X * P *vP2;
end
%fourier transform for f1
function p = P1f(om,S,X,tau,r,v,sigma,sigmar,sigmav,sigmaJ,rho,...
    gamma,alpha,beta,lambda,muJ,kappa,P)
i=1i;
p = real(exp(-i*log(X)*om).* cf55(om,S,X,tau,r,v,sigma,sigmar,...
    sigmav,sigmaJ,rho,gamma,alpha,beta,lambda,muJ,kappa,P)./...
    (i * om ));
end
% fourier transform for f2
```

```
function p = P2f(om,S,X,tau,r,v,sigma,sigmar,sigmav,sigmaJ,...
    rho,gamma,alpha,beta,lambda,muJ,kappa,P)
i=1i;
p = real(exp(-i*log(X)*om) .* cf56(om,S,X,tau,r,v,sigma,sigmar,...
    sigmav,sigmaJ,rho,gamma,alpha,beta,lambda,muJ,kappa,P)./...
    (i * om));
end
%characteristic function for f1
function cf = cf55(om,S,X,tau,r,v,sigma,sigmar,sigmav,sigmaJ,rho,...
    gamma,alpha,beta,lambda,muJ,kappa,P)
b1=sigmav.^2/2; b2=rho.*sigma.*sigmav.*(1i*om)-gamma;
b3=(-1i*om-om).^2*sigma^2/2;
d=sqrt(b2.^2-4.*b3.*b1); b11=b2+d; b12=b2-d; e0=exp(tau*d);
e00=exp(-tau*d);C = 1i*(om) .* (1-exp(-beta*tau))./beta;
B= 2.*(b2.^2-d.^2).*(e0-1)./2*b1*(b2+d-e0.*(b2-d));
A1=gamma*kappa*d*ln(((b2+d)*e0-b2+d))/(2*d))/...
   b1-tau*d*gamma*kappa*(b2+d)/(2*b1);
A2=(1i*om)*alpha*(e00-1+tau*beta)/beta^2;
A3=sigmar^2*((1i*om)^2/2-1i*om)/(2*beta^3)*((e00-2)^2-7-2*beta*tau)
j1=lambda*exp(1i*om*muj+(1i*om+1)^2*sigmaj^2/2)
j2=lambda*exp(tau*(1+1i*om)*exp(muj+sigmaj^2/2)+1i*om);
A4=J1-J2;
A=A1+A2+A3+A4;
J=lambda.*tau.*(exp(1i.*om.*muJ-(om.^2*sigmaJ.^2)./2)-1);
cf = exp(1i.*om *log(S)+ r.*C + v.*B+A+J);
end
% characteristic function for f2
function cf = cf56(om,S,X,tau,r,v,sigma,sigmar,sigmav,sigmaJ,rho,...
    gamma,alpha,beta,lambda,muJ,kappa,P)
b1=sigmav.^2/2; b2=rho.*sigma.*sigmav.*(1i*om)-gamma;
b3= (-1i*om-om).^2*sigma^2/2; d=sqrt(b2.^2-4.*b3.*b1);
```

```
b11=b2+d; b12=b2-d; e0=exp(tau*d);
e00=exp(-tau*d); C =(1i*(om)-1) .* (1-exp(-beta*tau))./beta;
B= 2.*(b2.^2-d.^2).*(e0-1)./2*b1*(b2+d-e0.*(b2-d));
A1=gamma*kappa*d*ln(((b2+d)*e0-b2+d))/(2*d))/...
    b1-tau*d*gamma*kappa*(b2+d)/(2*b1);
A2=(1i*om-1)*alpha*(e00-1+tau*beta)/beta^2;
A3=sigmar^2*((1i*om-1)^2/2-1i*om-1)/(2*beta^3)*...
      ((e00-2)^2-7-2*beta*tau)
j1=exp(1i*om*muj+(1i*om+1)^2*sigmaj^2/2)
j2=exp(tau*(1i*om*exp(muj+sigmaj^2/2)+1i*om);
A4=lambda*(J1-J2);
A1=(i*om.*alpha./beta^2).*(1-e00-tau);
A=A1+A2+A3+A4;
cf  = exp(1i.*om *log(S)+ r.*C + v.*B+A+J-ln(P));
end
```

A.2 European Call Option using Monte Carlo Simulation

```
function M=Monte[S0,r0,v0,K,NS,tau,alpha,beta,sigmar,sig,lambda,...
      muj,sigmaj,kappa,rhov,gamma,sigv]
% NS= number of simulation, tau= time to marturity.
% Initail value = S0 ,r0, v0.
% K= exercise price.
% parameter =alpha,beta,sigmar,sig,lambda,muj,sigmaj,kappa,rhov,...
gamma,sigv.
clc;
% Simulation of Stock price and interest rate,
N=tau; numean = exp(muj-sigmaj^2/2)-1;
gammar=0; T=N/252; dt=1/tau; TimeStep=dt;
sqrtdt = sqrt(dt);
for j=1:NS
      fdata4(1,j)=r0; fdata5(1,j)=r0;
```

```
    S(1,j)=S0;   V(1,j)=v0;
    DU = rand(N,1);  ldt = lambda*dt;
    ul = (1-ldt)/2;  ur = (1+ldt)/2;
    for  i=2:N
     fdata4(i,j)=alpha*TimeStep + (1+beta*TimeStep)*fdata4(i-1,j)+...
                 sqrt(sigmar2*fdata4(i-1,j)^(2*gammar)*...
                 TimeStep)*randn(1);
     a(i)=sigmarT2/betaT*(1-exp(-betaT*(1-i/N)));
     fdata5(i,j)=alphaT*dt + (1-betaT*dt)*fdata5(i-1,j)+dt*a(i-1)+...
                 sqrt(sigmarT2*fdata5(i-1,j)^(2*gamma)*dt)*randn(1);
     fdata51(i,j)=fdata5(i,j)/365;
     V1(i,j)=V(i-1,j)+(gamma*kappa-gamma*V(i-1,j))*dt+sigV*...
             sqrt(V(i-1,j))*sqrtdt*randn(1);
     V(i,j)=(V1(i,j))^2;
     k(i,j)=fdata51(i,j)-0.5*sig^2*sqrt(V(i,j))-lambda*numean ;
     d(i,j)=rhov*sqrtdt*randn(1)+sqrt(1-rhov^2)*sqrtdt*randn(1);
     S(i,j)= S(i-1,j)*(1+(k(i,j)*dt)+ sig*sqrt(V(i,j))*d(i,j));
     nu(i,j)=exp(sigmaj*randn(1)+muj)-1;
     if DU(i) <= ur && DU(i) >= ul % Get jump if prob. in [ul,ur]:
           S(i,j) = S(i,j) + nu(i,j)*S(i-1,j);
     end %if
    end %i
    P1(j)=sum(fdata51(1:N,j)*dt);
    P2(j)=exp(P1(j));
    MXS(j)=max(S(N,j)-K,0);
end%j
% Monte Carlo Simulation for Bond price and Call option
Bond=mean(P2);
C=Bond*MXS;
Call=Bond*mean(MXS);
% calculate Standard deviation and standard error
```

```
for i=1:NS
    CCall(i)=Call;
    BBond(i)=Bond;
end
diffB=P2-BBond; diffB2=diffB.^2;
diffC=C-CCall; diffC2=diffC.^2;
StdMSC=sqrt(sum(diffC2)/(NS-1));
StdMSCB=sqrt(sum(diffB2)/(NS-1));
ErrorB=StdMSCB/sqrt(NS); Error=StdMSC/sqrt(NS);
toc
% summary result
fprintf('\n==========================================================');
fprintf('\n     EUROPEAN CALL OPTION WITH SVJSI MODEL            ');
fprintf('\n==== =====================================================');
fprintf('\n Asset price at time 0 = %3.2f,... interest rate = %3.4f \n',...
        S0, r0 );
fprintf('\n Exercise price = %3.1f ,number of simulation = %3.0f \n',...
        K, NS);
fprintf('\n----------------------------------------------------------');
fprintf('\n Monte Carlo Standard deviation  standard error  \n');
fprintf('\n----------------------------------------------------------');
fprintf('\n Bond Price %f  %f   %f  %f  \n',Bond,StdMSCB,ErrorB );
fprintf('\n option Price %f %f %f  %f  \n' ,Call, StdMSC,Error);
fprintf('\n---------------------------------------------------------');
toc
```

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