GROUP CLASSIFICATION OF SECOND-ORDER DELAY ORDINARY DIFFERENTIAL EQUATIONS

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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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วัตถุประสงค์ของงานวิจัยนี้ เพื่อแสดงการจำแนกเชิงกลุ่มของกลุ่มลี (LIE GROUP CLASSIFICATION) ของสมการเชิงอนุพันธ์สามัญประวิงอันดับสองในรูปแบบ

$$y'' = f(x, y, y_\tau, y', y'_\tau)$$

โดยที่ $\tau > 0$ คือประวิง $y_\tau = y(x-\tau)$ และ $y'_\tau = y'(x-\tau)$ งานวิจัยนี้ยังได้พัฒนาระเบียบวิธีในการหาคำตอบของปัญหา และพบว่ากลุ่มของสมการเชิงอนุพันธ์สามัญประวิงอันดับสองทั้งหมดที่ยอมรับพีชคณิตของลี (ADMIT LIE ALGEBRA) ได้ถูกจำแนกออกเป็น 39 กลุ่ม โดยที่ตัวแทนสมการของกลุ่มเหล่านี้ได้แสดงไว้ในวิทยานิพนธ์นี้ด้วย

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PRAPART PUE-ON : GROUP CLASSIFICATION OF SECOND -ORDER DELAY ORDINARY DIFFERENTIAL EQUATIONS. THESIS ADVISOR : PROF. SERGEY MELESHKO, Ph.D., 121 PP.

DELAY ORDINARY DIFFERENTIAL EQUATION / DELAY DIFFERENTIAL INVARIANT / SYMMETRY GROUP / GROUP ANALYSIS

The purpose of this research is to give a complete Lie group classification of second-order delay ordinary differential equations of the form

$$y'' = f(x, y, y_\tau, y', y'_\tau)$$

where $\tau > 0$ is a delay, $y_\tau = y(x - \tau)$ and $y'_\tau = y'(x - \tau)$. The method for solving this problem was developed. All classes of second-order delay ordinary differential equations admitting a Lie algebra were obtained. The set of second-order delay ordinary differential equations admitting a Lie algebra consists of 39 classes. Representations of these equations are presented in the thesis.

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CHAPTER I

INTRODUCTION

In general, not only ordinary differential equations but also delay ordinary differential equations are used to describe various physical phenomena. Delay ordinary differential equations, or DODEs, are similar to ordinary differential equations, but their evolutions involve past values of the state variables. In this thesis, the following general simple form of second-order DODEs

$$y'' = f(x, y, y_\tau, y', y'_\tau) \tag{1.1}$$

is focused on, where $y = y(x), y' = y'(x), y_\tau = y(x-\tau)$ and $y'_\tau = y'(x-\tau)$.

DODEs play a major role in physical, biological and medical modeling: the two-body problem of electrodynamics (Driver, 1977), prey and predator population models (Driver), mixing of liquids(Driver), evolution equations of a single species (Gopalsamy, 1991, quoted in Kolmanovskii and Myshkis, 1992), coexistence of competitive micro-organisms (Freeman, So, Waltman, 1988, quoted in Kolmanovskii and Myshkis), mathematical models of the sugar quantity in blood (Shvitra, 1989, quoted in Kolmanovskii and Myshkis), models of arterial blood pressure regulation (Godin, Kolmanovskii and Stengold, 1990, quoted in Kolmanovskii and Myshkis), mathematical models of learning(Shimbell, 1950, quoted in Kolmanovskii and Myshkis), vision processes in the compound eye (Hadeler, 1976, quoted in Kolmanovskii and Myshkis), optimal advertising policies (Pauwels, 1977, quoted in Kolmanovskii and Myshkis), models of fishing processes (Kot, 1979, quoted in Kolmanovskii and Myshkis), river pollution control (Lee and Lietmann, 1988, quoted in Kolmanovskii and Myshkis), etc.

Although DODEs are widely applied to many branches of science, exact solutions are not yet known for most of them. Throughout the years, many methods for obtaining exact solutions of differential equations instead of approximating solutions have been developed. One of them is group analysis.

Group analysis was initially introduced in the 1870s by a Norwegian mathematician, Sophus Lie (Ovsiannikov, 1978). He found a new method for integrating differential equations. This method is universal and effective for solving nonlinear differential equations analytically. It involves the study of symmetries of differential equations, with the emphasis on using the symmetries to find solutions. The theory of group analysis has been applied to both ordinary and partial differential equations.

One of its applications to differential equations is the problem of group classification of differential equations. Group classification means to classify given differential equations with respect to arbitrary elements. The group classification problem of differential equation was first formulated by Lie (Ibragimov, 1996). He gave a classification of ordinary differential equations in terms of their symmetry groups, thereby identifying the full set of equations which could be solved or reduced to lower-order equations by this method. In 2002, group analysis was applied systematically to delay differential equations (Tanthanuch and Meleshko, 2002). The method for constructing and solving the determining equation are shown in Tanthanuch and Meleshko (2003).

Even though the group classification for second-order ordinary differential equation has been studied, group classification for DODEs has not been fully developed yet. This research deals with the classification problem of second-order delay ordinary differential equations.

The purpose of this thesis is to classify a family of second-order delay ordinary differential equations (1.1) according to their symmetries.

The thesis is designed as follows. Chapter II reviews the definitions of functional and delay differential equations and some of their applications. Moreover, existence theorem of a solution of DDEs is also presented. Chapter III provides an introduction to the concept of a group of point transformations, their corresponding infinitesimal generators and definition of symmetry group of DDEs. Complete group classification of second-order DODEs is revisited in Chapter IV. The conclusion of this thesis is presented in the last chapter.

CHAPTER II

FUNCTIONAL AND DELAY DIFFERENTIAL EQUATIONS

A more general type of differential equations called functional differential equation is frequently found in modern scientific and engineering research publications. Although this type of equation plays a key role in many branches, the theory for functional differential equation is still being developed.

In this chapter, definitions of functional differential equation, delay differential equation and some mathematical models which are described by these types of equations are given. The existence theory for delay differential equation is also presented.

2.1 Functional Differential Equations (FDEs)

Definition 2.1. (FDE). An equation involving functionals* of independent variables, dependent variables and derivatives of dependent variables with respect to one or more independent variables is called *a functional differential equation.*

Consider an FDE with aftereffect,

$$u^{(m)}(x) = f(x, u^{(m_1)}(x - h_1(x)), ..., u^{(m_k)}(x - h_k(x))), \tag{2.1}$$

where $u(x) \in \mathbb{R}^n$, $u^{(m_i)}$ is the m_i-order derivative of u with respect to x and all

*Some familiarity with the concept of "*functional*" and related concepts is assumed but a review is included in A.1, Appendix A. One may find the definition and its concepts from textbooks, e.g. Kreyszig (1978).

$m_i \geq 0,\ h_i(x) \geq 0,\ i = 1, ..., k.$

In the literature, equation (2.1) is called

- a functional differential equation of *retarded type* or *retarded differential equation* (RDE), if $\max\{m_1, ..., m_k\} < m$;
- a functional differential equation of *neutral type* (NDE), if $\max\{m_1, ..., m_k\} = m$; and
- a functional differential equation of *advanced type* (ADE), if $\max\{m_1, ..., m_k\} > m$.

FDEs are widely applicable in biology, physics, engineering and economics. Experience in mathematical modeling has shown that the evolution equations of actual process with aftereffect are almost exclusively RDEs and NDEs. The following are some examples of them.

Coexistence of competitive micro-organisms. The following model of competing micro-organisms surviving on a single nutrient and with delays in birth and death process has been described in (Freeman, So and Waltman, 1988, quoted in Kolmanovskii and Myshkis):

$$\begin{aligned}
\dot{x}_0(t) &= 1 - x_0(t) - x_1(t) f_1(x_0) - x_2(t) f_2(x_0), \\
\dot{x}_1(t) &= [f_1(x_0(t - \tau_1)) - 1] x_1(t), \\
\dot{x}_2(t) &= [f_2(x_0(t - \tau_2)) - 1] x_2(t).
\end{aligned}$$

Here x_0 is the nutrient concentration, $x_1,\ x_2$ are the concentrations of competing micro-organisms $\tau_i > 0$ are (constant) delays, and $f_i(0) = 0,\ f_i(x) > 0$ for $x > 0$.

Mathematical models of the sugar quantity in blood. FDEs can be efficiently used to describe various processes in living organizations. Various heredity models have been proposed to describe the functioning of the thyroid gland, the system of maintaining the sugar level in blood, and blood production. Certain parameters in these models can be regulated (temperature, diet, drugs, etc.) E.g., the control model for the sugar level in blood has the form (Shvitra, 1989, quoted in Kolmanovskii and Myshkis)

$$\begin{aligned}
\dot{x}_1(t) &= a_1\{a_2x_4(t) + a_3[a_2x_4(t) - a_4x_2(t)] - a_5x_1(t-\tau)\}x_1(t), \\
\dot{x}_2(t) &= a_6\{a_2x_4(t) + b_1u(t) - a_7[a_2x_4(t) - a_5x_1(t)] - a_4x_2(t)\}x_2(t), \\
\dot{x}_3(t) &= a_8\{a_5x_1(t) + b_2u(t) + a_9[a_5x_1(t) - a_4x_2(t)] - a_{10}x_3(t)\}x_3(t), \\
\dot{x}_4(t) &= a_{11}\{1 + u(t) + a_{12}[1 - a_4x_2(t)] - a_2x_4(t)\}x_4(t).
\end{aligned}$$

Here, $x_1(t)$ is the amount of insulin produced by the pancreas, $x_2(t)$ is the amount of active insulin in the blood, x_3 is the total amount of insulin in the blood, $x_4(t)$ is the amount of sugar in the blood (all at time t); $a_2,\ a_4,\ a_5,\ a_{10}$ are the averages of these amount; the delay τ characterizes the finite time needed for production of insulin, and a_1 is the rate of insulin production; $a_6,\ a_8,\ a_{11}$ reflect the increase of insulin, total amount of insulin and sugar in the blood; finally, $b_1 \geq 0,\ b_2 \geq 0,\ a_3,\ a_7,\ a_9,\ a_{12}$ are feedback coefficients. The control $u(t)$ is fulfilled by choice of a diet, and may affect the amount of sugar in the blood.

Models of lasers. (Stats, de Mars, Wilson and Tang, 1965, quoted in Kolmanovskii and Myshkis) FDEs are widely used to model the dynamic properties of a laser

$$\begin{aligned}
\dot{x}_1(t) &= vx_1(t)[x_2(t) - 1 - m - \alpha m x_1(t-\tau)] + vU_0, \\
\dot{x}_2(t) &= K_0 - K(t)[x_1(t) + 1],
\end{aligned}$$

where $x_1(t)$ is the radiation density and $x_2(t)$ the amplification coefficient. The other parameters are constants depending on the properties of the laser.

Mathematical models of learning.(Shimbell, 1950, quoted in Kolmanovskii and Myshkis) The following model has been proposed to describe the behavior of the central nervous system in a learning process

$$\begin{aligned}\dot{x}(t) &= K[x(t)-x(t-1)][N-x(t)], \quad t \geq 0,\\ x(t) &= 0, \quad (-1 \leq t < 0), \quad x(0) = x_0.\end{aligned}$$

Here, K and N are positive constants, $0 < x_0 < N$.

Model of survival of red blood cells. A model for the survival of red blood cells in an animal has been described (Wazewska-Czyzevsia and Lasota, 1988, quoted in Kolmanovskii and Myshkis) by the equation

$$\dot{x}(t) = -ax(t) + be^{-\gamma x(t-\tau)}, \quad t \geq t_0,$$

where $x(t)$ is the number of red blood cells at time t, a is the probability of death of a red blood cell, $b,\ \gamma > 0$ are constants related to the production of red blood cells per unit time, and the delay $\tau > 0$ is the time required to produce a red blood cells.

River pollution control. Let $z(t)$ and $q(t)$ be the concentrations per unit volume of biological oxygen demand (BOD) and dissolved oxygen (DO), respectively, at time t. It is assumed that the flow rate discount, water is well mixed, and there exists $\tau > 0$ such that BOD and DO concentrations entering at time t are equal to the corresponding concentrations τ time units ago. Using mass balance concentration, the following equations have been derived (Lie and Leitmann, 1989, quoted in Kolmanovskii and Myshkis)

$$\begin{aligned}\dot{z}(t) &= -k_1(t)z(t) + v^{-1}[Q_1(m+u_1(t)) + Qz(t-\tau) - (Q+Q_1)z(t)] + v_1(t),\\ \dot{q}(t) &= -k_3(t)z(t) + k_2(t)[q_0 - q(t)] + v^{-1}[Qq(t-\tau) - (Q+Q_1)q(t)] + u_2(t) + v_2(t).\end{aligned}$$

Here, $k_i(\cdot)$ denote the BOD decay rate, the BO re-action rate, and the BOD deoxygenation rate; q_0 is the DO saturation concentration; Q and Q_1 are the stream flow rate and the effluent flow rate; v is the constant volume of water under consideration; $u_i(t)$ are controls; $v_i(\cdot)$ are random disturbances affecting the rates of change of BOD and DO; and m is a constant.

Similarly to the classification of differential equations by order, we classify FDEs according to the order of the highest derivative appearing in the equation.

Definition 2.2. The *order* of a FDE is the order of the highest derivative of the unknown function entering in the equation, when written in the form of (2.1).

Definition 2.3. A *solution* of an FDE in some region $\mathcal{R}$ of the space of the independent variables is a function that has derivatives and functionals of derivatives appearing in the equation in some domain containing $\mathcal{R}$ and satisfies the equation everywhere in $\mathcal{R}$.

2.2 Delay Differential Equations (DDEs)

Definition 2.4. (DDE). *Delay differential equations with one independent variable*, or *functional differential equations of retarded type*, are of the form

$$u'(x) = f(x, u(g_1(x)), ..., u(g_q(x))), \tag{2.2}$$

where $x \in [x_0, \beta)$, $u : [\gamma, x] \mapsto \mathcal{D}$, $\mathcal{D}$ is an open subset in $\mathbb{R}^n$, u and f are n-vector-valued, sufficiently time differentiable functions, $f : [x_0, \beta) \times \mathcal{D}^q \mapsto \mathbb{R}^n$, and for each $\lambda = 1, ..., q$, $\gamma \leq g_\lambda(x) \leq x$, for $x_0 \leq x < \beta$.

Note that g_1 is usually chosen to be the identity mapping.

Definition 2.5. A *solution* of equation (2.2), with the initial condition $\theta(x)$ defined on $[\gamma, x_0]$, is a continuous function $u : [\gamma, \beta_1) \mapsto D$, for some $\beta_1 \in (x_0, \beta]$ such that

1. $u(x) = \theta(x)$ for $\gamma \leq x \leq x_0$, and

2. $u'(x) = f(x, u(g_1(x)), ..., u(g_q(x)))$ for $x_0 \leq x \leq \beta_1$.

Remark. The derivative of u at the point x_0 is considered only from the right-hand side.

Definitions 2.4 and 2.5 indicate that initial values of DDEs have to be satisfied for the whole interval considered. In other words, they are of *non-local differential equation* type.

2.3 Existence Theory of a Solution of a DDEs

Consider a delay differential equation system

$$u'(x) = f(x, u(g_1(x)), ..., u(g_q(x))). \tag{2.3}$$

By definition 2.4, we may assume that

$$x - \tau \leq g_\lambda(x) \leq x \quad \text{for} \quad x \geq x_0, \quad \lambda = 1, ..., q,$$

for some constant $\tau \geq 0$. The initial condition takes the form

$$u(x) = \theta(x) \quad \text{for} \quad x_0 - \tau \leq x \leq x_0,$$

here $\theta(x)$ is a given function. Note that system (2.3) is reduced to a system of ODEs if $\tau = 0$. It is assumed that f is defined on $[x_0, \beta) \times \mathcal{D}^q \mapsto \mathbb{R}^n$ for some $\beta > x_0$ and some open set $\mathcal{D} \subset \mathbb{R}^n$.

Since the notation of system (2.3) is cumbersome, it would be better to have a simpler notation.

If u is a function defined at least on $[x - \tau, x] \mapsto \mathbb{R}^n$, then we define a new function $u_x : [-\tau, 0] \mapsto \mathbb{R}^n$ by

$$u_x(\sigma) = u(x + \sigma) \quad \text{for} \ -\tau \leq \sigma \leq 0.$$

From another point of view, u_x is obtained by considering only $u(s)$ for $x-\tau \leq s \leq x$ and then translating this segment of u to the interval $[-\tau, 0]$. If u is a continuous function, then u_x is a continuous function on $[-\tau, 0]$.

Let real numbers $\tau \geq 0$ and x_0 be given and let $x_0 < \beta \leq \infty$. Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$, and let F be defined on $[x_0, \beta) \times \mathcal{C}_{\mathcal{D}} \mapsto \mathbb{R}^n$, where $\mathcal{C}_{\mathcal{D}}$ is the set of all continuous functions mapping $[-\tau, 0] \mapsto \mathcal{D}$, i.e. $\mathcal{C}_{\mathcal{D}} = \mathcal{C}([-\tau, 0], \mathcal{D})$. Define

$$F(x, u_x) \equiv f(x, u(g_1(x)), ..., u(g_q(x))).$$

Then system (2.3) can be written as

$$u'(x) = F(x, u_x). \tag{2.4}$$

Given any $\phi \in \mathcal{C}_{\mathcal{D}}$, we seek a continuous function $u : [x_0 - \tau, \beta_1) \mapsto \mathcal{D}$ for some $\beta_1 \in (x_0, \beta]$ such that system (2.4) is satisfied on $[x_0, \beta_1)$ and

$$u_{x_0} = \phi. \tag{2.5}$$

For the existence of solutions of system (2.4), it is sufficient to require the following conditions on F.

Definition 2.6. A function $F(x, u_x)$ satisfies the *Continuity Condition* if $F(x, u_x)$ is continuous with respect to x in $[x_0, \beta)$ for any given continuous function $u : [x_0 - \tau, \beta) \mapsto \mathcal{D}$.

If F satisfies the *Continuity Condition* then a continuous function $u : [x_0, \beta_1) \mapsto \mathcal{D}$ is a solution of equations (2.4) and (2.5) if and only if

$$u(x) = \begin{cases} \phi(x - x_0) & \text{for} \quad x_0 - \tau \leq x \leq x_0, \\ \phi(0) + \int_{x_0}^{x} F(s, u_s)ds & \text{for} \quad x_0 \leq x \leq \beta_1. \end{cases} \tag{2.6}$$

In order to define a *Lipschitz condition*, a means for measuring the magnitude of elements of $\mathcal{C}_{\mathcal{D}}$ is required.

For a function $\psi \in \mathcal{C}_{\mathcal{D}}$,

$$|\psi|_\tau = \sup_{-\tau \le \varrho \le 0} |\psi(\varrho)|.$$

Definition 2.7. Let $F : [x_0, \beta) \times \mathcal{C}_{\mathcal{D}} \mapsto \mathbb{R}^n$ and let $\mathcal{E}$ be a subset of $[x_0, \beta) \times \mathcal{C}_{\mathcal{D}}$. If there exists $K \ge 0$ so that

$$|F(x, \psi) - F(x, \bar{\psi})| \le K |\psi - \bar{\psi}|_\tau, \tag{2.7}$$

whenever (x, ψ) and $(x, \bar{\psi}) \in \mathcal{E}$, we say that F satisfies a *Lipschitz condition* (or F is *Lipschitzian*) on $\mathcal{E}$ with *Lipschitz constant* K.

Definition 2.8. A functional $F : [x_0, \beta) \times \mathcal{C}_{\mathcal{D}} \mapsto \mathbb{R}^n$ is *locally Lipschitzian* if for each given $(\bar{x}, \bar{\psi}) \in [x_0, \beta) \times \mathcal{C}_{\mathcal{D}}$ there exist numbers $a > 0$ and $b > 0$ such that

$$\mathcal{E} \equiv (\ [\bar{x} - a, \bar{x} + a] \cap [x_0, \beta)\ ) \times \{\psi \in \mathcal{C}_{\mathcal{D}} : |\psi - \bar{\psi}|_\tau \le b\}$$

is a subset of $[x_0, \beta) \times \mathcal{C}_{\mathcal{D}}$ and F is Lipschitzian on $\mathcal{E}$.

Remark. The Lipschitz constant for F depends on the particular set $\mathcal{E}$.

Theorem 2.1 (Local Existence, Driver, 1977)**.** *Let* $F : [x_0, \beta) \times \mathcal{C}_{\mathcal{D}} \mapsto \mathbb{R}^n$ *satisfy the Continuity Condition and be locally Lipschitzian. Then, for each* $\phi \in \mathcal{C}_{\mathcal{D}}$, *equations (2.4) and (2.5) have a unique solution on* $[x_0 - \tau, x_0 + \Delta)$ *for some* $\Delta > 0$.

CHAPTER III

GROUP ANALYSIS

Before moving on to the main discussion of this thesis in the next chapter, it is useful to review some basic concepts from group analysis. Group analysis was initially introduced in 1870 by a Norwegian mathematician, Sophus Lie. Lie group analysis provides general methods for integration of linear and nonlinear differential equations using their symmetries. It is a universal and effective method for solving nonlinear differential equations analytically.

The purpose of this chapter is to present preliminary knowledge of group analysis for differential equation: definition of a one-parameter Lie group and corresponding infinitesimal generator, prolongation formula, Lie-Bäcklund representation, Lie algebra of operators, definition of determining equation and symmetry group for delay differential equations.

3.1 Lie Group of Point Transformations

Let $x = (x_1, \ldots, x_n)$ be n-tuples of the independent variables and $u = (u^1, \ldots, u^m)$ be m-tuples of the dependent variables. Consider invertible transformations of x and u

$$\begin{aligned} \bar{x} &= (\bar{x}_1, \ldots, \bar{x}_n) = (\varphi_1^x(x, u; a), \ldots, \varphi_n^x(x, u; a)) = \varphi^x(x, u; a), \\ \bar{u} &= (\bar{u}^1, \ldots, \bar{u}^m) = (\varphi_1^u(x, u; a), \ldots, \varphi_m^u(x, u; a)) = \varphi^u(x, u; a), \end{aligned} \tag{3.1}$$

depending upon a real continuous parameter a, which lies in an open symmetric interval $\mathcal{S}$, with conditions

$$\begin{aligned} \varphi_i^x(x,u;0) &= x_i, \qquad i = 1,\ldots,n, \\ \varphi_\alpha^u(x,u;0) &= u^\alpha, \qquad \alpha = 1,\ldots,m. \end{aligned} \tag{3.2}$$

These transformations are assumed to be sufficiently differentiable with respect to the variables x_i and u^α, and to be analytic functions of the parameter a.

It is said that these transformations form a *one-parameter group* G if the successive action of two transformations is equivalent to the action of another transformation of the form (3.1), i.e.

$$\begin{aligned} \varphi^x(\bar{x},\bar{u};b) &= \varphi^x(\varphi^x(x,u;a),\varphi^u(x,u;a);b) = \varphi^x(x,u;a+b), \\ \varphi^u(\bar{x},\bar{u};b) &= \varphi^u(\varphi^x(x,u;a),\varphi^u(x,u;a);b) = \varphi^u(x,u;a+b). \end{aligned} \tag{3.3}$$

In practice, it often happens that the group property is valid only locally, i.e. only for $|a|$, $|b|$ and $|a|+|b|$ sufficiently small. In this case, G is referred to as a *local one-parameter transformation group.* In group analysis, local groups are used, which for brevity are simply called *groups.*

The transformations (3.1) are called *point transformations*, and the group G is called a *group of point transformations.* It is readily seen from formulas (3.2) and (3.3) that the inverse transformation can be obtained by changing the sign of the parameter:

$$x = \varphi^x(\bar{x},\bar{u},-a), \quad u = \varphi^u(\bar{x},\bar{u},-a) \tag{3.4}$$

Let T_a denote the transformation (3.1) of a point (x,u) into the point $(\bar{x},\bar{u})$, I denote the identity transformation, T_a^{-1} denote the transformation inverse to T_a, and T_bT_a denote the composition of two transformations. Then one may summarize properties (3.1)-(3.4) as follows:

A set G of transformations T_a is a ***group of point transformations*** *if the following hold:*

1. $T_0 = I \in G$,

2. $T_b T_a = T_{a+b} \in G, \quad a, b \in \mathcal{S}$,

3. If $a \in \mathcal{S}$ and $T_a\left((x,u)\right) = (x,u)$ for all (x,u), then $a = 0$.

The functions φ^x and φ^u can be represented via their Taylor series expansions with respect to the parameter a in the neighborhood of the expansion point 0 and thus the transformations in (3.1) can be written as follows:

$$\bar{x}_i = \varphi_i^x(x,u;a) = x_i + \xi_i(x,u)a + \cdots ,$$
$$\bar{u}^\alpha = \varphi_\alpha^u(x,u;a) = u^\alpha + \eta^\alpha(x,u)a + \cdots ,$$

or

$$\bar{x}_i \approx x_i + \xi_i(x,u)a, \qquad \bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x,u)a, \tag{3.5}$$

where

$$\xi_i(x,u) = \left.\frac{\partial \varphi_i^x(x,u;a)}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x,u) = \left.\frac{\partial \varphi_\alpha^u(x,u;a)}{\partial a}\right|_{a=0}.$$

Given an infinitesimal transformation (3.5), the corresponding group can be completely determined by the following system of differential equations, called *Lie equations*, with appropriate initial conditions:

$$\begin{aligned} \frac{d\varphi_i^x}{da} &= \xi_i(\varphi^x,\varphi^u), \quad \left.\varphi_i^x\right|_{a=0} = x_i, \\ \frac{d\varphi_\alpha^u}{da} &= \eta^\alpha(\varphi^x,\varphi^u), \quad \left.\varphi_\alpha^u\right|_{a=0} = u^\alpha. \end{aligned} \tag{3.6}$$

Consider the first-order differential operator

$$X = \xi_1(x,u)\frac{\partial}{\partial x_1} + \cdots + \xi_n(x,u)\frac{\partial}{\partial x_n} + \eta^1(x,u)\frac{\partial}{\partial u^1} + \cdots + \eta^m(x,u)\frac{\partial}{\partial u^m}. \tag{3.7}$$

Sophus Lie called the operator (3.7) a *symbol* of the infinitesimal transformation (3.5). In this thesis, the words *infinitesimal generator, infinitesimal operator, group generator, group operator* and *Lie operator* are used interchangeably.

The first-order differential operator (3.7) is written briefly as

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{3.8}$$

where the repeated index i means summation with respect to i from $i = 1$ to n and the repeated index α means summation with respect to α from $\alpha = 1$ to m.

3.2 Change of Variables

Let G be an one-parameter group of transformations

$$\tilde{x} = \varphi^x(x,y;a), \qquad \tilde{y} = \varphi^y(x,y;a)$$

with corresponding generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{3.9}$$

Consider an invertible (nonsingular) change of variables:

$$\bar{x} = h(x,y), \qquad \bar{y} = g(x,y), \tag{3.10}$$

and its inverse

$$x = \bar{h}(\bar{x},\bar{y}), \qquad y = \bar{g}(\bar{x},\bar{y}), \tag{3.11}$$

with the Jacobian $\Delta = \bar{h}_{\bar{x}}\bar{g}_{\bar{y}} - \bar{g}_{\bar{x}}\bar{h}_{\bar{y}} \neq 0$. Substituting (3.10) into (3.11), we obtain the identities

$$x = \bar{h}(h(x,y), g(x,y)), \qquad y = \bar{g}(h(x,y), g(x,y)). \tag{3.12}$$

Differentiating with respect to x and y, we have

$$1 = \bar{h}_{\bar{x}}(\bar{x},\bar{y})h_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) + \bar{h}_{\bar{y}}(\bar{x},\bar{y})g_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})),$$

$$0 = \bar{g}_{\bar{x}}(\bar{x},\bar{y})h_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) + \bar{g}_{\bar{y}}(\bar{x},\bar{y})g_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})),$$

$$0 = \bar{h}_{\bar{x}}(\bar{x},\bar{y})h_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) + \bar{h}_{\bar{y}}(\bar{x},\bar{y})g_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})),$$

$$1 = \bar{g}_{\bar{x}}(\bar{x},\bar{y})h_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) + \bar{g}_{\bar{y}}(\bar{x},\bar{y})g_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})).$$

Solving these equations for $h_x,\ h_y,\ g_x$ and g_y, one obtains

$$h_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) = \frac{\bar{g}_{\bar{y}}(\bar{x},\bar{y})}{\Delta(\bar{x},\bar{y})}, \qquad h_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) = -\frac{\bar{h}_{\bar{y}}(\bar{x},\bar{y})}{\Delta(\bar{x},\bar{y})}, \quad (3.13)$$

$$g_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) = -\frac{\bar{g}_{\bar{x}}(\bar{x},\bar{y})}{\Delta(\bar{x},\bar{y})}, \qquad g_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) = \frac{\bar{h}_{\bar{x}}(\bar{x},\bar{y})}{\Delta(\bar{x},\bar{y})}. \quad (3.14)$$

Under the change of variables (3.10) the differential operator (3.9) is transformed as follows:

$$\bar{X} = \bar{\xi}(\bar{x},\bar{y})\frac{\partial}{\partial \bar{x}} + \bar{\eta}(\bar{x},\bar{y})\frac{\partial}{\partial \bar{y}}. \quad (3.15)$$

Here $\bar{\xi}(\bar{x},\bar{y})$ and $\bar{\eta}(\bar{x},\bar{y})$ are obtained by the action of differential operator X on the function $h,\ g$, the results are written as a function of new variables $\bar{x},\ \bar{y}$, i.e.,

$$\begin{aligned}
\bar{\xi}(\bar{x},\bar{y}) &= X(h(x,y))\Big|_{x=\bar{h},\ y=\bar{g}} \\
&= \Big[\xi(x,y)h_x(x,y) + \eta(x,y)h_y(x,y)\Big]_{x=\bar{h},\ y=\bar{g}} \\
&= \Big[\xi(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))h_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) + \eta(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))h_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))\Big] \\
\bar{\eta}(\bar{x},\bar{y}) &= X(g(x,y))\Big|_{x=\bar{h},\ y=\bar{g}} \\
&= \Big[\xi(x,y)g_x(x,y) + \eta(x,y)g_y(x,y)\Big]_{x=\bar{h},\ y=\bar{g}} \\
&= \Big[\xi(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))g_x(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y})) + \eta(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))g_y(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))\Big].
\end{aligned}$$

Hence, from (3.13) and (3.14), $\bar{\xi},\ \bar{\eta}$ are rewritten as follows

$$\bar{\xi}(\bar{x},\bar{y}) = \xi(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))\frac{\bar{g}_{\bar{y}}(\bar{x},\bar{y})}{\Delta} - \eta(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))\frac{\bar{h}_{\bar{y}}(\bar{x},\bar{y})}{\Delta}, \quad (3.16)$$

$$\bar{\eta}(\bar{x},\bar{y}) = -\xi(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))\frac{\bar{g}_{\bar{x}}(\bar{x},\bar{y})}{\Delta} + \eta(\bar{h}(\bar{x},\bar{y}),\bar{g}(\bar{x},\bar{y}))\frac{\bar{h}_{\bar{x}}(\bar{x},\bar{y})}{\Delta}. \quad (3.17)$$

3.3 Prolongations

By definition, groups of point transformations act only on the space of (x,u) of $n+m$ variables. However, to apply these groups to differential equations, one needs the transformations of derivatives. Thus it is necessary to extend a group of point transformations acting on the (x,u)-space to groups of point transformations acting on the $(x, u, \underset{1}{u})$-space, $(x, u, \underset{1}{u}, \underset{2}{u})$-space, ..., $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u})$-space, $s \geq 1$, for a given differential equation with order s. These groups are called *the first prolongation group, the second prolongation group, ..., the s-times prolongation group*, respectively, where the transformations are of the form

$$
\begin{aligned}
\bar{x} &= \varphi^x(x, u; a) = x + \xi(x, u)a + \cdots\,, \\
\bar{u} &= \varphi^u(x, u; a) = u + \eta(x, u)a + \cdots\,, \\
\underset{1}{\bar{u}} &= \varphi^{\underset{1}{u}}(x, u, \underset{1}{u}; a) = \underset{1}{u} + \zeta^{(1)}(x, u, \underset{1}{u})a + \cdots\,, \\
&\;\vdots \\
\underset{s}{\bar{u}} &= \varphi^{\underset{s}{u}}(x, u, \underset{1}{u}, \ldots, \underset{s}{u}; a) = \underset{s}{u} + \zeta^{(s)}(x, u, \underset{1}{u}, \ldots, \underset{s}{u})a + \cdots\,.
\end{aligned}
$$

The prolongation transformation formulas* of the components $\{\bar{u}^\alpha_{,i}\}$ of $\underset{1}{\bar{u}}$ are determined by

$$
\begin{bmatrix} \bar{u}^\alpha_{,1} \\ \bar{u}^\alpha_{,2} \\ \vdots \\ \bar{u}^\alpha_{,n} \end{bmatrix}
=
\begin{bmatrix} (\varphi^{\underset{1}{u}})^\alpha_1(x, u, \underset{1}{u}; a) \\ (\varphi^{\underset{1}{u}})^\alpha_2(x, u, \underset{1}{u}; a) \\ \vdots \\ (\varphi^{\underset{1}{u}})^\alpha_n(x, u, \underset{1}{u}; a) \end{bmatrix}
= A^{-1}
\begin{bmatrix} D_1\varphi^u(x, u; a) \\ D_2\varphi^u(x, u; a) \\ \vdots \\ D_n\varphi^u(x, u; a) \end{bmatrix},
$$

where A^{-1} is the inverse (assumed to exist) of the matrix

$$
A = \begin{bmatrix} D_1\varphi^x_1 & D_1\varphi^x_2 & \cdots & D_1\varphi^x_n \\ D_2\varphi^x_1 & D_2\varphi^x_2 & \cdots & D_2\varphi^x_n \\ \vdots & \vdots & & \vdots \\ D_n\varphi^x_1 & D_n\varphi^x_2 & \cdots & D_n\varphi^x_n \end{bmatrix},
$$

*See more details in Ovsiannikov (1978)

and the prolongation transformations formulas of the components $\{\bar{u}^{\alpha}_{,i_1\cdots i_s}\}$ of $\underset{s}{\bar{u}}$ are determined by

$$\begin{bmatrix} \bar{u}^{\alpha}_{,i_1\cdots i_{s-1}1} \\ \bar{u}^{\alpha}_{,i_1\cdots i_{s-1}2} \\ \vdots \\ \bar{u}^{\alpha}_{,i_1\cdots i_{s-1}n} \end{bmatrix} = \begin{bmatrix} (\varphi^{\underset{s}{u}})^{\alpha}_{i_1\cdots i_{s-1}1}(x, u, \underset{1}{u}, \ldots, \underset{s}{u}; a) \\ (\varphi^{\underset{s}{u}})^{\alpha}_{i_1\cdots i_{s-1}2}(x, u, \underset{1}{u}, \ldots, \underset{s}{u}; a) \\ \vdots \\ (\varphi^{\underset{s}{u}})^{\alpha}_{i_1\cdots i_{s-1}n}(x, u, \underset{1}{u}, \ldots, \underset{s}{u}; a) \end{bmatrix}$$

$$= A^{-1} \begin{bmatrix} D_1[(\varphi^{\underset{s-1}{u}})^{\alpha}_{i_1\cdots i_{s-1}}(x, u, \underset{1}{u}, \ldots, \underset{s-1}{u}; a)] \\ D_2[(\varphi^{\underset{s-1}{u}})^{\alpha}_{i_1\cdots i_{s-1}}(x, u, \underset{1}{u}, \ldots, \underset{s-1}{u}; a)] \\ \vdots \\ D_n[(\varphi^{\underset{s-1}{u}})^{\alpha}_{i_1\cdots i_{s-1}}(x, u, \underset{1}{u}, \ldots, \underset{s-1}{u}; a)] \end{bmatrix}.$$

The formulas of the coefficients, $\zeta^{\alpha}_i, \ldots, \zeta^{\alpha}_{i_1\cdots i_s}$, of the infinitesimal generator are determined by

$$\begin{aligned} \zeta^{\alpha}_i &= D_i(\eta^{\alpha}) - u^{\alpha}_{,j} D_i(\xi_j), \\ \zeta^{\alpha}_{i_1 i_2} &= D_{i_2}(\zeta^{\alpha}_{i_1}) - u^{\alpha}_{,i_1 j} D_{i_2}(\xi_j), \\ &\vdots \\ \zeta^{\alpha}_{i_1\cdots i_s} &= D_{i_s}(\zeta^{\alpha}_{i_1\cdots i_{s-1}}) - u^{\alpha}_{,i_1\cdots i_{s-1} j} D_{i_s}(\xi_j). \end{aligned}$$

Thus, the first prolonged generator of (3.8) is

$$X^{(1)} = X + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_{,i}} = \xi_i \frac{\partial}{\partial x_i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_{,i}},$$

and the s-times prolonged generator is written recurrently as:

$$X^{(s)} = X^{(s-1)} + \zeta^{\alpha}_{i_1\cdots i_s} \frac{\partial}{\partial u^{\alpha}_{,i_1\cdots i_s}}.$$

3.4 Lie Algebras of Operators

The theory of Lie algebras is one of the well-developed fields of modern mathematics. A rigorous treatment of this subject can be found in the specialized literature.

Consider any pair of first-order linear partial differential operators

$$X_i = \xi_i(x,u)\frac{\partial}{\partial x} + \eta_i(x,u)\frac{\partial}{\partial u}, \quad X_j = \xi_j(x,u)\frac{\partial}{\partial x} + \eta_j(x,u)\frac{\partial}{\partial u}. \tag{3.18}$$

Definition 3.1. The *commutator* $[X_i, X_j]$ of operators (3.18) is the linear partial differential operator defined by the formula

$$[X_i, X_j] = X_i X_j - X_j X_i,$$

or equivalently

$$[X_i, X_j] = \big(X_i(\xi_j) - X_j(\xi_i)\big)\frac{\partial}{\partial x} + \big(X_i(\eta_j) - X_j(\eta_i)\big)\frac{\partial}{\partial u}. \tag{3.19}$$

Definition 3.2. (Lie algebra). Let L_r be an r-dimensional vector space spanned by r linearly independent operators of the form (3.18),

$$X = C_1 X_1 + C_2 X_2 + \cdots + C_r X_r,$$

$C_1, C_2, \ldots, C_r$ are constant. The space L_r is called a *Lie algebra* if it is closed under the commutator, $[X, Y] \in L_r$ whenever $X, Y \in L_r$. The operators $X_1, X_2, \ldots, X_r$ provide a basis of the Lie algebra L_r. We also say that L_r is a Lie algebra spanned by $X_1, X_2, \ldots, X_r$.

The Lie algebra is denoted by the same letter L, and the dimension $\dim L$ of the Lie algebra is the dimension of the vector space L. We shall use the symbol L_r to denote an $r-$dimensional Lie algebra.

It follows from (3.19) that the commutator is bilinear:

$$[c_1 X_1 + c_2 X_2, X] = c_1[X_1, X] + c_2[X_2, X],$$

$$[X, c_1 X_1 + c_2 X_2] = c_1[X, X_1] + c_2[X, X_2],$$

skew-symmetric:

$$[X_1, X_2] = -[X_2, X_1],$$

and satisfies the Jacobi identity:

$$[X_1,[X_2,X_3]]+[X_2,[X_3,X_1]]+[X_3,[X_1,X_2]]=0.$$

Definition 3.3. (Isomorphism). A linear one-to-one map f of a Lie algebra L onto a Lie algebra K is called an *isomorphism* (and L and K are said to be isomorphic) if

$$f([X_1,X_2]_L)=[f(X_1),f(X_2)]_K$$

where the indices L and K are used to denote the commutators in the corresponding algebras. An isomorphism of L onto itself is termed an *automorphism.*

Definition 3.4. (Subalgebra). Let L_r be a Lie algebra spanned by $X_1,X_2,\ldots,X_r$. A subspace L_s of the vector space L_r spanned by a subset of the basis operators $X_1,X_2,\ldots,X_s, s<r$, is called a *subalgebra* of L_r if $[X,Y]\in L_s$ for any $X,Y\in L_s$. Furthermore, L_s is called an *ideal* of L_r if $[X,Y]\in L_s$ whenever $X\in L_s, Y\in L_r$.

3.5 Lie-Bäcklund Representation

Let $\mathcal{A}$ denote the space of differentiable functions of all finite orders[†]. This space is a vector space with respect to the usual addition of functions. Furthermore, it has the important property of being closed under the differentiation given by $D_i=\dfrac{\partial}{\partial x_i}+u^\alpha_{,i}\dfrac{\partial}{\partial u^\alpha}+u^\alpha_{,ij}\dfrac{\partial}{\partial u^\alpha_{,j}}+\ldots.$

Consider an operator of the form

$$X=\xi_i\frac{\partial}{\partial x_i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}+\zeta^\alpha_i\frac{\partial}{\partial u^\alpha_{,i}}+\zeta^\alpha_{i_1i_2}\frac{\partial}{\partial u^\alpha_{,i_1i_2}}+\ldots, \tag{3.20}$$

where $\xi_i,\ \eta^\alpha\in\mathcal{A}$ are infinitely differentiable functions, and

$$\begin{aligned}\zeta^\alpha_i&=D_i(\eta^\alpha-\xi_ju^\alpha_{,j})+\xi_ju^\alpha_{,ji},\\ \zeta^\alpha_{i_1i_2}&=D_{i_2}D_{i_1}(\eta^\alpha-\xi_ju^\alpha_{,j})+\xi_ju^\alpha_{,ji_1i_2},\\ &\ldots\end{aligned}\tag{3.21}$$

[†]See more details in Ibragimov (1999)

Operator (3.20) with coefficients given by equations (3.20) and (3.21) is called a *Lie-Bäcklund operator*. In fact, the operator (3.20) is the infinite prolongation[‡] of

$$X = \xi_i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi_i, \ \eta^\alpha \in \mathcal{A}. \tag{3.22}$$

Lemma 3.1. *The Lie-Bäcklund operator (3.22) satisfies the commutation relation*

$$XD_i - D_i X = -D_i(\xi_j)D_j.$$

This is proved by straightforward computation.

Lemma 3.2. *Every operator*

$$X^* = \xi_i^* D_i = \xi_i^* \frac{\partial}{\partial x_i} + \xi_i^* u^\alpha_{,j} \frac{\partial}{\partial u^\alpha} + \xi_i^* u^\alpha_{,jj_1} \frac{\partial}{\partial u^\alpha_{,j_1}} + \dots \tag{3.23}$$

with arbitrary analytic coefficients ξ_i^* *is a Lie-Bäcklund operator. The set of operator (3.23) is an ideal in the Lie algebra of all Lie-Bäcklund operators with product* $[X, Y] \equiv XY - YX$.

It is often advantageous to work with the factor algebra of all Lie-Bäcklund operators by its ideal L^* of operators (3.23) rather than the full algebra. Accordingly, two Lie-Bäcklund operators, X and Y will be said to be *equivalent* whenever $X - Y \in L^*$. In particular, every operator (3.22) is equivalent to a Lie-Bäcklund operator with coordinates $\xi_i = 0$ $(i = 1, \dots, n)$; namely

$$X \sim Y = X - \xi_i D_i = (\eta^\alpha - \xi_i u_i^\alpha) \frac{\partial}{\partial u^\alpha} + \dots.$$

Definition 3.5. A Lie-Bäcklund operator (3.22) of the form

$$X = \eta^\beta \frac{\partial}{\partial u^\beta}, \quad \eta^\beta \in \mathcal{A}, \tag{3.24}$$

is called a *canonical Lie-Bäcklund operator*.

[‡]The concept of the prolongation group and prolonged generator has been given in Section 3.3

For such operators the prolongation formulas (3.21) acquire a simple form:

$$\zeta^{\alpha}_{i_1 \dots i_s} = D_{i_1} \dots D_{i_s}(\eta^{\alpha}). \tag{3.25}$$

From Lemma 3.1 it follows that the canonical Lie-Bäcklund operators commute with the differentiation operators D_i. Conversely, the condition that operator (3.20) (with $\xi_i = 0$) commutes with operator D_i implies that (3.25) are satisfied.

Although the shift from (3.22) to equivalent canonical operator (3.24) is convenient in many problems, there are cases in which it leads to a loss of geometric transparency. This is first of all true for groups of point transformation. For example, the infinitesimal generator $X = \dfrac{\partial}{\partial x_i}$ of the simplest one-parameter group of point transformations - the translations $\bar{x}_i = x_i + a$ along the x_i-axis - is reduced to the canonical form (3.24), namely $Y = u^{\alpha}_i \dfrac{\partial}{\partial u^{\alpha}} + \dots$.

3.6 Symmetry Group for Differential Equations

Lie groups are related with differential equations through the following idea.

Definition 3.6. (Admitted group). A *symmetry group of a system of differential equations* is a group of transformations mapping every solution to another solution of the same system. A symmetry group is also termed the *group admitted by the system*, or an *admitted group,* and that system of differential equations is said to be *invariant* under the symmetry group.

Consider a system of differential equations,

$$F(x, u, \underset{1}{u}, \cdots, \underset{s}{u}) = 0. \tag{3.26}$$

Let $u = v(x)$ be a solution of system (3.26) and let the transformations depending on a parameter a, $\bar{x} = \varphi^x(x, u; a)$, $\bar{u} = \varphi^u(x, u; a)$, belong to a group admitted by system (3.26). Therefore, by the definition of an admitted group, the transformed variables

$$\bar{x} = \varphi^x(x, \upsilon(x); a),$$
$$\bar{u} = \varphi^u(x, \upsilon(x); a),$$

must be another solution of system (3.26). Hence

$$F(\bar{x}, \bar{u}, \underset{1}{\bar{u}}, \cdots, \underset{s}{\bar{u}}) = 0, \tag{3.27}$$

whenever u satisfies system (3.26). This implies that system (3.27) is invariant with respect to the group parameter a:

$$\left.\frac{\partial F(\bar{x}, \bar{u}, \underset{1}{\bar{u}}, \cdots, \underset{s}{\bar{u}})}{\partial a}\right|_{a=0,\ (3.26)} \equiv 0. \tag{3.28}$$

Another representation of Equation (3.28) in generator form is

$$X^{(s)} F(\bar{x}, \bar{u}, \underset{1}{\bar{u}}, \cdots, \underset{s}{\bar{u}})\Big|_{(3.26)} = 0.$$

Definition 3.7. Equation (3.28) is called the *determining equation* of differential equation (3.26).

3.7 Group Classification Problem of DEs

Lie algebras connected by a change of variable are called *similar* or *equivalent*. When one equation is transformed into another by a change of variables, the algebras admitted by the two equations are similar.

The group classification of ordinary differential equation is based upon the enumeration of all possible nonequivalent Lie algebras of operators admitted by the chosen type of equations.

The investigation of the problem of group classification was carried out by Lie for second-order ordinary differential equations. He gave his classification in the complex variable domain. The result of the enumeration of all nonsimilar algebras (under complex changes of variables) and of invariant equations can be seen in Ibragimov (1996).

The great success in integration using symmetries provided Lie with an incentive to begin the classification of all ordinary differential equations of an arbitrary order in terms of symmetry groups.

There is a considerable literature on the group classification of differential equations while are of interest in physics. These results are presented in Ovsiannikov (1978), Ibragimov (1996) and the literature referenced there in.

For ordinary differential equations of second order with one dependent variable, group classification was obtained using the following strategy. First, all Lie algebras on the plane that were nonequivalent with respect to a change of the variables were constructed. Differential invariants of second-order prolongations were obtained. Lie algebras admitted second-order ODEs were chosen. Using the invariants of these algebras, the representation of second-order equations were obtained. These equations compose a group classification of second-order ordinary differential equations. This classification is presented in Table 3.1.

Lie group classification of second-order ODEs in two real variables domain up to change of variables. Let $p = \partial/\partial x$ and $q = \partial/\partial y$.

Table 3.1 Lie group classification of second-order ODEs in two real variables domain

No.	Lie algebra	Representative Equations
1	$X_1 = p$	$y'' = f(y, y')$
2	$X_1 = p,\ X_2 = q.$	$y'' = f(y')$
3	$X_1 = q,\ X_2 = xp + yq.$	$xy'' = f(y')$
4	$X_1 = p,\ X_2 = q,$	$y'' = Ce^{-y'}$
	$X_3 = xp + (x + y)q.$	
5	$X_1 = p,\ X_2 = q,$	$y'' = Cy'^{\frac{a-2}{a-1}},\ a \neq 0,\ \frac{1}{2},\ 2$
	$X_3 = xp + ayq.$	
6	$X_1 = p,\ X_2 = q,$	$y'' = C(1 + y'^2)^{\frac{3}{2}} e^{b \arctan y'}$
	$X_3 = (bx + y)p + (by - x)q.$	
7	$X_1 = q,\ X_2 = xp + yq,$	$xy'' = Cy'^3 - \frac{1}{2}y'$
	$X_3 = 2xyp + y^2q.$	
8	$X_1 = q,\ X_2 = xp + yq,$	$xy'' = y' + y'^3 + C(1 + y'^2)^{3/2}$
	$X_3 = 2xyp + (y^2 - x^2)q.$	
9	$X_1 = q,\ X_2 = xp + yq,$	$xy'' = y' - y'^3 + C(1 - y'^2)^{3/2}$
	$X_3 = 2xyp + (y^2 + x^2)q.$	
10	$X_1 = (1 + x^2)p + xyq,$	$y'' = C\left[\frac{1+y'^2+(y-xy')^2}{1+x^2+y^2}\right]^{3/2}$
	$X_2 = xyp + (1 + y^2)q,$	
	$X_3 = yp - xq.$	
11	$X_1 = p,\ X_2 = q,\ X_3 = xq,$	$y'' = 0$
	$X_4 = xp,\ X_5 = yp,\ X_6 = yq,$	
	$X_7 = x^2p + xyq,\ X_8 = xyp + y^2q.$	

Nesterenko's classification provides a classification of all Lie algebras in the space of two real variables (Nesterenko, 2006). The results are presented in the table below

Table 3.2 Classification of all finite dimensional Lie algebra on the real variable domain

No.	Lie algebra basis
1	∂_x
2	$\partial_x,\ \partial_y$
3	$\partial_x,\ y\partial_x$
4	$\partial_x,\ x\partial_x + y\partial_y$
5	$\partial_x,\ x\partial_x$
6	$\partial_y,\ x\partial_y,\ \xi_1(x)\partial_y$
7	$\partial_y,\ y\partial_y,\ \partial_x$
8	$e^{-x}\partial_y,\ \partial_x,\ \partial_y$
9	$\partial_y,\ \partial_x,\ x\partial_y$
10	$\partial_y,\ \partial_x,\ x\partial_x + (x+y)\partial_y$
11	$e^{-x}\partial_y,\ -xe^{-x}\partial_y,\ \partial_x$
12	$\partial_x,\ \partial_y,\ x\partial_x + y\partial_y$
13	$\partial_y,\ x\partial_y,\ y\partial_y$
14	$\partial_x,\ \partial_y,\ x\partial_x + ay\partial_y,\ 0 < \|a\| \leq 1, a \neq 1$
15	$e^{-x}\partial_y,\ e^{-ax}\partial_y,\ \partial_x,\ 0 < \|a\| \leq 1, a \neq 1$
16	$\partial_x,\ \partial_y,\ (bx+y)\partial_x + (by-x)\partial_y,\ b \geq 0$
17	$e^{-bx}\sin x\partial_y,\ e^{-bx}\cos x\partial_y,\ \partial_x,\ b \geq 0$
18	$\partial_x, x\partial_x + y\partial_y,\ (x^2-y^2)\partial_x + 2xy\partial_y$
19	$\partial_x + \partial_y,\ x\partial_x + y\partial_y,\ x^2\partial_x + y^2\partial_y$
20	$\partial_x,\ x\partial_x + \frac{1}{2}y\partial_y,\ x^2\partial_x + xy\partial_y$

No.	Lie algebra basis
21	$\partial_x,\ x\partial_x,\ x^2\partial_x$
22	$y\partial_x - x\partial_y,\ (1+x^2-y^2)\partial_x + 2xy\partial_y,\ 2xy\partial_x + (1+y^2-x^2)\partial_y$
23	$\partial_y,\ x\partial_y,\ \xi_1(x)\partial_y,\ \xi_2(x)\partial_y$
24	$\partial_x,\ x\partial_x,\ \partial_y,\ y\partial_y$
25	$e^{-x}\partial_y,\ \partial_x,\ \partial_y,\ y\partial_y$
26	$e^{-x}\partial_y,\ \ -xe^{-x}\partial_y,\ \partial_x,\ \partial_y$
27	$e^{-x}\partial_y,\ e^{-ax}\partial_y,\ \partial_x,\ \partial_y,\ 0 < \|a\| \leq 1, a \neq 1$
28	$e^{-bx}\sin x\partial_y,\ e^{-bx}\cos x\partial_y,\ \partial_x,\ \partial_y,\ b \geq 0$
29	$\partial_x,\ x\partial_x,\ y\partial_y,\ x^2\partial_x + xy\partial_y$
30	$\partial_x,\ \partial_y,\ x\partial_x,\ x^2\partial_x$
31	$\partial_y,\ \ -x\partial_y,\ \frac{1}{2}x^2\partial_y,\ \partial_x$
32	$e^{-bx}\partial_y,\ e^{-x}\partial_y,\ \ -xe^{-x}\partial_y,\ \partial_x$
33	$e^{-x}\partial_y,\ \ -x\partial_y,\ \partial_y,\ \partial_x$
34	$e^{-x}\partial_y,\ \ -xe^{-x}\partial_y,\ \frac{1}{2}x^2e^{-x}\partial_y,\ \partial_x$
35	$\partial_y,\ x\partial_y,\ \xi_1(x)\partial_y,\ y\partial_y$
36	$e^{-ax}\partial_y,\ e^{-bx}\partial_y,\ e^{-x}\partial_y,\ \partial_x,\ \ -1 \leq a < b < 1,\ ab \neq 0$
37	$e^{-ax}\partial_y,\ e^{-bx}\sin x\partial_y,\ e^{-bx}\cos x\partial_y,\ \partial_x,\ a > 0$
38	$\partial_x,\ \partial_y,\ x\partial_y,\ x\partial_x + (2y+x^2)\partial_y$
39	$\partial_y,\ \partial_x,\ x\partial_y,\ (1+b)x\partial_x + y\partial_y,\ \|b\| \leq 1$
40	$\partial_y,\ \ -x\partial_y,\ \partial_x,\ y\partial_y$
41	$\partial_x,\ \partial_y,\ x\partial_x + y\partial_y,\ y\partial_x - x\partial_y$
42	$\sin x\partial_y,\ \ \cos x\partial_y,\ y\partial_y,\ \partial_x$
43	$\partial_x,\ \partial_y,\ x\partial_x - y\partial_y,\ y\partial_x,\ x\partial_y$
44	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ y\partial_x,\ x\partial_y$
45	$\partial_x,\ \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y,\ (x^2-y^2)\partial_x - 2xy\partial_y,\ 2xy\partial_x - (y^2-x^2)\partial_y$

No.	Lie algebra basis
46	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ x^2\partial_x,\ y^2\partial_y$
47	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ y\partial_x,\ x\partial_y, x^2\partial_x + xy\partial_y,\ xy\partial_x + y^2\partial_y$
48	$\partial_y,\ x\partial_y,\ \xi_1(x)\partial_y, \dots,\ \xi_r(x)\partial_y,\ r \geq 3$
49	$y\partial_y,\ \partial_y,\ x\partial_y,\ \xi_1(x)\partial_y, \dots,\ \xi_r(x)\partial_y,\ r \geq 2$
50	$\partial_x,\ \eta_1\partial_y, \dots,\ \eta_r(x)\partial_y,\ r \geq 4$
51	$\partial_x,\ y\partial_y,\ \eta_1\partial_y, \dots,\ \eta_r(x)\partial_y,\ r \geq 3$
52	$\partial_x,\ \partial_y,\ x\partial_x + cy\partial_y,\ x\partial_y, \dots,\ x^r\partial_y,\ r \geq 2$
53	$\partial_x,\ \partial_y,\ x\partial_y, \dots,\ x^{r-1}\partial_y,\ x\partial_x + (ry + x^r)\partial_y,\ r \geq 3$
54	$\partial_x,\ x\partial_x,\ y\partial_y,\ \partial_y,\ x\partial_y, \dots,\ x^r\partial_y,\ r \geq 1$
55	$\partial_x,\ \partial_y,\ 2x\partial_x + ry\partial_y,\ x^2\partial_x + rxy\partial_y,\ x\partial_y,\ x^2\partial_y, \dots,\ x^r\partial_y,\ r \geq 1$
56	$\partial_x,\ x\partial_x,\ y\partial_y,\ x^2\partial_x + rxy\partial_y,\ \partial_y,\ x\partial_y, x^2\partial_y, \dots,\ x^r\partial_y,\ r \geq 0$

The functions $1,\ x,\ \xi_1, \dots, \xi_r$ are linearly independent. The functions $\eta_1, \dots, \eta_r$ form a fundamental system of solutions for an r-order linear ordinary differential equation with constant coefficients $\eta^{(r)}(x) + c_1\eta^{(n-1)}(x) + \dots + c_r\eta(x) = 0$.

3.8 Symmetry Group for DDEs

For delay differential equations, the definition of an admitted Lie group and the algorithm for constructing, and solving the determining equations were expressed in 2002 by S. Meleshko and J.Tanthanuch.

For the sake of simplicity, the definition of admitted Lie group for a delay differential equation with one independent variable is described.

Consider a system of delay differential equations (2.4),

$$\Xi(x, u) \equiv u' - F(x, u_x) = 0. \tag{3.29}$$

Let G be a one-parameter Lie group of transformations

$$\bar{x} = \varphi^x(x, u; a), \qquad \bar{u} = \varphi^u(x, u; a)$$

with the infinitesimal generator

$$X = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}.$$

Definition 3.8. (Admitted group). A one-parameter Lie group G of transformation (3.1) is a *symmetry group of the delay differential equations* or *symmetry group admitted by the delay differential equation* (3.29) if G satisfies

$$\left(\tilde{X}\Xi\right)((x, u(x))) = 0 \tag{3.30}$$

for any solution $u(x)$ of equation (3.29).

Here the operator $\tilde{X}$ is the prolongation of the canonical Lie-Bäcklund operator equivalent to the generator X given by

$$\tilde{X} = \zeta^u \partial_u + \zeta^{u_x} \partial_{u_x} + \dots,$$

where $\zeta^u = \eta - u_x\xi,\ \zeta^{u_x} = D_x\zeta^u$ and D_x is the total derivative with respect to x.

A symmetry group is also termed the *group admitted by the system*, or an *admitted group,* and the system of differential equations is said to be *invariant* under the symmetry group.

Definition 3.9. Equation (3.30) is called the *determining equation* for delay differential equation (3.29) .

CHAPTER IV

GROUP CLASSIFICATION OF SECOND-ORDER DELAY ORDINARY DIFFERENTIAL EQUATIONS

The purpose of this chapter is to give a complete classification of second-order delay ordinary differential equations of the form

$$y'' = f(x, y, y_\tau, y', y'_\tau) \tag{4.1}$$

admitting the Lie algebra.

4.1 Strategy for Obtaining a Complete Classification of DODEs

This section is devoted to explain the strategy for obtaining a complete classification of second-order DODEs (4.1) admitting a Lie group.

4.1.1 Properties of an Admitted Generator

Assume that the infinitesimal generator

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y \tag{4.2}$$

is admitted by a second-order DODE (4.1). The corresponding canonical Lie-Bäcklund operator has the form

$$X = \zeta(x, y, y')\partial_y, \tag{4.3}$$

where $\zeta = \eta - y'\xi$. For obtaining determining equations of second-order DODEs, one has to prolong the canonical Lie-Bäcklund operator to the six-dimensional space of variables $(x, y, y_\tau, y', y'_\tau, y'')$:

$$\tilde{X}_{\mathcal{B}} = \zeta^y \partial_y + \zeta^{y_\tau} \partial_{y_\tau} + \zeta^{y'} \partial_{y'} + \zeta^{y'_\tau} \partial_{y'_\tau} + \zeta^{y''} \partial_{y''}, \tag{4.4}$$

where

$$\begin{aligned}
\zeta^y(x, y, y') &= \eta(x, y) - y'\xi(x, y), \\
\zeta^{y_\tau}(x, y_\tau, y'_\tau) &= \zeta^y(x - \tau, y_\tau, y'_\tau) = \eta(x - \tau, y_\tau) - y'_\tau \xi(x - \tau, y_\tau), \\
\zeta^{y'}(x, y, y', y'') &= D(\zeta^y) = \eta_x(x, y) + [\eta_y(x, y) - \xi_x(x, y)]y' - \xi_y(x, y)(y')^2 - \xi(x, y)y'', \\
\zeta^{y'_\tau}(x, y_\tau, y'_\tau, y''_\tau) &= \zeta^{y'}(x - \tau, y_\tau, y'_\tau, y''_\tau) = \eta_x(x - \tau, y_\tau) + [\eta_y(x - \tau, y_\tau) \\
&\quad -\xi_x(x - \tau, y_\tau)]y'_\tau - \xi_y(x - \tau, y_\tau)(y'_\tau)^2 - \xi(x - \tau, y_\tau)y''_\tau, \\
\zeta^{y''}(x, y, y', y'', y''') &= D(\zeta^{y'}) = \eta_{xx}(x, y) + [2\eta_{xy}(x, y) - \xi_{xx}(x, y)]y' \\
&\quad +[\eta_{yy}(x, y) - 2\xi_{xy}(x, y)](y')^2 - \xi_{yy}(x, y)(y')^3 \\
&\quad +[\eta_y(x, y) - 2\xi_x(x, y)]y'' - 3\xi_y(x, y)y'y'' - \xi(x, y)y''',
\end{aligned}$$

D is the operator of the total derivative with respect to x, i.e. $D = \partial_x + y'\partial_y + \cdots$. The determining equation for the second-order DODE is

$$\tilde{X}_{\mathcal{B}}\Big(y'' - f(x, y, y_\tau, y', y'_\tau)\Big)\Big|_{(4.1)} = 0. \tag{4.5}$$

Equation (4.5) has to be satisfied by any solution of equation (4.1). Substituting $y''' = f_x + y'f_y + y'_\tau f_{y_\tau} + y'' f_{y'} + y''_\tau f_{y'_\tau}$, $y'' = f$ and $y''_\tau = f_\tau$, the determining equation (4.5) is rewritten as

$$\begin{gathered}
-\xi_{yy}(y')^3 + (\eta_{yy} - 2\xi_{xy} + \xi_y f_{y'})(y')^2 + \xi^\tau_{y_\tau} f_{y'_\tau}(y'_\tau)^2 + (2\eta_{xy} - \xi_{xx})y' + (\xi_x - \eta_y)f_{y'}y' \\
-3\xi_y f y' + \eta_{xx} - \eta_x f_{y'} + (\eta_y - 2\xi_x)f - \eta^\tau_x f_{y'_\tau} + (\xi^\tau_x - \eta^\tau_{y_\tau})f_{y'_\tau}y'_\tau - f_x\xi - f_y\eta \\
-\eta^\tau f_{y_\tau} + (\xi^\tau - \xi)f_{y_\tau}y'_\tau + (\xi^\tau - \xi)f_\tau f_{y'_\tau} = 0,
\end{gathered} \tag{4.6}$$

where $f_\tau = f(x - \tau, y_\tau, y_{2\tau}, y'_\tau, y'_{2\tau})$, $y_{2\tau} = y(x - 2\tau)$ and $y'_{2\tau} = y'(x - 2\tau)$.

By virtue of the Cauchy problem, one can account the variables x, y, y_τ, $y', y'_\tau, y_{2\tau}$ and $y'_{2\tau}$ in (4.6) as arbitrary variables.

For the case $f_{y'_\tau} \neq 0$, we can split the determining equation (4.6) with respect to $y'_{2\tau}$. This implies $\xi = \xi^\tau$.

If $f_{y'_\tau} = 0$, then the assumption of DODE implies f must depend on the delay terms, i.e. $f_{y_\tau} \neq 0$. Splitting (4.6) with respect to y'_τ, we also get $\xi = \xi^\tau$. This shows the periodic property of ξ, i.e.,

$$\xi(x, y) = \xi(x - \tau, y_\tau). \tag{4.7}$$

Because this property is satisfied for any solution of the Cauchy problem, then (4.7) implies function ξ does not depend on y, i.e., $\xi_y = 0$. Moreover, property (4.7) allows us to rewrite the determining equation (4.5) in the form

$$\bar{X}\Big(y'' - f(x, y, y_\tau, y', y'_\tau)\Big)\Big|_{(4.1)} = 0, \tag{4.8}$$

where

$$\begin{aligned}
\bar{X} &= \tilde{X}_{\mathcal{B}} + \xi D = \xi\partial_x + \eta^y\partial_y + \eta^{y_\tau}\partial_{y_\tau} + \eta^{y'}\partial_{y'} + \eta^{y'_\tau}\partial_{y'_\tau} + \eta^{y''}\partial_{y''}, \\
\eta^y(x, y) &= \eta(x, y), \\
\eta^{y_\tau}(x, y_\tau) &= \eta(x - \tau, y_\tau), \\
\eta^{y'}(x, y, y') &= \eta_x(x, y) + [\eta_y(x, y) - \xi_x(x, y)]y' - \xi_y(x, y)(y')^2, \\
\eta^{y'_\tau}(x, y_\tau, y'_\tau) &= \eta^{y'}(x - \tau, y_\tau, y'_\tau) = \eta_x(x - \tau, y_\tau) + [\eta_y(x - \tau, y_\tau) - \xi_x(x - \tau, y_\tau)]y'_\tau \\
&\quad -\xi_y(x - \tau, y_\tau)(y'_\tau)^2, \\
\eta^{y''}(x, y, y', y'') &= \eta_{xx}(x, y) + [2\eta_{xy}(x, y) - \xi_{xx}(x, y)]y' + [\eta_{yy}(x, y) - 2\xi_{xy}(x, y)](y')^2 \\
&\quad -\xi_{yy}(x, y)(y')^3 + [\eta_y(x, y) - 2\xi_x(x, y)]y'' - 3\xi_y(x, y)y'y'',
\end{aligned}$$

D is the operator of the total derivative with respect to x. The difference between the generator $\bar{X}$ and $\tilde{X}_{\mathcal{B}}$ is the following. The generator $\bar{X}$ acts in the space of variables $(x, y, y_\tau, y', y'_\tau, y'')$, whereas the coefficients of the operator $\tilde{X}_{\mathcal{B}}$ include the derivatives y''_τ and y'''.

Notice that equation (4.8) means the manifold defined by equation (4.1) is an invariant manifold of the generator $\bar{X}$. Because of the invariant manifold theorem, any invariant manifold can be represented through invariants of the generator $\bar{X}$.

Hence, for describing equations admitting the generator X, one needs to find all invariants of the generator $\bar{X}$.

Another property of admitted generator, which allows developing a method for classifying all second-order DODEs is the following. Direct calculations show that if two generators X_1 and X_2 are admitted by equation (4.1), then their commutator $[X_1, X_2]$ is also admitted by equation (4.1). This property allows stating that the set of infinitesimal generators admitted by equation (4.1) composes a Lie algebra on the real plane.

4.1.2 The Strategy

As it was explained in section 3.7, there is a complete description of all finite dimensional Lie algebras on the real space (Nesterenko, 2006). The classification is obtained up to a nonsingular change of the variables x and y, and consists of a list of 56 Lie algebras (See Table 3.2). Since the set of generators admitted by a second-order DODE composes a Lie algebra, then this algebra is equivalent to one of these 56 Lie algebras.

In order to complete classification of second-order DODEs, we need to carry out the following steps for each class of 56 Lie algebras:

(a) change the variables x and y

$$x = \bar{h}(\bar{x}, \bar{y}), \qquad y = \bar{g}(\bar{x}, \bar{y}), \tag{4.9}$$

(b) find invariants of the Lie algebra in the space of changed variables $(\bar{x}, \bar{y}, \bar{y}_\tau, \bar{y}', \bar{y}'_\tau, \bar{y}'')$,

(c) use the found invariants to form a second-order DODE.

Applying this strategy we will obtain representations of all second-order DODEs admitting a Lie group.

4.2 Illustrative Examples

This section gives examples which illustrate an application of the above strategy. Complete results of the classification are presented in the next section.

Here the notation L^n_j is used to denote the n-dimensional Lie algebra of the number j from Table 3.2.

Example 4.1. Let us consider a three-dimensional Lie algebra L^3_{10}, which is generated by the generators

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + (x+y)\partial_y. \tag{4.10}$$

Changing the variables, $x = \bar{h}(\bar{x}, \bar{y}),\ y = \bar{g}(\bar{x}, \bar{y})$ and using equation (3.16), the first components $\bar{\xi}_i$ are :

$$\bar{\xi}_1 = \frac{\bar{g}_{\bar{y}}}{\Delta}, \quad \bar{\xi}_2 = \frac{\bar{h}_{\bar{y}}}{\Delta}, \quad \bar{\xi}_3 = \frac{\bar{h}\bar{g}_{\bar{y}} - (\bar{h}+\bar{g})\bar{h}_{\bar{y}}}{\Delta},$$

which have to satisfy the conditions $(\bar{\xi}_i)_{\bar{y}} = 0$ and $\bar{\xi}_i(\bar{x}) = \bar{\xi}_i(\bar{x}-\tau)$ based on (4.7), (i=1,2,3). These conditions imply that $(\bar{\xi}_1)_{\bar{y}} = 0,\ (\bar{\xi}_2)_{\bar{y}} = 0,\ (\bar{\xi}_3)_{\bar{y}} = 0$. Equations $(\bar{\xi}_2)_{\bar{y}} = 0$ and $(\bar{\xi}_3)_{\bar{y}} = 0$ lead us to the restrictions $\bar{h}_{\bar{y}} = 0$ and $\bar{h}(\bar{x}) - \bar{h}(\bar{x}-\tau) = c$, where c is an arbitrary constant. Then $\Delta = \bar{h}_{\bar{x}}\bar{g}_{\bar{y}}$. Using equation (3.15), generators (4.10) become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{(\bar{h}+\bar{g})\bar{h}_{\bar{x}} - \bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}. \tag{4.11}$$

We consequently solve equations for invariants which are related with the prolonged generators $\bar{X}_1^{(2)}$, $\bar{X}_2^{(2)}$, $\bar{X}_3^{(2)}$:

$$\bar{X}_1^{(2)}J = 0, \quad \bar{X}_2^{(2)}J = 0, \quad , \bar{X}_3^{(2)}J = 0, \tag{4.12}$$

where $\bar{X}_i^{(2)}$, $(i = 1, 2, 3)$ is the second prolongation of the generator $\bar{X}_i$.

To find invariants with respect to $\bar{X}_1$ we have to solve the equation

$$\bar{X}_1^{(2)}J(\bar{x}, \bar{y}, \bar{y}_\tau, \bar{y}', \bar{y}'_\tau, \bar{y}'') = 0, \tag{4.13}$$

where

$$\bar{X}_1^{(2)} = \bar{\xi}_1(\bar{x})\partial_{\bar{x}} + \bar{\eta}_1(\bar{x}, \bar{y})\partial_{\bar{y}} + \bar{\eta}_1^{\bar{y}'}(\bar{x}, \bar{y}, \bar{y}')\partial_{\bar{y}'} + \bar{\eta}_1^{\bar{y}''}(\bar{x}, \bar{y}, \bar{y}')\partial_{\bar{y}''} \tag{4.14}$$
$$+\bar{\eta}_1(\bar{x} - \tau, \bar{y}_\tau)\partial_{\bar{y}_\tau} + \bar{\eta}_1^{\bar{y}'_\tau}(\bar{x} - \tau, \bar{y}_\tau, \bar{y}'_\tau)\partial_{\bar{y}'_\tau}.$$

For integrating equation (4.13) one has to solve the characteristic system of equations

$$\frac{d\bar{x}}{\bar{\xi}_1} = \frac{d\bar{y}}{\bar{\eta}_1} = \frac{d\bar{y}'}{\bar{\eta}_1^{\bar{y}'}} = \frac{d\bar{y}''}{\bar{\eta}_1^{\bar{y}''}} = \frac{d\bar{y}_\tau}{\bar{\eta}_1^\tau} = \frac{d\bar{y}'_\tau}{\bar{\eta}_1^{\bar{y}'_\tau}}. \tag{4.15}$$

This characteristic system is cumbersome to solve. However, one may note that the first part of this system (without last two equations containing the variables related with delay) is equivalent to the system which corresponds to the prolongation of the original generator X_1 with the variables (x, y, y', y''):

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0} = \frac{dy''}{0}. \tag{4.16}$$

Differential invariants of the last system are easily obtained, i.e. $y,\ y',\ y''$. Hence, we found three invariants of equation (4.13):

$$J_1 = \bar{g}(\bar{x}, \bar{y}), \quad J_2 = \frac{D(\bar{g}(\bar{x}, \bar{y}))}{D(\bar{h}(\bar{x}))}, \quad J_3 = \frac{D(J_2(\bar{x}, \bar{y}, \bar{y}'))}{D(\bar{h}(\bar{x}))}, \tag{4.17}$$

where D is the operator of the total derivative with respect to $\bar{x}$. The other two invariants are chosen as follows

$$J_1^\tau = J_1(\bar{x} - \tau, \bar{y}_\tau), \quad J_2^\tau = J_2(\bar{x} - \tau, \bar{y}_\tau, \bar{y}'_\tau). \tag{4.18}$$

Direct calculations show that (4.17)-(4.18) compose the universal differential invariant of the generator $\bar{X}_1^{(2)}$. Hence, the general solution equation (4.13) is

$$\Phi = \Phi(J_1, J_1^\tau, J_2, J_2^\tau, J_3). \tag{4.19}$$

At this step, the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ is an arbitrary function.

For solving the other two equations

$$\bar{X}_2^{(2)} J = 0, \quad \bar{X}_3^{(2)} J = 0, \tag{4.20}$$

we have to find the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ which satisfies the equations

$$\bar{X}_2^{(2)} \Phi(J_1, J_1^\tau, J_2, J_2^\tau, J_3) = 0, \tag{4.21}$$

$$\bar{X}_3^{(2)} \Phi(J_1, J_1^\tau, J_2, J_2^\tau, J_3) = 0. \tag{4.22}$$

Equation (4.21) becomes

$$\Phi_{y_1} + \Phi_{y_2} = 0. \tag{4.23}$$

The general solution of this equation is

$$\Phi = \psi(y_1 - y_2, y_3, y_4, y_5) \tag{4.24}$$

where the function $\psi(z_1, z_2, z_3, z_4)$ is an arbitrary function.

For solving equation (4.22), we have to find the function $\psi(z_1, z_2, z_3, z_4)$ which satisfies the equation

$$\psi_{z_2} + \psi_{z_3} + z_1\psi_{z_1} - z_4\psi_{z_4} = 0. \tag{4.25}$$

This equation was obtained by substituting $J = \psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$ into equation (4.22). The general solution of this equation is

$$\psi = H(z_2 - z_3, z_1 e^{-z_2}, z_4 e^{z_2}), \tag{4.26}$$

where H is an arbitrary function.

Thus, the universal invariant of the Lie algebra L_{10}^3 consists of the invariants

$$J_2 - J_2^\tau, \quad (J_1 - J_1^\tau)e^{-J_2}, \quad J_3 e^{J_2}. \tag{4.27}$$

The set of equations admitting the Lie algebra L_{10}^3 can be expressed as the form

$$J_3 = e^{-J_2} f(J_2 - J_2^\tau, (J_1 - J_1^\tau)e^{-J_2}). \tag{4.28}$$

Because of the meaning of the functions $J_1, J_1^\tau, J_2, J_2^\tau$ and J_3, we represent this equation in Table 4.1 as

$$y'' = e^{-y'} f(y' - y'_\tau, (y - y_\tau)e^{-y'}). \tag{4.29}$$

Example 4.2. The representation of second-order DODEs admitting L_{24}^4

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x, \quad X_4 = y\partial_y, \tag{4.30}$$

can be found as the follows. Changing the variables $x = \bar{h}(\bar{x}, \bar{y})$ and $y = \bar{g}(\bar{x}, \bar{y})$ under the condition $\bar{\xi}_i = \bar{\xi}_i^\tau$, $(i = 1, 2, 3, 4)$ leads to $\bar{h}_{\bar{y}} = 0$, $\bar{h}(\bar{x}) - \bar{h}(\bar{x} - \tau) = c$. The transformed generators are

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}. \tag{4.31}$$

Suppose $\psi(z_1, z_2, z_3, z_4)$ is an arbitrary function. Like the previous example, invariant function

$$\Phi = \psi(y_1 - y_2, y_3, y_4, y_5) \tag{4.32}$$

admitting generators X_1 and X_2 are obtained. Next, we will find the function $\psi(z_1, z_2, z_3, z_4)$ which satisfies

$$\bar{X}_3^{(2)}\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3) = 0, \tag{4.33}$$

$$\bar{X}_4^{(2)}\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3) = 0. \tag{4.34}$$

Equation (4.33) becomes

$$z_2\psi_{z_2} + z_3\psi_{z_3} - 2z_4\psi_{z_4} = 0. \tag{4.35}$$

The general solution of this equation is

$$\psi = H\Big(z_1, \frac{z_3}{z_2}, \frac{(z_2)^2}{z_4}\Big). \tag{4.36}$$

Here H is an arbitrary function. Lastly, for solving (4.34) we have to find function $H(v_1, v_2, v_3)$ which satisfies

$$v_1 H_{v_1} + v_3 H_{v_3} = 0. \tag{4.37}$$

This equation was obtained by substituting

$$\psi = H\Big(J_1 - J_1^{\tau}, \frac{J_2^{\tau}}{J_2}, \frac{(J_2)^2}{J_3}\Big). \tag{4.38}$$

into equation (4.34). The general solution of this equation is

$$H = G\Big(v_2, \frac{v_1}{v_3}\Big). \tag{4.39}$$

Here G is an arbitrary function. Thus the universal invariant of the Lie algebra L_{24}^4 consists of invariants

$$\frac{J_2^{\tau}}{J_2}, \quad \frac{J_3(J_1 - J_1^{\tau})}{(J_2)^2}.$$

The set of equations admitting the Lie algebra L_{24}^4 can be expressed the form

$$J_3 = \frac{(J_2)^2}{J_1 - J_1^{\tau}} f\Big(\frac{J_2^{\tau}}{J_2}\Big),$$

where f is an arbitrary function of $\frac{J_2^{\tau}}{J_2}$.

Because of the meaning of the functions $J_1, J_1^{\tau}, J_2, J_2^{\tau}$ and J_3, in Table 4.1, we represent this set of equations as

$$y'' = \frac{(y')^2}{y - y'_{\tau}} f\Big(\frac{y'_{\tau}}{y'}\Big). \tag{4.40}$$

4.3 Second-Order Differential Invariants

Here, we present the results of calculations which are collected in Table 4.1.

4.3.1 Lie Algebra L_1^1

The generator $X_1 = \partial_x$ in new variables has the representation

$$\bar{X}_1 = \frac{-\bar{g}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{\bar{g}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}}.$$

Differential invariants up to second-order of this generator $\bar{X}_1^{(2)}$ are defined in (4.17)-(4.18):

$$J_1(\bar{x}, \bar{y}), \quad J_1^{\tau}(\bar{x}, \bar{y}_{\tau}), \quad J_2(\bar{x}, \bar{y}, \bar{y}'), \quad J_2^{\tau}(\bar{x}, \bar{y}_{\tau}, \bar{y}'_{\tau}), \quad J_3(\bar{x}, \bar{y}, \bar{y}', \bar{y}''). \tag{4.41}$$

The set of equation admitting the generator $\bar{X}_1$ is

$$J_3 = f(J_1, J_1^{\tau}, J_2, J_2^{\tau}). \tag{4.42}$$

In table 4.1, this set of equations is written as

$$y'' = f(y, y_{\tau}, y', y'_{\tau}). \tag{4.43}$$

4.3.2 Lie Algebra L_2^2

This algebra is defined by the generators

$$X_1 = \partial_x, \quad X_2 = \partial_y.$$

After changing the variables, the generators become

$$\bar{X}_1 = \frac{-\bar{g}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{\bar{g}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.41). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ by letting

$$y_1 = J_1, \; y_2 = J_1^{\tau}, \; y_3 = J_2, \; y_4 = J_2^{\tau}, \; y_5 = J_3,$$

we obtain

$$\Phi_{y_1} + \Phi_{y_2} = 0.$$

Thus, the universal invariant of this algebra is

$$J_1 - J_1^\tau, \ J_2, \ J_2^\tau, \ J_3. \tag{4.44}$$

The set of equations admitting the generator L_2^2 is

$$J_3 = f(J_1 - J_1^\tau, J_2, J_2^\tau). \tag{4.45}$$

In table 4.1, this set of equations is written as

$$y'' = f(y - y_\tau, y', y'_\tau). \tag{4.46}$$

4.3.3 Lie Algebra L_3^2

This algebra is defined by the generators

$$X_1 = \partial_x, \ X_2 = y\partial_x$$

which after changing the variables, the generators become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{g}}{\bar{h}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.41). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ by letting

$$y_1 = J_1, \ y_2 = J_1^\tau, \ y_3 = J_2, \ y_4 = J_2^\tau, \ y_5 = J_3,$$

we obtain

$$(y_3)^2\Phi_{y_3} + (y_4)^2\Phi_{y_4} + 3y_3y_5\Phi_{y_5} = 0.$$

Thus, the universal invariant of this algebra is

$$J_1, \ J_1^\tau, \ \frac{1}{J_2} - \frac{1}{J_2^\tau}, \ \frac{J_3}{(J_2)^3}. \tag{4.47}$$

The set of equations admitting the generator L_3^2 is

$$J_3 = (J_2)^3 f\Big(J_1, J_1^\tau, \frac{1}{J_2} - \frac{1}{J_2^\tau}\Big). \tag{4.48}$$

In table 4.1, this set of equations is written as

$$y'' = (y')^3 f\Big(y, y_\tau, \frac{1}{y'} - \frac{1}{y'_\tau}\Big). \tag{4.49}$$

4.3.4 Lie Algebra L_4^2

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = x\partial_x + y\partial_y,$$

which after changing the variables become

$$\begin{aligned}\bar{X}_1 &= \frac{-\bar{g}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{\bar{g}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}}\\ \bar{X}_2 &= \frac{-\bar{h}\bar{g}_{\bar{y}} + \bar{g}\bar{h}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{\bar{h}\bar{g}_{\bar{x}} - \bar{g}\bar{h}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}}.\end{aligned}$$

Invariants of the first generator are (4.41). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ by letting

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$y_1\Phi_{y_1} + y_2\Phi_{y_2} - y_5\Phi_{y_5} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{J_1}{J_1^\tau},\ J_2,\ J_2^\tau,\ J_1 J_3. \tag{4.50}$$

The set of equations admitting Lie algebra L_4^2 is

$$J_3 = \frac{1}{J_1} f\Big(\frac{J_1}{J_1^\tau}, J_2, J_2^\tau\Big). \tag{4.51}$$

In table 4.1, this set of equations is written as

$$y'' = \frac{1}{y} f\Big(\frac{y}{y_\tau}, y', y'_\tau\Big). \tag{4.52}$$

4.3.5 Lie Algebra L_5^2

This algebra is generated by

$$X_1 = \partial_x,\ X_2 = x\partial_x$$

which after changing the variables, the generators become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.41). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$y_3\Phi_{y_3} + y_4\Phi_{y_4} + 2y_5\Phi_{y_5} = 0.$$

Thus, the universal invariant of this algebra is

$$J_1,\ J_1^\tau,\ \frac{J_2}{J_2^\tau},\ \frac{J_3}{(J_2)^2}. \tag{4.53}$$

The set of equations admitting the generator L_5^2 is

$$J_3 = (J_2)^2 f\Big(J_1, J_1^\tau, \frac{J_2^\tau}{J_2}\Big). \tag{4.54}$$

In table 4.1, this set of equations is written as

$$y'' = y'^2 f\Big(y, y_\tau, \frac{y'_\tau}{y'}\Big). \tag{4.55}$$

4.3.6 Lie Algebra L_6^3

This algebra is defined by the generators

$$X_1 = \partial_y,\ X_2 = x\partial_y,\ X_3 = \xi_1(x)\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\ \bar{X}_2 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\ \bar{X}_3 = \frac{\xi_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are $\bar{h}(\bar{x}),\ J_1 - J_1^\tau,\ J_2,\ J_2^\tau,\ J_3$. Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ by letting

$$y_1 = \bar{h},\ y_2 = J_1 - J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$c\Phi_{y_2} + \Phi_{y_3} + \Phi_{y_4} = 0,$$

where c is an arbitrary constant. The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function,

$$z_1 = y_1,\ z_2 = cy_3 - y_2,\ z_3 = y_3 - y_4,\ z_4 = y_5.$$

Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(\bar{h}, cJ_2 - J_1 + J_1^\tau, J_2 - J_2^\tau, J_3)$, we find

$$(\xi_1' c - \xi_1 + \xi_1^\tau)\psi_{z_2} + (\xi_1' - \xi_1^{\tau'})\psi_{z_3} + \xi_1''\psi_{z_4} = 0.$$

Thus, the universal invariant of this algebra is

$$\bar{h},\ (\xi_1' - \xi_1^{\tau'})(cJ_2 - J_1 + J_1^\tau) - (\xi_1' c - \xi_1 + \xi_1^\tau)(J_2 - J_2^\tau),\ (\xi_1' - \xi_1^{\tau'})J_3 - \xi_1''(J_2 - J_2^\tau).$$

The set of equations admitting the generator L_6^3 is

$$J_3 = \frac{f\Big(\bar{h}, (\xi_1' - \xi_1^{\tau'})(cJ_2 - J_1 + J_1^\tau) - (\xi_1' c - \xi_1 + \xi_1^\tau)(J_2 - J_2^\tau)\Big) + \xi_1''(J_2 - J_2^\tau)}{(\xi_1' - \xi_1^{\tau'})}.$$

In table 4.1, this set of equations is written as

$$y'' = \frac{f\Big(x, (\xi_1' - \xi_1^{\tau'})(cy' - y + y_\tau) - (\xi_1' c - \xi_1 + \xi_1^\tau)(y' - y_\tau')\Big) + \xi_1''(y' - y_\tau')}{(\xi_1' - \xi_1^{\tau'})}.$$

4.3.7 Lie Algebra L_7^3

This algebra is defined by

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = y\partial_y,$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$\Phi_{y_1} + \Phi_{y_2} = 0.$$

The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5.$$

Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$z_1\psi_{z_1} + z_2\psi_{z_2} + z_3\psi_{z_3} + z_4\psi_{z_4} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{J_1 - J_1^\tau}{J_2},\ \frac{J_2^\tau}{J_2},\ \frac{J_3}{J_2}. \tag{4.56}$$

The set of equations admitting the generator L_7^3 is

$$J_3 = J_2 f\Big(\frac{J_1 - J_1^\tau}{J_2}, \frac{J_2^\tau}{J_2}\Big). \tag{4.57}$$

In table 4.1, this set of equations is written as

$$y'' = y' f\Big(\frac{y - y_\tau}{y'}, \frac{y'_\tau}{y'}\Big). \tag{4.58}$$

4.3.8 Lie Algebra L_8^3

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = e^{-x}\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

the invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$(1-k)\psi_{z_1} - \psi_{z_2} - k\psi_{z_3} + z_4\psi_{z_4} = 0,$$

where $k > 0$ is constant. Thus, the universal invariant of this algebra is

$$k(J_1 - J_1^\tau) + (1-k)J_2^\tau,\ kJ_2 - J_2^\tau,\ J_2 + J_3. \tag{4.59}$$

The set of equations admitting the generator L_8^3 is

$$J_3 = f(kJ_2 - J_2^\tau, k(J_1 - J_1^\tau - J_2^\tau) + J_2^\tau) - J_2. \tag{4.60}$$

In table 4.1, this set of equations is written as

$$y'' = f\Big(ky' - y'_\tau, k(y - y_\tau - y'_\tau) + y'_\tau\Big) - y'. \tag{4.61}$$

4.3.9 Lie Algebra L_9^3

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariant of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$c\psi_{z_1} + \psi_{z_2} + \psi_{z_3} = 0,$$

where c is constant. Thus, the universal invariant of this algebra is

$$J_2 - J_2^\tau,\ cJ_2 - (J_1 - J_1^\tau),\ J_3. \tag{4.62}$$

The set of equations admitting the generator L_9^3 is

$$J_3 = f(J_2 - J_2^\tau,\ cJ_2 - J_1 + J_1^\tau). \tag{4.63}$$

In table 4.1, this set of equations is written as

$$y'' = f(y' - y'_\tau,\ cy' - y + y_\tau). \tag{4.64}$$

4.3.10 Lie Algebra L_{10}^3

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x + (x+y)\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{(\bar{h}+\bar{g})\bar{h}_{\bar{x}} - \bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$z_1\psi_{z_1} + \psi_{z_2} + \psi_{z_3} - z_4\psi_{z_4} = 0.$$

Thus, the universal invariant of this algebra is

$$J_2 - J_2^\tau,\ (J_1 - J_1^\tau)e^{-J_2},\ J_3e^{J_2}. \tag{4.65}$$

The set of equations admitting the generator L_{10}^3 is

$$J_3 = e^{-J_2} f\Big(J_2 - J_2^\tau,\ (J_1 - J_1^\tau)e^{-J_2}\Big). \tag{4.66}$$

In table 4.1, this set of equations is written as

$$y'' = e^{-y'} f\Big(y' - y'_\tau,\ (y - y_\tau)e^{-y'}\Big). \tag{4.67}$$

4.3.11 Lie Algebra L_{11}^3

This algebra is defined by

$$X_1 = \partial_x,\ X_2 = e^{-x}\partial_y,\ X_3 = -xe^{-x}\partial_y.$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}} \quad \bar{X}_2 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}} \quad \bar{X}_3 = \frac{-\bar{h}e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$y_1\Phi_{y_1} - y_3\Phi_{y_3} + k(\Phi_{y_2} + \Phi_{y_4}) + \Phi_{y_5} = 0.$$

The invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and

$$v_1 = y_1 + y_3,\ v_2 = y_3 + y_5,\ v_3 = ky_1 - y_2,\ v_4 = y_2 + y_4.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 + J_2, J_2 + J_3, kJ_1 - J_1^\tau, J_1^\tau + J_2^\tau)$, we find

$$\psi_{v_1} - \psi_{v_2} + kc\psi_{v_3} + k\psi_{v_4} = 0.$$

Thus, the universal invariant of this algebra is

$$J_3 + 2J_2 + J_1,\ k(J_1 + J_2) - (J_1^\tau - J_2^\tau),\ kc(J_1 + J_2) - kJ_1 + J_1^\tau.$$

The set of equations admitting the generator L_{11}^3 is

$$J_3 = f\Big(k(J_1 + J_2) - (J_1^\tau + J_2^\tau),\ kc(J_1 + J_2) - kJ_1 + J_1^\tau\Big) - (2J_2 + J_1).$$

In table 4.1, this set of equations is written as

$$y'' = f\Big(k(y + y') - (y_\tau + y'_\tau),\ kc(y + y') - ky + y_\tau\Big) - (2y' + y).$$

4.3.12 Lie Algebra L^3_{12}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x + y\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{-\bar{g}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{\bar{g}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}},$$
$$\bar{X}_3 = \frac{-\bar{h}\bar{g}_{\bar{y}} + \bar{g}\bar{h}_{\bar{y}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \frac{\bar{h}\bar{g}_{\bar{x}} - \bar{g}\bar{h}_{\bar{x}}}{\bar{g}_{\bar{x}}\bar{h}_{\bar{y}} - \bar{g}_{\bar{y}}\bar{h}_{\bar{x}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^{\tau},\ y_3 = J_2,\ y_4 = J_2^{\tau},\ y_5 = J_3,$$

the invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^{\tau}, J_2, J_2^{\tau}, J_3)$, we arrive at

$$z_1\psi_{z_1} - z_4\psi_{z_4} = 0.$$

Thus, the universal invariant of this algebra is

$$J_2,\ J_2^{\tau},\ (J_1 - J_1^{\tau})J_3. \tag{4.68}$$

The set of equations admitting the generator L^3_{12} is

$$J_3 = \frac{1}{(J_1 - J_1^{\tau})} f(J_2,\ J_2^{\tau}). \tag{4.69}$$

In table 4.1, this set of equations is written as

$$y'' = \frac{f(y',\ y'_{\tau})}{y - y_{\tau}}. \tag{4.70}$$

4.3.13 Lie Algebra L^3_{13}

This algebra is defined by

$$X_1 = \partial_y,\ X_2 = x\partial_y,\ X_3 = y\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are $\bar{h}(\bar{x}),\ \bar{g}-\bar{g}^\tau,\ J_2,\ J_2^\tau,\ J_3$. Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = \bar{h},\ y_2 = J_1 - J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$c\Phi_{y_2} + \Phi_{y_3} + \Phi_{y_4} = 0,$$

where c is an arbitrary constant. The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = y_1,\ z_2 = y_3 - y_4,\ z_3 = cy_3 - y_2,\ z_4 = y_5.$$

Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(\bar{h}, J_2 - J_2^\tau, cJ_2 - J_1 + J_1^\tau, J_3)$, we arrive at

$$z_2\psi_{z_2} + z_3\psi_{z_3} + z_4\psi_{z_4} = 0.$$

Thus, the universal invariant of this algebra is

$$\bar{h},\ \frac{cJ_2 - J_1 + J_1^\tau}{(J_2 - J_2^\tau)},\ \frac{J_3}{(J_2 - J_2^\tau)}. \tag{4.71}$$

The set of equations admitting the generator L^3_{13} is

$$J_3 = (J_2 - J_2^\tau) f\Big(\bar{h}, \frac{cJ_2 - J_1 + J_1^\tau}{(J_2 - J_2^\tau)}\Big). \tag{4.72}$$

In table 4.1 this set of equations is written as

$$y'' = (y' - y'_\tau) f\Big(x, \frac{cy' - y + y_\tau}{(y' - y'_\tau)}\Big). \tag{4.73}$$

4.3.14 Lie Algebra L^3_{14}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x + ay\partial_y, \quad 0 < |a| \le 1,\ a \neq 1$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + (-\frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \frac{a\bar{g}}{\bar{g}_{\bar{y}}})\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

the invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we obtain

$$az_1\psi_{z_1} + (a-1)z_2\psi_{z_2} + (a-1)z_3\psi_{z_3} - (a-2)z_4\psi_{z_4} = 0.$$

Thus, the universal invariant of this algebra is

$$J_2(J_1 - J_1^\tau)^{\frac{(1-a)}{a}},\ \frac{J_2^\tau}{J_2},\ J_3 J_2^{\frac{(2-a)}{(a-1)}}. \tag{4.74}$$

The set of equations admitting the generator L^3_{14} is

$$J_3 = J_2^{\frac{(a-2)}{(a-1)}} f\Big(\frac{J_2^\tau}{J_2}, J_2(J_1 - J_1^\tau)^{\frac{(1-a)}{a}}\Big). \tag{4.75}$$

In table 4.1, this set of equations is written as

$$y'' = y'^{\frac{(a-2)}{(a-1)}} f\Big(\frac{y'_\tau}{y'}, y'(y - y_\tau)^{\frac{(1-a)}{a}}\Big). \tag{4.76}$$

4.3.15 Lie Algebra L^3_{15}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = e^{-x}\partial_y,\ X_3 = e^{-ax}\partial_y,\quad 0 < |a| \neq 0,\ a \neq 1$$

which after changing variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3 = \frac{e^{-a\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$y_1\Phi_{y_1} - y_3\Phi_{y_3} + k(\Phi_{y_2} + \Phi_{y_4}) + \Phi_{y_5} = 0.$$

The invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and

$$v_1 = y_1 + y_3,\ v_2 = y_3 + y_5,\ v_3 = ky_1 - y_2,\ v_4 = y_2 + y_4.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 + J_2, J_2 + J_3, kJ_1 - J_1^\tau, J_1^\tau - J_2^\tau)$, we find

$$(1-a)\psi_{v_1} + a(a-1)\psi_{v_2} + (k - k^a)\psi_{v_3} + k^a(1-a)\psi_{v_4} = 0.$$

Thus, the universal invariant of this algebra is

$$J_3 + (1+a)J_2 + aJ_1,\ k^a(J_1 + J_2) - (J_1^\tau + J_2^\tau),$$
$$(k - k^a)(J_1 + J_2) - (1-a)(kJ_1 + J_1^\tau).$$

The set of equations admitting the generator L^3_{15} is

$$J_3 = f\Big(k^a(J_1 + J_2) - (J_1^\tau + y_2^\tau), (k - k^a)(J_1 + J_2) - (1-a)(kJ_1 - J_1^\tau)\Big)$$
$$-[(1+a)y' + ay].$$

In table 4.1, this set of equations is written as

$$y'' = f\Big(k^a(y+y') - (y_\tau + y'_\tau), (k-k^a)(y+y') - (1-a)(ky - y_\tau)\Big)$$
$$-((1+a)y' + ay).$$

4.3.16 Lie Algebra L^3_{16}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = (bx+y)\partial_x + (by-x)\partial_y.$$

After changing the variables under conditions

$$(\bar{\xi}_i)_{\bar{y}} = 0 \quad \text{and} \quad \bar{\xi}_i(\bar{x}) = \bar{\xi}_i(\bar{x}-\tau),\ i = 1,2,3. \tag{4.77}$$

It leads us to $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$. This contradicts to the assumption $\Delta \neq 0$.

4.3.17 Lie Algebra L^3_{17}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = e^{-bx}\sin x\partial_y,\ X_3 = e^{-bx}\cos x\partial_y, \quad b \geq 0$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{e^{-b\bar{h}}\sin(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{e^{-b\bar{h}}\cos(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$-k^b c_1 \Phi_{y_2} + \Phi_{y_3} + k^b(c_2 + bc_1)\Phi_{y_4} - 2b\Phi_{y_5} = 0.$$

The invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and

$$v_1 = y_1,\ v_2 = y_5 + 2by_3,\ v_3 = c_1y_4 + (c_2 + bc_1)y_2,\ v_4 = k^bc_1y_3 + y_2.$$

Next, Applying the generator $\bar{X}_3^{(2)}$ to the function

$$\psi(J_1, J_3 + 2bJ_2, c_1J_2^\tau + (c_2 + c_1b)J_1^\tau, k^bc_1J_2 + J_1^\tau,)$$

we arrive at

$$\psi_{v_1} - (b^2 + 1)\psi_{v_2} + k^b\psi_{v_3} + k^b(c_2 - bc_1)\psi_{v_4} = 0.$$

Thus, the universal invariant of this algebra is

$$J_3 + 2bJ_2 + (b^2 + 1)y, \tag{4.78}$$

$$I_1 = k^bJ_1 - [c_1J_2^\tau + (c_2 + bc_1)J_1^\tau], \tag{4.79}$$

$$I_2 = (c_2 - bc_1)[c_1J_2^\tau + (c_2 + bc_1)J_1^\tau] - [k^bc_1J_2 + J_1^\tau]. \tag{4.80}$$

The set of equations admitting the generator L_{17}^3 is

$$J_3 = f\Big(I_1, I_2\Big) - ((2bJ_2 + (b^2 + 1)J_1).$$

In table 4.1, this set of equations is written as

$$y'' = f\Big(I_1, I_2\Big) - (2by' + (b^2 + 1)y).$$

4.3.18 Lie Algebra L_{18}^3

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = x\partial_x + y\partial_y,\ X_3 = (x^2 - y^2)\partial_x + 2xy\partial_y.$$

After changing the variables under conditions (4.77). The results is also $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$. This contradicts to the assumption $\Delta \neq 0$.

4.3.19 Lie Algebra L^3_{19}

This algebra is defined by the generators

$$X_1 = \partial_x + \partial_y,\ X_2 = x\partial_x + y\partial_y,\ X_3 = x^2\partial_x + y^2\partial_y$$

which after changing the variables become

$$\begin{aligned}
\bar{X}_1 &= \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + (-\frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \frac{1}{\bar{g}_{\bar{y}}})\partial_{\bar{y}},\\
\bar{X}_2 &= \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + (-\frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \frac{\bar{g}}{\bar{g}_{\bar{y}}})\partial_{\bar{y}},\\
\bar{X}_3 &= \frac{\bar{h}^2}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + (-\frac{\bar{h}^2\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \frac{\bar{g}^2}{\bar{g}_{\bar{y}}})\partial_{\bar{y}}.
\end{aligned}$$

Differential invariant up to second-order of the first generator is

$$\bar{h} - \bar{g},\ \bar{h}^\tau - \bar{g}^\tau,\ J_2,\ J_2^\tau,\ J_3. \tag{4.81}$$

Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = \bar{h} - J_1,\ y_2 = \bar{h}^\tau - J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$y_1\Phi_{y_1} + y_2\Phi_{y_2} - y_5\Phi_{y_5} = 0.$$

The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = \frac{y_2}{y_1},\ z_2 = y_1y_5,\ z_3 = y_3,\ z_4 = y_4.$$

Applying the generator $\bar{X}_3^{(2)}$ to function $\psi(\frac{J_1^\tau}{J_1}, J_1J_3, J_2, J_2^\tau)$, one gets

$$z_1(1 - z_1)\psi_{z_1} - 2z_3\psi_{z_3} - 2z_1z_4\psi_{z_4} + (-3z_2 + 2z_3(z_3 - 1))\psi_{z_2} = 0.$$

Thus, the universal invariant of this algebra is

$$(\bar{h} - J_1)J_3(J_2)^{-3/2},\ J_2\Big(\frac{\bar{h} - J_1^\tau}{J_1 - J_1^\tau}\Big)^2,\ \frac{(J_1 - J_1^\tau)^2}{J_2^\tau(\bar{h} - J_1)^2} - 2J_2(J_2 + 1). \tag{4.82}$$

The set of equations admitting the generator L^3_{19} is

$$J_3 = \frac{(J_2)^{3/2}}{(\bar{h} - J_1)}\Bigg(f\Big(J_2(\frac{\bar{h} - J_1^\tau}{J_1^\tau - J_1})^2,\ \frac{(J_1^\tau - J_1)^2}{J_2^\tau(\bar{h} - J_1)^2}\Big) - 2J_2(J_2 + 1)\Bigg).$$

In table 4.1, this set of equations is written as

$$y'' = \frac{y'^{3/2}}{(x - y)}\Bigg(f\Big(y'(\frac{x - y_\tau}{y_\tau - y})^2,\ \frac{(y_\tau - y)^2}{y'_\tau(x - y)^2}\Big) - 2y'(y' + 1)\Bigg).$$

4.3.20 Lie Algebra L^3_{20}

This algebra is defined by

$$X_1 = \partial_x,\ X_2 = x\partial_x + \frac{1}{2}y\partial_y,\ X_3 = x^2\partial_x + xy\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \tfrac{1}{h_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{h_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},$$

$$\bar{X}_2 = \tfrac{\bar{h}}{h_{\bar{x}}}\partial_{\bar{x}} + (-\tfrac{\bar{h}\bar{g}_{\bar{x}}}{h_{\bar{x}}\bar{g}_{\bar{y}}} + \tfrac{\bar{g}}{2\bar{g}_{\bar{y}}})\partial_{\bar{y}},$$

$$\bar{X}_3 = \tfrac{\bar{h}^2}{h_{\bar{x}}}\partial_{\bar{x}} + (-\tfrac{\bar{h}^2\bar{g}_{\bar{x}}}{h_{\bar{x}}\bar{g}_{\bar{y}}} + \tfrac{\bar{h}\bar{g}}{\bar{g}_{\bar{y}}})\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$y_1\Phi_{y_1} + y_2\Phi_{y_2} - y_3\Phi_{y_3} - y_4\Phi_{y_4} - 3y_5\Phi_{y_5} = 0.$$

The invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and

$$v_1 = \frac{y_2}{y_1},\ v_2 = \frac{y_4}{y_3},\ v_3 = (y_1)^3y_5,\ v_4 = y_2y_3.$$

Then, Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(\frac{J_1^\tau}{J_1}, \frac{J_2^\tau}{J_2}, (J_1)^3J_3, J_1^\tau J_2)$, we find

$$v_4\psi_{v_4} + (v_1 - v_2)\psi_{v_2} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{J_1^\tau}{J_1},\ J_1^\tau J_2\Big(\frac{J_1^\tau}{J_1}-\frac{J_2^\tau}{J_2}\Big),\ (J_1)^3 J_3. \tag{4.83}$$

The set of equations admitting the generator L_{20}^3 is

$$J_3=(J_1)^{-3}f\Big(\frac{J_1^\tau}{J_1},\ J_2J_1^\tau(\frac{J_1^\tau}{J_1}-\frac{J_2^\tau}{J_2})\Big).$$

In table 4.1, this set of equations is written as

$$y''=y^{-3}f\Big(\frac{y_\tau}{y},\ y'y_\tau(\frac{y_\tau}{y}-\frac{y'_\tau}{y'})\Big).$$

4.3.21 Lie Algebra L_{21}^3

This algebra is defined by the generators

$$X_1=\partial_x,\ X_2=x\partial_x,\ X_3=x^2\partial_x.$$

After changing the variables, they become

$$\bar{X}_1=\frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}}-\frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2=\frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}}-\frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3=\frac{\bar{h}^2}{\bar{h}_{\bar{x}}}\partial_{\bar{x}}-\frac{\bar{h}^2\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1,y_2,y_3,y_4,y_5)$ with substituted

$$y_1=J_1,\ y_2=J_1^\tau,\ y_3=J_2,\ y_4=J_2^\tau,\ y_5=J_3,$$

we obtain

$$y_3\Phi_{y_3}+y_4\Phi_{y_4}+2y_5\Phi_{y_5}=0.$$

The invariant function is $\Phi=\psi(v_1,v_2,v_3,v_4)$ where ψ is an arbitrary function and

$$v_1=y_1,\ v_2=y_2,\ v_3=\frac{y_5}{(y_4)^2},\ v_4=\frac{y_5}{(y_3)^2}.$$

Then, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1,J_1^\tau,\frac{J_3}{(J_2^\tau)^2},\frac{J_3}{(J_2)^2})$, we found

$$v_3\psi_{v_3}+v_4\psi_{v_4}=0.$$

Thus, the universal invariant of this algebra is $J_1, J_1^\tau, \Big(\frac{J_2^\tau}{J_2}\Big)^2$, which has no second-order derivative term. Hence the set of equations admitting the generator L_{21}^3 cannot be constructed.

4.3.22 Lie Algebra L^3_{22}

This algebra is defined by the generators

$$X_1 = y\partial_x - x\partial_y,\ X_2 = (1+x^2-y^2)\partial_x + 2xy\partial_y,\ X_3 = 2xy\partial_x + (1+y^2-x^2)\partial_y.$$

After changing the variables under conditions

$$(\bar{\xi}_i)_{\bar{y}} = 0 \quad \text{and} \quad \bar{\xi}_i(\bar{x}) = \bar{\xi}_i(\bar{x}-\tau),\ i = 1,2,3.$$

It leads us to $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$ which contradicts to the assumption $\Delta \neq 0$.

4.3.23 Lie Algebra L^4_{23}

This algebra is defined by the generators

$$X_1 = \partial_y,\ X_2 = x\partial_y,\ X_3 = \xi_1(x)\partial_y,\ X_4 = \xi_2(x)\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\xi_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\xi_2(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^3_6, invariant of generator $X_1^{(2)},\ X_2^{(2)},\ X_3^{(2)}$ is an arbitrary function $G(w_1, w_2, w_3)$ where $w_1 = \bar{h},\ w_2 = (\xi_1' - \xi_1^{\tau'})J_3 - \xi_1''(J_2 - J_2^\tau),\ w_3 = (\xi_1' - \xi_1^{\tau'})(cJ_2 - J_1 + J_1^\tau) - (\xi_1'c - \xi_1 + \xi_1^\tau)(J_2 - J_2^\tau)$. Applying generator $X_4^{(2)}$ to function $G(w_1, w_2, w_3)$ with substituted

$$w_1 = \bar{h},\ w_2 = (\xi_1' - \xi_1^{\tau'})J_3 - \xi_1''(J_2 - J_2^\tau),$$
$$w_3 = (\xi_1' - \xi_1^{\tau'})(cJ_2 - J_1 + J_1^\tau) - (\xi_1'c - \xi_1 + \xi_1^\tau)(J_2 - J_2^\tau),$$

lead us to

$$\Big[\xi_1''(\xi_2'^\tau - \xi_2') + \xi_2''(\xi_1' - \xi_1'^\tau)\Big]G_{w_2}$$
$$+\Big[c(\xi_1'\xi_2'^\tau - \xi_1'^\tau\xi_2') + (\xi_1'^\tau - \xi_1')(\xi_2 - \xi_2^\tau) + (\xi_2' - \xi_2'^\tau)(\xi_1 - \xi_1^\tau)\Big]G_{w_3} = 0.$$

Thus, the universal invariant of this algebra is $\bar{h}$,

$$[\xi_1''(\xi_2'^\tau - \xi_2') + \xi_2''(\xi_1' - \xi_1'^\tau)][(\xi_1' - {\xi_1^\tau}')(cJ_2 - J_1 + J_1^\tau) - (\xi_1'c - \xi_1 + \xi_1^\tau)(J_2 - J_2^\tau)] -$$

$$[c(\xi_1'\xi_2'^\tau - \xi_1'^\tau\xi_2') + (\xi_1'^\tau - \xi_1')(\xi_2 - \xi_2^\tau) + (\xi_2' - \xi_2'^\tau)(\xi_1 - \xi_1^\tau)][(\xi_1' - {\xi_1^\tau}')J_3 - \xi_1''(J_2 - J_2^\tau)].$$

The set of equations admitting the generator L_{23}^4 is

$$J_3 = \frac{\Big(I_{12} - f(\bar{h})\Big)}{(\xi_1' - {\xi_1^\tau}')I_{13}} + \frac{\xi_1''(J_2 - J_2^\tau)}{(\xi_1' - {\xi_1^\tau}')},$$

where

$$I_{12} = [\xi_1''(\xi_2'^\tau - \xi_2') + \xi_2''(\xi_1' - \xi_1'^\tau)][(\xi_1' - {\xi_1^\tau}')(cJ_2 - J_1 + J_1^\tau)$$
$$-(\xi_1'c - \xi_1 + \xi_1^\tau)(J_2 - J_2^\tau)],$$
$$I_{13} = [c(\xi_1'\xi_2'^\tau - \xi_1'^\tau\xi_2') + (\xi_1'^\tau - \xi_1')(\xi_2 - \xi_2^\tau) + (\xi_2' - \xi_2'^\tau)(\xi_1 - \xi_1^\tau)].$$

In table 4.1, this set of equations is written as

$$y'' = \frac{\Big(I_{12} - f(x)\Big)}{(\xi_1' - {\xi_1^\tau}')I_{13}} + \frac{\xi_1''(y' - y_\tau')}{(\xi_1' - {\xi_1^\tau}')}.$$

4.3.24 Lie Algebra L_{24}^4

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x,\ X_4 = y\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$\Phi_{y_1} + \Phi_{y_2} = 0.$$

The invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and $v_1 = y_1 - y_2,\ v_2 = y_3,\ v_3 = y_4,\ v_4 = y_5$. Then, Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, one gets

$$v_2\psi_{v_2} + v_3\psi_{v_3} - 2v_4\psi_{v_4} = 0.$$

Solving for invariant function, one obtains $\psi = H(z_1, z_2, z_3)$ where H is an arbitrary function and $z_1 = v_1,\ z_2 = \frac{y_4}{y_3},\ z_3 = \frac{(y_3)^2}{y_5}$. Finally, applying the generator $\bar{X}_4^{(2)}$ to function $H(J_1 - J_1^\tau, \frac{J_2^\tau}{J_2}, \frac{(J_2)^2}{J_3})$,

$$z_1 H_{z1} + z_3 H_{z_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{(J_1 - J_1^\tau)J_3}{(J_2)^2},\ \frac{J_2^\tau}{J_2}. \tag{4.84}$$

The set of equations admitting the generator L_{24}^4 is

$$J_3 = \frac{(J_2)^2}{J_1 - J_1^\tau} f\Big(\frac{J_2^\tau}{J_2}\Big).$$

In table 4.1, this set of equations is written as

$$y'' = \frac{y'^2}{y - y_\tau} f\Big(\frac{y'_\tau}{y'}\Big).$$

4.3.25 Lie Algebra L_{25}^4

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = e^{-x}\partial_y,\ X_4 = y\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariant of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3.$$

The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and $z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5$. Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, one gets

$$(1-k)\psi_{z_1} - \psi_{z_2} - k\psi_{z_3} + \psi_{z_4} = 0.$$

Solving for invariant function, one obtains $\psi = H(v_1, v_2, v_3)$ where $v_1 = (k-1)y_3 - v,\ v_2 = ky_3 - y_4$ and $v_3 = y_3 + y_5$. Finally, applying the generator $\bar{X}_4^{(2)}$ to function $H((k-1)J_2 - J_1 + J_1^\tau, kJ_2 - J_2^\tau, J_2 + J_3)$, then

$$v_1 H_{v_1} + v_2 H_{v_2} + v_3 H_{z_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{J_3 + J_2}{kJ_2 - J_2^\tau},\ \frac{kJ_2 - J_2^\tau}{((k-1)J_2 - (J_1 - J_1^\tau))}. \tag{4.85}$$

The set of equations admitting the generator L_{25}^4 is

$$J_3 = (kJ_2 - J_2^\tau) f\Big(\frac{kJ_2 - J_2^\tau}{(k-1)J_2 - J_1 + J_1^\tau}\Big) - J_2.$$

In table 4.1, this set of equations is written as

$$y'' = (ky' - y'_\tau) f\Big(\frac{ky' - y'_\tau}{(k-1)y' - y + y_\tau}\Big) - y'.$$

4.3.26 Lie Algebra L_{26}^4

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = e^{-x}\partial_y,\ X_4 = -xe^{-x}\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{-\bar{h}e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^4_{25}, invariant of the generators $\bar{X}^{(2)}_1$, $\bar{X}^{(2)}_2$ and $\bar{X}^{(2)}_3$ is $\psi = H(v_1, v_2, v_3)$ where H is an arbitrary function and

$$v_1 = (k-1)y_3 - y_1 + y_2, \ v_2 = ky_3 - y_4, \ v_3 = y_3 + y_5.$$

Applying the generator $\bar{X}^{(2)}_4$ to function $H[(k-1)J_2 - J_1 + J^\tau_1, kJ_2 - J^\tau_2, J_2 + J_5]$, then

$$(kc - k + 1)H_{v_1} + kcH_{v_2} + H_{v_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\begin{aligned} & (kc - k + 1)(J_3 + J_2) - (k-1)J_2 - (J_1 - J^\tau_1), \\ I_3 = \ & kc(J^\tau_1 - J_1 - J_2 + J^\tau_2) + (k-1)(kJ_2 - J^\tau_2). \end{aligned}$$

The set of equations admitting the generator L^4_{26} is

$$J_3 = \frac{f(I_3) + (k-1)J_2 - J_1 + J^\tau_1}{(kc - k + 1)} - J_2.$$

In table 4.1, this set of equations is written as

$$y'' = \frac{f(I_3) + (k-1)y' - y + y_\tau}{(kc - k + 1)} - y'.$$

4.3.27 Lie Algebra L^4_{27}

This algebra is defined by the generators

$$X_1 = \partial_x, \ X_2 = \partial_y, \ X_3 = e^{-x}\partial_y, \ X_4 = e^{-ax}\partial_y, \quad 0 < |a| \neq 1, \ a \neq 1$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{e^{-a\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^4_{25}, invariant of the generators $\bar{X}^{(2)}_1$, $\bar{X}^{(2)}_2$ and $\bar{X}^{(2)}_3$ is $\psi = H(v_1, v_2, v_3)$ where H is an arbitrary function and

$$v_1 = (k-1)y_3 - v,\ v_2 = ky_3 - y_4,\ v_3 = y_3 + y_5.$$

Applying the generator $\bar{X}^{(2)}_4$ to function $H[(k-1)J_2 - J_1 + J_1^\tau, kJ_2 - J_2^\tau, J_2 + J_3]$, then

$$(k^a - ak + a - 1)H_{v_1} + a(k^a - k)H_{v_2} + a(a-1)H_{v_3} = 0.$$

Thus, the universal invariant of this algebra is

$$(k^a - k)(J_3 + J_2) - (a-1)[(kJ_2 - J_2^\tau)],$$

$$I_4 = (k^a - ak + a - 1)(kJ_2 - J_2^\tau) - a(k^a - k)[(k-1)J_2 - J_1 + J_1^\tau].$$

The set of equations admitting the generator L^4_{27} is

$$J_3 = \frac{1}{(k^a - k)}\Big(f(I_4) + (a-1)(kJ_2 - J_2^\tau)\Big) - J_2.$$

In table 4.1, this set of equations is written as

$$y'' = \frac{\Big(f(I_4) + (a-1)(ky' - y'_\tau)\Big)}{(k^a - k)} - y'.$$

4.3.28 Lie Algebra L^4_{28}

This algebra is defined by

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = e^{-bx}\sin x\partial_y,\ X_4 = e^{-bx}\cos x\partial_y, \quad b \geq 0$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{e^{-b\bar{h}}\sin(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{e^{-b\bar{h}}\cos(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}^{(2)}_2$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

the invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and $z_1 = y_1 - y_2,\ z_2 = y3,\ z_3 = y_4,\ z_4 = y_5$. Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$\psi_{z_2} + k^b(c_2 + bc_1)\psi_{z_3} - 2b\psi_{z_4} + k^b c_1 \psi_{z_1} = 0.$$

Solving for invariant function, one obtains $\psi = H(v_1, v_2, v_3)$ where H is an arbitrary function and $v_1 = y_5 + 2by_3,\ v_2 = c_1 y_4 - (c_2 + bc_1)(y_1 - y_2),\ v_3 = k^b c_1 y_3 - y_1 + y_2$. Finally, applying the generator $\bar{X}_4^{(2)}$ to function

$$H\Big(J_3 + 2bJ_2, c_1 J_2^\tau - (c_2 + bc_1)(J_1 - J_1^\tau), k^b c_1 J_2 - J_1 + J_1^\tau\Big),$$

then

$$-(b^2 + 1)H_{v_1} + [k^b - (bc_1 + c_2)]H_{v_2} + [k^b(c_2 - bc_1) - 1]H_{v_3} = 0.$$

Thus, the universal invariant of this algebra is

$$[k^b - (c_2 + bc_1)][J_3 + 2bJ_2] + (b^2 + 1)[c_1 J_2^\tau - (c_2 + bc_1)(J_1 - J_1^\tau)],$$

$$I_5 = [k^b(c_2 - bc_1) - 1][c_1 J_2^\tau - (c_2 + bc_1)(J_1 - J_1^\tau)]$$
$$+[k^b - (c_2 + bc_1)][k^b c_1 J_2 - (J_1 - J_1^\tau)].$$

The set of equations admitting the generator L_{28}^4 is

$$J_3 = \frac{1}{[k^b - (bc_1 + c_2)]}\Big(f(I_5) - (b^2 + 1)\big(c_1 J_2^\tau - (c_2 + bc_1)(J_1 - J_1^\tau)\big)\Big) - 2bJ_2.$$

In table 4.1, this set of equations is written as

$$y'' = \frac{f(I_5) - (b^2 + 1)[c_1 y'_\tau - (c_2 + bc_1)(y - y_\tau)]}{k^b - (bc_1 + c_2)} - 2by'.$$

4.3.29 Lie Algebra L_{29}^4

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = x\partial_x,\ X_3 = y\partial_y,\ X_4 = x^2\partial_x + xy\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},$$
$$\bar{X}_3 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{h}^2}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + (-\frac{\bar{h}^2\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \frac{\bar{h}\bar{g}}{\bar{g}_{\bar{y}}})\partial_{\bar{y}}.$$

From Lie algebra L_5^2, invariants function of the generators $\bar{X}_1^{(2)}$, $\bar{X}_2^{(2)}$ is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and

$$z_1 = J_1,\ z_2 = J_1^\tau,\ z_3 = \frac{J_2^\tau}{J_2},\ z_4 = \frac{J_3}{(J_2)^2}.$$

Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1, J_1^\tau, \frac{J_2^\tau}{J_2}, \frac{J_3}{(J_2)^2})$

$$z_1\psi_{z_1} + z_2\psi_{z_2} - z_4\psi_{z_4} = 0.$$

Solving for function ψ, we obtain $\psi = H(v_1, v_2, v_3)$ where H is an arbitrary function and $v_1 = \frac{z2}{z1}, v_2 = z_3,\ v_3 = z_1z_4$. Finally, applying the generator generator $\bar{X}_4^{(2)}$ to function $H(\frac{J_1^\tau}{J_1}, \frac{J_2^\tau}{J_2}, \frac{J_1J_3}{(J_2)^2})$, one gets

$$(v_1 - v_2)H_{v_2} - 2v_3H_{v_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{J_1^\tau}{J_1},\ \frac{J_1J_3}{(J_2)^2(\frac{J_1^\tau}{J_1} - \frac{J_2^\tau}{J_2})^2}. \tag{4.86}$$

The set of equations admitting the generator L_{29}^4 is

$$J_3 = f\Big(\frac{J_1^\tau}{J_1}\Big)\frac{(J_2)^2}{J_1}\Big(\frac{J_1^\tau}{J_1} - \frac{J_2^\tau}{J_2}\Big)^2.$$

In table 4.1, this set of equations is written as

$$y'' = f\Big(\frac{y_\tau}{y}\Big)\frac{y'^2}{y}\Big(\frac{y_\tau}{y} - \frac{y'_\tau}{y'}\Big)^2.$$

4.3.30 Lie Algebra L^4_{30}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x,\ X_4 = x^2\partial_x$$

which after changing the variables, they become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_4 = \frac{\bar{h}^2}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{h}^2\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain
$$\Phi_{y_1} + \Phi_{y_2} = 0.$$

The invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and

$$v_1 = y_1 - y_2,\ v_2 = y_3,\ v_3 = y_4,\ v_4 = y_5.$$

Then, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$v_2\psi_{v_2} + v_3\psi_{v_3} + 2v_4\psi_{v_4} = 0.$$

The invariant function is $\psi = H(z_1, z_2, z_3)$ where H is an arbitrary function and

$$z_1 = v_1,\ z_2 = \frac{v_4}{(v_3)^2},\ z_3 = \frac{v_4}{(v_2)^2}.$$

Finally, applying the generator $\bar{X}_4^{(2)}$ to the function $H(J_1 - J_1^\tau, \frac{J_3}{(J_2^\tau)^2}, \frac{J_3}{(J_2)^2})$, one gets

$$z_3H_{z_3} + z_2H_{z_2} = 0.$$

Thus, the universal invariant of this algebra is $J_1 - J_1^\tau,\ \left(\frac{J_2^\tau}{J_2}\right)^2$, which has no second-order derivative term. Hence, the set of equations admitting the generator L^4_{30} cannot be constructed.

4.3.31 Lie Algebra L^4_{31}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = -x\partial_y,\ X_4 = \frac{1}{2}x^2\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3 = \frac{-\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_4 = \frac{\bar{h}^2}{2\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$\Phi_{y_1} + \Phi_{y_2} = 0.$$

The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and $z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5$. Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$c\psi_{z_1} + \psi_{z_2} + \psi_{z_3} = 0.$$

Solving for function ψ, one obtains $\psi = H(v_1, v_2, v_3)$ where H is an arbitrary constant and $v_1 = z_1 - cz_2,\ v_2 = z_2 - z_3,\ v_3 = z_4$. Finally, applying the generator $X_4^{(2)}$ to function $H(J_1 - J_1^\tau - cJ_2, J_2 - J_2^\tau, J_3)$, then

$$v_3 H_{v_3} + cH_{v_2} - \frac{c^2}{2}H_{v_1} = 0.$$

Thus, the universal invariant of this algebra is

$$c(J_2 - J_2^\tau) + 2[(y - J_1^\tau) - cJ_2],\ J_2 - J_2^\tau - cJ_3.$$

The set of equations admitting the generator L^4_{31} is

$$cJ_3 = J_2 - J_2^\tau - f\Big(2(J_1 - J_1^\tau) - c(J_2 + J_2^\tau)\Big).$$

In table 4.1, this set of equations is written as

$$cy'' = y' - y'_\tau - f\Big(2(y - y_\tau) - c(y' + y'_\tau)\Big).$$

4.3.32 Lie Algebra L^4_{32}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = e^{-x}\partial_y,\ X_3 = -xe^{-x}\partial_y,\ X_4 = e^{-bx}\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3 = \frac{-\bar{h}e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_4 = \frac{e^{-b\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^3_{11}, invariants of the generator $X_1^{(2)}, X_2^{(2)}, X_3^{(2)}$ is $\Phi = \psi(w_1, w_2, w_3)$ where $w_1 = k(y_1 + y_3) - (y_2 - y_4)$, $w_2 = y_5 + 2y_3 + y_1$ and $w_3 = kc(y_1 + y_3) - ky_1 + y_2$. Applying the generator $\bar{X}_4^{(2)}$ to the function $\Phi(w_1, w_2, w_3)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$(b-1)(k^b - k)\psi_{w_1} + (b-1)^2\psi_{w_2} + (k^b - bck + ck - k)\psi_{w_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\begin{aligned}
&(b-1)^2[k(J_1 + J_2) - (J_1^\tau - J_2^\tau)] - (b-1)(k^b - k)[J_3 + 2J_2 + y],\\
I_6 &= (k^b - bck + ck - k)\Big(k(y + J_2) - (J_1^\tau - J_2^\tau)\Big)\\
&\quad -(b-1)(k^b - k)\Big(kc(y + J_2) - ky + J_1^\tau\Big).
\end{aligned}$$

The set of equations admitting the generator L^4_{32} is

$$J_3 = -\frac{1}{(b-1)(k^b - k)}\Big(f(I_6) - (b-1)^2\big(k(J_1 + J_2) - (J_1^\tau - J_2^\tau)\big)\Big) - [2J_2 + J_1].$$

In table 4.1, this set of equations is written as

$$y'' = -\frac{1}{(b-1)(k^b - k)}\Big(f(I_6) - (b-1)^2\big(k(y + y') - (y_\tau - y'_\tau)\big)\Big) - (2y' + y).$$

4.3.33 Lie Algebra L^4_{33}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = -x\partial_y,\ X_4 = e^{-x}\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{-\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$\Phi_{y_1} + \Phi_{y_2} = 0.$$

The invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and $z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5$. Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$c\psi_{z_1} + \psi_{z_2} + \psi_{z_3} = 0.$$

Solving for function ψ, one obtains $\psi = H(v_1, v_2, v_3)$ where H is an arbitrary function and $v_1 = z_1 - cz_2,\ v_2 = z_2 - z_3,\ v_3 = z_4$. Finally, applying the generator $X_4^{(2)}$ to function $H(J_1 - J_1^\tau - cJ_2, J_2 - J_2^\tau), J_3$, then

$$H_{v_3} + (c - k + 1)H_{v_1} + (k - 1)H_{z_2} = 0.$$

Thus, the universal invariant of this algebra is

$$(k-1)J_3 - J_2 + J_2^\tau,\ I_7 = (k-1)(J_1 - J_1^\tau - cJ_2) + (k - c - 1)(J_2 - J_2^\tau).$$

The set of equations admitting the generator L^4_{33} is

$$J_3 = \frac{1}{(k-1)}f(I_7) + (J_2 - y_2^\tau).$$

In table 4.1, this set of equations is written as

$$y'' = \frac{1}{(k-1)} f(I_7) + (y' - y'_\tau).$$

4.3.34 Lie Algebra L^4_{34}

This algebra is defined by

$$X_1 = \partial_x,\ X_2 = e^{-x}\partial_y,\ X_3 = -xe^{-x}\partial_y,\ X_4 = \frac{1}{2}x^2 e^{-x}\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{-\bar{h}e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{h}^2 e^{-\bar{h}}}{2\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^3_{11}, invariant of the first three generators $X_1^{(2)},\ X_2^{(2)},\ X_3^{(2)}$ is $\Phi(w_1, w_2, w_3)$ where $w_1 = k(y_1 + y_3) - (y_2 - y_4),\ w_2 = y_5 + 2y_3 + y_1$ and $w_3 = kc(y_1 + y_3) - ky_1 + y_2$. Applying the generator $\bar{X}_4^{(2)}$ to the function $\Phi(w_1, w_2, w_3)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$kc\Phi_{w_1} + \Phi_{w_2} + \frac{k^2}{2}c^2\Phi_{w_3} = 0.$$

Thus, the universal invariant of this algebra is

$$k(J_1 + J_2) - (J_1^\tau + J_2^\tau) - kc[J_3 + 2J_2 + J_1],$$

$$I_8 = c[k(J_1 + J_2) - (J_1^\tau + J_2^\tau)] - 2[kc(J_1 + J_2) + kJ_1 - J_1^\tau].$$

The set of equations admitting the generator L^4_{34} is

$$J_3 = -\frac{1}{kc}\Big(f(I_8) - k(J_1 + J_2) + (J_1^\tau + J_2^\tau)\Big) - (2J_2 + J_1).$$

In table 4.1, this set of equations is written as

$$y'' = -\frac{1}{kc}\Big(f(I_8) - k(y + y') + (y_\tau + y'_\tau)\Big) - (2y' + y).$$

4.3.35 Lie Algebra L^4_{35}

This algebra is defined by the generators

$$X_1 = \partial_y,\ X_2 = x\partial_y,\ X_3 = \xi_1(x)\partial_y,\ X_4 = y\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\xi_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are $\bar{h}(\bar{x}),\ J_1 - J_1^\tau,\ J_2,\ J_2^\tau,\ J_3$. Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = \bar{h},\ y_2 = J_1 - J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$\Phi_{y_3} + \Phi_{y_4} = 0.$$

The invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and

$$v_1 = y_1,\ v_2 = y_2,\ v_3 = y_3 - y_4,\ v_4 = y_5.$$

Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(\bar{h}, J_1 - J_1^\tau, J_2 - J_2^\tau, J_3)$, one finds

$$v_2\psi_{v_2} + v_3\psi_{v_3} + v_4\psi_{v_4} = 0.$$

The invariant function is $\psi = H(z_1, z_2, z_3)$ where H is an arbitrary function and

$$z_1 = v_1,\ z_2 = \frac{v_2}{v_3},\ z_3 = \frac{v_4}{v_2}.$$

Finally, applying generator $\bar{X}_4^{(2)}$ to the function $H(\bar{h}, \frac{J_1 - J_1^\tau}{J_2 - J_2^\tau}, \frac{J_3}{J_1 - J_1^\tau})$, one obtains

$$-(z_2)^2\xi_1' H_{z_2} + \xi_1'' H_{z_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\bar{h},\ \frac{\xi_1' J_3 - \xi_1''(J_2 - J_2^\tau)}{J_1 - J_1^\tau}. \tag{4.87}$$

The set of equations admitting the generator L^4_{35} is

$$J_3 = \frac{\xi_1''(J_2 - J_2^\tau) + (J_1 - J_1^\tau)f(\bar{h})}{\xi_1'}. \tag{4.88}$$

In table 4.1 this set of equations is written as

$$y'' = \frac{\xi_1''(x)(y' - y_\tau') + (y - y_\tau)f(x)}{\xi_1'(x)}. \tag{4.89}$$

4.3.36 Lie Algebra L^4_{36}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = e^{-x}\partial_y,\ X_3 = e^{-ax}\partial_y,\ X_4 = e^{-bx}\partial_y, \quad -1 \le a < b < 1,\ ab \ne 0$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{-\bar{h}e^{-\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \bar{X}_4 = \frac{e^{-b\bar{h}}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^3_{11}, invariant of the first two generators $X_1^{(2)}$, $X_2^{(2)}$ is $\Phi(w_1, w_2, w_3, w_4)$ where $w_1 = y_1 + y_3,\ w_2 = y_5 + y_3$, $w_3 = ky_1 - y_2$ and $w_4 = y_2 + y_4$. Applying the generator $\bar{X}_3^{(2)}$ to the function $\Phi(w_1, w_2, w_3, w_4)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we obtain

$$(1-a)\Phi_{w_1} + (1-a)k^a\Phi_{w_4} + a(a-1)\Phi_{w_2} + (k - k^a)\Phi_{w_3} = 0.$$

Solving for function Φ, one gets $\Phi = \psi(z_1, z_2, z_3)$ where ψ is an arbitrary function and $z_1 = k^a w_1 - w_4,\ z^2 = aw_1 + w_2,\ z_3 = (k - k^a)w_4 - (1-a)k^a w_3$. Finally, applying the generator $X_4^{(2)}$ to function $\psi(z_1, z_2, z_3)$ with substituted $y_1 = J_1$, $y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3$, then

$$(k^b - k_1)(b-1)\psi_{z_1} + (b-1)(b-a)\psi_{z_2} + [-k^b a k_1 - k^b b k + k^b b k_1 + k^b k + akk_1 - kk_1]\psi_{z_3} = 0.$$

Thus, the universal invariant of this algebra is

$$(b-1)(b-a)[k^a(J_1+J_2)-(J_1^\tau+J_2^\tau)]-(b-1)(k^b-k_1)[J_3+(1+a)J_2+aJ_1],$$

$$I_9=[-k^bak_1-k^bbk+k^bbk_1+k^bk+akk_1-kk_1][k^a(J_1+J_2)-(J_1^\tau+J_2^\tau)]$$

$$-(k^b-k_1)(b-1)[(k-k^a)(J_1^\tau+J_2^\tau)-(1-a)k^a(kJ_1-J_1^\tau)].$$

The set of equations admitting the generator L_{36}^4 is

$$J_3=\tfrac{1}{(k^b-kc)(b-1)}\Big((b-1)(b-a)(k^a(J_1+J_2)-(J_1^\tau+J_2^\tau))-f(I_9)\Big)$$
$$-(a(J_1+J_2)+J_2).$$

In table 4.1, this set of equations is written as

$$y''=\tfrac{1}{(k^b-kc)(b-1)}\Big((b-1)(b-a)(k^a(y+y')-(y_\tau+y'_\tau))-f(I_9)\Big)$$
$$-(a(y+y')+y').$$

4.3.37 Lie Algebra L_{37}^4

This algebra is defined by the generators

$$X_1=\partial_x,\ X_2=e^{-ax}\partial_y,\ X_3=e^{-bx}\sin x\partial_y,\ X_4=e^{-bx}\cos x\partial_y,\quad a>0$$

which after changing the variables become

$$\bar X_1=\frac{1}{\bar h_{\bar x}}\partial_{\bar x}-\frac{\bar g_{\bar x}}{\bar h_{\bar x}\bar g_{\bar y}}\partial_{\bar y},\quad \bar X_2=\frac{e^{-a\bar h}}{\bar g_{\bar y}}\partial_{\bar y},\quad \bar X_3=\frac{e^{-b\bar h}\sin(\bar h)}{\bar g_{\bar y}}\partial_{\bar y},\quad \bar X_4=\frac{e^{-b\bar h}\cos(\bar h)}{\bar g_{\bar y}}\partial_{\bar y}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar X_2^{(2)}$ to the function $\Phi(y_1,y_2,y_3,y_4,y_5)$ with substituted

$$y_1=J_1,\ y_2=J_1^\tau,\ y_3=J_2,\ y_4=J_2^\tau,\ y_5=J_3,$$

one obtains

$$\Phi_{y_1}+k^a\Phi_{y_2}-a\Phi_{y_3}-ak^a\Phi_{y_4}+a^2\Phi_{y_5}=0.$$

Solving for function Φ, we obtain $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function $v_1 = ay_1 + y_3,\ v_2 = k^a y_1 - y_2,\ v_3 = a^2 y_1 - y_5,\ v_4 = ay_2 + y_4$. Applying the generator $X_3^{(2)}$ to function $\psi(v_1, v_2, v_3, v_4)$ with substituted $y_1 = J_1,\ y_2 = J_1^\tau,$ $y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3$, then

$$\psi_{v_1} + k^b(c_2 + k^b c_1 \psi_{v_2} + 2b\psi_{v_3} + (b-a)c_1)\psi_{v_4} = 0.$$

Solving for function ψ , we reach $\psi = H(w_1, w_2, w_3)$ where H is an arbitrary function and $w_1 = v_3 - 2bv_1,\ w_2 = c_1 v_4 - [c_2 + (b-a)c_1]v_2,\ w_3 = k^b c_1 v_1 - v_2$. Finally, applying generator $X_4^{(2)}$ to function H with substituted $y_1 = J_1,\ y_2 = J_1^\tau,$ $y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3$, one finds

$$[(a-b)^2 + 1]H_{w_1} + [k^b - k^a(c_2 + (b-a)c_1)]H_{w_2} + [k^b(c_2 + c_1(a-b)) - k^a]H_{w_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\begin{aligned} &((a-b)^2+1)\Big[k^b c_1[aJ_1 + J_2] - [k^a J_1 - J_1^\tau]\Big] \\ &-[k^b(c_2 + (b-a)c_1) - k^a]\Big[a^2 J_1 - J_3 - 2b(aJ_1 + J_2)]\Big], \\ I_{10} = {}& k^b(c_2 + (a-b)c_1) - k^a)(c_1(aJ_1^\tau J_2^\tau) - [c_2 + (b-a)c_1][k^a J_1 - J_1^\tau]) \\ &-[k^b + k^a(c_1(a-b) - c_2)]. \end{aligned}$$

The set of equations admitting the generator L_{37}^4 is

$$J_3 = \frac{\Bigg(f(I_{10})((a-b)^2+1)\Big(k^b c_1[aJ_1 + J_2] - [k^a J_1 - J_1^\tau]\Big)\Bigg)}{[k^b(c_2 + (b-a)c_1) - k^a]} - \Big(2b(aJ_1 + J_2) + a^2 J_1\Big).$$

In table 4.1, this set of equations is written as

$$y'' = \frac{\Bigg(f(I_{10})((a-b)^2+1)\Big(k^b c_1[ay + y'] - [k^a y - y']\Big)\Bigg)}{[k^b(c_2 + (b-a)c_1) - k^a]} - \Big(2b(ay + y') + a^2 y\Big).$$

4.3.38 Lie Algebra L^4_{38}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_y,\ X_4 = x\partial_x + (2y + x^2)\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},$$
$$\bar{X}_3 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + (-\frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \frac{2\bar{g} + \bar{h}^2}{\bar{g}_{\bar{y}}})\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

the invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and $z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_4$. Then, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$\psi_{z_2} + \psi_{z_3} = 0.$$

Solving for invariant function, one obtains $\psi = H(v_1, v_2, v_3)$ where H is an arbitrary function and $v_1 = z_1,\ v_2 = z_2 - z_3,\ v_3 = z_4$. Applying the generator $\bar{X}_4^{(2)}$ to function $H(J_1 - J_1^\tau, J_2 - J_2^\tau, J_3)$, then

$$2v_1H_{v_1} + v_2H_{v_2} + 2H_{v_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{e^{J_3}}{(J_2 - J_2^\tau)^2},\ \frac{(J_2 - J_2^\tau)^2}{(J_1 - J_1^\tau)}. \tag{4.90}$$

The set of equations admitting the generator L^4_{38} is

$$J_3 = \ln\left((J_2 - J_2^\tau)^2 f\left(\frac{(J_2 - J_2^\tau)^2}{J_1 - J_1^\tau}\right)\right).$$

In table 4.1 this set of equations is written as

$$y'' = \ln\left((y'-y'_\tau)^2 f\Big(\frac{(y'-y'_\tau)^2}{y-y_\tau}\Big)\right).$$

4.3.39 Lie Algebra L^4_{39}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_y,\ X_4 = (1+b)x\partial_x + y\partial_y$$

which after changing the variables become

$$\begin{aligned}\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\\ \bar{X}_4 = \frac{(1+b)\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \Big(-(b+1)\frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \frac{\bar{g}}{\bar{g}_{\bar{y}}}\Big)\partial_{\bar{y}}.\end{aligned}$$

From Lie algebra L^4_{38}, invariant function of the generator $X^{(2)}_1$, $X^{(2)}_2$, $X^{(2)}_3$ is $\psi(z_1, z_2, z_3)$ where $z_1 = J_1 - J^\tau_1$, $z_2 = J_2 - J^\tau_2$ $z_3 = J_3$. Applying the generator $\bar{X}^{(2)}_4$ to function $\psi(J_1 - J^\tau_1, J_2 - J^\tau_2, J_3)$, then

$$z_1\psi_{z_1} - bz_2\psi_{z_2} - (1+2b)z_3\psi_{z_3} = 0.$$

Thus, the universal invariant of this algebra is

$$(J_1 - J^\tau_1)^b(J_2 - J^\tau_2),\ \frac{(J_3)^b}{(J_2 - J^\tau_2)^{1+2b}}. \tag{4.91}$$

The set of equations admitting the generator L^4_{38} is

$$J_3 = \Big[(J_2 - J^\tau_2)^{2b+1} f[(J_1 - J^\tau_1)^b(J_2 - J^\tau_2)]\Big]^{1/b}.$$

In table 4.1, this set of equations is written as

$$y'' = \Big[(y'-y'_\tau)^{2b+1} f[(y-y_\tau)^b(y'-y'_\tau)]\Big]^{1/b},$$

4.3.40 Lie Algebra L^4_{40}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = -x\partial_y,\ X_4 = y\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{-\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

the invariant function is $\Phi = \psi(z_1, z_2, z_3, z_4)$ where ψ is an arbitrary function and $z_1 = y_1 - y_2,\ z_2 = y_3,\ z_3 = y_4,\ z_4 = y_5$. Applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we find

$$\psi_{z_2} + \psi_{z_3} + c\psi_{z_1} = 0.$$

Solving for function ψ, we reach $\psi = H(v_1, v_2, v_5)$ where $v_1 = z_2 - cz_1,\ v_2 = z_2 - z_3, v_3 = z_4$ Then, applying the generator $X_4^{(2)}$ to function $H(J_2 - c(J_1 - J_1^\tau), J_2 - J_2^\tau, J_3)$, one obtains

$$v_1 H_{v_1} + v_2 H_{v_2} + v_3 H_{v_3} = 0.$$

Thus, the universal invariant of this algebra is

$$\frac{J_3}{J_2 - J_2^\tau},\ \frac{J_2 - c(J_1 - J_1^\tau)}{J_2 - J_2^\tau}.$$

The set of equations admitting the generator L^4_{40} is

$$J_3 = (J_2 - J_2^\tau) f\left(\frac{J_2 - c(J_1 - J_1^\tau)}{J_2 - J_2^\tau}\right).$$

In table 4.1, this set of equations is written as

$$y'' = (y' - y'_\tau) f\left(\frac{y' - c(y - y_\tau)}{y' - y'_\tau}\right).$$

4.3.41 Lie Algebra L^4_{41}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x + y\partial_y,\ X_4 = y\partial_x - x\partial_y.$$

After changing the variables under conditions

$$(\bar{\xi}_i)_{\bar{y}} = 0 \quad \text{and} \quad \bar{\xi}_i(\bar{x}) = \bar{\xi}_i(\bar{x} - \tau),\ i = 1, \ldots, 4$$

lead us to $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$. This contradicts to the assumption $\Delta \neq 0$.

4.3.42 Lie Algebra L^4_{42}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = y\partial_y,\ X_3 = \sin x\partial_y,\ X_4 = \cos x\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\sin(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\cos(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

Invariants of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we arrive at

$$y_1\Phi_{y_1} + y_2\Phi_{y_2} + y_3\Phi_{y_3} + y_4\Phi_{y_4} + y_5\Phi_{y_5} = 0.$$

Hence, the invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and $v_1 = \dfrac{y_2}{y_1},\ v_2 = \dfrac{y_4}{y_3},\ v_3 = \dfrac{y_3}{y_1},\ v_4 = \dfrac{y_5}{y_3}$. Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(\frac{J_1^\tau}{J_1}, \frac{J_2^\tau}{J_2}, \frac{J_2}{J_1}, \frac{J_3}{J_2})$, one finds

$$-c_1v_3\psi_{v_1} + (c_2 - v_2)\psi_{v_2} + v_3\psi_{v_3} - v_4\psi_{v_4} = 0.$$

Solving for function ψ, we arrive at $\psi = H(z_1, z_2, z_3)$ where H is an arbitrary function and $z_1 = v_1 + c_1 v_3,\ z_2 = v_3(c_2 - v_2),\ z_3 = v_3 v_4$. Then, applying the generator $X_4^{(2)}$ to function $H(z_1, z_2, z_3)$ with substituted $y_1 = J_1,\ y_2 = J_1^\tau,$ $y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3$, one obtains

$$(c_2 - z_1)H_{z_1} - (c_1 + z_2)H_{z_2} - (1 + z_3)H_{z_3} = 0.$$

Thus, the universal invariant of this algebra is

$$I_{11} = \frac{c_2 - \frac{J_1^\tau}{J_1} + c_1\frac{J_2}{J_1}}{c_1 + \frac{J_2}{J_1}(c_2 - \frac{J_2^\tau}{J_2})},\ \frac{1 + \frac{J_3}{J_1}}{c_1 + \frac{J_2}{J_1}(c_2 - \frac{J_2^\tau}{J_2})}.$$

The set of equations admitting the generator L_{42}^4 is

$$J_3 = J_1\left(f(I_{11})\left[c_1 + \left[\frac{J_2}{J_1}(c_2 - \frac{J_2^\tau}{J_2})\right]\right] - 1\right).$$

In table 4.1, this set of equations is written as

$$y'' = y\Big(f(I_{11})\Big[c_1 + \Big[\frac{y'}{y}(c_2 - \frac{y'_\tau}{y'})\Big]\Big] - 1\Big).$$

4.3.43 Lie Algebra L_{43}^5

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = y\partial_x,\ X_4 = x\partial_y,\ X_5 = x\partial_x - y\partial_y.$$

After changing the variables under conditions

$$(\bar{\xi}_i)_{\bar{y}} = 0 \quad \text{and} \quad \bar{\xi}_i(\bar{x}) = \bar{\xi}_i(\bar{x} - \tau),\ i = 1, \ldots, 5. \tag{4.92}$$

It leads us to $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$. This contradicts to the assumption $\Delta \neq 0$.

4.3.44 Lie Algebra L_{44}^5

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = y\partial_x,\ X_4 = x\partial_y,\ X_5 = x\partial_x.$$

After changing the variables under conditions (4.92). The results is also $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$ which contradicts to the assumption $\Delta \neq 0$.

4.3.45 Lie Algebra L^5_{45}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x + y\partial_y,$$
$$X_4 = y\partial_x - x\partial_y,\ X_5 = (x^2 - y^2)\partial_x - 2xy\partial_y.$$

After changing the variables under conditions (4.92). The results is also $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$ which contradicts to the assumption $\Delta \neq 0$.

4.3.46 Lie Algebra L^5_{46}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x,\ X_4 = y\partial_y,\ X_5 = x^2\partial_x,$$

which after changing the variables, they become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2 = \tfrac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3 = \tfrac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},$$
$$\bar{X}_4 = \tfrac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_5 = \tfrac{\bar{h}^2}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{h}^2\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^4_{30} (page.66), invariant of prolonged generators $\bar{X}_1^{(2)}, \bar{X}_2^{(2)}, \bar{X}_3^{(2)}, \bar{X}_4^{(2)}$ is an arbitrary function $G(w_1, w_2)$ where $w_1 = J_1 - J_1^\tau, w_2 = (\frac{J_2}{J_2^\tau})^2$. Applying generator $\bar{X}_5^{(2)}$ to function $G(J_1 - J_1^\tau, (\frac{J_2}{J_2^\tau})^2)$, one obtains

$$w_1 H_{w_1} = 0.$$

Thus the universal invariant of this algebra is $(\dfrac{J_2}{J_2^\tau})^2$, which has no second-order derivative term. Hence the set of equations admitting the generator L^5_{46} cannot be constructed.

4.3.47 Lie Algebra L^5_{47}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x,\ X_4 = y\partial_y,\ X_5 = y\partial_x.$$

After changing the variables under conditions (4.92). The results is also $\bar{h}_{\bar{y}} = \bar{g}_{\bar{y}} = 0$ which contradicts to the assumption $\Delta \neq 0$.

4.3.48 Lie Algebra L^5_{48}

This algebra is defined by the generators

$$X_1 = \partial_y,\ X_2 = x\partial_y,\ X_3 = \xi_1(x)\partial_y,\ X_4 = \xi_2(x)\partial_y,\ X_5 = \xi_3(x)\partial_y$$

which after changing the variables, they become

$$\bar{X}_1 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3 = \frac{\xi_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_4 = \frac{\xi_2(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_5 = \frac{\xi_3(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^4_{23} (page.58), invariant of prolonged generators $\bar{X}_1^{(2)}, \bar{X}_2^{(2)}, \bar{X}_3^{(2)}, \bar{X}_4^{(2)}$ is an arbitrary function $H(v_1, v_2)$ where $v_1 = \bar{h}$, and

$$v_2 = [\xi_1''(\xi_2'^{\tau} - \xi_2') + \xi_2''(\xi_1' - \xi_1'^{\tau})][(\xi_1' - \xi_1^{\tau'})(cJ_2 - J_1 + J_1^{\tau}) - (\xi_1'c - \xi_1 + \xi_1^{\tau})(J_2 - J_2^{\tau})] -$$
$$(c(\xi_1'\xi_2'^{\tau} - \xi_1'^{\tau}\xi_2') + (\xi_1'^{\tau} - \xi_1')(\xi_2 - \xi_2^{\tau}) + (\xi_2' - \xi_2'^{\tau})(\xi_1 - \xi_1^{\tau}))[(\xi_1' - \xi_1^{\tau'})J_3 - \xi_1''(J_2 - J_2^{\tau})].$$

Applying generator $X_5^{(2)}$ to function $H(v_1, v_2)$ lead to the universal invariant of this algebra is $\bar{h}$, which has no second-order derivative term. Hence, the set of equations admitting the generator L^5_{48} cannot be constructed.

4.3.49 Lie Algebra L^5_{49}

This algebra is defined by the generators

$$X_1 = \partial_y,\ X_2 = x\partial_y,\ X_3 = y\partial_y,\ X_4 = \xi_1(x)\partial_y,\ X_5 = \xi_2(x)\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\bar{h}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\xi_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_5 = \frac{\xi_2(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^4_{35} (page.71), invariant of prolonged generators $\bar{X}^{(2)}_1, \bar{X}^{(2)}_2$, $\bar{X}^{(2)}_3, \bar{X}^{(2)}_4$ is an arbitrary function $G(w_1, w_2)$ where $w_1 = \bar{h},\ w_2 = \frac{\xi_1' J_3 - \xi_1''(J_2 - J_2^\tau)}{J_1 - J_1^\tau}$. Applying generator $X^{(2)}_5$ to function $G(\bar{h}, \frac{\xi_1' J_3 - \xi_1''(J_2 - J_2^\tau)}{J_1 - J_1^\tau})$, we obtain $G_{w_2} = 0$. Thus the universal invariant of this algebra is $\bar{h}$, which has no second-order derivative term. Hence the set of equations admitting the generator L^5_{49} cannot be constructed.

4.3.50 Lie Algebra L^5_{50}

Let us consider Lie algebra defined by the generators

$$X_1 = \partial_x,\ X_2 = \eta_1(x)\partial_y,\ X_3 = \eta_2(x)\partial_y, \ldots,\ X_{r+1} = \eta_r(x)\partial_y$$

where the functions $\eta_1,\ \eta_2,\ \eta_3, \ldots, \eta_r$ form a fundamental system of solutions for an r-order ordinary differential equation with constant coefficients

$$\eta^{(r)}(x) + c_1\eta^{(r-1)}(x) + \ldots + c_{r-1}\eta'(x) + c_r\eta(x) = 0.$$

These Lie algebras are $L^3_8, L^3_9, L^3_{11}, L^3_{15}, L^3_{17}, L^4_{26}, L^4_{27}, L^4_{28}, L^4_{31}, L^4_{32}, L^4_{33}, L^4_{34}, L^4_{36}$, L^4_{37}, L^5_{50}

- Case $r = 2$, the Lie algebra defined by the generators

$$X_1 = \partial_x,\ X_2 = \eta_1(x)\partial_y,\ X_3 = \eta_2(x)\partial_y,$$

where $\eta_1,\ \eta_2$ satisfy the equation

$$\eta''(x) = -(c_1\eta'(x) + c_2\eta(x)). \tag{4.93}$$

In Table 1 these Lie algebras are $L^3_8, L^3_9, L^3_{11}, L^3_{15}$ and L^3_{17}. Changing the variables (3.10), the generators become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \tfrac{\eta_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \tfrac{\eta_2(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

The general solution of the function $\bar{X}_1^{(2)}J = 0$ is obtained similar to (4.19). Applying the generators $\bar{X}_2^{(2)}$, $\bar{X}_3^{(2)}$ to the function $J = \Psi(y_1, y_2, y_3, y_4, y_5, y_6)$ with

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,\ y_6 = J_3^\tau,$$

where $J_3^\tau = J_3(\bar{x}-\tau, \bar{y}_\tau, \bar{y}'_\tau, \bar{y}''_\tau)$, we obtain the system of differential equations

$$\eta_1\Psi_{y_1} + \eta'_1\Psi_{y_3} + \eta''_1\Psi_{y_5} + \eta_1^\tau\Psi_{y_2} + \eta_1'^\tau\Psi_{y_4} + \eta_1''^\tau\Psi_{y_6} = 0, \qquad (4.94)$$

$$\eta_2\Psi_{y_1} + \eta'_2\Psi_{y_3} + \eta''_2\Psi_{y_5} + \eta_2^\tau\Psi_{y_2} + \eta_2'^\tau\Psi_{y_4} + \eta_2''^\tau\Psi_{y_6} = 0, \qquad (4.95)$$

where $\eta_i^\tau = \eta_i(\bar{h}(\bar{x}-\tau)),\ \eta_i'^\tau = \eta'_i(\bar{h}(\bar{x}-\tau)),\ \eta_i''^\tau = \eta''_i(\bar{h}(\bar{x}-\tau)), (i = 1, 2)$. The variables y_6 is introduced for simplicity of representation of equations for invariant: for second-order delay ordinary differential equations $\Psi_{y_6} = 0$. Substituting η''_i and $\eta_i''^\tau$ found from (4.93) into (4.94)-(4.95), they become

$$\eta_1\Psi_{y_1} + \eta'_1\Psi_{y_3} - (c_1\eta'_1 + c_2\eta_1)\Psi_{y_5} + \eta_1^\tau\Psi_{y_2} + \eta_1'^\tau\Psi_{y_4} - (c_1\eta_1'^\tau + c_2\eta_1^\tau)\Psi_{y_6} = 0,$$

$$\eta_2\Psi_{y_1} + \eta'_2\Psi_{y_3} - (c_1\eta'_2 + c_2\eta_2)\Psi_{y_5} + \eta_2^\tau\Psi_{y_2} + \eta_2'^\tau\Psi_{y_4} - (c_1\eta_2'^\tau + c_2\eta_2^\tau)\Psi_{y_6} = 0.$$

In matrix form, these equations can be rewritten as

$$\Phi\vec{z} - \Psi_{y_5}\Phi\vec{c} + \Phi^\tau\vec{z}^\tau = 0. \qquad (4.96)$$

Here

$$\Phi = \begin{bmatrix} \eta_1 & \eta'_1 \\ \eta_2 & \eta'_2 \end{bmatrix}, \ \vec{c} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}, \ \vec{z} = \begin{bmatrix} \Psi_{y_1} \\ \Psi_{y_3} \end{bmatrix}, \ \vec{z}^\tau = \begin{bmatrix} \Psi_{y_2} \\ \Psi_{y_4} \end{bmatrix}, \ \Phi^\tau = \Phi(\bar{h}(\bar{x}-\tau)).$$

Since η_i composes a fundamental system of solutions of (4.93), Φ is a fundamental matrix, which has the properties $\Phi(\bar{h}(\bar{x}-\tau)) = \Phi(\bar{h}(\bar{x}))C,\ \det\Phi \neq 0$ with a nonsingular matrix $C = [c_{ij}]_{2\times 2}$ (Pontriagin, 1974). Multiplying (4.96) by Φ^{-1} , system (4.96) is rewritten

$$\vec{z} - \Psi_{y_5}\vec{c} + C\vec{z}^{\tau} = 0,$$

or these equations are

$$\Psi_{y_1} - c_2\Psi_{y_5} + c_{11}\Psi_{y_2} + c_{12}\Psi_{y_4} = 0,$$

$$\Psi_{y_3} - c_1\Psi_{y_5} + c_{21}\Psi_{y_2} + c_{22}\Psi_{y_4} = 0.$$

Since these equations have constant coefficients, one easily obtains the universal invariant

$$J_3 + c_1J_2 + c_2J_1,\ J_1^{\tau} - c_{11}J_1 - c_{21}J_2,\ J_2^{\tau} - c_{12}J_1 - c_{22}J_2.$$

The invariant equation has the form

$$J_3 = f\left(J_1^{\tau} - c_{11}J_1 - c_{21}J_2, J_2^{\tau} - c_{12}J_1 - c_{22}J_2\right) - (c_1J_2 + c_2J_1).$$

Because of the meaning of the functions $J_1, J_1^{\tau}, J_2, J_2^{\tau}$ and J_3, we present this equation as

$$y'' = f\left(y_{\tau} - c_{11}y - c_{21}y', y'_{\tau} - c_{12}y - c_{22}y'\right) - (c_1y' + c_2y) \qquad (4.97)$$

- Case $r = 3$, the Lie algebra is defined by

$$X_1 = \partial_x,\ X_2 = \eta_1(x)\partial_y,\ X_3 = \eta_2(x)\partial_y,\ X_4 = \eta_3(x)\partial_y,$$

where $\eta_1,\ \eta_2,\ \eta_3$ satisfy the equation

$$\eta'''(x) = -(c_1\eta''(x) + c_2\eta'(x) + c_3\eta(x)). \qquad (4.98)$$

In Table 1 these Lie algebras are $L^4_{26}, L^4_{27}, L^4_{28}, L^4_{31}, L^4_{32}, L^4_{33}, L^4_{34}, L^4_{36}$ and L^4_{37}. Changing the variables (3.10), the generators become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \tfrac{\eta_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \tfrac{\eta_2(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \tfrac{\eta_3(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

The general solution of the function $\bar{X}_1^{(2)}J = 0$ is obtained similar to (4.19). Applying the generators $\bar{X}_2^{(3)}$, $\bar{X}_3^{(3)}, \bar{X}_4^{(3)}$ to the function $J = \Psi(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$ with

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,$$

$$y_5 = J_3,\ y_6 = J_3^\tau y_7 = J_4,\ y_8 = J_4^\tau,$$

$$J_4 = \tfrac{D(J_3(\bar{x},\bar{y},\bar{y}',\bar{y}''))}{D(\bar{h}(\bar{x},\bar{y}))},\ J_4^\tau = J_4(\bar{x}-\tau, \bar{y}_\tau, \bar{y}'_\tau, \bar{y}''_\tau, \bar{y}'''_\tau),$$

we obtain system of differential equations

$$\eta_1\Psi_{y_1} + \eta_1'\Psi_{y_3} + \eta_1''\Psi_{y_5} + \eta_1'''\Psi_{y_7} + \eta_1^\tau\Psi_{y_2} + \eta_1'^\tau\Psi_{y_4} + \eta_1''^\tau\Psi_{y_6} + \eta_1'''^\tau\Psi_{y_8} = 0,$$

$$\eta_2\Psi_{y_1} + \eta_2'\Psi_{y_3} + \eta_2''\Psi_{y_5} + \eta_2'''\Psi_{y_7} + \eta_2^\tau\Psi_{y_2} + \eta_2'^\tau\Psi_{y_4} + \eta_2''^\tau\Psi_{y_6} + \eta_2'''^\tau\Psi_{y_8} = 0,$$

$$\eta_3\Psi_{y_1} + \eta_3'\Psi_{y_3} + \eta_3''\Psi_{y_5} + \eta_3'''\Psi_{y_7} + \eta_3^\tau\Psi_{y_2} + \eta_3'^\tau\Psi_{y_4} + \eta_3''^\tau\Psi_{y_6} + \eta_3'''^\tau\Psi_{y_8} = 0,$$

where $\eta_i^\tau = \eta_i(\bar{h}(\bar{x}-\tau))$, $\eta_i'^\tau = \eta_i'(\bar{h}(\bar{x}-\tau))$, $\eta_i''^\tau = \eta_i''(\bar{h}(\bar{x}-\tau))$, $\eta_i'''^\tau = \eta_i'''(\bar{h}(\bar{x}-\tau))$, $(i = 1, 2, 3)$. Here the variables y_6, y_7 and y_8 are introduced for simplicity of representation of equations for invariant: for second-order delay ordinary differential equations $\Psi_{y_6} = 0, \Psi_{y_7} = 0, \Psi_{y_8} = 0$. Substituting η_i''', and $\eta_i'''^\tau$ found from (4.98), the above system of equations in matrix form

$$\Phi\vec{z} - \Psi_{y_7}\Phi\vec{c} + \Phi^\tau\vec{z}^\tau = 0. \tag{4.99}$$

Here

$$\Phi = \begin{bmatrix} \eta_1 & \eta_1' & \eta_1'' \\ \eta_2 & \eta_2' & \eta_2'' \\ \eta_3 & \eta_3' & \eta_3'' \end{bmatrix},\ \vec{c} = \begin{bmatrix} c_3 \\ c_2 \\ c_1 \end{bmatrix},\ \vec{z} = \begin{bmatrix} \Psi_{y_1} \\ \Psi_{y_3} \\ \Psi_{y_5} \end{bmatrix},\ \vec{z}^\tau = \begin{bmatrix} \Psi_{y_2} \\ \Psi_{y_4} \\ \Psi_{y_6} \end{bmatrix},\ \Phi^\tau = \Phi(\bar{h}(\bar{x}-\tau)).$$

Since η_i composes a fundamental system of solutions of (4.98), Φ is a fundamental matrix, which has the properties $\Phi(\bar{h}(\bar{x}-\tau)) = \Phi(\bar{h}(\bar{x}))C$, $\det\Phi \neq 0$ with a nonsingular matrix $C = [c_{ij}]_{3\times 3}$. Multiplying (4.99) by Φ^{-1}, as in the previous case, system (4.99) is rewritten as

$$\vec{z} - \vec{c}\Psi_{y_7} + C\vec{z}^\tau = 0,$$

or

$$\Psi_{y_1} - c_3\Psi_{y_7} + c_{11}\Psi_{y_2} + c_{12}\Psi_{y_4} + c_{13}\Psi_{y_6} = 0,$$

$$\Psi_{y_3} - c_2\Psi_{y_7} + c_{21}\Psi_{y_2} + c_{22}\Psi_{y_4} + c_{23}\Psi_{y_6} = 0,$$

$$\Psi_{y_5} - c_1\Psi_{y_7} + c_{31}\Psi_{y_2} + c_{32}\Psi_{y_4} + c_{33}\Psi_{y_6} = 0.$$

Solving these equations and using the conditions $\Phi_{y_6} = \Phi_{y_7} = 0$, the universal invariant of this Lie algebra

$$J_1^\tau - c_{11}J_1 - c_{21}J_2 - c_{31}J_3,\ \ J_2^\tau - c_{12}J_1 - c_{22}J_2 - c_{32}J_3.$$

Since second-order delay ordinary differential equations are studied in this paper, one need to assume $(c_{31})^2 + (c_{32})^2 \neq 0$. The invariant equation has the form

$$\phi\Big(J_1^\tau - c_{11}J_1 - c_{21}J_2 - c_{31}J_3,\ \ J_2^\tau - c_{12}J_1 - c_{22}J_2 - c_{32}J_3\Big) = 0,$$

where $\phi(z_1, z_2)$ is an arbitrary function. Because of the meaning of the functions $J_1, J_1^\tau, J_2, J_2^\tau$ and J_3, we represent this equation as

$$\phi\Big(y_\tau - c_{11}y - c_{21}y' - c_{31}y'', y'_\tau - c_{12}y - c_{22}y' - c_{32}y''\Big) = 0.$$

- Case $r \geq 4$, in this case one can proceed in the same manner. The universal invariant of Lie algebra is

$$J_1^\tau - \sum_{i=1}^{r} c_{i1}J_i, \quad J_2^\tau - \sum_{i=1}^{r} c_{i2}J_i,$$

where J_i is the $y^{(i-1)}$ after change of variables. The set of equations admitting the generator L^5_{50} is

$$\phi\Big(J_1^\tau - \sum_{i=1}^{r} c_{i1}J_i,\ J_2^\tau - \sum_{i=1}^{r} c_{i2}J_i\Big) = 0,$$

where $\phi(z_1, z_2)$ is an arbitrary function with respect to

$$c_{i1}\phi_{z_1} + c_{i2}\phi_{z_2} = 0, \quad i = 4, \ldots, r.$$

Because of the meaning of the functions $J_1, J_1^\tau, J_2, J_2^\tau$ and J_3, we represent this equation as

$$\phi\Big(y_\tau - \sum_{i=1}^{r} c_{i1}y^{(i-1)},\ , y'_\tau - \sum_{i=1}^{r} c_{i2}y^{(i-1)}\Big) = 0.$$

4.3.51 Lie Algebra L^5_{51}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \eta_1(x)\partial_y,\ X_3 = \eta_2(x)\partial_y, \ldots,\ X_{r+1} = \eta_r(x)\partial_y,\ X_{r+2} = y\partial_y,$$

where the functions $\eta_i(x), i = 1, \ldots, r$ are defined as in Lie algebra L^5_{50}.

- Case $r = 2$, the Lie algebra defined by the generators

$$X_1 = \partial_x,\ X_2 = \eta_1(x)\partial_y,\ X_3 = \eta_2(x)\partial_y,\ X_4 = y\partial_y$$

where $\eta_1,\ \eta_2$ satisfy the equation (4.93). After changing the variables, they become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{\eta_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\eta_2(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_4 = \frac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L^5_{50}, invariant of prolonged generators $\bar{X}_1^{(2)}, \bar{X}_2^{(2)}, \bar{X}_3^{(2)}$ is an arbitrary function $G(z_1, z_2, z_3)$ where

$$z_1 = J_3 + c_1J_2 + c_2J_1,\ z_2 = J_1^\tau - c_{11}J_1 - c_{21}J_2,\ z_3 = J_2^\tau - c_{12}J_1 - c_{22}J_2.$$

Applying the generator $\bar{X}_4^{(2)}$ to the function

$$G(J_3 + c_1 J_2 + c_2 J_1, J_1^\tau - c_{11} J_1 - c_{21} J_2,\ J_2^\tau - c_{12} J_1 - c_{22} J_2),$$

we find

$$z_1 G_{z_1} + z_2 G_{z_2} + z_3 G_{z_3} = 0.$$

Thus the universal invariant function is

$$\frac{J_3 + c_1 J_2 + c_2 J_1}{J_2^\tau - c_{12} J_1 - c_{22} J_2}, \quad \frac{J_1^\tau - c_{11} J_1 - c_{21} J_2}{J_2^\tau - c_{12} J_1 - c_{22} J_2}.$$

The set of equation admitting this Lie algebra is

$$J_3 = (J_2^\tau - c_{12} J_1 - c_{22} J_2) f\Big(\frac{J_1^\tau - c_{11} J_1 - c_{21} J_2}{J_2^\tau - c_{12} J_1 - c_{22} J_2}\Big) - (c_1 J_2 + c_2 J_1).$$

Because of the meaning of the functions $J_1, J_1^\tau, J_2, J_2^\tau$ and J_3, we present this equation as

$$y'' = (y'_\tau - c_{12} y - c_{22} y') f\Big(\frac{y_\tau - c_{11} y - c_{21} y'}{y'_\tau - c_{12} y - c_{22} y'}\Big) - (c_1 y' + c_2 y). \qquad (4.100)$$

- Case $r \geq 3$, the Lie algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \eta_1(x)\partial_y, \ldots, X_{r+1} = \eta_r(x)\partial_y,\ X_{r+2} = y\partial_y,$$

which after changing the variables, they become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \tfrac{\eta_1(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \ldots, \bar{X}_{r+1} = \tfrac{\eta_r(\bar{h})}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_{r+2} = \tfrac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}.$$

From Lie algebra L_{50}^5, invariant of prolonged generators $\bar{X}_1^{(2)}, \bar{X}_2^{(2)}, \ldots, \bar{X}_{r+1}^{(2)}$ is an arbitrary function $G(z_1, z_2)$ where

$$z_1 = J_1^\tau - \sum_{i=1}^{r} c_{i1} J_i, \quad z_2 = J_2^\tau - \sum_{i=1}^{r} c_{i2} J_i.$$

Applying the generator $\bar{X}_{r+2}^{(2)}$ to the function

$$G(J_1^\tau - \sum_{i=1}^{r} c_{i1} J_i, J_2^\tau - \sum_{i=1}^{r} c_{i2} J_i),$$

we find

$$z_1 G_{z_1} + z_2 G_{z_2} = 0.$$

Thus the universal invariant function is

$$\frac{J_1^\tau - \sum_{i=1}^r c_{i1} J_i}{J_2^\tau - \sum_{i=1}^r c_{i2} J_i}.$$

The set of equation is written as

$$y_\tau - \sum_{i=1}^r c_{i1} y^{(i-1)} = c_5 (y'_\tau - \sum_{i=1}^r c_{i2} y^{(i-1)}),$$

where c_5 is an arbitrary constant with respect to

$$c_5 c_{j2} - c_{j1} = 0, \quad j = 4, \ldots, r+1.$$

4.3.52 Lie Algebra L_{52}^5

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_y,\ X_4 = x^2\partial_y,\ X_5 = x\partial_x + cy\partial_y$$

which after changing the variables, they become

$$\bar{X}_1 = \frac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \frac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \frac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \frac{\bar{h}}{\bar{g}_{\bar{x}}}\partial_{\bar{y}},$$
$$\bar{X}_4 = \frac{\bar{h}^2}{\bar{g}_{\bar{x}}}\partial_{\bar{y}}, \quad \bar{X}_5 = \frac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \left(\frac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} - \frac{c\bar{g}}{\bar{g}_{\bar{y}}}\right)\partial_{\bar{y}}.$$

Invariant function of the first generator are (4.17). Applying the second generator $\bar{X}_2^{(2)}$ to the function $\Phi(y_1, y_2, y_3, y_4, y_5)$ with substituted

$$y_1 = J_1,\ y_2 = J_1^\tau,\ y_3 = J_2,\ y_4 = J_2^\tau,\ y_5 = J_3,$$

we find

$$y_1 \Phi_{y_1} + y_2 \Phi_{y_2} = 0.$$

Hence, the invariant function is $\Phi = \psi(v_1, v_2, v_3, v_4)$ where ψ is an arbitrary function and $v_1 = y_1 - y_2,\ v_2 = y_3,\ v_3 = y_4,\ v_4 = y_5$. Next, applying the generator $\bar{X}_3^{(2)}$ to the function $\psi(J_1 - J_1^\tau, J_2, J_2^\tau, J_3)$, we obtain

$$\psi_{v_2} + \psi_{v_3} = 0.$$

Solving for function ψ, we arrive at $\psi = H(z_1, z_2, z_3)$ where H is an arbitrary function and $z_1 = v_1,\ z_2 = v_2 - v_3,\ z_3 = v_4$. Then, applying the generator $X_4^{(2)}$ to function $H(z_1, z_2, z_3)$ with substituted $y_1 = J_1, y_2 = J_1^\tau,\ y_3 = J_2, y_4 = J_2^\tau, y_5 = J_3$, one obtains

$$2H_{z_3} = 0.$$

The invariant function is $H = G(w_1, w_2)$ where G is an arbitrary function and $w_1 = z_1,\ w_2 = z_2$. Finally applying the generator $X_5^{(2)}$ to function $G(J_1 - J_1^\tau, J_2 - J_2^\tau)$, one gets

$$cw_1 G_{w_1} + (c-1)w_2 G_{w_2} = 0,$$

where c is an arbitrary constant. Thus, the universal invariant of this algebra is

$$(J_1 - J_1^\tau)^{(1-c)}(J_2 - J_2^\tau)^c,$$

which has no second-order derivative term. Hence, the set of equations admitting the generator L_{52}^5 cannot be constructed.

4.3.53 Lie Algebra L_{53}^5

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_y,\ X_4 = x^2\partial_y,\ X_5 = x\partial_x + (3y + x^3)\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \tfrac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \tfrac{\bar{h}}{\bar{g}_{\bar{x}}}\partial_{\bar{y}},$$
$$\bar{X}_4 = \tfrac{\bar{h}^2}{\bar{g}_{\bar{x}}}\partial_{\bar{y}}, \quad \bar{X}_5 = \tfrac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \Big(-\tfrac{\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}} + \tfrac{3\bar{g}+\bar{h}^3}{\bar{g}_{\bar{y}}}\Big)\partial_{\bar{y}}.$$

From Lie algebra L^5_{52}, invariant function of the generators $\bar{X}^{(2)}_1, \bar{X}^{(2)}_2, \bar{X}^{(2)}_3, \bar{X}^{(2)}_4$ is an arbitrary function $G(w_1, w_2)$ where $w_1 = J_1 - J^\tau_1, w_2 = J_2 - J^\tau_2$. Applying the generator $\bar{X}^{(2)}_5$ to the function $G(J_1 - J^\tau_1, J_2 - J^\tau_2)$, we find

$$w_1 G_{w_1} = 0.$$

Thus, the universal invariant of this algebra is $J_2 - J^\tau_2$, which has no second-order derivative term. Hence, the set of equations admitting the generator L^5_{53} cannot be constructed.

4.3.54 Lie Algebra L^5_{54}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_y,\ X_4 = y\partial_y,\ X_5 = x\partial_x$$

which after changing the variables become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \tfrac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \tfrac{\bar{h}}{\bar{g}_{\bar{x}}}\partial_{\bar{y}},$$
$$\bar{X}_4 = \tfrac{\bar{h}^2}{\bar{g}_{\bar{x}}}\partial_{\bar{y}}, \quad \bar{X}_5 = \tfrac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}}.$$

From Lie algebra L^5_{52}, invariant function of the generators $X^{(2)}_1, X^{(2)}_2, X^{(2)}_3$ is an arbitrary function $G(w_1, w_2, w_3)$ where $w_1 = J_1 - J^\tau_1, w_2 = J_2 - J^\tau_2, w_3 = J_3$. Applying the generator $\bar{X}^{(2)}_4$ to the function $G(J_1 - J^\tau_1, J_2 - J^\tau_2, J_3)$, we find

$$w_1 G_{w_1} + w_2 G_{w_2} + w_3 G_{w_3} = 0.$$

Invariant function is $G = V(z_1, z_2)$ where V is an arbitrary function and

$$z_1 = \frac{w_2}{w_1},\ z_2 = \frac{w_3}{w_2}.$$

Finally, applying the generator $X^{(2)}_5$ to function $V(\frac{J_2 - J^\tau_2}{J_1 - J^\tau_1}, \frac{J_3}{J_2 - J^\tau_2})$, we arrive at

$$z_1 V_{z_1} + z_2 V_{z_2} = 0.$$

Thus, the universal invariant of this algebra is $\dfrac{J_3 J_1}{(J_2 - J_2^\tau)^2}$. The set of equations admitting the generator L_{54}^5 is

$$J_3 = \frac{c_3(J_2 - J_2^\tau)^2}{J_1}, \tag{4.101}$$

where c_3 is arbitrary constant. In table 4.1 this set of equations is written as

$$y'' = \frac{c_3(y' - y'_\tau)^2}{y}. \tag{4.102}$$

4.3.55 Lie Algebra L_{55}^5

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_y,\ X_4 = 2x\partial_x + y\partial_y,\ X_5 = x^2\partial_x + xy\partial_y$$

which after changing the variables become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_2 = \tfrac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},\quad \bar{X}_3 = \tfrac{\bar{h}}{\bar{g}_{\bar{x}}}\partial_{\bar{y}},$$
$$\bar{X}_4 = \tfrac{2\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \Big(-\tfrac{2\bar{h}\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{x}}} + \tfrac{\bar{g}}{\bar{g}_{\bar{y}}}\Big)\partial_{\bar{y}},\quad \bar{X}_5 = \tfrac{\bar{h}^2}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} + \Big(-\tfrac{\bar{h}^2\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{x}}} + \tfrac{\bar{h}\bar{g}}{\bar{g}_{\bar{y}}}\Big)\partial_{\bar{y}}.$$

From Lie algebra L_{52}^5, invariant function of the generators $X_1^{(2)}, X_2^{(2)}, X_3^{(2)}$ is an arbitrary function $G(w_1, w_2, w_3)$ where $w_1 = J_1 - J_1^\tau, w_2 = J_2 - J_2^\tau, w_3 = J_3$. Applying the generator $\bar{X}_4^{(2)}$ to the function $G(J_1 - J_1^\tau, J_2 - J_2^\tau, J_3)$, we find

$$w_1 G_{w_1} - w_2 G_{w_2} - 3w_3 G_{w_3} = 0.$$

Invariant function is $G = V(z_1, z_2)$ where V is an arbitrary function and

$$z_1 = w_2 w_1,\ z_2 = w_1^3 w_3.$$

Finally, applying the generator $X_5^{(2)}$ to function $V\Big((J_1 - J_1^\tau)(J_2 - J_2^\tau), (J_1 - J_1^\tau)^3 J_3\Big)$, we arrive at

$$z_1 V_{z_1} = 0.$$

Thus, the universal invariant of this algebra is $(J_1 - J_1^{\tau})^3 J_3$. The set of equations admitting the generator L^5_{55} is

$$J_3 = \frac{c_4}{(J_1 - J_1^{\tau})^3}, \tag{4.103}$$

where c_4 is arbitrary constant. In table 4.1 this set of equations is written as

$$y'' = \frac{c_4}{(y - y_{\tau})^3}. \tag{4.104}$$

4.3.56 Lie Algebra L^5_{56}

This algebra is defined by the generators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = y\partial_y,\ X_4 = x\partial_x,\ X_5 = x^2\partial_x$$

which after changing the variables become

$$\bar{X}_1 = \tfrac{1}{\bar{h}_{\bar{x}}}\partial_{\bar{x}} - \tfrac{\bar{g}_{\bar{x}}}{\bar{h}_{\bar{x}}\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_2 = \tfrac{1}{\bar{g}_{\bar{y}}}\partial_{\bar{y}}, \quad \bar{X}_3 = \tfrac{\bar{g}}{\bar{g}_{\bar{y}}}\partial_{\bar{y}},$$
$$\bar{X}_4 = \tfrac{\bar{h}}{\bar{h}_{\bar{x}}}\partial_{\bar{x}}, \quad \bar{X}_5 = \tfrac{\bar{h}^2}{\bar{h}_{\bar{x}}}\partial_{\bar{x}}.$$

From Lie algebra L^3_7 (page.44), invariant function of the prolonged generators $\bar{X}_1^{(2)}, \bar{X}_2^{(2)}, \bar{X}_3^{(2)}$ is an arbitrary function $G(w_1, w_2, w_3)$ where $w_1 = \frac{J_1 - J_1^{\tau}}{J_2}, w_2 = \frac{J_2^{\tau}}{J_2}, w_3 = \frac{J_3}{J_2}$. Applying the generator $\bar{X}_4^{(2)}$ to the function $G(\frac{J_1 - J_1^{\tau}}{J_2}, \frac{J_2^{\tau}}{J_2}, \frac{J_3}{J_2})$, we find

$$w_1 G_{w_1} - w_3 G_{w_3} = 0.$$

Invariant function is $G = V(z_1, z_2)$ where V is an arbitrary function and

$$z_1 = w_2,\ z_2 = w_1 w_3.$$

Finally, applying the generator $X_5^{(2)}$ to function $V\left(\frac{J_2^{\tau}}{J_2}, \frac{(J_1 - J_1^{\tau})J_3}{(J_2)^2}\right)$, we arrive at

$$-2z_2 V_{z_2} = 0.$$

Thus, the universal invariant of this algebra is $\dfrac{J_2^{\tau}}{J_2}$, which has no second-order derivative term. Hence, the set of equations admitting the generator L^5_{56} cannot be constructed.

4.4 Group Classification of Second-Order DODEs

Table 4.1 Group classification of second-order DODEs on the domain of real space

No.		Lie algebra	Representation of second-order DODEs
1	L_1^1	∂_x	$y'' = f(y, y_\tau, y', y'_\tau)$
2	L_2^2	∂_x, ∂_y	$y'' = f(y - y_\tau, y', y'_\tau)$
3	L_3^2	$\partial_x, y\partial_x$	$y'' = y'^3 f(y, y_\tau, \frac{1}{y'} - \frac{1}{y'_\tau})$
4	L_4^2	$\partial_x, x\partial_x + y\partial_y$	$y'' = \frac{1}{y} f(\frac{y_\tau}{y}, y', y'_\tau)$
5	L_5^2	$\partial_x, x\partial_x$	$y'' = y'^2 f(y, y_\tau, \frac{y'_\tau}{y'})$
6	L_6^3	$\partial_y, x\partial_y, \xi_1(x)\partial_y$	$y'' = \frac{1}{(\xi'_1 - \xi_1^{\tau'})}\Big(f\Big(x, (\xi'_1 - \xi_1^{\tau'})(cy' - y + y_\tau) - (\xi'_1 c - \xi_1 + \xi_1^\tau)(y' - y'_\tau)\Big) + \xi''_1(y' - y'_\tau)\Big)$
7	L_7^3	$\partial_y, y\partial_y, \partial_x$	$y'' = y' f\Big(\frac{y - y_\tau}{y'}, \frac{y'_\tau}{y'}\Big)$
8	L_8^3	$e^{-x}\partial_y, \partial_x, \partial_y$	$y'' = f(ky' - y'_\tau, k(y - y_\tau - y'_\tau) + y'_\tau) - y'$
9	L_9^3	$\partial_y,\ \partial_x,\ x\partial_y$	$y'' = f(y' - y'_\tau,\ cy' - y + y_\tau)$
10	L_{10}^3	$\partial_y, \partial_x, x\partial_x + (x + y)\partial_y$	$y'' = e^{-y'} f(y' - y'_\tau,\ (y - y_\tau)e^{-y'})$
11	L_{11}^3	$e^{-x}\partial_y, -xe^{-x}\partial_y, \partial_x$	$y'' = f\Big(k(y + y') - (y_\tau + y'_\tau),\ kc(y + y') - ky + y_\tau\Big) - (2y' + y)$
12	L_{12}^3	$\partial_x, \partial_y, x\partial_x + y\partial_y$	$y'' = \frac{f(y',\ y'_\tau)}{y - y_\tau}.$
13	L_{13}^3	$\partial_y, x\partial_y, y\partial_y$	$y'' = (y' - y'_\tau) f\Big(x, \frac{cy' - y + y_\tau}{(y' - y'_\tau)}\Big).$
14	L_{14}^3	$\partial_x, \partial_y, x\partial_x + ay\partial_y$, $0 < \lvert a \rvert \le 1, a \ne 1$	$y'' = y'^{\frac{(a-2)}{(a-1)}} f\Big(\frac{y'_\tau}{y'}, y'(y - y_\tau)^{\frac{(1-a)}{a}}\Big)$
15	L_{15}^3	$e^{-x}\partial_y, e^{-ax}\partial_y, \partial_x$, $0 < \lvert a \rvert \le 1, a \ne 1$	$y'' = f\Big(k^a(y + y') - (y_\tau + y'_\tau), (k - k^a)(y + y') - (1 - a)(ky - y_\tau)\Big) - [(1 + a)y' + ay]$
16	L_{17}^3	$e^{-bx}\sin x\partial_y, e^{-bx}\cos x\partial_y, \partial_x$, $b \ge 0$	$y'' = f(I_1, I_2) - (2by' + (b^2 + 1)y)$
17	L_{19}^3	$\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y$	$y'' = \frac{y'^{3/2}}{(x - y)}\Big(f\big(y'(\frac{x - y_\tau}{y_\tau - y})^2, \frac{(y_\tau - y)^2}{y'_\tau(x - y)^2}\big) - 2y'(y' + 1)\Big)$

No.		Lie algebra	Representation of second-order DODEs
18	L^3_{20}	$\partial_x, x\partial_x + \frac{1}{2}y\partial_y, x^2\partial_x + xy\partial_y$	$y'' = y^{-3}f\Big(\frac{y_\tau}{y},\ y'y_\tau(\frac{y'_\tau}{y'} - \frac{y_\tau}{y})\Big)$
19	L^4_{24}	$\partial_x, x\partial_x, \partial_y, y\partial_y$	$y'' = \frac{y'^2}{(y-y_\tau)}f\Big(\frac{y'_\tau}{y'}\Big)$
20	L^4_{25}	$e^{-x}\partial_y, \partial_x, \partial_y, y\partial_y$	$y'' = (ky'-y'_\tau)f\Big(\frac{ky'-y'_\tau}{(k-1)y'-y+y_\tau}\Big)-y'$
21	L^4_{26}	$e^{-x}\partial_y, -xe^{-x}\partial_y, \partial_x, \partial_y$	$y'' = \frac{f(I_3)+(k-1)y'-y+y_\tau}{(kc-k+1)} - y'$
22	L^4_{27}	$e^{-x}\partial_y, e^{-ax}\partial_y, \partial_x, \partial_y,$ $0 < \vert a\vert \leq 1, a \neq 1$	$y'' = \frac{\Big(f(I_4)+(a-1)(ky'-y'_\tau)\Big)}{(k^a-k)} - y'.$
23	L^4_{28}	$e^{-bx}\sin x\partial_y, e^{-bx}\cos x\partial_y, \partial_x,$ $\partial_y, b \geq 0$	$y'' = \frac{f(I_5)-(b^2+1)[c_1y'_\tau-(c_2+bc_1)(y-y_\tau)]}{k^b-(bc_1+c_2)}-2by'$
24	L^4_{29}	$\partial_x, x\partial_x, y\partial_y, x^2\partial_x + xy\partial_y$	$y'' = f\Big(\frac{y_\tau}{y}\Big)\frac{y'^2}{y}\Big(\frac{y_\tau}{y} - \frac{y'_\tau}{y'}\Big)^2$
25	L^4_{31}	$\partial_y, -x\partial_y, \frac{1}{2}x^2\partial_y, \partial_x$	$cy'' = y' - y'_\tau - f\Big(2(y-y_\tau) - c(y'+y'_\tau)\Big)$
26	L^4_{32}	$e^{-bx}\partial_y, e^{-x}\partial_y, -xe^{-x}\partial_y, \partial_x$	$y'' = \frac{-1}{(b-1)(k^b-k)}\Big(f(I_6) - (b-1)^2[k(y$ $+y') - (y_\tau - y'_\tau)]\Big) - [2y'+y],$
27	L^4_{33}	$e^{-x}\partial_y, -x\partial_y, \partial_y, \partial_x$	$y'' = \frac{1}{(k-1)}f(I_7) + (y'-y'_\tau)$
28	L^4_{34}	$e^{-x}\partial_y,\ -xe^{-x}\partial_y,\ \frac{1}{2}x^2e^{-x}\partial_y,\ \partial_x$	$y'' = -\frac{1}{kc}f(I_8) - k(y+y') + (y_\tau + y'_\tau)$ $-(2y'+y)$
29	L^4_{35}	$\partial_y, x\partial_y, \xi_1(x)\partial_y, y\partial_y$	$y'' = \frac{\xi''_1(x)(y'-y'_\tau)+(y-y_\tau)f(x)}{\xi'_1(x)}$
30	L^4_{36}	$e^{-ax}\partial_y,\ e^{-bx}\partial_y,\ e^{-x}\partial_y,\ \partial_x,$ $-1 \leq a < b < 1,\ ab \neq 0$	$y'' = \frac{1}{(k^b-kc)(b-1)}\Big((b^2-ab+a-b)(k^a(y+$ $y') - (y_\tau + y'_\tau)) - f(I_9)\Big) - (a(y+y')+y')$
31	L^4_{37}	$e^{-ax}\partial_y, e^{-bx}\sin x\partial_y, \partial_x$ $e^{-bx}\cos x\partial_y,\ a > 0$	$y'' = \frac{\Big(f(I_{10})((a-b)^2+1)[k^bc_1[ay+y']-[k^ay-y']]\Big)}{[k^b(c_2+(b-a)c_1)-k^a]}$ $-(2b(ay+y')+a^2y)$
32	L^4_{38}	$\partial_x,\ \partial_y,\ x\partial_y,\ x\partial_x + (2y+x^2)\partial_y$	$y'' = \ln\Big((y'-y'_\tau)^2f\Big(\frac{(y'-y'_\tau)^2}{y-y_\tau}\Big)\Big)$
33	L^4_{39}	$\partial_y,\ \partial_x,\ x\partial_y,\ (1+b)x\partial_x + y\partial_y,$ $\vert b\vert \leq 1$	$y'' = \Big((y'-y'_\tau)^{2b+1}f[(y-y_\tau)^b(y'-y'_\tau)]\Big)^{1/b}$
34	L^4_{40}	$\partial_y,\ -x\partial_y,\ \partial_x,\ y\partial_y$	$y'' = (y'-y'_\tau)f(\frac{y'-c(y-y_\tau)}{y'-y'_\tau})$
35	L^4_{42}	$\sin x\partial_y,\ \cos x\partial_y,\ y\partial_y,\ \partial_x$	$y'' = y\Big(f(I_{11})(c_1 + [\frac{y'}{y}(c_2 - \frac{y'_\tau}{y'})]) - 1\Big)$

No.		Lie algebra	Representation of second-order DODEs
36	L_{50}^{r+1}	$\partial_x, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y,\ r \geq 4$	$\Phi_1(x, y, y_\tau, y', y'_\tau, y'')$
37	L_{51}^{r+2}	$\partial_x, y\partial_y, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y,\ r \geq 3$	$\Phi_2(x, y, y_\tau, y', y'_\tau, y'')$
38	L_{54}^5	$\partial_x,\ x\partial_x,\ y\partial_y,\ \partial_y,\ x\partial_y$	$y'' = \dfrac{c_3(y' - y'_\tau)^2}{y}$
39	L_{55}^5	$\partial_x, \partial_y, 2x\partial_x + y\partial_y, x\partial_y, x^2\partial_x + xy\partial_y$	$y'' = \dfrac{c_4}{(y - y_\tau)^3}$

Here c, c_3, c_4, c_5 is an arbitrary constant, $k = e^c,\ k_1 = kc,\ c_1 = \sin c,\ c_2 = \cos c$.

$$I_1 = k^b y - [c_1 y'_\tau + (c_2 + bc_1)y_\tau],$$

$$I_2 = (c_2 - bc_1)[c_1 y'_\tau + (c_2 + bc_1)y_\tau] - [k^b c_1 y' + y_\tau],$$

$$I_3 = kc(y_\tau - y - y' + y'_\tau) + (k-1)(ky' - y'_\tau),$$

$$I_4 = (k^a - ak + a - 1)(ky' - y'_\tau) - a(k^a - k)[(k-1)y' - y + y_\tau],$$

$$I_5 = [k^b(c_2 - bc_1) - 1][c_1 y'_\tau - (c_2 + bc_1)(y - y_\tau)] + [k^b - (c_2 + bc_1)][k^b c_1 y' - (y - y_\tau)],$$

$$I_6 = (k^b - bck + ck - k)\Big(k(y + y') - (y_\tau - y'_\tau)\Big) - (b-1)(k^b - k)\Big(kc(y + y') - ky + y_\tau\Big),$$

$$I_7 = (k-1)(y - y_\tau - cy') + (k - c - 1)(y' - y'_\tau),$$

$$I_8 = c[k(y + y') - (y_\tau + y'_\tau)] - 2[kc(y + y') + ky - y_\tau],$$

$$I_9 = [-k^{b+1}ac - k^{b+1}b + k^{b+1}bc + k^{b+1} + ack^2 - k^2c][k^a(y + y') - (y_\tau + y'_\tau)]$$
$$-(k^b - kc)(b-1)[(k - k^a)(y_\tau + y'_\tau) - (1-a)k^a(ky - y_\tau)],$$

$$I_{10} = k^b(c_2 + (a-b)c_1) - k^a(c_1(ay_\tau y'_\tau) - [c_2 + (b-a)c_1][k^a y - y_\tau])$$
$$-[k^b + k^a(c_1(a-b) - c_2)],$$

$$I_{11} = \frac{c_2 - \frac{y_\tau}{y} + c_1\frac{y'}{y}}{c_1 + \frac{y'}{y}(c_2 - \frac{y'_\tau}{y'})},$$

$$I_{12} = [\xi_1''(\xi_2'^\tau - \xi_2') + \xi_2''(\xi_1' - \xi_1'^\tau)][(\xi_1' - \xi_1^{\tau'})(cy' - y + y_\tau) - (\xi_1'c - \xi_1 + \xi_1^\tau)(y' - y'_\tau)],$$

$$I_{13} = [c(\xi_1'\xi_2'^\tau - \xi_1'^\tau\xi_2') + (\xi_1'^\tau - \xi_1')(\xi_2 - \xi_2^\tau) + (\xi_2' - \xi_2'^\tau)(\xi_1 - \xi_1^\tau)],$$

$$\Phi_1(x, y, y_\tau, y', y'_\tau, y'') = \phi\Big(y_\tau - \sum_{i=1}^r c_{i1}y^{(i-1)}, y'_\tau - \sum_{i=1}^r c_{i2}y^{(i-1)}\Big),\ y^0 = y,\ r \geq 4,$$

such that $\quad c_{j1}\phi_{z_1} + c_{j2}\phi_{z_2} = 0, \quad j = 4, ..., r,$

$$\Phi_2(x, y, y_\tau, y', y'_\tau, y'') = y_\tau - c_5 y'_\tau + \sum_{i=1}^r (c_5 c_{i2} - c_{i1})y^{(i-1)}, \quad r \geq 3,$$

such that $\quad c_5 c_{j2} - c_{j1} = 0, \quad j = 4, \ldots, r+1.$

CHAPTER V

CONCLUSIONS

In this research, we provide a complete group classification of second-order delay ordinary differential equations of the form

$$y'' = f(x, y, y_\tau, y', y'_\tau)$$

admitting a Lie group. The method for solving this problem was developed. Results are summarized in Table 4.1.

The algorithm for obtaining second-order DODEs which admit a given Lie group is as follow. First, for each Lie algebra on the real plane, change the variables, then find invariants of the Lie algebra in the space of new variables. Last, a second-order DODE can be formed by using the found invariants.

Results of this research could be extended to higher order DODEs.

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APPENDICES

APPENDIX A

SOME MATERIAL FOR REVIEW AND REFERENCE

A.1 Definition of a Functional

Mapping. Let X and Y be sets and $A \subset X$ be any nonempty subset. A *mapping* (or *transformation*) T from A into Y is obtained by associating with each $x \in A$ a single $y \in Y$, written $y = Tx$ and called the *image of x with respect to T*.

Operator. In Calculus, the real line $\mathbb{R}$ and real-valued functions on $\mathbb{R}$ (or on a subset of $\mathbb{R}$) are usually considered. Obviously, any such function is a *mapping* of its domain into $\mathbb{R}$. Generally we consider more general spaces, such as *metric spaces*, or *normed spaces*, and mappings of these spaces.

In the case of vector spaces and in particular, normed spaces, a mapping is called an *operator*.

Functional. A *functional* is an *operator* whose range lies on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$.

A.2 Inverse Function Theorem

Inverse function theorem (Lang, 1997)**.** *Let E and F be Euclidean spaces and U be open in E. Let $x_0 \in U$, and $f : U \mapsto F$ be a $\mathcal{C}^s$ map. Assume that the derivative $f'(x_0) : E \mapsto F$ is invertible. Then f is locally $\mathcal{C}^s$-invertible at x_0. If φ is its local inverse, and $y = f(x)$, then $\varphi'(x) = f'(x)^{-1}$.*

A.3 Invariants

Invariant. A function $F(x)$ is called an *invariant* of a continuous group G of transformations (3.1) if F remains unaltered where one moves along any path curve of the group G. For example, for a one-parameter group of transformations $T_a(x)$, F is an invariant if $F\left(T_a(x)\right) = F(x)$ identically for x and a in a neighborhood of $a = 0$.

A Basis of Invariants. *A one-parameter group G of transformations in $\mathbb{R}^n$ has precisely $n-1$ functionally independent invariants. Any set of independent invariants, $\psi_1(x), \ldots, \psi_{n-1}(x)$, is termed a **basis of invariants of G.** The basis is not unique. One can obtain basic invariants, the left-hand sides of $n-1$ first integrals*

$$\psi_1(x) = C_1, \ldots, \psi_{n-1}(x) = C_{n-1},$$

*from the **characteristic system** of equations*

$$X(F) \equiv \xi^i(x)\frac{\partial F(x)}{\partial x^i} = 0,$$

i.e.

$$\frac{\mathrm{d}x^1}{\xi^1(x)} = \cdots = \frac{\mathrm{d}x^n}{\xi^n(x)}.$$

An universal invariant $F(x)$ of G is given by the formula

$$F = \Phi(\psi_1(x), \ldots, \psi_{n-1}(x)).$$

See more details and proofs in Ibragimov (1999).

A.4 Periodic Linear Systems

Consider a linear system of n first-order ODE's in the matrix form

$$x'(t) = A(t)x(t) + b(t), \tag{A.1}$$

where $b(t)$ and $x(t)$ are column vectors of length n.

Periodic Linear System. *A linear system of ODE's (A.1) is called* ***a periodic linear system*** *with the period* $\tau \neq 0$ *if*

$$A(t+\tau) = A(t), \quad b(t+\tau) = b(t), \quad \forall t.$$

Theorem. *For any fundamental matrix* $\Phi(t)$ *of a periodic linear system of ODE's with period* τ *there is a constant nonsingular matrix* C *such that*

$$\Phi(t+\tau) = \Phi(t)C.$$

Remark. *The matrix* C *is called* ***a main matrix****.*

APPENDIX B

GROUP CLASSIFICATION OF LINEAR SECOND-ORDER DELAY ORDINARY DIFFERENTIAL EQUATION

In this chapter, a linear second-order delay ordinary differential equation

$$y''(x) + a(x)y'(x) + b(x)y'(x-\tau) + c(x)y(x) + d(x)y(x-\tau) = g(x), \qquad \text{(B.1)}$$

is studied. Here $b^2 + d^2 \neq 0$ and the initial conditions are

$$\begin{aligned} y(x) &= \chi(x), \qquad x \in (x_0 - \tau, x_0), \\ y'(x_0) &= y_0. \end{aligned}$$

The initial value problem (B.1) has a solution for any arbitrary value x_0 and any arbitrary given function $\chi(x), x \in (x_0 - \tau, x_0)$ (Driver, 1977).

Equation (B.1) can be simplified. Before discussing equation (B.1), let us consider a linear second-order ordinary differential equation

$$y''(x) + a(x)y'(x) + c(x)y(x) = g(x). \qquad \text{(B.2)}$$

Let y_p be a particular solution of (B.2). By changing variables $\tilde{x} = x$ and $\tilde{y} = y - y_p$, equation (B.2) becomes

$$\tilde{y}''(x) + a(x)\tilde{y}'(x) + c(x)\tilde{y}(x) + (y_p''(x) + a(x)y_p''(x) + c(x)y_p(x) - g(x)) = 0.$$

Because y_p is a particular solution of (B.2), the equation is reduced to

$$\tilde{y}''(\tilde{x}) + a(\tilde{x})\tilde{y}'(\tilde{x}) + c(\tilde{x})\tilde{y}(\tilde{x}) = 0. \qquad \text{(B.3)}$$

Moreover, the coefficient $a(\tilde{x})$ can be reduced by the change $\tilde{y} = v(\tilde{x})q(\tilde{x})$ with $q(\tilde{x})$ satisfying the equation $2q'(\tilde{x}) + a(\tilde{x})q(\tilde{x}) = 0$. In fact, Substituting $\tilde{y} = v(\tilde{x})q(\tilde{x})$ into (B.3), one gets

$$v'' + v\rho(\tilde{x}) = 0,$$

where $\rho(\tilde{x}) = \dfrac{(q'' + aq' + cq)}{q}$.

By the above technique, the coefficients $g(x)$ and $a(x)$ in (B.1) can be reduced. Thus equation (B.1) is able to be simplified to

$$y''(x) + b(x)y'(x-\tau) + c(x)y(x) + d(x)y(x-\tau) = 0. \tag{B.4}$$

We consider group classification of linear equation (B.4).

B.1 Constructing Determining Equation

Let G be an admitted Lie group of transformations

$$\bar{x} = \varphi^x(x, y; \epsilon), \qquad \bar{y} = \varphi^y(x, y; \epsilon)$$

and

$$\xi(x,y) = \frac{\partial \varphi^x(x,y;\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \qquad \eta(x,y) = \frac{\partial \varphi^y(x,y;\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0},$$

where ϵ is a real parameter. The determining equation is

$$\tilde{X}^{(2)}\Big(y''(x) + b(x)y'(x-\tau) + c(x)y(x) + d(x)y(x-\tau)\Big)\Big|_{(B.4)} = 0, \tag{B.5}$$

where

$$\begin{aligned}
\tilde{X}^{(2)} &= \zeta^{y}\partial_y + \zeta^{y_\tau}\partial_{y_\tau} + \zeta^{y'}\partial_{y'} + \zeta^{y'_\tau}\partial_{y'_\tau} + \zeta^{y''}\partial_{y''},\\
\zeta^{y}(x,y,y') &= \eta(x,y) - y'\xi(x,y),\\
\zeta^{y_\tau}(x,y_\tau,y'_\tau) &= \zeta^{y}(x-\tau,y_\tau,y'_\tau) = \eta(x-\tau,y_\tau) - y'_\tau\xi(x-\tau,y_\tau),\\
\zeta^{y'}(x,y,y',y'') &= \eta_x(x,y) + [\eta_y(x,y) - \xi_x(x,y)]y' - \xi_y(x,y)(y')^2 - \xi(x,y)y'',\\
\zeta^{y'_\tau}(x,y_\tau,y'_\tau,y''_\tau) &= \zeta^{y'}(x-\tau,y_\tau,y'_\tau,y''_\tau) = \eta_x(x-\tau,y_\tau) + [\eta_y(x-\tau,y_\tau)\\
&\quad -\xi_x(x-\tau,y_\tau)]y'_\tau - \xi_y(x-\tau,y_\tau)(y'_\tau)^2 - \xi(x-\tau,y_\tau)y''_\tau,\\
\zeta^{y''}(x,y,y',y'',y''') &= \eta_{xx}(x,y) + [2\eta_{xy}(x,y) - \xi_{xx}(x,y)]y' + [\eta_{yy}(x,y)\\
&\quad -2\xi_{xy}(x,y)](y')^2 - \xi_{yy}(x,y)(y')^3 + [\eta_y(x,y) - 2\xi_x(x,y)]y''\\
&\quad -3\xi_y(x,y)y'y'' - \xi(x,y)y''',
\end{aligned}$$

where $y_\tau = y(x-\tau)$, $y'_\tau = y'(x-\tau)$ and $y''_\tau = y''(x-\tau)$. Substituting $y''' = -by''_\tau - y'_\tau b' - cy' - yc' - dy'_\tau - y_\tau d'$, $y''_\tau = -(b^\tau y'_{2\tau} + c^\tau y_\tau + d^\tau y'_{2\tau})$, and $y'' = -by'_\tau - cy - dy_\tau$, the determining equation (B.5) becomes

$$\begin{gathered}
-\xi_{yy}(y')^3 + [\eta_{yy} - 2\xi_{xy}](y')^2 + [2\eta_{xy} - \xi_{xx} + 3c\xi_y y]y' - \xi^\tau_{y_\tau} b(y'_\tau)^2\\
+[b'\xi - b\eta_y + b\eta^\tau_{y_\tau} + 2b\xi_x - b\xi^\tau_x + d(\xi - \xi^\tau)]y'_\tau + bb^\tau(-\xi + \xi^\tau)y'_{2\tau}\\
+bd^\tau(-\xi + \xi^\tau)y_{2\tau} + 3(b+d)\xi_y y'_\tau y' + c'\xi y + d'y_\tau\xi + \eta_{xx} - \eta_y cy - \eta_y dy_\tau\\
+\eta^\tau_x b + 2\xi_x cy + 2\xi_x dy_\tau - bc^\tau\xi y_\tau + bc^\tau\xi^\tau y_\tau + c\eta + d\eta^\tau = 0,
\end{gathered}$$

where $\xi^\tau = \xi(x-\tau,y_\tau)$, $\eta^\tau = \eta(x-\tau,y_\tau)$, $y_{2\tau} = y(x-2\tau)$, $y'_{2\tau} = y'(x-2\tau)$, $b^\tau = b(x-\tau)$, $c^\tau = c(x-\tau)$ and $d^\tau = d(x-\tau)$. Because of the arbitrariness of x_0 and $\chi(x)$, the variables y, y_τ and their derivatives can be considered as arbitrary elements. Since the determining equation is written as a polynomial of variables and their derivatives, the coefficients of these variables in the equations must vanish.

B.2 Splitting Determining Equation

Consider the coefficients of the following variables,

$$y'_{2\tau} : bb^{\tau}(-\xi + \xi^{\tau}) = 0, \tag{B.6}$$

$$(y'_{\tau})^2 : -b\xi^{\tau}_{y_\tau} = 0, \tag{B.7}$$

$$y'_{\tau} : b'\xi - b(\eta_y - \eta^{\tau}_{y_\tau}) + 2b\xi_x - b\xi^{\tau}_x + d(\xi - \xi^{\tau}) = 0, \tag{B.8}$$

$$(y')^3 : -\xi_{yy} = 0, \tag{B.9}$$

$$(y')^2 : \eta_{yy} - 2\xi_{xy} = 0, \tag{B.10}$$

$$y' : 2\eta_{xy} - \xi_{xx} + 3\xi_y(cy + dy_\tau) = 0, \tag{B.11}$$

$$\begin{aligned} 1 &: \eta_{xx} + b\eta^{\tau}_x + c\eta + d\eta^{\tau} + (d'\xi - d\eta_y + 2d\xi_x - bc^{\tau}(\xi - \xi^{\tau}))y_\tau \\ &\quad +(c'\xi - c\eta_y + 2\xi_x c)y = 0, \end{aligned} \tag{B.12}$$

$$y'y'_{\tau} : 3\xi_y(b + d) = 0, \tag{B.13}$$

$$y_{2\tau} : bd^{\tau}(-\xi + \xi^{\tau}) = 0.$$

By equation (B.6), $\xi(x, y(x)) = \xi(x - \tau, y(x - \tau))$, i.e., ξ and ξ^{τ} are functions of x which implies that ξ does not depend to y, $\xi_y = \xi^{\tau}_y = 0$. This condition and equation (B.10) imply that η is a linear function with respect to y,

$$\eta(x, y) = \beta(x)y + \gamma(x),$$

where β, γ are arbitrary functions of x. Equations (B.8) and (B.11) are simplified to

$$b(\beta - \beta^{\tau}) = b'\xi + \xi' b, \tag{B.14}$$

$$\xi'' = 2\beta', \tag{B.15}$$

respectively. Substitute ξ, η into the determining equation, and then split the equation again with respect to y and y_τ. One finds

$$\beta'' = -c'\xi - 2c\xi', \tag{B.16}$$

$$\gamma'' = -b\gamma'_\tau - c\gamma - d\gamma_\tau, \tag{B.17}$$

$$d(\beta - \beta^\tau) = d'\xi + b\beta'^\tau + 2\xi' d. \tag{B.18}$$

By integrating (B.15), one finds $\beta = \xi'/2 + C_1$, where C_1 is an arbitrary constant. Since $\xi = \xi^\tau$, it implies $\beta = \beta^\tau$. Hence, integrating equation (B.14) one has

$$b\xi = C_2, \tag{B.19}$$

where C_2 is an arbitrary constant. Equation (B.18) is written as

$$d'\xi + 2\xi' d = -\frac{b}{2}\xi''. \tag{B.20}$$

The solution of this equation depends on the values of b and d:

- **Case** $b \neq 0, d \neq 0$.

  Substituting β into equation (B.16) and integrating yields

  $$\xi\xi'' - \frac{\xi'^2}{2} + 2c\xi^2 = C_3, \tag{B.21}$$

  where C_3 is an arbitrary constant.

  If $C_2 \neq 0$, then from equations (B.19), (B.20) and (B.21), one obtains

  $$\xi = \frac{C_2}{b},\ \eta = y\left(\frac{C_2}{2}(\frac{1}{b})' + C_1\right) + \gamma,$$
  $$c = \frac{1}{2}\left[C_5 b^2 - \frac{3}{2}(\frac{b'}{b})^2 + \frac{b''}{2b}\right],\ d = \frac{b'}{2} + C_4 b^2,$$

  where C_4 is an arbitrary constants, $C_5 = C_3/C_2$, and $\gamma(x)$ is an arbitrary solution of (B.4). Since $\xi = \xi^\tau$, the the coefficient b has to satisfy the same property, i.e., $b = b^\tau$. The infinitesimal generator obtained is

  $$X = C_1 y\partial_y + C_2\left(\frac{1}{b}\partial_x + \frac{y}{2}\left(\frac{1}{b}\right)'\partial_y\right) + \gamma\partial_y. \tag{B.22}$$

If $C_2 = 0$, then $\xi = 0$, $\eta = C_1 y + \gamma$ and all coefficients are arbitrary. The infinitesimal generator is

$$X = (C_1 y + \gamma)\partial_y. \tag{B.23}$$

- **Case** $b \neq 0, d = 0$.

Solving equations (B.18), (B.19), (B.20) and (B.16), one obtains $\beta^\tau = C_6$, $b\xi = C_2$, $\xi = C_7 x + C_8$, $c\xi^2 = C_9$, where C_6, C_7, C_8, C_9 are arbitrary constants. Since $\xi = \xi^\tau$, then $C_7 = 0$.

If $C_8 \neq 0$, then

$$c = \frac{C_9}{C_8^2}, \ b = \frac{C_2}{C_8}. \tag{B.24}$$

The infinitesimal generator of the admitted Lie group is

$$X = C_8\partial_x + (C_6 y + \gamma)\partial_y. \tag{B.25}$$

If $C_8 = 0$, then $\xi = 0$, $\eta = C_6 y + \gamma$, b and c are arbitrary, $\gamma(x)$ is an arbitrary solution of (B.4). The infinitesimal generator is

$$X = (C_6 y + \gamma)\partial_y.$$

- **Case** $b = 0, d \neq 0$.

From equation (B.20), one finds $\xi^2 d = C_{10}$, where C_{10} is an arbitrary constant. Hence,

$$\xi = \left(\frac{C_{10}}{d}\right)^{1/2}, \ \eta = -\left(\frac{C_{10}}{4}\frac{d'}{d^{3/2}} + C_1\right) y + \gamma. \tag{B.26}$$

If $C_{10} \neq 0$, then equation (B.21) implies

$$c = \frac{1}{2}\left[\frac{C_3}{C_{10}} d + \frac{d'}{2d} + \frac{1}{8}(\frac{d'}{d^2})^2\right]. \tag{B.27}$$

The infinitesimal generator obtained is

$$X = \frac{C_{10}}{d^{1/2}}\partial_x + \left(-\frac{C_{10}^{1/2} d'}{2d^{3/2}} + C_1 + \gamma\right)\partial_y. \tag{B.28}$$

If $C_{10} = 0$, then $\xi = 0, \beta = C_1, \eta = C_1 y + \gamma$ and the coefficients c and d are arbitrary functions. Hence, the infinitesimal generator is

$$X = (C_1 y + \gamma)\partial_y. \tag{B.29}$$

The result for the group classification of linear second-order DODEs (B.1) is expressed as the following.

Table B.1 Lie group classification of linear second-order DODEs

No.	$b(x), d(x), c(x)$	Generators
1	$b(x) \neq 0,$	$X_1 = y\partial_y,\ X_2 = \frac{1}{b}\partial_x + \frac{y}{2}\left(\frac{1}{b}\right)'\partial_y,$
	$d(x) = \frac{b'(x)}{2} + k_1 b^2(x),$	$X_3 = \gamma\partial_y$
	$c(x) = \frac{1}{2}[k_0 b^2 - \frac{3}{2}(\frac{b'}{b})^2 + \frac{b''}{2b}]$	
2	$b(x) \neq 0, d(x) = 0, c(x) = k_2$	$X_1 = \partial_x,\ X_2 = y\partial_y,\ X_3 = \gamma\partial_y$
3	$b(x) = 0, d(x) \neq 0,$	$X_1 = \frac{1}{d^{1/2}}\partial_x,\ X_2 = -\frac{d'}{2d^{3/2}}y\partial_y,$
	$c(x) = \frac{1}{2}[kd(x) + \frac{d'(x)}{2d(x)} + \frac{1}{8}(\frac{d'(x)}{d^2(x)})^2]$	$X_3 = \gamma\partial_y$

$k, k_0, k_1, k_2, C_1, C_2, C_3$ are arbitrary constants and $\gamma(x)$ is an arbitrary solution of (B.4).

APPENDIX C

GROUP CLASSIFICATION OF THE WAVE EQUATION WITH A DELAY

In this chapter, we focus on the wave equation with a delay

$$u_{tt}(t,x) - u_{xx}(t,x) = G(u^\tau), \tag{C.1}$$

where $u^\tau = u(t-\tau, x),\ \tau > 0$, and $G' = \frac{dG}{du^\tau} \neq 0$.

C.1 Constructing Determining Equation

Let G be an admitted Lie group of transformations

$$\bar{t} = \varphi^t(t,x,u;\epsilon), \qquad \bar{x} = \varphi^x(t,x,u;\epsilon), \qquad \bar{u} = \varphi^u(t,x,u;\epsilon)$$

and

$$\xi(t,x,u) = \frac{\partial \varphi^t(t,x,u;\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \qquad \eta(t,x,u) = \frac{\partial \varphi^x(t,x,u;\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0},$$
$$\zeta(t,x,u) = \frac{\partial \varphi^u(t,x,u;\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}.$$

For equation (C.1), the determining equation is

$$\tilde{Y}^{(2)}\Big(u_{tt} - u_{xx} - G(u^\tau)\Big)\Big|_{u_{tt}=u_{xx}+G(u^\tau)} = 0,$$

where

$$\tilde{Y}^{(2)} = \zeta^u\partial_u + \zeta^{u^\tau}\partial_{u^\tau} + \zeta^{u_t}\partial_{u_t} + \zeta^{u_x}\partial_{u_x} + \zeta^{u_{tt}}\partial_{u_{tt}} + \zeta^{u_{tx}}\partial_{u_{tx}} + \zeta^{u_{xx}}\partial_{u_{xx}},$$

$$\zeta^u = \zeta - \xi u_t - \eta u_x,$$

$$\zeta^{u^\tau} = \zeta^\tau - \xi^\tau u_t^\tau - \eta^\tau u_x^\tau,$$

$$\zeta^{u_t} = -\eta_t u_x - \eta_u u_t u_x - \xi_t u_t - \xi_u (u_t)^2 + \zeta_t + \zeta_u u_t - \eta u_{xt} - u_{tt}\xi,$$

$$\zeta^{u_x} = -\eta_u (u_x)^2 - \eta_x u_x - \xi_u u_t u_x - \xi_x u_t + \zeta_x + \zeta_u u_x - \eta u_{xx} - u_{xt}\xi,$$

$$\begin{aligned}\zeta^{u_{tt}} = & -2\eta_{tu}u_t u_x - \eta_{tt}u_x - 2\eta_t u_{xt} - \eta_{uu}u_x(u_t)^2 - 2\eta_u u_t u_{xt} - \eta_u u_x u_{tt} \\ & -2\xi_{tu}(u_t)^2 - \xi_{tt}u_t - 2\xi_t u_{tt} - \xi_{uu}(u_t)^3 - 3\xi_u u_t u_{tt} + 2\zeta_{tu}u_t + \zeta_{tt} + \zeta_{uu} \\ & +\zeta_{uu}(u_t)^2 + \zeta_u u_{tt} - \eta u_{xtt} - u_{ttt}\xi,\end{aligned}$$

$$\begin{aligned}\zeta^{u_{xx}} = & -2\eta_{xu}(u_x)^2 - \eta_{uu}(u_x)^3 - 3\eta_u u_x u_{xx} - \eta_{xx}u_x - 2\eta_x u_{xx} - 2\xi_{ux}u_t u_x \\ & \xi_{uu}u_t(u_x)^2 - \xi_u u_t u_{xx} - 2\xi_u u_x u_{xt} - \xi_{xx}u_t - 2\xi_x u_{xt} + 2\zeta_{xu}u_x + \zeta_{uu}(u_x)^2 + \zeta_u u_{xx} \\ & +\zeta_{xx} - \eta u_{xxx} - u_{xxt}\xi.\end{aligned}$$

The determining equation for equation (C.1) becomes

$$\begin{gathered} -2\eta_{tu}u_t u_x - \eta_{tt}u_x - 2\eta_t u_{xt} + 2\eta_{ux}(u_x)^2 - \eta_{uu}(u_t)^2 u_x + \eta_{uu}(u_x)^3 - \eta_u G u_x \\ -2\eta_u u_t u_{xt} + 2\eta_u u_x u_{xx} + \eta_{xx}u_x + 2\eta_x u_{xx} - G'\eta u_x^\tau + G'\eta^\tau u_x^\tau - G' u_t^\tau \xi \\ +G' u_t^\tau \xi^\tau - G'\zeta^\tau - 2\xi_{tu}(u_t)^2 - \xi_{tt}u_t - 2\xi_t G - 2\xi_t u_{xx} + 2\xi_{ux}u_t u_x - \xi_{uu}(u_t)^3 \\ +\xi_{uu}u_t(u_x)^2 - 3\xi_u G u_t - 2\xi_u u_t u_{xx} + 2\xi_u u_x u_{xt} + \xi_{xx}u_t + 2\xi_x u_{xt} + 2\zeta_{tu}u_t + \zeta_{tt} \\ -2\zeta_{ux}u_x + \zeta_{uu}(u_t)^2 - \zeta_{uu}(u_x)^2 + \zeta_u G - \zeta_{xx} = 0. \end{gathered}$$

This equation is written as a polynomial of u and u^τ and their derivatives. Since all coefficients are independent from these derivatives, these coefficients are equal to zero.

C.2 Splitting Determining Equation

Splitting with respect to the derivative terms $u_t^\tau, u_x^\tau, u_{xx}, u_{xt}, u_x, \ldots$, one finds that the coefficients of the polynomial vanish

$$u_t^\tau : G_{u^\tau}(\xi^\tau - \xi) = 0, \qquad u_x^\tau : G_{u^\tau}(\eta^\tau - \eta) = 0, \tag{C.2}$$

$$u_{xx} : 2(\eta_x - \xi_t) = 0, \qquad u_{xt} : 2(\xi_x - \eta_t) = 0, \tag{C.3}$$

$$(u_x)^3 : \eta_{uu} = 0, \qquad (u_t)^3 : -\xi_{uu} = 0, \tag{C.4}$$

$$(u_x)^2 : 2\eta_{ux} - \zeta_{uu} = 0, \qquad (u_t)^2 : -2\xi_{ut} + \zeta_{uu} = 0, \tag{C.5}$$

$$(u_t)^2 u_x : -\eta_{uu} = 0, \qquad u_t(u_x)^2 : \xi_{uu} = 0, \tag{C.6}$$

$$u_t u_{xt} : -2\eta_u = 0, \qquad u_x u_{xt} : 2\xi_u = 0, \tag{C.7}$$

$$u_t u_{xx} : -2\xi_u = 0, \qquad u_x u_{xx} : 2\eta_u = 0, \tag{C.8}$$

$$u_t u_x : -2\eta_{tu} + 2\xi_{ux} = 0, \tag{C.9}$$

$$u_x : \eta_{xx} - \eta_{tt} - 2\zeta_{ux} - \eta_u G = 0, \; u_t : \xi_{xx} - \xi_{tt} + 2\zeta_{ut} - 3\xi_u G = 0, \tag{C.10}$$

$$1 : -\zeta^\tau G_{u^\tau} - 2\xi_t G + \zeta_{tt} + \zeta_u G - \zeta_{xx} = 0. \tag{C.11}$$

From (C.2), one gets

$$\xi(t, x, u) = \xi(t - \tau, x, u(t - \tau, x)),$$

$$\eta(t, x, u) = \eta(t - \tau, x, u(t - \tau, x)),$$

which imply that $\xi_u = \eta_u = 0$. Substitute these into (C.5) then

$$\zeta(t, x, u) = \zeta_1(t, x)u + \zeta_2(t, x),$$

where $\zeta_1,\ \zeta_2$ are arbitrary functions. Solving (C.3) for ξ and η, and substituting it into (C.10), one obtains

$$\eta = \eta^\tau = \eta_1(t - x) + \eta_2(t + x),$$

$$\xi = \xi^\tau = \eta_2(t + x) - \eta_1(t - x),$$

$$\zeta(t, x, u) = K_1 u + \zeta_2(t, x),$$

where K_1 is an arbitrary constant. By the virtue of $\eta = \eta^\tau$, one obtains the periodic conditions for η_1 and η_2, i.e.,

$$\eta_1(t) = \eta_1(t-\tau),\ \eta_2(t) = \eta_2(t-\tau). \tag{C.12}$$

Hence, $\zeta^\tau = K_1 u^\tau + \zeta_2^\tau(t,x)$. Substituting these functions into the determining equation, one obtains

$$[K_1 u^\tau + \zeta_2^\tau]G_{u^\tau} + [2(\eta_2' - \eta_1') - K_1]G + [\zeta_{2,xx} - \zeta_{2,tt}] = 0, \tag{C.13}$$

where $\zeta_{2,xx} = \dfrac{\partial^2 \zeta_2}{\partial x^2}$ and $\zeta_{2,tt} = \dfrac{\partial^2 \zeta_2}{\partial t^2}$.

C.2.1 The kernel of admitted Lie groups

Assume that equation (C.13) is valid for an arbitrary function G. Without loss of generality, it is possible to consider the particular case

$$G(u^\tau) = \alpha_0 + \alpha_1 u^\tau + \alpha_2 (u^\tau)^2 + \alpha_3 (u^\tau)^3,$$

where $\alpha_0, \alpha_1, \alpha_2$ and α_3 are arbitrary constants. Substituting $G(u^\tau)$ into (C.13), the third degree polynomial with respect to u^τ is obtained,

$$\begin{gathered}2\alpha_3\Big[\eta_2' - \eta_1' + K_1\Big](u^\tau)^3 + \Big[3\alpha_3\zeta_2^\tau + \alpha_2[2(\eta_2' - \eta_1') + K_1]\Big](u^\tau)^2 \\ +2\Big[\alpha_2\zeta_2^\tau + \alpha_1(\eta_2' - \eta_1')\Big]u^\tau + \Big[\alpha_1\zeta_2^\tau + \alpha_0[2(\eta_2' - \eta_1') - K_1] + \zeta_{2,xx} - \zeta_{2,tt}\Big] = 0.\end{gathered}$$

Since u^τ is arbitrary, then the coefficients of the polynomials vanish:

$$\begin{aligned}(u^\tau)^3 &: \ 2\alpha_3[\eta_2' - \eta_1' + K_1] = 0, \\ (u^\tau)^2 &: \ 3\alpha_3\zeta_2^\tau + \alpha_2[2(\eta_2' - \eta_1') + K_1] = 0, \\ u^\tau &: \ 2\big[\alpha_2\zeta_2^\tau + \alpha_1(\eta_2' - \eta_1')\big] = 0, \\ 1 &: \ \alpha_1\zeta_2^\tau + \alpha_0[2(\eta_2' - \eta_1') - K_1] + \zeta_{2,xx} - \zeta_{2,tt} = 0.\end{aligned}$$

Hence, one gets

$$K_1 = 0, \quad \zeta_2 = \zeta_2^\tau = 0,$$

$$\eta_1(t-x) = C_{11}(t-x) + C_{12},$$

$$\eta_2(t+x) = C_{11}(t+x) + C_{22},$$

which imply

$$\xi(t,x,u) = C_1, \quad \eta(t,x,u) = C_2, \quad \zeta(t,x,u) = 0,$$

where $C_1, C_2, C_{11}, C_{12}, C_{22}$, are arbitrary constants. Thus, the kernel of admitted Lie group is defined by the infinitesimal generators

$$X_1 = \partial_t, \ X_2 = \partial_x. \tag{C.14}$$

C.2.2 Extensions of the kernel

Differentiating (C.13) with respect to u^τ, one obtains

$$[K_1 u^\tau + \zeta_2^\tau]G'' + 2(\eta_2' - \eta_1')G' = 0.$$

It can be written as

$$K_1\mathcal{A} + \zeta_2^\tau\mathcal{B} + 2(\eta_2' - \eta_1')\mathcal{C} = 0, \tag{C.15}$$

$$< K_1, \zeta_2^\tau, 2(\eta_2' - \eta_1') > \cdot < \mathcal{A}, \mathcal{B}, \mathcal{C} >= 0, \tag{C.16}$$

where $\mathcal{A} = u^\tau G''$, $\mathcal{B} = G''$, $\mathcal{C} = G'$. Analysis of equation (C.15) is similar to the analysis given for gas dynamics equation by Ovsiannikov (1978).

Let us consider the vector space $\mathbb{V} = \text{span}\{< \mathcal{A}, \mathcal{B}, \mathcal{C} >\}$.

Case dim($\mathbb{V}$)=3. If dim($\mathbb{V}$)=3, then the solution of (C.15) is

$$K_1 = 0, \ \zeta_2^\tau = 0, \ \eta_2' - \eta_1' = 0. \tag{C.17}$$

These imply that

$$\eta_1(t,x) = C_{11}(t-x) + C_{12}, \tag{C.18}$$

$$\eta_2(t,x) = C_{11}(t+x) + C_{22}, \tag{C.19}$$

where C_{11}, C_{12} and C_{22} are arbitrary constants. By the virtue of $\eta = \eta^\tau$, one gets $C_{11} = 0$. Thus relations (C.17)-(C.19) lead to the kernel

$$\xi(t,x,u) = C_3,\ \eta(t,x,u) = C_4,\ \zeta(t,x,u) = 0,$$

where C_3 and C_4 are arbitrary constants.

Case dim($\mathbb{V}$)=2. In this case, there exists a nonzero constant vector $< \alpha, \beta, \gamma >$ which is orthogonal to $\mathbb{V}$, i.e.

$$\alpha\mathcal{A} + \beta\mathcal{B} + \gamma\mathcal{C} = 0.$$

The equation can be rewritten as

$$(\alpha u^\tau + \beta)z' + \gamma z = 0, \tag{C.20}$$

where $z = G'$.

Case $\alpha = 0$. The assumption $\alpha = 0$ implies that $\beta \neq 0$ and

$$z = K_0 e^{-Ku^\tau},$$

where $K_0 \neq 0, K$ are arbitrary constants. Since the integration of function z depends on K, one needs to consider two subcases : $K = 0$ and $K \neq 0$.

Case $K = 0$. For this case, the function $G(u^\tau) = K_0 u^\tau + K_2$, where K_2 is constant. This function contradicts the condition dim($\mathbb{V}$)=0.

Case $K \neq 0$. In this case

$$G(u^\tau) = -\frac{K_0}{K} e^{-Ku^\tau} + K_4,$$

where K_4 is an arbitrary constant. Substituting $G(u^\tau)$ into (C.15), then split with respect to $u^\tau e^{-Ku^\tau}$ and e^{-Ku^τ}, we find

$$K_1 = 0,$$

$$\zeta_2^\tau = \tfrac{2}{K}[\eta_2'(t+x) - \eta_1'(t-x)],$$

$$\zeta_{2,xx} - \zeta_{2,tt} + 2K_4\Big(\eta_2'(t+x) - \eta_1'(t-x)\Big) = 0.$$

These equations give

$$K_4(\eta_2' - \eta_1') = 0.$$

Since the case $K_4 \neq 0$ leads to η_1 and η_2 are constants, which does not extend the kernel of admitted Lie group, then one needs to consider $K_4 = 0$.

For $K_4 = 0$ one obtains the admitted infinitesimal generator

$$X = (\eta_2 - \eta_1)\partial_t + (\eta_1 + \eta_2)\partial_x + \frac{2}{K}\left[\eta_2' - \eta_1'\right]\partial_u.$$

Case $\alpha \neq 0$. In this case, the general solution of (C.20) is

$$G' = K_{10}(\alpha u^\tau + \beta)^{-\frac{\gamma}{\alpha}}. \tag{C.21}$$

Further the integration depends on the value of α/γ.

Assuming that $\alpha \neq \gamma$, one finds

$$G = \frac{K_{10}}{\alpha - \gamma}(\alpha u^\tau + \beta)^{1-\frac{\gamma}{\alpha}} + K_{11},$$

where K_{11} is a constant. Substituting it into (C.13) and differentiating with respect to u^τ, one finds

$$\frac{\gamma K_1 u^\tau + \gamma \zeta_2^\tau}{\alpha u^\tau + \beta} - 2(\eta_2' - \eta_1') = 0.$$

Differentiate with respect to u^τ again,

$$\gamma(\zeta_2^\tau \alpha - \beta K_1) = 0.$$

Case $\gamma = 0$. This case implies that G is linear function with respect to u^τ, which leads to dim($\mathbb{V}$)$= 0$ and contradicts to the assumption.

Case $\gamma \neq 0$. In the case $\gamma \neq 0$. After splitting the determining equation with respect to u^τ, one finds

$$\eta_2' - \eta_1' = \frac{\gamma K_1}{2\alpha}.$$

From (C.12), one obtains that $K_1 = 0$, which does not give an extensions of the kernel.

Assuming that $\alpha = \gamma$, after splitting the determining equation with respect to u^τ, one gets

$$\eta_2' - \eta_1' = \frac{K_1}{2}.$$

Similar to the previous case, this case also does not give an extension of the kernel.

Case dim($\mathbb{V}$)=1. The assumption dim($\mathbb{V}$)=1 implies the existence of nonzero constant vector (α, β, γ) such that

$$< \mathcal{A}, \mathcal{B}, \mathcal{C} >= f(u^\tau) < \alpha, \beta, \gamma >, \tag{C.22}$$

where f is an arbitrary function. Without loss of generality, one can suppose that $\gamma = 1$. Then equation (C.22) gives

$$\beta u^\tau = \alpha,$$

which means that $\alpha = \beta = 0$. Hence, G' is constant which contradicts to the condition dim($\mathbb{V}$)=1.

Case dim($\mathbb{V}$) = 0. This means that $< \mathcal{A}, \mathcal{B}, \mathcal{C} >$ is a constant vector, say $< \alpha, \beta, \gamma >$. Then

$$u^\tau G'' = \alpha,$$

$$G'' = \beta,$$

$$G' = \gamma, \quad \gamma \neq 0.$$

These equations imply that $\alpha = \beta = 0$, and G is a linear function of u^τ,

$$G(u^\tau) = K_{15}u^\tau + K_{16}.$$

Substituting $G(u^\tau)$ into (C.13), and differentiating with respect to u^τ, one gets

$$\eta_2' - \eta_1' = 0.$$

This implies that $\eta(t,x) = C_1, \xi(t,x) = C_2$, where C_1, C_2 are constants. The remaining determining equation is

$$\zeta_{2,tt} - \zeta_{2,xx} = K_0\zeta_2^\tau - K_1K_{16}. \tag{C.23}$$

Hence, the infinitesimal generator is

$$X = C_1\partial_t + C_2\partial_x + [K_1u + \zeta_2(t,x)]\partial_u, \tag{C.24}$$

where $\zeta_2(t,x)$ is an arbitrary solution of (C.23).

The results for the previous calculations are presented in the following table.

Table C.1 Lie group classification of the wave equation with a delay

No.	$G(u^\tau)$	Generator
1	$G(u^\tau)$ is arbitrary	$X = c_1\partial_t + c_2\partial_x$
2	$G(u^\tau) = k_0u^\tau + k_1$	$X = c_1\partial_t + c_2\partial_x + (ku + \zeta_2(t,x))\partial_u$
3	$G(u^\tau) = k_0e^{ku^\tau}$	$X = (\eta_2 - \eta_1)\partial_t + (\eta_1 + \eta_2)\partial_x + \frac{2}{k}(\eta_2' - \eta_1')\partial_u$

Here $c_1, c_2, k \neq 0, k_0 \neq 0, k_1, \eta_1(t-x), \eta_2(t+x)$ are arbitrary and $\zeta_2(t,x)$ is an arbitrary solution of $\zeta_{tt} - \zeta_{xx} = k_0\zeta^\tau - kk_1$.

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