CALCULUS III

(103105)

WORKBOOK

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CONTENTS

Polar Coordinates

Recall: If (x, y) are the *Cartesian coordinates*, and

(r, θ) are the *polar coordinates* of a point then

$$\boxed{x = r\cos\theta} \quad \text{and} \quad \boxed{y = r\sin\theta}$$

Also,

$$\boxed{r = \sqrt{x^2 + y^2}} \quad \text{and} \quad \boxed{\tan\theta = \frac{y}{x}} \quad (\text{if } x \neq 0)$$

Exercise 1: Find the polar coordinates of the point $P(x, y)$.
Then sketch the point using the polar coordinates.

a) $P(x, y) = \left(1, \sqrt{3}\right)$.

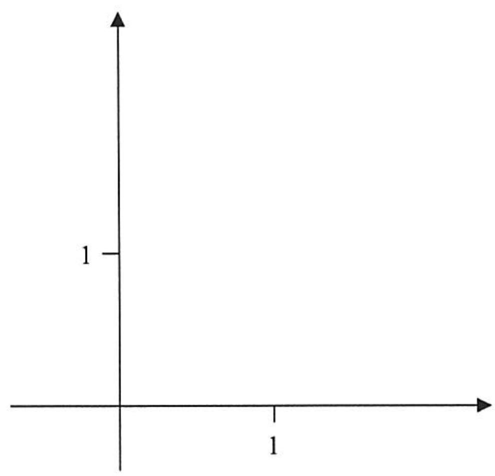
Solution: We use

$$r = \sqrt{\ldots\ldots + \ldots\ldots} = \sqrt{\ldots\ldots + \ldots\ldots} = \sqrt{\ldots\ldots} = \ldots\ldots$$

$$\tan\theta = \frac{y}{\ldots\ldots} = \frac{\sqrt{3}}{\ldots\ldots} = \ldots\ldots$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\ldots\ldots\right) = \ldots\ldots$$

$$\text{or} \quad \theta = \ldots\ldots + \pi = \ldots\ldots$$



Because the point P is in the …….. quadrant

$\Rightarrow \quad \theta = \ldots\ldots$

Answer: $(r, \theta) = (\ldots\ldots , \ldots\ldots)$

b) $P(x,y)=(-3,-3)$.

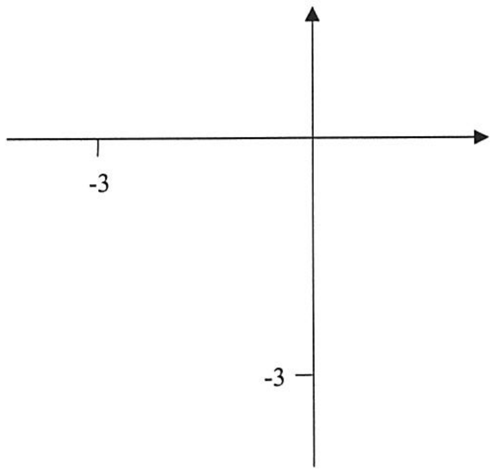
Solution: We use

$$r = \sqrt{\ldots\ldots + \ldots\ldots} = \sqrt{\ldots\ldots\ldots + \ldots\ldots\ldots} = \sqrt{\ldots\ldots} = \ldots\ldots$$

$$\tan\theta = \frac{y}{\ldots\ldots} = \frac{\quad}{\ldots\ldots} = \ldots\ldots$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\ldots\ldots\right) = \ldots\ldots$$

$$\text{or} \quad \theta = \ldots\ldots + \pi = \ldots\ldots$$



Because the point P is in the …….. quadrant

$$\Rightarrow \quad \theta = \ldots\ldots$$

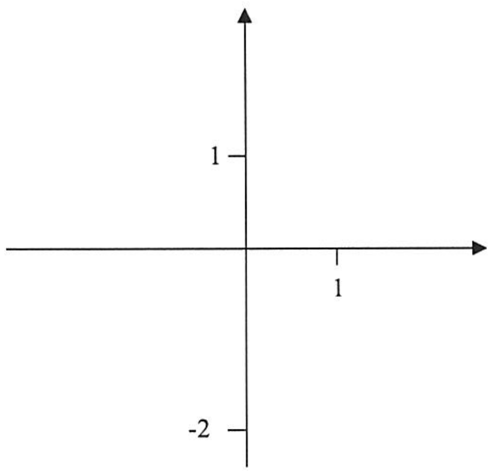
Answer: $(r,\theta) = (\ \ldots\ldots , \ldots\ldots\ )$

c) $P(x,y)=(0,-2)$.

Solution: We use

$$r = \sqrt{\ldots\ldots + \ldots\ldots} = \sqrt{\ldots\ldots + \ldots\ldots\ldots} = \sqrt{\ldots\ldots} = \ldots\ldots$$

$$\tan\theta = \frac{y}{\ldots\ldots} = \frac{\quad}{\ldots\ldots}$$



We cannot ……………………………………

Because the point P lies on the ……………………… - axis

$$\Rightarrow \quad \theta = \ldots\ldots$$

Answer: $(r,\theta) = (\ \ldots\ldots , \ldots\ldots\ )$

Exercise 2: Find the Cartesian coordinates (x, y) of the point $P(r,\theta)$. Then sketch the point using its polar coordinates.

a) $P(r,\theta)=\left(2,\dfrac{\pi}{4}\right).$

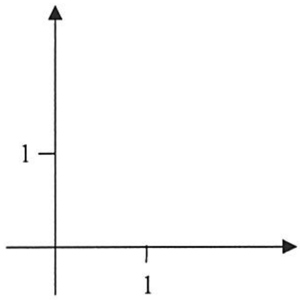
Solution: We use

x = ……… $\cos\theta$ = ……………………………

= ……………………… = ………

y = ……………… = ……………………………

= ……………………… = ………



Answer: $(x, y) = ($ ………… , ………… $)$

b) $P(r,\theta)=\left(-3,\dfrac{2\pi}{3}\right).$

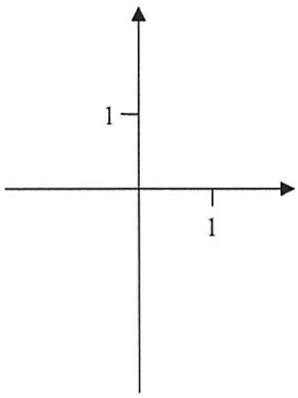
Solution: We use

x = ……… $\cos\theta$ = ……………………………

= ……………………… = ………

y = ……………… = ……………………………

= ……………………… = ………



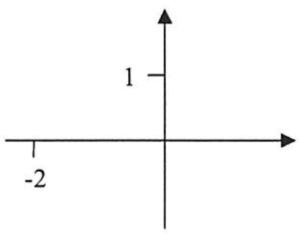
Answer: $(x, y) = ($ ………… , ………… $)$

c) $P(r,\theta)=(-2,0).$

Solution: We use

x = …… $\cos\theta$ = ……………………… = ………

y = …… $\sin\theta$ = ……………………… = ………



Answer: $(x, y) = ($ ………… , ………… $)$

Exercise 3: Change the equation to Cartesian coordinates, and sketch its graph.

a) $\theta = \dfrac{\pi}{6}$.

Solution: The equation

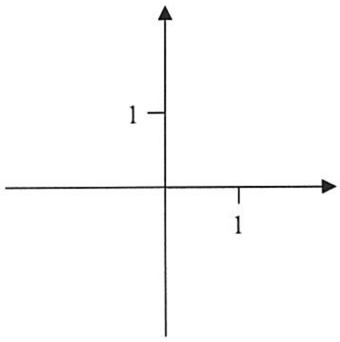
$$\tan\theta = \frac{}{\dots\dots\dots}$$

gives

$$y = \dots\dots\dots\dots\dots\dots$$

Here,

$$y = \dots\dots\dots\dots\dots\dots = \dots\dots\dots\dots$$



Answer: $y = \dots\dots\dots$ (The graph is a ……………………………………………)

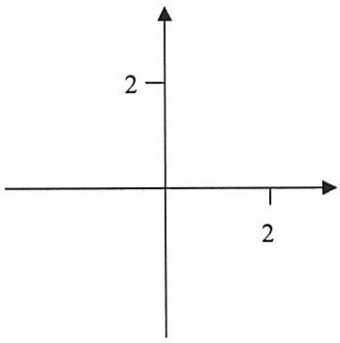
b) $r = 2$.

Solution: We use

$$r = \dots\dots\dots\dots\dots\dots\dots$$

$$\Rightarrow \sqrt{\dots\dots\dots\dots\dots\dots\dots\dots} = \dots\dots\dots$$

or $\dots\dots\dots\dots\dots\dots\dots\dots = \dots\dots\dots$



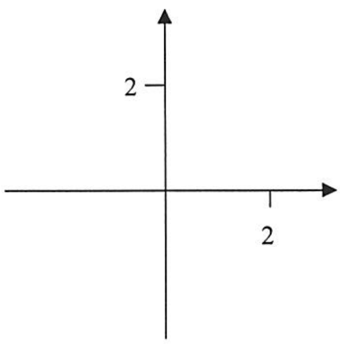
Answer: $\dots\dots\dots\dots\dots\dots = \dots\dots$ (The graph is a …………………………)

c) $r = 2\sec\theta$.

Solution: We multiply by ……………

$$\Rightarrow r\ \dots\dots\dots\dots = 2$$

$$\Rightarrow \dots\dots\dots\dots = 2$$



Answer: $\dots\dots\dots\dots = 2$

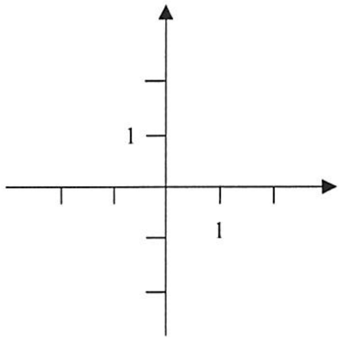
(The graph is a ………………………………)

d) $r^2 \sin 2\theta \;=\; 2\,.$ (1)

Solution: Recall that $x = r\cos\theta$ and $y = r\sin\theta\,.$

Can we discover these in (1) ?

Use the identity: $\sin 2\theta \;=\;$ ……………………………….



Then (1) becomes

r^2 ………………………. $\cos\theta \;=\; 2$

$(r$……………..$)\,(r$……………..$) \;=\; 1$

……………………………. $= 1$

<u>Answer:</u> $y \;=\;$ ………….. (The graph is a ………………………………)

e) $r \;=\; 8\sin\theta\,.$ (2)

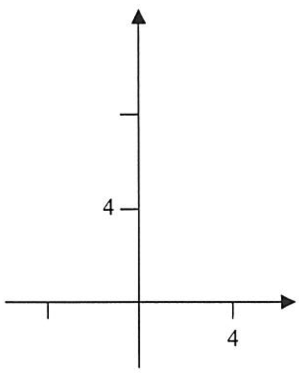
Solution: We multiply by r :

Then (2) becomes

………………. $= \;8r\sin\theta$

…………………………………. $= \;8$ ………

Complete the square.



……………………………………………. $= \;0$

……………………………………………. $=$ …..

<u>Answer:</u> $x^2 + (y -$……………$)^2 \;=\;$ ………

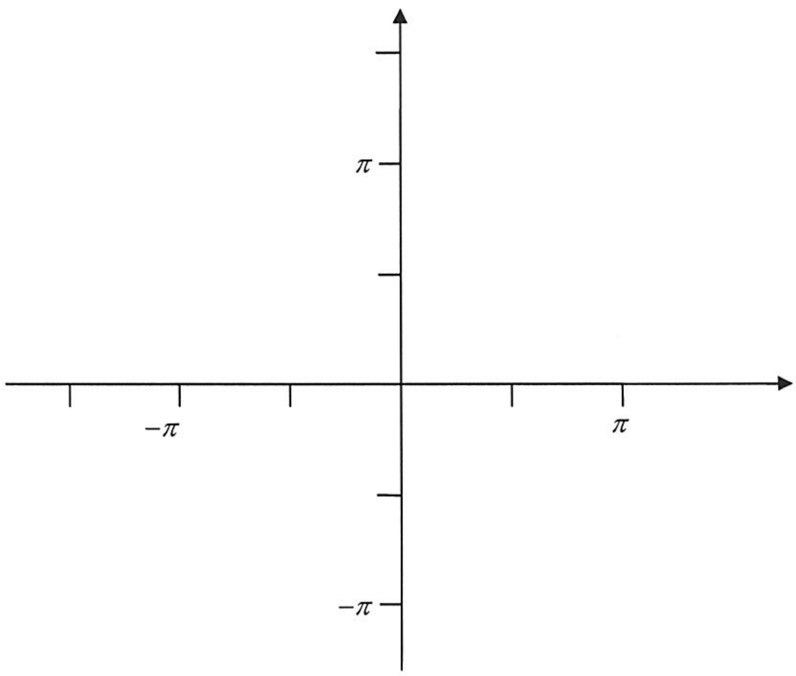
(The graph is a ………………………… with center ………………….. and radius ……….)

Exercise 4: Sketch the graph of the given equation in polar coordinates.

a) $r = \dfrac{\theta}{2} \qquad (\theta \geq 0)$.

Solution: We make a table of values and sketch :

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{5\pi}{2}$	3π
r												
$\approx r$		0.39		1.57			1.96			2.75		



(The graph is a ....)

b) $r = 1+\cos\theta$.

Solution: Let us test for *symmetry*:

About x-axis ?

$$r(-\theta) = 1+\cos(\ldots\ldots) = 1+\cos(\ldots\ldots) = \ldots\ldots\ldots$$

$$\Rightarrow \quad \ldots\ldots\ldots$$

About y-axis ?

$$r(\pi-\theta) = 1+\cos(\ldots\ldots\ldots) = 1_{\ldots}\cos(-\theta)$$

$$= 1_{\ldots}\cos(-\theta) = 1_{\ldots}\cos(\theta) \neq \ldots\ldots\ldots$$

$$\Rightarrow \quad \ldots\ldots\ldots$$

About origin ?

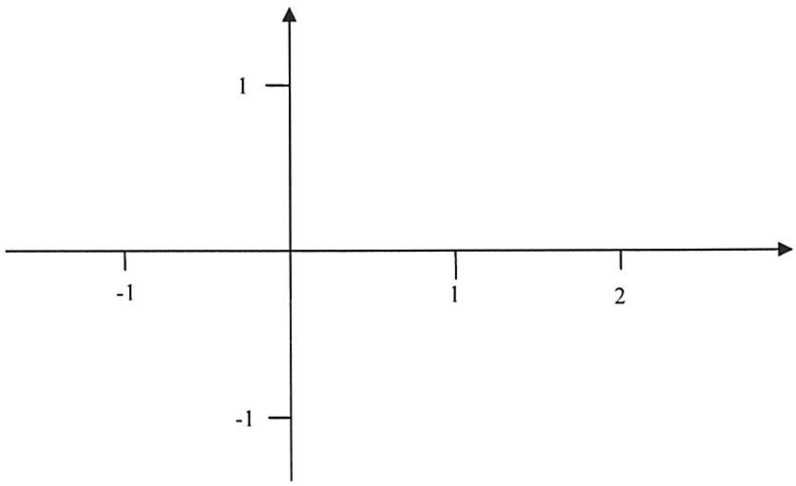
$$r(\theta+\pi) = 1+\cos(\ldots\ldots\ldots) = 1_{\ldots}\cos(\theta) \neq \ldots\ldots\ldots$$

$$\Rightarrow \quad \ldots\ldots\ldots$$

$\Rightarrow$ There is symmetry about ....

Table of values (by symmetry, $0 \le \theta \le \pi$ is enough) :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r		$1+\frac{\sqrt{3}}{2}$						$1-\frac{\sqrt{3}}{2}$	
$\approx r$	2	1.85					0.29		



(The graph is a ....)

c) $r = \sin 4\theta$.

Solution: Let us test for *symmetry*:

About x-axis ?

$$r(-\theta) = \sin(\ldots\ldots\ldots) = \ldots\ldots\ \sin(\ldots\ldots) = \ldots\ r(\theta)$$

$$\Rightarrow \quad \ldots\ldots\ldots$$

About y-axis ?

$$r(\pi - \theta) = \sin(\ldots\ldots\ldots\ldots\ldots) = \sin(\ldots\ldots\ldots\ldots\ldots)$$

$$= \sin(\ldots\ldots\ldots) = \ldots\ \sin(\ldots\ldots) = \ldots\ r(\theta)$$

$$\Rightarrow \quad \ldots\ldots\ldots$$

About origin ?

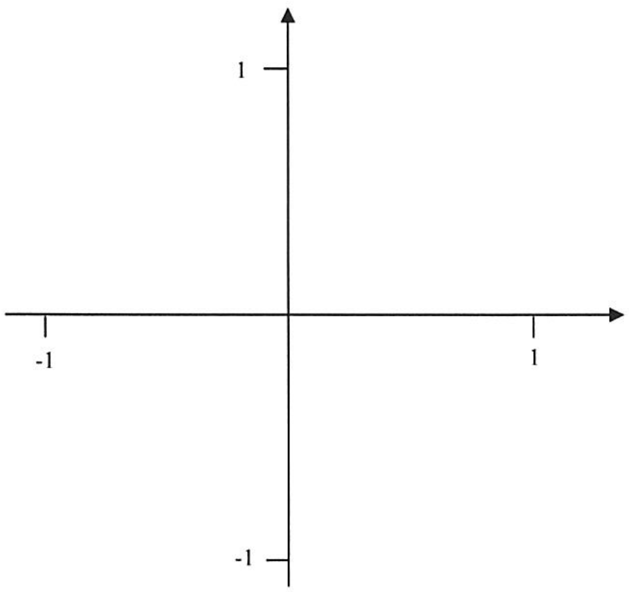
$$r(\theta + \pi) = \sin(\ldots\ldots\ldots\ldots\ldots) = \sin(\ldots\ldots\ldots\ldots\ldots)$$

$$= \sin(\ldots\ldots\ldots) = \ldots\ r(\theta) \quad \Rightarrow \quad \ldots\ldots\ldots$$

$\Rightarrow$ There is symmetry about ....

Table of values (because of the angle 4θ, $0 \le \theta \le \pi/2$ is sufficient) :

θ	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$	$\frac{5\pi}{16}$	$\frac{3\pi}{8}$	$\frac{7\pi}{16}$	$\frac{\pi}{2}$
r		$\frac{\sqrt{2}}{2}$						$-\frac{\sqrt{2}}{2}$	
$\approx r$		0.71						-0.71	



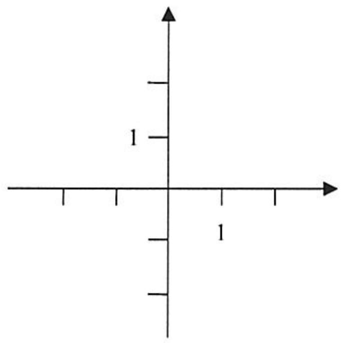
(The graph is a

....................................)

Exercise 4: Sketch the given set in the xy–plane.

a) $\left\{ (r,\theta) \;:\; 0 \le r \le 2,\; \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \right\}.$

Solution:



(The set is a ....)

b) $\left\{ (r,\theta) \;:\; 0 \le \theta \le \frac{\pi}{4},\; 0 \le r \le 3\sec\theta \right\}.$

Solution:

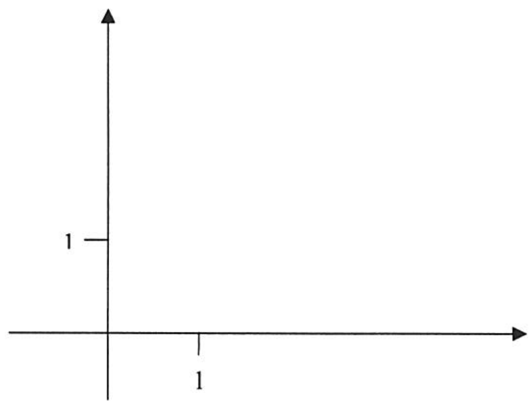
The range of r depends on the value of θ ! The largest value of r is

$$r = 3\sec\theta$$

r................ $= 3$

.... $= 3$

(a .... line)



(The set is a ....)

Additional Exercises:

1) Sketch the graph of the given polar equation

a) $r = -2$

b) $\theta = -\dfrac{\pi}{6}$

c) $r = 4\csc\theta$

d) $r = 8\cos 3\theta$

e) $r = 2 - 2\sin\theta$

f) $r = 2 + 4\sin\theta$

2) Sketch the equation and change it to polar form.

a) $x = -3$

b) $y = 2$

c) $x^2 = 8y$

d) $7 = 6x$

e) $xy = 8$

f) $y^2 - x^2 = 4$

3) Change the equation to Cartesian coordinates and sketch its graph.

a) $r = -\csc\theta$

b) $r = -2\sec\theta$

c) $r^2\cos 2\theta = 1$

d) $r(\sin\theta - 2\cos\theta) = 6$

e) $r(\sin\theta + r\cos^2\theta) = 1$

f) $r = \tan\theta$

Quadric Surfaces

Recall: Equations of some quadric surfaces:

$\boxed{x^2 + y^2 + z^2 = r^2}$ sphere with center $(0,0,0)$ and radius r

$\boxed{z = a(x^2 + y^2)}$ paraboloid

$\boxed{z = \sqrt{a(x^2 + y^2)}}$ cone

Another (non-quadric) surface is

$\boxed{ax + by + cz = d}$ plane with normal vector $a\vec{i} + b\vec{j} + c\vec{k}$

Exercise 1: Find the equation of the plane through the points $(2,0,0)$, $(0,1,0)$ and $(0,0,3)$.

Solution: These three points are very simple; we need not use the normal vector !

The *general* equation of a plane is

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = d$$

Substitute $(x, y, z) = (2,0,0)$:

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = d \quad \Rightarrow \quad a = \ldots\ldots\ldots\ldots\ldots$$

Substitute $(x, y, z) = (0,1,0)$:

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = d \quad \Rightarrow \quad b = \ldots\ldots\ldots\ldots\ldots$$

Substitute $(x, y, z) = \ldots\ldots\ldots\ldots\ldots$:

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = d \quad \Rightarrow \quad c = \ldots\ldots\ldots\ldots\ldots$$

We can choose d freely. Choose $d = \ldots\ldots\ldots\ldots\ldots$.

Then $a = \ldots\ldots\ldots\ldots\ldots$, $b = \ldots\ldots\ldots\ldots\ldots$, $c = \ldots\ldots\ldots\ldots\ldots$

<u>Answer:</u> The equation of the plane is $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$

Exercise 2: Sketch the part of the plane

$$4x + 2y + 3z = 4 \qquad (1)$$

which is in the *first octant.* Then find the *trace* of this plane in the xy-plane.

Solution: To sketch the plane, we need *3 points.*

Set $x = y = 0$: $3z =$ ……… $\Rightarrow$ $z =$ …………… (z-intercept)

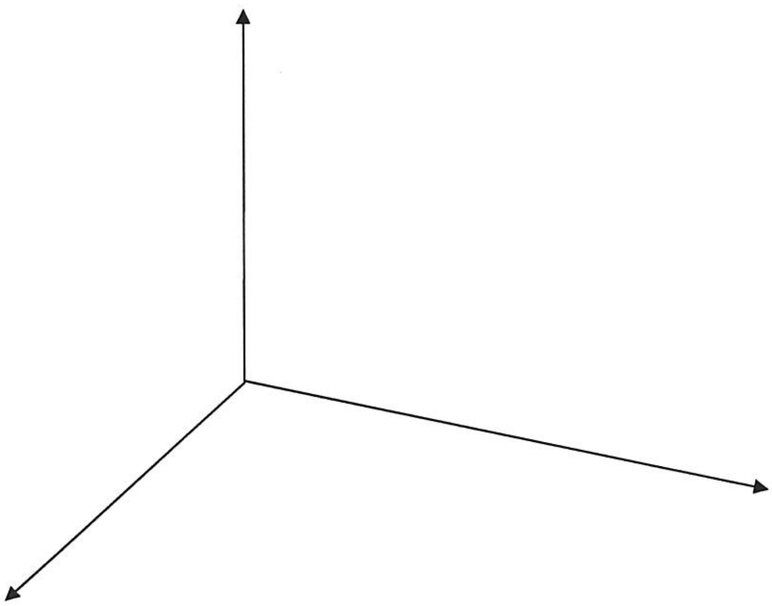
Set $x = z = 0$: ……… = ……… $\Rightarrow$ $y =$ …………… (y-intercept)

Set $y = z = 0$: ……… = ……… $\Rightarrow$ $x =$ …………… (x-intercept)

Now find the trace in the xy-plane. In (1), we set ………… $= 0$:

…………………………………………… $= 4$

<u>**Answer:**</u> The trace is the line …………………………………………… $= 2$



Exercise 3: Sketch and classify the graph of $z = 4 - x^2 - y^2$.

Solution: First find the traces in the coordinate planes:

Trace in yz-plane: (Set ……… $= 0$) $\Rightarrow$ $z =$ ………………….

(…………………………………)

Trace in xz-plane: (Set ……… $= 0$) $\Rightarrow$ $z =$ ………………….

(…………………………………)

Trace in horizontal plane $z = k$:

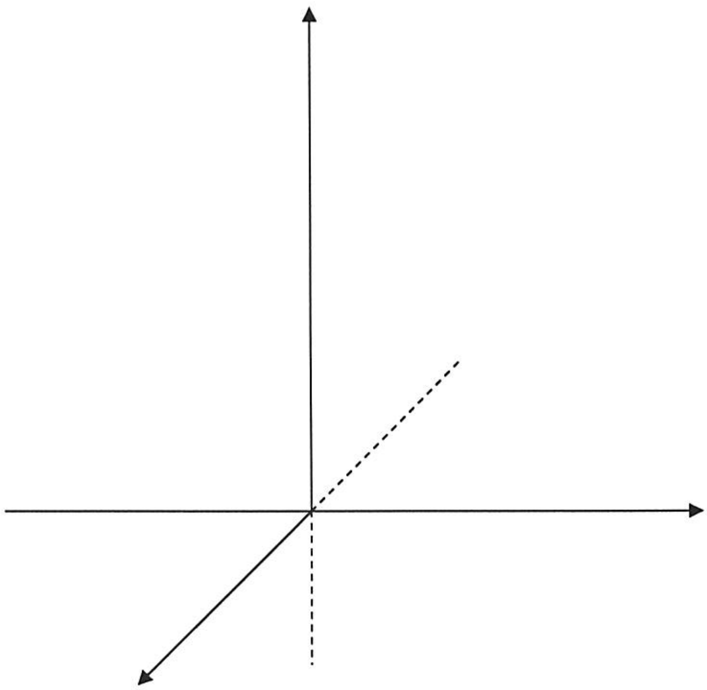
$\Rightarrow$ $k =$ …………………………….

$\Rightarrow$ $x^2 + y^2 =$ …………………… (……………………, radius …………)

Answer: The graph is a …………………………………………

First sketch the parabola $z =$ …………………. in the yz-plane.

Then rotate the parabola about the z-axis.



Exercise 4: Sketch the graph of $z = 1 - x^2$. (1)

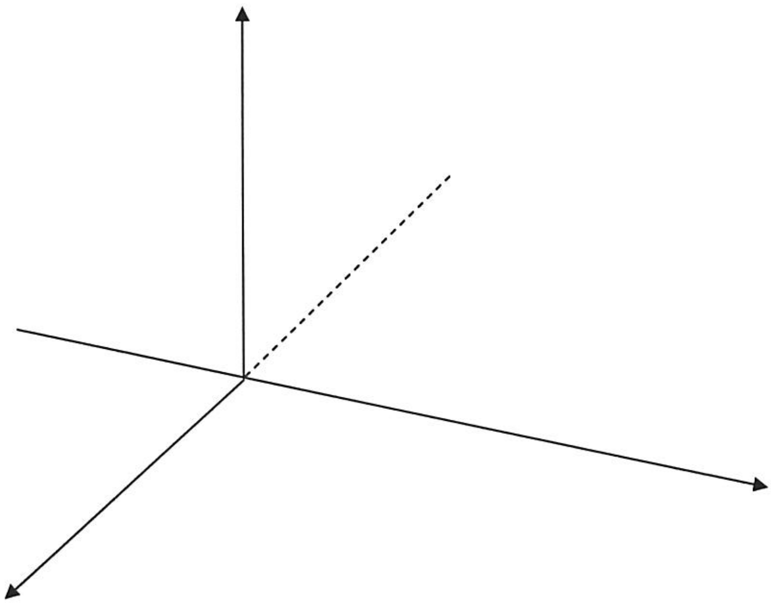
Solution: In *two dimensions*, the graph is a ....

In *three dimensions*, the graph is a ....

parallel to .... (the variable .... is missing)

First sketch the parabola $z = 1 - x^2$ in the xz-plane.

Then shift in direction of the y-axis.



Additional Exercises:

1) Sketch and classify the following surfaces.

a) $x^2 + y^2 = 9$

b) $x^2 = 9z$

c) $z = 3(x^2 + y^2)$

d) $z = \sqrt{3(x^2 + y^2)}$

e) $xz = 1$

f) $z^2 = \dfrac{x^2 + y^2}{4}$

2) Sketch the part of the plane which lies in the first octant. Then find its trace in the xy-plane.

a) $3x + 4y + 2z = 12$

b) $x + y + 2z = 1$

Iterated Integrals

Exercise 1: Compute the following iterated integrals:

a)
$$\int_0^1 \int_0^2 (x+y^2)\,dx\,dy \quad = \quad \int_0^1 \Big[\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \Big]_{x=\ldots\ldots}^{x=\ldots\ldots} dy$$

$$= \int_0^1 \Big(\big[\;\ldots\ldots\ldots\ldots\;\big] - \big[\;\ldots\ldots\ldots\ldots\;\big] \Big)\, dy$$

$$= \int_0^1 \Big(\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \Big)\, dy \quad = \quad \Big[\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \Big]_{\ldots\ldots}^{\ldots\ldots}$$

$$= \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad = \quad \underline{\underline{\ldots\ldots\ldots\ldots}}$$

b)
$$\int_0^\pi \int_0^2 x\sin(xy)\,dy\,dx \quad = \quad \int_0^\pi \Big[\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \Big]_{y=\ldots\ldots}^{y=\ldots\ldots} dx$$

> Recall: $\int \sin ky\,dy \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
>
> $\Rightarrow \quad \int \sin xy\,dy \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$= \int_0^\pi \Big(\;\ldots\ldots\ldots\ldots \;-\; \ldots\ldots\ldots\ldots\; \Big)\, dy$$

$$= \Big[\;\ldots\ldots\ldots\ldots\ldots\ldots \;-\; \ldots\ldots\ldots\ldots\ldots\ldots\; \Big]_{\ldots\ldots}^{\ldots\ldots}$$

$$= \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad = \quad \underline{\underline{\ldots\ldots\ldots\ldots}}$$

c) $$\int_0^1 \int_x^{x^2} \frac{y}{x}\, dy\, dx = \int_0^1 \frac{1}{x} \int_x^{x^2} y\;dy\,dx \qquad \text{(think: } x = \text{constant inside)}$$

$$= \int_0^1 \frac{1}{x}\left[\;\ldots\ldots\ldots\ldots\;\right]_{y=\ldots\ldots}^{y=\ldots\ldots} dx = \int_0^1 \frac{1}{x}\left(\;\ldots\ldots\ldots\ldots\ldots\ldots\;\right) dx$$

$$= \int_0^1 \left(\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\right) dx = \left[\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\right]_{\ldots\ldots}^{\ldots\ldots}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad = \underline{\underline{\ldots\ldots\ldots\ldots}}$$

d) $$\int_1^3 \int_3^2 \frac{1}{(x+y)^2}\, dy\, dx = \int_1^2 \int_{\ldots\ldots}^{\ldots\ldots} \frac{1}{\ldots\ldots}\, d_{\ldots}\, dx$$

$u =$ …………	If $y = 3$ then $u =$ …………
$du =$ …………	If $y = 4$ then $u =$ …………

$$= \int_1^2 \left[\;\ldots\ldots\ldots\;\right]_{u=\ldots\ldots}^{u=\ldots\ldots} dx = \int_1^2 \left(\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\right) dx$$

$$= \left[\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;\right]_{\ldots\ldots}^{\ldots\ldots}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad = \underline{\underline{\ldots\ldots\ldots\ldots}}$$

e) $$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{y^3} \sin\left(\frac{x}{y}\right) dx\,dy = \int_0^1 \int_{\dots\dots}^{\dots\dots} \dots\dots\dots \sin u \; du\,dy$$

$u = \dots\dots\dots$	If $x = 0$ then $u = \dots\dots\dots$
$du = \dots\dots\dots$	If $x = y^3$ then $u = \dots\dots\dots$

$$= \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \Big[\dots\dots\dots\dots\Big]_{u=\dots\dots}^{u=\dots\dots} dy = \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \Big(\dots\dots\dots\dots\dots\dots\Big) dy$$

$$= \Big[\dots\dots\dots\dots\dots\dots\dots\dots\Big]_{\sqrt{\pi}}^{\sqrt{2\pi}}$$

$$= \Big(\dots\dots\dots\dots\dots\Big) - \Big(\dots\dots\dots\dots\dots\Big) = \underline{\dots\dots\dots\dots}$$

f) $$\int_0^1 \int_0^{y^2} 3y^3 e^{xy} \; dx\,dy = \int_0^1 3y^3 \int_0^{y^2} \dots\dots\dots \; dx\,dy$$

Recall:	$\int e^{kx}\,dx = \dots\dots\dots\dots$ $\Rightarrow$	$\int e^{xy}\,dx = \dots\dots\dots\dots$

$$= \int_0^1 3y^3 \Big[\dots\dots\dots\Big]_{x=\dots\dots}^{x=\dots\dots} dy = \int_0^1 3y^3 \Big(\dots\dots - \dots\dots\Big) dy$$

$$= \int_0^1 \Big(\dots\dots\dots\dots - \dots\dots\dots\Big) dy = \Big[\dots\dots\dots - \dots\dots\Big]_0^1$$

$$\int 3y^2 e^{y^3}\,dy = \int e^{y^3}\,d(y^3) = \dots\dots\dots\dots$$

$$= \Big(\dots\dots\dots\Big) - \Big(\dots\dots\dots\Big) = \Big(\dots\dots\dots\Big) = \underline{\dots\dots\dots\dots}$$

g) $$\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dx\,dy \quad = \quad \int_0^1\int_{\ldots}^{\ldots} \frac{1}{2}\,\frac{\qquad}{\ldots\ldots}\,du\,dy$$

$u = \ldots\ldots\ldots\ldots$	If $x=0$ then $u = \ldots\ldots\ldots$
$du = \ldots\ldots\ldots\ldots$	If $x=1$ then $u = \ldots\ldots\ldots$

$$= \int_0^1\int_{\ldots}^{\ldots} y\,\frac{1}{2}\,\frac{\qquad}{\ldots\ldots}\,du\,dy \quad = \quad \int_0^1 \Big[\ldots\ldots\ldots\Big]_{u=\ldots\ldots}^{u=\ldots\ldots}\,dy$$

$$= \int_0^1 \Big(\ldots\ldots\ldots\ldots \quad - \quad \ldots\ldots\ldots\ldots\Big)\,dy$$

$u = \ldots\ldots\ldots$	$u = \ldots\ldots\ldots$
$du = \ldots\ldots\ldots$	$du = \ldots\ldots\ldots$

$$= \int_{\ldots}^{\ldots} \ldots\ldots\ldots\,du \quad - \quad \int_{\ldots}^{\ldots} \ldots\ldots\ldots\,du$$

$$= \Big[\ldots\ldots\ldots\Big]_{\ldots}^{\ldots} \quad - \quad \Big[\ldots\ldots\ldots\Big]_{\ldots}^{\ldots}$$

$$= \Big(\ldots\ldots\ldots\ldots\Big) \quad - \quad \Big(\ldots\ldots\ldots\ldots\Big)$$

$$= \ldots\ldots\ldots\ldots\ldots \quad = \quad \ldots\ldots\ldots\ldots\ldots \quad = \quad \underline{\underline{\ldots\ldots\ldots}}$$

Recall:

$$\boxed{\int_a^b \int_c^d f(x)\, g(y)\, dy\, dx \; = \; \int_a^b f(x)\, dx \; \cdot \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Constant limits | Product

Exercise 2: Compute the following iterated integrals:

a) $$\int_0^{\pi} \int_0^{\pi/2} \sin x \cos y \, dy\, dx \; = \; \int_0^{\pi} \ldots\ldots\ldots\ldots\ldots\, dx \cdot \int_0^{\pi/2} \ldots\ldots\ldots\ldots\ldots\, dy$$

$$= \Big[\ldots\ldots\ldots\ldots\ldots\Big]_0^{\pi} \cdot \Big[\ldots\ldots\ldots\ldots\ldots\Big]_{\ldots\ldots}^{\ldots\ldots}$$

$$= \Big(\ldots\ldots\ldots - \ldots\ldots\ldots\Big) \cdot \Big(\ldots\ldots\ldots - \ldots\ldots\ldots\Big)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; = \; \underline{\underline{\ldots\ldots\ldots}}$$

b) $$\int_0^1 \int_0^2 x y e^{x^2+y^2} \, dx\, dy \; = \; \int_0^1 \int_0^2 x y e^{x^2} \ldots\ldots\ldots\, dx\, dy$$

$$= \int_0^1 \ldots\ldots\ldots\ldots\ldots\, d\ldots\ldots \; \cdot \; \int_0^{\ldots\ldots} \ldots\ldots\ldots\ldots\ldots\, d\ldots\ldots$$

$$\boxed{\text{Use:} \quad \int 2xe^{x^2} dx \; = \; \int e^{x^2} d(x^2) \; = \; \ldots\ldots\ldots\ldots\ldots\ldots}$$

$$= \Big[\ldots\ldots\ldots\ldots\ldots\Big]_0^{\ldots\ldots} \cdot \Big[\ldots\ldots\ldots\ldots\ldots\Big]_0^{\ldots\ldots}$$

$$= \frac{1}{4}\Big(\ldots\ldots - \ldots\ldots\Big) \cdot \Big(\ldots\ldots - \ldots\ldots\Big) \; = \; \underline{\underline{\ldots\ldots\ldots\ldots\ldots\ldots}}$$

Additional Exercises:

1) Find the following iterated integrals:

a) $\displaystyle\int_0^1\int_0^2 (x+2)\,dy\,dx$

b) $\displaystyle\int_2^4\int_1^3 xy^3\,dx\,dy$

c) $\displaystyle\int_{-\pi}^{\pi}\int_{-e}^{e} dr\,d\theta$

d) $\displaystyle\int_0^3\int_0^1 y\sqrt{x+y^2}\,dy\,dx$

e) $\displaystyle\int_1^2\int_0^2 \frac{x}{(1+xy)^2}\,dy\,dx$

f) $\displaystyle\int_0^{\ln 2}\int_0^1 xye^{x^2y}\,dx\,dy$

g) $\displaystyle\int_{-\pi/4}^{\pi/4}\int_0^{\pi/4} \cos(x+y)\,dx\,dy$

h) $\displaystyle\int_1^e\int_0^{\pi/2} \frac{\sin y}{x}\,dy\,dx$

i) $\displaystyle\int_0^1\int_{x^2}^{x} xy^3\,dy\,dx$

j) $\displaystyle\int_1^2\int_y^{3-y} y\,dx\,dy$

k) $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{1-y^2/4}} y\,dx\,dy$

l) $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}}\int_0^{x^3} \sin\left(\frac{y}{x}\right)dy\,dx$

m) $\displaystyle\int_{\pi/2}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\left(\frac{y}{x}\right)dy\,dx$

n) $\displaystyle\int_0^{\pi/2}\int_0^{\sin y} e^x\cos y\,dx\,dy$

o) $\displaystyle\int_1^2\int_0^{y^2} e^{x/y^2}\,dx\,dy$

p) $\displaystyle\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x+y)\,dy\,dx$

q) $\displaystyle\int_0^1\int_0^x y\sqrt{x^2-y^2}\,dy\,dx$

Double Integrals over Rectangles

Recall: (Fubini's Theorem for Rectangles)

If $R = [a,b]\times[c,d] = \{ (x,y) \; : \; a \le x \le b, \;\; c \le y \le d \}$ then

$$\boxed{\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

Exercise 1: Find $\displaystyle\iint\limits_R 3x^2y\,dA$ where R is the rectangle $[0,2]\times[0,4]$.

Solution: We choose $dA = dy\,dx$ (there is no specific reason !)

$$\iint\limits_R 3x^2y\,dA = \int_0^{\dots\dots} \int_0^{\dots\dots} 3x^2y\,dy\,dx = \int_0^{\dots\dots} \dots\dots\dots\dots \,dx \cdot \int_0^{\dots\dots} \dots\dots\dots \,dy$$

$$= \Big[\,\dots\dots\dots\,\Big]_0^{\dots\dots} \cdot \Big[\,\dots\dots\dots\,\Big]_0^{\dots\dots} = (\dots - \dots)\cdot(\dots - \dots) = \underline{\dots\dots\dots}$$

Exercise 2: Find $\displaystyle\iint\limits_R (x\sin y - y\sin x)\,dA$ where R is the rectangle $\left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{3}\right]$.

Solution: We choose $dA = dx\,dy$ (there is no specific reason !)

$$\iint\limits_R (x\sin y - y\sin x)\,dA = \int_0^{\dots\dots\dots} \int_0^{\dots\dots\dots} (x\sin y - y\sin x)\,dx\,dy$$

$$= \int_0^{\dots\dots} \Big[\,\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\,\Big]_{x=0}^{x=\dots\dots} dy$$

$$= \int_0^{\dots\dots} \Big\{ (\,\dots\dots\dots\dots\dots\dots\dots\dots\dots\,) - (\,\dots\dots\dots\dots\dots\,) \Big\}\,dy$$

$$= \int_{0}^{\dots\dots} (\dots\dots\dots\dots\dots\dots) \, dy = \Big[\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \Big]_{y=0}^{\dots\dots}$$

$$= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots = \dots\dots\dots\dots\dots\dots = \dots\dots\dots\dots = \underline{\underline{\dots\dots\dots\dots}}$$

Exercise 3: Find $I = \int_{0}^{1} \int_{0}^{\sqrt{\pi}} y^3 \sin\left(xy^2\right) dy \, dx$.

Solution: The inner integral,

$$\int_{0}^{\sqrt{\pi}} y^3 \sin\left(xy^2\right) dy$$

is *very difficult* to evaluate ! We try to change the order of integration:

$$I = \int_{0}^{1} \int_{0}^{\sqrt{\pi}} y^3 \sin\left(xy^2\right) dy \, dx = \int_{0}^{\sqrt{\pi}} \int_{0}^{\dots\dots\dots} y^3 \sin\left(xy^2\right) d_{\dots\dots} \, dy$$

Recall: $\int \sin kx \, dx = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$$= \int_{0}^{\sqrt{\pi}} \Big[y^3 \dots\dots\dots\dots\dots\dots\dots\dots \Big]_{x=\dots\dots}^{x=\dots\dots} dy = \int_{0}^{\sqrt{\pi}} \left(\dots\dots - \dots\dots\dots\dots\dots\dots \right) dy$$

$$= \int_{0}^{\sqrt{\pi}} \left(\dots\dots\dots - \dots\dots\dots\dots\dots\dots \right) dy = \Big[\dots\dots\dots - \dots\dots\dots\dots\dots\dots\dots \Big]_{0}^{\dots\dots}$$

$$= (\dots\dots\dots\dots\dots\dots) - (\dots\dots\dots\dots\dots\dots) = \underline{\underline{\dots\dots\dots\dots}}$$

Exercise 4: Find the volume of the solid which lies above the rectangle $[0,1] \times [0,3]$ in the xy-plane, and below the plane $2x + y + 2z = 6$.

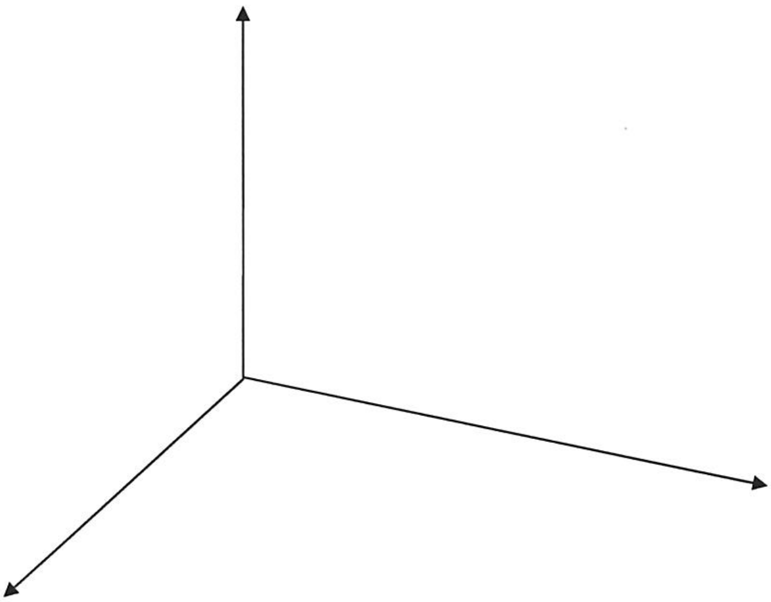
Solution: First sketch the solid: Find the intercepts with the coordinate axis.

x-intercept: set $y = z = 0$: ……… = ……… $\Rightarrow$ $x =$ ……

y-intercept: set ……=……$= 0$: ……… = ……… $\Rightarrow$ $y =$ ……

z-intercept: set ……=……$= 0$: ……… = ……… $\Rightarrow$ $z =$ ……

Now sketch the plane, and the solid.



The volume is

$$V = \iint_R z\, dA = \int_{\dots}^{\dots} \int_{\dots}^{\dots} \left(\dots\dots\dots\dots\dots\dots \right) dy\, dx$$

$$= \int_{\dots}^{\dots} \Big[\dots\dots\dots\dots\dots\dots \Big]_{y=\dots}^{y=\dots} dx$$

$$= \int_{\dots}^{\dots} \Big(\Big[\dots\dots\dots\dots\dots \Big] - \Big[\dots \Big] \Big) dx = \int_{\dots}^{\dots} \Big(\dots\dots\dots \Big) dx$$

$$= \Big[\dots\dots\dots\dots\dots \Big]_{\dots}^{\dots} = \Big(\dots\dots\dots \Big) - \Big(\dots \Big) = \underline{\underline{\dots\dots\dots}}$$

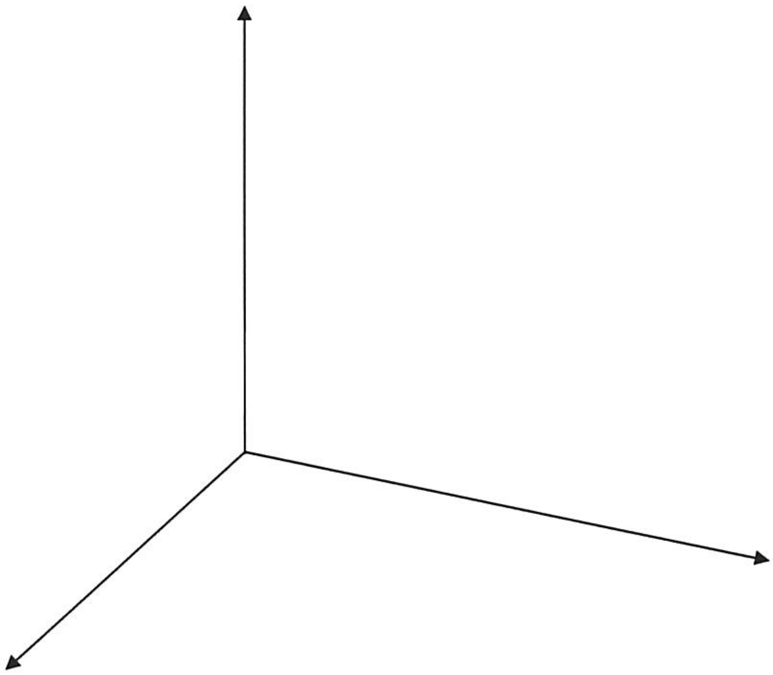
Exercise 5: Find the volume of the solid which lies above the rectangle

$$R = \{ (x,y) : 0 \le x \le 2,\ 0 \le y \le 3 \}$$

in the xy-plane, and below the surface $z = 9 - x^2$.

Solution: First sketch the solid:

The surface $z = 9 - x^2$ is a ....



The volume is

$$V = \iint_R z\, dA = \int_{\ldots\ldots}^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots\ldots\ldots \right) dx\, dy$$

$$= \int_0^{\ldots\ldots} \ldots\ldots\ldots\ldots\ldots\ldots \, dx \cdot \int_0^{\ldots\ldots} \ldots\ldots\ldots\ldots\ldots\ldots \, dy$$

$$= \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_0^{\ldots\ldots} \cdot \left(\ldots\ldots\ldots\ldots\ldots\ldots \right)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots \cdot \ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots\ldots}}$$

Additional Exercises:

1) Find $\iint\limits_R f(x,y)\,dA$ where R is as given

a) $\iint\limits_R 4x^3y\,dA, \quad R=[-1,2]\times[-2,2]$

b) $\iint\limits_R x\sqrt{1-x^2}\,dA, \quad R=[0,1]\times[-2,3]$

c) $\iint\limits_R (y+2x)\,dA, \quad R=[-1,2]\times[-1,4]$

d) $\iint\limits_R (12xy^2-8x^3)\,dA, \quad R=[1,2]\times[-1,2]$

2) Find the volume of the solid which lies above the rectangle R in the xy-plane, and below the given surface $z=f(x,y)$.

a) $R=[0,2]\times[0,2], \qquad z=8-x-2y$

b) $R=[1,3]\times[0,2], \qquad z=3x^3-3x^2y$

c) $R=[1,2]\times[1,2], \qquad z=\dfrac{1}{(x+y)^2}$

3) Evaluate the iterated integrals by changing the order of integration.

a) $\iint\limits_R 4x^3y\,dA, \quad R=[-1,2]\times[-2,2]$

b) $R=[0,1]\times[-2,3]$

Double Integrals over Arbitrary Regions

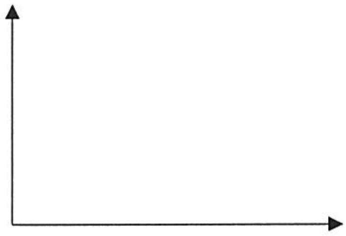
Recall:

Type I Region

$$a \le x \le b$$
$$g(x) \le y \le h(x)$$

$$\boxed{\iint_R f(x,y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx}$$

Type II Region

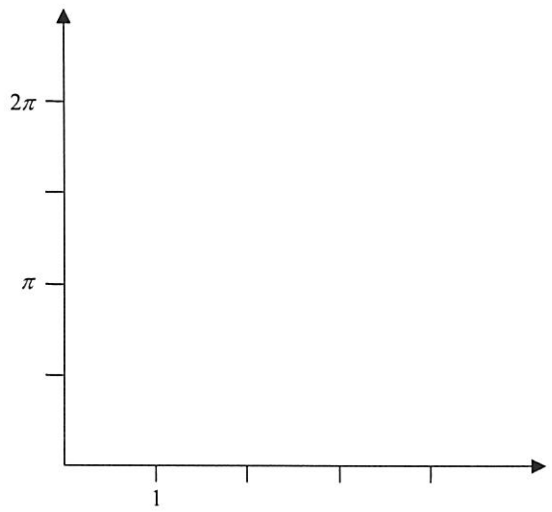


$$c \le y \le d$$
$$g(y) \le x \le h(y)$$

$$\boxed{\iint_R f(x,y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\, dx\, dy}$$

Exercise 1: Find $\displaystyle\iint_R x\cos(xy)\, dA$ where R is the region bounded by $x = 1$, $x = 2$, $y = \dfrac{2\pi}{x}$ and $y = \dfrac{\pi}{2}$.

Solution: First sketch the region R.



Type ....

$$\begin{cases} \ldots\ldots\ldots\ldots \le x \le \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \le y \le \ldots\ldots\ldots\ldots \end{cases}$$

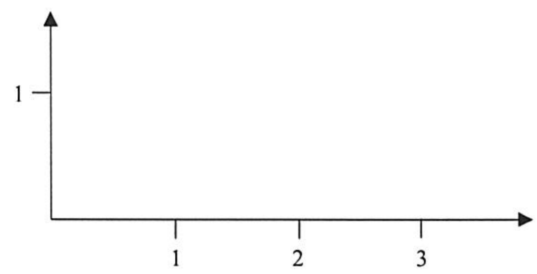
$$\iint_R x\cos(xy)\,dA \;=\; \int_{\dots\dots}^{\dots\dots}\int_{\dots\dots}^{\dots\dots} x\cos(xy)\,dy\,dx \;=\; \int_{\dots\dots}^{\dots\dots}\left[x\,\frac{\qquad\qquad}{x}\right]_{y=\dots\dots}^{y=\dots\dots} dx$$

$$= \int_{\dots\dots}^{\dots\dots}\Big(\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\Big)\,dx \;=\; \int_{\dots\dots}^{\dots\dots}\Big(\dots\dots\dots\dots\dots\dots\dots\dots\dots\Big)\,dx$$

$$= \Big[\dots\dots\dots\dots\dots\dots\dots\dots\dots\Big]_{\dots\dots}^{\dots\dots} \;=\; \dots\dots\dots\dots\dots\dots\dots\dots\dots \;=\; \underline{\underline{\dots\dots\dots\dots}}$$

Exercise 2: Find $\iint_R 2xy\,dA$ where R is the parallelogram with vertices $(0,0)$, $(1,1)$, $(2,0)$ and $(3,1)$.

Solution: Sketch the region R.



Type ....

$$\begin{cases} \dots\dots\dots \le \dots\dots \le \dots\dots\dots\dots \\ \dots\dots\dots \le \dots\dots \le \dots\dots\dots\dots \end{cases}$$

The lines on the left and right have equations:

$y = \dots\dots\dots\dots \quad \Rightarrow \quad x = \dots\dots\dots\dots\dots$

$y = \dots\dots\dots\dots \quad \Rightarrow \quad x = \dots\dots\dots\dots\dots$

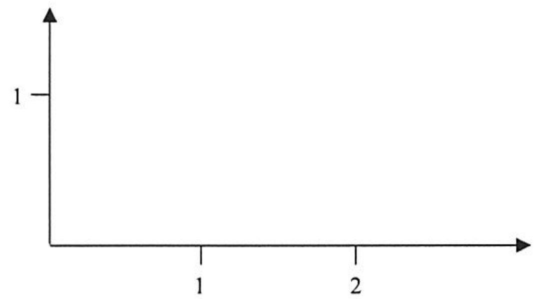
$$\iint_R 2xy\,dA \;=\; \int_{\dots\dots}^{\dots\dots}\int_{\dots\dots}^{\dots\dots} 2xy\; d_{\dots}\; d_{\dots} \;=\; \int_{\dots\dots}^{\dots\dots}\Big[\dots\dots\dots\dots\dots\Big]_{x=\dots\dots}^{x=\dots\dots} d_{\dots\dots}$$

$$= \int_{\dots\dots}^{\dots\dots}\Big(\dots\dots\dots\dots\dots\dots\dots\dots\Big)\,d_{\dots} \;=\; \int_{\dots\dots}^{\dots\dots}\Big(\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\Big)\,d_{\dots}$$

$$= \int_{\dots\dots}^{\dots\dots}\Big(\dots\dots\dots\dots\dots\Big)\,d_{\dots} \;=\; \Big[\dots\dots\dots\dots\dots\dots\dots\Big]_{\dots}^{\dots} \;=\; \dots\dots\dots\dots \;=\; \underline{\underline{\dots\dots\dots}}$$

Exercise 3: Find $\iint\limits_R \cos\left(y^2\right) dA$ where R is the triangle with vertices $(0,0)$, $(0,1)$ and $(2,1)$.

Solution: First sketch the region R.



The region R is type ....

Which one to choose ? Compare the two iterated integrals:

$$\underbrace{\int\int \cos\left(y^2\right) dy\, dx}_{\cdots\cdots\cdots\cdots\cdots\cdots} \qquad \text{and} \qquad \underbrace{\int\int \cos\left(y^2\right) dx\, dy}_{\cdots\cdots\cdots\cdots\cdots\cdots}$$

We choose type .... : $\begin{cases} \cdots\cdots\cdots \le \cdots\cdots\cdots \le \cdots\cdots\cdots \\ \cdots\cdots\cdots \le \cdots\cdots\cdots \le \cdots\cdots\cdots \end{cases}$

$$\iint\limits_R \cos\left(y^2\right) dA = \int_0^{\cdots\cdots} \int_0^{\cdots\cdots} \cos\left(y^2\right) dx\, dy = \int_0^{\cdots\cdots} \left[\cos\left(y^2\right)_{\cdots\cdots}\right]_{x=0}^{x=\cdots\cdots} dy$$

$$= \int_0^{\cdots\cdots} \left(\cdots\cdots\cdots\cdots\cdots\cdots\right) dy = \int_0^{\cdots\cdots} \left(\cdots\cdots\cdots\cdots\right) d(\cdots\cdots)$$

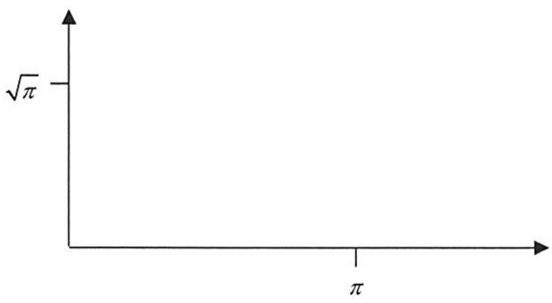
$$= \Big[\cdots\cdots\cdots\cdots\cdots\Big]_0^{\cdots\cdots} = \cdots\cdots\cdots\cdots\cdots\cdots = \underline{\underline{\cdots\cdots\cdots}}$$

Exercise 4: Find $\displaystyle\int_0^{\pi}\int_{\sqrt{x}}^{\sqrt{\pi}} \frac{\sin(y^2)}{y}\,dy\,dx$.

Solution: We can not integrate on the inside: $\displaystyle\int \frac{\sin(y^2)}{y}\,dy = ??$

Try to change the order of integration:

The given region is $\begin{cases} \ldots\ldots \le \ldots\ldots \le \ldots\ldots \\ \ldots\ldots \le \ldots\ldots \le \ldots\ldots \end{cases}$ (type ……..)



Change to type ………………… : $\begin{cases} \ldots\ldots \le \ldots\ldots \le \ldots\ldots \\ \ldots\ldots \le \ldots\ldots \le \ldots\ldots \end{cases}$

$$\int_0^{\pi}\int_{\sqrt{x}}^{\sqrt{\pi}} \frac{\sin(y^2)}{y}\,dy\,dx = \int_{\ldots}^{\ldots}\int_{\ldots}^{\ldots} \frac{\sin(y^2)}{y}\,dx\,dy = \int_{\ldots}^{\ldots} \frac{\sin(y^2)}{y}\Big[\ldots\Big]_{x=\ldots}^{x=\ldots}\,dy$$

$$= \int_{\ldots}^{\ldots} \frac{\sin(y^2)}{y}\left(\ldots\ldots\right)dy = \int_{\ldots}^{\ldots}\left(\ldots\ldots\right)d\left(y^2\right)$$

$$= \Big[\ldots\ldots\Big]_0^{\ldots} = \ldots\ldots = \ldots\ldots = \underline{\ldots}$$

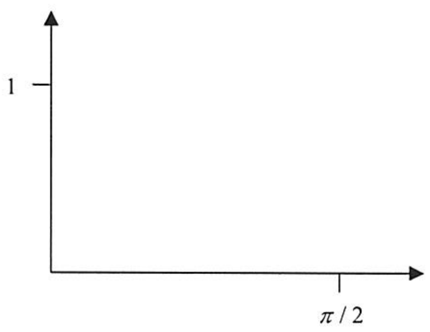
Exercise 5: Find $\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\pi/2} \sec^2(\cos x)\ dx\, dy$.

Solution: We can not integrate on the inside: $\displaystyle\int \sec^2(\cos x)\ dx = ??$

Try to change the order of integration:

The given region is $\begin{cases} \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \end{cases}$ (type ....)

Note: $x = \sin^{-1} y \iff y = \ldots\ldots\ldots\ldots$



Change to type ....:

$\begin{cases} \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \le \ldots\ldots\ldots\ldots \end{cases}$

$$\int_0^1 \int_{\sin^{-1} y}^{\pi/2} \sec^2(\cos x)\, dx\, dy = \int_{\ldots\ldots}^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} \sec^2(\cos x)\, dy\, dx$$

$$= \int_{\ldots\ldots}^{\ldots\ldots} \sec^2(\cos x) \Big[\ldots\Big]_{y=\ldots\ldots}^{y=\ldots\ldots} dy$$

$$= \int_{\ldots\ldots}^{\ldots\ldots} \sec^2(\cos x)\ (\ldots\ldots\ldots\ldots\ldots)\, dy \qquad \begin{cases} u = \ldots\ldots\ldots\ldots\ldots\ldots \\ du = \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

$$= \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots\ du = \Big[\ldots\ldots\ldots\Big]_{\ldots}^{\ldots} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots}}$$

Exercise 6: Find the *area* of the region R bounded by the curves

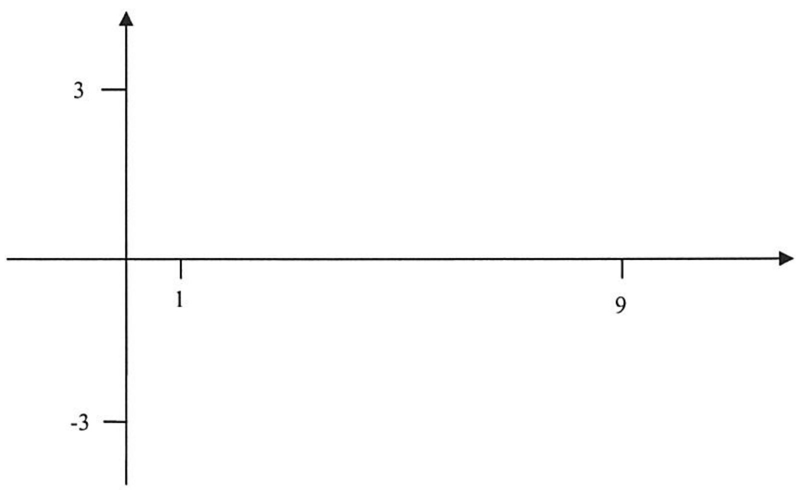
$$y^2 = 9 - x \qquad \text{and} \qquad y^2 = 9 - 9x.$$

Solution: Sketch the graphs:

$y^2 = 9 - x \;\Rightarrow\; x = \ldots\ldots\ldots\ldots\ldots\ldots$ (the graph is a ....)

$y^2 = 9 - 9x \;\Rightarrow\; x = \ldots\ldots\ldots\ldots\ldots\ldots$ (the graph is a ....)

(Think of y as the independent variable)



This is a type .... region: $\begin{cases} \ldots\ldots\ldots \le \ldots\ldots\ldots \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le \ldots\ldots\ldots \le \ldots\ldots\ldots \end{cases}$

The area is

$$A = \iint_R dA = \ldots\ldots \int_{\ldots\ldots}^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} dx\,dy = \ldots\ldots \int_{\ldots\ldots}^{\ldots\ldots} \Big[\ldots\ldots \Big]_{x=\ldots\ldots}^{x=\ldots\ldots} dy$$

By symmetry

$$= \ldots\ldots \int_{\ldots\ldots}^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) dy = \ldots\ldots \int_{\ldots\ldots}^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots\ldots\ldots \right) dy$$

$$= \ldots\ldots\ldots \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots\ldots}}$$

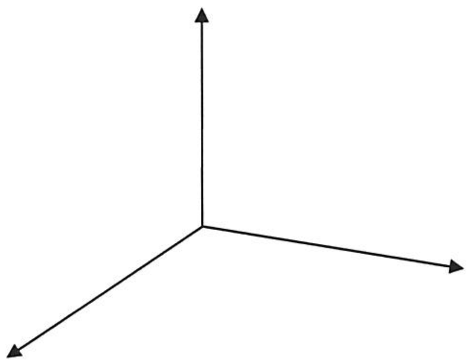
Exercise 7: Find the *volume* of the solid D which is bounded by the surfaces

$$x+z=1, \quad y=0, \quad z=0 \quad \text{and} \quad y^2 = x.$$

Solution: Sketch the surfaces:

$x+z=1$: the graph is a .... parallel to ....

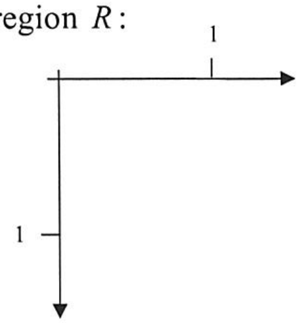
$y^2 = x$: the graph is a .... parallel to ....



z_{upper} =.....................

z_{lower} =.....................

Sketch the region R:



This is a type .... region:

$$\begin{cases} \ldots\ldots\ldots \le \ldots\ldots\ldots \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le \ldots\ldots\ldots \le \ldots\ldots\ldots \end{cases}$$

The volume is

$$V = \iint_R \left[z_{upper} - z_{lower} \right] dA = \iint_R \left[\ldots\ldots\ldots\ldots\ldots\ldots \right] dA$$

$$= \int_{\ldots\ldots}^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots \right) dy\, dx = \int_{\ldots\ldots}^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots \right) \Big[\ldots\ldots \Big]_{y=\ldots\ldots}^{y=\ldots\ldots} dx$$

$$= \int_{\ldots\ldots}^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots \right) \ldots\ldots\ldots\ldots \; dx = = \int_{\ldots\ldots}^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots\ldots\ldots \right) dx$$

$$= \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots}}$$

Additional Exercises:

1) Find the following integrals; the region R is bounded by the given curves.

a) $\iint\limits_R xy\, dA$, R is bounded by $y=0$, $x=2$ and $y=x^2$.

b) $\iint\limits_R x\cos(xy)\, dA$, R is bounded by $x=1$, $x=2$, $y=\frac{\pi}{2}$ and $y=x^2$.

c) $\iint\limits_R (x+y)\, dA$, R is bounded by $y=x^2$ and $y=\sqrt{x}$.

d) $\iint\limits_R (x-1)\, dA$, R is bounded by $y=x$ and $y=x^3$.

e) $\iint\limits_R x\cos(y)\, dA$, R is bounded by $y=0$, $y=\pi$ and $y=x$.

f) $\iint\limits_R (3x-2y)\, dA$, R is bounded by the circle $x^2+y^2=1$.

2) Evaluate each integral by exchanging the order of integration.

a) $\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx$

b) $\int_0^2 \int_{y/2}^1 \cos(x^2)\, dx\, dy$

c) $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$

d) $\int_1^3 \int_0^{\ln x} x\, dy\, dx$

e) $\int_0^1 \int_0^{\cos^{-1} x} x\, dy\, dx$

f) $\int_0^1 \int_{\sin^{-1} y}^{\pi/2} \sec^2(\cos x)\, dx\, dy$

3) Sketch the region R bounded by the graphs of the equations, and find its area using double integrals.

a) $y=1/x^2$, $y=-x^2$, $x=1$, $x=2$

b) $y=-x$, $x-y=4$, $y=-1$, $y=2$

c) $y=x$, $y=3x$, $x+y=4$

d) $y=e^x$, $y=\sin x$, $x=-\pi$, $x=\pi$

e) $y^2=-x$, $3y-x=4$

4) Find the volume of the solid that lies under the graph of $z = f(x, y)$ and above the region R in the xy-plane.

a) $z = x^2 + 3y^2$, R is bounded by the curves $y = x^2$ and $y = 1$.

b) $z = 9 - x^2$, R is in the first quadrant bounded by the curve $y^2 = 3x$.

c) $z = 4x^2 + y^2$, R is the polygon with vertices $(0,0)$, $(0,1)$, $(2,0)$, $(2,1)$.

d) $z = x^2 + 4y^2$, R is the polygon with vertices $(0,0)$, $(1,0)$, $(1,2)$.

e) $z = y + 3$, R is enclosed by the ellipse $4x^2 + y^2 = 9$.

5) Sketch the solid in the first octant bounded by the graphs of the equations, and find its volume.

a) $x^2 + z^2 = 9$, $y = 2x$, $y = 0$, $z = 0$

b) $z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$, $z = 0$

c) $2x + y + z = 4$, $x = 0$, $y = 0$, $z = 0$

d) $y^2 = z$, $y = x$, $x = 4$, $z = 0$

e) $z = x^2 + y^2$, $y = 4 - x^2$, $x = 0$, $y = 0$, $z = 0$

f) $x^2 + y^2 = 16$, $x = z$, $y = 0$, $z = 0$

Double Integrals in Polar Coordinates

Recall: If the region R is described in polar coordinates by

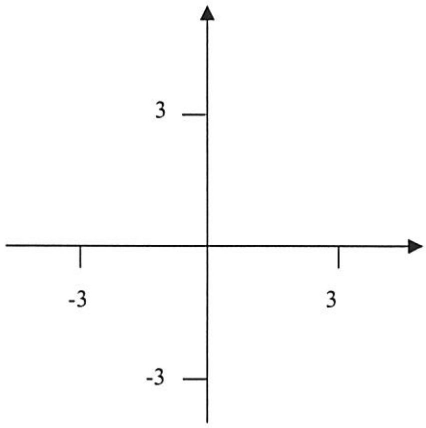
$$a \le \theta \le b$$
$$g(\theta) \le r \le h(\theta)$$

then

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Exercise 1: Find $\iint_R xy^2\,dA$ where R is the region bounded *to the right* by the circle $x^2 + y^2 = 9$, and to *the right* by the y-axis.

Solution: Sketch the region R:



Why should we use polar coordinates ?

....

$$R : \begin{cases} \ldots\ldots\ldots\ldots \le \theta \le \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \le r \le \ldots\ldots\ldots\ldots \end{cases}$$

$$\iint_R xy^2\,dA = \int_{\ldots}^{\ldots}\int_{\ldots}^{\ldots} (r\cos\theta)(r\ldots\ldots\ldots)^{\ldots\ldots} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \int_{\ldots}^{\ldots} \ldots\ldots\ldots\ldots \cos\theta\,d\theta \cdot \int_{\ldots}^{\ldots} \ldots\ldots\,dr = \int_{\ldots}^{\ldots} \ldots\ldots\,du \cdot \int_{\ldots}^{\ldots} \ldots\ldots\,dr$$

$$\boxed{u = \ldots\ldots\ldots \qquad du = \ldots\ldots\ldots\ldots}$$

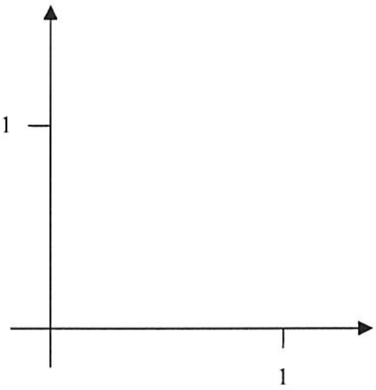
$$= \Big[\ldots\ldots\ldots\Big]_{\ldots}^{\ldots} \cdot \Big[\ldots\ldots\ldots\Big]_{\ldots}^{\ldots} = \ldots\ldots\ldots\ldots \cdot \ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots}}$$

Exercise 2: Find $\displaystyle\iint_R \frac{1}{\sqrt{x^2+y^2}}\, dA$ where R is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

Solution: Sketch the region R:

Why should we use polar coordinates ?

....



Change to polar coordinates:

$$x = 1 \;\Rightarrow\; \ldots\ldots\ldots\ldots\ldots\ldots = 1$$

$$\Rightarrow\; r = \ldots\ldots\ldots\ldots\ldots$$

$$R \;:\; \begin{cases} \ldots\ldots\ldots\ldots \le \theta \le \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \le r \le \ldots\ldots\ldots\ldots \end{cases}$$

$$\iint_R \frac{1}{\sqrt{x^2+y^2}}\, dA = \int_{\ldots\ldots}^{\ldots\ldots}\int_{\ldots\ldots}^{\ldots\ldots} \frac{1}{\ldots\ldots\ldots} \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{dA} = \int_{\ldots\ldots}^{\ldots\ldots}\int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots$$

$$= \int_{\ldots\ldots}^{\ldots\ldots} \Big[\ldots\ldots\ldots\Big]_{r=\ldots\ldots\ldots}^{r=\ldots\ldots\ldots} d\theta = \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\ldots\ldots\; d\theta$$

$$= \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big]_{\ldots\ldots}^{\ldots\ldots}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots\ldots\ldots}}$$

Exercise 3: Find $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{2+x^2+y^2}\, dx\, dy$.

Solution: Why should we use polar coordinates ?

1. ……………………………………………………………………………………………

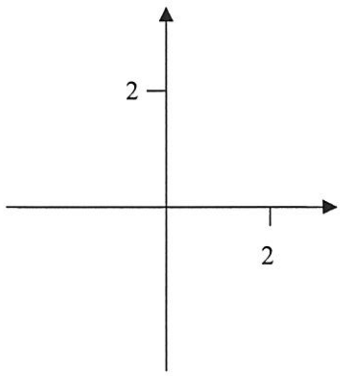
2. ……………………………………………………………………………………………

Sketch the region R:

$$R \ : \ \begin{cases} \ldots\ldots\ldots \le y \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le x \le \ldots\ldots\ldots \end{cases} \qquad \text{(type} \ \ldots\ldots\text{)}$$

In polar coordinates:

$$R \ : \ \begin{cases} \ldots\ldots \le \theta \le \ldots\ldots \\ \ldots\ldots \le r \le \ldots\ldots \end{cases}$$



$dA =$ ……………………………

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{2+x^2+y^2} \underbrace{dx\, dy}_{dA} = \int_{\ldots}^{\ldots} \int_{\ldots}^{\ldots} \frac{1}{2+\ldots} \underbrace{\ldots\ldots\ldots}_{dA}$$

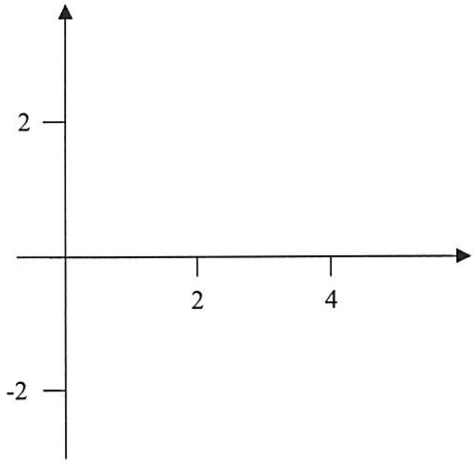
$$= \int_0^{\ldots} d\theta \cdot \int_0^{\ldots} \ldots\ldots \frac{2r}{\ldots\ldots\ldots} dr \qquad \boxed{\int \frac{f'}{f} = \ldots\ldots\ldots}$$

$$= \Big[\ldots \Big]_0^{\ldots} \cdot \Big[\frac{1}{2} \ldots\ldots\ldots\ldots \Big]_0^{\ldots}$$

$$= \ldots\ldots \cdot \ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots}}$$

Exercise 4: Find $\displaystyle\iint_R x^2 y\, dA$ where R is the region enclosed *above* by the circle $x^2 - 4x + y^2 = 0$ and *below* by the line $y = x$.

Solution: Sketch the circle and the line.



$x^2 - 4x + y^2 = 0$

$\Rightarrow \quad x^2 - 4x + \dots\dots\dots + y^2 = \dots\dots\dots$

$\Rightarrow \quad (x - \dots\dots\dots)^2 + y^2 = \dots\dots\dots$

circle, center .... radius ....

In polar coordinates:

$y = x \qquad \Rightarrow \qquad \theta = \dots\dots\dots$

$x^2 - 4x + y^2 = 0$

$\Rightarrow \quad r^2 - 4 \dots\dots\dots\dots\dots = 0$

$\Rightarrow \quad r \dots\dots\dots\dots\dots\dots\dots = 0$

Thus, $\quad R \; : \; \begin{cases} \dots\dots\dots \le \theta \le \dots\dots\dots \\ \dots\dots\dots \le r \le \dots\dots\dots \end{cases}$

$$\iint_R x^2 y\, dA = \int_{\dots}^{\dots} \int_{\dots}^{\dots} (\dots\dots\dots)^2 (\dots\dots\dots) \underbrace{\dots\dots\dots\dots}_{dA}$$

$$= \int_{\dots}^{\dots} \int_{\dots}^{\dots} \dots\dots\dots\dots\dots\dots\dots\dots \; dr\, d\theta$$

$$= \int_{\dots}^{\dots} \dots\dots\dots\dots\dots \Big[\dots \Big]_{r=\dots}^{r=\dots} d\theta = \int_{\dots}^{\dots} \dots\dots\dots\dots\dots\dots \; d\theta$$

$$= \int_{\dots}^{\dots} \frac{4^5}{5} \dots\dots\dots \; du = \left[\frac{4^5}{5} \dots\dots\dots \right]_{\dots}^{\dots} = \frac{4^5}{5} \cdot \frac{1}{\dots\dots\dots}$$

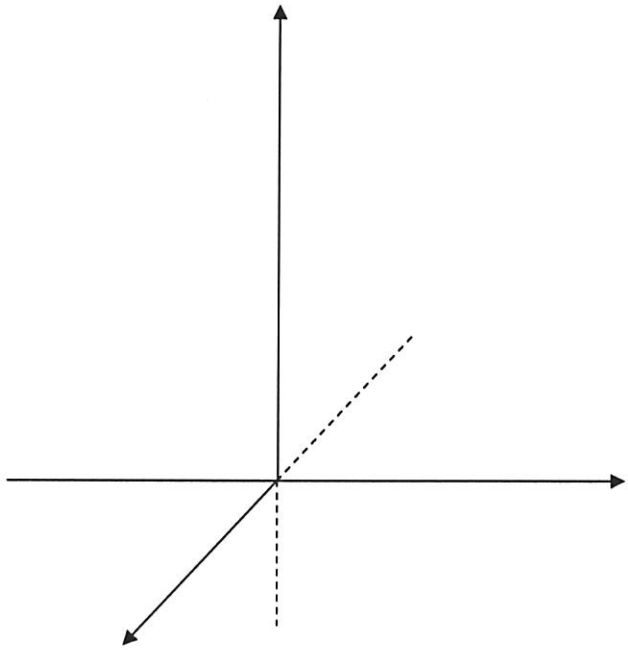
$\boxed{u = \dots\dots\dots \qquad du = \dots\dots\dots}$

$$= \dots\dots\dots\dots \qquad = \underline{\underline{\dots\dots\dots}}$$

Note: This integral can also be done in Cartesian coordinates ! Try.

Exercise 5: Find the *volume* of the solid which is bounded by the cylinder $x^2 + y^2 = 4$, the plane $y + z = 4$ and the plane $z = 0$.

Solution: Sketch the solid.



$z_{upper} = \dots\dots\dots\dots$

$z_{lower} = \dots\dots\dots\dots$

In polar coordinates,

$z_{upper} = \dots\dots\dots\dots$

$z_{lower} = \dots\dots\dots\dots$

$$R \ : \ \begin{cases} \dots\dots \leq \theta \leq \dots\dots \\ \dots\dots \leq r \leq \dots\dots \end{cases}$$

The volume is

$$V = \iint_R \left[z_{upper} - z_{lower} \right] dA = \iint_R [\dots\dots\dots - \dots\dots] \, dA$$

$$= \int_{\dots}^{\dots} \int_{\dots}^{\dots} (\dots\dots\dots\dots) \underbrace{\dots\dots\dots}_{dA} = \int_{\dots}^{\dots} \int_{\dots}^{\dots} (\dots\dots\dots\dots\dots) \, dr \, d\theta$$

$$= \int_{\dots}^{\dots} \Big[\dots\dots\dots\dots\dots\dots\dots\dots \Big]_{r=\dots}^{r=\dots} d\theta$$

$$= \int_{\dots}^{\dots} \Big(\dots\dots\dots\dots\dots\dots\dots\dots \Big) d\theta = \Big[\dots\dots\dots\dots\dots\dots \Big]_{\dots}^{\dots}$$

$$= \dots\dots\dots\dots\dots\dots\dots\dots = \underline{\underline{\dots\dots\dots}}$$

Note: the *average height* of the solid is $h = 4$. A cylinder of this height has volume

$$V = \pi r^2 h = \pi \cdot 2^2 \cdot 4 = 16\pi \,!$$

Additional Exercises:

1) Use polar coordinates to evaluate the integral.

a) $\iint_R (x^2+y^2)^{3/2}\, dA$ $\quad$ R is bounded by the circle $x^2+y^2=4$.

b) $\iint_R \frac{x^2}{x^2+y^2}\, dA$ $\quad$ R is bounded by the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ with $0<a<b$.

c) $\iint_R \sqrt{x^2+y^2}\, dA$ $\quad$ R is bounded by the triangle with vertices $(0,0)$, $(3,0)$ and $(3,3)$.

d) $\iint_R \sqrt{x^2+y^2}\, dA$ $\quad$ R is bounded by the semicircle $y=\sqrt{2x-x^2}$ and the line $y=x$.

e) $\iint_R x^2(x^2+y^2)^3\, dA$ $\quad$ R is bounded by the semicircle $y=\sqrt{1-x^2}$ and the x-axis.

2) Use polar coordinates to evaluate the integral.

a) $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-(x^2+y^2)}dy\, dx$

b) $\int_{0}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (x^2+y^2)^{3/2}\, dx\, dy$

c) $\int_{0}^{4}\int_{0}^{\sqrt{16-x^2}} \cos(x^2+y^2)\, dy\, dx$

d) $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\, dy\, dx$

e) $\int_{0}^{\sqrt{2}}\int_{y}^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\, dx\, dy$

f) $\int_{0}^{2}\int_{0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

g) $\int_{1}^{2}\int_{0}^{x} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$

h) $\int_{0}^{1}\int_{y}^{\sqrt{y}} \sqrt{x^2+y^2}\, dx\, dy$

3) Find the volume of the given solid D.

a) D is inside the sphere $x^2+y^2+z^2=25$ and outside the cylinder $x^2+y^2=9$.

b) D is bounded by the cone $z=\sqrt{x^2+y^2}$ and the cylinder $x^2+y^2=2x$.

c) D is inside the paraboloid $z=9-x^2+y^2$ and above the plane $z=5$.

d) D is inside the sphere $x^2+y^2+z^2=16$ and inside the cylinder $x^2+y^2=4y$.

Triple Integrals

Recall: If the solid D can be described as

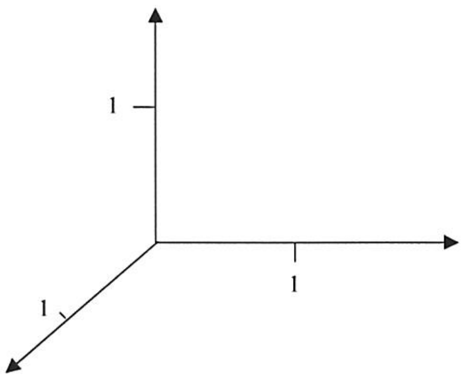
$$D = \{ (x,y,z) : (x,y) \in R, \quad z_{lower} \le z \le z_{upper} \}$$

then

$$\boxed{\iiint_D f(x,y,z)\,dV = \iint_R \int_{z_{lower}}^{z_{upper}} f(x,y,z)\,dz\,dA}$$

Exercise 1: Find $\iiint_D (x-y)\,dV$ where D is the *tetrahedron* with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Solution: Sketch the tetrahedron.



Find the equation of the *upper* plane:

The equation of a plane is: $ax +$ .... $= d$

Substitute $(x,y,z) = (1,0,0)$: .... $= d$ $\Rightarrow$ $a =$ ......

Substitute $(x,y,z) = (0,1,0)$: .... $= d$ $\Rightarrow$ $b =$ ......

Substitute $(x,y,z) =$ ....: .... $= d$ $\Rightarrow$ $c =$ ......

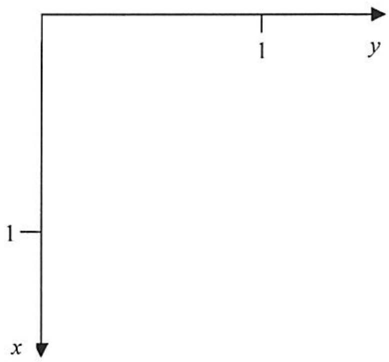
We can choose d freely. Choose $d =$ .... $\Rightarrow$ $a =$ ......, $b =$ ......, $c =$ ......

The equation of the plane is .... $=$ ....

or $z_{upper} =$ ....

Also, $z_{lower} =$ ....

Now sketch the region R:



R is a ....

Consider as type I:`

$$R : \begin{cases} \ldots\ldots\ldots\ldots \le x \le \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \le y \le \ldots\ldots\ldots\ldots \end{cases}$$

$$\iiint\limits_D (x-y)\,dV = \iint\limits_R \int\limits_{z_{lower}}^{z_{upper}} (x-y)\,dz\,dA = \int\limits_0^1 \int\limits_{\ldots\ldots}^{\ldots\ldots} \int\limits_{\ldots\ldots}^{\ldots\ldots\ldots} (x-y)\,dz\,dy\,dx$$

$$= \int\limits_0^1 \int\limits_{\ldots\ldots}^{\ldots\ldots} \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{z=\ldots\ldots\ldots}^{z=\ldots\ldots\ldots} dy\,dx$$

$$= \int\limits_0^1 \int\limits_{\ldots\ldots}^{\ldots\ldots} \Big[x\big(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\big) - y\big(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\big) \Big] dx$$

$$= \int\limits_0^1 \int\limits_{\ldots\ldots}^{\ldots\ldots} \big(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\big)\,dy\,dx$$

$$= \int\limits_0^1 \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{\ldots\ldots}^{\ldots\ldots} dx$$

$$= \int\limits_0^1 \big(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\big)\,dx$$

$$\boxed{u = \ldots\ldots\ldots\ldots \quad \Rightarrow \quad du = \ldots\ldots\ldots\ldots}$$

$$= \int\limits_{u=\ldots}^{u=\ldots} \big(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\big)\,du = \int\limits_{\ldots\ldots}^{\ldots\ldots} \big(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\big)\,du$$

$$= \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots\ldots}}$$

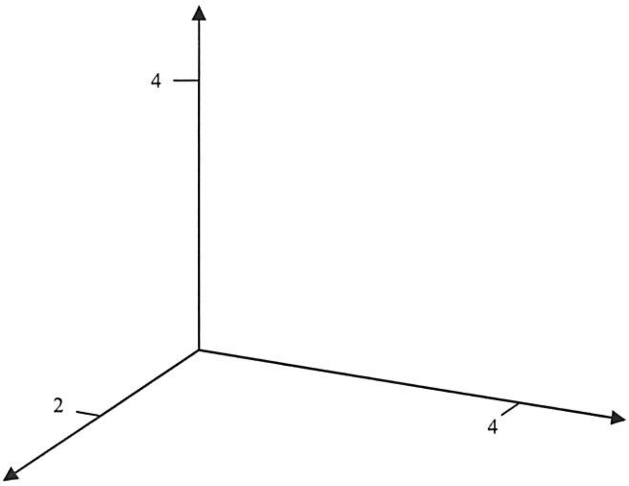
Exercise 2: Let D be the solid in the *first octant* bounded by the surfaces

$$z = 4 - x^2 \qquad \text{and} \qquad y = 4\ .$$

The *density* of the solid is $\delta(x, y, z) = 16 - xy$ (in g/m^3)

Find the *volume* and the *mass* of the solid D.

Solution: Sketch the solid.



$z_{upper} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$z_{lower} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \begin{cases} \ldots\ldots\ldots \le x \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le y \le \ldots\ldots\ldots \end{cases}$$

The volume is

$$V = \iiint_D 1\ dV = \iint_R \int_{z_{lower}}^{z_{upper}} 1\, dz\, dA = \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \int_0^{\ldots\ldots\ldots} 1\, dz\, dy\, dx$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \Big[\ldots\Big]_{z=0}^{z=\ldots\ldots\ldots} dy\, dx = \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \Big(\ldots\ldots\ldots\ldots\Big)\, dy\, dx$$

$$= \int_0^{\ldots\ldots} \Big(\ldots\ldots\ldots\ldots\Big)\, dx \cdot \int_0^{\ldots\ldots} \ldots\ldots\ dy = \Big[\ldots\ldots\ldots\ldots\Big]_0^{\ldots} \cdot \Big[\ldots\Big]_0^{\ldots}$$

$$= \Big(\ldots\ldots\ldots\Big)\Big(\ldots\ldots\ldots\Big) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots}}$$

The mass is

$$m = \iiint_D \delta(x,y,z)\,dV = \iint_R \int_{z_{lower}}^{z_{upper}} \left(\ldots\ldots\ldots\ldots\ldots\ldots \right) dz\,dA$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots \right) dz\,dy\,dx$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \Big[\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{z=0}^{z=\ldots\ldots} dy\,dx$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) dy\,dx$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) dy\,dx$$

$$= \int_0^{\ldots\ldots} \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{y=0}^{y=\ldots\ldots} dx$$

$$= \int_0^{\ldots\ldots} \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) dx$$

$$= \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_0^{\ldots\ldots}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \underline{\underline{\ldots\ldots\ldots\ldots}}$$

Additional Exercises:

1) Evaluate the given integrals.

a) $\iiint\limits_D xy\sin yz\, dV$ D is the cube $\{\ (x,y,z)\ :\ 0 \le x \le \pi,\ 0 \le y \le 1,\ 0 \le z \le \pi/6\ \}$.

b) $\iiint\limits_D y\, dV$ D is the solid bounded by the planes $y = x$, $y = 0$, $z = 0$ and the surface $z = 1 - x^2$.

c) $\iiint\limits_D x(y-z)\, dV$ D is the cube $[-1,1]\times[0,2]\times[0,3]$.

d) $\iiint\limits_D (x+y+z)\, dV$ D is the *tetrahedron* with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

2) Find the volume of the given solid D using a triple integral.

a) D is in the first octant, and enclosed by the plane $3x+6y+4z=12$.

b) D is in the first octant, and enclosed by the surfaces $y^2+z^2=1$ and $y=x$.

c) D is enclosed by the surfaces $z=\sqrt{y}$, $x+y=1$, $x=0$ and $z=0$.

d) D is enclosed by the surfaces $z=4-x^2$, $z=4-y$, $y=0$ and $z=0$.

e) D is enclosed by the surfaces $y^2+z^2=1$, $x+y+z=2$ and $x=0$.

f) D is enclosed by the surfaces $z=e^{x+y}$, $y=3x$, $x=2$, $y=0$ and $z=0$.

g) D is enclosed by the cylinders $x^2+y^2=1$ and $x^2+z^2=1$.

3) Consider the solid D enclosed by the given surfaces, with density $\delta(x,y,z)$.
Find the mass of the solid.

a) $x+2y+z=4$, $x=0$, $y=0$ and $z=0$, where $\delta(x,z,y)=x^2+y^2$.

b) $x+2y+z=4$, $x=0$, $y=0$ and $z=0$, where $\delta(x,z,y)=x^2+y^2$.

Triple Integrals in Cylindrical Coordinates

Recall: If the solid D can be described as

$$D = \left\{ (r,\theta,z) \;:\; a \le \theta \le b, \quad g(\theta) \le r \le h(\theta), \quad z_{lower} \le z \le z_{upper} \right\}$$

then

$$\boxed{\begin{aligned} \iiint\limits_D f(x,y,z)\, dV &= \iint\limits_R \int\limits_{z_{lower}}^{z_{upper}} f(x,y,z)\, dz\, dA \\ &= \int\limits_a^b \int\limits_{g(\theta)}^{h(\theta)} \int\limits_{z_{lower}}^{z_{upper}} f(r\cos\theta, r\sin\theta, z)\, \underbrace{r\, dz\, dr\, d\theta}_{dV} \end{aligned}}$$

Exercise 1: Find the *average value* of $f(x,y,z) = \dfrac{1}{(z+1)^2}$ on the solid D which is bounded by the surfaces

$$z = 2 - x^2 - y^2 \qquad \text{and} \qquad z = \sqrt{x^2 + y^2}\,.$$

Solution:

Graph is a	Graph is a ....

We use cylindrical coordinates.

Upper surface is the ....

$\Rightarrow \quad z_{upper} =$ ....

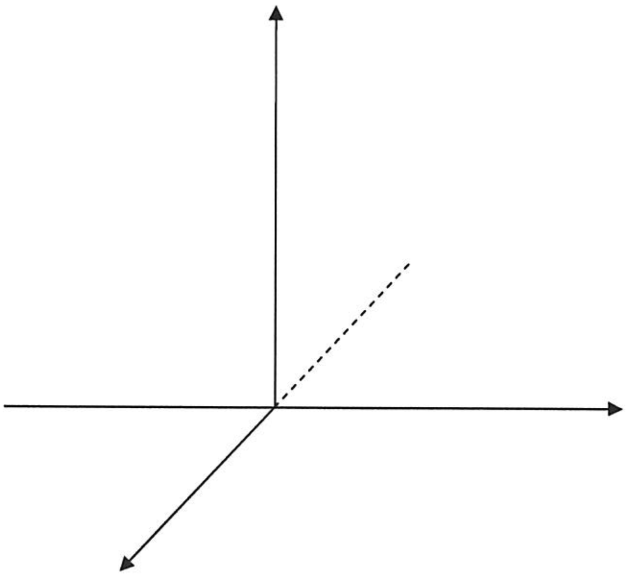
Lower surface is the ....

$\Rightarrow \quad z_{lower} =$ ....

In cylindrical coordinates:

$z_{upper} =$ ....

$z_{lower} =$ ....



Find the circle of intersection of the two surfaces:

$$z_{paraboloid} = z_{cone}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = r$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots) = 0$$

$$r = \ldots\ldots, \quad r = \ldots\ldots \quad \Rightarrow \quad r = \ldots\ldots \qquad\qquad R : \begin{cases} \ldots\ldots\ldots \le \theta \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le r \le \ldots\ldots\ldots \end{cases}$$

1. The integral is

$$\iiint_D f(x,y,z)\, dV = \iint_R \int_{z_{lower}}^{z_{upper}} \frac{1}{(z+1)^2}\, dz\, dA$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} \frac{1}{(z+1)^2} \underbrace{\ldots\ldots\ldots\ldots\; dr\, d\theta}_{dV}$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} r \left[\frac{-1}{\ldots\ldots\ldots\ldots} \right]_{z=\ldots\ldots}^{z=\ldots\ldots} dr\, d\theta$$

$$= \int_0^{\ldots\ldots} \int_0^{\ldots\ldots} r \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) dr\, d\theta$$

$$= \int_0^{\ldots\ldots} d\theta \cdot \int_0^{\ldots\ldots} \left\{ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right\} dr$$

$$= 2\pi \cdot \int_0^{\ldots\ldots} \left\{ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right\} dr$$

$$= 2\pi \cdot \Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_0^{\ldots\ldots}$$

$$= 2\pi \cdot \Big(\big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \big] - \big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \big] \Big)$$

$$= 2\pi \cdot \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right) = \pi \cdot \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right)$$

2. The volume is

$$V = \iiint\limits_D 1\, dV = \iint\limits_R \int_{z_{lower}}^{z_{upper}} 1\, dz\, dA = \int_0^{\dots\dots} \int_0^{\dots\dots} \int_{\dots\dots}^{\dots\dots} \underbrace{\dots\dots\dots\dots\, dr\, d\theta}_{dV}$$

$$= \int_0^{\dots\dots} \int_0^{\dots\dots} r \left[\dots \right]_{z=\dots\dots}^{z=\dots\dots} dr\, d\theta = \int_0^{\dots\dots} d\theta \cdot \int_0^{\dots\dots} \left(\dots\dots\dots\dots\dots\dots\dots\dots \right) dr$$

$$= 2\pi \left[\dots\dots\dots\dots\dots\dots\dots\dots \right]_0^{\dots} = 2\pi \left(\dots\dots\dots\dots\dots\dots\dots \right) = 2\pi_{\dots\dots} = \dots\dots$$

3. The average value is

$$f_{avg} = \frac{1}{V} \iiint\limits_D f(x,y,z)\, dV = \frac{\quad}{\dots\dots} \cdot \dots\dots\dots\dots\dots\dots = \frac{\qquad}{\dots\dots\dots\dots}$$

Additional Exercises:

1) Sketch the graph of the equation in three dimensions.

a) $r = 4$

b) $z = 4r^2$

c) $z = \sqrt{3}r$

d) $z = 4 - r^2$

e) $\theta = \pi / 4$

f) $r = -\csc\theta$

2) Evaluate the given integrals.

a) $\iiint\limits_D x^2\, dV$ — D is in the first octant, and bounded by the sphere $x^2 + y^2 + z^2 = 9$.

b) $\iiint\limits_D z^2\, dV$ — D is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 8$.

c) $\iiint\limits_D (x^2 + y^2 + z^2)\, dV$ — D is the bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

d) $\iiint\limits_D (x + y + z)\, dV$ — D is bounded by the cones $z = \sqrt{\dfrac{x^2 + y^2}{3}}$ and $z = \sqrt{3(x^2 + y^2)}$, and the cylinder $x^2 + y^2 = 9$.

3) Find the volume of the solid D using a triple integral in cylindrical coordinates.

a) D is bounded by the paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$ and the xy-plane.

b) D is enclosed by the surfaces $z = \sqrt{x^2 + y^2}$, $x^2 + y^2 = 4$ and $z = 8 - x^2 - y^2$.

c) D is outside the cylinder $x^2 + y^2 = 4$ and inside the sphere $x^2 + y^2 + z^2 = 9$.

d) D is in the first octant, and inside the surfaces $x^2 + y^2 = 4x$ and $x^2 + y^2 + z^2 = 16$.

4) Change to an integral in cylindrical coordinates, and integrate.

a) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z \, dz \, dy \, dx$$

b) $$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz \, dz \, dy \, dx$$

c) $$\int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} xy \, dz \, dx \, dy$$

d) $$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2yz \, dz \, dy \, dx$$

e) $$\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_1^{5-x^2-y^2} x^2yz \, dz \, dy \, dx$$

Triple Integrals in Spherical Coordinates

Recall: If the solid D can be described as

$$D = \{ (\rho,\phi,\theta) : a \le \theta \le b, \quad f_1(\theta) \le \phi \le f_2(\theta), \quad g_1(\phi,\theta) \le \rho \le g_2(\phi,\theta) \}$$

then

$$\boxed{\iiint\limits_D f(x,y,z)\, dV = \int_a^b \int_{f_1(\theta)}^{f_2(\theta)} \int_{g_1(\phi,\theta)}^{g_1(\phi,\theta)} f(\underbrace{\rho\sin\phi\cos\theta}_{x}, \underbrace{\rho\sin\phi\sin\theta}_{y}, \underbrace{\rho\cos\phi}_{z})\, \underbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}_{dV}}$$

Exercise 1: Find the spherical coordinates (ρ,ϕ,θ) of the point $P(x,y,z)$. Then sketch the point showing the spherical coordinates.

a) $P(x,y,z) = (1,0,1)$.

Solution: We use

$$\rho = \sqrt{\ldots\ldots + \ldots\ldots + \ldots\ldots} = \sqrt{\ldots\ldots + \ldots\ldots + \ldots\ldots} = \sqrt{\ldots} = \ldots\ldots$$

$$z = \rho\cos_{\ldots\ldots} \quad \Rightarrow \quad \cos\phi = \frac{z}{\ldots\ldots} = \frac{\quad}{\ldots\ldots} = \frac{\quad}{\ldots\ldots}$$

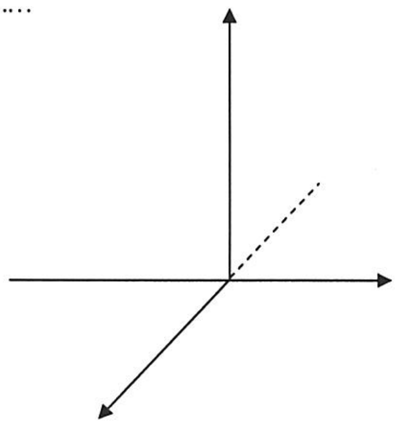
$$\Rightarrow \quad \phi = \cos^{-1}\left(\frac{\quad}{\quad}\right) = \ldots\ldots\ldots\ldots$$

$$\tan\theta = \frac{y}{\ldots\ldots} = \frac{\quad}{\ldots\ldots} = \ldots\ldots \quad \Rightarrow \quad \theta = \ldots\ldots \quad \text{or} \quad \theta = \ldots\ldots$$

Because $(x,y) = (\ldots,\ldots)$ $\quad \Rightarrow \quad \theta = \ldots\ldots$

Answer:

$(\rho,\phi,\theta) = (\ldots\ldots\ldots , \ldots\ldots\ldots , \ldots\ldots\ldots)$



b) $P(x,y,z)=\left(-1,-1,\sqrt{2}\right)$.

Solution: We use

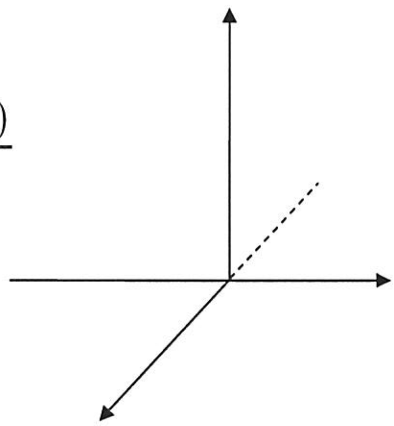
$$\rho \;=\; \sqrt{\ldots\ldots+\ldots\ldots+\ldots\ldots} \;=\; \sqrt{\ldots\ldots+\ldots\ldots+\ldots\ldots} \;=\; \sqrt{\ldots} \;=\; \ldots\ldots$$

$$z \;=\; \rho\cos_{\ldots\ldots} \qquad \Rightarrow \qquad \cos\phi \;=\; \frac{z}{\ldots\ldots} \;=\; \frac{\quad}{\ldots\ldots} \;=\; \frac{\quad}{\ldots\ldots}$$

$$\Rightarrow \qquad \phi \;=\; \cos^{-1}\left(\frac{\quad}{\quad}\right) \;=\; \ldots\ldots\ldots\ldots$$

$$\tan\theta \;=\; \frac{y}{\ldots\ldots} \;=\; \frac{\quad}{\ldots\ldots} \;=\; \ldots\ldots \quad \Rightarrow \quad \theta \;=\; \ldots\ldots \quad \text{or} \quad \theta \;=\; \ldots\ldots$$

Because $(x,y)=(\ldots\ldots,\ldots\ldots)$ $\quad$ ($\ldots\ldots\ldots$ quadrant) $\quad \Rightarrow \quad \theta \;=\; \ldots\ldots$



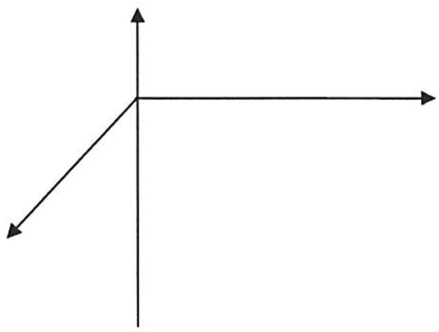
Answer:

$(\rho,\phi,\theta)=(\ \ldots\ldots\ldots\ ,\ \ldots\ldots\ldots,\ \ldots\ldots\ldots\ )$

Exercise 2: If $(\rho,\phi,\theta)=\left(3,\dfrac{3\pi}{4},\dfrac{\pi}{4}\right)$,

find (x,y,z).

Solution: Recall that



$$x \;=\; r\cos\theta \;=\; \rho\ \ldots\ldots\ldots\ldots\ \cos\theta$$

$$y \;=\; r\ \ldots\ldots\ldots\ldots \;=\; \rho\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$z \;=\; \rho\ \ldots\ldots\ldots\ldots\ldots$$

Therefore,

$$x \;=\; \ldots\ldots \cdot \sin_{\ldots\ldots\ldots\ldots} \cdot \cos_{\ldots\ldots\ldots} \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots$$

$$y \;=\; \ldots\ldots \cdot \sin_{\ldots\ldots\ldots\ldots} \cdot \sin_{\ldots\ldots\ldots} \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots$$

$$z \;=\; \ldots\ldots \cdot \cos_{\ldots\ldots\ldots\ldots} \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots\ldots\ldots$$

Answer: $(x,y,z)=(\ \ldots\ldots\ldots\ldots\ ,\ \ldots\ldots\ldots,\ \ldots\ldots\ldots\ldots\ )$

Exercise 3: Change each equation to *spherical* coordinates.

a) $x^2 + y^2 + z^2 = 25$.

Solution: This is the equation of ....

$\rho^2 \;=\; x^2 +$ .... gives $\rho^2 \;=\;$ ....

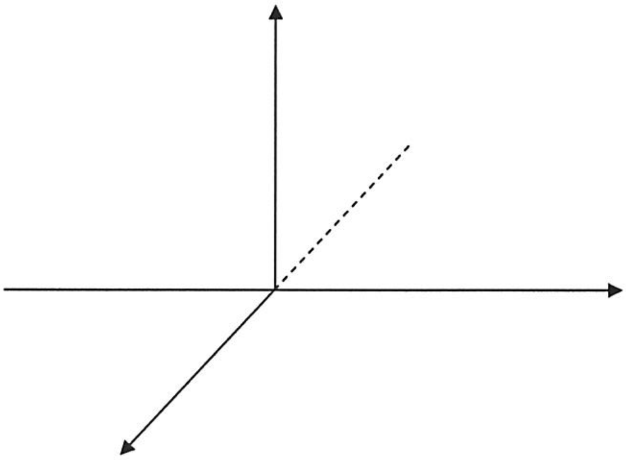
Answer: $\rho \;=\;$ ____________

b) $z = \sqrt{\dfrac{x^2 + y^2}{3}}$.

Solution: This is the equation of ....

Write as

$$z = \frac{1}{\ldots\ldots\ldots\ldots}\sqrt{x^2 + y^2}$$



By trigonometry,

$$\cot\phi \;=\; \frac{z}{\ldots\ldots\ldots} \;=\; \frac{1}{\ldots\ldots\ldots\ldots} \quad\Rightarrow\quad \phi \;=\; \cot^{-1}\left(\frac{1}{\quad}\right) \;=\; \ldots\ldots\ldots\ldots$$

Answer: $\phi \;=\;$ ____________

c) $z = 3$.

Solution: This is the equation of ....

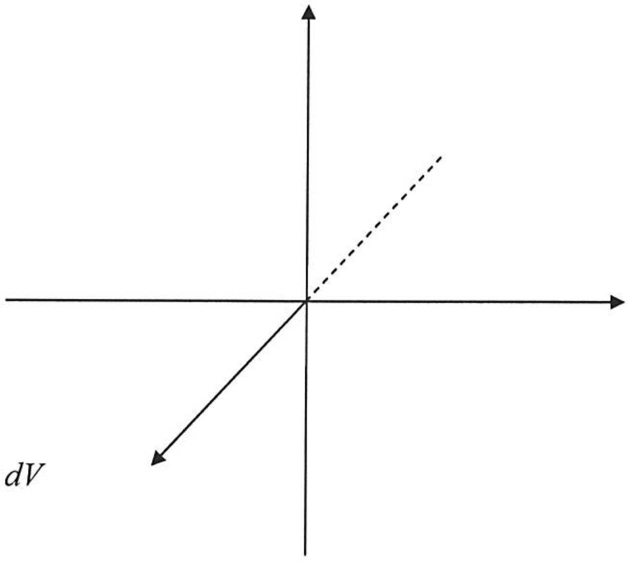
$z \;=\; \rho$.................. gives

$3 \;=\; \rho$.................. $\Rightarrow$ $\rho \;=\;$ ....

Answer: $\rho \;=\;$ ____________

Exercise 4: The temperature T at a point (x, y, z) inside a sphere of radius 4 centered at the origin is $T(x, y, z) = 100 + \sqrt[4]{x^2 + y^2 + z^2}$ (in $^{\circ}$C). Find the *average temperature* T_{avg} inside the sphere.

Solution: Sketch the sphere.



$$D : \begin{cases} \dots\dots \le \theta \le \dots\dots \\ \dots\dots \le \phi \le \dots\dots \\ \dots\dots \le \rho \le \dots\dots \end{cases}$$

Recall:

$$T_{avg} = \frac{1}{volume(D)} \iiint_D T(x, y, z)\, dV$$

1. The volume of a sphere of radius $\rho = 4$ is

$$V = volume(D) = \frac{4}{3}\pi\rho^3 = \dots\dots\dots\dots = \dots\dots$$

2. The average temperature is

$$T_{avg} = \frac{1}{V} \iiint_D \left(100 + \sqrt[4]{x^2 + y^2 + z^2}\right) dV$$

$$= \frac{1}{V}\left\{ \iiint_D 100\, dV + \iiint_D \dots\dots\dots\dots\dots\dots\, dV \right\}$$

$$= \frac{1}{V}\left\{ 100 \iiint_D dV + \int_0^{\dots}\int_0^{\dots}\int_0^{\dots} \dots\dots \underbrace{\dots\dots\dots\dots\dots\dots\, d\theta}_{dV} \right\}$$

$$= \frac{1}{V}\left\{ 100\,\dots\dots + \int_0^{\dots} d\theta \cdot \int_0^{\dots} \dots\dots\dots\, d\phi \cdot \int_0^{\dots} \dots\dots\dots\, d\rho \right\}$$

$$= \dots\dots + \frac{1}{V} \cdot \Big[\dots\dots\Big]_0^{\dots} \cdot \Big[\dots\dots\dots\Big]_0^{\dots} \cdot \Big[\dots\dots\dots\Big]_0^{\dots}$$

$$= \;\ldots\ldots\ldots\; + \frac{1}{V} \cdot \left[\ldots\ldots\ldots\ldots\right] \cdot \left[\ldots\ldots\ldots\ldots\ldots\right] \cdot \left[\ldots\ldots\ldots\ldots\ldots\right]$$

$$= \;\ldots\ldots\ldots\; + \frac{3}{\quad} \cdot \ldots\ldots \cdot \ldots\ldots \cdot \ldots\ldots\ldots \;=\; 100 + \frac{\quad}{\quad} \;=\; \underline{\underline{\ldots\ldots\ldots}}$$

Exercise 5: A solid D I bounded *below* by the plane $z = 1$, *sideways* by the cone $z = \sqrt{x^2 + y^2}$ and *above* by the sphere $x^2 + y^2 + z^2 = 8$. The *density* of the solid is $\delta(x, y, z) = \dfrac{1}{z}$ (in g / cm^3)

Find the mass of the solid D.

Solution: First sketch the solid.

Change to spherical coordinates:

$x^2 + y^2 + z^2 = 8$

(sphere, radius ……….)

$\Rightarrow \quad \rho = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

$z = \sqrt{x^2 + y^2}$

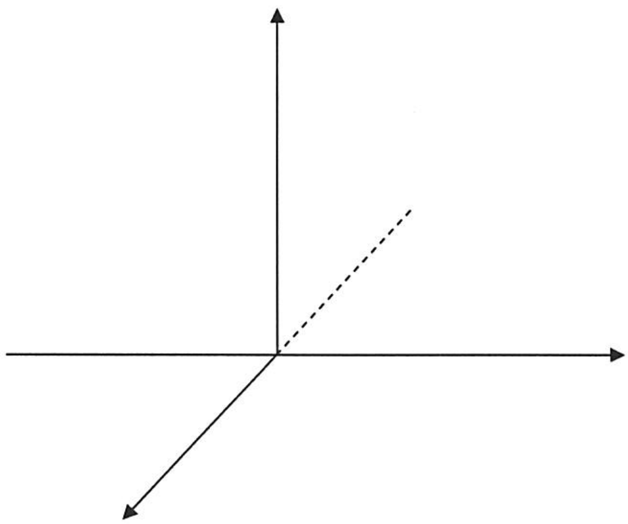
(cone, angle ……….)

$\Rightarrow \quad \theta = \ldots\ldots\ldots\ldots$

$z = 1$ (a ……………………)

$\Rightarrow \quad \rho \cos\phi = \ldots\ldots\ldots\ldots$

$\Rightarrow \quad \rho = \ldots\ldots\ldots\ldots\ldots$



$$D : \begin{cases} \ldots\ldots\ldots \le \theta \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le \phi \le \ldots\ldots \\ \ldots\ldots\ldots \le \rho \le \ldots\ldots \end{cases}$$

The mass is

$$m = \iiint_D \delta(x,y,z)\, dV = \int_0^{\cdots\cdots} \int_0^{\cdots\cdots} \int_{\cdots\cdots}^{\cdots\cdots} \frac{1}{\rho \cdots\cdots\cdots} \underbrace{\cdots\cdots\cdots\cdots\cdots\cdots\, d\phi\, d\theta}_{dV}$$

$$= \int_0^{\cdots\cdots} \int_0^{\cdots\cdots} \int_{\cdots\cdots}^{\cdots\cdots} \cdots\cdots\cdots\cdots\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\cdots\cdots} \int_0^{\cdots\cdots} \tan\phi \Big[\cdots\cdots \Big]_{\rho=\cdots\cdots}^{\rho=\cdots\cdots} d\phi\, d\theta$$

$$= \int_0^{\cdots\cdots} d\theta \cdot \int_0^{\cdots\cdots} \tan\phi \left(\cdots\cdots\cdots\cdots\cdots\cdots \right) d\phi$$

$$= \cdots\cdots \cdot \left\{ \int_0^{\cdots\cdots} \cdots\cdots\cdots\cdots\, d\phi \; - \int_0^{\cdots\cdots} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\, d\phi \right\}$$

$$\boxed{\begin{array}{l} u = \cdots\cdots\cdots\cdots \\ du = \cdots\cdots\cdots\cdots \end{array}}$$

$$= \cdots\cdots\cdots \left\{ \Big[\cdots\cdots\cdots\cdots\cdots \Big]_0^{\cdots\cdots} - \int_0^{\cdots\cdots} \cdots\cdots\cdots\cdots\, du \right\}$$

$$= \cdots\cdots\cdots \left\{ \Big[\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \Big] - \Big[\cdots\cdots\cdots \Big]_0^{\cdots\cdots} \right\}$$

$$= \cdots\cdots \left(\cdots\cdots\cdots\cdots\cdots\cdots\cdots \right) = \underline{\underline{\cdots\cdots\cdots}}$$

Exercise 6: Change to spherical coordinates and integrate:

$$I = \int_{0}^{\sqrt{2}} \int_{y}^{\sqrt{4-y^2}} \int_{0}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$

Solution: The solid of integration is:

$$D : \begin{cases} \ldots\ldots\ldots \le y \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le x \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le z \le \ldots\ldots\ldots \end{cases}$$

$x = y$ is a ....

$x = \sqrt{4-y^2}$ is a ....

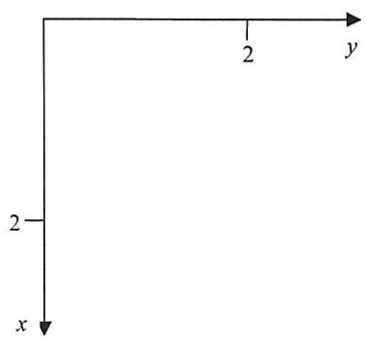
$z = \sqrt{4-x^2-y^2}$ is a ....

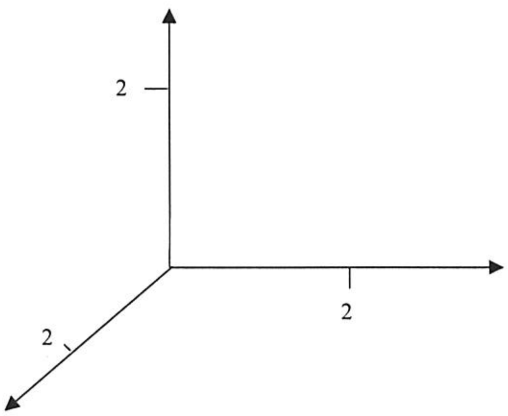
....

The projection in the xy-plane is:

$$R : \begin{cases} \ldots\ldots\ldots \le y \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le x \le \ldots\ldots\ldots \end{cases}$$

Sketch the region R and the solid D:





In spherical coordinates:
$$D : \begin{cases} 0 \le \theta \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le \phi \le \ldots\ldots\ldots \\ \ldots\ldots\ldots \le \rho \le \ldots\ldots\ldots \end{cases}$$

Therefore

$$I = \iiint_D \sqrt{x^2+y^2+z^2}\, dV = \int_0^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, d\phi\, d\theta}_{dV}$$

$$= \int_0^{\ldots\ldots} d\theta \cdot \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\ldots\, d\phi \cdot \int_{\ldots\ldots}^{\ldots\ldots} \ldots\ldots\, d\rho$$

$$= \ldots\ldots \cdot \Big[\ldots\ldots\ldots\Big]_{\ldots\ldots}^{\ldots\ldots} \cdot \Big[\ldots\ldots\Big]_{\ldots\ldots}^{\ldots\ldots} = \ldots\ldots \cdot (\ldots\ldots\ldots)\cdot(\ldots\ldots) = \underline{\ldots\ldots\ldots}$$

Additional Exercises:

1) Change the spherical coordinates to i) Cartesian coordinates, ii) cylindrical coordinates.

   a) $(4, \pi/6, \pi/2)$ b) $(1, 3\pi/4, 2\pi/3)$

2) Change the Cartesian coordinates to i) spherical coordinates, ii) cylindrical coordinates

   a) $(1,1,-2\sqrt{2})$ b) $(1,\sqrt{3},0)$

3) Sketch the graph of the equation in three dimensions

   a) $\rho = 4$ d) $\rho \sec\phi = 6$ g) $\theta = 0$

   b) $\phi = \pi/6$ e) $\tan\phi = 2$ h) $\phi = \pi/3$

   c) $\theta = \pi/4$ f) $\rho = 4\sec\phi$

4) Evaluate the given integrals.

   e) $\iiint_D x^2\, dV$ D is in the first octant, and bounded by the sphere $x^2+y^2+z^2=9$.

   f) $\iiint_D z^2\, dV$ D is bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=8$.

   g) $\iiint_D (x^2+y^2+z^2)\, dV$ D is the bounded by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$.

   h) $\iiint_D (x+y+z)\, dV$ D is in the first octant, and between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=16$.

5) Find the volume of the solid D using a triple integral in spherical coordinates.

a) D is inside the cone $z = \sqrt{3(x^2 + y^2)}$ and inside the sphere $x^2 + y^2 + z^2 = 16$.

b) D is outside the cylinder $x^2 + y^2 = 4$ and inside the sphere $x^2 + y^2 + z^2 = 9$.

c) D is outside the cone $z = \sqrt{x^2 + y^2}$, inside the sphere $x^2 + y^2 + z^2 = 9$ and above the xy-plane.

d) D is in the first octant, and inside the surfaces $x^2 + y^2 = 4x$ and $x^2 + y^2 + z^2 = 16$.

6) Change to an integral in spherical coordinates, and integrate.

a) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z \, dz \, dy \, dx$$

b) $$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz \, dz \, dy \, dx$$

c) $$\int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} xy \, dz \, dx \, dy$$

d) $$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 yz \, dz \, dy \, dx$$

e) $$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy$$

First Order Differential Equations

Recall: We have studied four types of first order differential equations:

1. **Separable Equation.** Can be written as

$$f(y)\,dy = g(x)\,dx$$

To solve it, integrate both sides,

$$\int f(y)\,dy = \int g(x)\,dx + C$$

2. **Linear Equation.** Can be written (in standard form) as

$$\frac{dy}{dx} + p(x)y = q(x)$$

To solve it, multiply by the integrating factor (I.F.) $e^{\int p(x)dx}$ and use the product rule backwards.

3. **Homogeneous Equation.** Can be written as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

To solve it, we change the *dependent* variable y and substitute

$$v = \frac{y}{x} \quad \Rightarrow \quad y = vx \quad \Rightarrow \quad \frac{dy}{dx} = x + v\frac{dv}{dx}$$

4. **Exact Equation.** Is of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad \text{where } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

It solution is

$$f(x,y) = C$$

where

$$f(x,y) = \int M(x,y)\,dx + g(y) \quad \text{or} \quad f(x,y) = \int N(x,y)\,dy + h(x)$$

Exercise 1: Solve $xy' - 4y = 0$.

Solution: This equation is .... and ....

First Method: Let us solve it as a .... equation: Write it as

$$x\frac{dy}{dx} - 4y = 0$$

Then

$$x\frac{dy}{dx} = \dots\dots\dots\dots$$

$$\frac{1}{\dots\dots} dy = \frac{4}{\dots\dots} dx$$

Integrate:

$$\int \frac{1}{\dots\dots} dy = \int \frac{4}{\dots\dots} dx + \dots\dots\dots$$

$$\dots\dots\dots\dots = \dots\dots\dots\dots\dots\dots + C = \dots\dots\dots\dots\dots + C$$

Exponentiate:

$$e^{\ln \dots\dots\dots} = e^{\ln \dots\dots\dots + C} = e^{\ln \dots\dots\dots} \cdot e^{C}$$

$$|y| = \dots\dots\dots\dots\dots$$

$$y = \pm \dots\dots\dots\dots \qquad (\text{ set } C_o = \dots\dots\dots)$$

Answer: $y = C_o \dots\dots\dots$

Second Method: Solve as a .... equation:

1. Bring to standard form:

$$\frac{dy}{dx} - \frac{4x}{x} y = 0 \qquad (*)$$

2. Find the I.F.

$$\int p(x)\, dx = \int \dots\dots\dots\dots\, dx = \dots\dots \int \dots\dots\dots\, dx$$

$$= \dots\dots\dots\dots\dots\dots = \ln\left(x^{\dots\dots}\right)$$

The I.F. is $e^{\ln \ldots\ldots\ldots\ldots} \;=\; \ldots\ldots\ldots\ldots\ldots\ldots$

3. Multiply (*) by $\ldots\ldots\ldots\ldots$:

$$x^{-4}\frac{dy}{dx} - \ldots\ldots\ldots\ldots\, y \;=\; 0$$

4. Product rule backwards:

$$\frac{d}{dx}\left[x^{-4}\ldots\ldots\ldots\ldots\right] \;=\; 0$$

5. Integrate:

$$x^{-4}\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots$$

Answer: $\underline{y \;=\; C_o\ldots\ldots\ldots}$

Check: $xy' - 4y \;=\; x(\ldots\ldots\ldots\ldots) - 4(\ldots\ldots\ldots\ldots) \;=\; \ldots\ldots\ldots\ldots$

Exercise 2: Solve $\sqrt{x}\,\frac{dy}{dx} - \sqrt{y} \;=\; x\sqrt{y}\,.$

Solution: Separable ? Try to separate the variables.

$$x^{1/2}\frac{dy}{dx} \;=\; xy^{1/2} + \ldots\ldots\ldots\ldots \;=\; (\ldots\ldots\ldots\ldots\ldots\ldots)\,y^{1/2}$$

We can separate the variables !

$$\ldots\ldots\ldots\ldots\ldots\, dy \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\, dx \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, dx$$

Integrate:

$$\int \ldots\ldots\ldots\ldots\ldots\, dy \;=\; \int \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, dx + C$$

$$\ldots\ldots\ldots\ldots\ldots \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C$$

$$y^{1/2} \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C/2$$

Answer: $\underline{y \;=\; \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots + C_o\right)^2}$ $\qquad (C_o = C/2)$

Exercise 3: Solve $x\dfrac{dy}{dx} - 3y = x^2 \quad (1).$

Solution: Separable ? ....

Homogeneous? ....

Linear? ....

1. Bring to standard form:

$$\frac{dy}{dx} - \ldots\ldots y = x^{\cdots\cdots} \qquad (2)$$

2. Find the I.F.

$$\int p(x)\,dx = \int \ldots\ldots\ldots\, dx = \ldots\ldots \int \ldots\ldots\ldots\, dx$$

$$= \ldots\ldots\ldots\ldots = \ln\left(x^{\cdots\cdots}\right)$$

The I.F. is $e^{\ln \ldots\ldots\ldots} = \ldots\ldots\ldots\ldots$

3. Multiply (2) by the I.F.:

$$x^{-3}\frac{dy}{dx} - \ldots\ldots\ldots\, y = x^{\cdots\cdots}$$

4. Product rule backwards:

$$\frac{d}{dx}\left[x^{-3}\ldots\ldots\ldots\right] = x^{\cdots\cdots}$$

5. Integrate:

$$x^{-3}\ldots\ldots\ldots = \int x^{\cdots\cdots}\,dx + C = \ldots\ldots\ldots + C$$

6. Solve for y: $y = x^{\cdots\cdots}\left(\ldots\ldots\ldots + C\right) = \ldots\ldots\ldots\ldots$

<u>**Answer:**</u> <u>$y = \ldots\ldots\ldots\ldots\ldots\ldots$</u>

<u>Check:</u> $\dfrac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots$

Substitute into (1):

$$x\left[\ldots\ldots\ldots\ldots\ldots\ldots\right] - 3\left[\ldots\ldots\ldots\ldots\ldots\ldots\right] = x^2$$

$$\ldots\ldots\ldots\ldots = x^2$$

Exercise 4: Solve $\left(x^3 + xy^2\right) dx = x^2 y\, dy$.

Solution: Separable ? ....

Linear? ....

Homogeneous? ....

Exact? ....

1. Write as

$$\frac{dy}{dx} = \frac{\qquad\qquad}{\ldots\ldots\ldots} = \frac{\qquad}{\ldots\ldots} + \frac{\qquad}{\ldots\ldots} = \frac{\quad}{\ldots} + \frac{\quad}{\ldots}$$

2. Set

$$v = \ldots\ldots\ldots \quad \Rightarrow \quad y = \ldots\ldots\ldots$$

$$\Rightarrow \quad \frac{dy}{dx} = v + \ldots\ldots\ldots\ldots$$

We obtain:

$$v + \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

3. Separate the variables:

$$\ldots\ldots\ldots\ldots\ldots\, dv = \ldots\ldots\ldots\ldots\ldots\, dx$$

$$\int \ldots\ldots\ldots\ldots\ldots\, dv = \int \ldots\ldots\ldots\ldots\ldots\, dx + C$$

$$\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots + C$$

$$v^2 = \ldots\ldots\ldots\ldots\ldots\ldots + 2C$$

$$v = \ldots\ldots\ldots\ldots\ldots\ldots + C_o \qquad \left(C_o = 2C\right)$$

5. Resubstitute:

$$\frac{y}{x} = \sqrt{\ldots\ldots\ldots\ldots\ldots + C_o}$$

<u>**Answer:**</u> <u>$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$</u>

Exercise 5: Solve the initial value problem

$$y' = 1 + x + y + xy, \qquad y(0) = 0.$$

Solution: Separable ? ……………………………………

Linear? ……………………………………

Homogeneous? ……………………………………

Exact? ……………………………………

1. Solve as a linear equation. Bring the equation to standard form:

$$\frac{dy}{dx} - [\ldots\ldots\ldots\ldots]\,y = 1 + x \qquad (*)$$

2. Find the I.F: $\int p(x)\,dx = \int \ldots\ldots\ldots\ldots\, dx = \ldots\ldots\ldots\ldots$

   The I.F. is $e^{\ldots\ldots\ldots\ldots}$

3. Multiply equation (*) by the I.F.:

$$\ldots\ldots\ldots\ldots \frac{dy}{dx} + \ldots\ldots\ldots\ldots \left[-(1+x)\right] y = \ldots\ldots\ldots\ldots$$

4. Product rule backwards:

$$\frac{d}{dx}\left[\ldots\ldots\ldots\ldots\, y\right] = \ldots\ldots\ldots\ldots$$

5. Integrate:

$$e^{-\left(x+\frac{x^2}{2}\right)} y = \int \ldots\ldots\ldots\ldots\, dx + C = \int \ldots\ldots\, du + C$$

$$\boxed{u = \ldots\ldots\ldots \quad \Rightarrow \quad du = \ldots\ldots\ldots}$$

$$= \ldots\ldots\ldots + C = C - \ldots\ldots\ldots$$

6. The general solution is: $y = \ldots\ldots\ldots\ldots$

7. Use the initial condition $y(0) = 0$: $0 = Ce^{\ldots\ldots} - \ldots\ldots \Rightarrow C = \ldots\ldots$

Answer: $y = \ldots\ldots\ldots\ldots$

Exercise 6: Solve $\left(2y^2 - 4x + 5\right) dx + \left(4 - 2y + 4xy\right) dy = 0$.

Solution: Separable ? ....

Linear? ....

Homogeneous? ....

Exact? $\dfrac{\partial M}{\partial y} = \ldots\ldots\ldots\ldots$, $\dfrac{\partial N}{\partial x} = \ldots\ldots\ldots\ldots$ $\Rightarrow$ ....

The solution is $f(x,y) = C$ where

$$\frac{\partial f}{\partial x} = M = 2y^2 - 4x + 5 \quad \text{and} \quad \frac{\partial f}{\partial y} = N = 4 - 2y + 4xy$$

To find f, let us integrate M:

$$f(x,y) = \int (\ldots\ldots\ldots\ldots\ldots\ldots)\, dx + g(\ldots\ldots)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + g(\ldots\ldots)$$

To find $g(y)$, use $\dfrac{\partial f}{\partial y} = N(x,y)$:

$$\frac{\partial f}{\partial y} = \ldots\ldots\ldots\ldots\ldots + g'(\ldots\ldots) = N(x,y) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Compare:

$$g'(y) = \ldots\ldots\ldots \Rightarrow g(y) = \int (\ldots\ldots\ldots\ldots)\, dy = \ldots\ldots\ldots\ldots$$

Therefore,

$$f(x,y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

<u>**Answer:**</u> <u>.................................... $= C$</u>

<u>Check:</u> The total differential of $f(x,y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= (\ldots\ldots\ldots\ldots\ldots\ldots)\, dx + (\ldots\ldots\ldots\ldots\ldots\ldots)\, dy$$

Exercise 7: Solve $\left(\sin x \sin y - xe^{y}\right) dy \;=\; \left(e^{y} + \cos x \cos y\right) dx$.

Solution: Separable ? ....

Linear? ....

Homogeneous? ....

Exact? Bring all terms to the left !

$$\underbrace{(\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;)}_{M(x,y)} dx \;+\; \underbrace{(\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;)}_{N(x,y)} dy \;=\; 0 \qquad (*)$$

$$\frac{\partial M}{\partial y} = \ldots\ldots\ldots\ldots\ldots\ldots \qquad \frac{\partial N}{\partial x} = \ldots\ldots\ldots\ldots\ldots\ldots \qquad \Rightarrow \qquad \ldots\ldots\ldots\ldots$$

The solution is $f(x,y) = C$ where $\dfrac{\partial f}{\partial x} = M(x,y)$ and $\dfrac{\partial f}{\partial y} = N(x,y)$

This time, let us integrate N:

$$f(x,y) \;=\; \int (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)\, dy \;+\; h(\ldots\ldots)$$

$$=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;+\; h(\ldots\ldots)$$

To find $h(x)$, use $\dfrac{\partial f}{\partial x} = M(x,y)$:

$$\frac{\partial f}{\partial x} \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \;+\; h'(\ldots\ldots) \;=\; M(x,y) \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Compare:

$$h'(x) \;=\; \ldots\ldots\ldots\ldots \qquad \Rightarrow \qquad h(x) \;=\; \ldots\ldots\ldots$$

Therefore,

$$f(x,y) \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

__Answer:__ $\underline{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;=\; C}$

__Check:__ The total differential of $f(x,y) \;=\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ is

$$df \;=\; \frac{\partial f}{\partial x}\, dx \;+\; \frac{\partial f}{\partial y}\, dy$$

$$=\; (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)\, dx \;+\; (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)\, dy$$

Additional Exercises:

1) Solve the following differential equations:

a) $y' = \sqrt[3]{64xy}$

b) $y' + y = 1$

c) $y \ln y \, dx - x \, dy = 0$

d) $xy' + y = \cos x$

e) $y' - 2y = e^{3x}$

f) $xyy' = y - 1$

g) $x\dfrac{dy}{dx} - 3y = x^4$

h) $(1+x^2)\, y' = \tan^{-1} x$

i) $(1+x^2)\, dy + (1+y^2)\, dx = 0$

j) $y' = y \sin x$

k) $y' + 2xy^2 = 0$

l) $xy' = y + 2(xy)^{1/2}$

m) $\tan\theta \, dr + 2r \, d\theta = 0$

n) $\dfrac{dy}{dx} - 2xy = 6xe^{x^2}$

o) $x^2 y' - 3xy - 2y^2 = 0$

p) $(x+y)y' = x - y$

q) $(x \ln x)y' + y = 3x^3$

r) $xy' = \sqrt{x^2 + y^2}$

s) $2xyy' = x^2 + 2y^2$

t) $y' + y = 2xe^{-x} + x^2$

u) $xy' = y + 2xe^{-y/x}$

v) $y - x + xy \cot x + xy' = 0$

w) $x \sin\dfrac{y}{x} y' = y \sin\dfrac{y}{x} + x$

x) $y' + y \cot x = 2x \csc x$

y) $(x^2 - 2y^2)\, dx + xy \, dy = 0$

z) $2\sqrt{x}\dfrac{dy}{dx} = \sqrt{1-y^2}$

2) Solve the following differential equations:

a) $(2x+3y)\, dx + (3x-4)\, dy = 0$

b) $(3x^2 - 2y^2)\, dx + (6y^2 - 4xy)\, dy = 0$

c) $(y^2 - 1)x \, dx + (x+2)y \, dy = 0$

d) $(\sin x \sin y - xe^y)\, dy = (e^y + \cos x \cos y)\, dx$

e) $(1+x^2)\, dy + 2xy \, dx = \cot x \, dx$

f) $(e^x \sin y + \tan y)\, dx + (e^x \cos y + x \sec^2 y)\, dy = 0$

g) $(x^2 + y/x)\, dx + (y^2 + \ln x)\, dy = 0$

h) $2x(1 + \sqrt{x^2 - y})\, dx = \sqrt{x^2 - y}\, dy$

i) $(\sin x \tan y + 1)\, dx - \cos x \sec^2 y \, dy = 0$

3) Solve in at least two different ways:

a) $y' + y\tan x = 0$

b) $y' - y\tan x = 0$

c) $(1+x)\dfrac{dy}{dx} = 4y$

d) $y' + 2xy = 0$

e) $xy' = 2x + 3y$

f) $xy' = x + y$

4) Solve the following initial value problems:

a) $y' = xe^x, \quad y(1) = 3$

b) $y' + 2y = 2, \quad y(0) = 1$

c) $y' = e^{3x-2y}, \quad y(0) = 0$

d) $xy' - y = x, \quad y(1) = 2$

e) $x(x^2 - 4)y' = 1, \quad y(1) = 0$

f) $xyy' = (x+1)(y+1), \quad y(1) = 0$

g) $y' = xe^x, \quad y(1) = 3$

h) $8\cos^2 y\, dx + \csc^2 x\, dy = 0, \quad y\left(\dfrac{\pi}{12}\right) = \dfrac{\pi}{4}$

i) $y' = (1-y)\cos x, \quad y(\pi) = 0$

j) $xy' + 3y = 2x^5, \quad y(2) = 1$

k) $(x^2 + 4)y' + 3xy = x, \quad y(0) = 1$

l) $(\sin x \tan y + 1)\, dx - \cos x \sec^2 y\, dy = 0, \quad y(0) = \dfrac{\pi}{4}$

Second Order Homogeneous Equations

Recall: Consider the *homogeneous* equation with *constant coefficients*,

$$y'' + \underset{\uparrow}{b}y' + \underset{\uparrow}{c}y = 0 \quad \text{-------------} \boxed{\text{constant}}$$

Its *characteristic equation*

$$\lambda^2 + b\lambda + c = 0$$

always has two solutions λ_1, λ_2:

Case 1: $\lambda_1 \neq \lambda_2$ are *real.* (real, distinct roots)

The solution is $\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$

Case 2: $\lambda_1 = \lambda_2 \ (= \lambda)$ (real, repeated roots)

The solution is $\boxed{y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}}$

Case 3: $\lambda_1, \lambda_2 = r \pm is$ where $s \neq 0$. (complex roots)

The solution is $\boxed{y = e^{rx}\left[c_1 \cos(sx) + c_2 \quad \sin(sx)\right]}$

Exercise 1: Find the *general solution* of the given homogeneous equations.

a) $y'' - 5y' + 6y = 0$.

Solution: The characteristic equation is

$$\lambda^2 - \text{..............................} = 0$$

$$(\lambda - \text{...........})(\lambda - \text{...........}) = 0$$

$$\lambda_1 = \text{...........}, \qquad \lambda_2 = \text{...........}$$

__Answer:__ $y =$ ....

b) $2y'' + 2y' - 3y = 0$.

Solution: The characteristic equation is

$$2\lambda^2 + \text{.......................} = 0 \qquad \text{(Difficult to factor !)}$$

$$\lambda_1, \lambda_2 = \frac{-2 \pm \sqrt{\qquad\qquad}}{2 \cdot 2} = \frac{-2 \pm \sqrt{\qquad}}{\text{.................}} = -\frac{1}{\text{.....}} \pm \frac{\sqrt{\quad}}{\text{.....}}$$

The general solution is $\quad y = c_1\, e^{\cdots\cdots\cdots} + c_2\, e^{\cdots\cdots\cdots}$

Answer: $\quad y = e^{-x/2}\left[\text{...}\right]$

c) $y'' + 2y' = 0$.

Solution: The characteristic equation is

$$\text{..............................} = 0$$

$$\lambda(\text{...............}) = 0 \quad \Rightarrow \quad \lambda_1 = \text{.....}, \qquad \lambda_2 = \text{.....}$$

The general solution is $\quad y = c_1\, e^{\cdots\cdots} + c_2\, e^{\cdots\cdots}$

Answer: $\quad y = \text{.....} + c_2\, e^{\cdots\cdots}$

d) $4y'' + 20y' + 25y = 0$.

Solution: The characteristic equation is

$$\text{....................................} = 0$$

$$(\lambda + \text{.....})^2 = 0$$

$$\lambda_1 = \lambda_2 = \text{.............} \qquad (\text{..................................}\text{. root})$$

The general solution is $\quad y = c_1 \text{ ....} + c_2 \text{ ....}$

Answer: $\quad y = e^{-5x/2}\left[\text{.....................}\right]$

e) $y'' - 4y' + 5y = 0$.

Solution: The characteristic equation is

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0 \qquad \text{(Difficult to factor !)}$$

$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{\qquad\qquad}}{2} = \frac{4 \pm \sqrt{\qquad}}{2} = \underbrace{2}_{r} \pm \underbrace{\ldots\ldots}_{s} \cdot i$$

<u>**Answer:**</u> $\underline{y = e^{\ldots\ldots\ldots}[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]}$

f) $y'' + 9y = 0$.

Solution: The characteristic equation is

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$\lambda^2 = \ldots\ldots\ldots\ldots \quad \Rightarrow \quad \lambda_1, \lambda_2 = \pm \underbrace{\ldots\ldots\ldots}_{s} \cdot i$$

<u>**Answer:**</u> $\underline{y = c_1 \ldots\ldots\ldots\ldots + c_2 \ldots\ldots\ldots\ldots\ldots}$

g) $y'' - 9y = 0$.

Solution: The characteristic equation is

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$\lambda^2 = \ldots\ldots\ldots\ldots \quad \Rightarrow \quad \lambda_1 = \ldots\ldots, \qquad \lambda_2 = \ldots\ldots$$

<u>**Answer:**</u> $\underline{y = c_1 \ldots\ldots\ldots\ldots + c_2 \ldots\ldots\ldots\ldots\ldots}$

Additional Exercises:

1) Solve the following differential equations:

a) $y''+y'-6y=0$

b) $y''+2y'+y=0$

c) $y''+8y=0$

d) $2y''-4y'+8y=0$

e) $y''-4y'+4y=0$

f) $y''-9y'+20y=0$

g) $2y''+2y'+3y=0$

h) $4y''-12y'+9y=0$

i) $y''+y'=0$

j) $y''-6y'+25y=0$

k) $4y''+20y'+25y=0$

l) $y''+2y'+3y=0$

m) $y''=4y$

n) $4y''-8y'+7y=0$

o) $2y''+y'-y=0$

p) $16y''-8y'+y=0$

q) $y''+4y'+5y=0$

r) $y''+4y'-5y=0$

2) Solve the following initial value problems:

a) $y''-5y'+6y=0, \quad y(1)=e^2, \quad y'(1)=3e^2$

b) $y''-6y'+5y=0, \quad y(0)=3, \quad y'(0)=11$

c) $y''-6y'+9y=0, \quad y(0)=0, \quad y'(0)=5$

d) $y''+4y'+5y=0, \quad y(0)=1, \quad y'(0)=0$

e) $y''+4y'+2y=0, \quad y(0)=-1, \quad y'(0)=2+3\sqrt{2}$

f) $y''+8y'-9y=0, \quad y(1)=2, \quad y'(1)=0$

The Method of Undetermined Coefficients

Recall: Consider a linear equation with *constant coefficients.*

$$y'' + \underbrace{b}_{}y' + \underbrace{c}_{}y = r(x) \quad \boxed{\text{constant}}$$

If the force function $r(x)$ is of special form, then *one* solution y_p looks like $r(x)$:

Case 1: $r(x) = a e^{mx}$. Then $\boxed{y_p = Ae^{mx}}$.

Case 2: $r(x) = ax^2 + bx + c$. Then $\boxed{y_p = Ax^2 + Bx + C}$.

Case 3: $r(x) = a\cos(\omega x) + bsin(\omega x)$ Then $\boxed{y_p = A\cos(\omega x) + Bsin(\omega x)}$.

Exercise 1: Solve $y'' + 4y' + 5y = 2e^{-4x}$ (1).

Solution: 1. Find y_h. The characteristic equation is

$$\lambda^2 + \ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$\lambda_1, \lambda_2 = \frac{-4 \pm \sqrt{\qquad}}{2} = \frac{-4 \pm \sqrt{\qquad}}{2} = \underbrace{-2}_{r} \pm \underbrace{\ldots\ldots}_{s} i$$

$$y_h = e^{\ldots\ldots}[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]$$

2. Find y_p. Since $r(x) = \ldots\ldots\ldots\ldots$

we choose $y_p = \ldots\ldots\ldots$ $\Rightarrow$ $y_p' = \ldots\ldots\ldots\ldots$, $y_p'' = \ldots\ldots\ldots\ldots$

Substitute into (1): $\ldots\ldots\ldots\ldots + 4[\ldots\ldots\ldots\ldots] + 5[\ldots\ldots\ldots] = 2e^{-4x}$

$$\ldots\ldots\ldots\ldots = 2e^{-4x}$$

Compare coefficients: $5A = \ldots\ldots$ $\Rightarrow$ $A = \ldots\ldots\ldots$

Therefore, $y_p = \ldots\ldots\ldots\ldots$

3. **Answer:** The general solution is $y = y_h + y_p$, or

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots$$

Exercise 2: Solve $y'' - 3y' + 2y = x^3$ (*).

Solution: 1. Find y_h: The characteristic equation is

$$\lambda^2 - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$(\lambda - \ldots\ldots\ldots)(\lambda - \ldots\ldots\ldots) = 0 \quad \Rightarrow \quad \lambda_1 = \ldots\ldots, \quad \lambda_2 = \ldots\ldots$$

$$y_h = c_1 \ldots\ldots + c_2 \ldots\ldots$$

2. Find y_p: Since $\underbrace{r(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 0}_{\text{polynomial degree } \ldots\ldots}$

we choose $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots Cx + D$

$$\Rightarrow \quad y_p' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \qquad y_p'' = \ldots\ldots\ldots\ldots\ldots\ldots$$

Substitute into (*):

$$\left[\ \ldots\ldots\ldots\ldots\ \right] - 3\left[\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \right] + 2\left[\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \right] = x^3$$

$$\ldots\ldots x^3 + \left[\ \ldots\ldots\ldots\ \right]x^2 + \left[\ \ldots\ldots\ldots\ldots\ldots\ldots\ \right]x + \left[\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \right] = x^3$$

Compare coefficients:

x^3: $2A = \ldots\ldots \Rightarrow A = \ldots\ldots\ldots$

x^2: $\ldots\ldots\ldots\ldots\ldots\ldots = 0 \Rightarrow B = \ldots\ldots\ldots = \ldots\ldots\ldots$

x: $\ldots\ldots\ldots\ldots\ldots\ldots = 0 \Rightarrow C = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots = \ldots\ldots$

const: $\ldots\ldots\ldots\ldots\ldots\ldots = 0 \Rightarrow D = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots = \ldots\ldots$

Therefore,

$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. **Answer:** The general solution is $y = y_h + y_p$, or

$$\underline{y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 3: Solve $y'' + 2y' + y = 3\cos 3x$ (*).

Solution: 1. Find y_h: The characteristic equation is

$$\lambda^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$(\lambda + \ldots\ldots)(\lambda + \ldots\ldots) = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \ldots\ldots\ldots$$

$$y_h = c_1 \ldots\ldots\ldots + c_2 \ldots\ldots\ldots\ldots\ldots$$

2. Find y_p: Since $r(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 0 \cdot \sin 3x$

we choose $y_p = \ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots$

$$\Rightarrow \quad y_p' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Rightarrow \quad y_p'' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Substitute into (*):

$$\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right] + 2\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right]$$

$$+ \left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right] = 3\cos 3x$$

$$\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right]\cos 3x + \left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right]\sin 3x = 3\cos 3x$$

Compare coefficients:

$\cos 3x$: $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$ (1)

$\sin 3x$: $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0 \quad \Rightarrow \quad 8B = \ldots\ldots\ldots, \quad B = \ldots\ldots\ldots$

Substitute $B = \ldots\ldots\ldots$ into (1),

$$-8A + 6\ldots\ldots\ldots = 3 \quad \Rightarrow \quad \ldots\ldots\ldots A = 3, \quad A = \ldots\ldots\ldots = \ldots\ldots\ldots$$

$$\Rightarrow \quad B = \ldots\ldots\ldots = \ldots\ldots\ldots$$

$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. **Answer:** The general solution is $y = y_h + y_p$, or

$$y = e^{-x}[\ldots\ldots\ldots\ldots\ldots] - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Exercise 4: Consider $y'' + y' - 6y = r(x)$.

Find the solution y_h of the related homogenous equation.
Then find the correct choice of y_p for the given force function $r(x)$.

Solution: <u>Find y_h:</u> The characteristic equation is

$$\lambda^2 + \ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$(\lambda + \ldots\ldots)(\lambda + \ldots\ldots) = 0 \quad \Rightarrow \quad \lambda_1 = \ldots\ldots\ldots, \quad \lambda_2 = \ldots\ldots\ldots$$

$$y_h = c_1 \ldots\ldots\ldots + c_2 \ldots\ldots\ldots\ldots$$

a) $r(x) = e^{-2x}$.

Solution: We choose $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Compare with y_h : ....

<u>Answer:</u> $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

b) $r(x) = e^{2x}$.

Solution: We choose $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Compare with y_h : ....

<u>Answer:</u> $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

c) $r(x) = e^{4x} + 2e^{-3x}$.

Solution: We choose $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Compare with y_h : ....

<u>Answer:</u> $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

d) $r(x) = \underbrace{x^2 + x}_{\ldots\ldots\ldots\ldots\ldots\ldots} + 2e^{-3x}$.

Solution: We choose $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Compare with y_h : ....

<u>Answer:</u> $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

e) $r(x) = \underbrace{x^2}_{\dots} e^{2x}$.

Solution: We choose y_p = ……………………………………

Compare with y_h : ……………………………………………………………………………………

……………………………………………………………………………………

<u>**Answer:**</u> y_p = ……………………………………………………

f) $r(x) = 2\cosh 3x$.

Solution: First write $r(x)$ = ………………………………….= ……………………..

We therefore choose y_p = ……………………………………

Compare with y_h : ……………………………………………………………………………………

<u>**Answer:**</u> y_p = ……………………………………………

Exercise 5: Consider $y'' + 4y = r(x)$.

Find the solution y_h of the related homogenous equation.
Then find the correct choice of y_p for the given force function $r(x)$.

Solution: <u>Find y_h:</u> The characteristic equation is

$\lambda^2 +$…………… $= 0$

$\lambda^2 =$ ……… $\Rightarrow$ $\lambda_1, \lambda_2 =$ ……………… = ………

$y_h = c_1$……………… $+ c_2$……………………

a) $r(x) = e^{2x} \cos x$.

Solution: We choose y_p = ………………………………………………

Compare with y_h : ……………………………………………………………

<u>**Answer:**</u> y_p = ……………………………………………………

b) $r(x) = \cos 2x - \sin 2x$.

Solution: We choose y_p = ………………………………………………

Compare with y_h : ……………………………………………………………

Answer: y_p = ………………………………………………………

c) $r(x) = \cos x + \cos 2x$.

Solution: We choose y_p = ………………………………………………

Compare with y_h : ……………………………………………………………

Answer: y_p = ………………………………………………………

d) $r(x) = e^x \cos 2x$.

Solution: We choose y_p = ………………………………………………

Compare with y_h : ……………………………………………………………

Answer: y_p = ………………………………………………………

e) $r(x) = x\cos 2x - \sin 2x$.

Solution: We choose y_p = ………………………………………………

Compare with y_h : ……………………………………………………………………………………

……………………………………………………………………………………

Answer: y_p = ………………………………………………………

f) $r(x) = \sinh 2x$.

Solution: First write $r(x)$ = ……………………………………

We therefore choose y_p = ………………………………………………

Compare with y_h : ……………………………………………………………………………………

Answer: y_p = ………………………………………………………

Additional Exercises:

1) Find the general solution:

a) $y''+4y'+5y=e^{3x}$

b) $2y''+2y'-4y=e^{-x}$

c) $y''+5y'+4y=\cos x$

d) $y''-4y=4x^2+4x+6$

e) $y''-3y'+2y=x^3$

f) $y''+2y'=3\sin x-\cos x$

2) Find the general solution:

a) $y''-2y'-3y=3e^{2x}$

b) $y''-2y'-3y=-3xe^{-x}$

c) $y''+2y'+5y=3\sin 2x$

d) $y''+2y'=3+4\sin 2x$

e) $y''+9y=x^2e^{3x}+6$

f) $y''+2y'+y=2e^{-x}$

g) $y''+y=3\sin 2x+x\cos 2x$

h) $2y''+3y'+y=x^2+3\sin x$

i) $y''+\omega_o^2 y=\cos\omega x \quad (\omega_o^2\neq\omega^2)$

j) $y''+\omega_o^2 y=\cos\omega_o x$

k) $y''-y'+4y=2\sinh x$

l) $y''-y'-2y=\cosh 2x$

3) Find the general solution:

a) $y''+4y=4\cos 2x+6\cos x+8x^2-4x$

b) $y''+9y=2\sin 3x+4\sin x-26e^{-2x}+27x^3$

4) Solve the following initial value problems:

a) $y''+y'-2y=2x, \quad y(0)=0, \quad y'(0)=1$

b) $y''+4y=x^2+3e^x, \quad y(0)=0, \quad y'(0)=2$

c) $y''-2y'+y=xe^x+4, \quad y(0)=1, \quad y'(0)=1$

d) $y''-2y'-3y=3xe^{2x}, \quad y(0)=1, \quad y'(0)=0$

e) $y''+4y=3\sin 2x, \quad y(0)=2, \quad y'(0)=-1$

f) $y''+2y'+5y=4e^{-x}\cos 2x, \quad y(0)=1, \quad y'(0)=0$

5) Specify the general form of y_p. No need to find the values of $A,B,C,\ldots$

a) $y''+3y'=2x^4+x^2e^{-3x}+\sin 3x$

b) $y''+y=x(1+\sin x)$

c) $y''-5y'+6y=e^x\cos 2x+e^{2x}(3x+4)\sin x$

d) $y''+2y'+2y=3e^{-x}+2e^{-x}\cos x+4e^{-x}x^2\sin x$

e) $y''-4y'+4y=2x^2+4xe^{2x}+x\sin 2x$

f) $y''+4y=x^2\sin 2x+(6x+7)\cos 2x$

g) $y''+3y'+2y=e^x(x^2+1)\sin 2x+3e^{-x}\cos x+4e^x$

h) $y''+2y'+5y=3xe^{-x}\cos 2x-2xe^{-2x}\cos x$

The Method of Variation of Parameters

Recall: Consider a linear equation with *arbitrary coefficients.*

$$a(x)y'' + b(x)y' + c(x)y = r(x)$$

variable

If the solution of the related homogeneous equation is

$$y_h = c_1 y_1 + c_2 y_2$$

Then we look for one solution y_p of the form

$$y_p = u(x)y_1(x) + v(x)y_2(x).$$

To find $u'(x)$ and $v'(x)$, we solve the system

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{r(x)}{a(x)} \end{bmatrix}.$$

Exercise 1: Find the general solution of $y'' + 4y = \tan x$.

Solution: 1. <u>Find y_h:</u> The characteristic equation is

$$\lambda^2 + \ldots\ldots = 0 \quad \Rightarrow \quad \lambda^2 = \ldots\ldots \quad \Rightarrow \quad \lambda_1, \lambda_2 = \ldots\ldots\ldots\ldots$$

$$y_h = c_1 \cos 2x + c_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. <u>Find y_p:</u>

Question: Can we expect $y_p = A \tan 2x$?

Answer: ....

Use variation of parameters. Set

$$y_p = u \underbrace{\cos 2x}_{y_1} + v \underbrace{\ldots\ldots\ldots\ldots}_{y_2}$$

and solve the system

$$\begin{bmatrix} y_1 & y_2 \\ \ldots\ldots & \ldots\ldots \end{bmatrix} \begin{bmatrix} \ldots\ldots \\ \ldots\ldots \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots\ldots \end{bmatrix}$$

Here,

$$\begin{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots\ldots \end{bmatrix}$$

Let us use Cramer's Rule. The determinant of this matrix is

$$W = \begin{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \end{vmatrix} = 2\cos^2 2x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

Then

$$u' = \frac{\begin{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots & \sin 2x \\ \ldots\ldots\ldots\ldots\ldots & 2\cos 2x \end{vmatrix}}{W} = \frac{\qquad\qquad\qquad}{2}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$v' = \frac{\begin{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \end{vmatrix}}{W} = \frac{\qquad\qquad\qquad}{\ldots\ldots\ldots} = \frac{1}{2}\ldots\ldots\ldots\ldots$$

Integrate

$$u = -\frac{1}{2}\int \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \, dx = -\frac{1}{2}\int \frac{\qquad\qquad}{\cos 2x}\, dx$$

$$= -\frac{1}{2}\int \frac{1 \; - \qquad}{\cos 2x}\, dx = -\frac{1}{2}\int\left[\sec 2x - \ldots\ldots\ldots\ldots\ldots\ldots\right] dx$$

$$= -\frac{1}{2}\left[\frac{1}{2}\ln\left|\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right| - \frac{1}{2}\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$= \frac{1}{4}\left[\sin 2x - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$v = \frac{1}{2}\int \ldots\ldots\ldots\ldots\ldots \, dx = -\frac{1}{4}\ldots\ldots\ldots\ldots\ldots$$

Therefore,

$$y_p = u\cos 2x + v\sin 2x$$

$$= \frac{1}{4}\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]\cos 2x - \frac{1}{4}\ldots\ldots\ldots\ldots \sin 2x$$

$$= -\frac{1}{4}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. **Answer:** The general solution is $y = y_h + y_p$, or

$$\underline{y = c_1\cos 2x + c_2\sin 2x - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

Exercise 2: Solve the IVP

$$y'' + 4y' + 4y = \frac{e^{-2x}}{1+x^2}, \qquad y(0)=1, \quad y'(0)=0.$$

Solution: 1. Find y_h: The characteristic equation is

$$\lambda^2 + \ldots\ldots\ldots\ldots\ldots\ldots = 0$$

$$(\lambda + \ldots\ldots)(\lambda + \ldots\ldots) = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 = \ldots\ldots\ldots \quad (\ldots\ldots\ldots\ldots\ldots \text{ root})$$

$$y_h = c_1 \ldots\ldots\ldots\ldots + c_2 \ldots\ldots\ldots\ldots\ldots\ldots$$

2. Find y_p: Question: Can we expect $y_p = A \frac{e^{-2x}}{1+x^2}$???

Answer: ....

Use variation of parameters. Set

$$y_p = u \underbrace{\ldots\ldots\ldots\ldots}_{y_1} + v \underbrace{\ldots\ldots\ldots\ldots}_{y_2}$$

and solve the system

$$\begin{bmatrix} e^{-2x} & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots\ldots \end{bmatrix}$$

Write as two equations:

$$e^{-2x} u' + \ldots\ldots\ldots\ldots\ldots\ldots v' = 0 \qquad (1)$$

$$-2e^{-2x} u' + \ldots\ldots\ldots\ldots\ldots\ldots v' = \ldots\ldots\ldots\ldots \qquad (2)$$

Cut e^{-2x}. (1) gives

$$u' = \ldots\ldots\ldots\ldots \qquad (3)$$

Substitute into (2):

$$-2(\ldots\ldots\ldots\ldots\ldots) + \ldots\ldots\ldots\ldots\ldots\ldots v' = \frac{1}{1+x^2}$$

$$\Rightarrow \quad v' = \ldots\ldots\ldots\ldots$$

Substitute into (3): $\quad u' = \ldots\ldots\ldots\ldots$

Integrate:

$$u = \int \ldots\ldots \frac{\ldots\ldots x}{1+x^2}\,dx = \ldots\ldots\ldots\ldots \qquad \left(\text{Use } \int \frac{f'}{f} = \ldots\ldots\ldots\right)$$

$$v = \int \frac{1}{\ldots\ldots\ldots\ldots}\,dx = \ldots\ldots\ldots\ldots$$

Therefore,

$$y_p = ue^{-2x} + vxe^{-2x}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3. The general solution is $y = y_h + y_p$, or

$$y = c_1 e^{-2x} + c_2 \ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y = e^{-2x}\left[c_1 + c_2 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

4. Use the initial conditions $y(0) = 1, \quad y'(0) = 0$

$y(0) = 1$ gives $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 1$

$$\Rightarrow \quad c_1 = \ldots\ldots$$

Differentiate y using the product rule.

$$y' = -2e^{-2x}\left[c_1 + c_2 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$+ \; e^{-2x}\left[\; c_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$y'(0) = 1$ gives $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 1$

$$\Rightarrow \quad c_2 = \ldots\ldots$$

<u>Answer:</u> $y = e^{-2x}\left[\ldots\ldots\ldots + x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]$

Additional Exercises:

1) Find the general solution:

a) $y'' + y = \tan x$

b) $y'' + 4y' + 4y = x^{-2}e^{-2x}$

c) $y'' + 9y = 9\sec^2 3x$

d) $y'' + 4y = \tan 2x$

e) $y'' + 2y' + y = e^{-x} \ln x$

f) $4y'' + y = 2\sec(x/2)$

g) $y'' + 4y = 3\csc 2x$

h) $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$

i) $y'' - 2y' - 3y = 64xe^{-x}$

j) $y'' + 2y' + 5y = e^{-x} \sec 2x$

k) $y'' - 3y' + 2y = (1+e^{-x})^{-1}$

2) Solve each equation twice: Using the method of undetermined coefficients, and using the method of variation of parameters.

a) $y'' - 5y' + 6y = 2e^x$

b) $y'' + 2y' + y = 3e^{-x}$

c) $y'' - y' - 2y = 2e^{-x}$

d) $4y'' - 4y' + y = 16e^{x/2}$

3) Show that the equation $y'' + y = f(x)$ has general solution

$$y = c_1 \cos x + c_2 \sin x + \int_{x_o}^{x} f(t) \sin(x-t)\, dt$$

The Laplace Transform

$f(x)$	$F(s)$
1	$\frac{1}{s}$
x^n	$\frac{n!}{s^{n+1}}$
e^{cx}	$\frac{1}{s-c}$
$\cos ax$	$\frac{s}{s^2+a^2}$
$\sin ax$	$\frac{a}{s^2+a^2}$
$e^{cx} f(x)$	$F(s-c)$
$x\, f(x)$	$-F'(s)$
$y'(x)$	$sY(s) - y(0)$
$y''(x)$	$s^2Y(s) - sy'(0) - y(0)$

Exercise 1: Find the Laplace transforms of the given functions.

a) $f(x) = 3x + 4e^{2x} - 2.$

Solution: $F(s) = 3 \frac{1}{\ldots\ldots\ldots\ldots} + 4 \frac{1}{\ldots\ldots\ldots\ldots} - 2 \frac{1}{\ldots\ldots\ldots\ldots}$

Answer: $F(s) =$ ……………………………………………

b) $f(x) = \sin 3x - 2\cos 3x + 2\sin 4x + \cos 4x.$

Solution:

$$F(s) = \frac{\qquad}{s^2 + \ldots\ldots} - 2\frac{\qquad}{s^2 + \ldots\ldots} + 2\frac{\qquad}{s^2 + \ldots\ldots} + \frac{\qquad}{s^2 + \ldots\ldots}$$

__Answer:__ $$F(s) = \frac{\qquad}{s^2 + \ldots\ldots} + \frac{\qquad}{s^2 + \ldots\ldots}$$

c) $f(x) = e^{2x}\cosh 3x + e^{3x}\sinh 2x.$

Solution: Write

$$f(x) = e^{2x}\frac{\qquad\qquad}{2} + e^{3x}\frac{\qquad\qquad}{2}$$

$$f(x) = \ldots\ldots\ldots e^{5x} - \ldots\ldots\ldots e^{x} + \ldots\ldots\ldots e^{-x}$$

Then

$$F(s) = \frac{1}{\ldots\ldots\; s - \ldots\ldots} - \frac{1}{\ldots\ldots\ldots\; s - \ldots\ldots} + \frac{1}{\ldots\ldots\ldots\; s - \ldots\ldots}$$

__Answer:__

$$F(s) = \frac{1}{s - \ldots\ldots} - \frac{1}{\ldots\ldots(s - \ldots\ldots)} + \frac{1}{\ldots\ldots(s + \ldots\ldots)}$$

d) $f(x) = e^{2x}\left(\cos 3x - 2\sin 3x\right).$

Solution: Because

$$\cos 3x - 2\sin 3x \;\leftrightarrow\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{\qquad}{s^2+9}$$

Then

$$e^{2x}\left(\cos 3x - 2\sin 3x\right) \;\leftrightarrow\; \frac{\qquad\qquad}{\left(s^2 - \ldots\ldots\ldots\right)^2 + \ldots\ldots\ldots}$$

__Answer__ $$F(s) = \frac{\qquad\qquad}{\left(s^2 - \ldots\ldots\ldots\right)^2 + \ldots\ldots\ldots}$$

e) $f(x) = x\sin x.$

Solution: Because $\sin x \leftrightarrow$ ....

Then $x\sin x \leftrightarrow -\dfrac{d}{ds}\dfrac{}{s^2+1} = -\dfrac{}{(s^2+1)^2}$

<u>Answer:</u> $F(s) = -\dfrac{}{(s^2+1)^2}$

Exercise 2: Find the inverse Laplace transforms of the given functions.

a) $F(s) = \dfrac{4}{s^2} - \dfrac{2}{s+3} + \dfrac{4s+1}{s^2+4}$.

Solution: Write

$$F(s) = \ldots\ldots \frac{1}{s^2} - \ldots\ldots \frac{1}{s-(\ldots\ldots\ldots)} + \ldots\ldots \frac{s}{s^2+4} + \ldots\ldots \frac{2}{s^2+4}$$

Then

$$f(x) = \ldots\ldots\ \ldots\ldots - \ldots\ldots\ \ldots\ldots + \ldots\ldots\ \ldots\ldots + \ldots\ldots\ \ldots\ldots$$

<u>Answer:</u> $f(x) = \ldots\ldots\ \ldots\ldots - \ldots\ldots\ \ldots\ldots + \ldots\ldots\ \ldots\ldots + \ldots\ldots\ \ldots\ldots$

b) $F(s) = \dfrac{12}{s^4}$.

Solution: By the table, $x^n \leftrightarrow \dfrac{}{\ldots\ldots\ldots\ldots\ldots}$

Write

$$F(s) = \frac{12}{s^4} = \frac{12}{\ldots\ldots\ldots}\frac{3!}{s^4} = \ldots\ldots\ldots \frac{3!}{s^4} \leftrightarrow \ldots\ldots\ldots\ldots\ldots$$

<u>Answer:</u> $f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

c) $F(s) = \dfrac{2s+6}{s^3+9s}$.

Solution: Do a partial fraction decomposition.

$$\frac{2s+6}{s^3+9s} = \frac{2s+6}{s(\;\ldots\ldots\;)} = \frac{A}{\ldots\ldots} + \frac{\qquad\qquad}{\ldots\ldots\ldots\ldots\ldots\ldots}$$

$$2s+6 = A(\ldots\ldots\ldots\ldots\ldots) + (\ldots\ldots\ldots\ldots\ldots)\ldots\ldots\ldots\ldots$$

$$= (\ldots\ldots\ldots\ldots\ldots)s^2 + (\ldots\ldots\ldots)s + (\ldots\ldots\ldots)$$

Compare coefficients:

$$\begin{array}{llll} s^2: & A+B = \ldots\ldots\ldots & \Rightarrow & B = \ldots\ldots\ldots = \ldots\ldots\ldots \\ s: & C = \ldots\ldots\ldots & & \\ \text{const}: & 9A = \ldots\ldots\ldots & \Rightarrow & A = \ldots\ldots\ldots \end{array}$$

$$\Rightarrow \; F(s) = \frac{\quad}{s} + \frac{\qquad}{s^2+9} = \frac{2}{3}\left[\frac{\quad}{s} - \frac{\qquad}{s^2+9} + \frac{\qquad}{s^2+9}\right]$$

<u>**Answer:**</u> $f(x) = \dfrac{2}{3}\left[\ldots\ldots\ldots - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$

d) $F(s) = \dfrac{s-1}{(s+2)^3}$.

Solution: The partial fraction decomposition is easy. Write

$$F(s) = \frac{s-1}{(s+2)^3} = \frac{(s+\ldots\ldots-\ldots\ldots)-1}{(s+2)^3} = \frac{(s+\ldots\ldots)-\ldots\ldots}{(s+2)^3}$$

$$= \frac{s+2}{(s+2)^3} - \frac{\qquad}{(s+2)^3} = \frac{1}{\ldots\ldots\ldots\ldots} - \frac{\qquad}{(s+2)^3}$$

We know (cut the 2):

$$\frac{1}{s^2} \leftrightarrow \ldots\ldots\ldots\ldots \quad \text{and} \quad \frac{3}{s^3} = \frac{3}{\ldots\ldots}\frac{2!}{s^3} \leftrightarrow \ldots\ldots\ldots\ldots$$

$$\Rightarrow \quad \frac{1}{(s+2)^2} - \frac{3}{(s+2)^3} \leftrightarrow \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

<u>**Answer:**</u> $f(x) = e^{2x}[\ldots\ldots\ldots - \ldots\ldots\ldots]$

e) $F(s) = \dfrac{4-11s}{\left(s^2+s\right)\left(s^2+4\right)}$.

Solution: Factor the denominator and do a partial fraction decomposition.

$$F(s) = \frac{4-11s}{\ldots(\ldots\ldots\ldots)\left(s^2+4\right)} = \frac{A}{\ldots} + \frac{}{\ldots\ldots\ldots} + \frac{}{\ldots\ldots\ldots}$$

Multiply by ....

$$4-11s = A(\ldots\ldots)(\ldots\ldots) + Bs(\ldots\ldots) + (Cs+D)s(\ldots\ldots)$$

$$= (\ldots\ldots\ldots\ldots)s^2 + (\ldots\ldots\ldots\ldots)s + (\ldots\ldots\ldots\ldots)$$

$s=0$: .... = ... $\Rightarrow$ $A =$ ....

$s=$......: .... = ... $\Rightarrow$ $B =$ ....

$s=$......: .... $= 10A + 5B +$ ....

.... = ... + ...

.... $= 2C + 2D$

.... $= C + D$ (1)

$s=$......: .... $= 24A + 16B +$ ....

.... = ... + ...

.... = ... + ...

.... $= 2C + D$ (2)

(1) − (2): $-3 =$ .... $\Rightarrow$ $C =$ ....

Then (1) gives $D = -1 -$ .... = ... $\Rightarrow$ $D =$ ....

$$\text{Thus, } F(s) = \frac{}{s} + \frac{}{s+1} + \frac{}{s^2+4}$$

$$= \frac{}{s} - \ldots\ldots\frac{}{s+1} + \ldots\ldots\frac{s}{s^2+4} - \ldots\ldots\frac{}{s^2+4}$$

<u>**Answer:**</u> $f(x) =$ .... − .... + ... + ...

f) $F(s) = \dfrac{3s+5}{s^2-6s+25}$.

Solution: Can we factor the denominator ?

$$b^2 - 4ac = \text{……………………} = \text{……….} < 0 \qquad \text{……………………………}$$

Therefore, complete the square.

$$F(s) = \frac{3s+5}{s^2-6s+25} = \frac{3s+5}{\left(s^2-6s+\text{……}\right)+25-\text{……}}$$

$$= \frac{3s+5}{(s-\text{……})^2+\text{……}}$$

$$= \frac{3(s-\text{……}+\text{……})+5}{(s-\text{……})^2+\text{……}} = \frac{3(s-\text{……})+\text{……}}{(s-\text{……})^2+\text{……}}$$

$$= 3\frac{s-\text{……}}{(s-\text{……})^2+\text{……}} + \frac{14}{\text{……}}\frac{\text{……}}{(s-\text{……})^2+\text{……}}$$

We know:

$$3\frac{s}{s^2+\text{……}} + \frac{7}{\text{……}}\frac{\text{……}}{s^2+\text{……}} \leftrightarrow 3\text{…………} + \frac{7}{\text{……}}\text{…………}$$

<u>**Answer:**</u> $f(x) = \underset{\text{………}}{}\left(\text{…………} + \text{…………}\right)$

Exercise 3: Solve the IVP by the Laplace transform method:

$$y'' - 3y' + 2y = 4x, \qquad y(0)=2, \quad y'(0)=-1 \qquad (*).$$

Solution: By the table: If

$$y(x) \leftrightarrow Y(s)$$

then

$$y'(x) \leftrightarrow sY(s) - \text{……….} = \text{…………………………}$$

$$y''(x) \leftrightarrow s^2Y(s) - \text{……….} - \text{……..} = \text{………………………}$$

The Laplace transform of equation (*) is

$$\left[s^2Y(s) - \ldots\ldots\ldots\ldots\ldots\ldots\right] - 3\left[sY(s) - \ldots\ldots\right] + \ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

Solve for $Y(s)$:

$$Y(s)\left[s^2 - \ldots\ldots\ldots\ldots\ldots\ldots\right] - \ldots\ldots\ldots\ldots + \ldots\ldots = \ldots\ldots\ldots\ldots\ldots$$

$$Y(s)\left[s^2 - \ldots\ldots\ldots\ldots\ldots\ldots\right] = \ldots\ldots\ldots\ldots - \ldots\ldots + \frac{4}{s^2}$$

$$= \frac{}{s^2} - \frac{}{s^2} + \frac{4}{s^2} = \frac{}{s^2}$$

$$Y(s) = \frac{}{s^2(\ldots\ldots\ldots\ldots\ldots\ldots\ldots)} = \frac{}{s^2(\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)}$$

$$= \frac{A}{s} + \frac{B}{\ldots\ldots} + \frac{C}{s-1} + \frac{D}{\ldots\ldots\ldots\ldots}$$

Multiply by ……………………………

$$2s^3 - 7s^2 + 4 = As(\ldots\ldots)(\ldots\ldots) + B(\ldots\ldots)(\ldots\ldots)$$
$$+ Cs^2(\ldots\ldots) + Ds^2(\ldots\ldots)$$

$s = 0$: $\quad \ldots\ldots\ldots = 4 \quad \Rightarrow \quad \boxed{B = \ldots\ldots}$

$s = 1$: $\quad \ldots\ldots\ldots = -1 \quad \Rightarrow \quad \boxed{C = \ldots\ldots}$

$s = 2$: $\quad \ldots\ldots\ldots = \ldots\ldots\ldots \quad \Rightarrow \quad \boxed{D = \ldots\ldots}$

$s = \ldots\ldots$: $\quad \ldots\ldots\ldots = \ldots\ldots\ldots \quad \Rightarrow \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

$$\boxed{A = \ldots\ldots}$$

Therefore,

$$Y(s) = \frac{3}{s} + \frac{}{\ldots\ldots} + \frac{1}{s-1} - \frac{2}{\ldots\ldots\ldots\ldots}$$

Answer: $\underline{y = \ldots\ldots\ldots + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots - \ldots\ldots\ldots\ldots}$

Exercise 4: Solve the IVP by the Laplace transform method:

$$y'' - 2y' + y = 2\cos x, \qquad y(0)=1, \quad y'(0)=-2 \qquad (*).$$

Solution: By the table: If

$$y(x) \leftrightarrow Y(s)$$

then

$$y'(x) \leftrightarrow sY(s) - \ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y''(x) \leftrightarrow s^2Y(s) - \ldots\ldots\ldots - \ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The Laplace transform of equation (*) is

$$\left[s^2Y(s) - \ldots\ldots\ldots\ldots\right] - 2\left[sY(s) - \ldots\ldots\right] + \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

Solve for $Y(s)$:

$$Y(s)\left[s^2 - \ldots\ldots\ldots\ldots\ldots\ldots\right] - \ldots\ldots\ldots + \ldots\ldots = \ldots\ldots\ldots\ldots$$

$$Y(s)\left[s^2 - \ldots\ldots\ldots\ldots\ldots\ldots\right] = \ldots\ldots\ldots - \ldots\ldots + \frac{2s}{s^2+1}$$

$$= \frac{(\ldots\ldots\ldots\ldots)(s^2+1)}{s^2+1} + \frac{2s}{s^2+1} = \frac{\qquad\qquad}{s^2+1}$$

$$Y(s) = \frac{\qquad\qquad}{(\ldots\ldots\ldots\ldots\ldots\ldots)(s^2+1)} = \frac{\qquad\qquad}{(\ldots\ldots\ldots)^2(s^2+1)}$$

$$= \frac{A}{\ldots\ldots\ldots} + \frac{\qquad}{(\ldots\ldots\ldots)^2} + \frac{\qquad}{\ldots\ldots\ldots\ldots}$$

Multiply by $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$s^3 - 4s^2 + 3s - 4 \;=\; A(\;\ldots\ldots\;)(\;\ldots\ldots\;) + B(\;\ldots\ldots\;) + (Cs + D)(\;\ldots\ldots\;)^2$$

$$= \; A(\;\ldots\ldots\ldots\ldots\ldots\ldots\;) + B(\;\ldots\ldots\ldots\;)$$

$$+ \; C(\;\ldots\ldots\ldots\ldots\ldots\;) + D(\;\ldots\ldots\ldots\ldots\;)$$

$$= \; (\ldots\ldots\ldots)s^3 + (\ldots\ldots\ldots\ldots\ldots)s^2 + (\ldots\ldots\ldots\ldots)s + \ldots\ldots\ldots\ldots$$

Set $s = 1$:

$$\ldots\ldots \;=\; \ldots\ldots\ldots\ldots \qquad \Rightarrow \qquad \boxed{B \;=\; \ldots\ldots} \qquad (1)$$

Compare coefficients:

$$s^3: \qquad \ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

$$s^2: \qquad \ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\overset{(1)}{\Rightarrow} \quad -2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3)$$

$$s: \qquad \ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4)$$

$$\text{const}: \qquad -4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\overset{(1)}{\Rightarrow} \quad -2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (5)$$

$$(4)-(2): \qquad \ldots\ldots = \ldots\ldots\ldots\ldots \qquad \Rightarrow \qquad \boxed{D \;=\; \ldots\ldots}$$

$$(5): \qquad A = \ldots\ldots\ldots\ldots \qquad \Rightarrow \qquad \boxed{A \;=\; \ldots\ldots}$$

$$(2): \qquad C = \ldots\ldots\ldots\ldots \qquad \Rightarrow \qquad \boxed{C \;=\; \ldots\ldots}$$

Therefore,

$$Y(s) \;=\; \frac{}{\ldots\ldots\ldots} - 2\frac{}{(s-1)^2} - \frac{}{s^2+1}$$

Note that $$\frac{1}{(s-1)^2} \;\leftrightarrow\; \ldots\ldots\ldots\ldots$$

Answer: $\underline{y = \ldots\ldots\ldots - 2\ldots\ldots\ldots - \ldots\ldots\ldots - \ldots\ldots\ldots}$

Additional Exercises:

1) Find the Laplace transforms:

a) $f(x) = 2x^3 - 4x^2 + 2x - e^{2x}$

b) $f(x) = (x-1)^2$

c) $f(x) = xe^{4x}$

d) $f(x) = x\cos 2x$

e) $f(x) = \sin^2 3x$

f) $x^2 \sin ax$

g) $xe^{-2x}\cos 4x$

h) $f(x) = x^5 + \cos 2x + 10$

i) $f(x) = x^3 e^{-x} + e^{2x}\cos 2x$

j) $f(x) = 4\sin x \cos x + 2e^{-x}$

2) Find the inverse Laplace transform:

a) $\dfrac{30}{s^4}$

b) $\dfrac{4}{s-1} + \dfrac{6}{s^2+9}$

c) $\dfrac{3}{s^2+5s}$

d) $\dfrac{-3}{s^4+s^2}$

e) $\dfrac{2s+1}{s(s^2+9)}$

f) $\dfrac{(s+1)^3}{(s-4)^4}$

g) $\dfrac{1}{s^2(s^2-a^2)}$

h) $\dfrac{1}{s(s+1)(s+2)}$

i) $\dfrac{2}{s^2+3s-4}$

j) $\dfrac{3s}{s^2-s-6}$

k) $\dfrac{2s-3}{s^2-4}$

l) $\dfrac{8s^2-4s+12}{s(s^2+4)}$

m) $\dfrac{1-2s}{s^2+4s+5}$

n) $\dfrac{2s-3}{9s^2-12s+20}$

o) $\dfrac{3s+5}{s^2-6s+25}$

p) $\dfrac{1}{s^3-5s^2}$

q) $\dfrac{1}{s^4-16}$

r) $\dfrac{1}{s(s^2-9)}$

s) $\dfrac{1}{(s^2+s-6)^2}$

t) $\dfrac{s^2-2s}{s^4+5s^2+4}$

3) Solve each following initial value problem using the Laplace transform method:

a) $y'' - y' - 6y = 0 \qquad y(0) = 1, \quad y'(0) = -1$

b) $y'' + 3y' + 2y = 0 \qquad y(0) = 1, \quad y'(0) = 0$

c) $y'' + y' - 2y = 2e^{-x} \qquad y(0) = 1, \quad y'(0) = 0$

d) $y'' + 4y = x^2 \qquad y(0) = 0, \quad y'(0) = 2$

e) $y'' - 2y' - 3y = \sin x \qquad y(0) = \frac{1}{2}, \quad y'(0) = \frac{1}{5}$

f) $y'' - 2y' + 2y = 0 \qquad y(0) = 0, \quad y'(0) = 1$

g) $y'' - 4y' + 4y = 0 \qquad y(0) = 1, \quad y'(0) = 1$

h) $y'' - 2y' - 2y = 0 \qquad y(0) = 2, \quad y'(0) = 0$

i) $y'' + 2y' + 5y = 0 \qquad y(0) = 2, \quad y'(0) = -1$

j) $y'' - 2y' + 2y = \cos x \qquad y(0) = 1, \quad y'(0) = 0$

k) $y'' - 2y' + 2y = e^{-x} \qquad y(0) = 0, \quad y'(0) = 1$

l) $y'' - 6y' + 8y = 2 \qquad y(0) = 0, \quad y'(0) = 0$

m) $y'' + 2y' + y = 4e^{-x} \qquad y(0) = 2, \quad y'(0) = -1$

n) $y'' - 4y = 3x \qquad y(0) = 0, \quad y'(0) = 0$