APPLICATION OF THE INTERMEDIATE INTEGRAL AND DIFFERENTIAL CONSTRAINT TECHNIQUES TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN TWO VARIABLES: MONGE-AMPERE, BENJAMIN-BONA-MAHONY AND KORTEWEG DE VRIES EQUATIONS

Miss Sommai Sungngoen

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Applied Mathematics

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การประยุกต์เทคนิคอินทิกรัลระหว่างกลางและเทคนิคเงื่อนไขบังคับเชิงอนุพันธ์กับสมการอนุพันธ์ย่อยไม่เชิงเส้นในสองตัวแปรในกรณีของสมการมอนจ์-แอมแปร์ สมการเบนจามิน-โบนา-มะโฮนี และสมการคอร์เดเวก เดอร์ วรีย์

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SOMMAI SUNGNGOEN: APPLICATION OF THE INTERMEDIATE INTEGRAL AND DIFFERENTIAL CONSTRAINT TECHNIQUES TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN TWO VARIABLES: THE MONGE-AMPERE, BENJAMIN-BONA-MAHONY AND KORTEWEG DE VRIES EQUATIONS THESIS ADVISOR: ASST. PROF. ARJUNA PETER CHAIYASENA, Ph. D.

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INTERMEDIATE INTEGRAL/DIFFERENTIAL CONSTRAINTS

This thesis is devoted to applying the intermediate integrals technique and the method of differential constraints to some partial differential equations, in particular, the Monge-Ampere, Benjamin-Bona-Mahony and Korteweg de Vries equations. It is discovered that the intermediate integral exists only for the Monge-Ampere equation. However, the other equations can be solved by the method of differential constraints.

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Sommai Sungngoen

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Introduction

One method for constructing exact solutions of the partial differential equations is the method of differential constraints proposed by Yanenko in 1964. The main idea of the method is at the foundation of group analysis and degenerated hodograph. With differential constraints, the initial system becomes an overdetermined system. Overdetermined systems of partial differential equations are systems in which the number of independent equations is greater than the number of the unknown functions. Finding a solutions of overdetermined system can be easier than finding a solution of the initial system of partial differential equations.

This research aims to find solutions of nonlinear partial differential equations in two variables by application of the intermediate integral technique and the method of differential constraints, through three major cases: the Monge-Ampere, Korteweg de Vries (KdV) and Benjamin-Bona-Mahony (BBM) equations, being nonlinear equations of second and third order respectively with two independent variables.

The intermediate integral technique is a special case of differential constraints, by assuming differential constraints with order less than order of original partial differential equations. Using this technique to find a solution of original partial differential equation means finding a solution of a lower order partial differential equation that may be easier to find.

1.1 Partial Differential Equations

A *partial differential equation*

$$G\left(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_1x_1}, u_{x_1x_2}, ...\right) = 0 \tag{1.1}$$

involves a function u of independent variables $x_1, ..., x_n,$ its partial derivatives and the *independent variables* $x_1, ..., x_n$. A partial differential equation is called *linear* if it is linear with respect to u and its partial derivatives. A partial differential equation is *quasilinear* if it is linear with respect to the highest order partial derivatives appearing in the equation. A partial differential equation is called *nonlinear* if it is not linear. For example, $a(x,t)u_x + b(x,t)u_t + c(x,t)u = d(x,t)$, where $a(x,t),\ b(x,t),\ c(x,t)$ and $d(x,t)$ are known functions and u is unknown function of the independent variables x and t, is a general form of a linear first order partial differential equation. The equation $u_t + uu_x = 0$ has the nonlinear term uu_x that makes it a quasilinear equation. The equation $u_t - uu_x + u_{xxx} = 0$ is also a quasilinear equation of third order. The equation ${u_x}^2 + {u_t}^2 = 1$ is an example of a nonlinear equation.

A function $u = \varphi(x_1, x_2, ..., x_n)$ that satisfies a given partial differential equation is called a *solution* of the partial differential equation. Obtaining such a solution for a given partial differential equation is called *solving* this equation, and the *integral hypersurface* of the equation is $u - \varphi(x_1, x_2, ..., x_n) = 0$. The general solution of an ordinary differential equation is known to be expressed through arbitrary constants. By using boundary conditions, a particular solution is then obtained. But for any partial differential equation, its general solution is expressed through arbitrary functions and for boundary conditions must be chosen that solution exits and unique. In this thesis, we study a second order partial differential equation with two independent variables, namely the Monge-

Ampere equation,

$$u_{xt}^2 - u_{xx}u_{tt} = a(x,t), \qquad (x,t) \in D; \tag{1.2}$$

and two third order partial differential equations in two independent variables, namely the Benjamin-Bona-Mahony (BBM) equation,

$$u_t = uu_x + u_{txx}, \qquad (x,t) \in D; \tag{1.3}$$

and the Korteweg de Vries (KdV) equation,

$$u_t = 6uu_x - u_{xxx}, \qquad (x,t) \in D; \tag{1.4}$$

where the independent variables x and t lie in some given domain D in R^2.

In 1779 and 1785, Lagrange studied the equation

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} + R = 0, \tag{1.5}$$

where P_i and R are functions of $x = (x_1, .., x_n)$ and u. He showed that this equation may be reduced to a system of ordinary differential equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{du}{R}. \tag{1.6}$$

The geometrical theory was studied by Monge, whose research began in 1770. He introduced the concepts of *characteristic curves* and *characteristic cone.* The characteristic curve is determined by a solution of equations (1.6), which corresponds to equation (1.5), and is defined as the curve in the xu-space. The solution can then be obtained from this characteristic curves.

Quasilinear Partial Differential Equations and Their Characteristic Curves

Suppose that the following quasilinear partial differential equation is given:

$$\sum_{i=1}^{n} p_i(x,u) \frac{\partial u}{\partial x_i} = Q(x,u), \qquad x = (x_1, ..., x_n).$$

Its characteristic curve is defined by a solution $x_i = x_i(t), u = u(t)$ of the system of ordinary differential equations

$$\frac{dx_i}{dt} = p_i(x,u), \qquad i = 1, ..., n; \quad \frac{du}{dt} = Q(x,u).$$

The characteristic curve passing through any point on the hypersurface $u = u(x)$ in $n+1$ dimensional xu-space and contained in the hypersurface is necessary and sufficient for $u = u(x)$ to be a solution of quasilinear equation.

Nonlinear Partial Differential Equations and Their Characteristic Strips

Consider the partial differential equation

$$F(x_1, ..., x_n, u, p_1, ..., p_n) = 0, \quad p_i = \frac{\partial u}{\partial x_i},$$

with the surface element (or hypersurface element) defines by the $(2n+1)$-dimensional vector $(x_1, ..., x_n, u, p_1, ..., p_n)$ such that a set $(x_1(t), ..., x_n(t), u(t), p_1(t), ..., p_n(t))$ of surface elements depends on a parameter t and satisfies the system of ordinary differential equations

$$\frac{dx_i}{dt} = F_{p_i}, \quad \frac{du}{dt} = \sum_{i=1}^{n} p_i F_{p_i}, \quad \frac{dp_i}{dt} = -(F_{x_i} + p_i F_u).$$

The last system is called a *characteristic strip* of the previous equation with characteristic curve $x_1(t), ..., x_n(t)$ and $u(t)$. In general, a solution of the partial differential equation can be obtained from characteristic strip, as initial values and the surface elements belonging to an $(n-1)$-dimensional, union the surface elements that satisfy $F(x_1, ..., x_n, u, p_1, ..., p_n) = 0$.

1.2 Cauchy Problem

One of the fundamental problems in the theory of partial differential equations is to find a solution (an *integral*) of a differential equation that satisfies initial conditions (*initial data*) specified for $t = 0$. The solution is required for $t \geq 0$. This problem with the initial data is called a *Cauchy problem*: it differs from boundary value problems in that the domain in which the desired solution must be defined is not specified in advance. However, Cauchy problems, like boundary value problems, are defined by imposing a limiting condition for the solution on (part of) the boundary of the domain of definition. The simplest Cauchy problem is to find a function $u(x)$ defined on the half-line $x \geq x_0$, satisfying a first order ordinary differential equation

$$\frac{du}{dx} = f(x, u), \tag{1.7}$$

where f is a given function and taking a specified value u_0 at x_0:

$$u(x_0) = u_0. \tag{1.8}$$

In geometrical terms, this means that, considering the family of integral curves of equation (1.7) in the (x, u) plane, one wishes to find the curve passing through the point (x_0, u_0). The existence of such a function (on the assumption that f is continuous for all x and continuously differentiable with respect to u) was proved by A.L. Cauchy (1820-1830) and generalized by E. Picard (1891-1896), who replaced differentiability by a Lipschitz condition with respect to u. Under those conditions, the Cauchy problem has a unique solution which, moreover, depends continuously on the initial data. For linear partial differential equations

$$Lu = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^\alpha u}{\partial x^\alpha} = f(x), \tag{1.9}$$

the Cauchy problem may be formulated as follows. In a certain region D of variables $x = (x_1, ..., x_n)$ it is required to find a solution satisfying initial conditions. This means that it takes on the specified values, together with its derivatives of order up to and including $m - 1$, on some $(n - 1)$ dimensional hypersurface S

in D. This hypersurface is know as the *carrier of the initial conditions (or the initial surface).* The initial conditions, the *Cauchy data* may be given in the form of derivatives of u with respect to the direction of the unit normal ν to S:

$$\left.\frac{\partial^k u}{\partial \nu^k}\right|_S = \phi_k, \quad o \leq k \leq m-1, \tag{1.10}$$

where the $\phi_k(x)$, $x \in S$ are known functions. The formulation of the Cauchy problem for nonlinear differential equations is similar. A concept related to the Cauchy problem is that of a noncharacteristic surface. If a non-singular coordinate transformation $x \to x'$ straightens out the surface S in a neighbourhood of x_0, that is, it transforms it into a part of the hyperplane $x'_n = 0$, then the coefficient of $(\partial/\partial x'_n)^m$ in the transformed equation (1.9) is proportional to

$$Q(x,\nu) = \sum_{|\alpha|=m} a_\alpha(x)\nu^\alpha, \quad \nu^\alpha = \nu_1^{\alpha_1}\cdots\nu_n^{\alpha_n}.$$

The surface S is said to be *noncharacteristic* at the point x_0 if

$$Q(x_0,\nu) \neq 0.$$

Cauchy problems are usually studied when the initial surface is a noncharacteristic surface, that is $Q(x_0,\nu) \neq 0$ for all $x_0 \in S$.

1.3 Cauchy Method

If the partial differential equation is a first order equation, then the solution can be found by using the *method of characteristics* (or *Cauchy method*). Let us illustrate the method of characteristics with a quasilinear differential equation in two independent variables x and t. Consider the quasilinear equation:

$$P(x,t,u)u_x + Q(x,t,u)u_t = R(x,t,u).$$

The characteristics are given by:

$$\frac{dx}{P} = \frac{dt}{Q} = \frac{du}{R}.$$

We solve any two of the following three ordinary differential equations:

$$\frac{dx}{P} = \frac{dt}{Q}, \quad \frac{dx}{P} = \frac{du}{R}, \quad \frac{dt}{Q} = \frac{du}{R}.$$

After solving two equations from previous system, one obtains the general solution of these equations as functions of x, t and u defined by

$$\xi(x, t, u) = c_1, \ \eta(x, t, u) = c_2.$$

The general solution of the original quasilinear equation is then $F(\xi, \eta) = 0$.

Simultaneously, if $P_1, P_2, ..., P_n$ are functions of n independent variables $x_1, x_2, ..., x_n$ such that

$$P_1 \frac{\partial u}{\partial x_1} + P_2 \frac{\partial u}{\partial x_2} + ... + P_n \frac{\partial u}{\partial x_n} = 0.$$

Then to find the solution of this equation is equivalent to solving the system of ordinary differential equations:

$$\frac{\partial x_1}{P_1} = \frac{\partial x_2}{P_2} = ... = \frac{\partial x_n}{P_n}.$$

If $f_1, f_2, ..., f_{n-1}$ are $n - 1$ independent integrals of this equation, then for an arbitrary function ϕ, $u = \phi(f_1, f_2, ..., f_{n-1})$ is a general solution of original equation.

The method of characteristics is extended to use with nonlinear first order partial differential equations in two independent variables. The most general form of a first order partial differential equation in two independent variables can be written as

$$F(x, t, u, u_x, u_t) = 0, \tag{1.11}$$

Let $p = u_x$ and $q = u_t$. Consider an integral surface $u = u(x, t)$ that satisfies equation (1.11). Its normal vector has the form $[u_x, u_t, -1] = [p, q, -1]$, and equation (1.11) requires that at the point (x, t, u), the components p and q of the normal vector satisfy the equation

$$F(x, t, u, p, q) = 0. \tag{1.12}$$

Each normal vector determines a tangent plane to the surface, and equation (1.12) is seen to generate a one parameter family of tangent planes which could be the integral surfaces at each point in (x, t, u) space. For instance, if equation (1.11) is $u_x u_t - 1 = 0$, then $F = pq - 1 = 0$ with $q = 1/p$ determines a one parameter family of normal vectors $[p, q, -1] = [p, 1/p, -1]$ at each point (x, t, u). These equations require that ${F_p}^2 + {F_q}^2 \neq 0$. Equation (1.12) can be considered as a relation between the point (x, t, u) on the integral surface u and the direction cosines of a tangent plane at that point. Therefore the tangent planes at all points of the surface form a one parameter family. In general, the tangent plane determined by p and q. They envelope a *Monge cone* on u whose vertex is (x, t, u). The tangent plane at point (x, t, u) on the integral surface u is tangent to this cone along one of the generating lines, G. The intersection of the Monge cones with the surface determines a field of a directions on the surface called *characteristic directions.* A curve on u whose tangents are all generating lines of this cone is a *characteristic curve.* Then the characteristic curve is given by the system of ordinary differential equations:

$$\frac{dx}{F_p} = \frac{dt}{F_q} = \frac{du}{pF_p + qF_q} = \frac{-dp}{F_x + pF_u} = \frac{-dq}{F_t + qF_u}.$$

This equation is called the *characteristic differential equation* or *Charpit subsidiary (auxiliary) equation* of partial differential equation (1.11). It determines not only x, t, u but also p and q. The set of these surface elements (x, t, u, p, q), *characteristic manifold,* is considered as a part of the integral surface with infinitesimal width, and in this case it is called a *characteristic strip.* The characteristic strip is represented by the equations

$$x = x(\lambda), \quad t = t(\lambda), \quad u = u(\lambda), \quad p = p(\lambda), \quad q = q(\lambda)$$

containing a parameter λ. On the integral surface $u = u(x, t)$,

$$\frac{du}{d\lambda} = \frac{du}{dx}\frac{dx}{d\lambda} + \frac{du}{dt}\frac{dt}{d\lambda}$$

and

$$du = pdx + qdt. \tag{1.13}$$

Equation (1.13), called the *strip condition* such that the previous equation must satisfy this condition. Generally, a single partial differential equation in one unknown function defined by

$$F(x, u, p) = 0, \tag{1.14}$$

with initial data

$$x_i = x_i(t),\ u = u(t), \qquad i = 1, 2, ..., n \tag{1.15}$$

is called a Cauchy problem. Here $x = (x_1, x_2, ..., x_n)$ are independent variables, $p = (p_1, p_2, ..., p_n),\ p_i = \frac{\partial u}{\partial x_i},\ i = 1, 2, ..., n$ and $t = (t_1, t_2, ..., t_{n-1})$ are parameters of the initial values. The functions $u(t),\ x_i(t)$ and $F(x, u, p)$ are continuously differentiable with a set $(x(t), u(t), p(t))$ of surface elements depending on a parameter t. The Cauchy problem for equation (1.14) consists of finding an integral surface passing through a given (n-1)-dimensional initial condition. The Cauchy's method uses the following characteristics:

$$\frac{dx_i}{ds} = F_{p_i},\quad \frac{du}{ds} = p_i F_{p_i},\quad \frac{dp_i}{ds} = -(F_u p_i + F_{x_i}), \qquad i = 1, 2, ..., n, \tag{1.16}$$

with initial data at the point $s = o:$

$$x = x(t),\ u = u(t),\ p = p(t),$$

where $x = x(t)$ and $u = u(t)$ are characteristic curves determined by equation (1.15). The initial data $p(t)$ are obtained from equation (1.14) and the tangent conditions:

$$F(x(t), u(t), p(t)) = 0,\ u_{t_k}(t) = p_i(t)\frac{\partial x_i}{\partial t_k}(t), \qquad (k = 1, 2, ..., n-1).$$

After solving the characteristic system, $u(s, t_1, \ldots, t_{n-1})$ and $x_i(s, t_1, \ldots, t_{n-1})$, $i = 1, \ldots, n$ are obtained. The solution $u = u(x)$ is discovered by the elimination of the parameters $s, t_1, \ldots, t_{n-1}$ from the equations $x = x(s,t)$ and $u = u(s,t)$. The condition

$$\triangle \equiv \frac{\partial(x_1, \ldots, x_n)}{\partial(s, t_1, ..., t_{n-1})} = \det \begin{pmatrix} F_{p_i} \\ \partial x_i / \partial t_k \end{pmatrix} \neq 0$$

is sufficient for this elimination. To find the function u as solution of the partial differential equation, we can use the following theorem.

Theorem 1.1 *The function* $u(x_1, x_2, ..., x_n)$ *is constructed by solving the Cauchy problem* (1.16) *with initial data* (1.14), (1.15), *satisfying the condition*

$$\Delta(0, t_1, ..., t_{n-1}) \neq 0,$$

gives the solution of the Cauchy problem (1.14), (1.15).

1.4 A Demonstration of the Intermediate Integral Technique

In fact, any system of partial differential equation may be reduced to a system of first order partial differential equations. After augmenting new unknown functions and all their partial derivatives, the new system must be complete (Hazewinkel, 1995). For example, consider the equation

$$F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0.$$

By introducing new unknown functions $v = u_x$ and $w = u_t$ this equation is reduced to the following system of first order equations.

$$\begin{aligned} F(x, t, u, v, w, v_x, v_t, w_t) &= 0, \\ u_x - v &= 0, \\ u_t - w &= 0, \\ v_t - w_x &= 0. \end{aligned}$$

where the last three equations are independent. One way to reduce the order of high order partial differential equations to lower order partial differential equations is the intermediate integral technique. Consider a general partial differential equation

$$G\left(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_1 x_1}, u_{x_1 x_2}, ..., u_{x_n x_n}\right) = 0. \tag{1.17}$$

Definition 1.1 *A first order differential equation*

$$v(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_n}) = 0 \tag{1.18}$$

is called an ***intermediate integral*** *of equation* (1.17) *if any solution of equation* (1.18) *is also a solution of equation* (1.17).

With the help of intermediate integrals, solving a partial differential equations is reduced to finding solutions of less order. For example, consider an intermediate integral of the second order differential equation

$$F(x, y, u, p, q, r, s, t) = 0, \tag{1.19}$$

where $r = u_{xx},\ s = u_{xy},\ t = u_{yy},\ p = u_x,\ q = u_y$. Any first order differential equation

$$v(x, y, u, p, q) = 0, \tag{1.20}$$

is called an intermediate integral of the equation (1.19) if and only if the solution of this equation is also a solution of equation (1.19). For simplicity, begin studying an intermediate integral of quasilinear differential equation

$$s + pA(x, y, u) + qB(x, y, u) + C(x, y, u) = 0. \tag{1.21}$$

Conditions for existence of intermediate integrals of this equation will now be obtained. After differentiating equation (1.20) with respect to x and y, we obtain a system of linear algebraic equations for the second order derivatives is obtained

$$\begin{aligned} s + pA(x,y,u) + qB(x,y,u) + C(x,y,u) &= 0, \\ v_p r + v_q s + v_u p + v_x &= 0, \\ v_p s + v_q t + v_u q + v_y &= 0. \end{aligned} \tag{1.22}$$

The general solution of equation (1.20) has one arbitrary function. Hence, if equation (1.20) is an intermediate integral, the solution of equation (1.22) should also have such arbitrariness. Thus

$$\begin{vmatrix} 0 & 1 & 0 \\ v_p & v_q & 0 \\ 0 & v_p & v_q \end{vmatrix} = -v_p v_q = 0.$$

Let $v_p \neq 0$, $v_q = 0$. Without loss of generality one can take $v = p + g(x,y,u)$ and from equation (1.22) one has

$$q(g_u - B) + g_y + gA - C = 0.$$

Since $u(x,y)$ is an arbitrary solution of equation (1.20), one obtains from the last equation:

$$g_u - B = 0, \quad g_y + Ag - C = 0. \tag{1.23}$$

Note that the last equation of system (1.22) is a linear combination of the first and second equations. System (1.23) is an overdetermined system for the function $g(x,y,u)$. From the compatibility condition of $g_{uy} = g_{yu}$ one gets

$$AB + B_y + A_u g - C_u = 0.$$

If $A_u = 0$, then

$$B_y + AB - C_u = 0,$$

and in this case, these conditions are sufficient for the existence of an intermediate integral. If $A_u \neq 0$, then

$$g = \frac{B_y + AB - C_u}{A_u}$$

Hence, after substituting the function g in the second equation of (1.23) one obtains

$$(\frac{B_y + AB - C_u}{A_u})_u - B = 0, \quad (\frac{B_y + AB - C_u}{A_u})_y + A\frac{B_y + AB - C_u}{A_u} - C = 0.$$

These conditions provide the existence of the intermediate integral for equation (1.21). The case $v_p = 0$ is similar.

To illustrate the method of intermediate integrals in a relatively simple case, we consider the wave equation

$$u_{tt} = c^2 u_{xx}, \tag{1.24}$$

where c is a constant. To use a differential constraint of first order, we assume that

$$u_t = \varphi(u_x, u, x, t). \tag{1.25}$$

Recall that this differential constraint is called an intermediate integral if any solution of equation (1.25) is also a solution of equation (1.24). Using the chain rule, one can derive u_{xt} and u_{tt} from u_t in equation (1.25)

$$u_{xt} = \varphi_{u_x} u_{xx} + \varphi_u u_x + \varphi_x\,; \tag{1.26}$$

$$u_{tt} = \varphi_{u_x} u_{xt} + \varphi_u \varphi + \varphi_t. \tag{1.27}$$

Substituting u_{xt} from equation (1.26) into equation (1.27), one gets

$$u_{tt} = \varphi_{u_x}^2 u_{xx} + \varphi_{u_x}\varphi_u u_x + \varphi_{u_x}\varphi_x + \varphi_u\varphi + \varphi_t\,. \tag{1.28}$$

But from equation (1.24), $u_{tt} - c^2 u_{xx} = 0$, therefore one obtains

$$(\varphi_{u_x}^2 - c^2) u_{xx} + \varphi_{u_x}\varphi_u u_x + \varphi_{u_x}\varphi_x + \varphi_u\varphi + \varphi_t = 0\,. \tag{1.29}$$

To justify our next steps, let impose the initial condition

$$u(x, t_0) = h(x), \tag{1.30}$$

Because the line $t = t_0$ is noncharacteristic for equation (1.25). For an arbitrary function $h(x)$, there exists a unique solution of the Cauchy problem (1.25) and (1.30). The function $h(x)$ can be chosen as

$$h(x) = \frac{\alpha}{2}(x - x_0)^2 + \beta(x - x_0) + \gamma, \tag{1.31}$$

where α, β and γ are arbitrary constants. Now fix an arbitrary $x = x_0$. Then in a neighborhood of the point (x_0, t_0), there exists a unique solution $u(x, t)$ such that at (x_0, t_0), one has

$$u(x_0, t_0) = \gamma,\ u_x(x_0, t_0,) = \beta,\ u_{xx}(x_0, t_0,) = \alpha. \tag{1.32}$$

We now consider all arguments involved explicitly; so that equation (1.29) becomes

$$\begin{aligned}(\varphi_{u_x}(\beta, \gamma, x_0, t_0) - c^2)\alpha + \varphi_{u_x}(\beta, \gamma, x_0, t_0)\varphi_u(\beta, \gamma, x_0, t_0)\beta \\ +\varphi_{u_x}(\beta, \gamma, x_0, t_0)\varphi_x(\beta, \gamma, x_0, t_0) + \varphi_u(\beta, \gamma, x_0, t_0)\varphi(\beta, \gamma, x_0, t_0) \\ +\varphi_t(\beta, \gamma, x_0, t_0) &= 0,\end{aligned} \tag{1.33}$$

for all α, β, γ, x_0 and t_0. One can now split the equation with respect to α, since the functions do not have α as an argument. The coefficient of α is

$$\varphi^2_{u_x}(\beta, \gamma, x_0, t_0) - c^2 = 0, \tag{1.34}$$

which implies that

$$\varphi(\beta, \gamma, x_0, t_0) = \mu\beta + a(\gamma, x_0, t_0), \tag{1.35}$$

where $\mu = \pm c$. Substituting this result into equation (1.33) one obtains

$$2\mu a_\gamma \beta + a a_\gamma + \mu a_x + a_t = 0. \tag{1.36}$$

Now split with respect to β with similar reasoning. Then one gets

$$2\mu a_\gamma = 0, \tag{1.37}$$

that is

$$a(\gamma, x_0, t_0) = a(x_0, t_0). \tag{1.38}$$

Since x_0, t_0 and β are arbitrary, substituting equation (1.38) into equation (1.36) one obtains

$$\mu\, a_x(x,t) + a_t(x,t) = 0. \tag{1.39}$$

The solution of equation (1.39) is

$$a(x,t) = g(x - \mu t). \tag{1.40}$$

Hence

$$\varphi = \mu u_x + \ g(x - \mu t), \tag{1.41}$$

where g is an arbitrary function. This implies that $u_t = \mu u_x + \ g(x - \mu t)$. One has now obtained the explicit form of u_t in equation (1.25). To obtain $u(x,t)$ as a solution to the original wave equation, let $\xi = x + \mu t$ and $\eta = x - \mu t$. Then $u_t = \mu u_\xi - \mu u_\eta$ and $u_x = u_\xi + u_\eta$. That is, $u_\eta = -\frac{1}{2\mu} g(\eta)$. Solving for u, one gets:

$$\begin{aligned} u &= -\frac{1}{2\mu}\int g(\eta)d(\eta) + \ \rho(\xi) \\ &= \phi(\eta) + \ \rho(\xi). \end{aligned} \tag{1.42}$$

where ϕ and ρ are arbitrary functions. Hence the method of the intermediate integrals technique yields the same solution as *d'Alembert's solution.*

1.5 Differential Constraints: An overview

Another method of finding exact particular solutions of partial differential equation is that of differential constraints. Its idea is the following. Given a system of differential equations,

$$F_i(x, u, U) = 0, \qquad i = 1, \ldots, n, \tag{1.43}$$

we augment differential equations

$$\Phi_k(x, u, U) = 0, \qquad k = 1, \ldots, q, \tag{1.44}$$

where $x = (x_1, \ldots, x_n)$ are the independent variables, $u = (u^1, \ldots, u^m)$ are the dependent variables and $U = (u_i^\alpha)$ is the set of derivatives $u_i^\alpha = \frac{\partial^{|j|} u^\alpha}{\partial x^j}$ with $j = (j_1, \ldots, j_n)$, $\alpha = 1, \ldots, m$, $\mid j \mid \leq q$; $\mid j \mid = j_1 + \cdots + j_n$.

Definition 1.2 *The system* (1.44) *is called the* ***differential constraint*** *to system* (1.43).

System (1.43)-(1.44) is an overdetermined system and has to be compatible. The differential constraints (1.44) are said to be admitted by system (1.43).

Definition 1.3 *A solution of system* (1.43) *satisfying* (1.44) *is called the solution characterized by the differential constraints* (1.44).

Using the method of differential constraints involves two stages. The first stage is to find the set of differential constraints (1.44) compatible with the overdetermined system. In the process of compatibility analysis, the overdetermined system (1.43), (1.44) can be supplemented by new equations. The second stage is to construct the solutions of the involutive overdetermined system. Since it has more conditions, then it should be easier to construct particular solutions of the system (1.43). A classification of differential constraints and their characteristic solutions can be carried out with respect to the functional arbitrariness of solutions of the overdetermined system (1.43), (1.44) and the order of highest derivatives included in the differential constraints (1.44).

Chapter II

Monge-Ampere Equation

The partial differential equation

$$u_{xt}^2 - u_{xx}u_{tt} = a(x,t). \qquad (2.1)$$

is called the *Monge-Ampere* equation. If $a(x,t) = 0$ then equation (2.1) is called *homogeneous*; otherwise it is called *nonhomogeneous.* If $a(x,t) \geq 0$, it is *hyperbolic.* If $a(x,t) < 0$, it is *elliptic.*

2.1 Homogeneous Monge-Ampere Equation

Finding the solutions of the Monge-Ampere equation by the intermediate integral technique, involves assuming the existence of the first order differential constraint

$$u_t = \varphi(u_x, u, x, t). \qquad (2.2)$$

Using this condition, one derives u_{xt} and u_{tt}:

$$u_{xt} = \varphi_{u_x}u_{xx} + \varphi_u u_x + \varphi_x, \qquad (2.3)$$

$$u_{tt} = \varphi_{u_x}u_{xt} + \varphi_u u_t + \varphi_t. \qquad (2.4)$$

By substituting equation (2.3) in equation (2.4), one obtains

$$u_{tt} = \varphi_{u_x}^2 u_{xx} + \varphi_{u_x}\varphi_u u_x + \varphi_{u_x}\varphi_x + \varphi_u u_t + \varphi_t. \qquad (2.5)$$

Substituting (2.3) and (2.5) into equation (2.1), it becomes:

$$\begin{aligned}\phi(x,t,u,u_x,u_{xx},\varphi_{u_x},...) &= (\varphi_{u_x}\varphi_u u_x + \varphi_{u_x}\varphi_x - \varphi_u\varphi - \varphi_t)u_{xx} \\ &\quad +2\varphi_u\varphi_x u_x + \varphi_u^2 u_x^2 + \varphi_x^2 \\ &= 0. \end{aligned} \tag{2.6}$$

Let us study the properties of solutions of equation (2.2). According to the definition of intermediate integral solutions, equation (2.6) has to be satisfied for any solution of equation (2.2). By imposing the initial condition

$$u(x,t_0) = h(x), \tag{2.7}$$

when $h(x)$ is an arbitrary function there exists a solution of Cauchy problem (2.2) and (2.7), since the line $t = t_0$ is noncharacteristic for equation (2.2). Again one can choose h as follows

$$h(x) = \frac{\alpha}{2}(x-x_0)^2 + \beta(x-x_0) + \gamma,$$

where α, β, γ and x_0 are arbitrary constants. Then in a neighborhood of the point (x_0,t_0) there exists a unique solution $u(x,t)$ such that at the point (x_0,t_0), we have:

$$u(x_0,t_0) = \gamma, \;\; u_x(x_0,t_0) = \beta, \;\; u_{xx}(x_0,t_0) = \alpha,$$

so that equation (2.6) at the point (x_0,t_0) becomes

$$\begin{aligned} &(\varphi_{u_x}(\beta,\gamma,x_0,t_0)\varphi_u(\beta,\gamma,x_0,t_0)\beta + \varphi_{u_x}(\beta,\gamma,x_0,t_0)\varphi_x(\beta,\gamma,x_0,t_0) \\ &-\varphi_u(\beta,\gamma,x_0,t_0)\varphi(\beta,\gamma,x_0,t_0) - \varphi_t(\beta,\gamma,x_0,t_0))\alpha + 2\varphi_u(\beta,\gamma,x_0,t_0) \\ &\varphi_x(\beta,\gamma,x_0,t_0)\beta + \varphi_u^2(\beta,\gamma,x_0,t_0)\beta^2 + \varphi_x^2(\beta,\gamma,x_0,t_0) = 0. \end{aligned} \tag{2.8}$$

Comparing (2.6) and (2.8), due to arbitrariness of x_0, t_0, α, β and γ, one can consider equation (2.6) as an equation $\phi(x,t,u,u_x,u_{xx},\varphi_{u_x},...)$ for the function

$\varphi(u_x, u, x, t)$ with independent variables $x,\ t,\ u,\ u_x,\ u_{xx}, ...$: hence the coefficient of u_{xx} must be 0 as well as the other terms. Thus we obtain:

$$\varphi_{u_x}\varphi_u u_x + \varphi_{u_x}\varphi_x - \varphi_u\varphi - \varphi_t = 0, \tag{2.9}$$

and

$$2\varphi_u\varphi_x u_x + \varphi_u^2 u_x^2 + \varphi_x^2 = 0. \tag{2.10}$$

Equation (2.10) can be rewritten as

$$(\varphi_u u_x + \varphi_x)^2 = 0, \tag{2.11}$$

that is

$$\varphi_u u_x + \varphi_x = 0. \tag{2.12}$$

After substituting (2.12) into equation (2.9) one obtains

$$\varphi_u\varphi + \varphi_t = 0. \tag{2.13}$$

The characteristic equations of quasiliner equation (2.12) are given by

$$\frac{d\varphi}{0} = \frac{du_x}{0} = \frac{du}{u_x} = \frac{dx}{1} = \frac{dt}{0}. \tag{2.14}$$

Invariants of characteristic system of equation (2.14) are $\varphi = c_1,\ u_x = c_2,$ $c_3 = u - u_x x,\ \ t = c_4$, where $c_1,\ c_2,\ c_3$ and c_4 are constants. Thus the general solution of equation (2.12) is $\Upsilon(\varphi, u_x, u - xu_x, t) = 0$. It can be written as

$$\varphi = \Psi(u_x, t, \zeta), \tag{2.15}$$

where $\zeta = u - xu_x$. Using equation (2.15), we obtain $\varphi_u = \Psi_\zeta$ and $\varphi_t = \Psi_t$. After substituting these values into equation (2.13), the following equation is obtained:

$$\Psi_\zeta\Psi + \Psi_t = 0. \tag{2.16}$$

To find the solution of this first order quasilinear partial differential equation (2.16) by the Cauchy method (see chapter I), the following are characteristic equations of equation (2.16):

$$\frac{dt}{ds} = 1, \ \ \frac{d\zeta}{ds} = \Psi, \ \ \frac{d\Psi}{ds} = 0.$$

The general solution of this system is $t = s + k_1, \ \ \zeta = \Psi s + k_3$ and $\Psi = k_2$, where $k_1, \ k_2$ and k_3 are constants. Now using the initial conditions $t = 0, \ \ \zeta = \tau$, and $\Psi(0, \tau) = g(\tau)$ at the point $s = 0$, we obtain:

$$k_1 = 0, \ \ k_2 = g(\tau), \ \ k_3 = \tau. \tag{2.17}$$

Hence

$$t = s, \ \ \zeta = \Psi s + \tau, \ \ \Psi = g(\tau). \tag{2.18}$$

From system (2.18), we obtain: $\zeta = g(\tau)t + \tau$ so $\tau = \zeta - g(\tau)t$. Knowing the function $g(\tau)$ and using the inverse function theorem, one obtains

$$\tau = f(\zeta, t). \tag{2.19}$$

Finally the solution of equation (2.16) is

$$\Psi = g(f(\zeta, t)), \tag{2.20}$$

an intermediate integral in the homogeneous case.

2.2 Nonhomogeneous Monge-Ampere Equation

In the nonhomogeneous case, we consider (2.1) with $a(x, t) = \pm 1$. It can be written as

$$u_{xt}^2 - u_{xx}u_{tt} = \pm 1. \tag{2.21}$$

First, assuming the existence of a differential constraint of first order as in equation (2.2), we find the same derivatives u_{xt} and u_{tt} in (2.3) and (2.5). After substituting these derivatives into equation (2.21), we obtain;

$$\begin{aligned}\Phi(x,t,u,u_x,u_{xx},\varphi_{u_x},...) &= (\varphi_{u_x}\varphi_u u_x+\varphi_{u_x}\varphi_x-\varphi_u\varphi-\varphi_t)u_{xx}\\ &\quad +2\varphi_u\varphi_x u_x+\varphi_u^2u_x^2+\varphi_x^2-\epsilon\\ &= 0. \end{aligned} \tag{2.22}$$

where $\epsilon=\pm1$ in equation (2.22). Let us study properties of solutions of equation (2.2). According to the definition of an intermediate integral solution of equation (2.22) has to be satisfied for any solution of equation (2.2). By imposing the initial condition

$$u(x,t_0)=h(x), \tag{2.23}$$

then $h(x)$ is an arbitrary function there exists solution of Cauchy problem (2.2) and (2.23), since the line $t=t_0$ is again noncharacteristic for equation (2.2). For example one can choose as follows

$$h(x)=\frac{\alpha}{2}(x-x_0)^2+\beta(x-x_0)+\gamma,$$

where α, β, γ and x_0 are arbitrary constants. Then in a neighborhood of the point (x_0,t_0) there exists a unique solution $u(x,t)$ such that at the point (x_0,t_0), we have:

$$u(x_0,t_0)=\gamma,\;\; u_x(x_0,t_0)=\beta,\;\; u_{xx}(x_0,t_0)=\alpha,$$

so that equation (2.22) at the point (x_0,t_0) becomes

$$\begin{aligned}&(\varphi_{u_x}(\beta,\gamma,x_0,t_0)\varphi_u(\beta,\gamma,x_0,t_0)\beta+\varphi_{u_x}(\beta,\gamma,x_0,t_0)\varphi_x(\beta,\gamma,x_0,t_0)\\ &-\varphi_u(\beta,\gamma,x_0,t_0)\varphi(\beta,\gamma,x_0,t_0)-\varphi_t(\beta,\gamma,x_0,t_0))\alpha+2\varphi_u(\beta,\gamma,x_0,t_0)\\ &\varphi_x(\beta,\gamma,x_0,t_0)\beta+\varphi_u^2(\beta,\gamma,x_0,t_0)\beta^2+\varphi_x^2(\beta,\gamma,x_0,t_0)-\epsilon \;=\; 0.\end{aligned} \tag{2.24}$$

Comparing (2.22) and (2.24), and since $x_0,\ t_0,\ \alpha,\ \beta$ and γ are arbitrary, one can consider equation (2.22) as an equation $\Phi(x,t,u,u_x,u_{xx},\varphi_{u_x},...)$ for the function

$\varphi(u_x, u, x, t)$ with the independent variables x, t, u, u_x, $u_{xx}, \ldots$. Hence the coefficient of u_{xx} must be 0 as well as the other terms. These properties can be written as the following pair of equations:

$$\varphi_{u_x}\varphi_u u_x + \varphi_{u_x}\varphi_x - \varphi_u\varphi - \varphi_t = 0. \tag{2.25}$$

and

$$\varphi_u^2 u_x^2 + \varphi_x^2 + 2\varphi_u u_x \varphi_x - \epsilon = 0. \tag{2.26}$$

The last equation can be rewritten as $(\varphi_u u_x + \varphi_x)^2 - \epsilon = 0$. It shows that further analysis of intermediate integral can be done only for $\epsilon = 1$. In this case equation (2.26) can be expressed as:

$$\varphi_u u_x + \varphi_x + 1 = 0, \tag{2.27}$$

or

$$\varphi_u u_x + \varphi_x - 1 = 0. \tag{2.28}$$

Note that equation (2.28) can be obtained from equation (2.27) by changing x to $-x$. Therefore we study only equation (2.27). Its characteristic equation is

$$\frac{du_x}{0} = \frac{du}{u_x} = \frac{dx}{1} = \frac{dt}{0} = \frac{d\varphi}{-1}. \tag{2.29}$$

The last equation has invariants of characteristic system: $u_x = h_1$, $t = h_2$, $u - u_x x = h_3$, $\varphi + x = h_4$, where h_1, h_2, h_3 and h_4 are arbitrary constants. Hence the general solution is

$$K(u_x, t, u - u_x x, \varphi + x) = 0$$

or

$$\varphi = F(u_x, t, \xi) - x, \tag{2.30}$$

where $\xi = u - u_x x$. From equation (2.25) and equation (2.26), we obtain

$$\varphi_{u_x} + \varphi_u \varphi + \varphi_t = 0. \tag{2.31}$$

Because of the representation of equation (2.30), the following derivatives are obtained

$$\varphi_{u_x} = F_{u_x} - xF_\xi,\ \varphi_u = F_\xi,\ \varphi_t = F_t.$$

After substituting them into equation (2.31), it becomes

$$F_{u_x} + F_\xi F + F_t = 0. \tag{2.32}$$

This quasilinear equation (2.32) can be solved by the Cauchy method:

$$\frac{dF}{ds} = 0,\ \frac{du_x}{ds} = 1,\ \frac{dt}{ds} = 1,\ \frac{d\xi}{ds} = F.$$

The general solution of this system are

$$u_x = s + c_1,\ t = s + c_2,\ F = c_3,\ \xi = Fs + c_4,$$

where $c_1,\ c_2,\ c_3$ and c_4 are arbitrary constants. After using initial values;

$$u_x = 0,\ t = 0,\ \xi = \tau,\ F = g(\tau).$$

At the point $s = 0$, one can find the constants

$$c_1 = 0,\ c_2 = 0,\ c_3 = g(\tau),\ c_4 = \tau.$$

Hence

$$u_x = s,\ t = s,\ \xi = Fs + \tau,\ F = g(\tau).$$

These imply that $\xi = g(\tau)t + \tau$ so that $\tau = \xi - g(\tau)t$. Knowledge of the function $g(\tau)$ and use of the inverse function theorem helps one to obtain the solution of equation (2.32):

$$F = g(f(\xi, t)). \tag{2.33}$$

From equation (2.30), we obtain:

$$\varphi = g(f(\xi, t)) - x \tag{2.34}$$

is an intermediate integral in this case.

Chapter III

KdV and BBM Equations

The solution of third order partial differential equations, *Korteweg de Vries* (KdV) and the *Benjamin-Bona-Mahony* (BBM) equations are found in this chapter through the technique of differential constraints.

3.1 Korteweg De Vries Equation

The KdV equation can be written as

$$u_t - 6uu_x + u_{xxx} = 0, \tag{3.1}$$

where x and $t \in \mathbf{R}^1$ with $u(x,t) \in \mathbf{R}^2$. It was proposed by D. Korteweg and G. de Vries to describe wave propagation on the surface of shallow water. The differential constraints method will be applied to solve it. Assume that

$$u_t = \varphi(u_x, u, x, t) \tag{3.2}$$

is a differential constraint of equation (3.1). By differentiation of equation (3.2) with respect to t and x, one derives $u_{tt},\ u_{xt},\ u_{ttt},\ u_{xtt}$ and u_{xxt} (on the computer using a symbolic calculation system). Note that all these derivatives are functions of $u_{xx},\ u_x,\ u,\ x$ and t. The derivative u_{xxx} can not be derived by this manner, but it can be obtained from equation (3.1)

$$u_{xxx} = -\varphi + 6uu_x. \tag{3.3}$$

Thus one can find all third order derivatives through u_{xx}, u_x, u, x and t. Since we are working with enough time continuously differentiable functions, we use the following consistency conditions:

$$
\begin{aligned}
u_{xxx,t} - u_{xxt,x} &= 0, \\
u_{xxt,t} - u_{xtt,x} &= 0, \quad \text{and} \\
u_{xtt,t} - u_{ttt,x} &= 0.
\end{aligned}
\tag{3.4}
$$

The commas in system (3.4) denote the fourth derivatives that can be taken; we start working with the first equation of system (3.4). The left side of this equation is a polynomial function with respect to u_{xx} :

$$P(u_{xx}) = A_1 u_{xx}^3 + A_2 u_{xx}^2 + A_3 u_{xx} + A_4 = 0,$$

where

$$A_1 = \ \varphi_{u_x u_x u_x}, \tag{3.5}$$

$$A_2 = \ u_x \varphi_{u u_x u_x} + \varphi_{u u_x} + \varphi_{u_x u_x x}, \tag{3.6}$$

$$A_3 = \ u_x^2 \varphi_{u u u_x} + u_x(2\varphi_{u u_x x} + \varphi_{uu} + 6\varphi_{u_x u_x} u) + \varphi_{ux} + \varphi_{u_x xx} - \varphi_{u_x u_x}\varphi, \tag{3.7}$$

and finally

$$
\begin{aligned}
A_4 = u_x^3 \varphi_{uuu} + 3u_x^2(6\varphi_{u u_x} u + \varphi_{uux} + 2\varphi_{u_x}) - 3u_x(\varphi_{u u_x}\varphi - \varphi_{uxx} \\
- 6\varphi_{u_x x} u + 2\varphi) + \varphi_t - 3\varphi_{u_x x}\varphi + \varphi_{xxx} - 6\varphi_x u.
\end{aligned}
\tag{3.8}
$$

In this study we consider the case where the consistency conditions do not produce new equations of constraint. Then all coefficients with respect to u_{xx} of this polynomial must be equal to zero. Hence $A_1 = A_2 = A_3 = A_4 = 0$. The solution of equation $A_1 = 0$ is

$$\varphi = \frac{1}{2} a(u,x,t) u_x^2 + b(u,x,t) u_x + c(u,x,t), \tag{3.9}$$

where $a,$ b and c are the functions of independent variables $u,$ x and t. Hence after substituting φ into equation $A_2 = 0$ one has

$$2a_u u_x + a_x + b_u = 0. \tag{3.10}$$

The equation $A_3 = 0$ can be rewritten as

$$
\begin{aligned}
&9u_x^3 a_{uu} - 3u_x^2(-5a_{ux} - 4b_{uu} + a^2) - 6u_x(-a_{xx} \\
&-3b_{ux} - c_{uu} + ab - 6au) - 6(-b_{xx} - c_{ux} + ac) = 0
\end{aligned} \tag{3.11}
$$

and equation $A_4 = 0$ becomes:

$$
\begin{aligned}
&u_x^5\, a_{uuu} - u_x^4(-3a_{uux} + 3a_u a - 2b_{uuu}) - u_x^3(-3a_{uxx} + 6a_u b \\
&-36a_u u + \ 3a_x a - 6b_{uux} + \ 3b_u a - 2c_{uuu} - 6a\ ) - u_x^2\ (6a_u c \\
&-a_t - a_{xxx} + 6a_x b - 30a_x u - 6b_{uxx} + 6b_u b - 36b_u u + 3b_x a \\
&-6c_{uux}) - 2u_x(\ 3a_x c + 3b_u c - a_t - a_{xxx} + 6a_x b - b_t - b_{xxx} \\
&+3b_x b - 12b_x u - 3c_{uxx} + 6c) - 2(\ 3b_x c - c_t - c_{xxx} + 6c_x u) = 0.
\end{aligned} \tag{3.12}
$$

Let us study equation (3.10) and (3.11). The left side of these equation is a polynomial function with respect to u_x. Since consistency conditions do not produce new equations, all coefficients with respect to u_x of this polynomial must be equal to zero. Therefore

$$a_u = 0, \quad a_x + b_u = 0, \tag{3.13}$$

$$4b_{uu} - a^2 = 0, \quad a_{xx} + 3b_{ux} + c_{uu} - ab + 6au = 0, \quad b_{xx} + c_{ux} - ac = 0. \tag{3.14}$$

After substituting the b_u found from system (3.13) into the first equation of system (3.14), we get

$$a = 0. \tag{3.15}$$

Hence $b_u = 0$: that is, the function b depends on independent variables x and t. Therefore the second equation of system (3.14) yields

$$c_{uu} = 0. \tag{3.16}$$

This means that

$$c = c_1(x,t)u + c_2(x,t), \tag{3.17}$$

where c_1 and c_2 are functions of independent variables x and t. The last equation of system (3.14) becomes:

$$b_{xx} + c_{1x} = 0. \tag{3.18}$$

Equation (3.12) can be rewritten as

$$\begin{gathered} u_x\left(-\ b_t - b_{xxx} +\ 3b_x b -\ 12 b_x u - 3c_{1xx} +\ 6c_1 u +\ 6c_2\right) +\ 3b_x c_2 \\ +3b_x c_1 u - c_{1t} u - c_{1xxx} u + 6c_{1x} u^2 - c_{2t} - c_{2xxx} + 6c_{2x} u = 0. \end{gathered} \tag{3.19}$$

Since in equation (3.19) the left side of this equation is a polynomial function with respect to u_x, the coefficients have to be equal to zero:

$$\begin{gathered} b_t + b_{xxx} -\ 3b_x b + 12 b_x u + 3c_{1xx} - 6c_1 u - 6c_2 = 0, \\ 3\ b_x c_2 + 3\ b_x c_1 u -\ c_{1t} u -\ c_{1xxx} u + 6c_{1x} u^2 - c_{2t} - c_{2xxx} + 6c_{2x} u = 0. \end{gathered} \tag{3.20}$$

The left side of system (3.20) is now a polynomial function with respect to u, therefore

$$\begin{gathered} 2b_x - c_1 = 0,\ \ b_t - 3b_x b - 6c_2 = 0,\ \ c_{1x} = 0, \\ 3b_x c_1 - c_{1t} + 6c_{2x} = 0,\ \ 3b_x c_2 - c_{2t} - c_{2xxx} = 0. \end{gathered} \tag{3.21}$$

The third equation of system (3.21) indicates that

$$c_1 = c_1(t), \tag{3.22}$$

with c_1 a function of only the independent variable t. Integrating the first equation of system (3.21), one has

$$b = b_1(t) + \frac{1}{2} c_1 x, \tag{3.23}$$

where b_1 is a function of the independent variable t. Hence the following equation is considered instead of the fourth equation of system (3.21):

$$2c_{1t} - 12c_{2x} - 3c_1^2 = 0. \tag{3.24}$$

Studying the last equation of system (3.21) one obtains

$$2c_{2t} - 3c_1 c_2 = 0. \tag{3.25}$$

Consideration of the second equation of system (3.21) yields the following:

$$4b_{1t} + 2c_{1t}x - 6b_1c_1 - 3c_1^2x - 24c_2 = 0. \tag{3.26}$$

The left side of equation (3.26) is a polynomial function with respect to x, hence

$$2c_{1t} - 3c_1^2 = 0, \;\; 4b_{1t} - 6b_1c_1 - 24c_2 = 0. \tag{3.27}$$

From equation (3.24) and using the first equation of system (3.27), we find that

$$c_2 = c_2(t), \tag{3.28}$$

Therefore c_2 is a function of one independent variable t. Integrating the first equation of system (3.27), we obtain:

$$c_1 = \frac{2}{2k_1 - 3t}, \tag{3.29}$$

with some constant k_1. We then integrate equation (3.25) to get:

$$c_2 = \frac{-k_2}{2k_1 - 3t}, \tag{3.30}$$

where k_2 is a constant. The second equation of system (3.27) becomes

$$\frac{b_{1t}(2k_1 - 3t) - 3b_1 + 6k_2}{2k_1 - 3t} = 0. \tag{3.31}$$

The general solution of equation (3.31) is

$$b_1 = \frac{-k_3 - 6k_2t}{2k_1 - 3t}, \tag{3.32}$$

with some constant k_3. Therefore the following differential constraint is obtained:

$$\varphi = \frac{-k_3 - 6k_2t + x}{2k_1 - 3t}u_x + \frac{2u - k_2}{2k_1 - 3t}. \tag{3.33}$$

Without loss of generality, by the transformations $\bar{x} = x - k_3$ and $\bar{t} = t - \frac{2}{3}k_1$, one can account that $k_1 = 0$ and $k_3 = 0$. But $\varphi = u_t$, so that the following equation is considered instead of equation (3.33) to find the function u:

$$3tu_t + (x - 6k_2t)u_x = k_2 - 2u. \tag{3.34}$$

This quasilinear equation can be solved by the Cauchy method (see chapter I) with its characteristic equation:

$$\frac{dt}{3t} = \frac{dx}{x - 6k_2 t} = \frac{du}{k_2 - 2u}. \tag{3.35}$$

At last, the solution of equation (3.34) is

$$u(x,t) = \frac{1}{2}k_2 + t^{-2/3}F((x + 3k_2 t)t^{-1/3}). \tag{3.36}$$

If $k_2 = 0$, then $u(x,t) = t^{-2/3}\hat{F}(xt^{-1/3})$ is a solution of the KdV equation (Ibragimov, editor, 1994), where $\hat{F}$ satisties equation

$$\hat{F}_{yyy} + \hat{F}\hat{F}_y - \frac{1}{3}y\hat{F}_y - \frac{2}{3}\hat{F} = 0.$$

3.2 Benjamin-Bona-Mahony Equation

Another equation of interest to this thesis is the Benjamin-Bona-Mahony (BBM) equation that can be written as

$$u_t = uu_x + u_{txx}, \tag{3.37}$$

where x and t are independent variables and u is a function of these independent variables. We shall use the method of differential constraints again. First, assume that

$$u_t = \varphi(u_x, u, x, t), \tag{3.38}$$

a differential constraint for equation (3.37). Without loss of generality one can assume $\varphi_u^2 + \varphi_{u_x}^2 \neq 0$. Otherwise, the method of differential constraints does not work. In this case, $u_t = \varphi(x,t)$ with no other restrictions for the function $\varphi(x,t)$: for any solution of the BBM equation one can select the function $\varphi(x,t)$. We can derive the derivatives u_{tx}, u_{tt}, u_{xxt}, u_{xtt} and u_{ttt}, as in the KdV equation. After substituting u_t and u_{txx} into equation (3.37), the following equation is obtained:

$$\begin{aligned} \varphi_{u_x} u_{xxx} + \ \varphi_{uu} u_x^2 + \ 2\varphi_{uu_x} u_{xx} u_x + \ \varphi_{u_x u_x} u_{xx}^2 & \\ +(\varphi_u + 2\varphi_{u_x x}) u_{xx} + (u + 2\varphi_{ux}) u_x + \varphi_{xx} - \varphi \ &= \ 0. \end{aligned} \tag{3.39}$$

This thesis studies the equation (3.39) in the case $\varphi_{u_x} \neq 0$. The derivative u_{xxx} can be obtained from equation (3.39)

$$\begin{aligned} u_{xxx} \ &= \ \tfrac{-1}{\varphi_{u_x}} \left(\varphi_{uu} u_x^2 + \ 2\varphi_{uu_x} u_{xx} u_x + \ \varphi_{u_x u_x} u_{xx}^2 \right. \\ &\quad \left. + (\varphi_u + 2\varphi_{u_x x}) u_{xx} + (u + 2\varphi_{ux}) u_x + \varphi_{xx} - \varphi\right). \end{aligned} \tag{3.40}$$

Hence, one can find all third order derivatives through u_{xx}, u_x, u, x and t. Since the function u is a sufficiently continuously differentiable function, we can have the following consistency conditions:

$$\begin{aligned} u_{xxx,t} - u_{xxt,x} \ &= \ 0, \\ u_{xxt,t} - u_{xtt,x} \ &= \ 0, \\ u_{xtt,t} - u_{ttt,x} \ &= \ 0. \end{aligned} \tag{3.41}$$

In system (3.41) as before, the commas denote differentiation. As with the KdV equation in previous section, we start working with the first consistency condition. The left side of this equation is a polynomial function with respect to u_{xx}. Since this thesis studies the case where consistency conditions do not produce new equations, all of coefficients with respect to u_{xx}^3, u_{xx}^2 and u_{xx} must be equal to zero. Therefore,

$$\frac{\varphi_{u_x u_x u_x} \varphi_{u_x} - \varphi_{u_x u_x}^2}{\varphi_{u_x}} = 0, \tag{3.42}$$

$$\begin{aligned} & u_x \left(- 2\, \varphi_{uu_x u_x} \varphi_{u_x}^2 + \ 2\, \varphi_{uu_x} \varphi_{u_x u_x} \varphi_{u_x} - \ \varphi_u \varphi_{u_x u_x u_x} \varphi_{u_x} + \ \varphi_u \varphi_{u_x u_x}^2 \right) \\ & \quad - \varphi_{uu_x u_x} \varphi_{u_x} \varphi + \ \varphi_{uu_x} \varphi_{u_x u_x} \varphi - \ 3\varphi_{uu_x} \varphi_{u_x}^2 \ + \ \varphi_u \varphi_{u_x u_x} \varphi_{u_x} \\ & \quad - \varphi_{tu_x u_x} \varphi_{u_x} + \varphi_{tu_x} \varphi_{u_x u_x} + 2\varphi_{u_x x} \varphi_{u_x u_x} \varphi_{u_x} \\ & \quad - \varphi_{u_x u_x u_x} \varphi_{u_x} \varphi_x - 2\varphi_{u_x u_x x} \varphi_{u_x}^2 + \varphi_{u_x u_x}^2 \varphi_x = 0, \end{aligned} \tag{3.43}$$

$$u_x^2\ (\ -2\ \varphi_{uu_xu_x}\varphi_u\varphi_{u_x} +\ 2\ \varphi_{uu_x}\varphi_u\varphi_{u_xu_x} -\ \varphi_{uuu_x}\varphi_{u_x}^2$$
$$+\varphi_{uu}\varphi_{u_xu_x}\varphi_{u_x}) + u_x(-2\varphi_{utu_x}\varphi_{u_x} + 2\varphi_{uu_x}\varphi_{tu_x} - 2\varphi_{uu_xx}\varphi_{u_x}^2$$
$$-2\varphi_{uu_xu_x}\varphi_{u_x}\varphi_x - 3\varphi_{uu_x}\varphi_u\varphi_{u_x} + 2\varphi_{uu_x}\varphi_{u_xu_x}\varphi_x + 2\varphi_{ux}\varphi_{u_xu_x}\varphi_{u_x}$$
$$-2\varphi_{uuu_x}\varphi_{u_x}\varphi + \varphi_u^2\varphi_{u_xu_x} +\ 2\varphi_u\varphi_{u_xx}\varphi_{u_xu_x} - 2\varphi_u\varphi_{u_xu_xx}\varphi_{u_x} \tag{3.44}$$
$$+3\varphi_{u_xu_x}\varphi_{u_x}u + 2\varphi_{uu_x}\varphi - 2\varphi_{uu}\varphi_{u_x}^2) - \varphi_{ut}\varphi_{u_x} + \varphi_{uu_x}\varphi_u\varphi$$
$$-\varphi_{u_xxx}\varphi_{u_x}^2 + 2\varphi_{uu_x}\varphi_{u_xx}\varphi - 3\varphi_{uu_x}\varphi_{u_x}\varphi_x - 2\varphi_{ux}\varphi_{u_x}^2 - \varphi_{uu}\varphi_{u_x}\varphi$$
$$+\varphi_u\varphi_{u_x} - 2\varphi_{uu_xx}\varphi_{u_x}\varphi + \varphi_u\varphi_{u_xu_x}\varphi_x +\ 2\varphi_{tu_x}\varphi_{u_xx} - 2\varphi_{tu_xx}\varphi_{u_x}$$
$$+2\ \varphi_{u_xx}\varphi_{u_xu_x}\varphi_x -\ 2\ \varphi_{u_xu_xx}\varphi_{u_x}\varphi_x + \varphi_{u_xu_x}\varphi_{u_x}\varphi_{xx} - 3\varphi_{u_xu_x}\varphi_{u_x}\varphi = 0,$$

and

$$u_x^3\ \varphi_u(-\ \varphi_{uuu_x}\varphi_{u_x} + \varphi_{uu}\varphi_{u_xu_x}) + u_x^2\ (-2\ \varphi_{uu_xx}\varphi_u\varphi_{u_x} + \varphi_{uu_x}\varphi_{uu}\varphi$$
$$+2\varphi_{uu_x}\varphi_{u_x}u + 2\varphi_{ux}\varphi_u\varphi_{u_xu_x} - \varphi_{uuu}\varphi_{u_x}\varphi - \varphi_{uut}\varphi_{u_x} - \varphi_{uuu_x}\varphi_{u_x}\varphi_x$$
$$-2\ \varphi_{uu}\varphi_u\varphi_{u_x} + \varphi_{uu}\varphi_{tu_x} + \varphi_{uu}\varphi_{u_xu_x}\varphi_x + \varphi_u\varphi_{u_xu_x}u + \varphi_{u_x}^2\ ) +\ u_x($$
$$-2\ \varphi_{utx}\varphi_{u_x} - 2\varphi_{uu_xx}\varphi_{u_x}\varphi_x + 2\varphi_{uu_x}\varphi_{ux}\varphi - 2\varphi_{uu_x}\varphi_{u_x}\varphi +\ \varphi_{uu_x}\varphi u$$
$$-2\ \varphi_{ux}\varphi_u\varphi_{u_x} + 2\ \varphi_{ux}\varphi_{tu_x} +\ 2\varphi_{ux}\varphi_{u_xu_x}\varphi_x - 2\ \varphi_{uux}\varphi_{u_x}\varphi -\ \varphi_{u_x}\varphi \tag{3.45}$$
$$-2\ \varphi_{uu}\varphi_{u_x}\varphi_x -\ \varphi_u\varphi_{u_xxx}\varphi_{u_x} +\ \varphi_u\varphi_{u_xu_x}\varphi_{xx} - \varphi_u\varphi_{u_xu_x}\varphi +\ \varphi_{tu_x}u$$
$$+\ 2\varphi_{u_xx}\varphi_{u_x}u +\ \varphi_{u_xu_x}\varphi_x u) + \varphi_{uu_x}\varphi_{xx}\varphi - \varphi_{uxx}\varphi_{u_x}\varphi - 2\varphi_{ux}\varphi_{u_x}\varphi_x$$
$$-\ \varphi_{uu_x}\ \varphi^2 -\ \varphi_{txx}\ \varphi_{u_x} -\ \varphi_{u_xxx}\varphi_{u_x}\varphi_x -\ 2\ \varphi_{u_xx}\ \varphi_{u_x}\ \varphi -\ \varphi_{u_x}\ \varphi_x u$$
$$+\varphi_{u_xu_x}\varphi_{xx}\varphi_x - \varphi_{u_xu_x}\varphi_x\varphi + \varphi_{tu_x}\varphi_{xx} + \varphi_t\varphi_{u_x} = 0.$$

To solve equation (3.42) for φ_{u_x}, let $V = \varphi_{u_x}$. Equation (3.42) can then be rewritten as:

$$\frac{V_{u_xu_x}}{V_{u_x}} - \frac{V_{u_x}}{V} = 0.$$

Integrating the last equation with respect to u_x, one gets:

$$\varphi_{u_xu_x} = \varphi_{u_x}a(u, x, t).$$

Again integrating this equation with respect to u_x yields:

$$\varphi_{u_x} = a(u,x,t)\varphi + b(u,x,t), \tag{3.46}$$

and after substituting equation (3.46) into equation (3.43) one gets

$$-4u_x a_u(a\varphi + b) + 2(-4a_u\varphi - a_t - 2a_x a\varphi - 2a_x b - 3b_u - 2\varphi_u a) = 0. \tag{3.47}$$

If $a \neq 0$, one can find φ_u from the last equation. Thus we study two cases: a) $a \neq 0$ and b) $a = 0$.

Let $a \neq 0$. It will be shown that in this case one obtains a contradiction. Consider the following derivative in equation (3.47).

$$\varphi_u = \frac{1}{2a}(-2u_x a_u(a\varphi + b) - 4a_u\varphi - a_t - 2a_x a\varphi - 2a_x b - 3b_u). \tag{3.48}$$

Differentiating equation (3.48) with respect to u_x yields

$$\varphi_{uu_x} = \frac{1}{a}(-u_x a_u a(a\varphi + b) - 3a_u a\varphi - 3a_u b - a_x a^2\varphi - a_x ab). \tag{3.49}$$

After differentiating equation (3.46) with respect to u one has

$$\varphi_{u_x u} = \frac{1}{2}(-2u_x a_u(a\varphi + b) - 2a_u\varphi - a_t - 2a_x a\varphi - 2a_x b - b_u). \tag{3.50}$$

The following mixed derivatives are equal. Hence,

$$\varphi_{uu_x} - \varphi_{u_x u} = \frac{1}{2a}(4a_u a\varphi + 6a_u b - a_t a - b_u a) = 0. \tag{3.51}$$

Differentiating equation (3.51) with respect to u_x gives

$$a_u a\varphi_{u_x} = 0. \tag{3.52}$$

After substituting φ_{u_x} in equation (3.46) into equation (3.52), it becomes

$$a_u a(a\varphi + b) = 0. \tag{3.53}$$

Since $a(a\varphi + b) \neq 0$, $a_u = 0$; so:

$$a = a(x,t). \tag{3.54}$$

Equation (3.51) becomes

$$a_t + b_u = 0. \tag{3.55}$$

Since the solution of equation (3.46) is

$$\varphi = e^{au_x} d(u, x, t) - \frac{b}{a}, \tag{3.56}$$

where d is a function of the independent variables u, x and t,

$$\varphi_u = \frac{1}{a}(a_t + e^{au_x} d_u a). \tag{3.57}$$

But, from equation (3.48), by using equation (3.55) one obtains

$$\varphi_u = \frac{1}{a}(a_t - e^{au_x} d a a_x). \tag{3.58}$$

Comparing φ_u in equation (3.57) and (3.58) gives

$$e^{u_x a}(a_x d + d_u) = 0. \tag{3.59}$$

Since $e^{u_x a}$ is not equal to zero, hence

$$d_u = -a_x d. \tag{3.60}$$

Therefore, equation (3.44) becomes:

$$\begin{aligned} &3e^{2u_x a} a^5 d^3 (a_{xx} - a) + e^{u_x a} u_x\, a^4 d^2 (2a_{tx} a - 2a_t a_x + \ 3a^2 u) + e^{u_x a} a^3 \\ &\quad (-3\ a_{tx} a d^2 + \ 2a_t a_x d^2 + a_t d_x a d - a_{xx} a b d^2 - 2a_x^2 b d^2 + \ 2a_x b_x a d^2 \\ &\quad -b_{xx} a^2 d^2 - 2d_{tx} a^2 d + 2d_t d_x a^2 + 3a^2 b d^2) + 2u_x a_t^2 a^3 d + a^2 \\ &\quad (-a_{tt} a d + 2a_t^2 d + 2a_t a_x b d - a_t b_x a d + a_t d_t a) = 0. \end{aligned} \tag{3.61}$$

Because the left side of equation (3.61) is a polynomial function with respect to e^{au_x}, all coefficients of e^{2au_x} and e^{au_x} must be equal to zero. Hence,

$$a^5 d^3 (a_{xx} - a) = 0, \tag{3.62}$$

$$u_x\ a^4d^2(2a_{tx}a - 2a_ta_x + \ 3a^2u)$$
$$+\ a^3(-3\ a_{tx}ad^2 + \ 2a_ta_xd^2 + \ a_td_xad - \ a_{xx}abd^2 \qquad (3.63)$$
$$-\ 2a_x^2bd^2 + \ 2a_xb_xad^2 - b_{xx}a^2d^2 - 2d_{tx}a^2d + 2d_td_xa^2 + 3a^2bd^2) = 0,$$

and

$$2u_xa_t^2a^3d + a^2(-a_{tt}ad + 2a_t^2d + 2a_ta_xbd - a_tb_xad + a_td_ta) = 0. \qquad (3.64)$$

Since $d \neq 0$, equation (3.62) gives

$$a_{xx} = a. \qquad (3.65)$$

In equation (3.63) the left side of the equation is a polynomial function with respect to u_x. Therefore, one obtains

$$a^4d^2(2a_{tx}a - 2a_ta_x + 3a^2u) = 0. \qquad (3.66)$$

Differentiating the last equation with respect to u one gets $a = 0$, which contradicts the assumption. Therefore $a = 0$.

For $a = 0$, equation (3.46) gives

$$\varphi_{u_x} = b(u, x, t). \qquad (3.67)$$

From equation (3.47), one obtains that b is a function of just the independent variables x and t. After integrating equation (3.67) with respect to u_x, one has

$$\varphi = b(x, t)u_x + c(u, x, t). \qquad (3.68)$$

Equation (3.44) can thus be rewritten as follows

$$-3u_xc_{uu}b^2 - 2b_{tx}b + 2b_tb_x + b_tc_u - \ b_{xx}b^2 - c_{ut}b - \ 2c_{ux}b^2 - \ c_{uu}bc = 0. \qquad (3.69)$$

The left side of equation (3.69) is a polynomial function with respect to u_x, hence,

$$c_{uu} = 0, \quad -2b_{tx}b + 2b_tb_x + b_tc_u - \ b_{xx}b^2 - c_{ut}b - \ 2c_{ux}b^2 = 0. \qquad (3.70)$$

Solving the first equation of system (3.70) one has

$$c = c_1(x,t)u + c_2(x,t), \tag{3.71}$$

where c_1 and c_2 being functions of independent variables x and t. Hence, equation (3.45) shows that

$$\begin{aligned} u_x\,(-\,b_{txx}b + b_t b_{xx} + \ 2b_t c_{1x} + \ b_t u - b_{xx}b_x b - \ b_{xx}bc_1 - \ 2b_x c_{1x} b \\ -2b_x b^2 + b_x bu - 2c_{1tx}b - c_{1xx}b^2 - 2c_{1x}bc_1 - bc_1 u - bc_2) - b_t c_2 \\ + b_t c_{1xx}u + \ b_t c_{2xx} - \ b_t c_1 u - b_{xx}c_{1x}bu - b_{xx}c_{2x}b - \ 2b_x bc_1 u \\ - \ 2b_x bc_2 - c_{1txx}bu + \ c_{1t}bu - c_{1xx}bc_1 u - \ c_{1xx}bc_2 - \ 2c_{1x}^2 bu \\ -2c_{1x}c_{2x}b - c_{1x}bu^2 + c_{2t}b - c_{2txx}b - c_{2x}bu = 0. \end{aligned} \tag{3.72}$$

The left side of equation (3.72) is a polynomial function with respect to u_x. Thus:

$$\begin{aligned} -b_{txx}b + b_t b_{xx} + 2b_t c_{1x} + b_t u - b_{xx}b_x b - b_{xx}bc_1 - 2b_x c_{1x}b \\ -2b_x b^2 + b_x bu - 2c_{1tx}b - c_{1xx}b^2 - 2c_{1x}bc_1 - bc_1 u - bc_2 = 0, \end{aligned} \tag{3.73}$$

$$\begin{aligned} - \ b_t c_2 + \ b_t c_{1xx}u + \ b_t c_{2xx} - \ b_t c_1 u - \ b_{xx}c_{1x}bu - \ b_{xx}c_{2x}b \\ - \ 2b_x bc_1 u - 2b_x bc_2 - c_{1txx}bu + c_{1t}bu - c_{1xx}bc_1 u - c_{1xx}bc_2 \\ -2c_{1x}^2 bu - \ 2c_{1x}c_{2x}b - c_{1x}bu^2 + c_{2t}b - c_{2txx}b - c_{2x}bu = 0. \end{aligned} \tag{3.74}$$

The left side of equation (3.73) and (3.74) are polynomial functions with respect to u, so:

$$b_t + b_x b - bc_1 = 0, \quad -b_{txx}b + b_t b_{xx} - b_{xx}b_x b - b_{xx}bc_1 - 2b_x b^2 - bc_2 = 0, \tag{3.75}$$

$$\begin{aligned} c_{1x} = 0, \quad -b_t c_1 - 2b_x bc_1 + \ c_{1t}\,b - \ c_{2x}\,b = 0, \\ b_t c_{2xx} - b_{xx}c_{2x}b - 2b_x bc_2 + \ c_{2t}\,b - b_t c_2 - c_{2txx}b = 0. \end{aligned} \tag{3.76}$$

The first equation of system (3.75) shows that

$$c_1 = \frac{b_t + b_x b}{b}. \tag{3.77}$$

The second equation of system (3.75) yields:

$$c_2 = \frac{1}{b}\left(-b_{txx}b + b_t b_{xx} - 2b_{xx}b_x b - b_{xx}b_t - 2b_x b^2\right). \tag{3.78}$$

After replacing c_1 by equation (3.77) in the first equation of system (3.76), one has

$$b_{tx} = \frac{b_t b_x - b_{xx}b^2}{b}. \tag{3.79}$$

Note that from the first equation of system (3.76), $c_1 = c_1(t)$ is a function of only independent variable t. The second equation of system (3.70) and the second equation of system (3.76) can be rewritten as:

$$b_{tt} = \frac{2b_t^2 + 2b_{xx}b^3}{b}, \tag{3.80}$$

$$b_{xx} = \frac{2b_t b_x}{3b^2}. \tag{3.81}$$

The next step is to obtain the function b through considering the following two consistency conditions:

$$b_{xx,t} - b_{tx,x} = 0, \quad b_{tx,t} - b_{tt,x} = 0, \tag{3.82}$$

yielding:

$$b_t b_x^2 = 0, \quad b_t^2 b_x = 0.$$

Hence, the function b can be considered in two cases: *a*) $b_t = 0$ and *b*) $b_x = 0$.

Case 2.1 $b_t = 0$.

In this case

$$b = b(x). \tag{3.83}$$

Equation (3.77) gives the representation:

$$c_1 = b_x, \tag{3.84}$$

and from equation (3.78),

$$c_2 = -2b_{xx}b_x - 2b_x b. \tag{3.85}$$

Using equation (3.83), equation (3.80) becomes

$$b_{xx} = 0. \tag{3.86}$$

Solving equation (3.86), the following is obtained

$$b = k_1 x + k_2, \tag{3.87}$$

where k_1 and k_2 are constants. The third equation of system (3.76) leads to:

$$k_1 = 0. \tag{3.88}$$

Thus, $c_1 = 0$, $c_2 = 0$ and

$$\varphi = k_2 u_x. \tag{3.89}$$

This concludes the consideration of consistency conditions of system (3.41). The last step in this case is finding the function u itself. After solving the differential constraint $u_t = k_2 u_x$, one obtains that

$$u(x,t) = g(k_2 t + x) \tag{3.90}$$

is a solution of the BBM equation for this case, where g satisfies equation

$$k_2 g''' + gg' - k_2 g' = 0.$$

Case 2.2 $b_x = 0.$

In this case b is a function that depends on the independent variable t: $b = b(t)$. Hence from equation (3.77),

$$c_1 = \frac{b_t}{b} \tag{3.91}$$

Equation (3.78) shows that

$$c_2 = 0. \tag{3.92}$$

Hence

$$c = c_1 u, \tag{3.93}$$

and from equation (3.80),

$$b_{tt} = \frac{2b_t^2}{b}. \tag{3.94}$$

The last equation can be rewritten

$$\frac{b_{tt}}{b_t} = \frac{2b_t}{b}. \tag{3.95}$$

After integrating equation (3.95) with respect to t, one gets

$$b_t = b^2 k_3, \tag{3.96}$$

where k_3 is a constant. If $k_3 = 0$ then equation (3.96) shows that b is a constant. Hence the solution can be found in the case $b_t = 0$. If $k_3 \neq 0$ then equation (3.96) can be integrated to obtain

$$b = \frac{-1}{j + k_3 t}, \tag{3.97}$$

where j is a constant. Thus

$$c_1 = \frac{-k_3}{j + k_3 t}. \tag{3.98}$$

Therefore the function φ, defined by equation (3.68), is

$$\varphi = \frac{-1}{j + k_3 t} u_x + \frac{-k_3}{j + k_3} u. \tag{3.99}$$

The differential constraint (3.38) becomes:

$$(j + k_3 t) u_t + u_x = -k_3 u. \tag{3.100}$$

By using the Cauchy method (see chapter I), with the following characteristic equations

$$\frac{dt}{j + k_3 t} = \frac{dx}{1} = \frac{du}{-k_3 u}, \tag{3.101}$$

we obtain the following two invariants

$$\Gamma_1 = \frac{1}{k_3}\, ln \ \mid j + k_3 t \mid - x, \quad \Gamma_2 = u e^{k_3 x}.$$

They are discovered by

$$\frac{dt}{j+k_3t}=\frac{dx}{1},\quad \frac{dx}{1}=\frac{du}{-k_3u},$$

respectively. By changing $\hat{t}=t+\dfrac{j}{k_3}$ and $\alpha=\dfrac{1}{k_3}$ one can rewrite

$$\Gamma_1=x-\alpha\, ln\ \mid \hat{t}\mid .$$

Hence the general solution of equation (3.100) and the solution of the BBM equation in this case, is $H(\xi, ue^{k_3x})=0$, which can be written as

$$u(x,t)=G(\xi)e^{-k_3x}, \tag{3.102}$$

where $\xi=x-\alpha\, ln\ \mid \hat{t}\mid$, and G satisfies:

$$G'''-2k_3G''+(k_3^2-1)G'-k_3\beta GG'+k_3^2\beta G^2=0,$$

here $\beta=e^{-k_3x}t$.

Chapter IV

Conclusion

4.1 Thesis Summary

The thesis is devoted to applying the intermediate integrals technique and the method of differential constraints to some partial differential equations.

Firstly, the Monge-Ampere equation

$$u_{xt}^2 - u_{xx}u_{tt} = a(x,t),$$

where a is a constant, was studied, applying the intermediate integral technique. Without loss of generality, the cases $a = \pm 1$ or $a = 0$ are sufficient. The intermediate integral concerned has the form

$$u_t = \varphi(u_x, u, x, t). \tag{4.1}$$

Full analysis of such intermediate integral is done (section 2.1) and 2.2)).

When the same technique was applied to the KdV equation, calculations showed that there was no intermediate integral of first order for this equation. Therefore, a more general method; the method of differential constraints, was applied. This method was also applied to the BBM equation.

For the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0.$$

It was found that the differential constraint of first order (4.1) yielded

$$u_t = -\frac{x - 6k_2t}{3t}u_x + \frac{k_2 - 2u}{3t},$$

where k_2 is a constant, leading to the solution $u(x,t) = t^{2/3}\hat{F}(xt^{-1/3})$, when $k_2 = 0$ and $\hat{F}$ satisfies

$$\hat{F}_{yyy} + \hat{F}\hat{F}_y - \frac{1}{3}y\hat{F}_y - \frac{2}{3}\hat{F} = 0.$$

For the Benjamin-Bona-Mahony equation,

$$u_t - uu_x - u_{txx} = 0,$$

the method of differential constraints gives the following two types of differential constraints (4.1).

The first differential constraint is

$$u_t = k_2 u_x,$$

where k_2 is a constant. After solving this equation for $u(x,t)$, one has $u(x,t) = g(k_2 t + x)$ is a solution of the BBM equation, where g satisfies equation

$$k_2 g''' + gg' - k_2 g' = 0.$$

The second differential constraint is

$$u_t = \frac{-1}{j + k_3 t}u_x + \frac{-k_3}{j + k_3}u,$$

where j and k_3 are constants. It is solved to obtain the solution $u(x,t) = G(\xi)e^{-k_3 x}$, where $\xi = x - \alpha \, ln \ \mid \hat{t} \mid$, and G satisfies:

$$G''' - 2k_3 G'' + (k_3^2 - 1)G' - k_3\beta GG' + k_3^2\beta G^2 = 0,$$

where $\beta = e^{-k_3 x}t$.

4.2 Applications and Comments

The method of differential constraints can be applied to any partial differential equation. The intermediate integral technique can be considered as a particular case of the method of differential constraints. Obtaining solutions by the intermediate integral technique is easier than the method of differential constraints.

But sometimes solution of partial differential equations cannot be obtained by this technique. In this case, one has to use more general techniques. For example, in the study of KdV and BBM we obtained the negative results: there are no intermediate integral of first order for these equation. Then the more general method, differential constraints, was applied.

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Appendix

Appendix A

KdV Reduce Code

```
% KdV EQUATION %
% The reduce program for solving the " KdV " equation by the
% differential constraint method.
depend phi, ux, q, x, t;
depend f, q, x, t, ux, uxx, uxxx, uxxxx;

% Define the following operator;
dx := df(f,x) + df(f,q)*ux + df(f,ux)*uxx + df(f,uxx)*uxxx $
dt := df(f,t) + df(f,q)*ut + df(f,ux)*uxt + df(f,uxx)*uxxt$

factor ux, uxx, uxxx;

% Assume the existence of the following " differential constraint ":
ut := phi;

% Derive the following derivatives by using previous assumption.
utt := sub(f = ut, dt)$
uxt := sub(f = ut, dx)$

uttt := sub(f=utt, dt)$
uxtt := sub(f=utt, dx)$
uxxt:= sub(f=uxt, dx)$

% The following is the " KdV " equation.
uxxx:= -ut +6*q*ux ;

% Consider the following three constraints:
ss1:= sub(f=uxxx,dt) -  sub(f=uxxt,dx)$
ss2:= sub(f=uxxt,dt) - sub(f=uxtt,dx)$
ss3:= sub(f=uxtt,dt) - sub(f=uttt,dx)$

ss:= ss1$
ssq:= df(ss,uxx,3);
j:=df(ssq,df(phi,ux,3));
df(phi,ux,3):= df(phi,ux,3) - ssq/j;
ssq;

% After solving for " phi " we obtain:
depend a, q, x, t;
depend b, q, x, t;
depend c, q, x, t;
phi:=1/2*(a*ux^2) + b*ux +c;
ss:=ss1$
df(ss,uxx,ux,3);

% Solving for " a " we obtain:
depend a1,x,t;
depend a2,x,t;
a:=q*a1+a2;
```

```
ssq:=df(ss,uxx,2,ux);

% Hence we obtain;
a1:=0;
ss:=ss1$
ssq:=df(ss,uxx,2);
df(b,q):= -df(a2,x);
ss:=ss1$
ssq:=df(ss,ux,2,uxx);

% Solving for " a2 " we obtain:
a2:=0;
ss:=ss1$
ssq:=df(ss,uxx,ux);
j:=df(ssq,df(c,q,2));
df(c,q,2):=df(c,q,2)-ssq/j;
ssq;

% After solving for " c " we obtain:
depend c1, x, t;
depend c2, x, t;
c:=c1*q + c2;
ss:=ss1$
ssq:=df(ss,ux,q);
df(b,x):=1/2*c1;

ssq;
ss:=ss1$
ssq:=df(ss,uxx);
j:=df(ssq,df(c1,x));
df(c1,x):=df(c1,x)-ssq/j;
ssq;

% Solving for " c1 " we obtain:
depend c10, t;
c1:=c10;

%  Solving for " b ", we obtain:
depend  b1, q, t;
b:=1/2*c10*x + b1;
ss:=ss1$
ssq:=df(ss,uxx,2);
j:=df(ssq,df(b1,q));
df(b1,q):=df(b1,q) - ssq/j;
ssq;

% After solving for " b1 " we obtain;
depend b10, t;
b1:=b10;
ss:=ss1$
ssq:=df(ss,q);
df(c2,x):=1/12*(2*df(c10,t) - 3*c10^2);
ss:=ss1;

% Consider the other therm of  " ux ", we have:
df(c2,t):=3/2*c10*c2;
ss:=ss1;
ssq:=df(ss,ux,x);

% After solving for " c10 ", we let  " k1 " is a constant, obtaining:
c10:=2*(1/(-3*t+2*k1));
```

```
df(c2,t);

% After solving for " c2 ", we obtain:
depend c20, x;
c2:=(c20)/(-2*k1+3*t);
ss:=ss1$
ssq:=df(ss,q);
j:=df(ssq,df(c20,x));
df(c20,x):=df(c20,x)-ssq/j;
ssq;

% After solving for " c20 ", we let " k20 " is a constant so that:
c20:=k20;
ss:=ss1;
ssq:=df(ss,ux);

% After solving for " b10 ", we let " k2 " is a constant then:
b10:=(6*k20*t+k2)/(-2*k1+3*t);
ssq;
ss:=ss1;
ss:=ss2;
ss:=ss3;

% Hence we obtain the following differential constraint:

phi;

end;
```

Appendix B

BBM Reduce Code

The reduce program to obtain the solution of the BBM equation has three program, they consider the function $u_t = \varphi(u_x, u, x, t)$, where $\varphi_{u_x} = a(u, x, t)\varphi + b(u, x, t)$ in two case: the first case is $a \neq 0$, obtained the contradiction. The second case is $a = 0$, consider the function $\varphi = b(x, t)u_x + c(u, x, t)$ in two case: case $b_t = 0$ and $b_x = 0$ respectively.

Consideration $\varphi_{u_x} \neq 0$, where $\varphi_{u_x} = a(u, x, t)\varphi + b(u, x, t)$.

B.1 Case $a \neq 0$

```
      % THE BENJAMIN-BONA-MAHONY EQUATION %
% The reduce program for finding the solution
% of the Benjamin-Bona-Mahony equation
% case " df(phi,ux) is not equal to zero ":

depend phi, ux, q, x, t;
depend f, q, x, t, ux, uxx, uxxx, uxxxx;

% Define the following operator:
dx := df(f,x) + df(f,q)*ux + df(f,ux)*uxx +
df(f,uxx)*uxxx + df(f,uxxx)*uxxxx$
dt := df(f,t) + df(f,q)*ut + df(f,ux)*uxt +
df(f,uxx)*uxxt + df(f,uxxx)*uxxxt$

factor ux, uxx, uxxx, uxxxx;

% Asuming the following " differential constraint ":
ut := phi;

% Using this differential constaint to find the following
% two second derivatives:
utt := sub(f = ut, dt);
uxt := sub(f = ut, dx);

% and the following three third derivatives:
uttt := sub(f=utt, dt)$
uxtt := sub(f=utt, dx)$
uxxt:= sub(f=uxt,dx)$
```

```
% Substituting derivatives " ut " and " uxxt " from previous
% step into the following " BBM equation " to obtain:

ss:= ut - q*ux - uxxt$

j:=df(ss,uxxx);
% The following derivative " uxxx " is discovered from
% the last equation:
uxxx:=uxxx-ss/j;
ss;

% Defind the following two operator:
dx := df(f,x) + df(f,q)*ux + df(f,ux)*uxx+ df(f,uxx)*uxxx$
dt := df(f,t) + df(f,q)*ut + df(f,ux)*uxt + df(f,uxx)*uxxt$

% Consider the following consistency conditions:
ss1:= sub(f=uxxx,dt) -  sub(f=uxxt,dx)$
ss2:= sub(f=uxxt,dt) -  sub(f=uxtt,dx)$
ss3:= sub(f=uxtt,dt) -  sub(f=uttt,dx)$

ss:=ss1$
ssq:=df(ss,uxx,3);

% After solving for " phi " obtaining;
depend a, q, x, t;
depend b, q, x, t;
phi_ux:=df(phi,ux):=a*phi + b;
ssq;

ss:=ss1$
ssq:=df(ss,uxx,2);

ss:=df(ss1,uxx,2);
sss:=ss1$

%%%% The case " a neq 0 " %%%%

j:=df(ss,df(phi,q));
phi_q:=df(phi,q):=df(phi,q)-ss/j;
ss;

% Consider the following constraint:
ss:=df(df(phi,ux),q)-df(df(phi,q),ux);
ss:= num ss;
ssq:=df(ss,ux);
j:=df(ssq,df(a,q));
df(a,q):=df(a,q)-ssq/j;
ssq;
ss;

j:=df(ss,df(b,q));
df(b,q):=df(b,q)-ss/j;
ss;

ss1:=ss1$

% Solving for " phi " to obtain:
depend c1,q,x,t;
phi:=c1*e**(a*ux)-b/a;

% Consider the following results:
```

```
phi_ux-df(phi,ux);
phi_q-df(phi,q);
df(c1,q):=-c1*df(a,x);
phi_q-df(phi,q);

ss1:=num ss1$
factor e**(a*ux);
df(ss1,uxx,e**(ux*a),2);
df(a,x,2):=a;
df(ss1,uxx,e**(ux*a),2);
ssq:=df(ss1,uxx,e**(ux*a));
ssq:=df(ssq,ux)/c1**2;
ssq:=df(ssq,q);

%% The last step show that " a " must be zero %%

end;
```

B.2 Case $a = 0$

This thesis study two cases

B.2.1 Case $b_t = 0$

```
               % THE BENJAMIN-BONA-MAHONY EQUATION %

% The reduce program for solving the " BBM equation "
%  case "df(phi,ux) is not equal to zero" but "a=0":

depend phi, ux, q, x, t;
depend f, q, x, t, ux, uxx, uxxx, uxxxx;

% Define the following operators:
dx := df(f,x) + df(f,q)*ux + df(f,ux)*uxx + df(f,uxx)*uxxx
+ df(f,uxxx)*uxxxx$
dt := df(f,t) + df(f,q)*ut + df(f,ux)*uxt + df(f,uxx)*uxxt
+ df(f,uxxx)*uxxxt$

factor ux, uxx, uxxx, uxxxx;

% First, assume the existence of first order differential
% constraint :
ut := phi;

% Next we find the derivative by using previous assumption,
%  the following derivatives is obtained:

utt := sub(f = ut, dt);
uxt := sub(f = ut, dx);

uttt := sub(f=utt, dt)$
uxtt := sub(f=utt, dx)$
```

```
uxxt:= sub(f=uxt,dx)$

%  Substituting these derivatives in the following " BBM equation ",
%  obtaining;

ss:= ut - q*ux - uxxt$

j:=df(ss,uxxx);
% We consider the derivative " uxxx "  from previous equation.
uxxx:=uxxx-ss/j;
ss;

dx := df(f,x) + df(f,q)*ux + df(f,ux)*uxx+ df(f,uxx)*uxxx$
dt := df(f,t) + df(f,q)*ut + df(f,ux)*uxt + df(f,uxx)*uxxt$

% Consider the following consistency conditions:
ss1:= sub(f=uxxx,dt) -  sub(f=uxxt,dx)$
ss2:= sub(f=uxxt,dt) -  sub(f=uxtt,dx)$
ss3:= sub(f=uxtt,dt) -  sub(f=uttt,dx)$

ss:=ss1$
ssq:=df(ss,uxx,3);

% After solving for phi obtaining;
depend a, q, x, t;
depend b, q, x, t;
phi_ux:=df(phi,ux):=a*phi + b;
ssq;

ss:=ss1$
ssq:=df(ss,uxx,2);

%% Consider  " a = 0". %%
a:=0;
ss:=ss1$
ssq:=df(ss,uxx,2);
j:=df(ssq,df(b,q));
df(b,q):=df(b,q)-ssq/j;
ssq;

% After solving for  " phi " we obtain;
depend c,q,x,t;
phi:=b*ux+c;
phi_ux-df(phi,ux);
ss:=ss1$
ssq:=df(ss,uxx,ux);
j:=df(ssq,df(c,q,2));
df(c,q,2):=df(c,q,2)-ssq/j;
ssq;

% After solving for " c2 " we obtain;
depend c1, x, t;
depend c2, x, t;
c:=c1*q+c2;
ssq;
ss:=ss1$
ssq:=df(ss,ux,q);
j:=df(ssq,c1);
c1:=c1-ssq/j;
ssq;
```

```
ss:=ss1$
ssq:=df(ss,ux);
j:=df(ssq,c2);
c2:=c2-ssq/j;
ssq;

ss:=ss1$
ssq:=df(ss,q,2);
j:=df(ssq,df(b,t,x));
df(b,t,x):=df(b,t,x) - ssq/j;
ssq;

ss:=ss1$
ssq:=df(ss,uxx);
j:=df(ssq,df(b,t,2));
df(b,t,2):=df(b,t,2)-ssq/j;
ssq;

ss:=ss1$
ssq:=df(ss,q);
j:=df(ssq,df(b,x,2));
df(b,x,2):=df(b,x,2)-ssq/j;
ssq;

% Consider the following two constraints:
sss1:= df(b,x,2,t) - df(b,t,x,2);
sss2:= df(b,t,x,t) - df(b,t,2,x);

% To find "b" we consider 2 case:
% case 1;
df(b,t):=0;
ss:=sss1;

% This impies that:
depend b1, x;
b:=b1;
c1;
c2;

ss:=ss1$
ssq:=df(ss,uxx);
j:=df(ssq,df(b,x,2));
df(b,x,2):=df(b,x,2)-ssq/j;
ssq;

% Let k1, k2 are constant.
b:=k1*x + k2;
ssq;

ss:=ss1;
k1:=0;
ss;

% Hence the differential constraint is:

phi;

% After solving for " u(x,t) "  from the differential
% constraint we obtain the solution of original equation
% the " BBM equation " is " u(x,t)=G((k2*t+x)/k2) ",
% where " G "  is arbitrary function.
```

```
end;
```

B.2.2 Case $b_x = 0$

```
 % THE BENJAMIN-BONA-MAHONY EQUATION %
% The reduce program for solving the " BBM equation "
%  by differential constraint.

depend phi, ux, q, x, t;
depend f, q, x, t, ux, uxx, uxxx, uxxxx;

% Define the following operators:
dx := df(f,x) + df(f,q)*ux + df(f,ux)*uxx + df(f,uxx)*uxxx
+ df(f,uxxx)*uxxxx$
dt := df(f,t) + df(f,q)*ut + df(f,ux)*uxt + df(f,uxx)*uxxt
+ df(f,uxxx)*uxxxt$

factor ux, uxx, uxxx, uxxxx;

% First, assume the existence of first
% order differential constraint:

ut := phi;

% Next we find the derivative by using previous assumption,
%  we obtain:

utt := sub(f = ut, dt);
uxt := sub(f = ut, dx);

uttt := sub(f=utt, dt)$
uxtt := sub(f=utt, dx)$
uxxt:= sub(f=uxt,dx)$

%  Substituting these derivatives in the following " BBM equation ",
%  obtaining:

ss:= ut - q*ux - uxxt$

j:=df(ss,uxxx);
% We consider the derivative " uxxx "  from previous equation.
uxxx:=uxxx-ss/j;
ss;

dx := df(f,x) + df(f,q)*ux + df(f,ux)*uxx+ df(f,uxx)*uxxx$
dt := df(f,t) + df(f,q)*ut + df(f,ux)*uxt + df(f,uxx)*uxxt$

% Consider the following consistency conditions:
ss1:= sub(f=uxxx,dt) -  sub(f=uxxt,dx)$
ss2:= sub(f=uxxt,dt) -  sub(f=uxtt,dx)$
ss3:= sub(f=uxtt,dt) -  sub(f=uttt,dx)$

ss:=ss1$
ssq:=df(ss,uxx,3);

% After solving for phi obtaining;
```

```
depend a, q, x, t;
depend b, q, x, t;
phi_ux:=df(phi,ux):=a*phi + b;
ssq;

ss:=ss1$
ssq:=df(ss,uxx,2);

% Consider case " a = 0 ".
a:=0;
ss:=ss1$
ssq:=df(ss,uxx,2);
j:=df(ssq,df(b,q));
df(b,q):=df(b,q)-ssq/j;
ssq;

% After solving for  " phi " we obtain:
depend c,q,x,t;
phi:=b*ux+c;
phi_ux-df(phi,ux);
ss:=ss1$
ssq:=df(ss,uxx,ux);
j:=df(ssq,df(c,q,2));
df(c,q,2):=df(c,q,2)-ssq/j;
ssq;

% After solving for " c2 " we obtain:
depend c1, x, t;
depend c2, x, t;
c:=c1*q+c2;
ssq;
ss:=ss1$
ssq:=df(ss,ux,q);
j:=df(ssq,c1);
c1:=c1-ssq/j;
ssq;

ss:=ss1$
ssq:=df(ss,ux);
j:=df(ssq,c2);
c2:=c2-ssq/j;
ssq;

ss:=ss1$
ssq:=df(ss,q,2);
j:=df(ssq,df(b,t,x));
df(b,t,x):=df(b,t,x) - ssq/j;
ssq;

ss:=ss1$
ssq:=df(ss,uxx);
j:=df(ssq,df(b,t,2));
df(b,t,2):=df(b,t,2)-ssq/j;
ssq;

ss:=ss1$
ssq:=df(ss,q);
j:=df(ssq,df(b,x,2));
df(b,x,2):=df(b,x,2)-ssq/j;
ssq;
```

```
% Consider the the following two constraints:
sss1:= df(b,x,2,t) - df(b,t,x,2);
sss2:= df(b,t,x,t) - df(b,t,2,x);

 % Consider case 2:
ss:=sss1;
df(b,x):=0;
ss;

% Solving for " b " to obtained:
depend b2, t;
b:=b2;
ssq;
c1;
c2;
df(b2,t,2);

% Solving for " b1 "  we let " h, k " are constants and obtaining:

b2:= -1/(h+k*t);

ss:=ss1;
ss:=ss2;
ss:=ss3;

% Hence the following differential constraint is obtained:

phi;

% After solving for " u(x,t) " we obtain
%" u(x,t)=e^(-kx)*g([(h + kt)*e^(-kx)]/k) ",
% where " g "  is arbitrary function.

end;
```

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