Invariants of linear parabolic differential equations

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Abstract

The paper is dedicated to construction of invariants for the parabolic equation

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0.$$

We consider the equivalence group given by point transformations and find all invariants up to seventh-order, i.e. the invariants involving the derivatives up to seventh-order of the coefficients a, b and c with respect to the independent variables t, x.

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1. Introduction

We consider the standard linear second-order parabolic partial differential equations in two independent variables:

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0, \quad a(t,x) \neq 0. \tag{1}$$

Recall that the well-known group of equivalence transformations for Eq. (1) (given in [1]), i.e. the changes of variables t, x and u that do not change the form of Eq. (1), is composed of the linear transformation of the dependent variable,

$$\bar{u} = \sigma(t,x)u, \tag{2}$$

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and the following change of the independent variables:

$$\bar{t} = \phi(t), \qquad \bar{x} = \psi(t,x), \tag{3}$$

where $\sigma(t,x)$, $\phi(t)$ and $\psi(t,x)$ are arbitrary functions obeying the invertibility conditions, $\sigma(t,x) \neq 0$, $\phi'(t) \neq 0$ and $\psi_x(t,x) \neq 0$. The form invariance of Eq. (1) means that the transformations (2), (3) map Eq. (1) into an equation of the same form:

$$\bar{u}_{\bar{t}} + \bar{a}(\bar{t},\bar{x})\bar{u}_{\bar{x}\bar{x}} + \bar{b}(\bar{t},\bar{x})\bar{u}_{\bar{x}} + \bar{c}(\bar{t},\bar{x})\bar{u} = 0. \tag{1'}$$

Eqs. (1) and (1′) connected by an equivalence transformation are called *equivalent equations.*

An *invariant of* Eq. (1) is a function

$$J(a,b,c,a_t,a_x,b_t,b_x,c_t,c_x,a_{tt},a_{tx},a_{xx},\ldots,c_{xx},\ldots)$$

that remains unaltered under the equivalence transformations (2) and (3). It means that J has the same value for equivalent Eqs. (1) and (1′):

$$J(a,b,c,a_t,\ldots,c_{xx},\ldots) = J(\bar{a},\bar{b},\bar{c},\bar{a}_{\bar{t}},\ldots,\bar{c}_{\bar{x}\bar{x}},\ldots).$$

If J is invariant only under the transformation (2) it is termed a *semi-invariant* [2]. The *order* of an invariant (or semi-invariant) J is identified with the highest order of derivatives of a, b, c involved in J.

Semi-invariants of hyperbolic equations (termed the Laplace invariants) have been known since the 1770s. Recently there have been considerable interest in invariants of parabolic equations. The first step toward solving the problem of invariants for parabolic equations was made in [2] where the semi-invariant of the second-order

$$K = 2c_x a^2 - b_t a - b_{xx}a^2 - b_x ba + b_x a_x a + \frac{1}{2}b^2 a_x + ba_t + ba_{xx}a - ba_x^2 \tag{4}$$

was found. It was also shown there that K and the coefficient $a(t,x)$ provide a basis of semi-invariants. This solves the problem of semi-invariants. Namely, any semi-invariant J of an arbitrary order involves only a and K together with their derivatives of an appropriate order, i.e.

$$J = J(a,a_t,a_x,a_{tt},a_{tx},a_{xx},\ldots,K,K_t,K_x,K_{tt},K_{tx},K_{xx},\ldots). \tag{5}$$

Furthermore, it follows from this result that the invariants of Eq. (1) with respect to the general equivalence group can be obtained by subjecting the functions (5) to the condition of invariance under the change (3) of the independent variables.

The method and result of [2] were used in [3] for investigating invariants and invariant equations up to fifth-order with respect to the joint transformations (2) and (3). It has been shown in [3] that Eq. (1) has no invariants up to fifth-order and that it has precisely one invariant equation of the fifth-order, namely the equation

$$\lambda = 0. \tag{6}$$

The quantity λ is defined by

$$\begin{aligned}\lambda = {} & 4a(2aK_{xx} - 5a_xK_x) - 12K(aa_{xx} - 2a_x^2) + a_x(4aa_{tt} - 9a_x^4) - 12a_ta_x(a_t + 2a_x^2) \\ & + 4a(3a_t + 6a_x^2 - 5aa_{xx})a_{tx} + 2aa_x(16a_ta_{xx} - 12aa_{xx}^2 + 15a_x^2a_{xx}) - 4a^2a_{ttx} - 12a^2a_xa_{txx} \\ & - 4a^2a_{xxx}(2a_t - 4aa_{xx} + 3a_x^2) + 8a^3a_{txxx} - 4a^4a_{xxxxx}.\end{aligned} \tag{7}$$

and is termed a *relative invariant* due to the invariance of Eq. (6) with respect to the equivalence transformations (2) and (3). It is demonstrated in [3] that Eq. (6) provides a necessary and sufficient condition for Eq. (1) to be equivalent to the heat equation.

In the present paper, we find all invariants and invariant equations of the sixth- and seventh-orders. Since $\lambda = 0$ singles out the heat equation and all equations equivalent to the heat equation, we exclude these equations and assume in what follows that $\lambda = 0$. Under this assumption, we prove the following result.

Theorem. *An arbitrary Eq. (1) with* $\lambda = 0$ *has one invariant of the sixth-order*:

$$\Lambda_1 = \frac{2a\lambda_x - 5\lambda a_x}{\lambda^{6/5}}, \tag{8}$$

and one invariant of the seventh-order:

$$\Lambda_2 = \frac{2a^2\lambda_{xx} - 9aa_x\lambda_x + 5(3a_x^2 - aa_{xx})\lambda}{\lambda^{7/5}}. \tag{9}$$

Furthermore, there are additional invariants of the seventh-order in the following particular cases.

(A) *The family of Eq. (1) obeying the invariant conditions*

$$5\Lambda_2 - 3\Lambda_1^2 = 0, \quad \Lambda_1 \neq 0 \tag{10}$$

has the invariant

$$\Lambda_3 = \frac{a}{\lambda^{8/5}}\left[a_x\lambda_t + 2a_t\lambda_x - \frac{12}{5\lambda}a\lambda_t\lambda_x + 2a\lambda_{tx} - 5\lambda a_{tx}\right]. \tag{11}$$

(B) *The family of Eq. (1) defined by two invariant equations*

$$\Lambda_1 = 0, \qquad \Lambda_2 = 0 \tag{12}$$

has the invariant

$$\begin{aligned}\Lambda_4 = \frac{1}{4\lambda^{9/5}}&\left[10\lambda a^2(3a_xa_{xxx} - 2aa_{xxxx} + 3a_{xx}^2 - 4a_{txx}) + 5\lambda a(8a_ta_{xx} - 8a_{tt} + 16a_xa_{tx} - 15a_x^2a_{xx} - 8K_x)\right.\\ &+ 2\lambda(50a_t^2 - 4a_ta_x^2 + 15a_x^4 + 40a_xK) + a\lambda_x(8a_xa_t + 6aa_xa_{xx} - 4a^2a_{xxx} - 8aa_{tx} - 3a_x^3 - 8K)\\ &\left. - 40aa_t\lambda_t + 8a^2\lambda_{tt}\right] - \frac{3}{5\lambda^{14/5}}(2a\lambda_t - 5\lambda a_t)^2.\end{aligned} \tag{13}$$

(C) *The family of Eq. (1) obeying the invariant conditions*

$$\Lambda_1 = 0, \qquad \Lambda_2 \neq 0 \tag{14}$$

has the invariant

$$\Lambda_5 = 4\Lambda_2\Lambda_4 - 3\Lambda_3^2. \tag{15}$$

2. Generalities

2.1. Equivalence group

For obtaining invariants we use the Lie approach. This approach consists of finding an equivalence group of point transformations, and finding its invariants by solving a system of homogeneous linear equations. Let us recall the method for obtaining an equivalence group. Consider a parabolic equation (1). Since the functions a, b, c depend on the independent variables t, x only, the equivalence group should leave invariant the equations

$$a_u = 0, \qquad b_u = 0, \qquad c_u = 0. \tag{16}$$

Let the generator of a one-parameter equivalence group be

$$X^e = \xi^t\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \zeta^u\frac{\partial}{\partial u} + \zeta^a\frac{\partial}{\partial a} + \zeta^b\frac{\partial}{\partial b} + \zeta^c\frac{\partial}{\partial c}, \tag{17}$$

where the coefficients $\xi^t, \ldots, \zeta^c$ may depend, in general, depend on the variables t, x, u, a, b, c. The coefficients of the prolonged operator

$$\widetilde{X}^e = X^e + \zeta^{u_t}\frac{\partial}{\partial u_t} + \zeta^{u_x}\frac{\partial}{\partial u_x} + \zeta^{u_{xx}}\frac{\partial}{\partial u_{xx}} + \zeta^{a_u}\frac{\partial}{\partial a_u} + \zeta^{b_u}\frac{\partial}{\partial b_u} + \zeta^{c_u}\frac{\partial}{\partial c_u}$$

are defined by the prolongation formulae

$$\zeta^{u_t} = D^e_t\zeta^u - u_t D^e_t\xi^t - u_x D^e_t\xi^x, \qquad \zeta^{u_x} = D^e_x\zeta^u - u_t D^e_x\xi^t - u_x D^e_x\xi^x,$$
$$\zeta^{u_{xx}} = D^e_x\zeta^{u_x} - u_{xt} D^e_x\xi^t - u_{xx} D^e_x\xi^x, \qquad \zeta^{a_u} = D_u\zeta^a - a_t D_u\xi^t - a_x D_u\xi^x,$$
$$\zeta^{b_u} = D_u\zeta^b - b_t D_u\xi^t - b_x D_u\xi^x, \qquad \zeta^{c_u} = D_u\zeta^b - c_t D_u\xi^t - c_x D_u\xi^x.$$

Here the operators D^e_t, D^e_x are operators of the total derivatives with respect to t and x, respectively, where the space of the independent variables consists of t and x,

$$D^e_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + (a_t + u_t a_u)\frac{\partial}{\partial a} + (b_t + u_t b_u)\frac{\partial}{\partial b} + (c_t + u_t c_u)\frac{\partial}{\partial c} + \cdots,$$
$$D^e_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + (a_x + u_x a_u)\frac{\partial}{\partial a} + (b_x + u_x b_u)\frac{\partial}{\partial b} + (c_x + u_x c_u)\frac{\partial}{\partial c} + \cdots$$

The operators D_t, D_x and D_u are operators of total derivatives with respect to t, x and u, where the space of the independent variables consists of t, x, and u,

$$D_t = \frac{\partial}{\partial t} + a_t\frac{\partial}{\partial a} + b_t\frac{\partial}{\partial b} + c_t\frac{\partial}{\partial c} + \cdots,$$
$$D_x = \frac{\partial}{\partial x} + a_x\frac{\partial}{\partial a} + b_x\frac{\partial}{\partial b} + c_x\frac{\partial}{\partial c} + \cdots,$$
$$D_u = \frac{\partial}{\partial u} + a_u\frac{\partial}{\partial a} + b_u\frac{\partial}{\partial b} + c_u\frac{\partial}{\partial c} + \cdots$$

Because of (16) and the definitions of $\zeta^{a_u}, \zeta^{b_u}, \zeta^{c_u}$, one can split the part of determining equations

$$\zeta^{a_u} = 0, \qquad \zeta^{b_u} = 0, \qquad \zeta^{c_u} = 0$$

with respect to a_t, a_x, b_t, b_x, c_t, c_x. Consequently the coefficients ξ^t, ξ^x, ζ^a, ζ^b and ζ^c do not depend on u.

Solving the determining equations

$$\widetilde{X}^e F|_{(1),(2)} = 0,$$

one finds

$$\xi^t = p, \quad \xi^x = q, \quad \zeta^u = u\sigma(t,x), \quad \zeta^a = 2aq_t - ap_t, \quad \zeta^b = aq_{xx} + bq_x - bp_t + q_t - 2a\sigma_x,$$
$$\zeta^c = -cp_t - \sigma_t - a\sigma_{xx} - b\sigma_x,$$

with arbitrary functions $p = p(t)$, $q = q(t,x)$, $\sigma = \sigma(t,x)$. Hence, we arrive at the following generator of the equivalence group:

$$X^e = p\frac{\partial}{\partial t} + q\frac{\partial}{\partial x} + u\sigma\frac{\partial}{\partial u} + a(2q_x - p_t)\frac{\partial}{\partial a} + (aq_{xx} + bq_x - bp_t + q_t - 2a\sigma_x)\frac{\partial}{\partial b} - (cp_t + \sigma_t + a\sigma_{xx} + b\sigma_x)\frac{\partial}{\partial c}. \tag{18}$$

This manuscript is devoted to constructing differential invariants of the equivalence group. For obtaining nth-order invariants we use the infinitesimal test

$$\widetilde{X}^e(J) = 0,$$

where J depends on a, b, c and their derivatives up to order n. Notice that for relative invariants the infinitesimal test is

$$\widetilde{X}^e(J_k)|_S = 0, \quad k = 1, \ldots, s,$$

where S is a manifold defined by equations $J_k = 0$, $k = 1, \ldots, s$.

2.2. Semi-invariants and the representation of invariants

Recall that the generator for finding semi-invariants is (see [2])

$$X^e = u\sigma\frac{\partial}{\partial u} - (2a\sigma_x)\frac{\partial}{\partial b} - (\sigma_t + a\sigma_{xx} + b\sigma_x)\frac{\partial}{\partial c}, \tag{19}$$

and that Eq. (1) has the following semi-invariants up to the second-order (see Section 1)

$$a, \quad a_t, \quad a_x, \quad a_{tt}, \quad a_{tx}, \quad a_{xx}, \quad K,$$

where K is given by Eq. (4):

$$K = 2c_x a^2 - b_t a - b_{xx}a^2 - b_x ba + b_x a_x a + \frac{1}{2}b^2 a_x + ba_t + ba_{xx}a - ba_x^2.$$

Furthermore, the invariants of the equivalence group defined by the generator (18) are in the class of functions J of the form (4) involving, in general, the derivatives of a up to the order n, and derivatives of the function $K(t,x)$ are up to the order $n-2$. Accordingly, the generator (18) is rewritten in the form

$$X^e = a(2q_x - p_t)\frac{\partial}{\partial a} + \zeta^K\frac{\partial}{\partial K},$$

where

$$\zeta^K = q_{tx}aa_x - q_{xxxx}a^3 - q_{xxx}a_x a^2 - 2q_{txx}a^2 + 3q_x K - q_{tt}a + q_t(a_t + a_{xx}a - a_x^2) - 3p_t K.$$

The coefficients of the prolonged operator

$$\widetilde{X}^e = X^e + \zeta^{a_t}\frac{\partial}{\partial a_t} + \zeta^{a_x}\frac{\partial}{\partial a_x} + \cdots + \zeta^{K_t}\frac{\partial}{\partial K_t} + \zeta^{K_x}\frac{\partial}{\partial K_x} + \cdots \tag{20}$$

are defined by the prolongation formulae, e.g.

$$\zeta^{a_t} = D_t\zeta^a - a_t D_t\xi^t - a_x D_t\xi^x, \qquad \zeta^{a_x} = D_x\zeta^a - a_t D_x\xi^t - a_x D_x\xi^x,$$
$$\zeta^{K_t} = D_t\zeta^K - K_t D_t\xi^t - K_x D_t\xi^x, \qquad \zeta^{K_x} = D_x\zeta^K - K_t D_x\xi^t - K_x D_x\xi^x,$$

where

$$D_t = \frac{\partial}{\partial t} + a_t\frac{\partial}{\partial a} + K_t\partial_K + \cdots, \qquad D_x = \frac{\partial}{\partial x} + a_x\frac{\partial}{\partial a} + K_x\partial_K + \cdots$$

For finding invariants one has to apply the following procedure. Let us consider an invariant of order n, where it is assumed that J depends on the variable a, its derivatives up to nth-order, the function K and its derivatives up to $(n-2)$ order. Invariants can be obtained by solving the equations

$$\widetilde{X}^e(J) = 0,$$

and relative invariants by solving the equations

$$\widetilde{X}^e(J_k)|_S = 0.$$

3. Sixth-order invariants

This section is devoted to finding sixth-order differential invariants. Let

$$J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}, \ldots, a_{xxxxxx}, K, K_t, K_x, K_{tt}, K_{tx}, K_{xx}, \ldots, K_{xxxx})$$

be a sixth-order differential invariant.

The prolonged operator $\widetilde{X}^e$ is defined by (20). Splitting the equations $\widetilde{X}^e(J) = 0$ with respect to p, q and its derivatives, one obtains a system of 43 linear homogeneous equations. Some of these equations are of the following two types. The first type is

$$J_x + \sum_{i=1}^{n} a_i J_{y_i} = 0, \tag{21}$$

where $J = J(x, y_1, y_2, \ldots, y_n)$, and the coefficients a_i $(i = 1, 2, \ldots, n)$ are linear functions of the independent variables $y_1, y_2, \ldots, y_{i-1}$ which have the form

$$a_i = \sum_{k=1}^{i-1} \beta_{i,k}(x) y_k + \gamma_i(x).$$

The characteristic system for Eq. (21) is

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y_1}{\gamma_1(x)} = \frac{\mathrm{d}y_2}{\beta_{2,1}(x)y_1 + \gamma_2(x)} = \frac{\mathrm{d}y_3}{\beta_{3,1}(x)y_1 + \beta_{3,2}(x)y_2 + \gamma_3(x)} = \cdots$$

From the characteristic system one can obtain the general solution of (21).

The second type of equations is

$$xJ_x + \sum_{i=1}^{n} k_i y_i J_{y_i} = 0, \tag{22}$$

where k_i $(i = 1, 2, \ldots, n)$ are constant. The general solution of (22) is

$$J = J(J_1, J_2, \ldots, J_n), \quad \text{where } J_i = \frac{y_i}{x^{k_i}}, \ (i = 1, 2, \ldots, n).$$

The calculations for obtaining the system of equations for finding invariants and solving its equations are cumbersome. For these calculations we therefore used the Reduce programs developed for solving the linearization problem of third-order ordinary differential equation [5].

After solving the equations of the first and second types the system is reduced to the following system of equations

$$\frac{\partial J}{\partial J_2} J_1 = 0, \qquad 6\frac{\partial J}{\partial J_3} J_3 + 5\frac{\partial J}{\partial J_1} J_1 + 7\frac{\partial J}{\partial J_2} J_2 = 0, \qquad \frac{\partial J}{\partial J_2} J_3 = 0, \tag{23}$$

where

$$J_1 = \lambda/(8a^5), \qquad J_2 = \left(2\frac{\partial \lambda}{\partial t} a - 5\frac{\partial a}{\partial t}\lambda\right) \Big/ (16a^7), \qquad J_3 = \left(2\frac{\partial \lambda}{\partial x} a - 5\frac{\partial a}{\partial x}\lambda\right) \Big/ (8a^6),$$

and $J = (J_1, J_2, J_3)$.

If $\lambda = 0$, then $J_1 = J_2 = J_3 = 0$. This case was studied in [3]. If $\lambda \neq 0$, then $J_1 \neq 0$. Because of the first equation of (23), J does not depend on J_2. Solving the second equation of (23), one obtains the only invariant J_3^5/J_1^6. This invariant was also obtained in [4] as an invariant with respect to contact transformations.

4. Seventh-order invariants

Similar to the previous section the system for finding invariants of seventh-order is reduced to the following equations

$$24\frac{\partial J}{\partial J_6} J_2 + 6\frac{\partial J}{\partial J_4} J_3 + 5\frac{\partial J}{\partial J_2} J_1 = 0, \tag{24}$$

$$9\frac{\partial J}{\partial J_6} J_6 + 7\frac{\partial J}{\partial J_5} J_5 + 6\frac{\partial J}{\partial J_3} J_3 + 8\frac{\partial J}{\partial J_4} J_4 + 5\frac{\partial J}{\partial J_1} J_1 + 7\frac{\partial J}{\partial J_2} J_2 = 0, \tag{25}$$

$$3\frac{\partial J}{\partial J_6} J_4 + 2\frac{\partial J}{\partial J_4} J_5 + \frac{\partial J}{\partial J_2} J_3 = 0, \tag{26}$$

where

$$J_4 = (-5\lambda_t a_x a - 4\lambda_x a_t a - 5a_{tx}a\lambda + 15a_t a_x \lambda + 2\lambda_{tx}a^2)/(8a^8),$$
$$J_5 = (-9\lambda_x a_x a - 5a_{xx}a\lambda + 15a_x^2\lambda + 2\lambda_{xx}a^2)/(8a^7),$$
$$\begin{aligned}J_6 = {} & (-40a_{txx}a^2\lambda - 4a_{xxx}\lambda_x a^3 + 30a_{xxx}a_x a^2\lambda - 40\lambda_t a_t a \\ & + \lambda_x a(-8a_{tx}a + 8a_t a_x + 6a_{xx}a_x a - 3a_x^3 - 8K) + 80a_{tx}a_x a\lambda \\ & - 40a_{tt}a\lambda + 100a_t^2\lambda + 40a_t a_{xx}a\lambda - 80a_t a_x^2\lambda - 20a_{xxxx}a^3\lambda \\ & + 30a_{xx}^2a^2\lambda - 75a_{xx}a_x^2 a\lambda + 30a_x^4\lambda + 80a_x K\lambda - 40K_x a\lambda + 8\lambda_{tt}a^2)/(32a^9).\end{aligned}$$

Taking the Poisson bracket of Eqs. (24) and (26), one obtains the equation

$$\frac{\partial J}{\partial J_6}J_3 = 0. \tag{27}$$

Assuming $\lambda \neq 0$, Eqs. (24) and (25) can be solved. The remaining Eqs. (26) and (27) are reduced to the equations

$$2\frac{\partial J}{\partial J_{10}}(5J_8 - 3J_7^2) + 15\frac{\partial J}{\partial J_9}J_{10} = 0, \tag{28}$$
$$\frac{\partial J}{\partial J_9}J_7 = 0, \tag{29}$$

where $J = J(J_7, J_8, J_9, J_{10})$, and

$$J_7 = \frac{J_3}{J_1^{6/5}}, \quad J_8 = \frac{J_5}{J_1^{7/5}}, \quad J_9 = \frac{5J_1J_6 - 12J_2^2}{5J_1^{14/5}}, \quad J_{10} = \frac{5J_1J_4 - 6J_2J_3}{5J_1^{13/5}}.$$

Since Eqs. (28) and (29) contain no derivatives with respect to J_7 and J_8, the variables J_7 and J_8 are invariants.
If $J_7 \neq 0$, then J does not depend on J_9, and Eq. (28) becomes

$$2\frac{\partial J}{\partial J_{10}}(5J_8 - 3J_7^2) = 0.$$

This equation shows that there is the additional invariant J_{10} which is obtained for $(5J_8 - 3J_7^2) = 0$.
If $J_7 = 0$, then one needs only to solve Eq. (28) which becomes

$$10\frac{\partial J}{\partial J_{10}}J_8 + 15\frac{\partial J}{\partial J_9}J_{10} = 0.$$

If $J_8 \neq 0$, this equation yields the invariant

$$J_{11} = J_9 - \frac{3}{4}\frac{J_{10}^2}{J_8}.$$

The assumption $J_8 = 0$ leads to the analysis of the equation

$$J_{10}\frac{\partial J}{\partial J_9} = 0.$$

If $J_{10} = 0$ then one only obtains the invariant J_9.

Conditions			Additional invariant
$J_7 \neq 0$	$5J_8 - 3J_7^2 \neq 0$		No
	$5J_8 - 3J_7^2 = 0$		J_{10}
$J_7 = 0$	$J_8 \neq 0$		J_{11}
	$J_8 = 0$	$J_{10} \neq 0$	No
		$J_{10} = 0$	J_9

Remark. The invariants J_7, J_8, J_{10} and J_{11} are equal, up to immaterial constant factors, to the invariants (8), (9), (11),..., respectively. Namely:

$$J_7 = 8^{1/5}\Lambda_1, \quad J_8 = 8^{2/5}\Lambda_2, \quad J_{10} = 8^{3/5}\Lambda_3, \quad J_9 = 8^{4/5}\Lambda_4, \quad J_{11} = 8^{4/5}\frac{\Lambda_5}{4\Lambda_2}.$$

5. Conclusion

This paper is devoted to finding sixth and seventh-order differential invariants of linear second-order parabolic partial differential equation (1) under an action of the equivalence group of point transformations. We found one sixth-order differential invariant J_3^5/J_1^6. Seventh-order invariants are J_7, J_8. Other functions J_9, J_{10} and J_{11} are invariants for particular cases. We have also found invariants of eighth and ninth-order, but the result is too cumbersome, and it is not presented in the paper.

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